Unit 6.2 Dynamic Programming, II

Algorithms

EE3980

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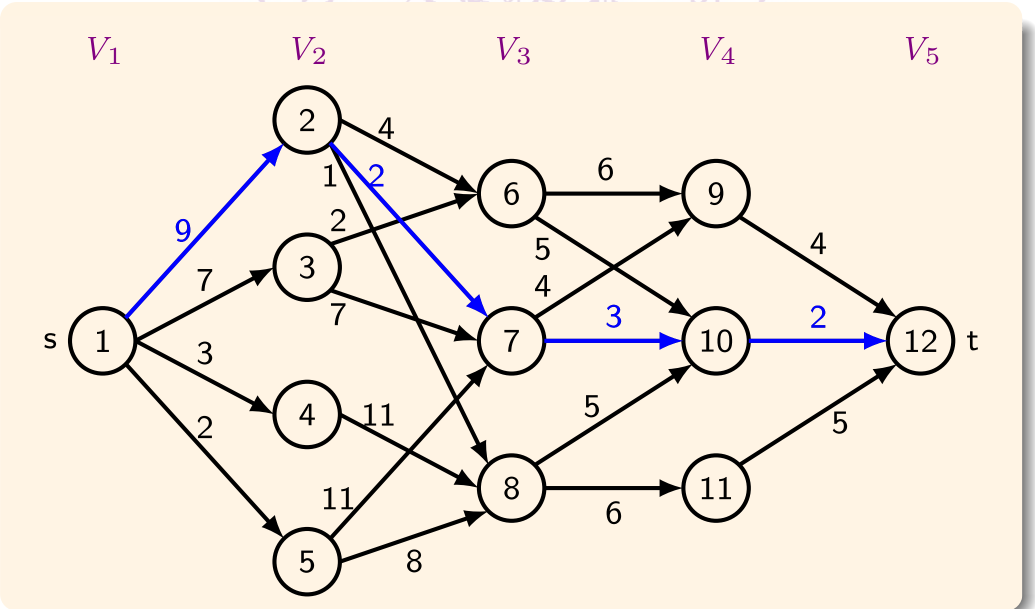
Multi-Stage Graphs

- A multistage graph $G = (V, E)$ is a directed graph.
  - Vertices are partitioned into $k > 2$ disjoint sets V_i, $1 \le i \le k$.
  - If $\langle u, v\rangle \in E$, then $u \in V_i$ and $v \in V_{i+1}$ for some i, $1 \le i < k$.
  - The sets V_1 and V_k both have only one vertex.
  - Vertex $s \in V_1$ is the source and $t \in V_k$ is the sink.
  - The cost of a path from s to t is the sum of the costs of the edges on the path.
  - The multistage graph problem is to find the minimum-cost path from s to t.

