2b)

Suppose R = # of rolls made by seller at the beginning of the day

Suppose P = profit per roll sold

Suppose L = loss per roll discarded

Let X = Demand for rolls from customers. $X \sim U[0,17)$

We want $P_T(X,R)$ which depends on whether $X \geq R$ or $X \leq R$

Case 1: $X \leq R$ (Sell fewer rolls than you made)

So you sell X rolls and make profit XP on them and have $R-X$ left over that you must discard each costing you in total $(R-X)\cdot L$

$$P_T(X,R) = XP - (R-X)L$$

Case 2: $X \geq R$ (Sell out of rolls)

So you sell all of R rolls you made making profit RP and you dont have any rolls to discard. However, you practically lose out on the profit that you could have made with better planning, $(X-R)P$.

$$P_T(X,R) = RP - (X-R)P$$

Combining both equations we get

$$P_T(X,R) = P\cdot\min(X,R) - L\cdot\max(0,R-X) - P\cdot\max(0,X-R)$$

Goal: To determine R so that average daily profit is as large as possible.

$$P_T(R) = \int_0^\infty P_T(x,R)\cdot f(x)\,dx$$

$$= \int_0^\infty \left(P\min(x,R) - L\max(0,R-x) - P\max(0,x-R)\right)f(x)\,dx$$

$$= \int_0^\infty P\min(x,R)f(x)\,dx \qquad \text{(First term)}$$

$$- \int_0^\infty L\max(0,R-x)f(x)\,dx \qquad \text{(Second term)}$$

$$- \int_0^\infty P\max(0,x-R)f(x)\,dx \qquad \text{(Third term)}$$

The first term: $\int_0^\infty P\min(x,R)\, f(x)dx$

$$= P\int_0^R x\cdot f(x)dx + PR\int_R^\infty f(x)dx$$

$$= P\int_0^R x f(x)dx + PR\left(1 - \int_0^R f(x)dx\right)$$

$$= P\int_0^R x f(x)dx - PR\int_0^R f(x)dx + PR \qquad \text{(First term)}$$

The second term: $-\int_0^\infty L\max(0,R-x)\, f(x)dx$

$$= -\int_0^R L\cdot\max(0,R-x)f(x)dx - \int_R^\infty L\cdot\max(0,R-x)f(x)dx$$

$$= -L\int_0^R (R-x)f(x)dx - L\int_R^\infty (0)\cdot f(x)dx = L\int_0^R (x-R)f(x)dx$$

$$= L\int_0^R x f(x)dx - LR\int_0^R f(x)dx \qquad \text{(Second term)}$$

The third term: $-\int_0^\infty P\max(0, x-R) f(x)\,dx$

$$= -\int_0^R P\max(0, x-R) f(x)\,dx - \int_R^\infty P\max(0, x-R) f(x)\,dx$$

$$= -P\int_0^R (0) f(x)\,dx - P\int_R^\infty (x-R) f(x)\,dx = P\int_R^\infty (R-x) f(x)\,dx$$

$$= PR\int_R^\infty f(x)\,dx - P\int_R^\infty x f(x)\,dx$$

$$= PR\left(1-\int_0^R f(x)\,dx\right) - P\left(E[X] - \int_0^R x f(x)\,dx\right)$$

$$= P\int_0^R x f(x)\,dx - PR\int_0^R f(x)\,dx + PR - P\cdot E[X] \qquad \text{(Third term)}$$

Putting everything back together we have

$$P_T(R) = 2P\int_0^R x f(x)dx + L\int_0^R x f(x)dx$$

$$-LR\int_0^R f(x)dx - 2PR\int_0^R f(x)dx + (2PR - P\cdot E[X])$$

$$= \int_0^R (2Px + Lx - LR - 2PR) f(x)dx + (2PR - P\cdot E[X])$$

$$= \int_0^R (2P(x-R) + L(x-R)) f(x)dx + P(2R - E[X])$$

$$= P(2R - E[X]) + (2P+L)\int_0^R (x-R) f(x)dx$$

$$\boxed{P_T(R) = P(2R - E[X]) + (2P+L)\int_0^R (x-R) f(x)dx}$$

Now we want to find R such that $\frac{dP_T(R)}{dR} = 0$

$$\frac{dP_T(R)}{dR} = \frac{d}{dR}\left(P(2R - E[X]) + (2P+L)\int_0^R (x-R)f(x)dx\right)$$

$$= 2P + (2P+L)\frac{d}{dR}\left[\int_0^R (x-R)f(x)dx\right]$$

$$= 2P + (2P+L)\left[(R-R)f(R)\frac{dR}{dR} - (0-R)f(0)\frac{d0}{dR} + \int_0^R \frac{\partial(x-R)}{\partial R}f(x)dx\right]$$

