Unit 5.2 The Greedy Method, II

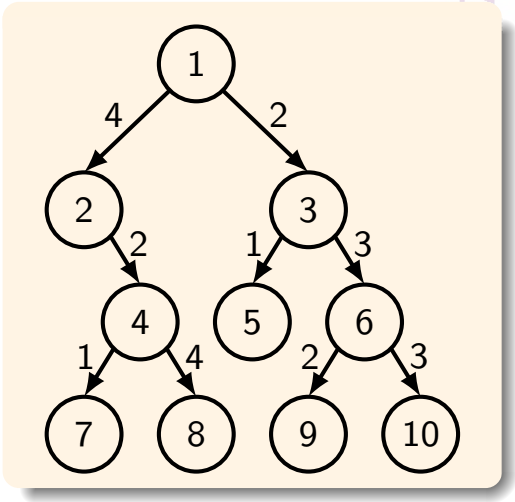
Algorithms

EE3980

Apr. 30, 2020

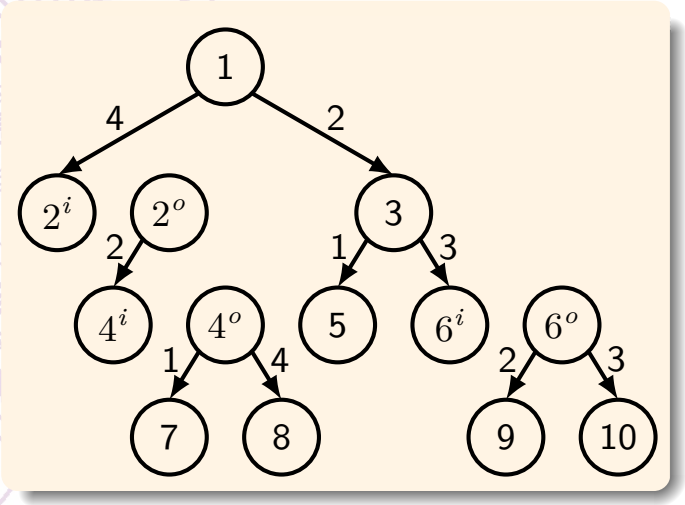
Tree Vertex Splitting Problem

Original tree T



$d(T) = 10.$

Tree with vertices splatted T/X



$d(T/X) = 5.$

Tree Vertex Splitting Problem – Definition

- $T = (V, E, w)$ is weighted directed tree.
  - V is the vertex set, E is the edge set, and w is weight function for the edges.
  - $w(i, j)$ is defined if the edge $\langle i, j \rangle \in E$; $w(i, j)$ is undefined if $\langle i, j \rangle \notin E$.
  - A source vertex is a vertex with in-degree 0.
  - A sink vertex is a vertex with out-degree 0.
  - For any path P in the tree, its delay, $d(P)$, is defined to be the sum of the weights on the path.
  - The delay of the tree, $d(T)$, is the maximum of all the path delays.
- T/X is the forest resulted from splitting every vertex u in $X \subseteq V$ into two nodes u^i and u^o such that all the edges $\langle i, u \rangle$ are replaced by $\langle i, u^i \rangle$ and all the edges $\langle u, j \rangle$ are replaced by $\langle u^o, j \rangle$.
- The Tree Vertex Splitting Problem (TVSP) is to find a set $X \subseteq V$ with minimum cardinality for which $d(T/X) \leq \delta$ for some specified tolerance δ.
  - Note that a TVSP has solution only if the maximum edge weight is less than or equal to δ.
  - Any $X \subseteq V$ with $d(T/X) \leq \delta$ is a feasible solution.
  - The optimal solution is the feasible X with the minimum number of vertices.

Tree Vertex Splitting Problem – Algorithm

Algorithm 5.2.1. TVS

```
   // Find the minimum set X for vertex splitting.
   // Input: tree T, maximum edge weight δ
   // Output: solution X.
 1 Algorithm TVS(T, δ, X)
 2 {
 3       if (T ≠ ∅) then {
 4             d[T] := 0;
 5             for each child v of T do {
 6                   TVS(v, δ, X);
 7                   d[T] := max(d[T], d[v] + w(T, v));
 8             }
 9             if ((T is not the root ) and (d(T) + w(parent(T), T) > δ)) then {
10                   X := X ∪ {T};
11                   d[T] := 0;
12             }
13       }
14 }
```

- Note that d is a global array that stores the $delay$ for each vertex.

Tree Vertex Splitting Problem – Algorithm II

Algorithm 5.2.2. TVS1

```
   // Tree vertex splitting with tree stored in an array tree[1 : n].
   // Input: root i, maximum edge weight δ
   // Output: solution X.
 1 Algorithm TVS1(i, δ, X)
 2 {
 3       if (tree[i] ≠ 0) then {
 4            if (2×i > N) then d[i] := 0 ; // i is a leaf.
 5            else {
 6                 TVS1(2×i, δ, X) ;
 7                 d[i] := max(d[i], d[2×i] + w[2×i]) ;
 8                 if (2×i + 1 ≤ N) then {
 9                      TVS1(2×i + 1, δ, X) ;
10                      d[i] := max(d[i], d[2×i + 1] + w[2×i + 1]) ;
11                 }
12            }
13            if ((i ≠ 1) and (d[i] + w[i] > δ)) then {
14                 X := X ∪ {i} ;
15                 d[i] := 0 ;
16            }
17       }
18 }
```


Tree Vertex Splitting Problem – Complexity and Optimality

- In this version the directed binary tree is stored in an array $tree$
- The weight is stored in array w and $w[i]$ is the weight of the parent of vertex i to vertex i.
- Array d is still the $delay$ of each vertex.

