Mid-term Exam of Matrix Theory (2014)

Preferentially Selected Five Questions ($5 \times 20'$)

Q1. Given $A \in P^{n\times n}$, consider the following questions.
1) If A is invertible, prove that A^{-1} can be represented by the polynomial of A with degree less than n.
2) For any positive integer $k \in N$, prove that A^k can be represented by the polynomial of A with degree less than n.
3) Especially $A = \begin{pmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{pmatrix}$, find the representative polynomials of A^{-1} and A^{2014} as mentioned in 1) and 2).

Q2. Denote $\mathbf{A}$ a linear transformation in R^3, $\alpha_1, \alpha_2, \alpha_3$ the basis of R^3. Suppose that the representation matrix of A with respect to $\alpha_1, \alpha_2, \alpha_3$ is $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}$.
1) Show that $\beta_1 = \alpha_1, \beta_2 = \alpha_1 + \alpha_2, \beta_1 = \alpha_1 + \alpha_2 + \alpha_3$ also form a basis of R^3.
2) Determine the representative matrix of $\mathbf{A}$ with respect to $\beta_1, \beta_2, \beta_2$.
3) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

Q3. Denote $R[x]_3$ to be the vector space of zero and polynomials with degree less than 3.
1) Determine the dimension of $R[x]_3$ and give a basis of $R[x]_3$.
2) Define the linear transformation $\mathbf{D}$ on $R[x]_3$,

$$\mathbf{D}(f(x)) = f'(x), \quad \forall f(x) \in R[x]_3.$$

Show $R(\mathbf{D})$ and $\ker(\mathbf{D})$.
3) Prove that $\mathbf{D}$ is not diagonalizable.
4) Define the inner product on $R[x]_3$,

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx, \quad \forall f(x), g(x) \in R[x]_3,$$

please Gram-Schmidt orthogonalize the basis given in 1).

Q4. 1) To the best of your knowledge about $\lambda-$matrix, determine if the following two matrices are similar or not, and give reason,

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & a & 0 \\ 0 & 2 & a \\ 0 & 0 & 2 \end{pmatrix}.$$

2) Denote $V = \{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in R^{2\times 2} | a_{11} = a_{22}\}$.

i) Find a basis of V and show the dimension.

ii) Arbitrarily given $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ in V, define

$$(A, B) = a_{11}b_{11} + 2a_{12}b_{12} + a_{21}b_{21}.$$

Please show that (A, B) is an inner product on V.

Q5. Given $A \in C^{m\times n}$ and $b \in C^m$, please prove

1) there exists a real number $\alpha > 0$ such that $A^H A + \alpha I$ is nonsingular;

2) the solution to the least square problem $\min\limits_{x\in C^n}\{\|Ax-b\|^2+\alpha\|x\|^2\}$ is $x^* = (A^H A+\alpha I)^{-1}A^H b$, where $\|\cdot\|$ stands for the 2−norm in C^m.

Q6. Given $A \in R^{n\times n}$, summarize the necessary and sufficient conditions of A to be diagonalizable, and prove at least one of them. Determine if the matrix A given in Q2 is diagonalizable or not. If yes, please explain why, if not, please give the Jordan canonical form of A.