Unit 6.3 Dynamic Programming III

Algorithms

EE3980

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String Editing Problem

- Given two strings $X =$"$x_1 x_2 \cdots x_n$" and $Y =$"$y_1 y_2 \cdots y_m$", where x_i, $1 \le i \le n$, and y_j, $1 \le j \le m$, are members of a finite set of symbols known as the alphabet.
- The string editing problem is to transform X into Y using the following editing operations with corresponding cost and to find the sequence of operations that minimizes the total cost.
  - Delete the symbol x_i from X with cost $D(x_i)$,
  - Insert the symbol y_j to Y with cost $I(y_j)$,
  - Change the symbol x_i of X into y_j with cost $C(x_i, y_j)$.
  - Note that keep x_i to become y_j has no cost.
- Example, $X =$"*elate*" and $Y =$"*later*". Total cost to transform X into Y is $D(e) + I(r)$.

Step	X	Y	Cost
1	$elate$		$D(e)$
2	$elate$	l	0
3	$elate$	la	0
4	$elate$	lat	0
5	$elate$	$late$	0
6	$elate$	$later$	$I(r)$
			$D(e) + I(r)$

String Editing — Algorithm

Algorithm 6.3.1. Wagner Fischer Algorithm

```
// Transform X[n] into Y[m] with minimum cost using matrix M[n, m].
// Input: int n, m, strings X[n], Y[m], cost D[n], I[m], C[n, m]
// Output: min cost matrix M[n, m].
Algorithm WagnerFischer(n, m, X, Y, D, I, C, M)
{
    M[0,0] := 0;
    for i := 1 to n do M[i,0] := M[i-1,0] + D(X[i]);
    for j := 1 to m do M[0,j] := M[0,j-1] + I(Y[j]);
    for i := 1 to n do {
        for j := 1 to m do {
            if (X[i] = Y[j]) then m1 := M[i-1,j-1];
            else m1 := M[i-1,j-1] + C(X[i], Y[j]);
            m2 := M[i-1,j] + D(X[i]);
            m3 := M[i,j-1] + I(Y[j]);
            M[i,j] := min(m1, m2, m3);
        }
    } // When done, M[n, m] contains the minimum cost of the transformation
}
```


String Editing — Example

- Example. Given $X =$"*elate*", $Y =$"*later*" and the cost functions $D(x) = 1,\ I(y) = 1,\ C(x, y) = 2,\ x, y \in \{A, \cdots, Z, a, \cdots, z\},\ x \neq y$.

		l	a	t	e	r
	0	1	2	3	4	5
e	1	2	3	4	3	4
l	2	1	2	3	4	5
a	3	2	1	2	3	4
t	4	3	2	1	2	3
e	5	4	3	2	1	2

Matrix M of WagnerFischer algorithm.

- Thus the transformation sequence is

Step	operation		Y
1	Delete	e	
2	Keep	l	l
3	Keep	a	la
4	Keep	t	lat
5	Keep	e	$late$
6	Insert	r	$later$

- And the total cost is $D(e) + I(r) = 2$.
- After WagnerFischer algorithm, the following algorithm traces the M matrix to generate the transformation sequence.
  - Note that array T has the transformation sequence but is in reverse order.

String Editing — Transformation Trace

Algorithm 6.3.2. Trace

```
   // Trace the matrix M[n, m] to find the transformation operations.
   // Input: int n, m, cost D[n], I[m], C[n, m] and M[n, m]
   // Output: T[n + m] transformation.
1  Algorithm Trace(n, m, M, D, I, C, T)
2  {
3        i := n; j := m; k := 0;
4        while (i > 0 and j > 0) do {
5              else if (i > 0 and j > 0 and (M[i, j] = M[i − 1, j − 1] + C(X[i], Y[j])) then {
6                    T[k] := 'C'; i := i − 1; j := j − 1; k := k + 1; // Change X[i] to Y[j].
7              }
8              else if (j = 0 or (M[i, j] = M[i, j − 1] + I(Y[j]))) { // Add Y[j].
9                    T[k] := 'I'; j := j − 1; k := k + 1;
10             }
11             else if (i = 0 or (M[i, j] = M[i − 1, j] + D(X[i]))) then { // Delete X[i].
12                   T[k] := 'D'; i := i − 1; k := k + 1;
13             }
14             else { // No changes.
15                   T[k] := '−'; i := i − 1; j := j − 1; k := k + 1;
16             }
17       } // Array T has the transformation sequence but is in reverse order.
18 }
```

- In this algorithm, it is assumed that $C > I > D$.
  - If the assumption is not correct then modifications are necessary to ensure correct results.

