Unit 3.3 More on Divide and Conquer Algorithms

Algorithms

EE/NTHU

Apr. 9, 2020

Selection Algorithm

- The selection problem is to find the k'th smallest element of an array A and place all elements less than or equal to $A[k]$ in $A[1:k-1]$ and the rest in $A[k+1:n]$.
- Divide and conquer can be applied to this problem as well.
  - The Partition algorithm can be effective in this selection problem.

Algorithm 3.3.1. Selection

```
    // Partition the array into A[1 : k−1] ≤ A[k] ≤ A[k+1 : n].
    // Input: A, int n, k
    // Output: k-th smallest element of A, and with A rearranged.
Algorithm Select1(A, n, k)
{
    low := 1 ; high := n+1 ; A[n+1] := ∞ ; // Initialize ranges and j.
    repeat { // Loop until value found by Partition is k.
        j := Partition(A, low, high) ;
        if (j > k) then high := j; // j ≠ k then update range for search.
        else if (j < k) then low := j+1 ;
    } until (j = k) ;
    return A[k];
}
```

Selection Algorithm – Complexity

- Note that the Partition$(A, low, high)$ decrease the range of the array A by at least 1.
- Thus, the worst-case complexity of the Select1 algorithm is $\mathcal{O}(n^2)$.
- Let $T_A^k(n)$ be the average time to find the k'th smallest element in $A[1:n]$
  - The average is taken over all $n!$ different permutations.
- Define

$$T_A(n) = \frac{1}{n}\sum_{k=1}^{n} T_A^k(n) \tag{3.3.1}$$

$$R(n) = \max_k T_A^k(n) \tag{3.3.2}$$

  - $T_A(n)$ is the average execution time of Select1 algorithm,
  - And it is obvious that $T_A(n) \leq R(n)$.

Selection Algorithm – Complexity, II

Theorem 3.3.2.

The average execution time $T_A(n)$ of Select1 algorithm is $\mathcal{O}(n)$.

Proof. The complexity of Partition algorithm is $\mathcal{O}(n)$ and hence there is a constant c such that

$$T_A^k(n) \leq c \cdot n + \frac{1}{n}\Big(\sum_{i=1}^{k-1} T_A^k(n-i) + \sum_{i=k+1}^{n} T_A^k(i-1)\Big).$$

Define $R(n) = \max_k T_A^k(n)$, then

$$R(n) \leq c \cdot n + \frac{1}{n}\max_k\Big(\sum_{i=1}^{k-1} R(n-i) + \sum_{i=k+1}^{n} R(i-1)\Big),$$

$$= c \cdot n + \frac{1}{n}\max_k\Big(\sum_{i=n-k+1}^{n-1} R(i) + \sum_{i=k}^{n-1} R(i)\Big).$$

To show by induction that $R(n) \leq 4c \cdot n$.

Selection Algorithm – Complexity, III

For $n = 2$,

$$R(n) \le 2 \cdot c + \frac{1}{2} \max\Big(R(1), R(1)\Big) = 2 \cdot c + \frac{1}{2} \max(4c, 4c)$$
$$= 2 \cdot c + 2 \cdot c < 4 \cdot c \cdot n$$

Next assume $R(n) \le 4 \cdot c \cdot n$ for $2 \le n < m$.
For $n = m$,

$$R(m) \le c \cdot m + \frac{1}{m} \max_k \Big(\sum_{i=m-k+1}^{m-1} R(i) + \sum_{i=k}^{m-1} R(i) \Big).$$

Since $R(n)$ is a nondecreasing function of n, the term

$$\sum_{i=m-k+1}^{m-1} R(i) + \sum_{i=k}^{m-1} R(i)$$

is maximum when $k = m/2$ when m is even, and $k = (m+1)/2$ when m is odd.

Selection Algorithm – Complexity, IV

When m is even

$$R(m) \le c \cdot m + \frac{2}{m} \sum_{i=m/2}^{m-1} R(i) = c \cdot m + \frac{8c}{m} \frac{(3m/2 - 1)(m/2)}{2}$$
$$= c \cdot m + c(3m - 2) = 4 \cdot c \cdot m - 2c < 4 \cdot c \cdot m$$

Similarly, when m is odd

$$R(m) \le c \cdot m + \frac{2}{m} \sum_{i=(m+1)/2}^{m-1} R(i) = c \cdot m + \frac{8c}{m} \frac{(3m-1)/2 \cdot (m-1)/2}{2}$$
$$= c \cdot m + \frac{c}{m}(3m^2 - 4m + 1) = 4 \cdot c \cdot m - 4c + c/m < 4 \cdot c \cdot m$$

Since $T_A(n) \le R(n)$, therefore $T_A(n) \le 4 \cdot c \cdot n$ and $T_A(n)$ is $\mathcal{O}(n)$. □

- The space complexity of the `Select1` algorithm is $\mathcal{O}(n)$ for the array A.
- The `Select1` algorithm can also be randomized as `RQuickSort` algorithm.
  - The expected time complexity is still $\mathcal{O}(n)$.
  - But the average performance is expected to be better.

Selection Algorithm – Complexity, V

- The execution time of Select1 in the worst-case is $\mathcal{O}(n^2)$.
  - The worst-case can happen if the partition element, $A[low]$, is close to extreme.
  - If the partition element is close to the median, $A[(low+high)/2]$, then the number of iterations can be reduced significantly.
  - Using this argument, the selection algorithm is modified to have worst-case linear time complexity.
- The array A is divided into subarrays each has r elements
  - $\lceil n/r \rceil$ groups
  - A small r is usually prefered ($r = 5$, for example).
- Then the median of each group is found and move to the front array A
- The median of the medians, mm, is then found using Partition function
- Now, this mm can be used to partition array A.
- Since mm is used for each partition step in the selection algorithm, worst-case linear time can be guaranteed.
- Note that though Select2 is worst-case linear, it has a much larger coefficient, as compared to Select1, thus for small to median-size problems, Select2 may not be faster in execution.
  - Select2 returns the position j such that $A[j]$ is the k'th smallest element.

