PHY 362K Homework 6

Xiaohui Chen
EID: xc2388

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1

(a)

The wave function is shown in Figure 1. From Figure 1. We can see that the wave function approaches 0 at both $z = 0$ and at infinity. Also, it has a local maximum value, which indicates the object getting to the highest point. Therefore this wave function is a suitable approximation

The wave function is $\psi(z) = Aze^{-\alpha z^2}$

Since the wave function is normalized, we get that $\int_0^\infty |\psi(z)|^2 dz = A^2 \int_0^\infty z^2 e^{-2\alpha z^2} dz = 1$

Use Mathematica, $A^2 \int_0^\infty z^2 e^{-2*\alpha z^2} dz = \frac{\sqrt{\frac{\pi}{2}} A^2}{8\alpha^{3/2}} = 1$

$\therefore A = 2\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{4}}$

Therefore the wave function is $\psi(z) = 4\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{2}} ze^{-\alpha z^2}$

$\langle H \rangle = \langle T \rangle + \langle V \rangle$

$\langle T \rangle = -\frac{\hbar^2}{2m} 4\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{2}} \int_0^\infty ze^{-\alpha z^2} \frac{d^2 ze^{-\alpha z^2}}{dz^2} dz = \frac{3\alpha\hbar^2}{2m}$ (using Mathematica)

$\langle V \rangle = 4mg\left(\frac{2\alpha}{\pi}\right)^{\frac{3}{2}} \int_0^\infty z^3 e^{-2\alpha z^2} dz = mg\sqrt{\frac{2}{\pi\alpha}}$ (using Mathematica)

$\therefore \langle H \rangle = \frac{3\alpha\hbar^2}{2m} + mg\sqrt{\frac{2}{\pi\alpha}}$

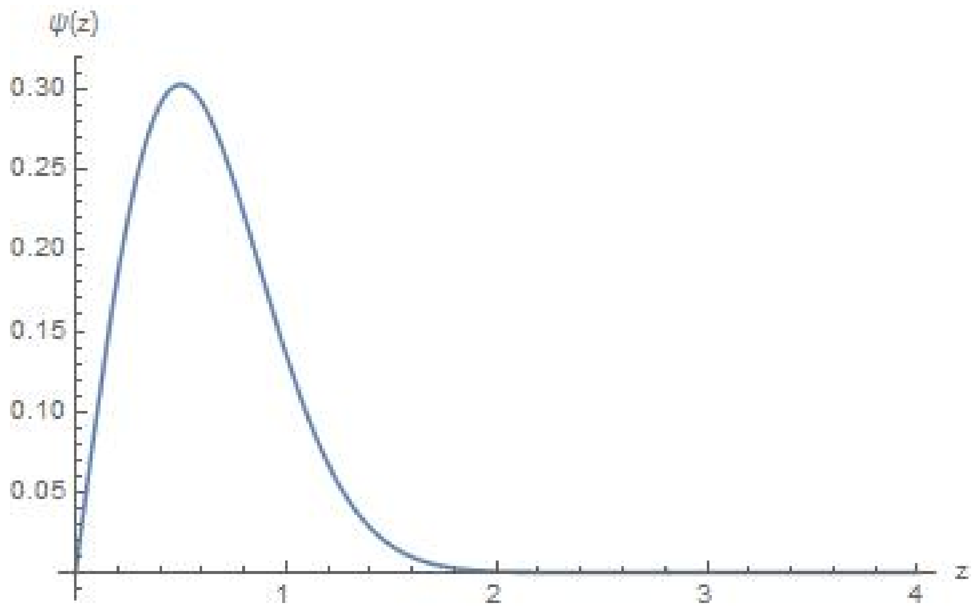

Figure 1: Plot of $\psi(z)$

$\frac{d}{dz}\langle H\rangle = \frac{3\hbar^2}{2m} - \frac{gm}{\sqrt{2\pi}\alpha^{3/2}}$