String Editing — Complexities

- Algorithm WagnerFischer
  - for loop, lines 6–13, executes $n \times m$ times
  - for loops, lines 6,7, execute n and m times, separately
  - Overall time complexity $\mathcal{O}(mn)$
- Algorithm Trace while loop, lines 4-21, executes at most $(m + n)$ times
  - Time complexity $\mathcal{O}(m + n)$
- The longest common substring problem
  - Given two strings, X and Y, find a common substring Z such that Z has the most number of characters.
  - Example, $X =$"elate" and $Y =$"later" the longest common substring is $Z =$"late". Z has 4 characters.
  - The WagnerFischer algorithm can be used to find the longest common substring.
  - The Trace algorithm needs to be modified to find and print out the common substring.

0/1 Knapsack Problem

- The 0/1 knapsack problem is a variation of the knapsack problem.
  - Given n objects, each with profit p_i and weight w_i, $1 \le i \le n$, to be placed into a sack that can hold maximum of m weight. However, there is an additional constraint that each object must be placed as a whole into the sack, or not at all. That is, find x_i, $1 \le i \le n$, such that

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{n} p_i x_i, \\
\text{subject to} \quad & \sum_{i=1}^{n} w_i x_i \le m, \\
\text{and} \quad & x_i = 0 \text{ or } 1, \qquad 1 \le i \le n.
\end{aligned}
\tag{6.3.1}
$$

- Let $f_n(m)$ be the optimal solution to n-object 0/1 knapsack problem.
- For the n'th object it can either be placed into the sack or not, thus

$$
f_n(m) = \max\Big(f_{n-1}(m), f_{n-1}(m - w_n) + p_n\Big). \tag{6.3.2}
$$

  - $f_n(m)$ must be the larger of the following two cases
  - n-th object is not placed into the sack, $x_n = 0$,
    - In this case, $f_n(m) = f_{n-1}(m)$.
  - n-th object is placed into the sack, $x_n = 1$,
    - In this case, $f_n(m) = f_{n-1}(m - w_n) + p_n$.

0/1 Knapsack — Recursive Algorithm

- Using Eq. (6.3.2) a recursive version of the 0/1 knapsack algorithm can be formulated.

Algorithm 6.3.3. Recursive DKP

```
// Find the solution array x[n] for the 0/1 knapsack problem.
// Input: int n, profit p[n], weight w[n], m
// Output: Solution x[n].
Algorithm DKPr(n, p, w, m, x)
{
      if (n = 1) then {
            if (m ≥ w[1]) then {
                  x[1] := 1 ; return p[1] ;
            }
            else {
                  x[1] := 0 ; return 0 ;
            }
      }
      f1 := DKPr(n − 1, p, w, m, x) ; // object n not placed
      if (m ≥ w[n]) then // placing n'th object
            f2 := DKPr(n − 1, p, w, m − w[n], x) + p[n] ;
      else f2 := 0 ; // no room for additional objects
      if (f1 > f2) then {
            x[n] := 0 ; return f1 ;
      }
      else {
            x[n] := 1 ; return f2 ;
      }
}
```

0/1 Knapsack — Example

- Given 3 objects, $(p_1, p_2, p_3) = (1, 2, 5)$, $(w_1, w_2, w_3) = (2, 3, 4)$, and $m = 6$. Find the optimal 0/1 knapsack solution, (x_1, x_2, x_3), $x_i = 0$ or $x_i = 1$, $1 \le i \le 3$, that maximizes the profit,

$$P = \sum_{i=1}^{3} p_i x_i.$$

- The function DKPr is invoked by calling $P =$DKPr$(3, p, w, 6, x)$
  - And the calling sequence of the function is

```
    // DKPr calling sequence
DKPr(3, p, w, 6, x)
    DKPr(2, p, w, 6, x) // object 3 not placed
        DKPr(1, p, w, 6, x) // object 2 not placed
            P := 1 ; x := (1, 0, 0) ;
        DKPr(1, p, w, 3, x) // object 2 placed
            P := 3 ; x := (1, 1, 0) ;
    DKPr(2, p, w, 2, x) // object 3 placed
        DKPr(1, p, w, 2, x) // object 2 not placed
            P := 6 ; x := (1, 0, 1) ;
        DKPr(1, p, w, −1, x) // object 2 placed
            P := −∞ ; x := (0, 1, 1) ;
Maximum profit P := 6, x := (1, 0, 1).
```


0/1 Knapsack — Complexity

- Note that function DKPr is invoked 7 times
  - All possible combinations of $x_i = 0$ and $x_i = 1$, $1 \le i \le n$ are tested for the maximum profit.
- The time complexity of DKPr algorithm is $\mathcal{O}(2^n)$.
- Line 11 of DKPr algorithm can eliminate unnecessary function calls
  - If there is no room for object n then it is not necessary to call DKPr further.
- The worst-case complexity of DKPr remains as $\mathcal{O}(2^n)$.