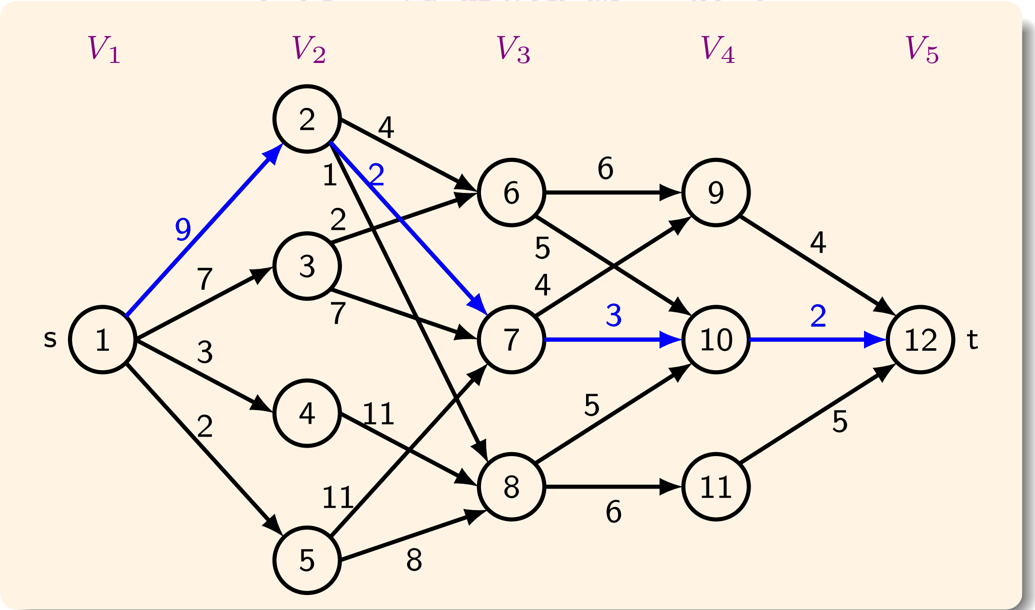


Multi-Stage Graphs — Example

- Since edges connect only consecutive stages, $\langle u, v\rangle \in E$, $u \in V_i$ and $v \in V_{i+1}$, minimum cost path from source s is

$$cost(1,1) = \min_{\langle 1,j\rangle \in E} \{c(1,j) + cost(2,j)\} \qquad (6.2.1)$$

where $cost(a,b)$ is the minimum cost of vertex b at stage a and $c(i,j)$ is the edge cost of $\langle i,j\rangle$.



Multi-Stage Graphs – Recursive Algorithm

- Note that Eq. (6.2.1) can be generalized to

$$cost(r,i) = \min_{\langle i,j\rangle \in E} \{c[i,j] + cost(r+1,j)\} \qquad (6.2.2)$$

- Therefore a recursive algorithm to solve the multistage graph problem is

Algorithm 6.2.1. Recursive Multistage Graph

```
// Find minimum cost path p of n-vertices multistage graph for vertex i.
// Input: n, cost matrix c, vertex i
// Output: mincost, path p.
Algorithm MSGraph_R(n, c, i, p)
{
    if (i = n) then { // sink vertex
        p[i] := 0;
        return 0;
    } // Otherwise, find the minimum cost path to the sink.
    mincost := ∞; // initialize.
    for all j such that ⟨i, j⟩ ∈ E do { // check all out-going edges
        if (c[i, j]+ MSGraph_R(n, c, j, p) < mincost) then { // smaller cost.
            mincost := c[i, j]+ MSGraph_R(n, c, j, p); p[i] := j;
        }
    }
    return mincost;
}
```

Multi-Stage Graphs – Recursive Algorithm Analysis

- The vertices of the graph is assumed to be ordered from 1 to n.
  - Vertex 1 is the source vertex and n is the sink vertex.
- Matrix $c[i, j]$ is the cost of the edge $\langle i, j\rangle$.
- After completion the array $p[1:n]$ is the minimum-cost path from source vertex to sink vertex.
- This function is invoked by MSGraph_R$(n, c, 1, p)$ at the top level and it returns the minimum path cost and the path array p.
- Though coding of this recursive version of the algorithm is straightforward, the execution efficiency can be improved.
  - For any vertex j, $j \neq 1$, with more than one edge $\langle i, j\rangle \in E$, MSGraph_R$(n, c, j, p)$ can be called more than once.
  - This inefficiency can be corrected by the following algorithms.

Multi-Stage Graphs — Top-Down Approach

Algorithm 6.2.2. Multistage Graph Top-Down Approach

```
// Find minimum cost path p of n-vertices multistage graph for vertex i.
// Input: n, cost matrix c, vertex i
// Output: mincost, path p, mincost table d.
Algorithm MSGraph_TD(n, c, i, d, p)
{
    if (i = n) then { // sink vertex
        p[i] := 0;
        d[i] := 0;
        return 0;
    }
    // Otherwise, find the minimum cost path to the sink.
    mincost := ∞; // initialize.
    for all j such that ⟨i, j⟩ ∈ E do { // check all out-going edges
        if (d[j] < 0) then
            d[j] := MSGraph_TD(n, c, j, d, p); // eval min cost for j.
        if (c[i, j]+ d[j] < mincost) then { // smaller cost.
            mincost := c[i, j]+ d[j];
            p[i] := j;
        }
    }
    d[i] := mincost; // record min cost for vertex i.
    return mincost;
}
```