In order to get the minimum value of $\langle H\rangle$, we have to let $\frac{d}{dz}\langle H\rangle = 0$. Therefore $\alpha = \frac{\sqrt[3]{\frac{2}{\pi}}}{3^{2/3}\left(\frac{\hbar^2}{gm^2}\right)^{2/3}}$

Plug in the value of α and we get $E_{gs} \approx \langle H\rangle_{min} = \sqrt[3]{\frac{81mg^2\hbar^2}{4\pi}}$

The Mathematica code used in this part is shown below:

```
Simplify[Integrate[
  A^2*z^2*Exp[-2*\[Alpha]*z^2], {z, 0,
    Infinity}], \[Alpha] \[Element] Reals && \[Alpha] > 0]


Simplify[Solve[(A^2 Sqrt[\[Pi]/2])/(8 \[Alpha]^(3/2)) == 1, A], A
    \[Element] Reals && A > 0]


A := (2 2^(3/4) \[Alpha]^(3/4))/\[Pi]^(1/4)
```

```
\[Psi][z_] := A*z*Exp[-\[Alpha]*z^2]


Simplify[-(\[HBar]^2/(2*m))*
  Integrate[\[Psi][z]*D[\[Psi][z], {z, 2}], {z, 0,
    Infinity}], \[Alpha] \[Element] Reals && \[Alpha] > 0]


T[\[Alpha]_] := (3 \[Alpha] \[HBar]^2)/(2 m)


Simplify[m*g*
  Integrate[
   z*\[Psi][z]*\[Psi][z], {z, 0, Infinity}], \[Alpha] \[Element]
   Reals && \[Alpha] > 0]


V[\[Alpha]_] := (g m Sqrt[2/\[Pi]])/Sqrt[\[Alpha]]
H[\[Alpha]_] := T[\[Alpha]] + V[\[Alpha]]
D[H[\[Alpha]], {\[Alpha], 1}]


Simplify[Solve[-((g m)/(Sqrt[2 \[Pi]] \[Alpha]^(3/2))) + (3 \[HBar
   ]^2)/(2 m) == 0, \[Alpha]], \[Alpha] >
   0 && \[Alpha] \[Element] Reals]


H[(2/\[Pi])^(1/3)/(3^(2/3) (\[HBar]^2/(g m^2))^(2/3))]
```

(b)

In classical region, $E > V(x)$

At the turning point, $E = V(x)$. Therefore the turning point is $a = \frac{E}{mg}$

Let $p(z) = \sqrt{2m(E - mgz)}$

In classical region, we have $\psi_1(z) = \frac{C}{\sqrt{p}}\left[\sin\phi(z) + \cos\phi(z)\right]$, where $\phi(z) = \frac{1}{\hbar}\int_0^z p(z')dz'$

$\because \psi_1(0) = 0$

$\therefore \cos\phi(0) = 0$

$\psi_1(z) = \frac{C}{\sqrt{p(z)}}\sin\left[\frac{1}{\hbar}\int_0^z p(z')dz'\right] = \frac{C}{\sqrt{p(z)}}\cos\left[\frac{1}{\hbar}\int_0^z p(z')dz' - \frac{\pi}{2}\right]$ for $0 < z < a$

At the turning point, according to the connection formulas shown in textbook, we have

$\psi_2(z) = \frac{C'}{\sqrt{p(z)}}\sin\left[\frac{1}{\hbar}\int_z^a p(z')dz' + \frac{\pi}{4}\right] = \frac{C'}{\sqrt{p(z)}}\cos\left[\frac{1}{\hbar}\int_z^a p(z')dz' - \frac{\pi}{4}\right]$ for $0 < z < a$

Since we need $\psi_1(z) = \psi_1(z)$, we have

$\frac{1}{\hbar}\int_0^z p(z')dz' - \frac{\pi}{2}\frac{1}{\hbar}\int_z^a p(z')dz' - \frac{\pi}{4} = n\pi$