Selection Algorithm – Worst-case Linear

Algorithm 3.3.3. Selection – Worst-case Linear

```
// Partition the array into A[1 : k − 1] ≤ A[k] ≤ A[k + 1 : n].
// Input: A, int k, low, high, r
// Output: k-th element of A.
Algorithm Select2(A, k, low, high, r)
{
    j := k + low − 1 ;
    while (k ≠ j − low) do {
        n := high − low + 1 ;
        if (n ≤ r) then { // small array
            InsertionSort(A, low, high) ;
            return low + k ;
        }
        for i := 1 to ⌈ n/r⌉ do { // find median of each group and move to front
            InsertionSort(A, low + (i − 1) ∗ r, low + i ∗ r − 1) ;
            Swap(low + i − 1, low + (i − 1) ∗ r + ⌈ r/2⌉ −1) ;
        }
        j := Select2(A, ⌈⌊ n/r⌋/2⌉, low, low + ⌈ n/r⌉ −1) ; // find median of medians
        Swap(low, j) ; // move median of median to A[low]
        j := Partition(A, low, high + 1) ;
        if (k < j − low) high := j; // reduce to A[low : j]
        else if (k > j − low) { // reduce to A[j + 1 : high]
            k := k − (j − low + 1) ;
            low := j + 1 ;
        }
    }
    return j;
}
```

Matrix Multiplication

- Given two $n \times n$ matrix $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A}[i,j] \in \mathbb{R}$, $\mathbf{B}[i,j] \in \mathbb{R}$, $1 \le i,j \le n$, then $n \times n$ matrix $\mathbf{C}$ is the product of $\mathbf{A}$ and $\mathbf{B}$, ($\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$),

$$\mathbf{C}[i,j] = \sum_{k=1}^{n} \mathbf{A}[i,k] \times \mathbf{B}[k,j], \qquad 1 \le i,j \le n. \tag{3.3.3}$$

- Note that to calculate $\mathbf{C}[i,j]$, one needs n multiplications and $n-1$ additions.
- Thus to calculate $\mathbf{C}$, which has n^2 elements, the time complexity is $\Theta(n^3)$.

Matrix Multiplication – Divide and Conquer

- Suppose $n = 2^k$, we can apply divide and conquer approach to matrix multiplication problem.
- Divide each matrix into 4 submatrices with $\frac{n}{2} \times \frac{n}{2}$ dimensions each, then

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \tag{3.3.4}$$

where

$$\begin{aligned} \mathbf{C}_{11} &= \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} \\ \mathbf{C}_{12} &= \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{C}_{21} &= \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} \\ \mathbf{C}_{22} &= \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{aligned} \tag{3.3.5}$$

- To calculate matrix $\mathbf{C}$, we need
  - Eight matrix multiplications $(\frac{n}{2} \times \frac{n}{2})$,
  - Four matrix additions ($\mathcal{O}(n^2)$ complexity due to n^2 elements in $\mathbf{C}$).
- Let $T(n)$ be the complexity, then

$$T(n) = \begin{cases} b, & n \le 2 \\ 8 \cdot T(n/2) + c \cdot n^2, & n > 2. \end{cases} \tag{3.3.6}$$

where b and c are constants.

Matrix Multiplication – Divide and Conquer, II

- If $n = 2^k$

$$\begin{aligned}
T(n) &= 8\,T(n/2) + c \cdot n^2 \\
&= 8\left[8\,T(n/4) + c \cdot \left(\frac{n}{2}\right)^2\right] + c \cdot n^2 \\
&= 8^2\,T(n/4) + c \cdot n^2\,(2+1) \\
&= 8^3\,T(n/8) + c \cdot n^2\,(4+2+1) \\
&= 8^{k-1}\,T(n/2^{k-1}) + c \cdot n^2 \sum_{i=0}^{k-2} 2^i \\
&= 2^{3k-3}\,b + c \cdot n^2 \left(2^{k-1}\right) \\
&= \frac{n^3}{8} b + c \cdot n^2 \left(\frac{n}{2}\right) \\
&= \left(\frac{b}{8} + \frac{c}{2}\right) n^3 \\
&= \mathcal{O}(n^3)
\end{aligned}$$

- Thus, this divide and conquer approach does not improve the computational complexity

Strassen's Matrix Multiplication

- Given Equations (3.3.4) and (3.3.5), define the following

$$\begin{array}{lcl}
\mathbf{P} & = & (\mathbf{A}_{11} + \mathbf{A}_{22})(\mathbf{B}_{11} + \mathbf{B}_{22}) \\
\mathbf{Q} & = & (\mathbf{A}_{21} + \mathbf{A}_{22})\mathbf{B}_{11} \\
\mathbf{R} & = & \mathbf{A}_{11}(\mathbf{B}_{12} - \mathbf{B}_{22}) \\
\mathbf{S} & = & \mathbf{A}_{22}(\mathbf{B}_{21} - \mathbf{B}_{11}) \\
\mathbf{T} & = & (\mathbf{A}_{11} + \mathbf{A}_{12})\mathbf{B}_{22} \\
\mathbf{U} & = & (\mathbf{A}_{21} - \mathbf{A}_{11})(\mathbf{B}_{11} + \mathbf{B}_{12}) \\
\mathbf{V} & = & (\mathbf{A}_{12} - \mathbf{A}_{22})(\mathbf{B}_{21} + \mathbf{B}_{22})
\end{array} \tag{3.3.7}$$

Then

$$\begin{array}{lcl}
\mathbf{C}_{11} & = & \mathbf{P} + \mathbf{S} - \mathbf{T} + \mathbf{V} \\
\mathbf{C}_{12} & = & \mathbf{R} + \mathbf{T} \\
\mathbf{C}_{21} & = & \mathbf{Q} + \mathbf{S} \\
\mathbf{C}_{22} & = & \mathbf{P} + \mathbf{R} - \mathbf{Q} + \mathbf{U}
\end{array} \tag{3.3.8}$$

- To find matrix $\mathbf{C}$, we need 7 matrix multiplications of $\frac{n}{2} \times \frac{n}{2}$ and 18 matrix additions.
- Since matrix multiplications, $\mathcal{O}(n^3)$, is more expensive than matrix addition, $\mathcal{O}(n^2)$, for large n this approach might be more efficient.

