Problem Set 1

Econometrics III

Jackson Bunting, Attila Gyetvai, Peter Horvath, Leonardo Salim Saker Chaves*

September 10, 2016

Problem 1. Verify that the uniform metric defined as $d(f,g) = \sup_{t\in\mathbb{T}} |f(t) - g(t)|,\ f,g \in l^\infty(\mathbb{T})$ is indeed a metric.

Solution: Let $f(t), g(t), h(t) \in l^\infty(\mathbb{T})$. We will show that the uniform metric satisfies the definition of a metric. That is,

1. $\sup_{t\in\mathbb{T}} |f(t) - g(t)| = 0$ if and only if $f(t) = g(t)$ since $|\cdot| \geq 0$ and $|a| = 0$ if and only if $a = 0$
2. $|f(t) - g(t)| = |g(t) - f(t)|$, therefore $\sup_{t\in\mathbb{T}} |f(t) - g(t)| = \sup_{t\in\mathbb{T}} |g(t) - f(t)|$
3. $|f(t) - g(t)| = |f(t) - h(t) + h(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|$ by triangle inequality. Therefore, $\sup_{t\in\mathbb{T}} |f(t) - g(t)| \leq \sup_{t\in\mathbb{T}} |f(t) - h(t)| + \sup_{t\in\mathbb{T}} |h(t) - g(t)|$

Problem 2.

Problem 3. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of random variables with mean μ. Suppose that $Cov(X_t, X_s) = 0 \forall t \neq s$ and $Var(X_t) \leq Kt^{1/2}$ for some constant $K > 0$. Show that $\bar{X}_t \xrightarrow{\mathbb{P}} \mu$.

Solution: Consider any $\epsilon > 0$. Using Markov inequality we get

$$\begin{aligned} P(|\bar{X}_t - \mu| > \epsilon) &\leq \frac{\mathbb{E}(|\bar{X}_t - \mu|^2)}{\epsilon^2} = \frac{V(\bar{X}_t)}{\epsilon^2} \\ &\leq \frac{\sum_{t=1}^T V(X_t)}{T^2\epsilon^2} = \frac{K\sum_{t=1}^T t^{1/2}}{T^2\epsilon^2} \\ &\leq \frac{KT^{3/2}}{T^2\epsilon^2} \xrightarrow{t\to\infty} 0 \end{aligned}$$

Then $\bar{X}_t - \mu \xrightarrow{\mathbb{P}} 0$ which is equivalent to what we wanted to show.

*Department of Economics, Duke University

Problem 4.

Problem 5. Suppose that $\sup_{t\in\mathcal{T}} \mathbb{E}(|X_t|^{1+\delta}) < \infty$ for some $\delta > 0$. Then $\{X_t : t \in \mathcal{T}\}$ is uniformly integrable. (Hint: Markov's inequality)

Solution: Notice that $\frac{|X_t|}{M} \geq I[|X_t| > M] \implies \frac{|X_t|^\delta}{M^\delta} \geq I[|X_t| > M]$.
Using that $|X_t| \geq 0$,

$$\begin{aligned}
&\frac{|X_t|^{1+\delta}}{M^\delta} \geq |X_t| I[|X_t| > M] \forall\ t \in \mathcal{T} \\
&\implies \sup_{t\in\mathcal{T}} \mathbb{E}(|X_t|^{1+\delta})(1/M^\delta) \geq \sup_{t\in\mathcal{T}} \mathbb{E}(|X_t| I[|X_t| > M]) \forall\ M \in \mathbb{N} \\
&\implies \lim_{M\to\infty} \sup_{t\in\mathcal{T}} \mathbb{E}(|X_t|^{1+\delta})(1/M^\delta) = 0 \geq \lim_{M\to\infty} \sup_{t\in\mathcal{T}} \mathbb{E}(|X_t| I[|X_t| > M]) \geq 0
\end{aligned}$$

The inequalities above give that X_t is uniformly integrable.

Problem 6.