Multi-Stage Graphs — Top-down Approach, II

- Before the top-down multistage algorithm is called, the array $d[i]$, which stores the minimum cost from vertex i to sink, should be initialized to $-\infty$.
- The algorithm should be called from `main` function by
  `MSGraph_TD`$(n, c, 1, d, p)$;
  where n is the number of vertices of the graph,
  $c[1:n, 1:n]$ is a matrix such that $c[i,j]$ is the edge cost connecting vertices i and j,
  1 is the source vertex,
  $d[1:n]$ is an array such that $d[i]$ records the min cost from vertex i to sink,
  $p[1:n]$ is an array such that $p[i]$ records the next vertex from vertex i along the min cost path to the sink.
- In this top-down algorithm each vertex is processed once on lines 11-12.
- Each edge should be visited once, line 10
- The overall time complexity is $\mathcal{O}(|V| + |E|)$
- This is more efficient than the recursive version.
- The array (or table) d reduces the number of recursive calls and improves the efficiency significantly.
  - This is one of the key in dynamic programming approach.

Multi-Stage Graphs – Bottom-Up Approach

Algorithm 6.2.3. Multistage Graph Bottom-Up Approach

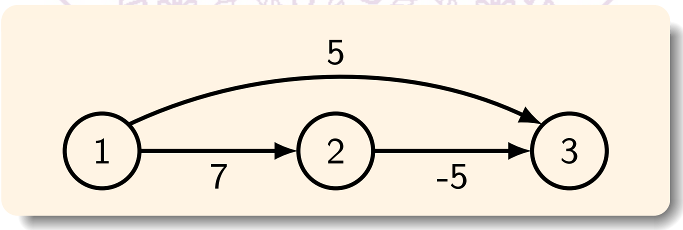
```
   // Find minimum cost path p of n-vertices multistage graph.
   // Input: n, cost matrix c
   // Output: path p, mincost table d.
Algorithm MSGraph_BU(n, c, d, p)
{
     d[n] := 0; // sink vertex.
     for r := n − 1 to 1 step −1 do { // for n − 1 stages.
          for each vertex i ∈ V_r do { // All vertices in stage r.
               d[i] := ∞;
               for each ⟨ i, j ⟩ ∈ E do { // All edges from vertex i.
                    if (c[i, j] + d[j] < d[i]) { // Smaller cost.
                         d[i] := c[i, j] + d[j]; // Record min cost.
                         p[r] := j; // Record path.
                    }
               }
          }
     }
}
```

Multi-Stage Graphs – Bottom-Up Approach, Analysis

- This bottom-up multistage algorithm is non-recursive.
- It should be called by `MSGraph_BU`(n, c, d, p),
  where n is the number of vertices of the graph,
  $c[1:n, 1:n]$ is a matrix such that $c[i,j]$ is the edge cost connecting vertices i and j,
  $d[1:n]$ is an array such that $d[i]$ records the min cost from vertex i to sink,
  $p[1:n]$ is an array such that $p[i]$ records the next vertex from vertex i along the min cost path to the sink.
- This algorithm has the same complexities, time and space, as the top-down approach.
- Similar table, array d, is used to improve the efficiency of the algorithm.

Single-Source Shortest Paths: General Weights

- The single-source shortest paths problem is revisited to allow negative weights for some edges.
  - However, no cycle of negative length is allowed.
  - Cycle of negative length can lead to $-\infty$ path length.
- Example



- The greedy algorithm `ShortestPaths` can fail in this case.
  - If vertex 1 is the source
  - It generates path $\langle 1, 3\rangle$ with weight 5 as the shortest path
  - But path $\langle 1, 2, 3\rangle$ has the weight of 2.
  - This example shows that we need consider paths through other intermediate vertices.