0/1 Knapsack — Dynamic Programming Approach

Algorithm 6.3.4. 0/1 Knapsack

// Find the solution array x for the 0/1 knapsack problem.
// `Input:` int n, profit p, weight w, m
// `Output:` Solution x.

`Algorithm` $DKP(n, p, w, m, x)$
{
    $S_0^1 := \{(0,0)\}$;
    `for` $i := 1$ `to` $n-1$ `do` {
        $S_1^i := \{(p+p_i, w+w_i) | (p,w) \in S_0^i$ `and` $w + w_i \le m\}$;
        $S_0^{i+1} :=$ `MergePurge`(S_0^i, S_1^i) ;
    }
    $(px, wx) :=$ last pair in S_0^n ;
    $(py, wy) := (p' + p_n, w' + w_n)$ where w' is the largest w' for any pairs
        $(p', w') \in S_0^n$ such that $w' + w_n \le m$;
    `if` $(px > py)$ `then` $x_n := 0$;
    `else` $x_n := 1$;
    `TraceBack` $x_{n-1}, \cdots, x_1$;
}

0/1 Knapsack — Example Revisited

- Given 3 objects, $(p_1, p_2, p_3) = (1, 2, 5)$, $(w_1, w_2, w_3) = (2, 3, 4)$, and $m = 6$. Find the optimal 0/1 knapsack solution, (x_1, x_2, x_3), $x_i = 0$ or $x_i = 1$, $1 \le i \le 3$, that maximizes the profit, $P = \sum_{i=1}^{3} p_i x_i$.
- The sets of feasible solutions are derived as the following.

$$
\begin{aligned}
S_0^1 &= \{(0,0)\} \\
S_1^1 &= \{(1,2)\} \\
S_0^2 &= \{(0,0),(1,2)\} \\
S_1^2 &= \{(2,3),(3,5)\} \\
S_0^3 &= \{(0,0),(1,2),(2,3),(3,5)\}
\end{aligned}
$$

- The last pair in S^2 is $(p_x, w_x) = (3, 5)$, and $(p_y, w_y) = (6, 6)$.
- Thus the optimal solution $\sum p_i x_i = 6$ and $\sum w_i x_i = 6$.
  - Since $p_x \not> p_y$, $x_3 = 1$.
  - Note that $(p_y, w_y) - (5, 4) = (1, 2) \notin S_1^2$, thus $x_2 = 0$.
  - Trace back again, $(1, 2) \in S_1^1$, therefore $x_1 = 1$.
  - Finally we have $(x_1, x_2, x_3) = (1, 0, 1)$ and $\sum p_i x_i = 6$, $\sum w_i x_i = 6$.

0/1 Knapsack — Properties

- Note that lines 9, 10 of Algorithm (6.3.4) actually requires to evaluate S_1^n.
- For the last example, we have

$$S_1^3 = \{(5,4),(6,6)\}.$$

since $(7,7)$ and $(8,9)$ both have $w + w_n \not\leq m$.

- And the optimal solution can be found when S_0^3 and S_1^3 are merged together which is

$$S_0^4 = \{(0,0)(1,2)(2,3),(3,5),(5,4),(6,6)\}.$$

- Note that comparing $(3,5)$ and $(5,4)$, the former has smaller profit, $3 < 5$, but larger weight, $5 > 4$, thus it is not a likely solution.
- The former, $(3,5)$, is dominated by the latter, $(5,4)$.
- When merging two feasible sets, the dominated solutions should be purged.
- Of course, by definition, the solutions with weight larger than m are also purged.