Strassen's Matrix Multiplication, II

- The recurrence relation for the computation time $T(n)$ is

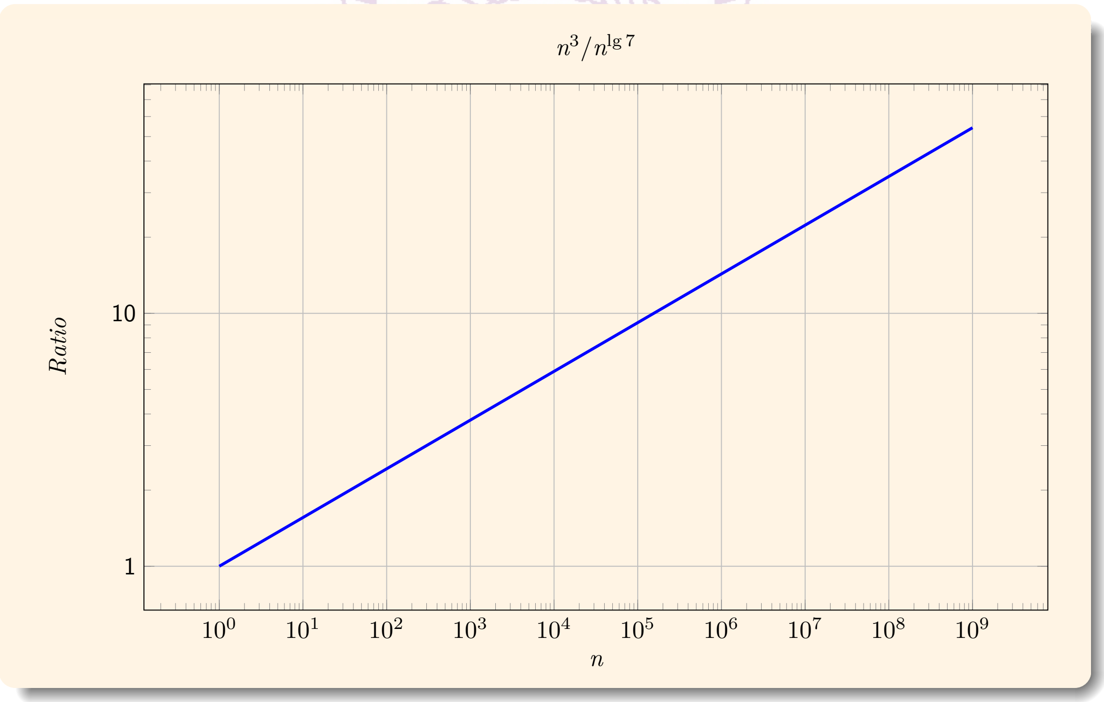
$$T(n) = \begin{cases} b, & n \le 2, \\ 7 \cdot T(n/2) + c \cdot n^2, & n > 2. \end{cases} \tag{3.3.9}$$

  where b and c are two constants.
- If $n = 2^k$, then

$$\begin{aligned} T(n) &= 7 \cdot T(n/2) + c \cdot n^2 \\ &= 7^2 \cdot T(n/4) + 7 \cdot c \cdot (n/2)^2 + c \cdot n^2 \\ &= 7^2 \cdot T(n/4) + c \cdot n^2 (7/4 + 1) \\ &= 7^{k-1} \cdot T(n/2^{k-1}) + c \cdot n^2 \sum_{i=0}^{k-2} (7/4)^i \\ &= 7^{k-1} \cdot b + c \cdot n^2 \left((7/4)^{k-1} - 1 \right) / (3/4) \\ &\approx n \frac{\lg 7}{7} \cdot b + c' n^{\lg 4 + \lg 7 - \lg 4} \\ &= \mathcal{O}(n^{\lg 7}) = \mathcal{O}(n^{2.807}) \end{aligned}$$

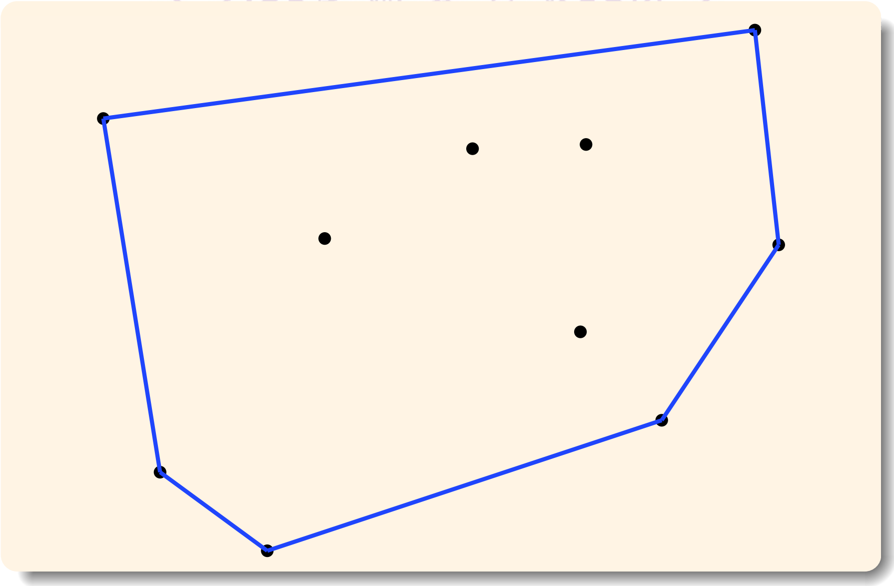
Strassen's Matrix Multiplication, III

- Compared to direct matrix multiplication, $\mathcal{O}(n^3)$, Strassen's approach can be faster for large n.
  - But, coding is much more complex



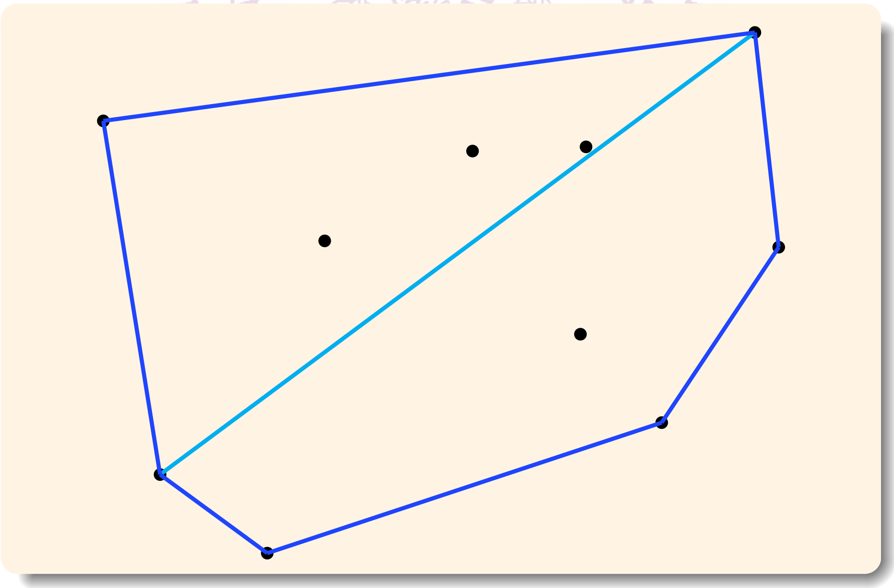
Convex Hull Problem

- Given a set S that contains points on a 2-D plane, the convex hull is defined as the smallest convex polygon that contains all the points in S.
- A polygon is convex if for any two points p_1, p_2 inside of of the polygon, the straight line segment connecting p_1 and p_2 is fully contained in the polygon.
- The vertices of the convex hull of a set S is a subset of S.
  - But, not necessarily a proper subset.