Problem 7. (Characteristic Function) Let X_n and X be $\mathbb{R}^d$-valued random variables. Suppose that $X_n \Rightarrow X$. Show that for any $\theta \in \mathbb{R}^d$, we have the following: (a) $\mathbb{E}\left[\cos(\theta' X_n)\right] \to \mathbb{E}[\cos(\theta' X)]$ (b) $\mathbb{E}[\sin(\theta' X_n)] \to \mathbb{E}[\sin(\theta' X)]$; (c) the characteristic function of X_n converges to the characteristic function of X.

Solution:
(a) $f(x) = \cos(x)$ is a continuous bounded function on $\mathbb{R}$. Invoking the definition of weak convergence, we get that $\mathbb{E}\left[\cos(\theta' X_n)\right] \to \mathbb{E}[\cos(\theta' X)]$ since $X_n \Rightarrow X$.

(b) $f(x) = \sin(x)$ is a continuous bounded function on $\mathbb{R}$. Invoking the definition of weak convergence, we get that $\mathbb{E}[\sin(\theta' X_n)] \to \mathbb{E}[\sin(\theta' X)]$ since $X_n \Rightarrow X$.

(c) The characteristic function of Y random variable is $\phi_Y(t) = \mathbb{E}\left[\exp(itY)\right] = \mathbb{E}(\cos(tY) + i\sin(tY))$. From (a) and (b), it follows that $\phi_{X_n}(t) = \mathbb{E}(\cos(tX_n) + i\sin(tX_n)) = \mathbb{E}(\cos(tX_n)) + i\mathbb{E}(\sin(tX_n)) \to \mathbb{E}(\cos(tX)) + i\mathbb{E}(\sin(tX)) = \phi_X(t)$.

Problem 8. (Laplace Transform) Let X_n and X be $\mathbb{R}_+$-valued random variables. Suppose that $X_n \Rightarrow X$. Show that for any $\theta \geq 0$, we have $\mathbb{E}(\exp(-\theta X_n)) \to \mathbb{E}(\exp(-\theta X))$

Solution:
$f(x) = \exp(-\theta x)$ is a bounded continuous function on $\mathbb{R}_+$ if $\theta \geq 0$. Therefore, we can use the definition of weak convergence to get $\mathbb{E}(\exp(-\theta X_n)) \to \mathbb{E}(\exp(-\theta X))$ as $X_n \Rightarrow X$.

Problem 9. Let X_n and X be real-valued random variables with $X_n \xrightarrow{a.s.} X$. Let $r \in \mathbb{R}$ with $\mathbb{P}(X = r) = 0$. Show that (a) $1_{X_n \le r} \xrightarrow{a.s.} 1_{X \le r}$; (b) $\mathbb{P}(X_n \le r) \to \mathbb{P}(X \le r)$. (Hint: bounded convergence theorem). Show (b) but assume $X_n \Rightarrow X$ instead of $X_n \xrightarrow{a.s.} X$. (Hint: Almost sure representation)

Solution:
(a) $f(x) = 1_{x \le r}$ is only discontinuous at r, otherwise it is continuous on the real line. However, $\mathbb{P}(X = r) = 0$, therefore we can apply continuous mapping theorem. That is, if $X_n \xrightarrow{a.s.} X$, then $f(X_n) \xrightarrow{a.s.} f(X)$ for $f(x) = 1_{x \le r}$.

(b) From (a), we know that $1_{X_n \le r} \xrightarrow{a.s.} 1_{X \le r}$ and $|1_{X_n \le r}| \le Y = 2$ for every $n \in \mathbb{N}$ (i.e. Y is some integrable variable). So we can invoke dominated convergence theorem to get that $\mathbb{P}(X_n \le r) = \mathbb{E}\left[1_{X_n \le r}\right] \xrightarrow{a.s.} \mathbb{E}\left[1_{X \le r}\right] = \mathbb{P}(X \le r)$.

