PHY 362K Homework 4

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1

(a)

From this question we know that if there is no perturbation, the Hamiltonian is $H = \frac{\vec{p}^2}{2m^e} - \frac{e^2}{4\pi\epsilon_0 r}$

$\therefore E_n^{(0)} = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}$

The perturbed Hamiltonian is $H' = -\mu_{lz}B$

There is no spin, therefore $H' = -\mu_B B\frac{l_z}{\hbar}$ where $\mu_B = \frac{e\hbar}{2m_e}$

Therefore, the first order perturbed Energy, is $E_B^1 = \langle H'\rangle = -\frac{\mu_B B}{\hbar}\hbar m_l = -\mu_B B m_l$ where $\mu_B = \frac{e\hbar}{2m_e} \approx 9.273 * 10^{-24} J/T$

The total energy is $E_n = E_n^{(0)} + E_B^1 = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} - \mu_B B m_l$

(b)

The energy level diagram is shown in Figure 1. Note that the value between the consecutive spacings is $\mu_B B$

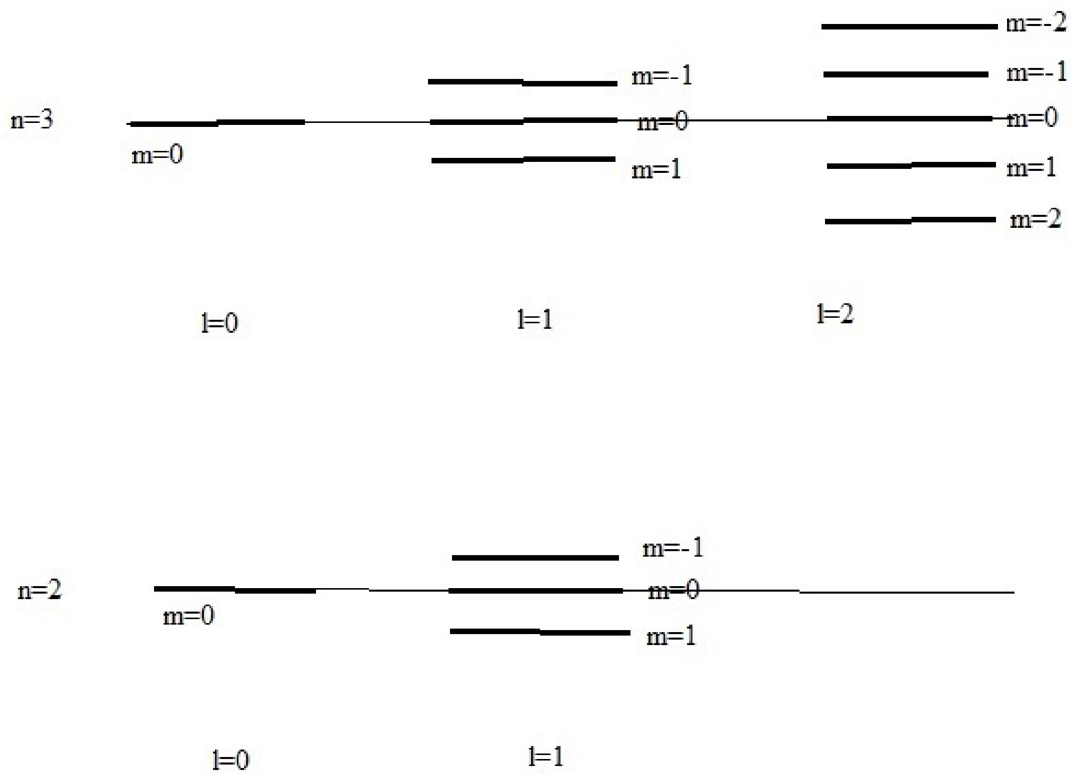

Figure 1: The energy level diagram of "Zeeman energy"

