Unit 5.2 The Greedy Method, II

Algorithms

EE3980

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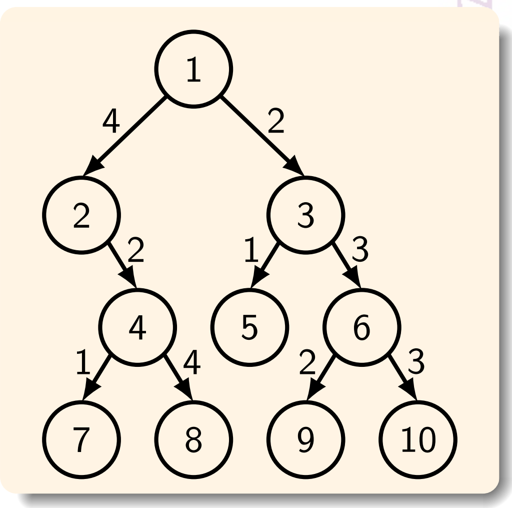
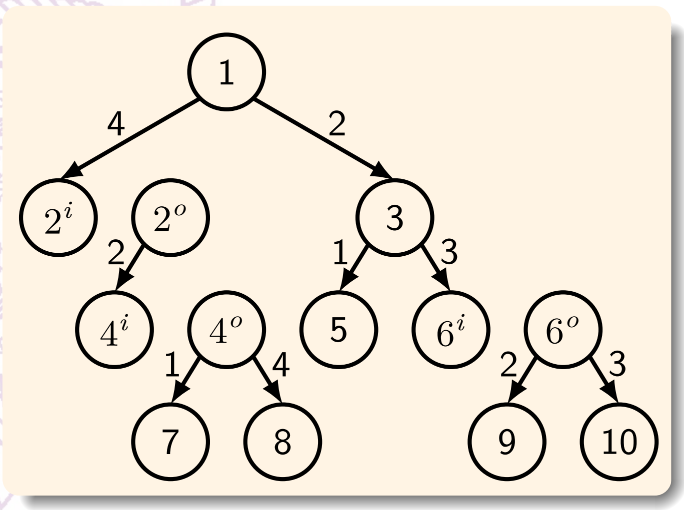
Tree Vertex Splitting Problem

Original tree T



$d(T) = 10.$

Tree with vertices splatted T/X



$d(T/X) = 5.$

Tree Vertex Splitting Problem – Definition

- $T = (V, E, w)$ is weighted directed tree.
  - V is the vertex set, E is the edge set, and w is weight function for the edges.
  - $w(i, j)$ is defined if the edge $\langle i, j \rangle \in E$; $w(i, j)$ is undefined if $\langle i, j \rangle \notin E$.
  - A source vertex is a vertex with in-degree 0.
  - A sink vertex is a vertex with out-degree 0.
  - For any path P in the tree, its delay, $d(P)$, is defined to be the sum of the weights on the path.
  - The delay of the tree, $d(T)$, is the maximum of all the path delays.
- T/X is the forest resulted from splitting every vertex u in $X \subseteq V$ into two nodes u^i and u^o such that all the edges $\langle i, u \rangle$ are replaced by $\langle i, u^i \rangle$ and all the edges $\langle u, j \rangle$ are replaced by $\langle u^o, j \rangle$.
- The Tree Vertex Splitting Problem (TVSP) is to find a set $X \subseteq V$ with minimum cardinality for which $d(T/X) \leq \delta$ for some specified tolerance δ.
  - Note that a TVSP has solution only if the maximum edge weight is less than or equal to δ.
  - Any $X \subseteq V$ with $d(T/X) \leq \delta$ is a feasible solution.
  - The optimal solution is the feasible X with the minimum number of vertices.

Tree Vertex Splitting Problem – Algorithm

Algorithm 5.2.1. TVS

```
// Find the minimum set X for vertex splitting.
// Input: tree T, maximum edge weight δ
// Output: solution X.
Algorithm TVS(T, δ, X)
{
    if (T ≠ ∅) then {
        d[T] := 0;
        for each child v of T do {
            TVS(v, δ, X);
            d[T] := max(d[T], d[v] + w(T, v));
        }
        if ((T is not the root ) and (d(T) + w(parent(T), T) > δ)) then {
            X := X ∪ {T};
            d[T] := 0;
        }
    }
}
```

- Note that d is a global array that stores the *delay* for each vertex.

Tree Vertex Splitting Problem – Algorithm II

Algorithm 5.2.2. TVS1

```
    // Tree vertex splitting with tree stored in an array tree[1 : n].
    // Input: root i, maximum edge weight δ
    // Output: solution X.
Algorithm TVS1(i, δ, X)
{
    if (tree[i] ≠ 0) then {
        if (2×i > N) then d[i] := 0; // i is a leaf.
        else {
            TVS1(2×i, δ, X);
            d[i] := max(d[i], d[2×i] + w[2×i]);
            if (2×i + 1 ≤ N) then {
                TVS1(2×i + 1, δ, X);
                d[i] := max(d[i], d[2×i + 1] + w[2×i + 1]);
            }
        }
        if ((i ≠ 1) and (d[i] + w[i] > δ)) then {
            X := X ∪ {i};
            d[i] := 0;
        }
    }
}
```


Tree Vertex Splitting Problem – Complexity and Optimality

- In this version the directed binary tree is stored in an array $tree$
- The weight is stored in array w and $w[i]$ is the weight of the parent of vertex i to vertex i.
- Array d is still the $delay$ of each vertex.
- The time complexity of Algorithm TVS is $\Theta(n)$.
    - Every vertex of T is traversed once.