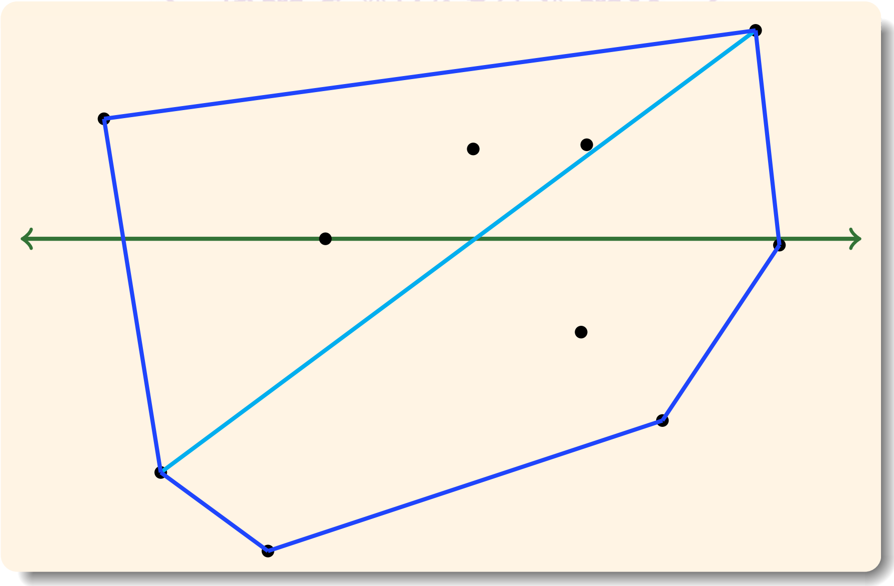
Convex Hull – Direct Implementation

- The convex hull of S can be found using the definition above
  - For any $p_1 \in S$, if p_1 is inside the triangle formed by $p_2, p_3, p_4 \in S$, with $p_1 \notin \{p_2, p_3, p_4\}$, then p_1 is not a vertex of the convex hull.
- This direct implementation has the time complexity of $\mathcal{O}(n^4)$.
  - n points to be tested, n^3 for all possible triangles.



Convex Hull – Direct Implementation, II

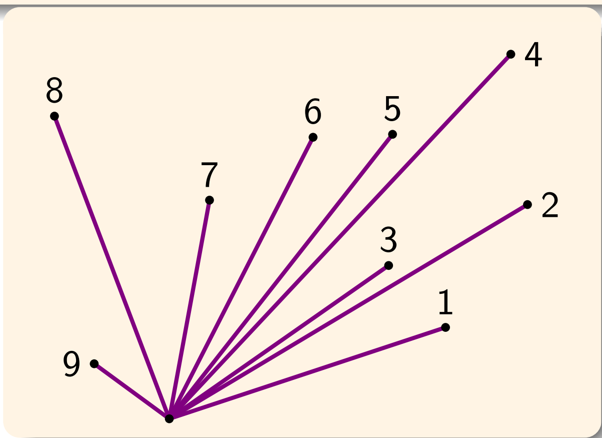
- To test if a point p_1 is inside of a triangle $\triangle p_2 p_3 p_4$
  - Let L be the horizontal line passing through $p_1 = (x_1, y_1)$, note that L can be described by the linear equation $y = y_1$, then check if L intersects with any of the line segments, $\overline{p_2 p_3}$, $\overline{p_3 p_4}$, $\overline{p_4 p_2}$. If not, then p_1 is outside of $\triangle p_2 p_3 p_4$. Otherwise let (x_a, y_1) and (x_b, y_1) be the intersect points, if $x_a \le x_1 \le x_b$ then p_1 is inside of the triangle. (Note that it is possible $x_a = x_b$.)