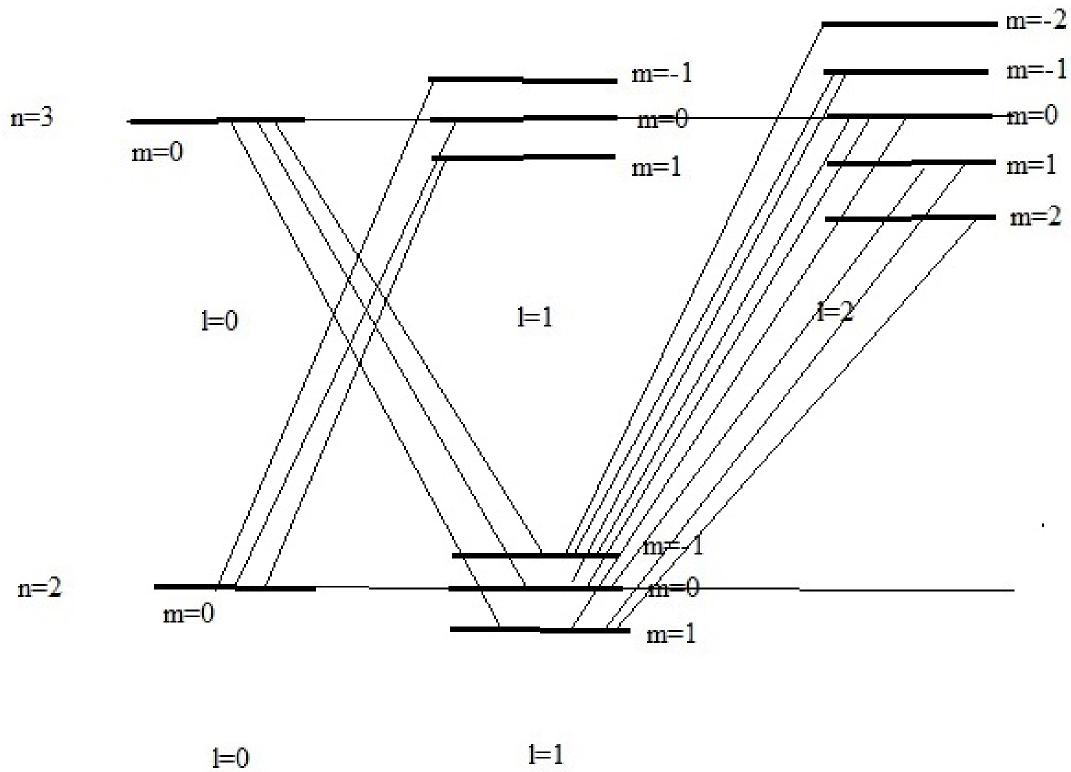

Figure 2: The energy level diagram of "Zeeman energy" with energy transition

(c)

The allowed transitions are shown in Figure 2

(d)

Since $\Delta l = \pm 1$ and $\Delta m_l = 0$ or ± 1, according to the diagram, there are 15 allowed transitions.

Since There are three distinct energies

$$\Delta E_1 = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\left(\frac{1}{3^2} - \frac{1}{2^2}\right) + \mu_B B \text{ when } \Delta m_l = 1$$

$$\Delta E_2 = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\left(\frac{1}{3^2} - \frac{1}{2^2}\right) - \mu_B B \text{ when } \Delta m_l = -1$$

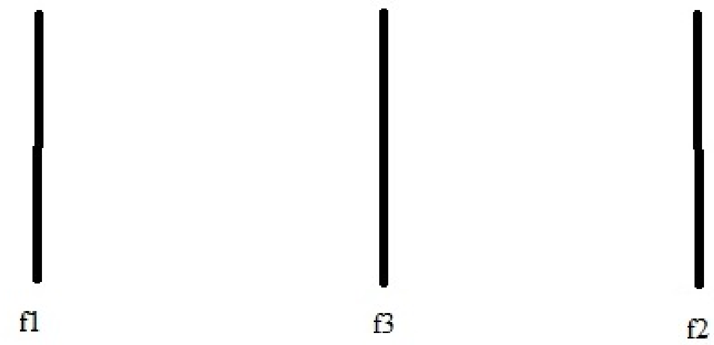

Figure 3: Emission spectrum

$\Delta E_3 = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\left(\frac{1}{3^2} - \frac{1}{2^2}\right)$ when $\Delta m_l = 0$

Therefore there are three distinct frequencies

$f_1 = \frac{\Delta E_1}{h}$

$f_2 = \frac{\Delta E_2}{h}$

$f_3 = \frac{\Delta E_3}{h}$

The spectrum is shown in Figure 3

(e)

When $B = 10T$, the energy spacing is $\Delta E_{Zeeman} = \mu_B B = 9.273 * 10^{-24} * 10J = 9.273 * 10^{-23}J$