Theorem 5.2.3.

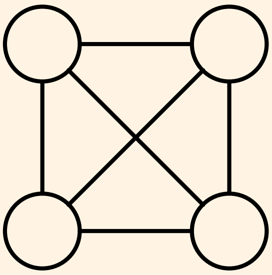
Algorithm TVS finds a minimum cardinality set X such that $d(T/X) \leq \delta$ on any tree T, provided that no edge of T has weight greater than δ.

- Proof please see textbook [Horowitz], pp. 225 - 226.

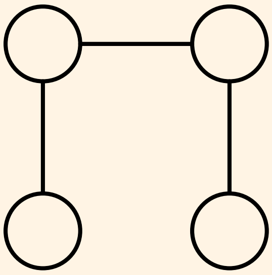
Minimum-Cost Spanning Trees

Definition 5.2.4.

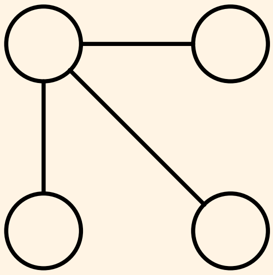
Let $G = (V, E)$ be an undirected connected graph. A sub-graph $T = (V, E')$ with $E' \subseteq E$ is a spanning tree of G if and only if T is a tree.



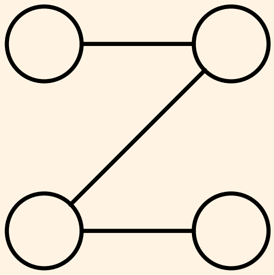
Undirected graph G.



Spanning tree T_1.



Spanning tree T_2.

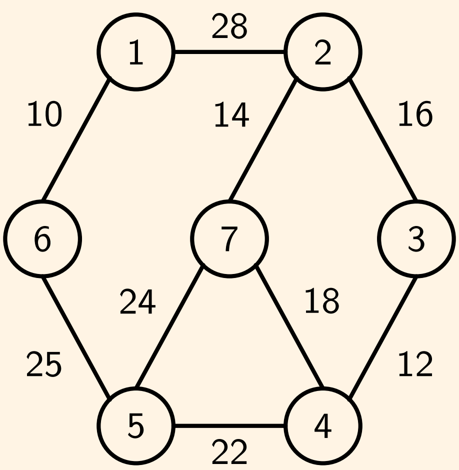


Spanning tree T_3.

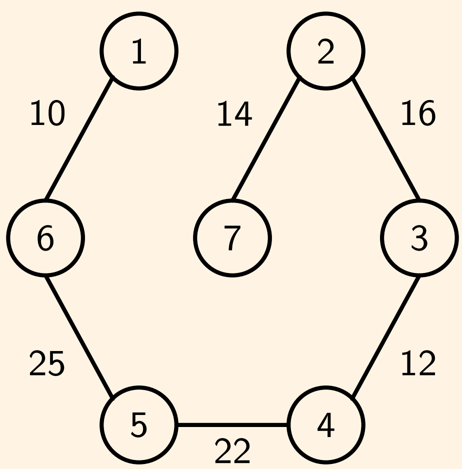
- Notes
  - Spanning tree is not unique.
  - Spanning trees have $n - 1$ edges $(n = |V|.)$

Minimum-Cost Spanning Tree, Example

- In addition, there is a cost function associated with each edge, $w : E \rightarrow \mathbb{R}$.
- The cost of a tree is the sum of the costs of the tree edges.
- A feasible solution of the minimum-cost spanning tree of a undirected graph G is any spanning tree T of G.
- The optimal solution is a spanning tree with the minimum cost.



An undirected graph, G.



Minimum-cost spanning tree, T.

Minimum-Cost Spanning Tree, Generic Algorithm

- Using the greedy methodology, let T be a subset of a spanning tree, at each step an edge (u, v) is added to T to maintain the feasibility of the solution.
- An edge, (u, v), is safe to a set of edges T if $T \cup \{(u, v)\}$ is still a subset of a spanning tree.
- The generic algorithm for the minimum-cost spanning tree then is:

Algorithm 5.2.5. Generic minimum-cost spanning tree

// Given a graph $G(V, E)$ with cost function w find minimum cost spanning tree.
// `Input:` V, E, n, w
// `Output:` minimum cost tree T.
1 **Algorithm** MCST(V, E, n, w, T)
2 {
3     $T := \emptyset$;
4   **while** $(|T| < n - 1)$ **do** {
5   **select** an edge $(u, v) \in E$ {
6   **if** (u, v) is safe to T **then** $T := T \cup (u, v)$;
7             $E := E - \{(u, v)\}$;
8   }
9   }
10 }

- The key is in line 5, how to select an edge.

