Problem Set 1

Econometrics III

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Problem 1. Verify that the uniform metric defined as $d(f,g) = \sup_{t\in\mathbb{T}} |f(t) - g(t)|,\ f, g \in l^\infty(\mathbb{T})$ is indeed a metric.

Solution: Let $f(t), g(t), h(t) \in l^\infty(\mathbb{T})$. We will show that the uniform metric satisfies the definition of a metric. That is,

1. $\sup_{t\in\mathbb{T}} |f(t) - g(t)| = 0$ if and only if $f(t) = g(t)$ since $|\cdot| \geq 0$ and $|a| = 0$ if and only if $a = 0$
2. $|f(t) - g(t)| = |g(t) - f(t)|$, therefore $\sup_{t\in\mathbb{T}} |f(t) - g(t)| = \sup_{t\in\mathbb{T}} |g(t) - f(t)|$
3. $|f(t) - g(t)| = |f(t) - h(t) + h(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|$ by triangle inequality. Therefore, $\sup_{t\in\mathbb{T}} |f(t) - g(t)| \leq \sup_{t\in\mathbb{T}} |f(t) - h(t)| + \sup_{t\in\mathbb{T}} |h(t) - g(t)|$

Problem 2.

Problem 3. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of random variables with mean μ. Suppose that $Cov(X_t, X_s) = 0 \forall t \neq s$ and $Var(X_t) \leq Kt^{1/2}$ for some constant $K > 0$. Show that $\bar{X}_t \xrightarrow{\mathbb{P}} \mu$.

Solution: Consider any $\epsilon > 0$. Using Markov inequality we get

$$\begin{aligned}
P(|\bar{X}_t - \mu| > \epsilon) &\leq \frac{\mathbb{E}(|\bar{X}_t - \mu|^2)}{\epsilon^2} = \frac{V(\bar{X}_t)}{\epsilon^2} \\
&\leq \frac{\sum_{t=1}^T V(X_t)}{T^2\epsilon^2} = \frac{K\sum_{t=1}^T t^{1/2}}{T^2\epsilon^2} \\
&\leq \frac{KT^{3/2}}{T^2\epsilon^2} \xrightarrow{t\to\infty} 0
\end{aligned}$$

Then $\bar{X}_t - \mu \xrightarrow{\mathbb{P}} 0$ which is equivalent to what we wanted to show.

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Problem 4.

Problem 5. Suppose that $\sup_{t\in\mathcal{T}} \mathbb{E}(|X_t|^{1+\delta}) < \infty$ for some $\delta > 0$. Then $\{X_t : t \in \mathcal{T}\}$ is uniformly integrable. (Hint: Markov's inequality)

Solution: Notice that $\frac{|X_t|}{M} \geq I[|X_t| > M] \implies \frac{|X_t|^\delta}{M^\delta} \geq I[|X_t| > M]$.
Using that $|X_t| \geq 0$,

$$\begin{aligned}
&\frac{|X_t|^{1+\delta}}{M^\delta} \geq |X_t| I[|X_t| > M] \forall\, t \in \mathcal{T} \\
&\implies \sup_{t\in\mathcal{T}} \mathbb{E}(|X_t|^{1+\delta})(1/M^\delta) \geq \sup_{t\in\mathcal{T}} \mathbb{E}(|X_t| I[|X_t| > M]) \forall\, M \in \mathbb{N} \\
&\implies \lim_{M\to\infty} \sup_{t\in\mathcal{T}} \mathbb{E}(|X_t|^{1+\delta})(1/M^\delta) = 0 \geq \lim_{M\to\infty} \sup_{t\in\mathcal{T}} \mathbb{E}(|X_t| I[|X_t| > M]) \geq 0
\end{aligned}$$

The inequalities above give that X_t is uniformly integrable.

Problem 6. Suppose that $|X_t| \leq Y$ for some integrable random variable Y and all $t \in \mathbb{T}$. Then $\{X_t : t \in \mathbb{T}\}$ is uniformly integrable.

