The Poisson Process

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1. A Stationary Poisson Process Having Constant Rate λ

Consider a sequence of random events such as the arrival of units at a shop, or customers arriving at a bank, or web-site hits on the internet. These events may be described by a counting function $N(t)$ (defined for all $0 \leq t$), which equals the *number of events* that occur in the closed interval $[0, t]$. We assume that $t = 0$ is the point at which the observations begin, whether or not an arrival occurs at that instant. Note that $N(t)$ is a *random variable* and the possible values of $N(t)$ (i.e., its range space) are the non-negative integers: $0, 1, 2, 3, \ldots$.

A counting process $N(t)$ is called a *Poisson process* with *mean rate* (per unit time) λ if the following assumptions are fulfilled.

A1: *Arrivals occur one at a time*: This implies that the probability of 2 or more arrivals in a very small (i.e., infinitesimal) time interval Δt is zero *compared to* the probability of 1 or less arrivals occurring in the same time interval Δt.

A2: $N(t)$ *has stationary increments*: The distribution of the numbers of arrivals between t and $t + \Delta t$ depends only on the length of the interval Δt and not on the starting point t. Thus, arrivals are completely random without rush or slack periods. In addition, the probability that a *single arrival* occurs in a small time interval Δt is proportional to Δt and given by $\lambda \Delta t$ where λ is the mean arrival rate (per unit time).

A3: $N(t)$ *has independent increments*: The numbers of arrivals during non-overlapping time intervals are independent random variables. Thus, a large or small number of arrivals in one time interval has no effect on the number of arrivals in subsequent time intervals. Future arrivals occur completely at random, independent of the number of arrivals in past time intervals.

Given that arrivals occur according to a Poisson process, (i.e., meeting the three assumptions A1, A2, and A3), let us derive an expression for the probability that n arrivals ($n = 0, 1, 2, 3, \ldots,$) occur in the time interval $[0, t]$. We shall denote this probability by $P_n(t)$, so that

$$P_n(t) = \Pr(N(t) = n)$$

for $n = 0, 1, 2, 3, \ldots$, and, of course

$$\sum_{n=0}^{\infty} P_n(t) = 1$$

for all time t. First let us consider computing $P_0(t)$ which is the probability that no arrivals occur in the time interval $[0, t]$. From the above equation we may write this as

$$P_0(t) = 1 - P_1(t) - \sum_{n=2}^{\infty} P_n(t).$$

For a small time interval $[0, \Delta t]$, this becomes

$$P_0(\Delta t) = 1 - P_1(\Delta t) - \sum_{n=2}^{\infty} P_n(\Delta t).$$

From assumption A2, we may say that $P_1(\Delta t) \simeq \lambda \Delta t$ for some arrival rate λ, and from assumption A1, we have $P_n(\Delta t) \simeq 0$ for $n \geq 2$. This leads to $P_0(\Delta t) \simeq 1 - \lambda \Delta t$, for small Δt. Then we may write that

$$P_0(t + \Delta t) = \Pr(N(t + \Delta t) = 0) = \Pr((N(t) = 0) \cap (N(t + \Delta t) - N(t) = 0))$$

which can be written as

$$P_0(t + \Delta t) = \Pr(N(t) = 0) \Pr(N(t + \Delta t) - N(t) = 0)$$

from assumption A3, since $[0, t]$ and $[t, t + \Delta t]$ are non-overlapping time intervals. Therefore we have

$$P_0(t + \Delta t) \simeq P_0(t) P_0(\Delta t) = P_0(t)(1 - \lambda \Delta t)$$

which yields

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} \simeq -\lambda P_0(t).$$

In the limit as $\Delta t \to 0$, this becomes

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} \equiv \frac{dP_0(t)}{dt} = -\lambda P_0(t).$$