Minimum-Cost Spanning Tree, Prim's Algorithm

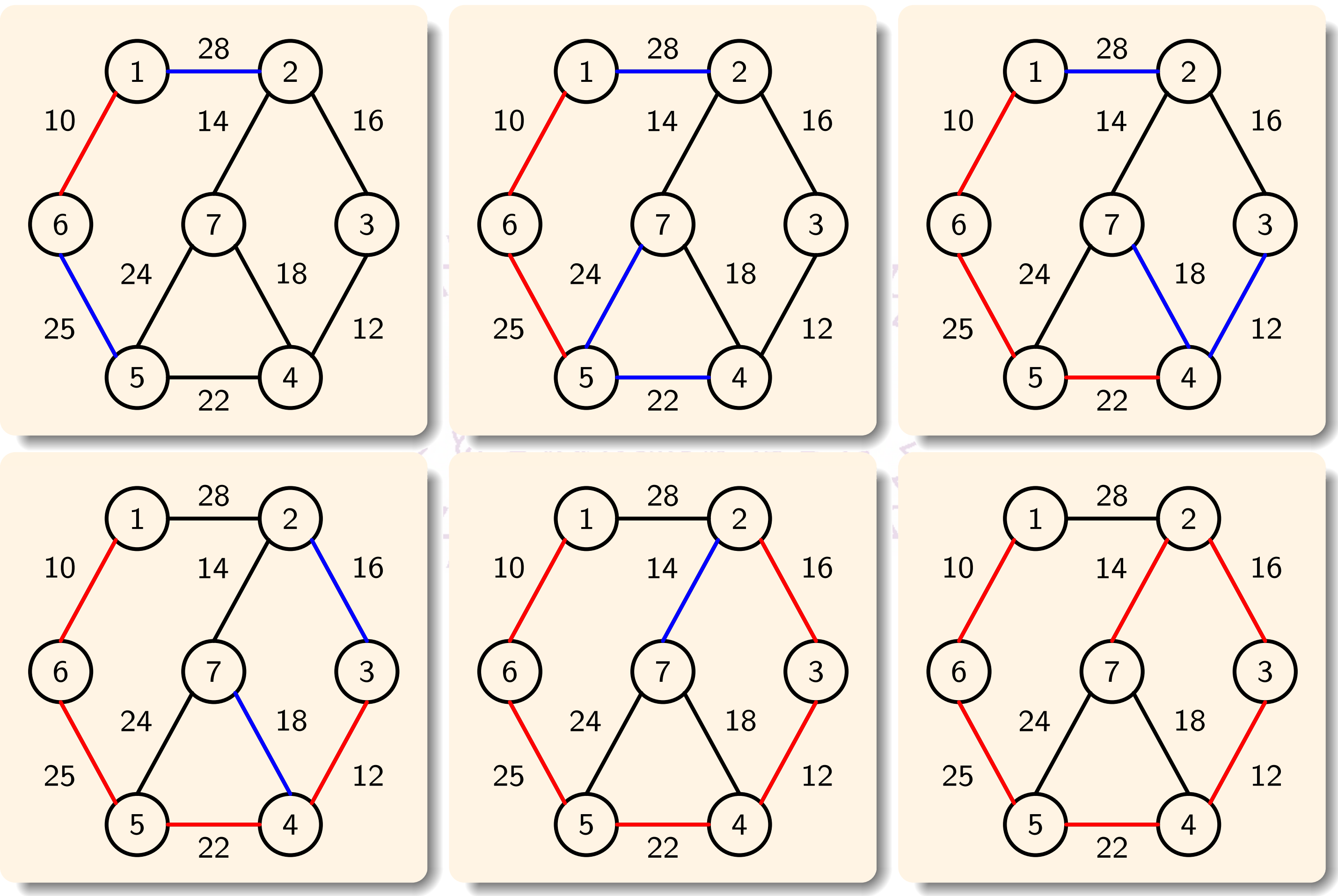
Algorithm 5.2.6. Prim

// Given a graph $G(V, E)$ with cost function w find minimum cost spanning tree.
// `Input:` V, E, n, w
// `Output:` minimum cost tree T and $mincost$.
Algorithm Prim(V, E, n, w, T)
{
    Find edge $(k, \ell) \in E$ with the minimum cost ;
    $mincost := w[k, \ell]$; // $mincost$ set to minimum edge cost.
    $T[1, 1] := k$; // Add (k, ℓ) to spanning tree.
    $T[1, 2] := \ell$;
    for $i := 1$ **to** n **do** // Init $near$ array for every vertices.
        if $(w[i, \ell] < w[i, k])$ **then** $near[i] := \ell$;
        else $near[i] := k$;
    $near[k] := near[\ell] := 0$; // Vertices already in the spanning tree.
    for $i := 2$ **to** $(n - 1)$ **do** {
        Find j such that $near[j] \neq 0$ **and** $w[j, near[j]]$ is minimum ;
        $T[i, 1] := j$; // Add minimum cost $near$ edge to tree.
        $T[i, 2] := near[j]$;
        $mincost := mincost + w[j, near[j]]$; // Update $mincost$.
        $near[j] := 0$; // Reset $near$ array for selected vertex.
        for $k := 1$ **to** n **do** // update $near$ array for the other unselected vertices.
            if $((near[k] \neq 0)$ **and** $(w[k, near[k]] > w[k, j]))$ **then** $near[k] := j$;
    }
    return $mincost$;
}