The wave number is $\bar{v} = \frac{\Delta E_{Zeeman}}{hc} = \frac{9.273*10^{-23}}{6.63*10^{-34}*3*10^8} * 10^{-2}cm^{-1} = 4.66cm^{-1}$

$\Delta E_{nominal} = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\left(\frac{1}{3^2} - \frac{1}{2^2}\right) = 13.6 * 1.6 * 10^{-19} * \frac{5}{36}J = 3.022 * 10^{-19}J$

$f_{splitting} = \frac{\Delta E_{Zeeman}}{h}$

$f_{nominal} = \frac{\Delta E_{nominal}}{h}$

$\therefore \frac{f_{splitting}}{f_{nominal}} = \frac{\Delta E_{Zeeman}}{\Delta E_{nominal}} = \frac{9.273*10^{-23}}{3.022*10^{-19}} = 3.068 * 10^{-4}$

2

(a)

We know that $n = 3$ and $s = \frac{1}{2}$. Therefore j can be either $l + \frac{1}{2}$ or $l - \frac{1}{2}$.

We also know that $m_j = -j, -j+1, \ldots, j$

So the possible states are:

$3^2 s_{\frac{1}{2},\frac{1}{2}}$, $3^2 s_{\frac{1}{2},-\frac{1}{2}}$

$3^2 p_{\frac{1}{2},\frac{1}{2}}$, $3^2 p_{\frac{1}{2},-\frac{1}{2}}$

$3^2 p_{\frac{3}{2},\frac{3}{2}}$, $3^2 p_{\frac{3}{2},\frac{1}{2}}$, $3^2 p_{\frac{3}{2},-\frac{1}{2}}$, $3^2 p_{\frac{3}{2},-\frac{3}{2}}$

$3^2 d_{\frac{3}{2},\frac{3}{2}}$, $3^2 d_{\frac{3}{2},\frac{1}{2}}$, $3^2 d_{\frac{3}{2},-\frac{1}{2}}$, $3^2 d_{\frac{3}{2},-\frac{3}{2}}$

$3^2 d_{\frac{5}{2},\frac{5}{2}}$, $3^2 d_{\frac{5}{2},\frac{3}{2}}$, $3^2 d_{\frac{5}{2},\frac{1}{2}}$, $3^2 d_{\frac{5}{2},-\frac{1}{2}}$, $3^2 d_{\frac{5}{2},-\frac{3}{2}}$, $3^2 d_{\frac{5}{2},-\frac{5}{2}}$

(b)

The states of $|n, l, j, m_j\rangle$ are:

$|3, 0, \frac{1}{2}, \frac{1}{2}\rangle$, $|3, 0, \frac{1}{2}, -\frac{1}{2}\rangle$

$|3, 1, \frac{1}{2}, \frac{1}{2}\rangle$, $|3, 1, \frac{1}{2}, -\frac{1}{2}\rangle$

$|3, 1, \frac{3}{2}, \frac{3}{2}\rangle$, $|3, 1, \frac{3}{2}, \frac{1}{2}\rangle$, $|3, 1, \frac{3}{2}, -\frac{1}{2}\rangle$, $|3, 1, \frac{3}{2}, -\frac{3}{2}\rangle$

$|3, 2, \frac{3}{2}, \frac{3}{2}\rangle$, $|3, 2, \frac{3}{2}, \frac{1}{2}\rangle$, $|3, 2, \frac{3}{2}, -\frac{1}{2}\rangle$, $|3, 2, \frac{3}{2}, -\frac{3}{2}\rangle$

$|3, 2, \frac{5}{2}, \frac{5}{2}\rangle$, $|3, 2, \frac{5}{2}, \frac{3}{2}\rangle$, $|3, 2, \frac{5}{2}, \frac{1}{2}\rangle$, $|3, 2, \frac{5}{2}, -\frac{1}{2}\rangle$, $|3, 2, \frac{5}{2}, -\frac{3}{2}\rangle$, $|3, 2, \frac{5}{2}, -\frac{5}{2}\rangle$

The total number of states is 18, which is the same as the one in part (a)

(c)

The linear combination is written as $|n, l, m_l, m_s\rangle = \sum_{j=l+s, m_j=m_l+m_s} c_{l,s,m_l,m_s} |l, m_l\rangle |s, m_s\rangle$, where c_{l,s,m_l,m_s} is the coefficient