Solving this differential equation for $P_0(t)$, we write

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) \qquad \text{or} \qquad \frac{dP_0(t)}{P_0(t)} = -\lambda dt$$

which leads to

$$\int \frac{dP_0(t)}{P_0(t)} = -\lambda \int dt \qquad \text{or} \qquad \ln(P_0(t)) = -\lambda t + C_1$$

or simply

$$P_0(t) = e^{C_1 - \lambda t} = e^{C_1} e^{-\lambda t} = A e^{-\lambda t}$$

with $A = e^{C_1}$. Now $P_n(0) = 0$ for $n \geq 1$, since no arrivals can occur if no time has elapsed and hence

$$P_0(0) = 1 - \sum_{n=1}^{\infty} P_n(0) = 1 - 0 = 1.$$

Therefore we may say that $P_0(0) = Ae^{-\lambda 0} = A = 1$, and so we finally find that

$$P_0(t) = e^{-\lambda t} = \Pr(N(t) = 0)$$

for all time t. Next we consider $P_n(t)$ for $n \geq 1$, and note that

$$P_n(t+\Delta t) = \Pr(N(t+\Delta t) = n)$$

which can occur if and only if one of the following $n+1$ events has occurred:

- $N(t) = 0$ and $N(t+\Delta t) - N(t) = n$, or
- $N(t) = 1$ and $N(t+\Delta t) - N(t) = n-1$, or
- $N(t) = 2$ and $N(t+\Delta t) - N(t) = n-2$, and so on until,

- $N(t) = n - 1$ and $N(t + \Delta t) - N(t) = 1$, or
- $N(t) = n$ and $N(t + \Delta t) - N(t) = 0$.

This leads to

$$
\begin{aligned}
P_n(t + \Delta t) &= \Pr(N(t + \Delta t) = n) \\
&= \sum_{x=0}^{n} \Pr((N(t) = x) \cap (N(t + \Delta t) - N(t) = n - x)) \\
&= \sum_{x=0}^{n} \Pr(N(t) = x) \Pr(N(t + \Delta t) - N(t) = n - x) \\
&\simeq \sum_{x=0}^{n} P_x(t) P_{n-x}(\Delta t) \\
&= \sum_{x=0}^{n-2} P_x(t) P_{n-x}(\Delta t) + P_{n-1}(t) P_1(t) + P_n(t) P_0(\Delta t) \\
&\simeq \sum_{x=0}^{n-2} P_x(t)(0) + P_{n-1}(t)(\lambda \Delta t) + P_n(t)(1 - \lambda \Delta t)
\end{aligned}
$$

since (assumption A1) $P_{n-x}(\Delta t) \simeq 0$ for small Δt and $x \leq n - 2$, $P_1(\Delta t) \simeq \lambda \Delta t$, and $P_0(\Delta t) \simeq 1 - \lambda \Delta t$. Therefore we see that

$$P_n(t + \Delta t) \simeq P_{n-1}(t)(\lambda \Delta t) + P_n(t)(1 - \lambda \Delta t)$$

which may be re-written as

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} \simeq \lambda P_{n-1}(t) - \lambda P_n(t).$$

In the limit as $\Delta t \to 0$, this yields

$$\lim_{\Delta t \to 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} \equiv \frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t)$$

for $n \geq 1$. If we write this as

$$\frac{dP_n(t)}{dt} + \lambda P_n(t) = \lambda P_{n-1}(t)$$

and multiply both sides by $e^{\lambda t}$, we get

$$e^{\lambda t} \frac{dP_n(t)}{dt} + \lambda e^{\lambda t} P_n(t) = \lambda e^{\lambda t} P_{n-1}(t)$$

or simply

$$\frac{d}{dt}(e^{\lambda t}P_n(t)) = \lambda e^{\lambda t}P_{n-1}(t),$$

which (after integrating) leads to

$$e^{\lambda t}P_n(t) = C_2 + \int_0^t \lambda e^{\lambda z}P_{n-1}(z)dz$$

or simply

$$e^{\lambda t}P_n(t) = \lambda \int_0^t e^{\lambda z}P_{n-1}(z)dz$$

since $P_n(0) = 0$ for $n \geq 1$ results in $C_2 = 0$. This leads to

$$P_n(t) = \lambda e^{-\lambda t}\int_0^t e^{\lambda z}P_{n-1}(z)dz$$

for $n = 1, 2, 3, \ldots$. Consequently we have

$$P_1(t) = \lambda e^{-\lambda t}\int_0^t e^{\lambda z}P_0(z)dz = \lambda e^{-\lambda t}\int_0^t e^{\lambda z}e^{-\lambda z}dz = \lambda e^{-\lambda t}\int_0^t dz$$

or $P_1(t) = (\lambda t)e^{-\lambda t}$. For $n = 2$, we have

$$P_2(t) = \lambda e^{-\lambda t}\int_0^t e^{\lambda z}P_1(z)dz = \lambda e^{-\lambda t}\int_0^t e^{\lambda z}(\lambda z)e^{-\lambda z}dz = \lambda^2 e^{-\lambda t}\int_0^t zdz$$