Minimum-Cost Spanning Tree, Prim's Algorithm II

- In Algorithm Prim
  1. The edge with the minimum cost is first selected as the initial tree
  2. The array near keeps the node already selected in the tree with the smallest single-edge cost for each node
  3. Among the all the near edges, the minimum is selected and the node added to the tree
  4. Array near is then updated and go back to step 3 until all nodes have been selected
- The time complexity is dominated by
  - Finding the minimum-cost edge on line 3, $\mathcal{O}(|E|) \approx \mathcal{O}(n^2)$
  - Loop on lines 7-9, $\mathcal{O}(n)$
  - Loop on lines 11-19
    - Inner loops line 12 and lines 17-18
    - Complexity $\mathcal{O}(n^2)$
  - Overall complexity is $\mathcal{O}(n^2)$
- The time complexity can be improved to $\mathcal{O}((n + |E|) \lg n)$
  - If the non-selected vertices are stored in a red-black tree

Minimum-Cost Spanning Tree, Prim's Algorithm Example



Kruskal's Algorithm – High Level

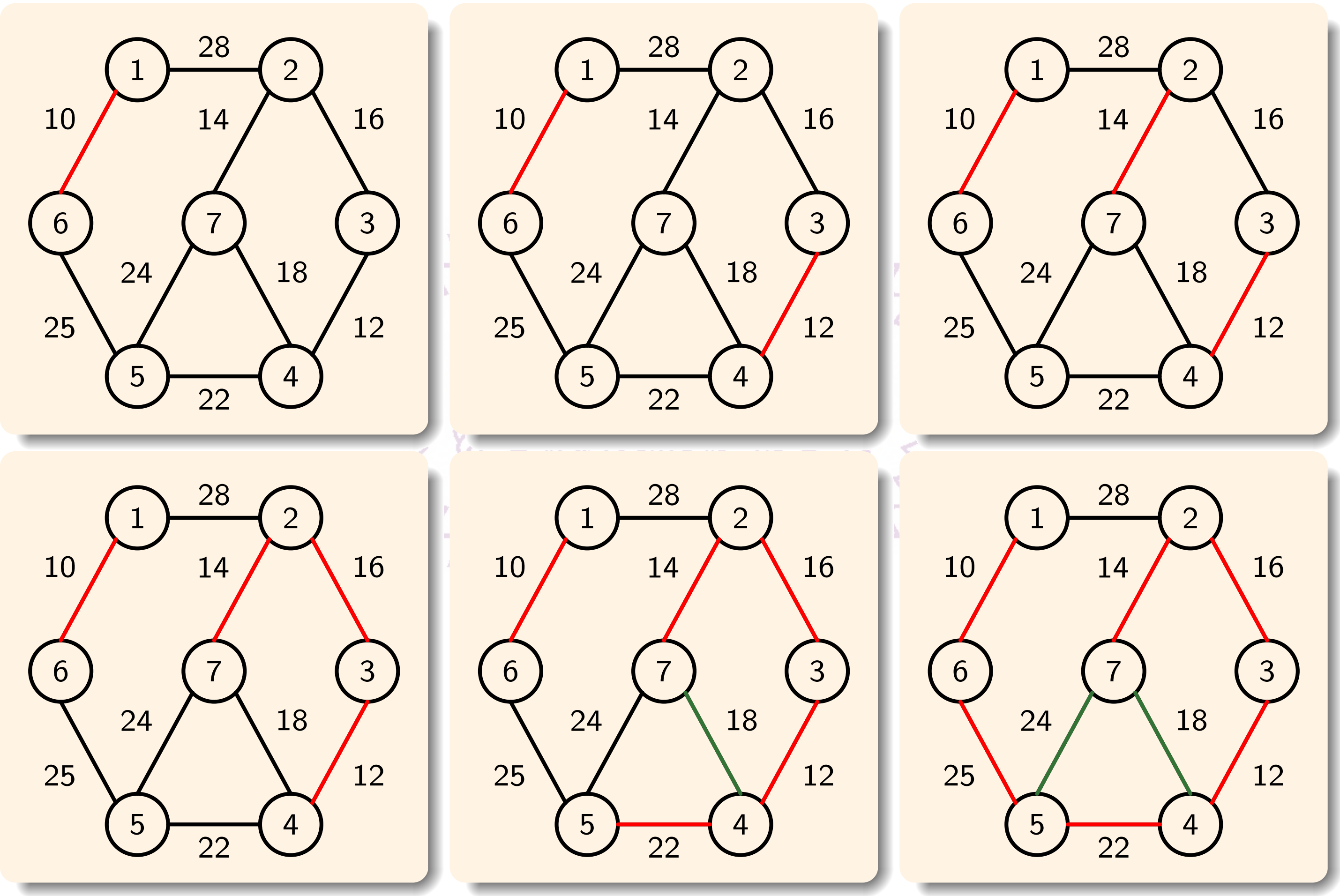
- A different approach to finding the minimum-cost spanning tree
- High level description of the algorithm

Algorithm 5.2.7. Kruskal's Algorithm

// Given a graph $G(V, E)$ with cost function w find minimum cost spanning tree.
// `Input:` V, E, n, w
// `Output:` minimum cost tree T.