- The time complexity of Algorithm TVS is $\Theta(n)$.
  - Every vertex of T is traversed once.

Theorem 5.2.3.

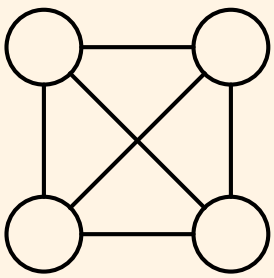
Algorithm TVS finds a minimum cardinality set X such that $d(T/X) \leq \delta$ on any tree T, provided that no edge of T has weight greater than δ.

- Proof please see textbook [Horowitz], pp. 225 - 226.

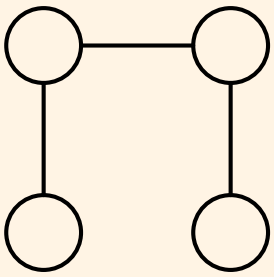
Minimum-Cost Spanning Trees

Definition 5.2.4.

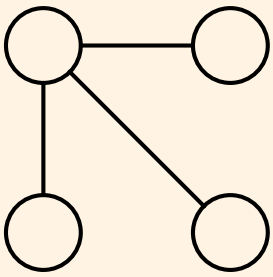
Let $G = (V, E)$ be an undirected connected graph. A sub-graph $T = (V, E')$ with $E' \subseteq E$ is a spanning tree of G if and only if T is a tree.



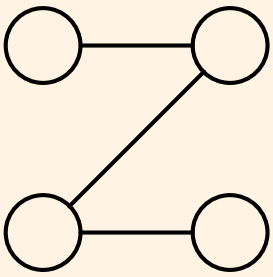
Undirected graph G.



Spanning tree T_1.



Spanning tree T_2.

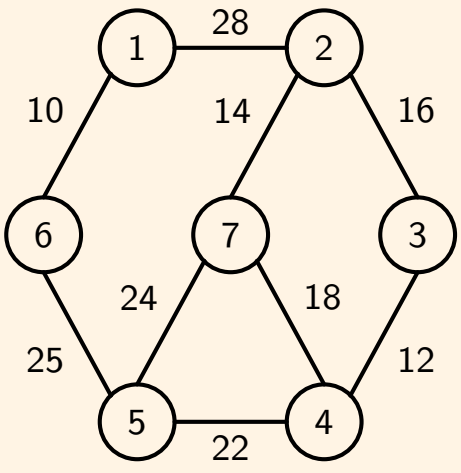


Spanning tree T_3.

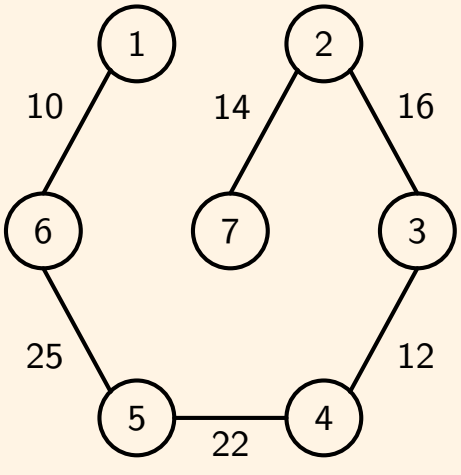
- Notes
  - Spanning tree is not unique.
  - Spanning trees have $n - 1$ edges ($n = |V|$.)

Minimum-Cost Spanning Tree, Example

- In addition, there is a cost function associated with each edge, $w : E \to \mathbb{R}$.
- The cost of a tree is the sum of the costs of the tree edges.
- A feasible solution of the minimum-cost spanning tree of a undirected graph G is any spanning tree T of G.
- The optimal solution is a spanning tree with the minimum cost.



An undirected graph, G.



Minimum-cost spanning tree, T.

Minimum-Cost Spanning Tree, Generic Algorithm

- Using the greedy methodology, let T be a subset of a spanning tree, at each step an edge (u, v) is added to T to maintain the feasibility of the solution.
- An edge, (u, v), is safe to a set of edges T if $T \cup \{(u, v)\}$ is still a subset of a spanning tree.
- The generic algorithm for the minimum-cost spanning tree then is:

Algorithm 5.2.5. Generic minimum-cost spanning tree

```
   // Given a graph G(V, E) with cost function w find minimum cost spanning tree.
   // Input: V, E, n, w
   // Output: minimum cost tree T.
 1 Algorithm MCST(V, E, n, w, T)
 2 {
 3       T := ∅;
 4       while (|T| < n − 1) do {
 5             select an edge (u, v) ∈ E {
 6                   if (u, v) is safe to T then T := T ∪ (u, v);
 7                   E := E − {(u, v)};
 8             }
 9       }
10 }
```

- The key is in line 5, how to select an edge.