(c) Since $X_n \Rightarrow X$, by almost sure representation, $\exists \left((\tilde{X}_n)_{n \in \mathbb{N}} \text{ and } \tilde{X}\right)$, such that $X_n \sim \tilde{X}_n, X \sim \tilde{X}$ and $\tilde{X}_n \xrightarrow{a.s.} \tilde{X}$. From (a) and (b), we know that for $\tilde{X}_n$ and $\tilde{X}$, that $\mathbb{P}(\tilde{X}_n \le r) \to \mathbb{P}(\tilde{X} \le r)$. Since $X_n \sim \tilde{X}_n, X \sim \tilde{X}$, we have that $\mathbb{P}(\tilde{X}_n \le r) = \mathbb{P}(X_n \le r)$ and $\mathbb{P}(\tilde{X} \le r) = \mathbb{P}(X \le r)$.

Problem 10. Let $a_n = o_p(1)$. Interpret and prove that $O_p(a_n) = o_p(1)$.

Solution: If I divide X_n by $a_n = o_p(1)$, it has to be the case that $X_n = o_p(1)$. The proof is as it follows:

$$X_n = O_p(a_n) \iff \frac{X_n}{a_n} = O_p(1);$$
$$X_n = \frac{X_n}{a_n}\, a_n = O_p(1)\, o_p(1) = o_p(1)$$

The first line is just the definition of $O_p(a_n)$. The second line comes from the o_p–O_p identities in the notes.

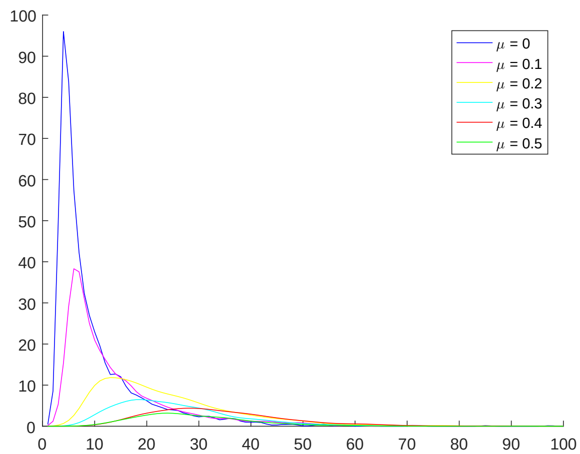
Problem 11. Interpret and prove that $e^{o_p(1)} - 1 = o_p(1)$ and $(O_p(1))^{\sqrt{2}} = O_p(1)$.

Solution: If you pick a r.v. $X = o_p(1)$, then a continuous transformation will also be $o_p(1)$. The proof is a straightforward consequence of CMT.
For the second claim, the interpretation is that you preserve $O_p(1)$ property when you take a continuous function of this r.v.. To prove, let $X_n = O_p(1)$. By definition, $\exists M > 0; \sup_{n \in \mathbb{N}} P(|X_n| > M) < \epsilon)$. Notice that $\{\omega \in \Omega : |X_n(\omega)| > M\} = \{\omega \in \Omega : |X_n(\omega)|^{\sqrt{2}} > M^{\sqrt{2}}\}$ because it is a monotonic transformation. Thus, $\exists \eta = M^{\sqrt{2}} > 0; \sup_{n \in \mathbb{N}} P(|X_n^{\sqrt{2}}| > \eta) < \epsilon)$.

Problem 12. Let $f(x) = \exp(x^2)$. Let $(X_i)_{1 \le i \le n}$ be iid $N(\mu, 1)$ variables. Simulate such a sequence with $n = 100$ and $\mu = 0.3$. Compute $f(\bar{X}_n)$ where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Repeat this for 2000 times. Plot the distribution of $f(\bar{X})$ in the simulation. What is the asymptotic distribution of $f(\bar{X})$? Is the asymptotic distribution a good approximation to the empirical distribution? Repeat for $\mu = 0$, 0.1, 0.2, 0.3, 0.4, 0.5. Discuss the results.