$\therefore \frac{1}{\hbar}\int_0^a p(z)dz = (n + \frac{3}{4})\pi$

$\int_0^a \sqrt{2m(E - mgz)}dz = (n + \frac{3}{4})\hbar\pi$

Use Mathematica, we therefore find

$E = \frac{1}{3}\sqrt[3]{9\pi^2\hbar^2 g^2 m}\left(n + \frac{3}{4}\right)^{2/3}$

The Mathematica code used in this part is shown below:

```
Solve[Integrate[
   Sqrt[2*m*(En - m*g*z)], {z, 0,
    En/(m*g)}] == (n + 3/4)*\[HBar]*\[Pi], En]
```

(c)

The time independent Schrodinger equation is

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dz^2} + V(x)\psi = E\psi$

The Schrodinger equation in this question is therefore

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dz^2} + mgz\psi = E\psi$

$\because x = \frac{z}{b}$

$\therefore dz^2 = b^2 dx^2$

The Schrodinger equation becomes $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + b^2 mgz\psi = b^2 E\psi$

This is equivalent to $-\frac{1}{2}\frac{d^2\psi}{dx^2} + \frac{m^2 g b^3}{\hbar^2} x\psi = \frac{mb^2}{\hbar^2} E\psi$

$\because b = \left(\frac{\hbar^2}{m^2 g}\right)^{1/3}$

$-\frac{1}{2}\frac{d^2\psi}{dx^2} + \frac{m^2 g}{\hbar^2}\frac{\hbar^2}{m^2 g} x\psi = \frac{mb^2}{\hbar^2} E\psi$

$\therefore -\frac{1}{2}\frac{d^2\psi}{dx^2} + x\psi = \epsilon E\psi$, where $\epsilon = \frac{E}{\hbar^2/mb^2}$

(d)

When $\epsilon \approx 1.85576$, $n = 1$, the plot is shown in Figure 2. In this case, $E = 1.85576\frac{\hbar^2}{mb^2} = 1.85576\sqrt[3]{mg^2\hbar^2}$

When $\epsilon \approx 3.24464$, $n = 2$, the plot is shown in Figure 3. In this case, $E = 3.24464\sqrt[3]{mg^2\hbar^2}$

When $\epsilon \approx 4.38491$, $n = 3$, the plot is shown in Figure 5. In this case, $E = 4.38491\sqrt[3]{mg^2\hbar^2}$

When $\epsilon \approx 16.29999$, $n = 20$, the plot is shown in Figure 5. In this case, $E = 16.29999\sqrt[3]{mg^2\hbar^2}$

The Mathematica code used in this part is shown below:

```
Clear[\[Epsilon], \[Psi]]

pfun = ParametricNDSolveValue[{-(1/2)*\[Psi]''[x] +
```