Solution:

$$\begin{aligned}
|X_t| &\leq Y \\
\implies |X_t| 1_{\{|X_t| \geq M\}} &\leq Y 1_{\{Y \geq M\}} \\
\implies \mathbb{E}\left[|X_t| 1_{\{|X_t| \geq M\}}\right] &\leq \mathbb{E}\left[Y 1_{\{Y \geq M\}}\right] \\
\implies \sup_{t\in\mathbb{T}} \mathbb{E}\left[|X_t| 1_{\{|X_t| \geq M\}}\right] &\leq \sup_{t\in\mathbb{T}} \mathbb{E}\left[Y 1_{\{Y \geq M\}}\right] = \mathbb{E}\left[Y 1_{\{Y \geq M\}}\right] \\
\implies \lim_{M\to\infty} \sup_{t\in\mathbb{T}} \mathbb{E}\left[|X_t| 1_{\{|X_t| \geq M\}}\right] &\leq \underbrace{\lim_{M\to\infty} \mathbb{E}\left[Y 1_{\{Y \geq M\}}\right]}_{=0 \text{ as } Y \text{ is integrable}} \\
\implies \lim_{M\to\infty} \sup_{t\in\mathbb{T}} \mathbb{E}\left[|X_t| 1_{\{|X_t| \geq M\}}\right] &= 0
\end{aligned}$$

The second line follows from the fact that the indicator function is stricly increasing. The third line follows from the fact that expectation is strictly increasing.

Problem 7. (Characteristic Function) Let X_n and X be $\mathbb{R}^d$-valued random variables. Suppose that $X_n \Rightarrow X$. Show that for any $\theta \in \mathbb{R}^d$, we have the following: (a) $\mathbb{E}\left[\cos(\theta' X_n)\right] \to \mathbb{E}[\cos(\theta' X)]$ (b) $\mathbb{E}[\sin(\theta' X_n)] \to \mathbb{E}[\sin(\theta' X)]$; (c) the characteristic function of X_n converges to the characteristic function of X.

Solution:

(a) $f(x) = \cos(x)$ is a continuous bounded function on $\mathbb{R}$. Invoking the definition of weak convergence, we get that $\mathbb{E}\left[\cos(\theta' X_n)\right] \to \mathbb{E}[\cos(\theta' X)]$ since $X_n \Rightarrow X$.

(b) $f(x) = \sin(x)$ is a continuous bounded function on $\mathbb{R}$. Invoking the definition of weak convergence, we get that $\mathbb{E}[\sin(\theta' X_n)] \to \mathbb{E}[\sin(\theta' X)]$ since $X_n \Rightarrow X$.

(c) The characteristic function of Y random variable is $\phi_Y(t) = \mathbb{E}\left[\exp(itY)\right] = \mathbb{E}(\cos(tY) + i\sin(tY))$. From (a) and (b), it follows that $\phi_{X_n}(t) = \mathbb{E}(\cos(tX_n) + i\sin(tX_n)) = \mathbb{E}(\cos(tX_n)) + i\mathbb{E}(\sin(tX_n)) \to \mathbb{E}(\cos(tX)) + i\mathbb{E}(\sin(tX)) = \phi_X(t)$.

Problem 8. (Laplace Transform) Let X_n and X be $\mathbb{R}_+$-valued random variables. Suppose that $X_n \Rightarrow X$. Show that for any $\theta \geq 0$, we have $\mathbb{E}(\exp(-\theta X_n)) \to \mathbb{E}(\exp(-\theta X))$

Solution:
$f(x) = \exp(-\theta x)$ is a bounded continuous function on $\mathbb{R}_+$ if $\theta \geq 0$. Therefore, we can use the definition of weak convergence to get $\mathbb{E}(\exp(-\theta X_n)) \to \mathbb{E}(\exp(-\theta X))$ as $X_n \Rightarrow X$.

Problem 9. Let X_n and X be real-valued random variables with $X_n \xrightarrow{a.s.} X$. Let $r \in \mathbb{R}$ with $\mathbb{P}(X = r) = 0$. Show that (a) $1_{X_n \leq r} \xrightarrow{a.s.} 1_{X \leq r}$; (b) $\mathbb{P}(X_n \leq r) \to \mathbb{P}(X \leq r)$. (Hint: bounded convergence theorem). Show (b) but assume $X_n \Rightarrow X$ instead of $X_n \xrightarrow{a.s.} X$. (Hint: Almost sure representation)

Solution:
(a) $f(x) = 1_{x \leq r}$ is only discontinuous at r, otherwise it is continuous on the real line. However, $\mathbb{P}(X = r) = 0$, therefore we can apply continuous mapping theorem. That is, if $X_n \xrightarrow{a.s.} X$, then $f(X_n) \xrightarrow{a.s.} f(X)$ for $f(x) = 1_{x \leq r}$.