or simply

$$P_2(t) = \frac{e^{-\lambda t}(\lambda t)^2}{2}.$$

It is easy to see that in general,

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

for $n \geq 0$. Therefore we see that if arrivals occur according to a Poisson process, meeting the three assumptions A1, A2, and A3, the probability that $N(t)$ is equal to n (i.e., the probability that n arrivals occur in the time interval $[0, t]$) is given by

$$\Pr(N(t) = n) = P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

for $t \geq 0$ and $n = 0, 1, 2, 3, \ldots$. This is a *Poisson distribution* with parameter $\alpha = \lambda t$ which has mean and variance both given by

$$E(N(t)) = V(N(t)) = \alpha = \lambda t.$$

Note that for any times t and s with $s < t$, assumption A2 implies that the random variable $N(t) - N(s)$, representing the number of arrivals in the interval $[s, t]$, is also Poisson with parameter $\lambda(t - s)$ so that

$$\Pr(N(t) - N(s) = n) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^n}{n!}$$

for $n = 0, 1, 2, 3, \ldots$.

2. The Poisson Process Having Rate λ And The Exponential Distribution

Let Δt_1, Δt_2, Δt_3, ..., represent successive interarrival times so that Δt_1 is the time for the first arrival, $\Delta t_1 + \Delta t_2$ is the time for the second arrival, and so on. Since the first arrival occurs after time t if and only if there are no arrivals in the interval $[0, t]$, we see that

$$\Pr(\Delta t_1 > t) = \Pr(N(t) = 0) = e^{-\lambda t}$$

and so

$$\Pr(\Delta t_1 \leq t) = 1 - \Pr(\Delta t_1 > t) = 1 - e^{-\lambda t}$$

which is the cdf of an exponential distribution with parameter λ. More generally, *all interarrival times* Δt_1, Δt_2, Δt_3, ..., are *exponentially distributed and independent* with parameter λ, and hence mean $1/\lambda$.

Conversely, if the interarrival times Δt_1, Δt_2, Δt_3, ..., are exponentially distributed and independent with parameter λ, then $N(t)$ will be a Poisson process.

3. Properties of a Poisson Process

Random Splitting: Consider a Poisson process $N(t)$ having rate λ, as represented by the left side of figure 5.25 on page 192 of the text. Suppose that each time an event occurs it is classified as either type 1 (with probability p_1), type 2 (with probability p_2), ..., type k (with probability p_k), where, of course

$$p_1 + p_2 + p_3 + \cdots + p_k = 1,$$

then if $N_j(t)$ for $j = 1, 2, 3, \ldots, k$, represents the random variable of the number of type j events occurring in $[0, t]$, it is true that

$$N(t) = N_1(t) + N_2(t) + \cdots + N_k(t)$$

and $N_j(t)$ for $j = 1, 2, 3, \ldots, k$, are all Poisson processes having rates λp_j, respectively. Furthermore each of these is independent of each other. For example, if the arrival of customers in a bank is a Poisson process with parameter λ. We may break these customers up into disjoint classes (i.e., male and female, or customers younger than 30, between 31 and 50, and older than 51), and the separate classes will all form a Poisson process with rates given by λp, where p is the probability that a particular class exists. For example, $p = 1/2$ in the case of male and female classes.

Random Pooling: Consider k different independent Poisson processes $N_j(t)$ for $j = 1, 2, 3, \ldots, k$, having rates λ_j, respectively, as represented by the left side of figure 5.26 on page 193 of the text. If

$$N(t) = N_1(t) + N_2(t) + \cdots + N_k(t)$$

then it is true that $N(t)$ is a Poisson process with rate

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k.$$

4. A Nonstationary Poisson Process Having Rate Function $\lambda_{\text{NSPP}}(t)$

If assumption A1 and A3 above are maintained, but assumption A2 is relaxed, then we have a *Non-Stationary Poisson Process* (NSPP), which is characterized by a rate that is not constant, but rather is a function of time t, so that $\lambda = \lambda_{\text{NSPP}}(t)$, which is known as an *arrival-rate function*, and it gives *the arrival rate AT time t*. This is useful for simulations in which the arrival rate varies during the period of interest, including, meal times for restaurants, phone calls during business hours, and orders for pizza delivery around 6:00 PM.