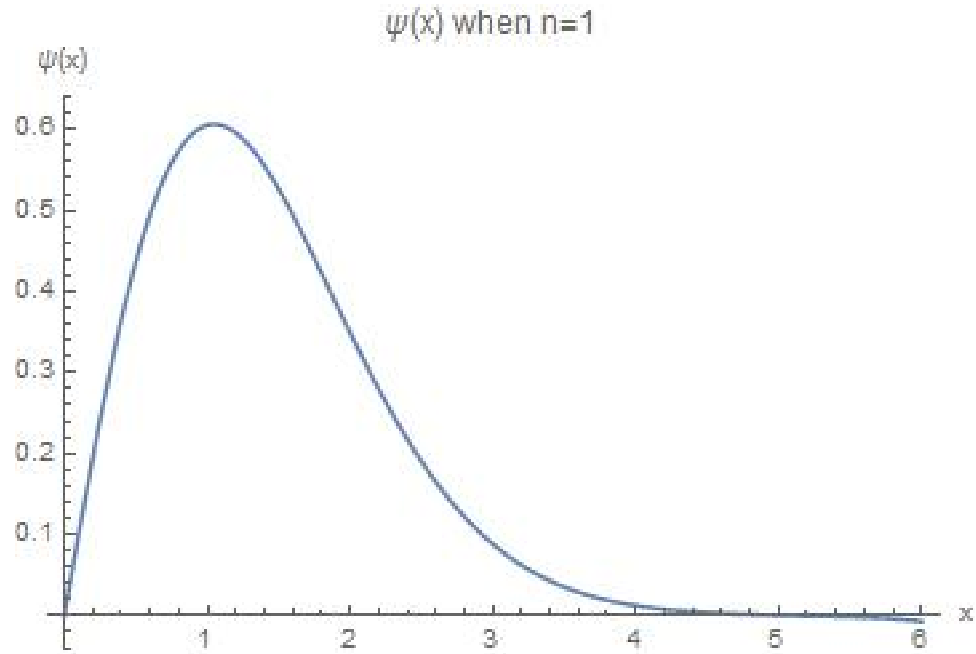

Figure 2: Plot of $\psi(x)$ when $n = 1$

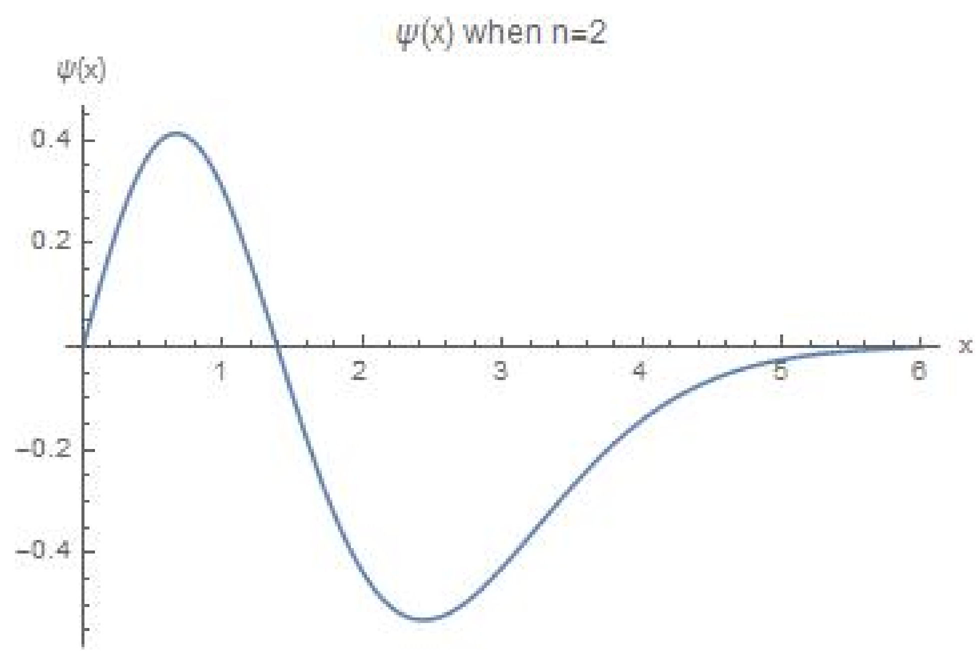

Figure 3: Plot of $\psi(x)$ when $n = 2$

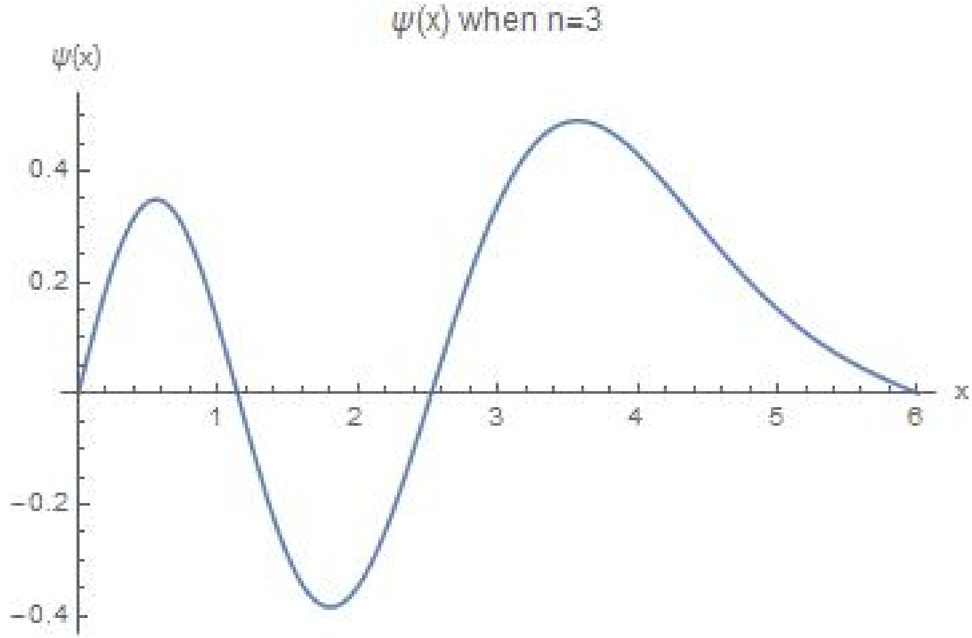

Figure 4: Plot of $\psi(x)$ when $n = 3$

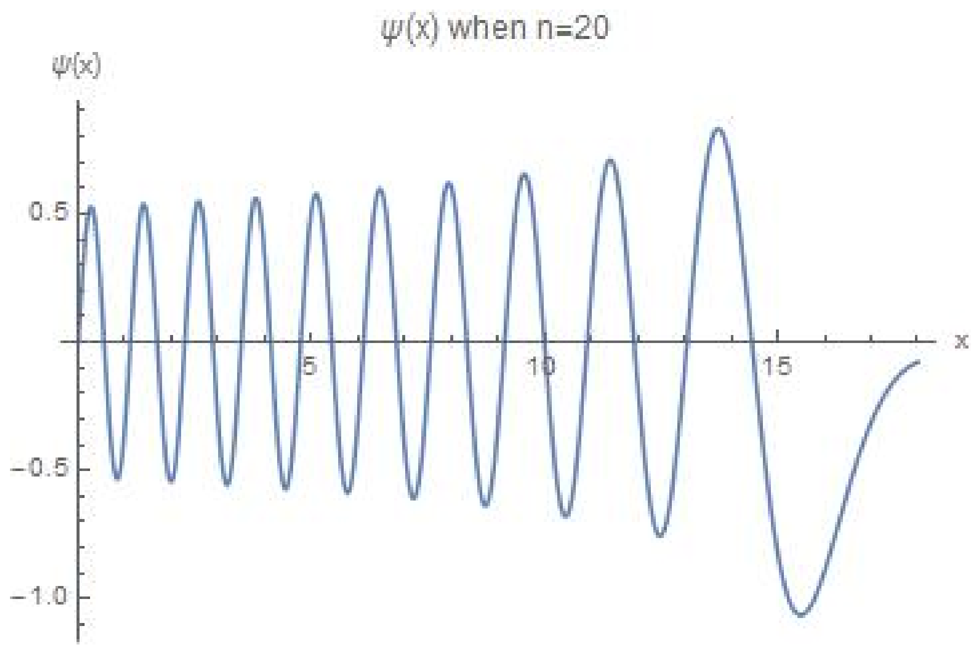

Figure 5: Plot of $\psi(x)$ when $n = 20$

```
    x*\[Psi][x] == \[Epsilon]*\[Psi][x], \[Psi][0] ==
   0, \[Psi]'[0] == 3}, \[Psi], {x, 0, 20}, \[Epsilon]]

Plot[pfun[1.85576][x], {x, 0, 6}, PlotLabel -> "\[Psi](x) when n=1
  ",
 AxesLabel -> {"x", "\[Psi](x)"}]

Plot[pfun[3.24464][x], {x, 0, 6}, PlotLabel -> "\[Psi](x) when n=2
  ",
 AxesLabel -> {"x", "\[Psi](x)"}]

Plot[pfun[4.38491][x], {x, 0, 6}, PlotLabel -> "\[Psi](x) when n=3
  ",
 AxesLabel -> {"x", "\[Psi](x)"}]

Plot[pfun[16.29999][x], {x, 0, 18},
 PlotLabel -> "\[Psi](x) when n=20", AxesLabel -> {"x", "\[Psi](x)
   "}]
```

(e)

