1) $E(S) = \int_0^3 \frac{2s^2(3-s)}{9} = \frac{3}{2} \Rightarrow \mu = \frac{2}{3}$, $\lambda = \frac{3}{2}$

$E(S^2) = \int_0^3 \frac{2s^3(3-s)}{9} \Rightarrow \sigma^2 = \frac{9}{20}$

a) We need $\rho = \frac{\lambda}{c\mu} < 1$ so $c > \frac{\lambda}{\mu} \Rightarrow c > \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

So the smallest number of servers must be at least 3.

b) $\rho = \frac{\lambda}{3\mu} \Rightarrow \rho = \frac{3}{4}$

$$P_0 = \left(\sum_{n=0}^{2} \frac{(9/4)^n}{n!} + \frac{(9/4)^3}{3!\left(\frac{1}{4}\right)}\right)^{-1} = \frac{8}{107}$$

$$L_{Q,M/M/3} = \frac{(9/4)^4 (8/107)}{3(3!)(1/4)^2} = 1.703$$

$$W_{Q,M/M/3} = \frac{1}{\lambda} L_{Q,M/M/3} = 1.1355 \text{ min}$$

$$L_Q = 1.703 \cdot \left(\frac{1+\mu^2\sigma^2}{2}\right) = 1.703 \cdot \frac{3}{5} = 1.0218 \text{ customers}$$

$$W_Q = 1.1355\left(\frac{3}{5}\right) = 0.6813 \text{ mins}$$

$$W = W_Q + \frac{1}{\mu} = 2.181 \text{ mins} \qquad L = \lambda W = 3.272 \text{ customers}$$

2) We want to compute $\Pr(X \leq 336)$ where $X = \max(X_1, X_2) + X_3 + X_4$

$$\Pr(X \leq 336) = \Pr(\max(X_1, X_2) + X_3 + X_4 \leq 336)$$

$$= \Pr(X_1 + X_3 + X_4 \leq 336)\Pr(X_1 > X_2)$$
$$+ \Pr(X_2 + X_3 + X_4 \leq 336)\Pr(X_2 > X_1)$$

$$= \frac{1}{2}\Pr(X_1 + X_3 + X_4 \leq 336)$$
$$+ \frac{1}{2}\Pr(X_2 + X_3 + X_4 \leq 336)$$

Since $X_1 + X_2 + X_4$ has the same distribution of $X_2 + X_3 + X_4$

$$P(X \leq 336) = \Pr(X_1 + X_3 + X_4 \leq 336)$$

$$= \Pr(S \leq 336) \text{ where } S = X_1 + X_3 + X_4 \sim Er(k=3, \theta = \tfrac{\lambda}{3})$$

$$= 1 - e^{-336 \cdot \lambda} \cdot \sum_{i=0}^{2} \frac{(336 \cdot \lambda)^i}{i!}$$

For $\lambda = \frac{1}{150}$,

$$P(X \leq 336) = 1 - e^{-336/150} \cdot \sum_{i=0}^{2} \frac{(336/150)^i}{i!} = 0.388$$

correct answer is 0.26 (from monte carlo)
so not sure what is wrong here

3)

$$f(x) = \begin{cases} \dfrac{e^x}{3}, & x < 0 \\ \dfrac{1}{3}, & 0 < x < 1 \\ \dfrac{1}{3x^2}, & x > 1 \end{cases}$$

$$F(x) = \begin{cases} \frac{1}{3}\int_{-\infty}^{x} e^t\,dt = \frac{1}{3}e^x & x < 0 \\ \frac{1}{3}\left(\int_{-\infty}^{0} e^t\,dt + \int_{0}^{x} 1\,dt\right) = \frac{x+1}{3} & 0 < x < 1 \\ \frac{1}{3}\left(\int_{-\infty}^{0} e^t\,dt + \int_{0}^{1} dt + \int_{1}^{x} \frac{1}{t^2}\,dt\right) = 1 - \frac{1}{3x}, & x > 1 \end{cases}$$

$$F(x) = \begin{cases} \frac{1}{3}e^x, & x < 0 \\ \dfrac{x+1}{3}, & 0 < x < 1 \\ 1 - \frac{1}{3x}, & x > 1 \end{cases}$$

$$F(x) = R \Rightarrow X = \begin{cases} \ln(3R), & R < \frac{1}{3} \\ 3R - 1, & \frac{1}{3} < R < \frac{2}{3} \\ \dfrac{1}{3(1-R)}, & R > \frac{2}{3} \end{cases}$$

$$\Pr(a < X < b) = F(b) - F(a)$$

$$= \begin{cases} \frac{1}{3}e^{b}, & b < 0 \\ \frac{1}{3}(b+1), & 0 < b < 1 \\ \frac{1}{3}\left(3 - \frac{1}{b}\right), & b > 1 \end{cases} - \begin{cases} \frac{1}{3}e^{a}, & a < 0 \\ \frac{1}{3}(a+1), & 0 < a < 1 \\ \frac{1}{3}\left(3 - \frac{1}{a}\right), & a > 1 \end{cases}$$

$$\boxed{\Pr(a < X < b) = \begin{cases} \frac{1}{3}\left(e^{b} - e^{a}\right) & a < 0,\ b < 0 \quad a < b \\ \frac{1}{3}\left(b+1-e^{a}\right) & a < 0,\ 0 < b < 1 \quad a < b \\ \frac{1}{3}(b-a) & 0 < a < 1,\ 0 < b < 1 \quad a < b \\ \frac{1}{3}\left(3 - \frac{1}{b} - e^{a}\right) & a < 0,\ b > 1 \quad a < b \\ \frac{1}{3}\left(2 - a - \frac{1}{b}\right) & 0 < a < 1,\ b > 1,\ a < b \\ \frac{1}{3}\left(\frac{1}{a} - \frac{1}{b}\right) & a > 1,\ b > 1 \quad a < b \end{cases}}$$

4)

$$f_Y(y) = P(Y = y) = P(X^2 + 2X = y)$$

$$= P(X^2 + 2X - y = 0)$$

$$= P\left(X = -\left(\sqrt{y+1} + 1\right)\right) + P\left(X = \sqrt{y+1} - 1\right)$$

$$= f_X\left(-\left(\sqrt{y+1} + 1\right)\right) + f_X\left(\sqrt{y+1} - 1\right)$$

$$= \begin{cases} 0 & -\left(\sqrt{y+1} + 1\right) < 0 \\ \lambda e^{\lambda\left(\sqrt{y+1} + 1\right)} & -\left(\sqrt{y+1} + 1\right) > 0 \end{cases} + \begin{cases} 0 & \sqrt{y+1} - 1 < 0 \\ \lambda e^{-\lambda\left(\sqrt{y+1} - 1\right)} & \sqrt{y+1} - 1 > 0 \end{cases}$$

$$= \begin{cases} 0 & y > 0 \\ \lambda e^{\lambda\left(\sqrt{y+1} + 1\right)} & y \text{ imaginary} \end{cases} + \begin{cases} 0 & y \text{ imaginary} \\ \lambda e^{\lambda\left(1 - \sqrt{y+1}\right)} & y > 0 \end{cases}$$

$$= \boxed{\begin{cases} 0 & y < 0 \\ \lambda e^{\lambda\left(1 - \sqrt{y+1}\right)} & y > 0 \\ \lambda e^{\lambda\left(\sqrt{y+1} + 1\right)} & y \text{ imaginary} \end{cases}}$$