0/1 Knapsack — Dynamic Algorithm

Algorithm 6.3.5. 0/1 Knapsack

```
struct PW {
      double p, w; // for profit and weight of each object
}
Algorithm DKnap(n, p, w, x, m)
// p and w are arrays of n profits and weight; m capacity, x solution.
{
      b[0] := 0 ; pair[1].p := 0 ; pair[1].w := 0 ; // S_0^1
      t := 1 ; h := 1 ; // start and end of S_0^1
      b[1] := next := 2 ; // next free spot in pair array
      for i := 1 to n do { // generate S_0^{i+1}
            k := t ;
            u := Largest(pair, t, h, w[i], m) ; // largest u, pair[u].w + w[i] ≤ m.
            for j := t to u do { // generate S_1^i and merge
                  pp := pair[j].p + p[i] ; ww := pair[j].w + w[i] ;
                  while ((k ≤ h) and (pair[k].w ≤ ww)) do {
                        pair[next].p := pair[k].p ; pair[next].w := pair[k].w ;
                        next := next + 1 ; k := k + 1 ;
                  }
                  if ((k ≤ h) and (pair[k].w = ww)) then {
                        if (pp < pair[k].p) then pp := pair[k].p ; // new entry dominated
                        k := k + 1 ;
                  }
                  if (pp > pair[next − 1].p) then { // new entry is dominating
                        pair[next].p := pp ; pair[next].w := ww ;
                        next := next + 1 ;
                  }
```

0/1 Knapsack — Dynamic Algorithm, II

```
27              while ((k ≤ h) and (pair[k].p ≤ pair[next − 1].p)) do k := k + 1;
28          }
29          while (k ≤ h) do { // merge remaining terms from S_1^i
30              pair[next].p := pair[k].p; pair[next].w := pair[k].w;
31              next := next + 1; k := k + 1;
32          }
33          t := h + 1; h := next − 1; b[i + 1] := next; // initialize for S_0^{i+1}
34      }
35      TraceBack(n, p, w, m, pair, x); // find solution x
36 }
```

- In the above algorithm
  - $pair$ is an array to store all feasible solutions, $S_0^i, 0 \le i \le n$.
  - b is an array to store the indices of S_0^i in $pair$ array
  - Function Largest$(pair, t, h, w[i], m)$ finds the largest u satisfying

$$pair[u].w + w[i] \le m, \qquad t \le u \le h$$

  - The **for** loop of lines 10–34 generates $S_0^i, 1 \le i \le n$.
  - First S_0^{i-1} is copied into S_0^i
  - Then S_1^{i-1} is generated and merged into S_0^i
  - Lines 19-26 remove dominated entries

0/1 Knapsack — Example

- Given 3 objects, $(p_1, p_2, p_3) = (1, 2, 5)$, $(w_1, w_2, w_3) = (2, 3, 4)$, and $m = 6$. Find the optimal 0/1 knapsack solution, (x_1, x_2, x_3), $x_i = 0$ or $x_i = 1$, $1 \le i \le 3$, that maximizes the profit, $P = \sum_{i=1}^{3} p_i x_i$.
- After executing the algorithm DKnap, we have

	1	2	3	4	5	6	7	8	9	10	11	12
$pair[\,].p$	0	0	1	0	1	2	3	0	1	2	5	6
$pair[\,].w$	0	0	2	0	2	3	5	0	2	3	4	6
	$\uparrow$ $b[0]$	$\uparrow$ $b[1]$		$\uparrow$ $b[2]$				$\uparrow$ $b[3]$				

- Note that $(p, w) = (3, 5) \in S_0^3$ but not S_0^4 since it is dominated by $(5, 4)$.
- The last entry, $(pp, ww) = (6, 6)$, is the optimal solution.

0/1 Knapsack — Example

- To find if each object is placed into the sack or not, $x[i], 1 \le i \le n$.
- One starts from $i = n$ and trace back to 1.
  - The optimal solution is (pp, ww),
  - If $(pp, ww) \in S_0^n$ then $x[n] = 0$
    - $(pp_{n-1}, ww_{n-1}) = (pp, ww)$.
  - Otherwise $x[n] = 1$,
    - $(pp_{n-1}, ww_{n-1}) = (pp - p[n], ww - w[n])$.
- Repeat checking for S_0^{n-i} and update (pp_{n-i}, ww_{n-i}), one finds the solution $x[i], 1 \le i \le n$.
- For the last example,
  - $(6,6) \notin S^2$, thus $x[3] = 1$,
  - $(1,2) \in S^1$, and $x[2] = 0$,
  - $(1,2) \notin S^0$, thus $x[1] = 1$.
  - Optimal solution $x = (1,0,1)$, $(p,w) = (6,6)$.

