1)

$$f(x) = \begin{cases} Ae^{\lambda_1 x}, & x < 0 \\ Ae^{-\lambda_2 x}, & x > 0 \end{cases}$$

$$\int_{-\infty}^{0} Ae^{\lambda_1 x}\,dx + \int_{0}^{\infty} Ae^{-\lambda_2 x}\,dx = 1$$

$$A\left(\frac{1}{\lambda_1}\right) + A\left(\frac{1}{\lambda_2}\right) = 1$$

$$A\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) = 1$$

$$A = \frac{1}{\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)} = \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\int_{-\infty}^{0} xe^{\lambda_1 x}\,dx + \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\int_{0}^{\infty} xe^{-\lambda_2 x}\,dx$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}\right) = \left(\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}\right)\left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 \lambda_2^2}\right)$$

$$= \frac{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)}{(\lambda_1 + \lambda_2)(\lambda_1^2 \lambda_2^2)}$$

$$E(X) = \frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}\int_{-\infty}^{0} x^2 e^{\lambda_1 x}\,dx + \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}\int_{0}^{\infty} x^2 e^{-\lambda_2 x}\,dx$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}\left(\frac{2}{\lambda_1^3} + \frac{2}{\lambda_2^3}\right)$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}\left(\frac{2(\lambda_1^3 + \lambda_2^3)}{(\lambda_1 \lambda_2)^3}\right)$$

$$= \frac{2(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2)}{(\lambda_1\lambda_2)^2}$$

$$V(X) = \frac{2(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2)}{(\lambda_1\lambda_2)^2} - \frac{(\lambda_1 - \lambda_2)^2}{\;}$$

$$= \frac{2\lambda_1^2 - 2\lambda_1\lambda_2 + 2\lambda_2^2 - (\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2)}{(\lambda_1\lambda_2)^2}$$

$$\boxed{V(X) = \frac{\lambda_1^2 + \lambda_2^2}{(\lambda_1\lambda_2)^2}}$$

2) $X \sim Exp(\lambda) \qquad Y \sim U[a, b)$

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x > 0 \end{cases} \qquad f_Y(y) = \begin{cases} 0 & y < a \\ \frac{1}{b-a} & a < y < b \\ 0 & y > b \end{cases}$$

Let $Z = X + Y$

$$f_Z(z) = P(Z = z)$$
$$= P(X + Y = z)$$
$$= P(Y = z - x \mid X = x)$$
$$= \int_{-\infty}^{\infty} f_Y(z-x) f_X(x) dx$$
$$= \begin{cases} \frac{\lambda}{b-a} \int_a^b e^{-\lambda x} dx, & a < z - x < b \text{ and } x > 0 \\ 0, & x < 0 \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{e^{-\lambda a} - e^{-\lambda b}}{b-a}, & a + x < z < b + x, \ x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$F_Z(z) = \begin{cases} 0 & z < a+x \\ \displaystyle\int_{a+x}^{z} \frac{e^{-\lambda a} - e^{-\lambda b}}{b-a}\, dt, & a+x < z < b+x \\ \displaystyle\int_{a+x}^{b+x} \frac{e^{-\lambda a} - e^{-\lambda b}}{b-a}\, dt, & z > b+x \end{cases}$$

$$F_Z(z) = \begin{cases} 0 & , z < a+x \\ \dfrac{(z-a-x)(e^{-\lambda a} - e^{-\lambda b})}{b-a} & , a+x < z < b+x \\ e^{-\lambda a} - e^{-\lambda b} & , z > b+x \\ 0 & z < x < 0 \end{cases}$$

