Problem Set 1

Econometrics III

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Problem 1.

Problem 2.

Problem 3. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of random variables with mean μ. Suppose that $Cov(X_t, X_s) = 0 \forall t \neq s$ and $Var(X_t) \leq Kt^{1/2}$ for some constant $K > 0$. Show that $\bar{X}_t \xrightarrow{\mathbb{P}} \mu$.

Solution: Consider any $\epsilon > 0$. Using Markov inequality we get

$$\begin{aligned} P(|\bar{X}_t - \mu| > \epsilon) &\leq \frac{\mathbb{E}(|\bar{X}_t - \mu|^2)}{\epsilon^2} = \frac{V(\bar{X}_t)}{\epsilon^2} \\ &\leq \frac{\sum_{t=1}^T V(X_t)}{T^2\epsilon^2} = \frac{K\sum_{t=1}^T t^{1/2}}{T^2\epsilon^2} \\ &\leq \frac{KT^{3/2}}{T^2\epsilon^2} \xrightarrow{t\to\infty} 0 \end{aligned}$$

Then $\bar{X}_t - \mu \xrightarrow{\mathbb{P}} 0$ which is equivalent to what we wanted to show.

Problem 4.

Problem 5. Suppose that $\sup_{t\in\mathcal{T}} \mathbb{E}(|X_t|^{1+\delta}) < \infty$ for some $\delta > 0$. Then $\{X_t : t \in \mathcal{T}\}$ is uniformly integrable. (Hint: Markov's inequality)

Solution: Notice that $\sup_{t\in\mathcal{T}} \mathbb{E}(|X_t| I_{[|X_t|>M]}) \leq \sup_{t\in\mathcal{T}} \mathbb{E}(|X_t|) \sup_{t\in\mathcal{T}} P(X_t > M)$.

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From Markov inequality we have

$$\begin{aligned} P(|X_t| > M) &\leq \frac{\mathbb{E}(|X_t|^{1+\delta})}{M^{1+\delta}} \\ &< \frac{\eta}{M^2} \stackrel{M\to\infty}{\longrightarrow} 0 \end{aligned}$$

Moreover, assuming $\sup_{t\in\mathcal{T}} \mathbb{E}(|X_t|^{1+\delta}) < \infty$ implies that $\sup_{t\in\mathcal{T}} \mathbb{E}(|X_t|) < \infty$.
All of it implies that $\lim_{M\to\infty} \sup_{t\in\mathcal{T}} \mathbb{E}(|X_t| I_{[X_t|>M]}) \leq 0$. Since $|X_t| I_{[X_t|>M]} \geq 0$, we get the uniform integrability of $\{X_t : t \in \mathcal{T}\}$.

Problem 6.

Problem 7.

Problem 8.

Problem 9.

Problem 10. Let $a_n = o_p(1)$. Interpret and prove that $O_p(a_n) = o_p(1)$.

Solution: If I divide X_n by $a_n = o_p(1)$, it has to be the case that $X_n = o_p(1)$. The proof is as it follows:

$$\begin{aligned} X_n = O_p(a_n) &\iff \frac{X_n}{a_n} = O_p(1); \\ X_n &= \frac{X_n}{a_n}\, a_n = O_p(1)\, o_p(1) = o_p(1) \end{aligned}$$

The first line is just the definition of $O_p(a_n)$. The second line comes from the o_p–O_p identities in the notes.

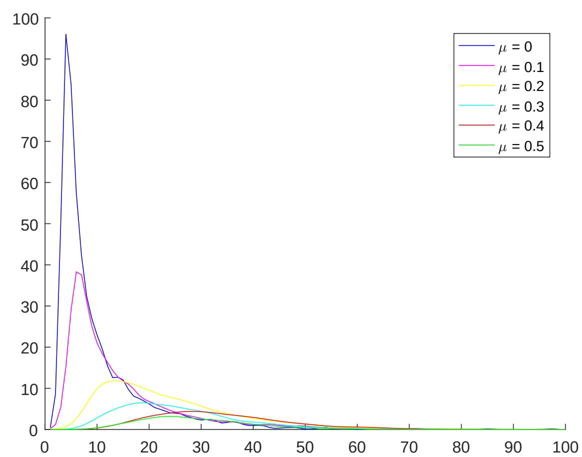
Problem 11. Interpret and prove that $e^{o_p(1)} - 1 = o_p(1)$ and $(O_p(1))^{\sqrt{2}} = O_p(1)$.

Solution: If you pick a r.v. $X = o_p(1)$, then a continuous transformation will also be $o_p(1)$. The proof is a straightforward consequence of CMT.
For the second claim, the interpretation is that you preserve $O_p(1)$ property when you take a continuous function of this r.v.. To prove, let $X_n = O_p(1)$. By definition, $\exists M > 0; \sup_{n\in\mathbb{N}} P(|X_n| > M) < \epsilon)$. Notice that $\{\omega \in \Omega : |X_n(\omega)| > M\} = \{\omega \in \Omega : |X_n(\omega)|^{\sqrt{2}} > M^{\sqrt{2}}\}$ because it is a monotonic transformation. Thus, $\exists \eta = M^{\sqrt{2}} > 0; \sup_{n\in\mathbb{N}} P(|X_n^{\sqrt{2}}| > \eta) < \epsilon)$.

Problem 12. Let $f(x) = \exp(x^2)$. Let $(X_i)_{1\le i\le n}$ be iid $N(\mu, 1)$ variables. Simulate such a sequence with $n = 100$ and $\mu = 0.3$. Compute $f(\bar{X})$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Repeat this for 2000 times. Plot the distribution of $f(\bar{X})$ in the simulation. What is the asymptotic distribution of $f(\bar{X})$? Is the asymptotic distribution a good approximation to the empirical distribution? Repeat for $\mu = 0$, 0.1, 0.2, 0.3, 0.4, 0.5. Discuss the results.

Solution: We plot the simulated empirical distributions below. For $\mu = 0$, the distribution is skewed to the right, resembling the χ^2-distribution. As μ increases (especially after $\mu \ge 0.4$), the distribution gets closer and closer to normal.

Figure 1: Simulated empirical distributions



Problem 13.