`Algorithm` `KruskalH`(V, E, n, w, T)
{
    $T := \emptyset$;
    `while` ((T has less than $(n-1)$ edges) `and` $(E \neq \emptyset)$) `do` {
        Find the edge $(u, v) \in E$ with the minimum cost ;
        `Delete`(u, v) from E;
        `if` (u, v) does not create a cycle in T `then` $T := T \cup (u, v)$;
        `else` discard (u, v);
    }
}

Kruskal's Algorithm – Example



Kruskal's Algorithm

Algorithm 5.2.8. Kruskal's Algorithm

```
   // Given a graph G(V, E) with cost function w find minimum cost spanning tree.
   // Input: V, E, n, w
   // Output: minimum cost tree T and mincost.
1  Algorithm Kruskal(V, E, n, w, T)
2  {
3      Construct a min heap from the edge costs using Heapity ;
4      for i := 1 to n do parent[i] := −1 ; // Enable cycle checking
5      i := 0 ;
6      mincost := 0 ;
7      while ((i < n − 1) and ( heap not empty )) do {
8          delete a minimum cost edge (u, v) from the heap ;
9          Adjust the heap ;
10         j := Find(u) ; // using parent array
11         k := Find(v) ;
12         if (j ≠ k) then {
13             i := i + 1 ;
14             T[i, 1] := u ;
15             T[i, 2] := v ;
16             mincost := mincost + w[u, v] ;
17             Union(j, k) ; // modify parent array
18         }
19     }
20     if (i ≠ n − 1) then write("No spanning tree") ;
21     else return mincost ;
22 }
```


Kruskal's Algorithm – Complexity and Optimality

- The time complexity of Kruskal algorithm is dominated by the while loop, lines 7-19, – $\mathcal{O}(|E|)$
  - Line 8 finding minimum cost edge, $\mathcal{O}(1)$
  - Line 9 Adjust the heap, $\mathcal{O}(\lg |E|)$
  - Overall complexity $\mathcal{O}(|E| \lg |E|)$.

Theorem 5.2.9.

Kruskal's algorithm (Algorithm 5.2.8) generates a minimum-cost spanning tree for every undirected connected graph G.

- Proof please see textbook [Horowitz], p. 244.

Minimum-Cost Spanning Tree, Properties

- A different approach to prove Kruskal's algorithm.
- We define the following terms.
  - A cut $(S, V - S)$ of an undirected graph $G = (V, E)$ is a partition of V, i.e., $S \in V$.
  - An edge $(u, v) \in E$ is said to cross the cut $(S, V - S)$ if one of its end points is in S and the other in $V - S$.
  - A cut is said to respect a set T of edges if no edges in T crosses the cut.
  - An edge is said to be a light edge crossing a cut if its cost is the minimum of any edge crossing the cut.

Theorem 5.2.10.

Let $G = (V, E)$ be a connected, undirected graph with a cost function w defined on E. Let T be a subset of E that is subset of a spanning tree of G, let $(S, V - S)$ be any cut of G that respects T, and let (u, v) be a light edge crossing $(S, V - S)$. Then, edge (u, v) is safe for T.

- Proof please see textbook [Cormen], pp. 627-628.

Minimum-Cost Spanning Tree, Properties, II

Corollary 5.2.11.

Let $G = (V, E)$ be a connect, undirected graph with cost function w defined on E. Let T be a subset of E that is included in a minimum spanning tree of G, and let $C = (V_C, E_C)$ be a connected component (tree) in the forest $G_T = (V, T)$. If (u, v) is a light edge connecting C to some other component in G_T, then (u, v) is safe for T.

- Proof please see textbook [Cormen], pp. 629.
- Algorithm `Prim` can be shown to be a special case of Theorem (5.2.10), and it also returns an optimal solution.

Matroid Theory and Greedy Method

- Matroid theory explains why greedy method solves the maximum/minimum subset problems.
  - The theory was developed by Hassler Whitney, American Mathematician, in 1935 by generalizing the structure of linear independence in vector space
  - Jack Edmonds, American Computer Scientist, applied to greedy algorithms
  - Reference: Bernhard Korte and Jens Vygen, *Combinatorial Optimization – theory and algorithms*, 4th edition, Springer, 2008.
- The concept of independence system is generalized from vector space.

Independent Systems

Definition 5.2.12. Independence System

Let S be a finite set and $\mathcal{I} = \{X : X \subseteq S\}$, then the set system $(S, \mathcal{I})$ is an independence system if

(M1) $\emptyset \in \mathcal{I}$;

(M2) If $Y \in \mathcal{I}$ and $X \subseteq Y$ then $X \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called independent, the elements of $2^S \setminus \mathcal{I}$ dependent. Minimal dependent sets are called circuits, maximal independent sets are called bases. For $X \subseteq S$, the maximal independent subsets of X are called bases of X.