The key to working with a NSPP is the *expected number of arrivals BY time t* (i.e., in the time interval from 0 to t) defined as

$$\Lambda_{\mathrm{NSPP}}(t) = \int_0^t \lambda_{\mathrm{NSPP}}(s)ds.$$

Note that when $\lambda_{\mathrm{NSPP}}(s) = \lambda$ (a constant), this gives $\Lambda_{\mathrm{NSPP}}(t) = \lambda t$, which is what we had earlier for a *Stationary Poisson Process* (SPP). To be useful as a arrival-rate function, $\lambda_{\mathrm{NSPP}}(t)$ must be nonnegative and integrable. Note also that

$$\bar{\lambda}_{\mathrm{NSPP}}(t) = \frac{1}{t}\int_0^t \lambda_{\mathrm{NSPP}}(s)ds = \frac{1}{t}\Lambda_{\mathrm{NSPP}}(t)$$

gives an *average arrival rate function* for the NSPP over the first t units of time. Dropping the subscript NSPP, we see that

$$\Lambda(t) = \int_0^t \lambda(s)ds = \bar{\lambda}t.$$

Now let $N(t)$ be the arrival function for SPP and let let $\mathcal{N}(t)$ be the arrival function for NSPP. The fundamental assumption for working with NSPPs is that

$$\Pr(\mathcal{N}(t) = n|\Lambda(t)) = \Pr(N(t) = n|\bar{\lambda}) \qquad \text{with} \qquad \bar{\lambda} = \frac{1}{t}\int_0^t \lambda(s)ds,$$

and since

$$\Pr(N(t) = n|\bar{\lambda}) = \frac{e^{-\bar{\lambda}t}(\bar{\lambda}t)^n}{n!}$$

we see that

$$\Pr(\mathcal{N}(t) = n|\Lambda(t)) = \frac{e^{-\bar{\lambda}t}(\bar{\lambda}t)^n}{n!} \qquad \text{with} \qquad \bar{\lambda} = \frac{1}{t}\int_0^t \lambda(s)ds$$

or just

$$\Pr(\mathcal{N}(t) = n|\Lambda(t)) = \frac{e^{-\Lambda(t)}(\Lambda(t))^n}{n!} \qquad \text{with} \qquad \Lambda(t) = \int_0^t \lambda(s)ds.$$

If the time interval is $[a, b]$ instead of $[0, t]$, then we have

$$\Pr(\mathcal{N}(b) - \mathcal{N}(a) = n|\Lambda(t)) = \Pr(N(b) - N(a) = n|\bar{\lambda})$$

with

$$\bar{\lambda} = \frac{1}{b-a}\int_a^b \lambda(s)ds,$$

and since

$$\Pr(N(b) - N(a) = n|\bar{\lambda}) = \frac{e^{-\bar{\lambda}(b-a)}(\bar{\lambda}(b-a))^n}{n!}$$

we have

$$\Pr(\mathcal{N}(b) - \mathcal{N}(a) = n|\Lambda) = \frac{e^{-\Lambda}\Lambda^n}{n!} \qquad \text{with} \qquad \Lambda = \int_a^b \lambda(s)ds$$

Let us illustrate this idea with an example.

Example - A Non-Stationary Poisson Process

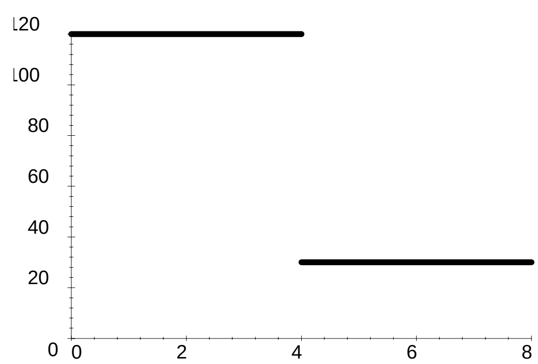
Suppose that arrivals to a post office occur at a rate of 2 per minute from 8:00 AM until 12:00 PM, then drop to 1 every 2 minutes until the post office closes at 4:00 PM. Determine the probability distribution on the number of arrivals between 11:00 AM and 2:00 PM. To solve this we let $t = 0$ correspond to 8:00 AM. Then the situation could be modeled as a NSPP with rate function