Minimum-Cost Spanning Tree, Prim's Algorithm

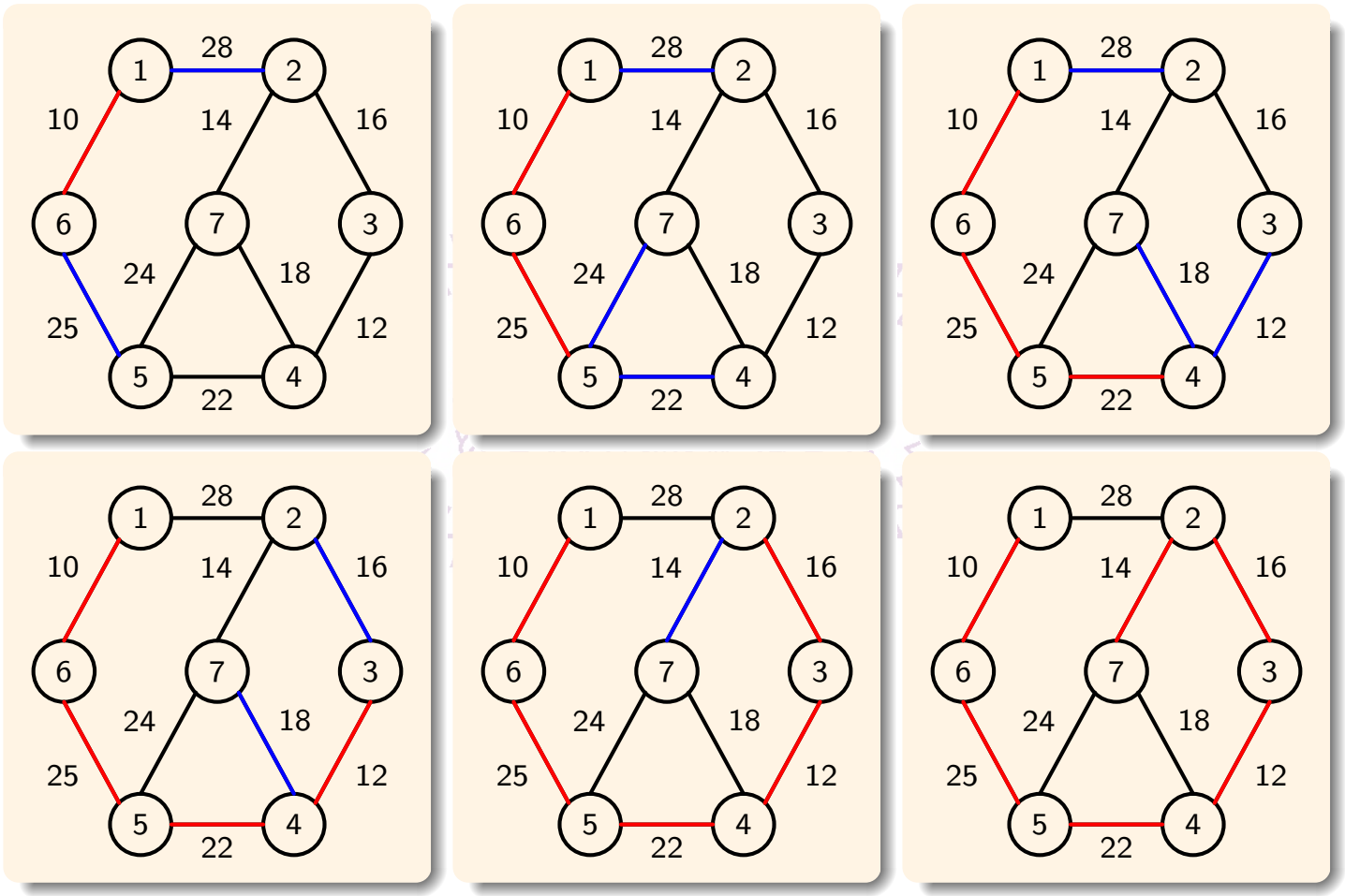
Algorithm 5.2.6. Prim

```
   // Given a graph G(V, E) with cost function w find minimum cost spanning tree.
   // Input: V, E, n, w
   // Output: minimum cost tree T and mincost.
 1 Algorithm Prim(V, E, n, w, T)
 2 {
 3       Find edge (k, ℓ) ∈ E with the minimum cost ;
 4       mincost := w[k, ℓ]; // mincost set to minimum edge cost.
 5       T[1, 1] := k; // Add (k, ℓ) to spanning tree.
 6       T[1, 2] := ℓ;
 7       for i := 1 to n do // Init near array for every vertices.
 8             if (w[i, ℓ] < w[i, k]) then near[i] := ℓ;
 9             else near[i] := k;
10       near[k] := near[ℓ] := 0; // Vertices already in the spanning tree.
11       for i := 2 to (n − 1) do {
12             Find j such that near[j] ≠ 0 and w[j, near[j]] is minimum ;
13             T[i, 1] := j; // Add minimum cost near edge to tree.
14             T[i, 2] := near[j];
15             mincost := mincost + w[j, near[j]]; // Update mincost.
16             near[j] := 0; // Reset near array for selected vertex.
17             for k := 1 to n do // update near array for the other unselected vertices.
18                   if ((near[k] ≠ 0) and (w[k, near[k]] > w[k, j])) then near[k] := j;
19       }
20       return mincost;
21 }
```