- The set $\mathcal{I}$ can be defined by its property, instead of listing all elements.

Definition 5.2.13.

Let $(S, \mathcal{I})$ be an independence system. For $X \subseteq S$ we define the rank of X by

$$r(X) = \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$

Examples of Independence Systems

- Example M1:
  Let S_V be the set of columns of a matrix $\mathbf{A}$ and $\mathcal{I}_V = \{X \subseteq S_V :$ the column vectors in X are linearly indepdent$\}$, then the set system $(S_V, \mathcal{I}_V)$ is an independence system.
  (1) It is apparent $\emptyset \in \mathcal{I}$.
  (2) If $Y \in \mathcal{I}$ then any subset $X \subseteq Y$ also contains independent column vectors.
- Example M2:
  Given a undirected graph $G(V, E)$, let $S_G = E$, the set of all edges, and $\mathcal{I}_G = \{Y : Y \subseteq E$ and Y is a forest$\}$, then the set system $(S_G, \mathcal{I}_G)$ is an independence system.
  (1) It is apparent $\emptyset \in \mathcal{I}$.
  (2) If $Y \in \mathcal{I}$, then Y is a forest, and any subset $X \subseteq Y$ is also a forest.
- Example M3:
  Given any finite set S_U, let k be an integer, $k \le |S_U|$, and $\mathcal{I} = \{Y : Y \subseteq S_U$ and $|Y| \le k\}$, then the set system $(S_U, \mathcal{I}_U)$ is an independence system.
  (1) It is apparent $\emptyset \in \mathcal{I}$.
  (2) If $Y \in \mathcal{I}$, then $|Y| \le k$. Any $X \subseteq Y$ has $|X| \le |Y| \le k$.

Matroid

Definition 5.2.14. Matroid

An independence system $(S, \mathcal{I})$ is a matroid if

(M3) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there is an $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

- Vector Matroid: The independence system $(S_V, \mathcal{I}_V)$ in Example M1 is a matroid.
  - (M3) property is observed $(S_V, \mathcal{I}_V)$.
- Graphic Matroid: The independence system $(S_G, \mathcal{I}_G)$ in Example M2 is a matroid.
  - (M3) property is observed. If $|X| > |Y|$ and for every $x \in X$ either $x \in Y$ or $Y \cup \{x\}$ forms a cycle. In either case, both vertices of the edge x belong to the same connected component in Y. The number of such edge cannot exceed $|Y|$ while maintaining a forest perperty, thus $|X| \le |Y|$. This contradicts to the assumption $|X| > |Y|$.
- Uniform Matroid: The independence system $(S_U, \mathcal{I}_U)$ in Example M3 is a matroid.
  - (M3) property is observed. If $|X| > |Y|$ then there is an $x \in X \setminus Y$ and then $|Y \cup \{x\}| = |Y| + 1 \le |X| \le k$.

Matroid, II

Theorem 5.2.15.

Let $(S, \mathcal{I})$ be an independence system. Then the following statements are equivalent:

(M3) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there is an $x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{I}$.

(M3′) If $X, Y \in \mathcal{I}$ and $|X| = |Y + 1|$, then there is an $x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{I}$.

(M3″) For each $X \subseteq S$, all bases of X have the same cardinality.

Proof. It is easy to see (M3) $\Leftrightarrow$ (M3′) and (M3) $\Rightarrow$ (M3″). To prove (M3″) $\Rightarrow$ (M3), let $X, Y \in \mathcal{I}$ and $|X| > |Y|$. By (M3″), Y cannot be a basis of $X \cup Y$. So there must be an $x \in (X \cup Y) \setminus Y = X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$. □

- Thus, an independence system can also be shown to be a matroid using either property (M3′) or (M3″).
- For the graphic matroid, it is known that a spanning tree of a connect graph $G(V, E)$ has $|V| - 1$ edges. Thus, the rank $r(S_G) = |S_G| - 1$ if G is connected.

Weighted Matroid and Optimization Problems

Definition 5.2.16. Weighted Matroid

A matroid $(S, \mathcal{I})$ is weighted if it is associated with a weight function $w : S \to \mathbb{R}^+$. The weight function w extends to subsets of S by summation:

$$w(X) = \sum_{x \in X} w(x) \qquad \text{for any } X \subseteq S. \tag{5.1}$$

- Maximization problem of independence systems
  Given an independence system $(S, \mathcal{I})$ and the weight function $w : S \to \mathbb{R}^+$, find an $X \in \mathcal{I}$ such that $w(X) = \sum_{x \in X} w(x)$ is maximum.
- A corresponding minimization can be formulated
  - Solution algorithms can also be derived.