	Numerical	WKB	Variational	WKB Error %	Variational Error %
$n=1$	$1.85576\sqrt[3]{mg^2\hbar^2}$	$0.48407\sqrt[3]{9\pi^2\hbar^2g^2m}$	$\sqrt[3]{\frac{81mg^2\hbar^2}{4\pi}}$	16.3859%	0.2851%
$n=2$	$3.24464\sqrt[3]{mg^2\hbar^2}$	$0.65429\sqrt[3]{9\pi^2\hbar^2g^2m}$	N/A	10.0258%	N/A
$n=3$	$4.38491\sqrt[3]{mg^2\hbar^2}$	$0.80457\sqrt[3]{9\pi^2\hbar^2g^2m}$	N/A	18.1314%	N/A
$n=20$	$16.29999\sqrt[3]{mg^2\hbar^2}$	$2.51704\sqrt[3]{9\pi^2\hbar^2g^2m}$	N/A	31.1003%	N/A

2

(a)

We let $|\psi_{gs}\rangle = |\psi_1\rangle$

Then $\langle\psi|\psi_1\rangle = 0$

Since we know that $\psi = \sum\limits_{n=1}^{\infty} C_n\psi_n$, $\langle\psi|\psi_1\rangle = \sum\limits_{n=1}^{\infty} C_n\langle\psi_n|\psi_1\rangle = \sum\limits_{n=1}^{\infty} C_n\delta_{n1}$

$\because \langle\psi|\psi_1\rangle = 0$

$\therefore C_1 = 0$

We also know that $\langle H\rangle = \langle\psi|H|\psi\rangle = \sum\limits_{n=1}^{\infty} |C_n|^2 E_n$