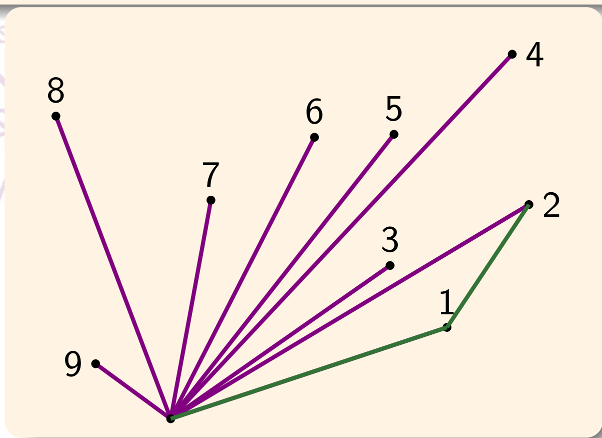


Convex Hull – Graham's Scan

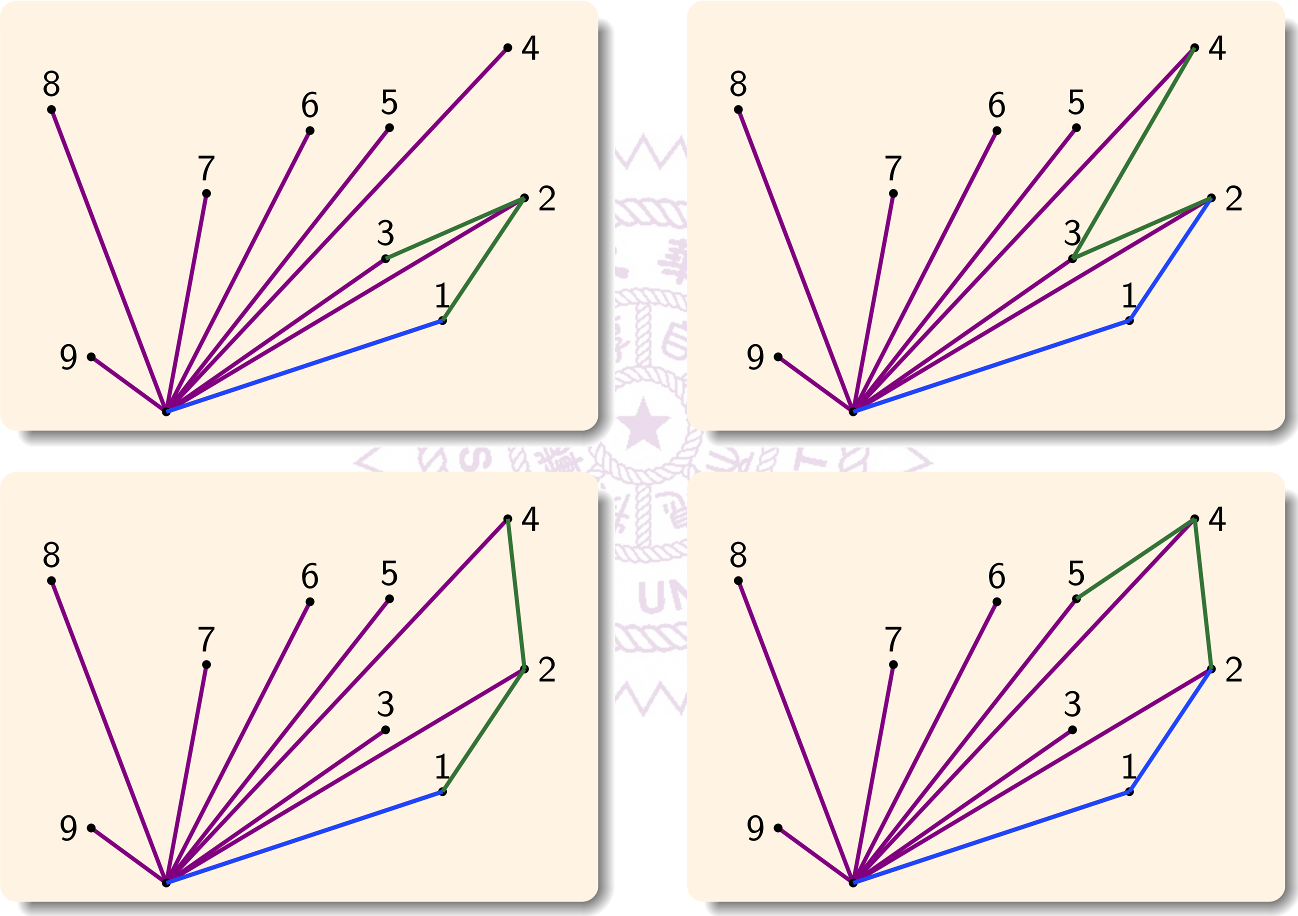
Algorithm 3.3.4. Convex Hull

```
// Find the convex hull of a set of points.
// Input: ptslist linked list of points
// Output: convex hull of ptslist.
Algorithm ConvexHull(ptslist)
{
    Let the first element of ptrlist has the smallest y coordinate ;
    Sort(ptrlist) ; // Sort by angle between p and x-axis.
    Scan(ptrlist) ;
    PrintList(ptrlist) ;
}
```

