1)

$$f_X(x) = \begin{cases} 0 & , x < 0 \\ A(e^{-\alpha x} + e^{-\beta x}) & , x > 0 \end{cases}$$

1a) We start with the fact $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} A(e^{-\alpha x} + e^{-\beta x})\,dx = 1$$

$$A\left(\int_{0}^{\infty} e^{-\alpha x}\,dx + \int_{0}^{\infty} e^{-\beta x}\,dx\right) = 1$$

We find $\boxed{A = \frac{1}{\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)} = \frac{\alpha\beta}{\beta + \alpha}}$

1b) $$M_X(t) = E(e^{Xt}) = \int_{-\infty}^{\infty} e^{xt} f_X(x)\,dx$$

$$= \int_{-\infty}^{0} e^{xt}\cdot(0)\,dx + \int_{0}^{\infty} e^{xt}\cdot\left(\frac{\alpha\beta}{\beta+\alpha}\right)\left(e^{-\alpha x} + e^{-\beta x}\right)dx$$

$$= \left(\frac{\alpha\beta}{\beta+\alpha}\right)\left(\int_{0}^{\infty} e^{x(t-\alpha)}\,dx + \int_{0}^{\infty} e^{x(t-\beta)}\,dx\right)$$

$$= \left(\frac{\alpha\beta}{\beta+\alpha}\right)\left(-\frac{1}{t-\alpha} - \frac{1}{t-\beta}\right)$$

$$M_X(t) = -\left(\frac{\alpha\beta}{\beta+\alpha}\right)\left(\frac{1}{t-\alpha} + \frac{1}{t-\beta}\right)$$

1c) $$M_X'(t) = -\left(\frac{\alpha\beta}{\beta+\alpha}\right)\left(-\frac{1}{(t-\alpha)^2} - \frac{1}{(t-\beta)^2}\right)$$

$$= \left(\frac{\alpha\beta}{\beta+\alpha}\right)\left(\frac{1}{(t-\alpha)^2} + \frac{1}{(t-\beta)^2}\right)$$

$$E(X) = M_X'(0) = \frac{\alpha\beta}{\beta+\alpha}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)$$

$$= \frac{\alpha\beta}{\beta+\alpha}\left(\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}\right)$$

$$\boxed{E(X) = \frac{\alpha+\beta}{\alpha\beta}}$$

$$M_x''(t) = \frac{d}{dt}\left[\left(\frac{\alpha\beta}{\beta+\alpha}\right)\left(\frac{1}{(t-\alpha)^2} + \frac{1}{(t-\beta)^2}\right)\right]$$

$$= \frac{-2\alpha\beta}{\beta+\alpha}\left(\frac{1}{(t-\alpha)^3} + \frac{1}{(t-\beta)^3}\right)$$

$$E(X^2) = M_x''(0) = \frac{2\alpha\beta}{\beta+\alpha}\left(\frac{1}{\alpha^3} + \frac{1}{\beta^3}\right)$$

$$= \frac{2\alpha\beta}{\beta+\alpha}\left(\frac{\alpha^3+\beta^3}{(\alpha\beta)^3}\right) = \frac{2(\alpha^2-\alpha\beta+\beta^2}{(\alpha\beta)^2}$$

$$V(X) = E(X^2) - E(X)^2$$

$$= \frac{2(\alpha^2-\alpha\beta+\beta^2)}{(\alpha\beta)^2} - \left(\frac{\alpha+\beta}{\alpha\beta}\right)^2$$

$$= \frac{2\alpha^2-2\alpha\beta+2\beta^2-\alpha^2-2\alpha\beta-\beta^2}{\alpha\beta^2}$$

$$\boxed{V(X) = \frac{\alpha^2-4\alpha\beta+\beta^2}{(\alpha\beta)^2}}$$

2)

$$\lambda(t) = \begin{cases} 6t, & 0 \le t \le 4 \\ 48-6t, & 4 \le t \le 8 \end{cases}$$

$$P(N(t) > 60 \mid 3 < t < 6)$$

$$= 1 - P(N(t) \le 60 \mid 3 < t < 6)$$

$$= 1 - \sum_{n=0}^{60} \frac{e^{-\int_3^6 \lambda(t)\,dt} \left(\int_3^6 \lambda(t)\,dt\right)^n}{n!}$$

$$= 1 - \sum_{n=0}^{60} \frac{e^{-\left(\int_3^4 6t\,dt + \int_4^6 48-6t\,dt\right)} \left(\int_3^4 6t\,dt + \int_4^6 48-6t\,dt\right)^n}{n!}$$

$$= 1 - \sum_{n=0}^{60} \frac{e^{-57}(57)^n}{n!}$$

$$= \boxed{0.315309}$$

3)