Using the Clebsch-Gordan coefficients table, we get $|3, 2, \frac{3}{2}, -\frac{3}{2}\rangle = \sqrt{\frac{1}{5}} |2, -1\rangle |\frac{1}{2}, -\frac{1}{2}\rangle - \sqrt{\frac{4}{5}} |2, -2\rangle |\frac{1}{2}, \frac{1}{2}\rangle$

(d)

Since $m_j = -\frac{3}{2}$, the measurement of J_z is $\hbar m_j = -\frac{3}{2}\hbar$ with possibility 1

(e)

From part (c) we know that the linear combination is $|3, 2, \frac{3}{2}, -\frac{3}{2}\rangle = \sqrt{\frac{1}{5}} |2, -1\rangle |\frac{1}{2}, -\frac{1}{2}\rangle - \sqrt{\frac{4}{5}} |2, -2\rangle |\frac{1}{2}, \frac{1}{2}\rangle$

The eigenvalue of L_z is $\hbar m_l$

Therefore the measurements are $-\hbar$ with probability $\frac{1}{5}$ and $-2\hbar$ with probability $\frac{4}{5}$

(f)

The eigenvalue of S_z is $\hbar m_s$

$\therefore \langle S_z \rangle = -\frac{1}{5} * \frac{1}{2}\hbar + \frac{4}{5} * \frac{1}{2}\hbar = \frac{3}{10}\hbar$

(g)

From the linear combination we can know that $\psi_{r,l,m_l}(r, \theta, \phi) = \sqrt{\frac{1}{5}} R_{32}(r) Y_{2-1}(\theta, \phi) - \sqrt{\frac{4}{5}} R_{32}(r) Y_{2-2}(\theta, \phi)$

We know that $R_{32}(r) = \frac{4}{81\sqrt{30}} * a_0^{-\frac{3}{2}} \left(\frac{r}{a_0}\right)^2 e^{-\frac{r}{3a_0}}$, $Y_{2-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}} \sin(\theta)\cos(\theta)e^{-i\phi}$ and $Y_{2-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}} \sin^2(\theta)e^{-2i\phi}$

$a_0 = 0.5292 * 10^{-10}$

Plug in the functions above to Mathematica, we get:

$Probability = \psi_{r,l,m_l}(2a_0, \frac{\pi}{3}, \frac{\pi}{4})^* \psi_{r,l,m_l}(2a_0, \frac{\pi}{3}, \frac{\pi}{4})(0.002a_0)^3 = 9.5178 * 10^{-14}$

The Mathematica code is shown below:

```
a0 := 0.5292*10^(-10)

R32[r_] := 4/(81*Sqrt[30]) *a0^(-3/2) *(r/a0)^2*Exp[-r/(3*a0)]

Y2neg1[theta_, phi_] :=
 Sqrt[15/(8*Pi)]*Sin[theta]*Cos[theta]*Exp[-I*phi]

Y2neg2[theta_, phi_] :=
 Sqrt[15/(32*Pi)]*(Sin[theta])^2 * Exp[-2*I*phi]

Psi[r_, theta_, phi_] :=
 Sqrt[1/5]*R32[r]*Y2neg1[theta, phi] -
  Sqrt[4/5]*R32[r]*Y2neg2[theta, phi]

Conjugate[Psi[2*a0, Pi/3, Pi/4]]*Psi[2*a0, Pi/3, Pi/4]*(0.002*a0)
   ^3
```

3

(a)

We know that $\vec{j}^2 = (\vec{l} + \vec{s})^2 = \vec{l}^2 + 2\vec{l} \cdot \vec{s} + \vec{s}^2$

Therefore $\vec{l}\cdot\vec{s} = \frac{1}{2}(\vec{j}^2 - \vec{l}^2 - \vec{s}^2)$

(b)

The expected value of $\vec{l}\cdot\vec{s}$ is $\langle\vec{l}\cdot\vec{s}\rangle = \frac{1}{2}(\langle\vec{j}^2\rangle - \langle\vec{l}^2\rangle - \langle\vec{s}^2\rangle) = \frac{1}{2}(j(j+1)\hbar^2 - l(l+1)\hbar^2 - s(s+1)\hbar^2)$

Plug in the values of j,l and s in Mathematica, we get

$$\vec{l}\cdot\vec{s} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\hbar^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hbar^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\hbar^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\hbar^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\hbar^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\hbar^2 \end{pmatrix}$$