Single-Source Shortest Paths: General Weights

- With the possibility of negative weights, paths with more segments may have smaller weights, and thus we need to try all paths between a pairs of vertices.
- A shortest path should not include a positive cycle either, since the cycle can be removed to obtain a shorter path.
- A shortest path should not include a cycle with 0 weight, again this cycle can be removed to obtain a shortest path.
  - Thus, a shortest path should not have any cycles.
- Any shortest paths has at most $n-1$ edges, $n = |V|$.
- Let $d^{(k)}[u]$ be the path weight from source vertex v_0 to vertex u through k edges.
  - Note that $d^{(1)}[u] = W[v_0, u]$ if $\langle v_0, u\rangle \in E$ and $W[v_0, u]$ is the weight of the edge.
- Then we have

$$d^{(k)}[u] = \min \left\{ d^{(k-1)}[u], \min_{i \in V} \left\{ d^{(k-1)}[i] + W[i,u] \right\} \right\}. \qquad (6.2.3)$$

  And $k \le n-1$.
- This leads to the dynamic programming algorithm shown next.

Bellman and Ford Algorithm

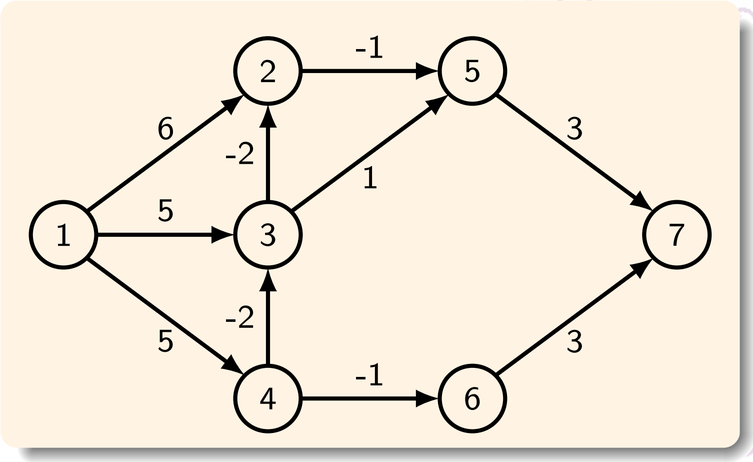
Algorithm 6.2.4. BellmanFord

```
   // Generate shortest paths, d[1 : n], from v with edge weight W[1 : n, 1 : n].
   // Input: n: |V|, source v, weight W
   // Output: distance d[1 : n].
 1 Algorithm BellmanFord(n, v, W, d)
 2 {
 3     for i := 1 to n do
 4         d[i] := W[v, i];
 5     for k := 2 to n − 1 do
 6         for each u such that u ≠ v and u has incoming edges do
 7             for each ⟨i, u⟩ ∈ E do
 8                 if (d[u] > d[i] + W[i, u]) then
 9                     d[u] := d[i] + W[i, u];
10 }
```

- If W is kept in a matrix form
  - Lines 6-9 takes $\mathcal{O}(n^2)$ time
  - Overall complexity is $\mathcal{O}(n^3)$
- If W is kept in a list form
  - Lines 6-9 takes $\mathcal{O}(e)$ time ($e = |E|$)
  - Overall complexity is $\mathcal{O}(ne)$
  - Efficiency can still be improved further.

Bellman and Ford Algorithm — Example

- Given the graph on the left, and $v = 1$ then we have shortest paths to all other vertices as shown on the right.

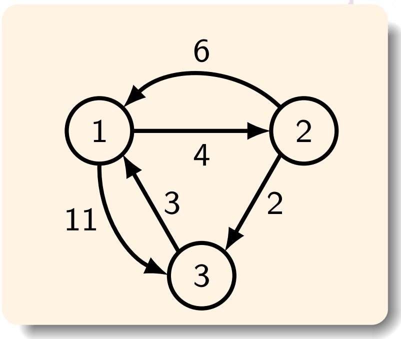


	$d^{(k)}[\,]$						
k	1	2	3	4	5	6	7
1	0	6	5	5	∞	∞	∞
2	0	3	3	5	5	4	∞
3	0	1	3	5	2	4	7
4	0	1	3	5	0	4	5
5	0	1	3	5	0	4	3
6	0	1	3	5	0	4	3

- Note that when $d^{(5)}[\cdot] = d^{(6)}[\cdot]$, hence the loop can be terminated.
- The shortest paths can also be printed, if the path information is kept.
- Correctness of the Bellman and Ford algorithm can be found in textbook [Cormen], pp. 652-654.