3a) $$P(0) = \frac{(6R)^2 - \pi(3R)^2}{(6R)^2} = \frac{36R^2 - 9\pi R^2}{36R^2} = 1 - \frac{\pi}{4}$$

$$P(10) = \frac{\pi(3R)^2 - \pi(2R)^2}{(6R)^2} = \frac{5\pi R^2}{36R^2} = \frac{5\pi}{36}$$

$$P(20) = \frac{\pi(2R)^3 - \pi R^2}{(6R)^2} = \frac{3\pi R^2}{36R^2} = \frac{\pi}{12}$$

$$P(30) = \frac{\pi R^2}{(6R)^2} = \frac{\pi}{36}$$

Score(S)	0	10	20	30
P(S)	$1-\frac{\pi}{4}$	$\frac{5\pi}{36}$	$\frac{\pi}{12}$	$\frac{\pi}{36}$

3b) Let S_1 and S_2 be the scores of the first and second throws. We want to compute $P(S_1+S_2=s)$ for

$s = \{0, 10, 20, 30, 40, 50, 60\}$

$$P(S_1+S_2=0) = P(S_1=0)P(S_2=0) = \left(1-\frac{\pi}{4}\right)^2 = 0.0461$$

$$P(S_1+S_2=10) = P(S_1=0)P(S_2=10) + P(S_1=10)P(S_2=0)$$
$$= 2\left(1-\frac{\pi}{4}\right)\left(\frac{5\pi}{36}\right) = 0.1873$$

$$P(S_1+S_2=20) = P(S_1=0)P(S_2=20) + P(S_1=20)P(S_2=0)$$
$$+P(S_1=10)P(S_2=10)$$
$$= 2\left(1-\frac{\pi}{4}\right)\left(\frac{\pi}{12}\right) + \left(\frac{5\pi}{36}\right)^2 = 0.3021$$

$$P(S_1+S_2=30) = P(S_1=0)P(S_2=30) + P(S_1=30)P(S_2=0)$$
$$+P(S_1=10)P(S_2=20) + P(S_1=20)P(S_2=10)$$
$$= 2\left(1-\frac{\pi}{4}\right)\left(\frac{\pi}{36}\right) + 2\left(\frac{5\pi}{36}\right)\left(\frac{\pi}{12}\right) = 0.2659$$

$$P(S_1+S_2=40) = P(S_1=30)P(S_2=10) + P(S_1=10)P(S_2=30)$$
$$+P(S_1=20)P(S_2=20)$$
$$= 2\left(\frac{\pi}{36}\right)\left(\frac{5\pi}{36}\right) + \left(\frac{\pi}{12}\right)^2 = 0.1447$$

$$P(S_1+S_2=50) = P(S_1=30)P(S_2=20) + P(S_1=20)P(S_2=30)$$
$$= 2\left(\frac{\pi}{36}\right)\left(\frac{\pi}{12}\right) = 0.0457$$

$$P(S_1+S_2=60) = P(S_1=30)P(S_2=30) = \left(\frac{\pi}{36}\right)^2 = 0.0076$$

4) $X_j \sim U[0,1)$ for $j = 1,2,3,4$

$$X = \max(X_1, X_2) \qquad Y = \min(X_3, X_4)$$

$$F_X(x) = F_{X_1}(x) F_{X_2}(x) \quad F_Y(y) = 1 - (1 - F_{X_3}(y))(1 - F_{X_4}(y))$$

$$f_Y(y) = \frac{d}{dy}\left[1 - (1 - F_{X_3}(y))(1 - F_{X_4}(y))\right] = \begin{cases} 0 & y < 0 \\ 2(1-y) & 0 < y < 1 \\ 0 & y > 0 \end{cases}$$

$$f_X(x) = \frac{d}{dx}\left[F_{X_1}(x) F_{X_2}(x)\right] = \begin{cases} 0 & x < 0 \\ 2x & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

Let $Z = X + Y$

$$f_Z(z) = P(Z = z) = P(X + Y = z)$$

$$= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy = \int_0^1 f_X(z-y) f_Y(y)\, dy$$

$$= \int_0^1 2(z-y) \cdot 2(1-y)\, dy$$

$$f_Z(z) = 2z - \frac{2}{3} \quad \text{for } 0 < z < 2, \ 0 \text{ otherwise}$$

I know this is wrong but can't figure out why.