The Mathematica code is shown below:

```
j := {1/2, 1/2, 1/2, 1/2, 3/2, 3/2, 3/2, 3/2}

l := {0, 0, 1, 1, 1, 1, 1, 1}

s := 1/2

f[x_, y_] :=

 KroneckerDelta[x,

    y]*0.5*((Part[j, x] + 1)*Part[j, x]*\[HBar]^2 - (Part[l, x] +

        1)*

      Part[l, x]*\[HBar]^2 - (s + 1)*s*\[HBar]^2)
```

```
Table[f[x, y], {x, 1, 8}, {y, 1, 8}] // MatrixForm
```

(c)

We know that $l_\pm = \vec{l_x} \pm i\vec{l_y}$ and $s_\pm = \vec{s_x} \pm i\vec{s_y}$

$\frac{1}{2}(l_+s_- + l_-s_+) + l_zs_z = \frac{1}{2}[(l_x + il_y)(s_x - is_y) + (l_x - il_y)(s_x + is_y)] + l_zs_z = \frac{1}{2}[l_xs_x - il_xs_y + il_ys_x + l_ys_y] + l_zs_z = l_xs_x + l_ys_y + l_zs_z = \vec{l} \cdot \vec{s}$

(d)

The expected value of $\vec{l} \cdot \vec{s}$ is $\frac{1}{2}(j(j+1)\hbar^2 - l(l+1)\hbar^2 - s(s+1)\hbar^2)$

Plug in the values of j,l and s in Mathematica, we get

$$\vec{l} \cdot \vec{s} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\hbar^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\hbar^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\hbar^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\hbar^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\hbar^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\hbar^2 \end{pmatrix}$$

The Mathematica code is shown below:

```
l1 := {0, 0, 1, 1, 1, 1, 1, 1}

s := 1/2

g[x_, y_] :=
```

```
KroneckerDelta[x,
   y]*0.5*((Part[l1, x] + 1/2 + 1)*(Part[l1, x] +
        1/2)*\[HBar]^2 - (Part[l1, x] + 1)*
     Part[l1, x]*\[HBar]^2 - (s + 1)*s*\[HBar]^2)


Table[g[x, y], {x, 1, 8}, {y, 1, 8}] // MatrixForm
```

4

(a)

$\vec{L} \cdot \vec{S} = L_x S_x + L_y S_y + L_z S_z$

$[\vec{L} \cdot \vec{S}, L_x] = [L_x S_x + L_y S_y + L_z S_z, L_x] = [L_x S_x, Lx] + [L_y S_y, Lx] + [L_z S_z, Lx] = [L_x, L_x]S_x + L_x[S_x, L_x] + L_y[S_y, L_x] + S_y[L_y, L_x] + [L_z, L_x]S_z + L_z[S_z, L_x] = i\hbar L_y S_z - i\hbar L_z S_y = i\hbar(\vec{L} \times \vec{S})_x$

Similarly, we get:

$[\vec{L} \cdot \vec{S}, L_y] = [L_x S_x + L_y S_y + L_z S_z, L_y] = i\hbar(\vec{L} \times \vec{S})_y$

$[\vec{L} \cdot \vec{S}, L_z] = [L_x S_x + L_y S_y + L_z S_z, L_z] = i\hbar(\vec{L} \times \vec{S})_z$

According to the three equations above, we get $[\vec{L} \cdot \vec{S}, \vec{L}] = i\hbar(\vec{L} \times \vec{S})$

(b)

$\vec{L} \cdot \vec{S} = L_x S_x + L_y S_y + L_z S_z$

$[\vec{L} \cdot \vec{S}, S_x] = [L_x S_x + L_y S_y + L_z S_z, S_x] = [L_x S_x, S_x] + [L_y S_y, S_x] + [L_z S_z, S_x] = [L_x, S_x]S_x + L_x[S_x, S_x] + L_y[S_y, S_x] + [L_y, S_x]S_y + [L_z, S_x]S_z + L_z[S_z, S_x] = i\hbar S_y L_z - i\hbar S_z L_y = i\hbar(\vec{S} \times \vec{L})_x$

Similarly, we get:

$[\vec{L} \cdot \vec{S}, S_y] = i\hbar(\vec{S} \times \vec{L})_y$

$[\vec{L} \cdot \vec{S}, S_z] = i\hbar(\vec{S} \times \vec{L})_z$

According to the three equations above, we get $[\vec{L} \cdot \vec{S}, \vec{S}] = i\hbar(\vec{S} \times \vec{L})$

(c)

We know that $\vec{J} = \vec{L} + \vec{S}$

Therefore $[\vec{L} \cdot \vec{S}, \vec{J}] = [\vec{L} \cdot \vec{S}, \vec{L} + \vec{S}] = [\vec{L} \cdot \vec{S}, \vec{L}] + [\vec{L} \cdot \vec{S}, \vec{S}] = i\hbar(\vec{L} \times \vec{S}) + i\hbar(\vec{S} \times \vec{L}) = i\hbar(\vec{L} \times \vec{S}) - i\hbar(\vec{L} \times \vec{S}) = 0$

(d)

Since L^2 commutes with all components of $\vec{L}$ and $\vec{S}$, we get:

$[\vec{L} \cdot \vec{S}, L^2] = [L_xS_x + L_yS_y + L_zS_z, L^2] = [L_xS_x, Lx] + [L_yS_y, Lx] + [L_zS_z, Lx] = [L_x, L^2]S_x + L_x[S_x, L^2] + L_y[S_y, L^2] + S_y[L_y, L^2] + [L_z, L^2]S_z + L_z[S_z, L^2] = 0$

(e)

Since S^2 commutes with all components of $\vec{L}$ and $\vec{S}$, we get:

$[\vec{L} \cdot \vec{S}, S^2] = [L_xS_x + L_yS_y + L_zS_z, S^2] = [L_xS_x, Lx] + [L_yS_y, Lx] + [L_zS_z, Lx] = [L_x, S^2]S_x + L_x[S_x, S^2] + L_y[S_y, S^2] + S_y[L_y, S^2] + [L_z, S^2]S_z + L_z[S_z, S^2] = 0$

(f)

We know that $J^2 = L^2 + S^2 + 2\vec{L} \cdot \vec{S}$

$\therefore [\vec{L}\cdot\vec{S}, J^2] = [\vec{L}\cdot\vec{S}, L^2+S^2+2\vec{L}\cdot\vec{S}] = [\vec{L}\cdot\vec{S}, L^2] + [\vec{L}\cdot\vec{S}, S^2] + 2[\vec{L}\cdot\vec{S}, \vec{L}\cdot\vec{S}] = 0$

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(a)

$E_n^{(1)} = \langle\psi_n|H'|\psi_n\rangle$

We know that $H = H_0 + H'$

We can let $H(\lambda) = H_0$ when $\lambda = \lambda_0$ where λ_0 is some value

Then $H' = H(\lambda_0 + d\lambda) - H_0 = H(\lambda_0 + d\lambda) - H(\lambda_0) = Hd\lambda$ when $\lambda \to \lambda_0$

Therefore $H' = \frac{\partial H}{\partial\lambda}$

$E_n = E_n^{(0)} + E_n^{(1)}$

By the same approach, we can get $E_n^1 = E_n d\lambda$ when $\lambda \to \lambda_0$

Therefore $E_n^{(1)} = \frac{\partial E_n}{\partial\lambda}$

$\therefore \frac{\partial E_n}{\partial\lambda} = \langle\psi_n|\frac{\partial H}{\partial\lambda}|\psi_n\rangle$

(b)

For an one-dimensional harmonic oscillator, we have:

$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2$

$T = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

$V = \frac{1}{2}m\omega^2x^2$

$E_n = (n+\frac{1}{2})\hbar\omega$

(i) When $\lambda = \omega$

$\frac{\partial E_n}{\partial \omega} = (n+\frac{1}{2})\hbar$

$\frac{\partial H}{\partial \omega} = m\omega x^2$

Using Feynman-Hellmann theorem:

$\frac{\partial E_n}{\partial \omega} = \langle\psi_n|\frac{\partial H}{\partial \omega}|\psi_n\rangle$

$(n+\frac{1}{2})\hbar = \langle\psi_n|\frac{\partial H}{\partial \omega}|m\omega x^2|\psi_n\rangle$

$\frac{1}{2}(n+\frac{1}{2})\hbar\omega = \langle\psi_n|\frac{\partial H}{\partial \omega}|\frac{1}{2}m\omega^2 x^2|\psi_n\rangle = \langle\psi_n|\frac{\partial H}{\partial \omega}|\frac{1}{2}V|\psi_n\rangle$