$$\lambda(t) = \begin{cases} 2 \text{ customers/minute}, & \text{for} \quad 0 \text{ hours} \le t < 4 \text{ hours} \\ \\ 1/2 \text{ customers/minute}, & \text{for} \quad 4 \text{ hours} \le t \le 8 \text{ hours} \end{cases}$$

where time t is in units of *hours*. The first thing we must do is get the units to mesh properly. Therefore, let us change $\lambda(t)$ to be on a per hour rate and so we write

$$\lambda(t) = \begin{cases} 120 \text{ customers/hour}, & \text{for} \quad 0 \text{ hours} \le t < 4 \text{ hours} \\ \\ 30 \text{ customers/hour}, & \text{for} \quad 4 \text{ hours} \le t \le 8 \text{ hours} \end{cases}$$

A plot of this is shown in the figure below.



Plot of $\lambda(t)$ versus time t (in hours)

This says that the expected number of arrivals by time t (in hours) is

$$\Lambda(t) = \int_0^t \lambda(s)ds = \int_0^t 120ds = 120t.$$

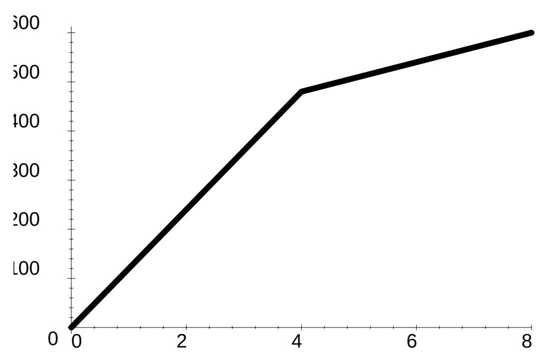
for 0 hours $\leq t \leq$ 4 hours, and

$$\begin{aligned}\Lambda(t) &= \int_0^t \lambda(s)ds = \int_0^4 \lambda(s)ds + \int_4^t \lambda(s)ds \\ &= \int_0^4 120ds + \int_4^t 30ds = 360 + 30t\end{aligned}$$

for 4 hours $\leq t \leq$ 8 hours, and so

$$\Lambda(t) = \begin{cases} 120t, & \text{for} \quad 0 \text{ hours} \leq t < 4 \text{ hours} \\ 360 + 30t, & \text{for} \quad 4 \text{ hours} \leq t \leq 8 \text{ hours} \end{cases}.$$

A plot of this is shown in the figure below.

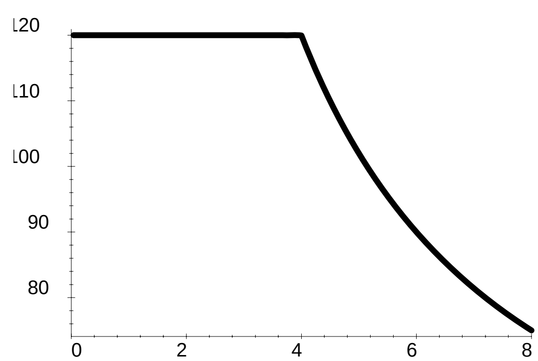


Plot of $\Lambda(t)$ versus time t (in hours)

We also note that

$$\bar{\lambda}(t) = \frac{1}{t}\Lambda(t) = \begin{cases} 120, & \text{for} \quad 0 \text{ hours} \leq t < 4 \text{ hours} \\ \\ 30 + 360/t, & \text{for} \quad 4 \text{ hours} \leq t \leq 8 \text{ hours} \end{cases}$$

and a plot of this is shown in the figure below.



Plot of $\bar{\lambda}(t)$ versus time t (in hours)

Now since 2:00 PM and 11:00 AM correspond to $t = 6$ hours and $t = 3$ hours, respectively, we have

$$\Lambda = \int_3^6 \lambda(s)ds = \int_3^4 \lambda(s)ds + \int_4^6 \lambda(s)ds$$

or

$$\Lambda = \int_3^4 120 ds + \int_4^6 30 ds = 180 \text{ customers},$$

so that

$$\Pr(\mathcal{N}(6) - \mathcal{N}(3) = n) = \frac{e^{-180}(180)^n}{n!}$$

for $n = 0, 1, 2, 3, ...$, which is a Poisson Distribution with $\alpha = 180$.