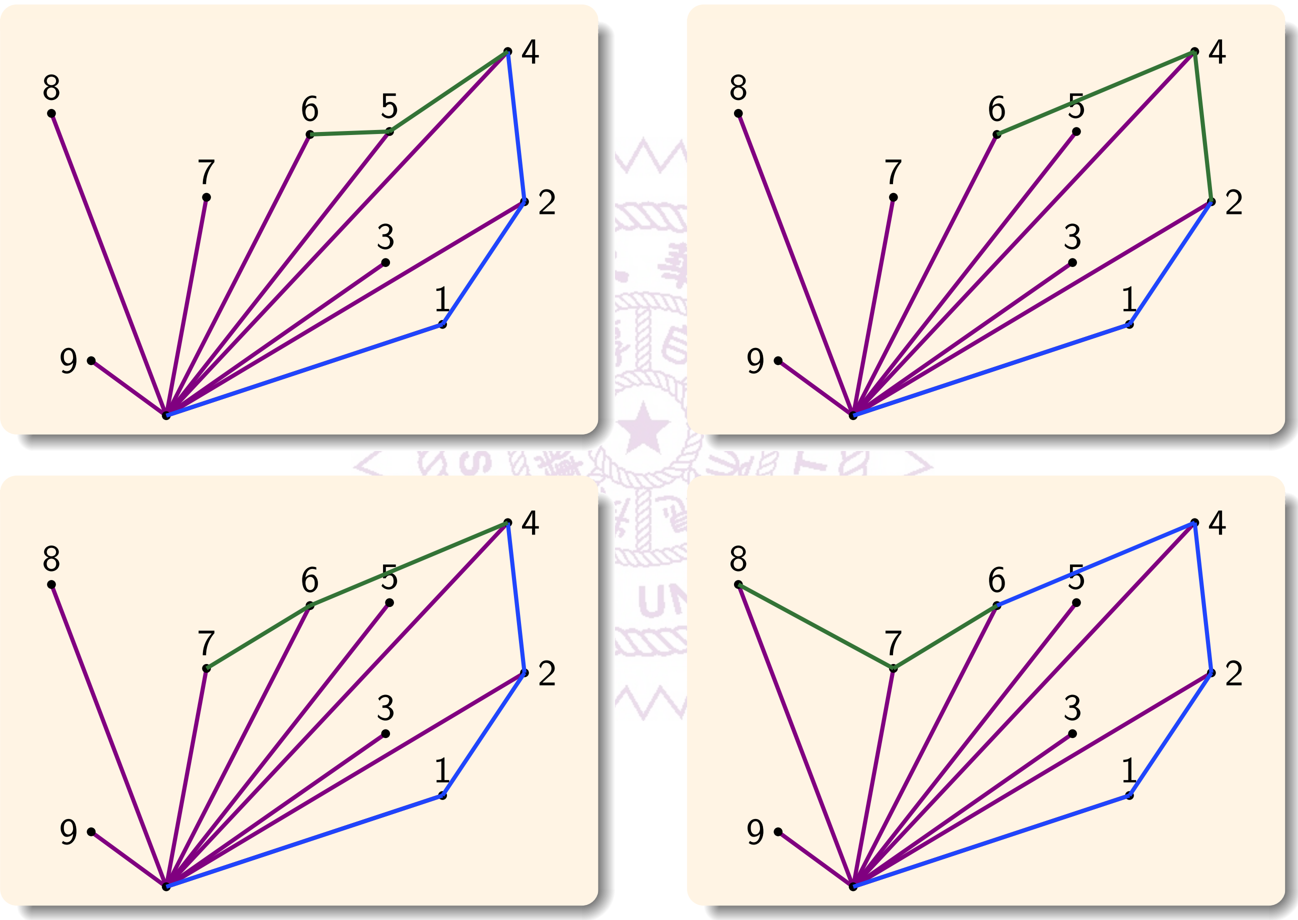




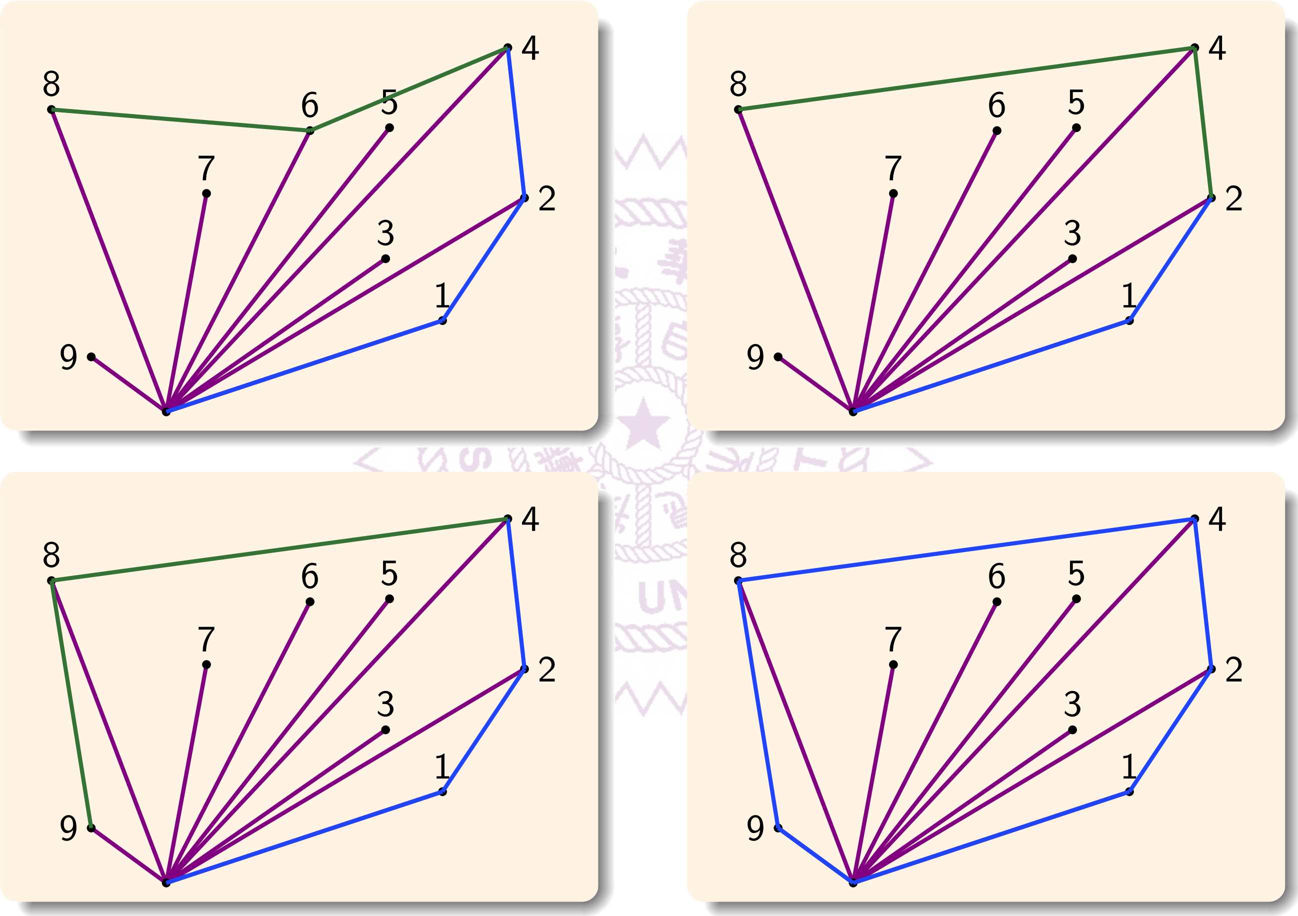
Convex Hull – Graham's Scan, II



Convex Hull – Graham's Scan, III



Convex Hull – Graham's Scan, IV



Convex Hull – Graham's Scan Algorithm

Algorithm 3.3.5. Graham's Scan Algorithm

```
// Remove internal points while doing convex hull.
// Input: list, linked list of points
// Output: none, list modified.
Algorithm Scan(list)
{
    p1 := list; p2 := p1→next;
    while (p2→next ≠ NULL) do { // For all points on the sorted list.
        p3 := p2→next;
        if ( Area(p1, p2, p3) > 0) then // p1p2p3 (vector) turning left, accept p2.
            p1 := p1→next;
        else {
            p1→next := p3; // Remove p2 from the list.
            p3→prev := p1;
            delete p2;
            p1 := p1→prev; // Backtrack p1.
        }
        p2 := p1→next;
    }
}
```

Convex Hull – Graham's Scan Algorithm, II

- In the preceding algorithm, the points are in linked list form consists

```
struct Point {
    double x,y;
    struct Point *next, *prev;
}
```

  - Note that this is a double linked list.
- Let $p_1(x_1, y_1)$, $p_2(x_2, y_2)$ and $p_3(x_3, y_3)$ be three points in a plane the function `Area`(p_1, p_2, p_3) is defined as

$$\det \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{3.3.10}$$

- It can be shown that
  - If the area is positive then p_3 is located to the left of the vector $\overrightarrow{p_1 p_2}$.
  - If the area is negative then p_3 is located to the right of the vector $\overrightarrow{p_1 p_2}$.
  - If the area is zero then p_3 is colinear with $\overrightarrow{p_1 p_2}$.

Convex Hull – Graham's Scan Algorithm, Complexity

- The Algorithm (3.3.4) consists of 3 steps
  - (line 4) finding the first element with the smallest y coordinate can be done in $\mathcal{O}(n)$ time,
  - (line 5) sort by the angle can be done in $\mathcal{O}(n \lg n)$ time,
  - (line 6) Graham's Scan can be done in $\mathcal{O}(n)$ time.
- Thus, the time complexity is $\mathcal{O}(n \lg n)$.

Quick Hull Algorithm

- Divide and conquer approach can be used to find the convex hull.