Minimum-Cost Spanning Tree, Prim's Algorithm II

- In Algorithm `Prim`
  1. The edge with the minimum cost is first selected as the initial tree
  2. The array `near` keeps the node already selected in the tree with the smallest single-edge cost for each node
  3. Among the all the `near` edges, the minimum is selected and the node added to the tree
  4. Array `near` is then updated and go back to step 3 until all nodes have been selected
- The time complexity is dominated by
  - Finding the minimum-cost edge on line 3, $\mathcal{O}(|E|) \approx \mathcal{O}(n^2)$
  - Loop on lines 7-9, $\mathcal{O}(n)$
  - Loop on lines 11-19
    - Inner loops line 12 and lines 17-18
    - Complexity $\mathcal{O}(n^2)$
  - Overall complexity is $\mathcal{O}(n^2)$
- The time complexity can be improved to $\mathcal{O}((n + |E|) \lg n)$
  - If the non-selected vertices are stored in a red-black tree

Minimum-Cost Spanning Tree, Prim's Algorithm Example



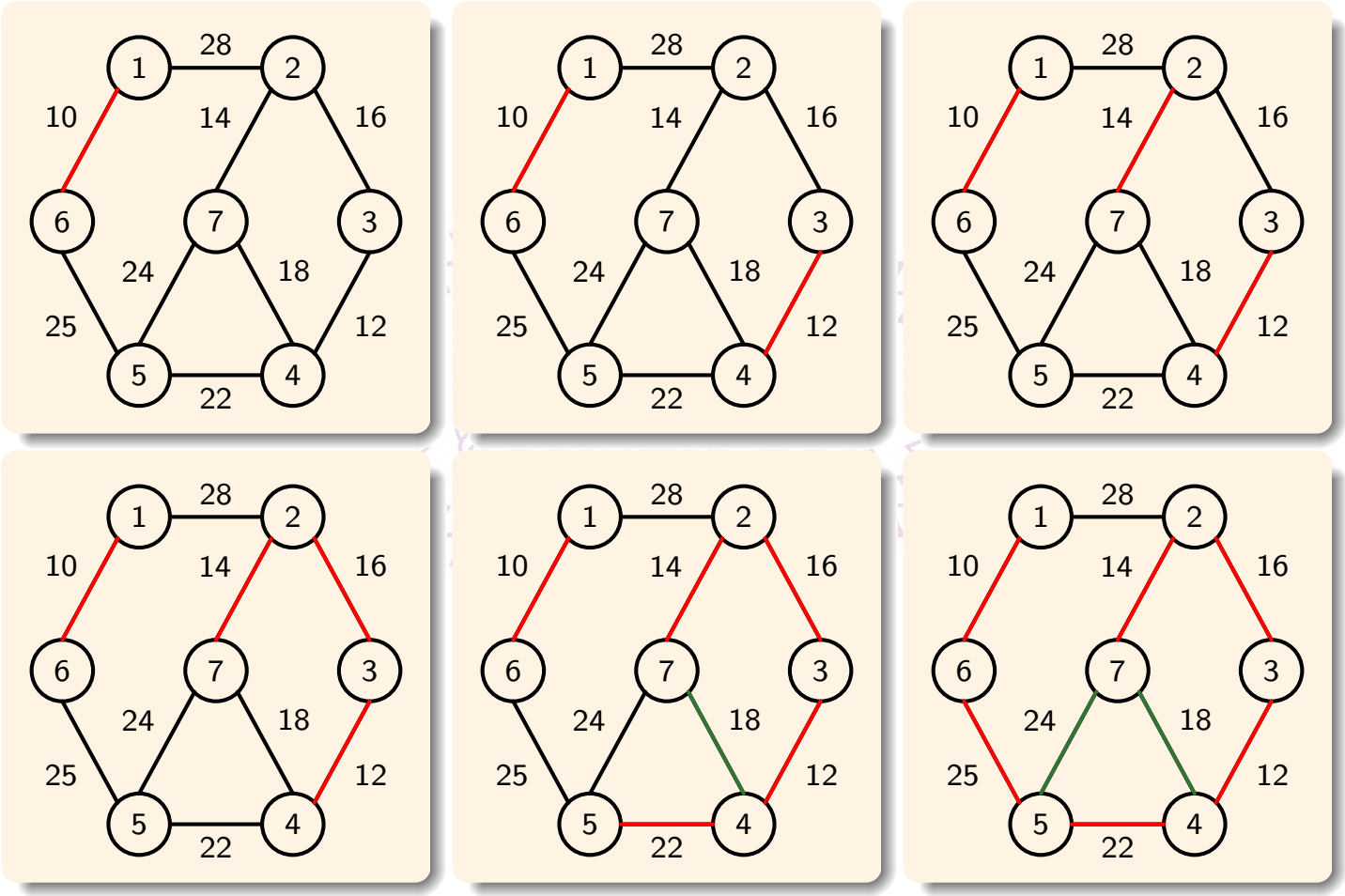
Kruskal's Algorithm – High Level

- A different approach to finding the minimum-cost spanning tree
- High level description of the algorithm