0/1 Knapsack — Complexity

- Let the space needed to store S_0^i in *pair* be $|S_0^i|$, then

$$|S_0^i| \le 2^{i-1}$$

  And the total space needed for *pair* is

$$\sum_{i=1}^{n} |S_0^i| \le \sum_{i=1}^{n} 2^{i-1} = 2^n - 1$$

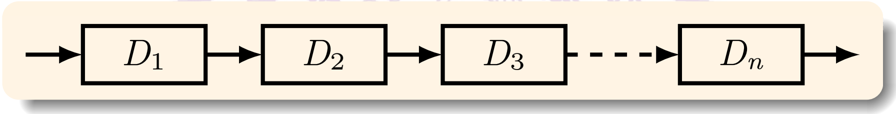
- Thus the space complexity is $\mathcal{O}(2^n)$
- The time needed to generate S_0^i is $\Theta(S_0^{i-1})$, therefore the total time to generate all pairs is

$$\sum_{i=1}^{n} |S_0^{i-1}| \le \sum_{i=1}^{n-1} 2^{i-1} = 2^{n-1} - 1$$

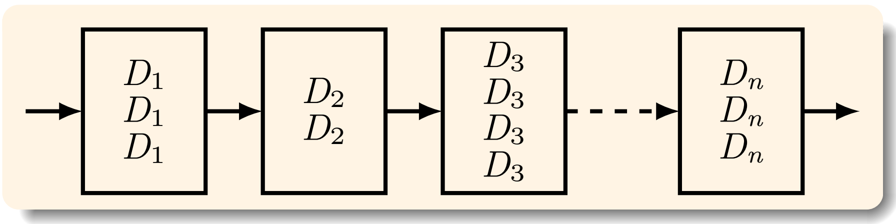
  and the time complexity is $\mathcal{O}(2^n)$.
- The time complexity of the `Traceback` function is $\mathcal{O}(n^2)$ since it involves n searches in the range $b[i]$ and $b[i+1]$.
  - Each search can take $\log(|S_0^i|) = \log(2^{i-1}) = (i-1)\log 2$.
  - Total time is $\sum_{i=1}^{n} (i-1)\log 2 = \mathcal{O}(n^2)$.

System Reliability

- Suppose a system is composed of n stages of devices connected in series.
  - Let r_i be the reliability of device D_i – the probability that device D_i function normally.
  - Then the reliability of the system is $\prod_{i=1}^{n} r_i = r_1 r_2 \cdots r_n$.



- To improve the reliability of the system, one can replace stage i by multiple, m_i, devices connected in parallel.
  - Then the reliability of stage i becomes $\phi_i(m_i) = 1 - (1 - r_i)^{m_i}$.
  - The system reliability becomes $\prod_{i=1}^{n} \phi_i(m_i)$.



Reliability Design Problem

- Assuming device D_i costs c_i each piece, and the total cost of the entire system is c, the reliability design problem is to find the multiplicity of each device, m_i for each D_i such that

$$\begin{aligned} \text{maximize} \quad & \prod_{i=1}^{n} \phi_i(m_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} c_i m_i \le c \\ \text{and} \quad & m_i \in N \text{ and } m_i \ge 1, \quad 1 \le i \le n. \end{aligned} \tag{6.3.3}$$

- Since $m_i \ge 1$ and $\sum c_i = c$, we can define

$$u_i = \lfloor (c + c_i - \sum_{j=1}^{n} c_j)/c_i \rfloor \tag{6.3.4}$$

- And the reliability design problem can be reformulated as

$$\begin{aligned} \text{maximize} \quad & \prod_{i=1}^{n} \phi_i(m_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} c_i m_i \le c \\ \text{and} \quad & 1 \le m_i \le u_i. \end{aligned} \tag{6.3.5}$$

Reliability Design Problem, II

- Given the n stages and the total cost of the optimal solution is $f_n(c)$, then the multiplicity, m_n, for stage n should be determined by

$$f_n(c) = \max_{m_n=1}^{u_n} \Big(\phi_n(m_n) \cdot f_{n-1}(c - c_n m_n) \Big) \tag{6.3.6}$$

  It is also assumed that $f_0(c) = 1$ for any c.
- Then this problem is similar to the 0/1 knapsack problem and the dynamic programming approach can be used to find the solution of the problem.
- Example, 3 devices, D_1, D_2 and D_3, with $r_1 = 0.9$, $r_2 = 0.8$ $r_3 = 0.5$, $c_1 = 30$, $c_2 = 15$, $c_3 = 20$, and the total cost $c \leq 105$. (It can derived that $u_1 = 2$, $u_2 = 3$ and $u_3 = 3$).

Summary

- String editing problem
  - $\mathcal{O}(mn)$
- 0/1 knapsack problem
  - $\mathcal{O}(2^n)$
- System reliability design
  - Large time complexity