Algorithm 3.3.6. QuickHull

// Find Convex Hull for points in *list*.
// Input: *list*, linked list of points
// Output: *CHull* convex hull of *list*.
Algorithm QuickHull(*list*, *CHull*)
{
  Find $p_1 \in$ *list* with the smallest x coordinate .
  Find $p_2 \in$ *list* with the largest x coordinate .
  Let $X_1 := \{p \mid \texttt{Area}(p_1, p_2, p) > 0\}$. // Upper half.
  Let $X_2 := \{p \mid \texttt{Area}(p_1, p_2, p) < 0\}$. // Lower half.
  Hull($p_1, p_2, X_1, \mathit{UpperHull}$) ; // Create upper hull.
  Hull($p_2, p_1, X_2, \mathit{LowerHull}$) ; // Lower hull.
  $\mathit{CHull} :=$ Merge($\mathit{UpperHull}, \mathit{LowerHull}$) ; // Merge them.
}

- Finding p_1 and p_2 takes $\mathcal{O}(n)$ time.
- Finding X_1 and X_2 takes $\mathcal{O}(n)$ time.
- Merge takes no more than $\mathcal{O}(n)$ time.
- The time complexity can be dominated by Hull function.

Quick Hull Algorithm, II

Algorithm 3.3.7. QuickHull

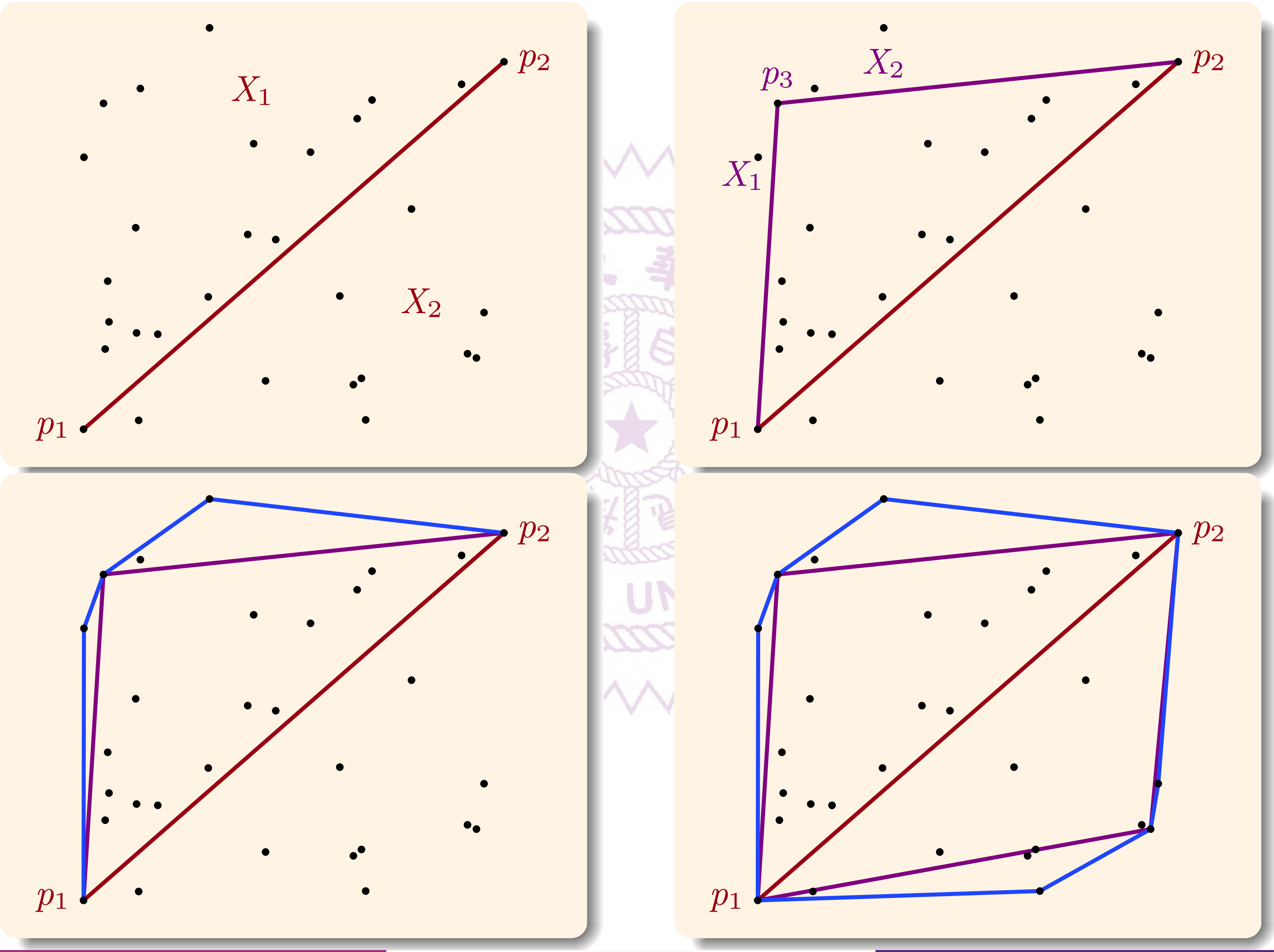
// Find Convex Hull for p_1, p_2 and *list*.
// Input: p_1, p_2, and *list*
// Output: Convex Hull, *CHull*.
Algorithm Hull(p_1, p_2, *list*, *CHull*)
{
  Find $p_3 \in$ *list* with the largest $|\texttt{Area}(p_1, p_2, p_3)|$;
  Let $X_1 := \{p \mid \texttt{Area}(p_1, p_3, p) > 0\}$. // All points left to $\overrightarrow{p_1 p_3}$/
  if $(X_1 = \emptyset)$ then $H_1 := \{p_1, p_3\}$; // No more points.
  else HULL(p_1, p_3, X_1, H_1) ; // Recursive call if more points.
  Let $X_2 := \{p \mid \texttt{Area}(p_3, p_2, p) > 0\}$.
  if $(X_2 = \emptyset)$ then $H_2 := \{p_3, p_2\}$;
  else HULL(p_3, p_2, X_2, H_2) ;
  $\mathit{CHull} :=$ Merge(H_1, H_2) ; // Combine those two hulls.
}

- Finding p_3, X_1, and X_2 take $\mathcal{O}(m)$ time, if *list* has m points.
- Thus, if $T(m)$ is the time for HULL algorithm we have

$$T(m) = T(m_1) + T(m_2) + \mathcal{O}(m), \tag{3.3.11}$$

  where $m_1 + m_2 \leq m$.
- This recurrence relationship is the same as QuickSort.
  - Worst-case complexity is $\mathcal{O}(m^2)$, and average-case is $\mathcal{O}(m \lg m)$.