Algorithm 5.2.7. Kruskal's Algorithm

```
   // Given a graph G(V, E) with cost function w find minimum cost spanning tree.
   // Input: V, E, n, w
   // Output: minimum cost tree T.
 1 Algorithm KruskalH(V, E, n, w, T)
 2 {
 3       T := ∅;
 4       while ((T has less than (n − 1) edges ) and (E ≠ ∅)) do {
 5           Find the edge (u, v) ∈ E with the minimum cost ;
 6           Delete(u, v) from E;
 7           if (u, v) does not create a cycle in T then T := T ∪ (u, v);
 8           else discard (u, v);
 9       }
10 }
```


Kruskal's Algorithm – Example



Kruskal's Algorithm

Algorithm 5.2.8. Kruskal's Algorithm

```
   // Given a graph G(V, E) with cost function w find minimum cost spanning tree.
   // Input: V, E, n, w
   // Output: minimum cost tree T and mincost.
 1 Algorithm Kruskal(V, E, n, w, T)
 2 {
 3       Construct a min heap from the edge costs using Heapity;
 4       for i := 1 to n do parent[i] := −1; // Enable cycle checking
 5       i := 0;
 6       mincost := 0;
 7       while ((i < n − 1) and ( heap not empty )) do {
 8             delete a minimum cost edge (u, v) from the heap ;
 9             Adjust the heap ;
10             j := Find(u); // using parent array
11             k := Find(v);
12             if (j ≠ k) then {
13                   i := i + 1;
14                   T[i, 1] := u;
15                   T[i, 2] := v;
16                   mincost := mincost + w[u, v];
17                   Union(j, k); // modify parent array
18             }
19       }
20       if (i ≠ n − 1) then write("No spanning tree");
21       else return mincost;
22 }
```


Kruskal's Algorithm – Complexity and Optimality

- The time complexity of `Kruskal` algorithm is dominated by the `while` loop, lines 7-19, – $\mathcal{O}(|E|)$
  - Line 8 finding minimum cost edge, $\mathcal{O}(1)$
  - Line 9 `Adjust` the heap, $\mathcal{O}(\lg|E|)$
  - Overall complexity $\mathcal{O}(|E|\lg|E|)$.

Theorem 5.2.9.

Kruskal's algorithm (Algorithm 5.2.8) generates a minimum-cost spanning tree for every undirected connected graph G.

- Proof please see textbook [Horowitz], p. 244.

Minimum-Cost Spanning Tree, Properties

- A different approach to prove Kruskal's algorithm.
- We define the following terms.
  - A cut $(S, V-S)$ of an undirected graph $G=(V,E)$ is a partition of V, i.e., $S \in V$.
  - An edge $(u,v) \in E$ is said to cross the cut $(S, V-S)$ if one of its end points is in S and the other in $V-S$.
  - A cut is said to respect a set T of edges if no edges in T crosses the cut.
  - An edge is said to be a light edge crossing a cut if its cost is the minimum of any edge crossing the cut.

Theorem 5.2.10.

Let $G=(V,E)$ be a connected, undirected graph with a cost function w defined on E. Let T be a subset of E that is subset of a spanning tree of G, let $(S, V-S)$ be any cut of G that respects T, and let (u,v) be a light edge crossing $(S, V-S)$. Then, edge (u,v) is safe for T.

- Proof please see textbook [Cormen], pp. 627-628.

Minimum-Cost Spanning Tree, Properties, II

Corollary 5.2.11.

Let $G=(V,E)$ be a connect, undirected graph with cost function w defined on E. Let T be a subset of E that is included in a minimum spanning tree of G, and let $C=(V_C, E_C)$ be a connected component (tree) in the forest $G_T=(V,T)$. If (u,v) is a light edge connecting C to some other component in G_T, then (u,v) is safe for T.

- Proof please see textbook [Cormen], pp. 629.

- Algorithm `Prim` can be shown to be a special case of Theorem (5.2.10), and it also returns an optimal solution.

Matroid Theory and Greedy Method

- Matroid theory explains why greedy method solves the maximum/minimum subset problems.
  - The theory was developed by Hassler Whitney, American Mathematician, in 1935 by generalizing the structure of linear independence in vector space
  - Jack Edmonds, American Computer Scientist, applied to greedy algorithms
  - Reference: Bernhard Korte and Jens Vygen, *Combinatorial Optimization – theory and algorithms*, 4th edition, Springer, 2008.
- The concept of independence system is generalized from vector space.

Independent Systems

Definition 5.2.12. Independence System

Let S be a finite set and $\mathcal{I} = \{X : X \subseteq S\}$, then the set system $(S, \mathcal{I})$ is an independence system if

(M1) $\emptyset \in \mathcal{I}$;

(M2) If $Y \in \mathcal{I}$ and $X \subseteq Y$ then $X \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called independent, the elements of $2^S \setminus \mathcal{I}$ dependent. Minimal dependent sets are called circuits, maximal independent sets are called bases. For $X \subseteq S$, the maximal independent subsets of X are called bases of X.

- The set $\mathcal{I}$ can be defined by its property, instead of listing all elements.

Definition 5.2.13.

