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Homework 2 - Serial-Link Manipulator Kinematics

1

In order to derive the Forward Kinematics, one should inspect the constants and variables on the constituent parts of the robot. There are 3 links, which have fixed lengths a_1, a_2 and a_3. We can easily see them as constant translations that have to be included to the Transformation operation. Also, there are 3 **revolute** joints, which can rotate, hence manipulate their own orientations. These rotations are directly controlled with θ_1, θ_2 and θ_3 angles. In order to map points in frame E to frame O, we should just calculate the all rotations and translations, starting from the frame O up until frame E. For the math, see below:

$$f(\theta_1, \theta_2, \theta_3) = R(\theta_1) * T_x(a_1) * R(\theta_2) * T_x(a_2) * R(\theta_3) * T_x(a_3)$$

I want to clarify some possible confussions in here. Since this is a planar robot, there is only one rotational dimension. Hence, rotation operation R refers to it. Moreover, even though the link lengths seem as somewhat combination of $x - y$ dimensions, all rotational operations just before the translational operations T_x, align new x frame to the link direction exactly. Therefore, all T_x operations remain valid.

Writing all as homogeneous transformation matrices in a row and applying matrix multiplication from right-to-left:

$$\begin{bmatrix} cos(\theta_3) & -sin(\theta_3) & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & 0 \\ sin(\theta_2) & cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta_1) & -sin(\theta_1) & 0 \\ sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = {}^{O}T_E$$

Writing the result directly:

$${}^{O}T_E = \begin{bmatrix} cos(\theta_3 + \theta_2 + \theta_1) & -sin(\theta_3 + \theta_2 + \theta_1) & a_3.cos(\theta_3 + \theta_2 + \theta_1) + a_2.cos(\theta_2 + \theta_1) + a_1.cos(\theta_1) \\ sin(\theta_3 + \theta_2 + \theta_1) & cos(\theta_3 + \theta_2 + \theta_1) & a_3.sin(\theta_3 + \theta_2 + \theta_1) + a_2.sin(\theta_2 + \theta_1) + a_1.sin(\theta_1) \\ 0 & 0 & 1 \end{bmatrix}$$

Consequently, these chains of rotations and translations produce a homogeneous transformation matrix, which maps points from E to O. Because, while the function above pushes the operations from left-to-right, the frame E pops the operations from right-to-left and applies in that order. It is now clear that this is an exact homogeneous transformation from E to O, which can be denoted as ${}^{O}T_{E}$.

2

For simplicity, assume ${}^{O}T_{E}^{*} = [x_d, y_d, \theta_d]$, which is the desired pose in SE(2). From the previous problem, we know that ${}^{O}T_{E}^{*} = f(\theta_1^*, \theta_2^*, \theta_3^*)$. This directly brings us the relation $(\theta_1^*, \theta_2^*, \theta_3^*) = f^{-}1({}^{O}T_{E}^{*})$. Since $\theta_1^*, \theta_2^*, \theta_3^* \in S^1$, the function $f^{-}1$ is properly set as $SE(2) \rightarrow S^1 \times S^1 \times S^1$. Now the problem reduces to derivation of inverse kinematic equations. In the previous question, the equations:

$$x_d = a_3.cos(\theta_3 + \theta_2 + \theta_1) + a_2.cos(\theta_2 + \theta_1) + a_1.cos(\theta_1)$$
$$y_d = a_3.sin(\theta_3 + \theta_2 + \theta_1) + a_2.sin(\theta_2 + \theta_1) + a_1.sin(\theta_1)$$
$$\theta_d = \theta_1 + \theta_2 + \theta_3,$$

were derived. Finding $(\theta_1, \theta_2, \theta_3)$ is a rigorous process through the equations above and might not have an analytical solution at all.

Let's call the last joint as $w(wrist)$. Since we both know θ_d and a_3, applying inverse transformations to x_d, y_d will produce x_w, y_w for further usage, and wrist position will be very useful on the IK calculation. Below is the computation procedure:

$$\begin{bmatrix} 1 & 0 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta_d & -sin\theta_d & 0 \\ sin\theta_d & cos\theta_d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & x_w \\ 0 & 1 & y_w \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} cos\theta_d & -sin\theta_d & x_d \\ sin\theta_d & cos\theta_d & y_d \\ 0 & 0 & 1 \end{bmatrix}$$

The relations at the end are:

$$x_d = cos\theta_d x_w - sin\theta_d y_w + a_3,$$

$$y_d = sin\theta_d x_w + cos\theta_d y_w$$

x_w and y_w can be straightforwardly computed from above equations. They can be also written in terms of a_1, a_2, θ_1 and θ_2. Below are those equations (further applying necessary Trigonometric Identities):

$$x_w = a_1 cos\theta_1 + a_2(cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2)$$

$$y_w = a_1 sin\theta_1 a_2(cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2)$$

Taking squares and isolating $cos\theta_2$:

$\frac{x_w^2 + y_w^2 - a_1^2 - a_2^2}{2a_1a_2} = k = cos\theta_2$, Observe that right hand side of the equation will not produce an expression that involves variable. All terms at lhs are known in advance. Hence $\theta_2 = arccos(k)$.