$$= 2P - (2P+L)\int_0^R f(x)dx$$

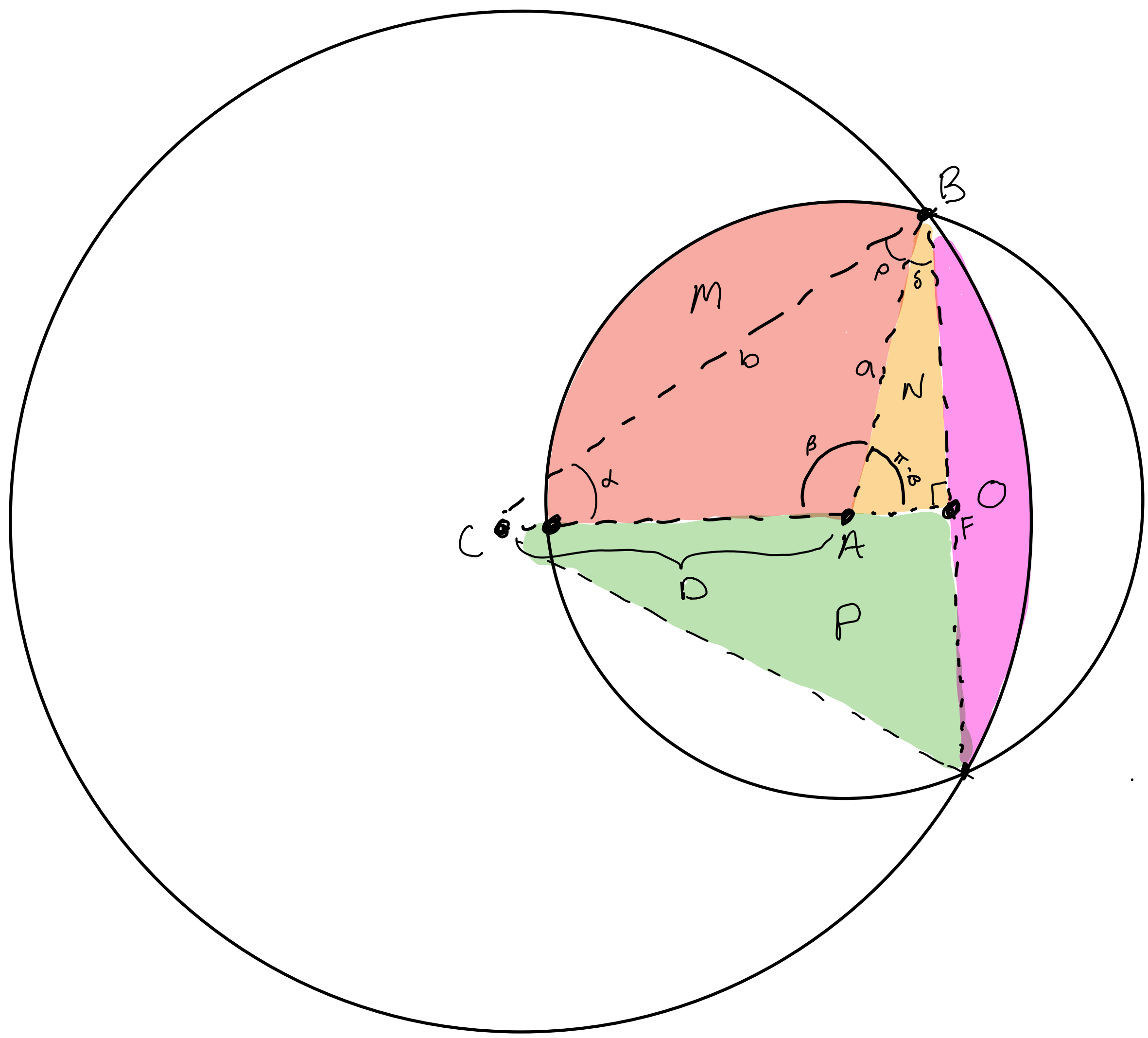
Solve for R in

$$2P - (2P+L)F(R) = 0$$

where $F(R) = \int_0^R f(x)dx$, the cdf evaluated at $x = R$

$$\boxed{F(R) = \frac{2P}{2P+L}}$$

5)



$$\text{Area}(M) = \frac{1}{2}a^2\beta$$ for β in radians

$$\text{Area}(N) = \frac{1}{2}|AF| \cdot |FB|$$

$$\sin(\pi-\beta) = \frac{|FB|}{a} \Rightarrow |FB| = a\sin(\pi-\beta)$$

$$\cos(\pi-\beta) = \frac{|AF|}{a} \Rightarrow |AF| = a\cos(\pi-\beta)$$

$$\text{Area}(N) = \frac{1}{2}a^2\sin(\pi-\beta)\cos(\pi-\beta)$$

$$2\,\text{Area}(P) + \text{Area}(O) = \frac{1}{2} b^2 (2\alpha)$$

where $\text{Area}(P) = \frac{1}{2}\left(D + |AF|\right)|FB| = \dfrac{a\sin(\pi-\beta)\left(D + a\cos(\pi-\beta)\right)}{2}$

$$\boxed{\text{Area}(O) = b^2\alpha - a\sin(\pi-\beta)\left(D + a\cos(\pi-\beta)\right)}$$

$$\begin{aligned}
\text{Area}(\text{Overlap}) &= 2\left(\text{Area}(M) + \text{Area}(N)\right) - \text{Area}(O)\\
&= 2\left(\tfrac{1}{2}a^2\beta + \tfrac{1}{2}a^2\sin(\pi-\beta)\cos(\pi-\beta)\right)\\
&\quad - b^2\alpha - a\sin(\pi-\beta)\left(D + a\cos(\pi-\beta)\right)\\
&= \beta a^2 + \cancel{a^2\sin(\pi-\beta)\cos(\pi-\beta)}\\
&\quad - aD\sin(\pi-\beta) - \cancel{a^2\sin(\pi-\beta)\cos(\pi-\beta)} - \alpha b^2\\
&= \beta a^2 - aD\sin(\pi-\beta) - \alpha b^2\\
&= \beta a^2 - aD\sin(\beta) - \alpha b^2
\end{aligned}$$

Law of sines:

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} \implies a\sin(\beta) = b\sin(\alpha)$$

Thus

$$\boxed{\text{Area}(\text{Overlap}) = \beta a^2 - bD\sin(\alpha) - \alpha b^2}$$