Let $(S, \mathcal{I})$ be an independence system. For $X \subseteq S$ we define the rank of X by

$$r(X) = \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$

Examples of Independence Systems

- Example M1:
  Let S_V be the set of columns of a matrix $\mathbf{A}$ and $\mathcal{I}_V = \{X \subseteq S_V :$ the column vectors in X are linearly indepdent$\}$, then the set system $(S_V, \mathcal{I}_V)$ is an independence system.
  (1) It is apparent $\emptyset \in \mathcal{I}$.
  (2) If $Y \in \mathcal{I}$ then any subset $X \subseteq Y$ also contains independent column vectors.
- Example M2:
  Given a undirected graph $G(V, E)$, let $S_G = E$, the set of all edges, and $\mathcal{I}_G = \{Y : Y \subseteq E$ and Y is a forest$\}$, then the set system $(S_G, \mathcal{I}_G)$ is an independence system.
  (1) It is apparent $\emptyset \in \mathcal{I}$.
  (2) If $Y \in \mathcal{I}$, then Y is a forest, and any subset $X \subseteq Y$ is also a forest.
- Example M3:
  Given any finite set S_U, let k be an integer, $k \leq |S_U|$, and $\mathcal{I} = \{Y : Y \subseteq S_U$ and $|Y| \leq k\}$, then the set system $(S_U, \mathcal{I}_U)$ is an independence system.
  (1) It is apparent $\emptyset \in \mathcal{I}$.
  (2) If $Y \in \mathcal{I}$, then $|Y| \leq k$. Any $X \subseteq Y$ has $|X| \leq |Y| \leq k$.

Matroid

Definition 5.2.14. Matroid

An independence system $(S, \mathcal{I})$ is a matroid if

(M3) If X, $Y \in \mathcal{I}$ and $|X| > |Y|$, then there is an $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

- Vector Matroid: The independence system $(S_V, \mathcal{I}_V)$ in Example M1 is a matroid.
  - (M3) property is observed $(S_V, \mathcal{I}_V)$.
- Graphic Matroid: The independence system $(S_G, \mathcal{I}_G)$ in Example M2 is a matroid.
  - (M3) property is observed. If $|X| > |Y|$ and for every $x \in X$ either $x \in Y$ or $Y \cup \{x\}$ forms a cycle. In either case, both vertices of the edge x belong to the same connected component in Y. The number of such edge cannot exceed $|Y|$ while maintaining a forest perperty, thus $|X| \leq |Y|$. This contradicts to the assumption $|X| > |Y|$.
- Uniform Matroid: The independence system $(S_U, \mathcal{I}_U)$ in Example M3 is a matroid.
  - (M3) property is observed. If $|X| > |Y|$ then there is an $x \in X \setminus Y$ and then $|Y \cup \{x\}| = |Y| + 1 \leq |X| \leq k$.

Matroid, II

Theorem 5.2.15.

Let $(S, \mathcal{I})$ be an independence system. Then the following statements are equivalent:

(M3) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there is an $x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{I}$.

(M3′) If $X, Y \in \mathcal{I}$ and $|X| = |Y + 1|$, then there is an $x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{I}$.

(M3″) For each $X \subseteq S$, all bases of X have the same cardinality.

Proof. It is easy to see (M3) $\Leftrightarrow$ (M3′) and (M3) $\Rightarrow$ (M3″). To prove (M3″) $\Rightarrow$ (M3), let $X, Y \in \mathcal{I}$ and $|X| > |Y|$. By (M3″), Y cannot be a basis of $X \cup Y$. So there must be an $x \in (X \cup Y) \setminus Y = X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$. □

- Thus, an independence system can also be shown to be a matroid using either property (M3′) or (M3″).
- For the graphic matroid, it is known that a spanning tree of a connect graph $G(V, E)$ has $|V| - 1$ edges. Thus, the rank $r(S_G) = |S_G| - 1$ if G is connected.

Weighted Matroid and Optimization Problems

Definition 5.2.16. Weighted Matroid

A matroid $(S, \mathcal{I})$ is weighted if it is associated with a weight function $w : S \to \mathbb{R}^+$. The weight function w extends to subsets of S by summation:

$$w(X) = \sum_{x \in X} w(x) \qquad \text{for any } X \subseteq S. \tag{5.1}$$

- Maximization problem of independence systems
  Given an independence system $(S, \mathcal{I})$ and the weight function $w : S \to \mathbb{R}^+$, find an $X \in \mathcal{I}$ such that $w(X) = \sum_{x \in X} w(x)$ is maximum.
- A corresponding minimization can be formulated
  - Solution algorithms can also be derived.

Greedy Algorithms

- Two types of algorithms possible
  The first one is

Algorithm 5.2.17. Best-In Greedy Algorithm

```
   // Given (S, I) and w : S → R find X ∈ I such that w(X) is maximum.
   // Input: (S, I) and w.
   // Output: X
 1 Algorithm Best-In-Greedy(S, I, w)
 2 {
 3       Sort S into nonincreasing order by w ;
 4       X := ∅ ; // Initialize to empty set.
 5       for each x ∈ S in order do { // Try all elements.
 6           if (X ∪ {x} ∈ I) then { // Maintain independence then add.
 7               X := X ∪ {x} ;
 8           }
 9       }
10       return X;
11 }
```


Greedy Algorithms, II

Theorem 5.2.18.

The best-in greedy algorithm (5.2.17) solves the independence system $(S, \mathcal{I})$ maximization problem correctly if $(S, \mathcal{I})$ is a matroid.