Greedy Algorithms

- Two types of algorithms possible
  The first one is

Algorithm 5.2.17. Best-In Greedy Algorithm

```
// Given (S, I) and w : S → ℝ find X ∈ I such that w(X) is maximum.
// Input: (S, I) and w.
// Output: X
Algorithm Best-In-Greedy(S, I, w)
{
      Sort S into nonincreasing order by w ;
      X := ∅; // Initialize to empty set.
      for each x ∈ S in order do { // Try all elements.
            if (X ∪ {x} ∈ I) then { // Maintain independence then add.
                  X := X ∪ {x};
            }
      }
      return X;
}
```


Greedy Algorithms, II

Theorem 5.2.18.

The best-in greedy algorithm (5.2.17) solves the independence system $(S, \mathcal{I})$ maximization problem correctly if $(S, \mathcal{I})$ is a matroid.

Proof. By induction. The first x in the ordered S with $\{x\} \in \mathcal{I}$ is apparently the solution for any $X \subset S$ with $r(X) = 1$. Suppose the best solution has been found for any $X \subset S$ and $r(X) = k$ and $k < r(S)$, then there is a $Y \in \mathcal{I}$ such that $r(Y) = r(X) + 1$, further more there is $x \in S$ with $Y = X \cup \{x\}$ and x is in the remaining S. The x with largest $w(x)$ is apparently the choice, which would be picked first by the algorithm. Hence, this Y is the solution for $r(X) + 1$. By induction, the theorem is proven. □

- Note that the maximization problem is defined for independence systems, and the algorithm works if the system is a matroid.
- The graph minimum spanning tree problem can be formulated as a minimization problem for a matroid system.
- The Kruskal's Algorithm is a best-in greedy algorithm.

Greedy Algorithms, III

- Alternative solution

Algorithm 5.2.19. Worst Out Greedy Algorithm

// Given $(S, \mathcal{I})$ and $w: S \to \mathbb{R}$ find a basis X of S such that $w(X)$ is maximum.
// `Input:` $(S, \mathcal{I})$ and w.
// `Output:` X
`Algorithm Worst-Out-Greedy`$(S, \mathcal{I}, w)$
{
    Sort S into nondecreasing order by w ;
    $X := S$; // Initialize to entire set.
    `for` each $x \in S$ in order `do` { // Try all elements.
        `if` $(r(X \setminus \{x\}) = r(X))$ `then` { // rank unchanged.
            $X := X \setminus \{x\}$;
        }
    }
    `return` X;
}

- This algorithm can generate optimal solution if the given system is a matroid, and the proof is similar to the best-in greedy algorithm.

Job Sequencing with Deadlines

- Given a set of n jobs to be processed on one machine.
  - Each job takes 1 time unit to process.
  - Associated with job i, $1 \le i \le n$, there is a deadline d_i and profit p_i.
  - If job i is completed by d_i then p_i is earned.
- A feasible solution is a subset J of jobs that each job in J can be completed by its deadline.
  - The value of the subset J is $\sum_{i \in J} p_i$.
- An optimal solution is a feasible solution with the maximum value.

- Example, $n = 4$, $\{p_1, p_2, p_3, p_4\} = \{100, 10, 15, 27\}$, $\{d_1, d_2, d_3, d_4\} = \{2, 1, 2, 1\}$.
- Feasible solutions are

	Feasible solution	Processing sequence	Value
1	$\{1, 2\}$	2,1	110
2	$\{1, 3\}$	1,3 or 3,1	115
3	$\{1, 4\}$	4,1	127
4	$\{2, 3\}$	2,3	25
5	$\{3, 4\}$	4,3	42
6	$\{1\}$	1	100
7	$\{2\}$	2	10
8	$\{3\}$	3	15
9	$\{4\}$	4	27

- Solution 3 is optimal.

Job Sequencing with Deadlines – Algorithm

- Applying greedy method to job sequencing problem

Alrogithm 5.2.20. Job Sequencing – Greedy Method

```
// Solve job scheduling problem with jobs sorted in non-increasing profit.
// Input: int n, deadline d[1 : n], profit p[1 : n]
// Output: Optimal sequence J[1 : k].
Algorithm JSgreedy(n, d, p, J)
{
      J := {1} ; // init to highest profit job
      for i := 2 to n do { // check every job
            if (J ∪ {i}  is feasible  ) then
                  J := J ∪ {i} ;
      }
}
```

- Example
  Ordered sequency: $\{1,4,3,2\}$, $p = \{100,27,15,10\}$,
  step 1: $J = \{1\}$, $p = 100$,
  step 2: $J = \{1,4\}$, $p = 127$,
  step 3: reject job 3, not feasible, step 4: reject job 2, not feasible.

Job Sequencing with Deadlines – Algorithm Optimality

- Optimality of the algorithm follows from the following theorem.

Theorem 5.2.21.

Given a job sequencing problem Algorithm `JSgreedy` (Algorithm 5.2.20) generates an optimal solution.

- Note that there can be groups of jobs that only some of them can be selected as $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, and $\{3,4\}$ in the example. Follows the algorithm, $\{1,4\}$ are selected. The only possibility that this solution is not optimal is that $P\{1,4\}$ is smaller than $P\{2,3\}$. But, job 4 has higher profit than either job 2 or 3. Thus, the selection by `JSgreedy` is the optimal solution.
- Unit execution time of each job is an important factor.

Proof can be found in textbook [Horowitz] pp. 229 - 230.