(b) From (a), we know that $1_{X_n \leq r} \xrightarrow{a.s.} 1_{X \leq r}$ and $|1_{X_n \leq r}| \leq Y = 2$ for every $n \in \mathbb{N}$ (i.e. Y is some integrable variable). So we can invoke dominated convergence theorem to get that $\mathbb{P}(X_n \leq r) = \mathbb{E}\left[1_{X_n \leq r}\right] \xrightarrow{a.s.} \mathbb{E}\left[1_{X \leq r}\right] = \mathbb{P}(X \leq r)$.

(c) Since $X_n \Rightarrow X$, by almost sure representation, $\exists \left((\tilde{X}_n)_{n \in \mathbb{N}} \text{ and } \tilde{X}\right)$, such that $X_n \sim \tilde{X}_n, X \sim \tilde{X}$ and $\tilde{X}_n \xrightarrow{a.s.} \tilde{X}$. From (a) and (b), we know that for $\tilde{X}_n$ and $\tilde{X}$, that $\mathbb{P}(\tilde{X}_n \leq r) \to \mathbb{P}(\tilde{X} \leq r)$. Since $X_n \sim \tilde{X}_n, X \sim \tilde{X}$, we have that $\mathbb{P}(\tilde{X}_n \leq r) = \mathbb{P}(X_n \leq r)$ and $\mathbb{P}(\tilde{X} \leq r) = \mathbb{P}(X \leq r)$.

Problem 10. Let $a_n = o(1)$. Interpret and prove that $O_p(a_n) = o_p(1)$.

Solution: If I divide X_n by $a_n = o(1)$, it has to be the case that $X_n = o_p(1)$. The proof is the following:

$$X_n = O_p(a_n) \iff \frac{X_n}{a_n} = O_p(1);$$
$$X_n = \frac{X_n}{a_n}\, a_n = O_p(1)\, o(1) \implies O_p(1) o_p(1) = o_p(1)$$

The first line is just the definition of $O_p(a_n)$. The second line comes from the fact that pointwise convergence implies convergence in probability, and the o_p–O_p identities in the notes.

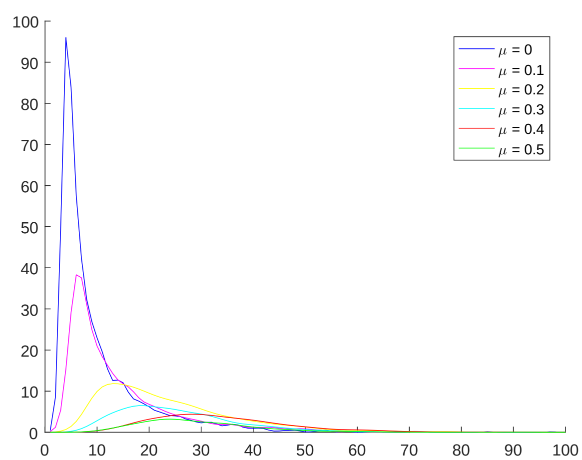
Problem 11. Interpret and prove that $e^{o_p(1)} - 1 = o_p(1)$ and $(O_p(1))^{\sqrt{2}} = O_p(1)$.

Solution: If you pick a r.v. $X = o_p(1)$, then a continuous transformation will also be $o_p(1)$. The proof is a straightforward consequence of CMT.
For the second claim, the interpretation is that you preserve $O_p(1)$ property when you take a continuous function of this r.v.. To prove, let $X_n = O_p(1)$. By definition, $\exists M > 0; \sup_{n\in\mathbb{N}} P(|X_n| > M) < \epsilon)$. Notice that $\{\omega \in \Omega : |X_n(\omega)| > M\} = \{\omega \in \Omega : |X_n(\omega)|^{\sqrt{2}} > M^{\sqrt{2}}\}$ because it is a monotonic transformation. Thus, $\exists \eta = M^{\sqrt{2}} > 0; \sup_{n\in\mathbb{N}} P(|X_n^{\sqrt{2}}| > \eta) < \epsilon)$.

