1)

a) $$G(y) = \begin{cases} 0, & y < 0 \\ \int_0^y e^{-t}\,dt = 1 - e^{-y}, & y > 0 \end{cases}$$

$$\frac{f(x)}{g(x)} = \frac{2x(1+x)e^{-2x}}{e^{-x}} \qquad x > 0$$

$$\frac{d}{dx}\left(\frac{2x(1+x)e^{-2x}}{e^{-x}}\right) = e^{-x}(-2x^2 + 2x + 2)$$

$$e^{-x}(-2x^2 + 2x + 2) = 0 \Rightarrow x_{max} = \frac{1}{2}(1 \pm \sqrt{5})$$

$$\frac{f(x_{max})}{g(x_{max})} = c = \frac{(1+\sqrt{5})\left(1 + \frac{1}{2}(1+\sqrt{5})\right)e^{-(1+\sqrt{5})}}{e^{-\frac{1}{2}(1+\sqrt{5})}}$$

$$c \simeq 1.68$$

$$G(y) = 1 - e^{-y} = R \Rightarrow 1 - R = e^{-y} \Rightarrow Y = -\ln(1-R)$$

So our algorithm will be the following:

① Generate $R_1 \sim U[0,1)$ and compute $Y = -\ln(1-R_1)$

② Generate $R_2 \sim U[0,1)$ and if $R_2 \leq \frac{2Y(1+Y)e^{-2Y}}{e^{-Y}} \cdot \frac{1}{1.68}$, set $X = Y$, otherwise go to ①.

b) $Pr(\text{Rejection}) \cdot Pr(\text{Acceptance}) = \left(1 - \frac{1}{c}\right)\left(\frac{1}{c}\right) \approx 0.241$

2)

a) We want L_Q and W_Q of a $M/G/2$ queue.

$\lambda = 6$ customers/hour, $\mu = 4$ customers/hours
$\sigma^2 = 0$

so $\rho = \frac{\lambda}{c\mu} = \frac{6}{2\cdot 4} = 0.75$ and correction $= \frac{1}{2}$

$$P_0 = \left(\sum_{n=0}^{c-1} \frac{(\rho c)^n}{n!} + \frac{(\rho c)^c}{c!(1-\rho)}\right)^{-1} = \left(\sum_{n=0}^{1}\left(\frac{(1.5)^n}{n!}\right) + \frac{(1.5)^2}{2(0.25)}\right)^{-1}$$

$$= (2.5 + 4.5)^{-1} = \frac{1}{7}$$

$$L_{Q,M/M/2} = \frac{(c\rho)^{c+1} P_0}{c(c!)(1-\rho)^2} = \frac{(2\cdot 0.75)^3\left(\frac{1}{7}\right)}{2\cdot 2(0.25)^2} = 1.92857 \text{ customers}$$

$$W_{Q,M/M/2} = \frac{L_{Q,M/M/2}}{\lambda} = \frac{1.92857}{6} = 0.321429 \text{ hours}$$

Consequently,

$$L_{Q,M/G/2} = L_{Q,M/M/2} \cdot \text{correction} = 1.92857\cdot\left(\frac{1}{2}\right) \approx 0.964 \text{ cust.}$$

$$W_{Q,M/G/2} = W_{Q,M/M/2} \cdot \text{correction} = 0.321429\left(\frac{1}{2}\right) \approx 0.161 \text{ hrs}$$

b) $W_{M/G/2} = W_{Q,M/G/2} + \frac{1}{\mu} = 0.410714$ hours

$L_{M/G/2} = \lambda W_{M/G/2} = 6(0.410714) = 2.46429$ customers

3) We have a $M/G/c/\infty$ with $\lambda = 625$ customers/hour and $E(S) = 4$ hours $\Rightarrow \mu = 0.25$ customers/hour

We want to solve for c given q so that $P(L \leq c) \geq q$, i.e.

$$P(L \leq c) = \sum_{n=0}^{c} e^{-\alpha} \frac{\alpha^n}{n!} \geq q \quad \text{where } \alpha = \frac{\lambda}{\mu} = \frac{625}{0.25} = 2500$$

$$\Rightarrow \sum_{n=0}^{c} e^{-2500} \frac{(2500)^n}{n!} \geq 0.975 \text{ and solve for } c.$$