$\because C_1 = 0$

$\therefore \langle H\rangle = \sum\limits_{n=2}^{\infty} |C_n|^2 E_n$

Since the excited state energies are larger than the ground state energies, $\langle H\rangle \geq \sum\limits_{n=2}^{\infty} |C_n|^2 E_{gs}$

$\sum\limits_{n=2}^{\infty} |C_n|^2 \leq 1$

$\therefore \langle H\rangle \geq E_{gs}$

(b)

We know that $\int_{-\infty}^{\infty} |\psi(x)|^2 = 1$

Using Mathematica, we get that $\int_{-\infty}^{\infty} A^2 x^2 e^{-2bx^2} = \frac{\sqrt{\frac{\pi}{2}}A^2}{4b^{3/2}}$

$\therefore A^2 = 4b\sqrt{\frac{2b}{\pi}}$

As for the harmonic oscillator, $H = -\frac{\hbar^2}{2m}\frac{d}{dx^2} + \frac{1}{2}m\omega^2 x^2$

$\langle T \rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \psi^*(x)\frac{d^2\psi(x)}{dx^2}dx = \frac{3b\hbar^2}{2m}$

$\langle V \rangle = \frac{1}{2}m\omega^2 \int_{-\infty}^{\infty} \psi^*(x)x^2\psi(x)dx = \frac{3m\omega^2}{8b}$