Problem 12. Let $f(x) = \exp(x^2)$. Let $(X_i)_{1\le i\le n}$ be iid $N(\mu, 1)$ variables. Simulate such a sequence with $n = 100$ and $\mu = 0.3$. Compute $f(\bar{X}_n)$ where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Repeat this for 2000 times. Plot the distribution of $f(\bar{X})$ in the simulation. What is the asymptotic distribution of $f(\bar{X})$? Is the asymptotic distribution a good approximation to the empirical distribution? Repeat for $\mu = 0$, 0.1, 0.2, 0.3, 0.4, 0.5. Discuss the results.

Solution: We calculate the asymptotic distribution of $f(\bar{X}_n)$ using CLT and the delta method. Since $X_i \sim N(\mu, 1)$, it follows from the CLT that $\sqrt{n}(\bar{X}_n - \mu) \Rightarrow N(0, 1)$. By the delta method, $\sqrt{n}(\exp(\bar{X}_n^2) - \exp(\mu^2)) \Rightarrow N(0, (2\mu\exp(\mu^2))^2)$. When $\mu = 0.3$, $\sqrt{n}(\exp(\bar{X}_n^2) - \exp(0.3^2)) \Rightarrow N(0, 0.43)$.
However, note that when $\mu = 0$, the asymptotic distribution becomes degenerate in which case we use a version of the delta method that relies on a second-order Taylor expansion: $\sqrt{n}(\exp(\bar{X}_n^2) - \exp(\mu^2)) \Rightarrow \frac{1}{2}\cdot 2\exp(\mu^2)(2\mu^2+1)\cdot\chi^2(1)$. Hence in the case of $\mu = 0$, $\sqrt{n}(\exp(\bar{X}_n^2)) \Rightarrow \chi^2(1)$.
We plot the simulated distributions below. As the above analysis suggests, the empirical distribution with $\mu = 0$ resembles the χ^2-distribution. As μ increases, the empirical distributions look more and more normal.

Figure 1: Simulated empirical distributions



Problem 13. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of iid random variables with mean μ and variance σ^2. Suppose that for some $\delta > 0$, $\mathbb{E}\left[|X_i|^{2+\delta}\right] < \infty$. Use Lindeberg's CLT to show that $n^{1/2}(\frac{1}{n}\sum_{i=1}^n X_i - \mu) \implies N(0, \sigma^2)$.

Solution: Initially consider $X_{n,i} = \frac{X_i - \mu}{\sqrt{n}}$. Now, we need to verify the assumptions of Lindeberg's CLT.

1. $\{X_{n,i}\}$ independent, real-valued, and zero mean.
   Since we are applying a continuous function to rv's which are iid by assumption, independence will be preserved. Moreover, considering $X_i : \Omega \mapsto \mathbb{R}$, we will have $X_{n,i}$ real-valued. And notice that $\mathbb{E}[X_{n,i}] = \mathbb{E}[(X_i - \mu)/\sqrt{n}] = 0$.

2. $\sum_{i=1}^n V(X_{n,i} = \frac{\sum_{i=1}^n \sigma^2}{n} = \sigma^2$

3. $\lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}[X_{n,i}^2 I[|X_{n,i}| > \epsilon]] = 0$
   Using an argument similar to Q5, we have $\frac{|X_{n,i}|^{2+\delta}}{\epsilon^\delta} \geq I[|X_{n,i}| > \epsilon]$. Therefore,

   $$\tfrac{\mathbb{E}(|X_i - \mu|^{2+\delta}}{\epsilon^\delta n^{1+\delta/2}}) \geq \mathbb{E}(|X_{n,i}|^2 I[|X_{n,i} > \epsilon])\forall\ i \in \mathbb{N}$$

   Using iid assumption and $\mathbb{E}\left[|X_i|^{2+\delta}\right] < \infty$ we have:

   $$\begin{aligned} \frac{\mathbb{E}(|X_i - \mu|^{2+\delta})}{\epsilon^\delta n^{\delta/2}} &\geq \sum_{i=1}^n \mathbb{E}(|X_{n,i}|^2 I[|X_{n,i} > \epsilon]) \\ \implies \lim_{n\to\infty} \frac{\mathbb{E}(|X_i - \mu|^{2+\delta})}{\epsilon^\delta n^{\delta/2}} &= 0 \geq \lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}(|X_{n,i}|^2 I[|X_{n,i} > \epsilon]) \geq 0 \end{aligned}$$

Finally, we can apply Lindeberg's CLT to get $n^{1/2}(\frac{1}{n}\sum_{i=1}^n X_i - \mu) = \sum_{i=1}^n X_{n,i} \implies N(0, \sigma^2)$.