Proof. By induction. The first x in the ordered S with $\{x\} \in \mathcal{I}$ is apparently the solution for any $X \subset S$ with $r(X) = 1$. Suppose the best solution has been found for any $X \subset S$ and $r(X) = k$ and $k < r(S)$, then there is a $Y \in \mathcal{I}$ such that $r(Y) = r(X) + 1$, further more there is $x \in S$ with $Y = X \cup \{x\}$ and x is in the remaining S. The x with largest $w(x)$ is apparently the choice, which would be picked first by the algorithm. Hence, this Y is the solution for $r(X) + 1$. By induction, the theorem is proven. □

- Note that the maximization problem is defined for independence systems, and the algorithm works if the system is a matroid.
- The graph minimum spanning tree problem can be formulated as a minimization problem for a matroid system.
- The Kruskal's Algorithm is a best-in greedy algorithm.

Greedy Algorithms, III

- Alternative solution

Algorithm 5.2.19. Worst Out Greedy Algorithm

```
// Given (S, I) and w : S → R find a basis X of S such that w(X) is maximum.
// Input: (S, I) and w.
// Output: X
1  Algorithm Worst-Out-Greedy(S, I, w)
2  {
3        Sort S into nondecreasing order by w ;
4        X := S; // Initialize to entire set.
5        for each x ∈ S in order do { // Try all elements.
6             if (r(X \ {x}) = r(X)) then { // rank unchanged.
7                  X := X \ {x} ;
8             }
9        }
10       return X;
11 }
```

- This algorithm can generate optimal solution if the given system is a matroid, and the proof is similar to the best-in greedy algorithm.

Job Sequencing with Deadlines

- Given a set of n jobs to be processed on one machine.
  - Each job takes 1 time unit to process.
  - Associated with job i, $1 \leq i \leq n$, there is a deadline d_i and profit p_i.
  - If job i is completed by d_i then p_i is earned.
- A feasible solution is a subset J of jobs that each job in J can be completed by its deadline.
  - The value of the subset J is $\sum_{i \in J} p_i$.
- An optimal solution is a feasible solution with the maximum value.

- Example, $n = 4$, $\{p_1, p_2, p_3, p_4\} = \{100, 10, 15, 27\}$, $\{d_1, d_2, d_3, d_4\} = \{2, 1, 2, 1\}$.
- Feasible solutions are

	Feasible solution	Processing sequence	Value
1	$\{1, 2\}$	2,1	110
2	$\{1, 3\}$	1,3 or 3,1	115
3	$\{1, 4\}$	4,1	127
4	$\{2, 3\}$	2,3	25
5	$\{3, 4\}$	4,3	42
6	$\{1\}$	1	100
7	$\{2\}$	2	10
8	$\{3\}$	3	15
9	$\{4\}$	4	27

- Solution 3 is optimal.

Job Sequencing with Deadlines – Algorithm

- Applying greedy method to job sequencing problem

Alrogithm 5.2.20. Job Sequencing – Greedy Method

```
   // Solve job scheduling problem with jobs sorted in non-increasing profit.
   // Input: int n, deadline d[1 : n], profit p[1 : n]
   // Output: Optimal sequence J[1 : k].
1  Algorithm JSgreedy(n, d, p, J)
2  {
3        J := {1}; // init to highest profit job
4        for i := 2 to n do { // check every job
5              if (J ∪ {i} is feasible ) then
6                    J := J ∪ {i};
7        }
8  }
```

- Example
  Ordered sequency: $\{1, 4, 3, 2\}$, $p = \{100, 27, 15, 10\}$,
  step 1: $J = \{1\}$, $p = 100$,
  step 2: $J = \{1, 4\}$, $p = 127$,
  step 3: reject job 3, not feasible, step 4: reject job 2, not feasible.

Job Sequencing with Deadlines – Algorithm Optimality

- Optimality of the algorithm follows from the following theorem.

Theorem 5.2.21.

Given a job sequencing problem Algorithm JSgreedy (Algorithm 5.2.20) generates an optimal solution.

- Note that there can be groups of jobs that only some of them can be selected as $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, and $\{3, 4\}$ in the example. Follows the algorithm, $\{1, 4\}$ are selected. The only possibility that this solution is not optimal is that $P\{1, 4\}$ is smaller than $P\{2, 3\}$. But, job 4 has higher profit than either job 2 or 3. Thus, the selection by JSgreedy is the optimal solution.
- Unit execution time of each job is an important factor.

Proof can be found in textbook [Horowitz] pp. 229 - 230.

Job Sequencing with Deadlines – Algorithm

Alrogithm 5.2.22. Job Sequencing

```
   // Solve job scheduling problem with jobs sorted in non-increasing profit.
   // Input: int n, deadline d[1 : n], profit p[1 : n]
   // Output: Optimal sequence J[1 : k].
 1 Algorithm JS(n, d, p, J)
 2 {
 3       d[0] := J[0] := 0 ; // to facilitate while loop stopping
 4       J[1] := 1 ; k := 1 ; // init to highest profit job
 5       for i := 2 to n do { // check every job
 6             r := k ;
 7             while ((d[J[r]] > d[i]) and (d[J[r]] ≠ r)) do // find time slot job i fits
 8                   r := r − 1 ;
 9             if ((d[J[r]] ≤ d[i]) and (d[i] > r)) then { // insert i into J
10                   for q := k to (r + 1) step −1 do J[q + 1] := J[q] ; // move jobs to make room
11                   J[r + 1] := i ; // assign job
12                   k := k + 1 ;
13             }
14       }
15 }
```

- Line 3 creates a stopping condition for the `while` loop on line 7.
- The worst-case time complexity of JS algorithm is $\mathcal{O}(n^2)$.
  - Outer loop, lines 5–14
  - Inner loops, lines 7–8 and line 10.
- The space complexity of JS algorithm is $\mathcal{O}(n)$ for arrays J, p, and d.