$\therefore \langle V\rangle = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$

(ii) When $\lambda = \hbar$

$\frac{\partial E_n}{\partial \hbar} = \langle\psi_n|\frac{\partial H}{\partial \hbar}|\psi_n\rangle$

$(n+\frac{1}{2})\omega = \langle\psi_n| - \frac{\hbar}{m}\frac{d^2}{dx^2}|\psi_n\rangle$

$\frac{\hbar}{2}(n+\frac{1}{2})\omega = \langle\psi_n| - \frac{\hbar^2}{2m}\frac{d^2}{dx^2}|\psi_n\rangle = \langle\psi_n|T|\psi_n\rangle$

$\therefore \langle T\rangle = \frac{\hbar}{2}(n+\frac{1}{2})\omega$

(iii) When $\lambda = m$

$\frac{\partial E_n}{\partial m} = \langle\psi_n|\frac{\partial H}{\partial m}|\psi_n\rangle$

$0 = \langle\psi_n|\frac{\hbar^2}{2m^2}\frac{d^2}{dx^2} + \frac{1}{2}\omega^2 x^2|\psi_n\rangle$

$0 = -\frac{1}{m}\langle\psi_n|m\frac{\hbar^2}{2m^2}\frac{d^2}{dx^2}|\psi_n\rangle + \frac{1}{m}\langle\psi_n|m\frac{1}{2}\omega^2 x^2|\psi_n\rangle$

$\langle V\rangle - \langle T\rangle = 0$

$\therefore \langle V\rangle = \langle T\rangle$

The results are consistent with Problem 2.12 and Problem 3.31

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(a)

We know that $H = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$

$E_n = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2(j_{max}+l+1)^2}$

Let $\lambda = e$

$\frac{\partial E_n}{\partial e} = \langle\psi_n|\frac{\partial H}{\partial e}|\psi_n\rangle$

$-\frac{me^3}{8\pi^2\epsilon_0^2\hbar^2(j_{max}+l+1)^2} = \langle\psi_n| - \frac{e}{2\pi\epsilon_0}\frac{1}{r}|\psi_n\rangle$

$\frac{me^2}{4\pi\epsilon_0\hbar^2(j_{max}+l+1)^2} = \langle\psi_n|\frac{1}{r}|\psi_n\rangle$

$\therefore \langle\frac{1}{r}\rangle = \frac{me^2}{4\pi\epsilon_0\hbar^2(j_{max}+l+1)^2} = -\frac{8\pi\epsilon_0}{e^2}E_n = -\frac{8\pi\epsilon_0}{e^2}\frac{E_1}{n^2} = \frac{8\pi\epsilon_0}{e^2n^2}\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right] = \frac{e^2m}{4\pi\epsilon_0\hbar^2}\frac{1}{n^2}$

Let $a_0 = \frac{4\pi\epsilon_0\hbar}{me^2}$, then we get $\langle\frac{1}{r}\rangle = \frac{1}{a_0n^2}$

(b)

Let $\lambda = l$

$\frac{\partial E_n}{\partial l} = \langle\psi_n|\frac{\partial H}{\partial l}|\psi_n\rangle$

$\frac{me^4}{16\pi^2\epsilon_0^2\hbar^2=(j_{max}+l+1)^3}\langle\psi_n|\frac{\hbar^2}{2m}\frac{2l+1}{r^2}|\psi_n\rangle$

$\left(-\frac{2}{j_{max}+l+1}E_n\right) = \left(-\frac{2}{n}\right)E_n = \langle\psi_n|\frac{\hbar^2}{2m}\frac{2l+1}{r^2}|\psi_n\rangle$

$\left(-\frac{2}{j_{max}+l+1}\right)E_n = \left(-\frac{2}{n}\right)E_n = \langle\psi_n|\frac{\hbar^2}{2m}\frac{2l+1}{r^2}|\psi_n\rangle$

$\therefore \langle\frac{1}{r^2}\rangle = \left(-\frac{4m}{\hbar^2n(2l+1)}\right)E_n$

Let $a_0 = \frac{4\pi\epsilon_0\hbar}{me^2}$, then we get $\langle\frac{1}{r^2}\rangle = \frac{1}{a_0^2n^3\left(l+\frac{1}{2}\right)}$