Solution: We calculate the asymptotic distribution of $f(\bar{X}_n)$ using CLT and the delta method. Since $X_i \sim N(\mu, 1)$, it follows from the CLT that $\sqrt{n}(\bar{X}_n - \mu) \Rightarrow N(0,1)$. By the delta method, $\sqrt{n}(\exp(\bar{X}_n^2) - \exp(\mu^2)) \Rightarrow N(0, (2\mu\exp(\mu^2))^2)$. When $\mu = 0.3$, $\sqrt{n}(\exp(\bar{X}_n^2) - \exp(0.3^2)) \Rightarrow N(0, 0.43)$.
However, note that when $\mu = 0$, the asymptotic distribution becomes degenerate in which case we use a version of the delta method that relies on a second-order Taylor expansion: $\sqrt{n}(\exp(\bar{X}_n^2) - \exp(\mu^2)) \Rightarrow \frac{1}{2} \cdot 2\exp(\mu^2)(2\mu^2+1) \cdot \chi^2(1)$. Hence in the case of $\mu = 0$, $\sqrt{n}(\exp(\bar{X}_n^2)) \Rightarrow \chi^2(1)$.
We plot the simulated distributions below. As the above analysis suggests, the empirical distribution with $\mu = 0$ resembles the χ^2-distribution. As μ increases, the empirical distributions look more and more normal.

Figure 1: Simulated empirical distributions



Problem 13. Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of iid random variables with mean μ and variance σ^2. Suppose that for some $\delta > 0$, $\mathbb{E}\left[|X_i|^{2+\delta}\right] < \infty$. Use Lindeberg's CLT to show that $n^{1/2}(\frac{1}{n}\sum_{i=1}^n X_i - \mu) \implies N(0, \sigma^2)$.
Solution: Initially consider $X_{n,i} = \frac{X_i - \mu}{\sqrt{n}}$. Now, we need to verify the assumptions of Lindeberg's CLT.

1. $\{X_{n,i}\}$ independent, real-valued, and zero mean.
   Since we are applying a continuous function to rv's which are iid by assumption, independence will be preserved. Moreover, considering $X_i : \Omega \mapsto \mathbb{R}$, we will have $X_{n,i}$ real-valued. And notice that $\mathbb{E}[X_{n,i}] = \mathbb{E}[(X_i - \mu)/\sqrt{n}] = 0$.

2. $\sum_{i=1}^n V(X_{n,i} = \frac{\sum_{i=1}^n \sigma^2}{n} = \sigma^2$

3. $\lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}[X_{n,i}^2 I[|X_{n,i}| > \epsilon]] = 0$
   Using an argument similar to Q5, we have $\frac{|X_{n,i}|^{2+\delta}}{\epsilon^\delta} \geq I[|X_{n,i}| > \epsilon]$. Therefore,

$$\frac{\mathbb{E}(|X_i-\mu|^{2+\delta}}{\epsilon^\delta n^{1+\delta/2}}) \geq \mathbb{E}(|X_{n,i}|^2 I[|X_{n,i} > \epsilon])\forall\ i \in \mathbb{N}$$

Using iid assumption and $\mathbb{E}\left[|X_i|^{2+\delta}\right] < \infty$ we have:

$$\frac{\mathbb{E}(|X_i-\mu|^{2+\delta})}{\epsilon^\delta n^{\delta/2}} \geq \sum_{i=1}^{n} \mathbb{E}(|X_{n,i}|^2 I[|X_{n,i} > \epsilon])$$
$$\implies \lim_{n\to\infty} \frac{\mathbb{E}(|X_i-\mu|^{2+\delta})}{\epsilon^\delta n^{\delta/2}} = 0 \geq \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}(|X_{n,i}|^2 I[|X_{n,i} > \epsilon]) \geq 0$$

Finally, we can apply Lindeberg's CLT to get $n^{1/2}(\frac{1}{n}\sum_{i=1}^n X_i - \mu) = \sum_{i=1}^n X_{n,i} \implies N(0,\sigma^2)$.