4b) $F_Z(z) = \begin{cases} 0 & z < 0 \\ z^2 - \frac{2}{3}z & 0 < z < 2 \\ 1 & z > 2 \end{cases}$ ← I know this is wrong but this is what I would compute if it were right

Interval	Probability
$[0, 0.2)$	$F_Z(0.2) - F_Z(0) =$
$[0.2, 0.4)$	$F_Z(0.4) - F_Z(0.2) =$
$[0.4, 0.6)$	$F_Z(0.6) - F_Z(0.4) =$
$[0.6, 0.8)$	$F_Z(0.8) - F_Z(0.6) =$
$[0.8, 1.0)$	$F_Z(1) - F_Z(0.8) =$
$[1, 1.2)$	$F_Z(1.2) - F_Z(1) =$
$[1.2, 1.4)$	$F_Z(1.4) - F_Z(1.2) =$
$[1.4, 1.6)$	$F_Z(1.6) - F_Z(1.4) =$
$[1.6, 1.8)$	$F_Z(1.8) - F_Z(1.6) =$
$[1.8, 2)$	$F_Z(2) - F_Z(1.8) =$

4c) Since my pdf isn't correct I will instead compute $E[Z]$ by linearity of expectaion.

$$E[Z] = E[X+Y] = E[X] + E[Y]$$

$$= \int_0^1 x f_X(x)dx + \int_0^1 y f_Y(y)dy$$

$$= \int_0^1 2x^2 dx + \int_0^1 2y(1-y)$$

$$= \frac{2}{3} + \frac{1}{3}$$

$$\boxed{E[Z] = 1}$$

$$V[Z] = V[X+Y] = V[X] + V[Y]$$ since X, Y are ind.

$$= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2$$

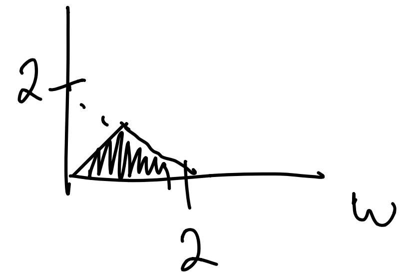
$$= \int_0^1 2x^3 dx - \left(\frac{2}{3}\right)^2 + \int_0^1 2y^2(1-y)dy - \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{2} + \frac{1}{6} - \frac{4}{9} - \frac{1}{9}$$

$$\boxed{V[Z] = \frac{1}{9}}$$

4d) Let $W = X + Y$ → notice x and y independent

$$f_W(w) = P(X + Y = w)$$



$$= \begin{cases} 0 & w < 0 \\ w & 0 < w < 1 \\ 2 - w & 1 < w < 2 \\ 0 & w > 2 \end{cases}$$

sum of two independent uniform variables

$$E[W] = E[X+Y]$$
$$= E[X] + E[Y]$$
$$= \int_0^1 x f_X(x)\,dx + \int_0^1 y f_Y(y)\,dy$$
$$= \frac{2}{3} + \frac{1}{3}$$
$$E[W] = 1$$

$$V[W] = E[W^2] - E[W]^2 = E[(X+Y)^2] - 1$$
$$E[(X+Y)^2] = E[X^2 + 2XY + Y^2]$$
$$= E[X^2] + 2E[XY] + E[Y^2]$$
$$= \int_0^1 2x^3\,dx + \int_0^1 2y^2(1-y)\,dy + 2\left(\frac{1}{4}\right)$$
$$= \frac{1}{2} + \frac{1}{6} + \frac{1}{2} = \frac{7}{6}$$
$$V[W] = \frac{7}{6} - 1 = \frac{1}{6}$$

5) Let T be the time it takes to grade each problem on an exam and $T \sim \exp(\lambda = 10 \text{mins})$

a) $P(4T > 50) = P\left(T > \frac{50}{4}\right) = 1 - P\left(T \le \frac{25}{2}\right)$

$$= 1 - \left(1 - e^{-10\left(\frac{25}{2}\right)}\right) = \boxed{e^{-125}}$$

b) Let t be the travel time from home to office

Let $X = \dfrac{t - 40}{5}$

$$P\left(X \le \frac{t-40}{5}\right) = 0.975$$

$$\Phi\left(\frac{t-40}{5}\right) = 0.975$$

$$\int_{-\infty}^{\frac{t-40}{5}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0.975$$

$t = 49.8$ mins (49 mins 48 seconds)

So you should leave no later than $\boxed{8:10:12 \text{ AM}}$