Quick Hull Example



Time Complexity of Divide and Conquer Algorithms

- In Algorithm DandC (Algorithm 3.1.1) the problem is divided into k subproblems; each solved recursively; then the results are combined to form the final solution.
- The execution time can be assumed to have a general recurrence equation as

$$T(n) = a \cdot T(n/k) + f(n). \tag{3.3.12}$$

where $f(n)$ is the time to divide problem into k subsets and to combine the subsets to form the final solution.
- Let $n = k^m$, then

$$\begin{aligned}
T(n) &= a \cdot T(n/k) + f(n) = a \cdot T(k^{m-1}) + f(k^m) \\
&= a \cdot \left(a \cdot T(k^{m-2}) + f(k^{m-1})\right) + f(k^m) = a^2 \cdot T(k^{m-2}) + a \cdot f(k^{m-1}) + f(k^m) \\
&= a^m \cdot T(1) + \sum_{i=0}^{m-1} a^i \cdot f(k^{m-i}) = a^m \left(T(1) + \sum_{i=0}^{m-1} f(k^{m-i})/a^{m-i}\right) \\
&= a^{\log_k n} \left(T(1) + \sum_{i=0}^{m-1} f(k^{m-i})/a^{m-i}\right) = n^{\log_k a} \left(T(1) + \sum_{i=1}^{\log_k n} f(k^i)/a^i\right)
\end{aligned}$$

Time Complexity of Divide and Conquer Algorithms, II

- Let's assume further that $f(n) = \Theta(n^d) \approx n^d$, then

$$T(n) = n^{\log_k a}\left(T(1) + \sum_{i=1}^{\log_k n} f(k^i)/a^i\right) = n^{\log_k a}\left(T(1) + \sum_{i=1}^{\log_k n} k^{d\cdot i}/a^i\right)$$
$$= n^{\log_k a}\left(T(1) + \sum_{i=1}^{\log_k n} (k^d/a)^i\right) \quad (3.3.13)$$

- If $k^d = a$, or $d = \log_k a$, then

$$T(n) = n^{\log_k a}\left(T(1) + \sum_{i=1}^{\log_k n} (k^d/a)^i\right) = n^{\log_k a}\left(T(1) + \sum_{i=1}^{\log_k n} 1\right)$$
$$= n^{\log_k k^d}\left(T(1) + \log_k n\right) = n^d\left(T(1) + \log_k n\right) = \Theta(n^d \log_k n) \quad (3.3.14)$$

Time Complexity of Divide and Conquer Algorithms, III

- If $k^d \neq a$, then

$$T(n) = n^{\log_k a}\left(T(1) + \sum_{i=1}^{\log_k n} (k^d/a)^i\right) = n^{\log_k a}\left(T(1) + \frac{(k^d/a)^{1+\log_k n} - 1}{(k^d/a) - 1}\right)$$

- In case of $k^d > a$, or $d > \log_k a$, then

$$T(n) = n^{\log_k a}\left(T(1) + \frac{(k^d/a)^{1+\log_k n} - 1}{(k^d/a) - 1}\right) = \Theta(n^{\log_k a}\frac{(k^d/a)^{1+\log_k n}}{(k^d/a)})$$
$$= \Theta(n^{\log_k a}(k^d/a)^{\log_k n}) = \Theta(a^{\log_k n}(k^d/a)^{\log_k n})$$
$$= \Theta(k^{d\log_k n}) = \Theta(k^{\log_k n^d}) = \Theta(n^d)$$

- In case of $k^d < a$, or $d < \log_k a$, then

$$T(n) = n^{\log_k a}\left(T(1) + \frac{(k^d/a)^{1+\log_k n} - 1}{(k^d/a) - 1}\right) = n^{\log_k a}\cdot\Theta(1) = \Theta(n^{\log_k a})$$

Master Method

- Combine those three equations together, we have

Theorem 3.3.8. Master Method

Let $T(n)$ be an eventually nondecreasing function that satisfies the recurrence relationship

$$\begin{aligned} T(n) &= a \cdot T(n/k) + f(n), \qquad \text{for } n = k^i, i = 1, 2, \ldots \\ T(1) &= c. \end{aligned}$$

where $a \geq 1$, $k \geq 2$, $c > 0$. If $f(n) = \Theta(n^d)$ where $d \geq 0$, then

$$T(n) = \begin{cases} \Theta(n^d), & \text{if } d > \log_k a, \\ \Theta(n^d \lg n), & \text{if } d = \log_k a, \\ \Theta(n^{\log_k a}), & \text{if } d < \log_k a. \end{cases}$$

Master Method – Example

- Example 1, Algorithm MaxMin

$$T(n) = 2\,T(n/2) + 2$$

$a = 2$, $k = 2$, $d = 0$, and $d < \log_2 2 = 1$.
Using Master Method, we have $T(n) = \Theta(n^{\log_2 2}) = \Theta(n)$.

- Example 2, Algorithm MaxSubArray

$$T(n) = 2\,T(n/2) + n$$

$a = 2$, $k = 2$, $d = 1$, and $d = \log_2 2$.
Using Master Method, we have $T(n) = \Theta(n \lg n)$.

- Example 3, MatrixMultiplication

$$T(n) = 8\,T(n/2) + n^2$$

$a = 8$, $k = 2$, $d = 2$, and $d < \log_2 8 = 3$.
Using Master Method, we have $T(n) = \Theta(n^{\log_2 8}) = \Theta(n^3)$.

Master Method – Example, II

- Example 4,

$$T(n) = 2\,T(n/2) + n^2$$

$a = 2$, $k = 2$, $d = 2$ and $d > \log_2 2 = 1$.
Using Master Method, we have $T(n) = \Theta(n^d) = \Theta(n^2)$.

This can also be derived as follows.

$$\begin{aligned} T(n) &= 2\,T(n/2) + n^2 \\ &= 2\Big(2\,T(n/4) + (n/2)^2\Big) + n^2 \\ &= 4\,T(n/4) + n^2(1 + 1/2) \\ &= 2^m\,T(n/2^m) + n^2 \sum_{i=0}^{m-1} 1/2^i \\ &= n + n^2 \cdot 2 \cdot (1 - 2^{-m}) \\ &= \Theta(n^2) \end{aligned}$$

- Thus, the Master method can be effective to find the complexity of divide and conquer algorithms.

Summary

- Selection problem
- Matrix multiplication
  - Strassen's matrix multiplication
- Convex hull problem
  - Graham's scan algorithm
  - Quick hull algorithm
- Time complexity of divide and conquer algorithms
  - Master method