Job Sequencing with Deadlines – Algorithm

Alrogithm 5.2.22. Job Sequencing

```
    // Solve job scheduling problem with jobs sorted in non-increasing profit.
    // Input: int n, deadline d[1 : n], profit p[1 : n]
    // Output: Optimal sequence J[1 : k].
 1 Algorithm JS(n, d, p, J)
 2 {
 3       d[0] := J[0] := 0 ; // to facilitate while loop stopping
 4       J[1] := 1 ; k := 1 ; // init to highest profit job
 5       for i := 2 to n do { // check every job
 6             r := k;
 7             while ((d[J[r]] > d[i]) and (d[J[r]] ≠ r)) do // find time slot job i fits
 8                   r := r − 1 ;
 9             if ((d[J[r]] ≤ d[i]) and (d[i] > r)) then { // insert i into J
10                   for q := k to (r + 1) step −1 do J[q + 1] := J[q] ; // move jobs to make room
11                   J[r + 1] := i; // assign job
12                   k := k + 1 ;
13             }
14       }
15 }
```

- Line 3 creates a stopping condition for the `while` loop on line 7.
- The worst-case time complexity of JS algorithm is $\mathcal{O}(n^2)$.
  - Outer loop, lines 5–14
  - Inner loops, lines 7–8 and line 10.
- The space complexity of JS algorithm is $\mathcal{O}(n)$ for arrays J, p, and d.

Job Sequencing with Deadlines – Algorithm correctness

Theorem 5.2.23.

Given a job sequencing problem Algorithm JS (Algorithm 5.2.22) correctly generates the optimal solution.

- Proof can be found in textbook [Horowitz] pp. 230 - 232.
- Example: $n = 5$, Jobs={A, B, C, D, E}, $p[\,] = \{20, 15, 10, 5, 1\}$, and $d[\,] = \{2, 2, 1, 3, 3)\}$. Then, the execution sequence of the algorithm is as following.

i	$d[i]$	action	$J[\,]$	$d[J[\,]]$	$p[J[\,]]$	k
1	2	init to A	$\{A\}$	$\{2\}$	$\{20\}$	1
2	2	accepting B	$\{A, B\}$	$\{2, 2\}$	$\{20, 15\}$	2
3	1	rejecting C	$\{A, B\}$	$\{2, 2\}$	$\{20, 15\}$	2
4	3	accepting D	$\{A, B, D\}$	$\{2, 2, 3\}$	$\{20, 15, 5\}$	3
5	3	rejecting E	$\{A, B, D\}$	$\{2, 2, 3\}$	$\{20, 15, 5\}$	3

Job Sequencing with Deadlines – Matroid Formulation

- The job sequencing with deadline can be shown to be a matroid.
  The set $\mathcal{S}$ contains all the jobs, and a set A of jobs are independent if there is a schedule such that all jobs in A are done before their deadlines.

Lemma 5.2.25.

For any set of jobs A, the following statements are equivalent.

1. The set A is independent.
2. Let $N_t(A)$ denote the number of jobs completed before time t, then for $t = 0, 1, 2, \ldots, n$, we have $N_t(A) \le t$.
3. If the tasks in A are scheduled in order of monotonically increasing deadlines, the all jobs in A are completed before their deadlines.

Theorem 5.2.26.

If $\mathcal{S}$ is a set of unit-time jobs with deadlines, and $\mathcal{I}$ is the set of all independent sets of tasks, then the corresponding system $(\mathcal{S}, \mathcal{I})$ is a matroid.

- Since the job sequencing problem is a matroid, the greedy algorithm can be applied and it results in an optimal solution.

Container-Loading Problem and Matroid

- Container loading problem
  - n containers with weight $w_i > 0$, $1 \le i \le n$
  - Capacity c
  - Find $x_i \in \{0, 1\}$, $1 \le i \le n$ such that

$$\text{Maximize:} \quad \sum_{i=1}^{n} x_i,$$

$$\text{Subject to:} \quad \sum_{i=1}^{n} x_i \cdot w_i \le c.$$

- Example: $n = 8$, $(w_1, \ldots, w_8) = (100, 200, 50, 90, 150, 50, 20, 80)$, $c = 400$.
  Let $S = \{w_i : 1 \le i \le 8\}$, $\mathcal{I} = \{T \subseteq S : \sum_{t_i \in T} w(t_i) \le c\}$.
  - It can be shown that $(S, \mathcal{I})$ is an independence system.
    Since if $T \in \mathcal{I}$, any subset of T has total weight less than c.
  - But $(S, \mathcal{I})$ is not matroid.
    - Example, $T_1 = \{100, 50, 50\} \in \mathcal{I}$, $T_2 = \{200\} \in \mathcal{I}$,
      $|T_1| > |T_2|$ but there is no $t \in T_1$ such that $T_2 \cup \{t\} \in \mathcal{I}$.
  - Greedy method still works for this problem.
  - Madroid is a necessary but not a sufficient condition for greedy method.

Summary

- Tree vertex splitting problem.
- Minimum-cost spanning tree problem.
- The theory of Matroid.
- Job sequencing with deadlines.