In order to compute θ_1, some more relations are necessary to be added. Those are:

$x_w = k_1.cos\theta_1 - k_2sin\theta_1$

$y_w = k_1.sin\theta_1 + k_2cos\theta_1$, where

$k_1 = a_1 + a_2cos\theta_2, \quad k_2 = a_2sin\theta_2.$

Note that $\gamma + \theta_1$ is the ccw angle between Frame {0} and Frame {W}. It is straightforward to compute this combined angle from x_w and y_w magnitudes.$\rightarrow$ $\gamma + \theta_1 = arctan(y_w, x_w)$. Adding two more relations will be enought to solve for θ_1. Those are:

$k_1 = rcos\gamma$

$k_2 = rsin\gamma$, where r is simply the length between wrist and origin. **Final equations**:

$\theta_1 = arctan(y_w, x_w) - arctan(k_2, k_1)$
$\theta_2 = arccos(k)$
$\theta_3 = \theta_d - \theta_1 - \theta_2.$

3

Following the hint, I will start with the derivation of $c'(\gamma)$. Since $c(\gamma)$ is a column vector in 2D, derivation must be applied to both units in x and y dimensions. Below is the derivative:

$$c'(\gamma) = \begin{bmatrix} -10\pi sin(2\pi\gamma) + 2\pi cos(14\pi\gamma) + 10\pi cos(12\pi\gamma)cos(2\pi\gamma) \\ 10\pi cos(2\pi\gamma) + 2\pi sin(14\pi\gamma) + 10\pi cos(12\pi\gamma)sin(2\pi\gamma) \end{bmatrix}$$

Now, let's inspect this expression. $c'(\gamma)$ is the vector equation of the tangent to the curve for any given γ. Aligning this with $\hat{E_y}$ simplifies the question further. Since the end effector is $\in SE(2)$, the $\hat{E_x}$ must be orthogonal to $\hat{E_y}$. We already know that, two orthogonal vectors have 0 dot-product. Moreover, since this is a right-handed coordinate system, $\hat{E_x}$ will have an orientation according to angle:

$arctan(c'(\gamma)(2), c'(\gamma)(1)) - \pi/2$, where $c'(\gamma)(x)$ refers to MATLAB type indexing.

Having computed the orientations of end-effector coordinate frames for any given γ, let's get back to the beginning of the problem. Assuming the orientation of the end-effector is not important, we can easily conclude that:

$$^{O}T_E(\gamma) = \begin{bmatrix} I_{2x2} & c(\gamma) \\ 0 & 1 \end{bmatrix}$$

Actually, translational part will remain as is. Only addition will be done through replacement of I_{2x2} with a $R(\theta)$ that assures the orthogonality of end-effector to the curve. In fact, the computation of θ is already done above:

$$\theta = arctan(c'(\gamma)(2), c'(\gamma)(1)) - \pi/2$$

The final form of $^{O}T_E(\gamma)$ is below. Since, θ and $c(\gamma)$ is already known, I'm writing the compact form.

$$^{O}T_E(\gamma) = \begin{bmatrix} R(\theta)_{2x2} & c(\gamma)_{2x1} \\ 0_{1x2} & 1 \end{bmatrix}, \text{ where } R(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

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In order to find minimum a_1, the maximum distance on the curve from the origin should be calculated. Since the parametric formula is in our hands, computing $\frac{\frac{dy}{d\gamma}}{\frac{dx}{d\gamma}} = 0$ for γ will approximately solve the problem. Of course, γ will have many values from the equation above. Carefully picking one that corresponds to a maximum value is enough. Hence, I'm picking $\gamma = 0.31831(0.673542 + \pi)$ from Wolfram website. The rest is straightforward. Replacing γ with above magnitude in $c(\gamma)$ produces the maximum of the curvature:

$\begin{bmatrix} 0.2651 \\ 1.1650 \end{bmatrix}$, basically a point on the curve, which has the maximum distance from the origin.

Let's compute the magnitude of this length, which is $\sqrt{(0.02651)^2 + (1.1650)^2} = 1.1948$. So, minimum $a_1 = 1.1948 - 0.5 - 0.2 = 0.4948$.