Since c is going to be large we can approximate it using a normal distribution

$$e^{-2500} \sum_{n=0}^{c} \frac{(2500)^n}{n!}$$ is cdf of poisson evaluated at c with parameter 2500

Thus,

$$e^{-2500} \sum_{n=0}^{c} \frac{(2500)^n}{n!} \approx P\left(\frac{X - 2500}{\sqrt{2500}} \leq \frac{c - 2500}{\sqrt{2500}} \right)$$

$$= P\left(Z \leq \frac{c-2500}{\sqrt{2500}}\right)$$

$$= \Phi\left(\frac{c-2500}{\sqrt{2500}}\right)$$

Now we want c such that

$$\Phi\left(\frac{c-2500}{\sqrt{2500}}\right) \geq 0.975$$

Using table, c must be (approximately) at least 2598.

4)

$$f(x) = \begin{cases} 0 & , \quad x<0 \\ p & , \quad x=0 \\ (1-p)\lambda e^{-\lambda x} & , \quad x>0 \end{cases} \qquad \text{for } \begin{array}{l} 0<p<1 \\ \lambda>0 \end{array}$$

$$F(x) = \begin{cases} 0 & , \quad x<0 \\ p & , \quad x=0 \\ p + \int_0^x (1-p)\lambda e^{-\lambda t}\,dt = p + (p-1)\left(1-e^{-\lambda x}\right) & , \quad x>0 \end{cases}$$

a)

Set $F(x) = R$ where $R \sim U[0,1)$

$$X = \begin{cases} 0 & , \quad 0<R<p \\ -\frac{1}{\lambda}\ln\left(1-\frac{(R-p)}{(p-1)}\right) & , \quad p<R<1 \end{cases}$$

b) $\lambda=2$ $p=0.4$ $R=0.33$

Since $R \leq p$, $X=0$

$\lambda=2$ $p=0.4$ $R=0.66$

Since $R>p$, $X = -\frac{1}{2}\ln\left(1-\frac{(0.66-0.4)}{(0.4-1)}\right)$

$X = -0.18$

5)

$$f(x|\lambda) = \begin{cases} 0 & x < 1 \\ \lambda x^{-(\lambda+1)} & x > 1 \end{cases} \qquad \lambda > 0$$

$$f'(x|\lambda) = x^{-(\lambda+1)} - \lambda x^{-(\lambda+1)} \ln(x) = x^{-(\lambda+1)}(1 - \lambda \ln(x))$$

$$\frac{f'(x|\lambda)}{f(x|\lambda)} = \frac{x^{-(\lambda+1)}(1-\lambda\ln(x))}{\lambda x^{-(\lambda+1)}} = \frac{1}{\lambda} - \ln(x)$$

solve,

$$\sum_{i=1}^{N} \left(\frac{1}{\lambda_{ML}} - \ln(x)\right) = 0 \Rightarrow \frac{N}{\lambda_{ML}} = \sum_{i=1}^{N} \ln(x)$$

$$\text{so} \quad \frac{1}{\lambda_{ML}} = \frac{1}{N} \sum_{i=1}^{N} \ln(x) \Rightarrow \lambda_{ML} = \frac{1}{\langle \ln(x) \rangle}$$

we have $\langle \ln(X) \rangle = \frac{1}{5}\left(\ln(2) + \ln(3) + \ln(4) + \ln(5) + \ln(6)\right)$

$$= 1.31585$$

Thus, $\boxed{\lambda_{ML} = 0.759965}$

b)
$$E(X) = \int_1^\infty x \cdot \lambda \cdot x^{-(\lambda+1)}\,dx = \int_1^\infty \lambda x^{-\lambda}\,dx = \frac{1}{\lambda-1}$$

If we choose $E(X) = \langle x \rangle$

then $$\frac{2+3+4+5+6}{5} = \frac{1}{\lambda_{EV}-1}$$

or $$4 = \frac{1}{\lambda_{EV}-1} \Rightarrow \lambda_{EV} - 1 = \frac{1}{4}$$

$$\Rightarrow \boxed{\lambda_{EV} = \frac{5}{4}}$$