PHY 362K Homework 1

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1

(a)

The normalized wave function satisfies the condition that $\int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = 1$

Therefore, $\int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = A^2 \int_{-\infty}^{\infty} \left(1 + 3\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{\hbar}x^2} dx = \frac{11\sqrt{\pi}}{2\sqrt{\frac{m\omega}{\hbar}}} A^2 = 1$

$A^2 = \frac{2}{11}\sqrt{\frac{m\omega}{\pi\hbar}}$ and so $A = \sqrt{\frac{2}{11}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

(b)

From (a) we can get $\Psi(x,0) = \sqrt{\frac{2}{11}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(1 + 3\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2} = \sqrt{\frac{2}{11}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} + \sqrt{\frac{18}{11}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2}$

We know that $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$ and $\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{2}\sqrt{\frac{m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2}$

Therefore we can know that $\Psi(x,0)$ can be represented as a linear combination of $\psi_0(x)$ and $\psi_1(x)$, which is $\Psi(x,0) = c_0\psi_0(x) + c_1\psi_1(x)$

$\therefore c_0 = \sqrt{\frac{2}{11}}$ and $c_1 = \sqrt{\frac{9}{11}}$

$\because E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

$\therefore E_0 = \frac{1}{2}\hbar\omega$ and $E_1 = \frac{3}{2}\hbar\omega$

$\langle E\rangle = |c_0|^2E_0 + |c_1|^2E_1 = \frac{1}{11}\hbar\omega + \frac{27}{22}\hbar\omega = \frac{29}{22}\hbar\omega$

Therefore the particle has energy $\frac{1}{2}\hbar\omega$ with probability $\frac{2}{11}$ and has energy $\frac{3}{2}\hbar\omega$ with probability $\frac{9}{11}$. The expected value of energy is $\frac{29}{22}\hbar\omega$

(c)

$$\Psi(x,t) = \frac{2}{11}\psi_0(x)e^{-\frac{iE_0}{\hbar}t} + \frac{9}{11}\psi_1(x)e^{-\frac{iE_1}{\hbar}t} = \frac{2}{11}\psi_0(x)e^{-\frac{i\omega}{2}t} + \frac{9}{11}\psi_1(x)e^{-\frac{i3\omega}{2}t}$$

where $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$ and $\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2}$

2

We can let $|a_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|a_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $|a_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

We also let $B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}$

$B|a_1\rangle = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_4 \\ b_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$B|a_2\rangle = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b_2 \\ b_5 \\ b_8 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$$B|a_3\rangle = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b_3 \\ b_6 \\ b_9 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\therefore B = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

3

(a)

From the equation of wave function for hydrogen, we get

$|2,1,1\rangle = -\sqrt{\left(\frac{2}{a}\right)^3 \frac{1}{4*6^3}} e^{-\frac{r}{2a}} \left(\frac{r}{a}\right) 6\sqrt{\frac{3}{8\pi}} \sin(\theta) e^{i\phi} = -\sqrt{\frac{1}{64\pi a^3}} \left(\frac{r}{a}\right) e^{-\frac{r}{2a}} \sin(\theta) e^{i\phi}$

Since $|2,1,-1\rangle$ only differs with $|2,1,1\rangle$ in the term $Y_l^m(\theta,\phi)$, we can know that

$|2,1,-1\rangle = \sqrt{\frac{1}{64\pi a^3}} \left(\frac{r}{a}\right) e^{-\frac{r}{2a}} \sin(\theta) e^{-i\phi}$

Therefore $\langle\vec{r}|\psi\rangle = -\sqrt{\frac{1}{80\pi a^3}} \left(\frac{r}{a}\right) e^{-\frac{r}{2a}} \sin(\theta) e^{i\phi} - \sqrt{\frac{1}{320\pi a^3}} \left(\frac{r}{a}\right) e^{-\frac{r}{2a}} \sin(\theta) e^{-i\phi}$

where $a \equiv 0.0529nm$

(b)

In both cases $\phi = 0$. Therefore the wave function can be simplified to

$\langle\vec{r}|\psi\rangle = -\left(\sqrt{\frac{1}{320\pi a^3}} + \sqrt{\frac{1}{80\pi a^3}}\right) \left(\frac{r}{a}\right) e^{-\frac{r}{2a}} \sin(\theta)$

(1) When the coordinate is (0.1nm,0,0), $r = 0.1$ and $\theta = 0$

Therefore $\sin(\theta) = 0$ and $\langle\vec{r}|\psi\rangle = 0$

The volume $V = 0.1 * 0.1 * 0.1nm^3 = 10^{-30}m$

The probability density $\langle\psi|\vec{r}\rangle\langle\vec{r}|\psi\rangle = 0$ and so the probability $Pr = \langle\psi|\vec{r}\rangle\langle\vec{r}|\psi\rangle * V = 0$

(2) When the coordinate is (0,-0.1,0), $r = 0.1$ and $\theta = \frac{3\pi}{2}$

Therefore $\sin(\theta) = -1$ and $\langle\vec{r}|\psi\rangle = -\left(\sqrt{\frac{1}{80\pi a^3}} + \sqrt{\frac{1}{320\pi a^3}}\right)\left(\frac{1000}{529}\right)e^{-\frac{5000}{529}}$