3)

a) $P(A \text{ wins}) = p + (1-p)(1-q)p + (1-p)(1-q)(1-p)(1-q)p + \ldots$

which is a geometric sequence with common ratio $r = (1-p)(1-q) < 1$ and $a_0 = p$. Thus $P(A \text{ wins}) = \frac{a_0}{1-r} = \frac{p}{1-((1-p)(1-q))} = \frac{p}{q+p-pq}$

$$P(B \text{ wins}) = (1-p)q + (1-p)(1-q)(1-p)q + (1-p)(1-q)(1-p)(1-q)(1-p)q + \ldots$$

$$= (1-p)\left(q + (1-q)(1-p)q + (1-q)(1-p)(1-q)(1-p)q + \ldots\right)$$

$$= (1-p)\left(\frac{q}{q+p-pq}\right)$$

b) For the game to be fair,

$$P(A \text{ wins}) = P(B \text{ wins})$$

or

$$\frac{p}{q+p-pq} = (1-p)\left(\frac{q}{q+p-pq}\right) \Rightarrow q = \frac{p}{(1-p)}$$ for a fair game

c) Let N be the number of flips in a game.

when A wins on n^{th} flip the sequence looks like

$$P(A \text{ wins on the } n^{th} \text{ toss}) = \left((1-p)(1-q)\right)^{\frac{n-1}{2}} p$$

and A can only win when n is odd

when B wins on n^{th} flip the sequence looks like

$$P(B \text{ wins on the } n^{th} \text{ toss}) = (1-p)\left((1-q)(1-p)\right)^{\frac{n-2}{2}} q$$

and B can only win when n is even

$$P(N=n) = P(A \text{ wins} \mid N=n \text{ is odd}) + P(B \text{ wins} \mid N=n \text{ is even})$$

$$= \begin{cases} \left((1-p)(1-q)\right)^{\frac{n-1}{2}} p & , n \text{ odd} \\ (1-p)\left((1-q)(1-p)\right)^{\frac{n-2}{2}} q & , n \text{ even} \end{cases}$$

$$E(N) = \sum_{n=1}^{\infty} n P(N=n)$$

$$= p \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} n \left((1-p)(1-q)\right)^{\frac{n-1}{2}} + q(1-p) \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} n \left((1-p)(1-q)\right)^{\frac{n-2}{2}}$$

Let $x = (1-p)(1-q)$, so the first sum is

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} n x^{\frac{n-1}{2}} = 1 + 3x + 5x^2 + 7x^3 + \dots$$

$$= \sum_{n=0}^{\infty} (2n+1) x^n = \sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty} n x^n = \frac{1}{1-x} + \frac{2x}{(x-1)^2}$$

$(|x|<1)$

And the second sum is

$$\sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} n x^{\frac{n-2}{2}} = 2 + 4x + 6x^2 + 8x^3 + \dots$$

$$= \sum_{n=1}^{\infty} 2n x^{n-1} = \frac{2}{(x-1)^2} \qquad (|x|<1)$$

Since $x = (1-p)(1-q) < 1$ indeed,

$$E(N) = p\left(\frac{1}{1-(1-p)(1-q)} + \frac{2(1-p)(1-q)}{((1-p)(1-q)-1)^2}\right) + q(1-p)\left(\frac{2}{((1-p)(1-q)-1)^2}\right)$$

$$\boxed{E(N) = \frac{p-2}{pq-p-q}}$$

(simplified with Wolfram Alpha)

Lets check this:

When $p=1$ we would expect $E(N)=1$ since A always wins on the first try

$$E(N) = \frac{1-2}{q-1-q} = \frac{-1}{-1} = 1 \quad \checkmark$$

When $q=1$ $p=0$, we would expect $E(N)=2$ since A wont win on the first toss and B will always win on the second toss

$$E(N) = \frac{0-2}{0 \cdot 1-0-1} = 2 \quad \checkmark$$

If $p=q=m$ then we would expect N have a geometric distributio with parameter m and thus an expected value of $\frac{1}{m}$.

$$E(N) = \frac{m-2}{m^2-m-m} = \frac{m-2}{m^2-2m} = \frac{m-2}{m(m-2)} = \frac{1}{m} \quad \checkmark$$