Job Sequencing with Deadlines – Algorithm correctness

Theorem 5.2.23.

Given a job sequencing problem Algorithm JS (Algorithm 5.2.22) correctly generates the optimal solution.

- Proof can be found in textbook [Horowitz] pp. 230 - 232.
- Example: $n = 5$, Jobs=$\{$A, B, C, D, E$\}$, $p[\,] = \{20, 15, 10, 5, 1\}$, and $d[\,] = \{2, 2, 1, 3, 3)\}$. Then, the execution sequence of the algorithm is as following.

i	$d[i]$	action	$J[\,]$	$d[J[\,]]$	$p[J[\,]]$	k
1	2	init to A	$\{A\}$	$\{2\}$	$\{20\}$	1
2	2	accepting B	$\{A, B\}$	$\{2, 2\}$	$\{20, 15\}$	2
3	1	rejecting C	$\{A, B\}$	$\{2, 2\}$	$\{20, 15\}$	2
4	3	accepting D	$\{A, B, D\}$	$\{2, 2, 3\}$	$\{20, 15, 5\}$	3
5	3	rejecting E	$\{A, B, D\}$	$\{2, 2, 3\}$	$\{20, 15, 5\}$	3

Job Sequencing with Deadlines – Matroid Formulation

- The job sequencing with deadline can be shown to be a matroid.
  The set $\mathcal{S}$ contains all the jobs, and a set A of jobs are independent if there is a schedule such that all jobs in A are done before their deadlines.

Lemma 5.2.25.

For any set of jobs A, the following statements are equivalent.

1. The set A is independent.
2. Let $N_t(A)$ denote the number of jobs completed before time t, then for $t = 0, 1, 2, \ldots, n$, we have $N_t(A) \leq t$.
3. If the tasks in A are scheduled in order of monotonically increasing deadlines, the all jobs in A are completed before their deadlines.

Theorem 5.2.26.

If $\mathcal{S}$ is a set of unit-time jobs with deadlines, and $\mathcal{I}$ is the set of all independent sets of tasks, then the corresponding system $(\mathcal{S}, \mathcal{I})$ is a matroid.

- Since the job sequencing problem is a matroid, the greedy algorithm can be applied and it results in an optimal solution.

Container-Loading Problem and Matroid

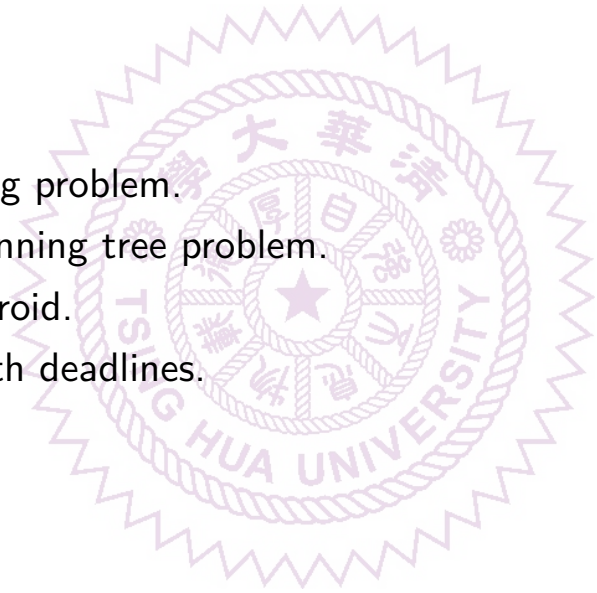
- Container loading problem
  - n containers with weight $w_i > 0$, $1 \leq i \leq n$
  - Capacity c
  - Find $x_i \in \{0, 1\}$, $1 \leq i \leq n$ such that

$$\text{Maximize:} \quad \sum_{i=1}^{n} x_i,$$

$$\text{Subject to:} \quad \sum_{i=1}^{n} x_i \cdot w_i \leq c.$$

- Example: $n = 8$, $(w_1, \ldots, w_8) = (100, 200, 50, 90, 150, 50, 20, 80)$, $c = 200$.
  Let $S = \{w_i : 1 \leq i \leq 8\}$, $\mathcal{I} = \{T \subseteq S : \sum_{t_i \in T} w(t_i) \leq c\}$.
  - It can be shown that $(S, \mathcal{I})$ is an independence system.
    Since if $T \in \mathcal{I}$, any subset of T has total weight less than c.
  - But $(S, \mathcal{I})$ is not matroid.
    - Example, $T_1 = \{100, 50, 50\} \in \mathcal{I}$, $T_2 = \{200\} \in \mathcal{I}$, $|T_1| > |T_2|$ but there is no $t \in T_1$ such that $T_2 \cup \{t\} \in \mathcal{I}$.
  - Greedy method still works for this problem.
  - Madroid is a necessary but not a sufficient condition for greedy method.

Summary



- Tree vertex splitting problem.
- Minimum-cost spanning tree problem.
- The theory of Matroid.
- Job sequencing with deadlines.