The probability density $\langle\psi|\vec{r}\rangle\langle\vec{r}|\psi\rangle = \left(\sqrt{\frac{1}{80\pi a^3}} + \sqrt{\frac{1}{320\pi a^3}}\right)^2 e^{-\frac{0.1}{0.0529}} * (\frac{0.1}{0.0529})^2 \approx 3.264 * 10^{28}$

Therefore the probability $Pr = \langle\psi|\vec{r}\rangle\langle\vec{r}|\psi\rangle * V \approx= 0.0326$

(c)

The eigenvalue of L_z operator is $\hbar m$

We know that $|\psi\rangle = \frac{2}{\sqrt{5}}|2,1,1\rangle - \sqrt{\frac{1}{5}}|2,1,-1\rangle$

Therefore, the measurement of L_z is $\hbar$ with probability $\frac{4}{5}$ and $-\hbar$ with probability $\frac{1}{5}$

4

i. $|\gamma\rangle = |\psi\rangle + i|\phi\rangle = \begin{pmatrix} -i \\ i \end{pmatrix} + i\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -i-1 \\ i+i \end{pmatrix} = \begin{pmatrix} -i-1 \\ 2i \end{pmatrix}$

ii. $\langle\psi| = \begin{pmatrix} i & -i \end{pmatrix}$

iii. $\langle\gamma| = \begin{pmatrix} i+1 & -2i \end{pmatrix}$

iv. $\langle\psi|\phi\rangle = \begin{pmatrix} i & -i \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = -1-i$

v. $\langle\phi|\psi\rangle = \begin{pmatrix} -i & 1 \end{pmatrix}\begin{pmatrix} -i \\ i \end{pmatrix} = -1+i$

vi. $A^+ = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}$

vii. $A|\psi\rangle = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} -i \\ i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

viii. $\langle\psi|A = \begin{pmatrix} i & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & i \end{pmatrix}$

ix. $\langle\psi|A^+ = \begin{pmatrix} i & -i \end{pmatrix} \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 1 \end{pmatrix}$

x. $(AB)^+ = \left(\begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)^+ = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & -i \\ -1 & 0 \end{pmatrix}$

xi. $\langle\psi|A|\phi\rangle = \begin{pmatrix} i & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = 2i$

xii. $\langle\psi|A^+|\phi\rangle = \begin{pmatrix} i & -i \end{pmatrix} \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = 2$

xiii. $\langle\phi|A|\psi\rangle = \begin{pmatrix} -i & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} -i \\ i \end{pmatrix} = 2$

xiv. $|\phi\rangle\langle\psi| = \begin{pmatrix} i \\ 1 \end{pmatrix} \begin{pmatrix} i & -i \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ i & -i \end{pmatrix}$

xv. $|\phi\rangle\langle\psi|B = \begin{pmatrix} i \\ 1 \end{pmatrix} \begin{pmatrix} i & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ i & i \end{pmatrix}$

5

(a)

Normalizing $\Phi(p,0)$ means $\int_{-\infty}^{\infty} |\Phi(p,0)|^2 dp = 1$

$\therefore \int_{-\infty}^{\infty} |\Phi(p,0)|^2 dp = N^2 \int_{-\infty}^{\infty} e^{-\frac{2a}{\hbar}|p|} = N^2 \frac{\hbar}{a} = 1$

Therefore $N = \sqrt{\frac{a}{\hbar}}$

(b)

According to the Fourier Transformation,

$\Psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{\infty} \Phi(p,0) e^{i\frac{p}{\hbar}x} dp$

$\Phi(p,0) = \sqrt{\frac{a}{\hbar}} e^{-\frac{a|p|}{\hbar}}$

Therefore, $\Psi(x,0) = \sqrt{\frac{2a^3}{\pi}} \frac{1}{a^2+x^2}$

(c)

$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x,0)|^2 dx = 0$

$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\Psi(x,0)|^2 dx = \frac{a^2}{2}$

$\Delta x_{rms} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{a}{\sqrt{2}}$

$\langle p \rangle = \int_{-\infty}^{\infty} p |\Phi(p,0)|^2 dx = 0$

$\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 |\Phi(p,0)|^2 dx = \frac{\hbar^2}{2a^2}$

$\Delta p_{rms} = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar}{\sqrt{2}a}$

$\therefore \Delta x_{rms} \Delta p_{rms} = \frac{\hbar}{2} \geq \frac{\hbar}{2}$

Hence the uncertainty principle preserves

(d)

Yes, the momentum space wave function changes with time

The Fourier Transform gives:

$$\Phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{\infty} \psi(x) e^{-i\frac{p}{\hbar}x} dx$$

$$\therefore \Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{\infty} \psi(x) e^{-i(\frac{p}{\hbar}x + \frac{p^2 t}{2m\hbar})} dx$$

The probability does not change with time since the probability density does not integrate with time

(e)

Yes, the position space wave function changes with time

The Inverse Fourier Transform gives:

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{\infty} \Phi(p,0) e^{i\frac{p}{\hbar}x} dp$$

$$\therefore \Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{\infty} \Phi(p,0) e^{i(\frac{p}{\hbar}x - \frac{p^2 t}{2m\hbar})} dp$$

The probability does not change with time since the probability density does not integrate with time