$\therefore \langle H \rangle = \langle T \rangle + \langle V \rangle = \frac{3m\omega^2}{8b} + \frac{3b\hbar^2}{2m}$

$\frac{d\langle H \rangle}{db} = \frac{3\hbar^2}{2m} - \frac{3m\omega^2}{8b^2}$

In order to find the minimum value, we have to let $\frac{d\langle H \rangle}{db} = 0$

Therefore, $b = \frac{m\omega}{2\hbar}$

Plug in the value of b, we get that $\langle H \rangle_{min} = \frac{3\omega\hbar}{2}$

The exact value of the first excited state is $E_1 = \frac{3}{2}\hbar\omega$. Theretofore, $\langle H \rangle_{min} = E_1$

The Mathematica code used in this part is shown below:

```
Clear[\[Psi], A, b]

\[Psi][x_] := A*x*Exp[-b*x^2]

Simplify[Integrate[Abs[\[Psi][x]]^2, {x, -Infinity, Infinity}],

 A \[Element] Reals && b \[Element] Reals && A > 0 && b > 0]



Solve[(A^2 Sqrt[\[Pi]/2])/(4 b^(3/2)) == 1, A]



T[x_] := -(\[HBar]^2/(2*m))*
```

```
  Integrate[\[Psi][x]*D[\[Psi][x], {x, 2}], {x, -Infinity,
    Infinity}]
V[x_] := (1/2)*m*\[Omega]^2*
  Integrate[x^2*\[Psi][x]*\[Psi][x], {x, -Infinity, Infinity}]
En[x_] := T[x] + V[x]


D[En[x], {b, 1}]


Solve[-((3 m \[Omega]^2)/(8 b^2)) + (3 \[HBar]^2)/(2 m) == 0, b]


b := (m \[Omega])/(2 \[HBar])
En[x]
```

3

(a)

We know that the probability of tunneling is $T \approx e^{-2\gamma}$

Here $\gamma \equiv \frac{1}{\hbar}\int_0^a |p(x)|dx$ and $p(x) = \sqrt{2m(V_0 - e\mathcal{E}x - E)} = \sqrt{2m(W - e\mathcal{E}x)}$

The end point a is given by $W = e\mathcal{E}a$

$\therefore a = \frac{W}{e\mathcal{E}}$

In this case, $\gamma = \frac{1}{\hbar}\int_0^{\frac{W}{e\mathcal{E}}} \sqrt{2m(W - e\mathcal{E}x)}dx = \frac{2\sqrt{2}W\sqrt{mW}}{3e\mathcal{E}\hbar}$

$\therefore T(\mathcal{E}) = e^{-\mathcal{E}_0/\mathcal{E}}$, where $\mathcal{E}_0 = \frac{4}{3}\frac{\sqrt{2m}}{\hbar}\frac{W^{3/2}}{e}$

The Mathematica code used in this part is shown below:

```
\[Gamma] :=
 1/\[HBar]*
  Integrate[
   Sqrt[2 m*(W - e*\[CapitalEpsilon]*x)], {x, 0,
    W/(e*\[CapitalEpsilon])}]
```

(b)

The mass of electron is $m = 9.10938291 * 10^{-31} kg$

Using Mathematica, we can get $\mathcal{E}_0 = 6.62971 * 10^{10}$

Since $\frac{\mathcal{E}_0}{\mathcal{E}} = 50$, we get that $\mathcal{E} = \frac{\mathcal{E}_0}{50} = 1.32594 * 10^9 V \cdot m^{-1}$

(c)

From part (a) we know that the endpoint a is given by $a = \frac{W}{e\mathcal{E}}$

$\therefore L = \frac{4.55*1.6*10^{-19}}{1.6*10^{-19}*1.32594*10^9} m = 3.4315 * 10^{-9} m$