All-Pairs Shortest Paths

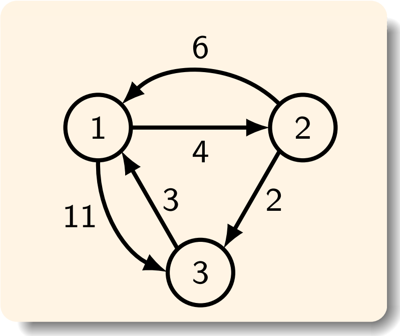
- Given a directed graph $G = (V, E)$ with n vertices and a weight function $w : E \to \mathbb{R}$, define the weight matrix, $W[1:n, 1:n]$, as
  - $W[i, i] = 0$, $1 \le i \le n$,
  - $W[i, j] = w(i, j)$, if $\langle i, j\rangle \in E$,
  - $W[i, j] = \infty$, if $\langle i, j\rangle \notin E$.
- The all-pairs shortest path problem is to determine a matrix D such that $D[i, j]$ is the weight of the shortest path from vertex i to vertex j.
- One can apply the single source shortest path algorithm n times to find all-pairs shortest paths.
  - Time complexity is $\mathcal{O}(n^4)$ since the single source shortest path algorithm has the complexity of $\mathcal{O}(n^3)$.
- $w[i, j]$ can be negative but no negative cycle exists.



$$\begin{bmatrix} 0 & 4 & 11 \\ 6 & 0 & 2 \\ 3 & \infty & 0 \end{bmatrix}$$

Weight matrix, W.

All-Pairs Shortest Paths – Formulation



- As shown on the left, the edge weight from vertex 2 to 1 is 6.
- However, there is a path $\langle 2, 3, 1 \rangle$ with small path weight, 5.
- Thus, to find the minimum path we need consider paths through all intermediate vertices.

- Let $D^{(0)} = W$, where W is the weight matrix defined above.
- Let $D^{(k)}[i, j]$ be the minimum cost path with intermediate vertices no more than vertex k, then

$$D^{(k)}[i, j] = \min\{D^{(k-1)}[i, j], D^{(k-1)}[i, k] + D^{(k-1)}[k, j]\}. \tag{6.2.4}$$

- Since there are only $n = |V|$ vertices in the graph, $D^{(n)}[i, j]$ is the minimum weight between any pair of vertices, i and j, $1 \le i, j \le n$.
- This formulation lends itself to a dynamic programming approach to solve the all-pair shortest path problem.

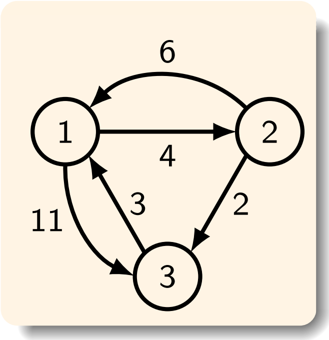
All-Pairs Shortest Paths – Algorithm

Algorithm 6.2.5. All-Pairs Shortest Paths

```
   // Find all-pairs shortest paths and store them in matrix D[1 : n, 1 : n].
   // Input: n: |V|, weight W
   // Output: distance D.
1  Algorithm AllPairs(n, W, D)
2  {
3      for i := 1 to n do // Create D^(0).
4          for j := 1 to n do
5              D[i, j] := W[i, j];
6      for k := 1 to n do // Loop through all D^(k).
7          for i := 1 to n do
8              for j := 1 to n do
9                  if (D[i, j] > D[i, k] + D[k, j]) then
10                     D[i, j] := D[i, k] + D[k, j];
11 }
```

- Using D to store all $D^{(k)}$ for better space efficiency.
- Space complexity remains as $\Theta(n^2)$.
- The time complexity is $\mathcal{O}(n^3)$.
  - Triple loop on lines 6-10.

All-Pairs Shortest Paths – Example



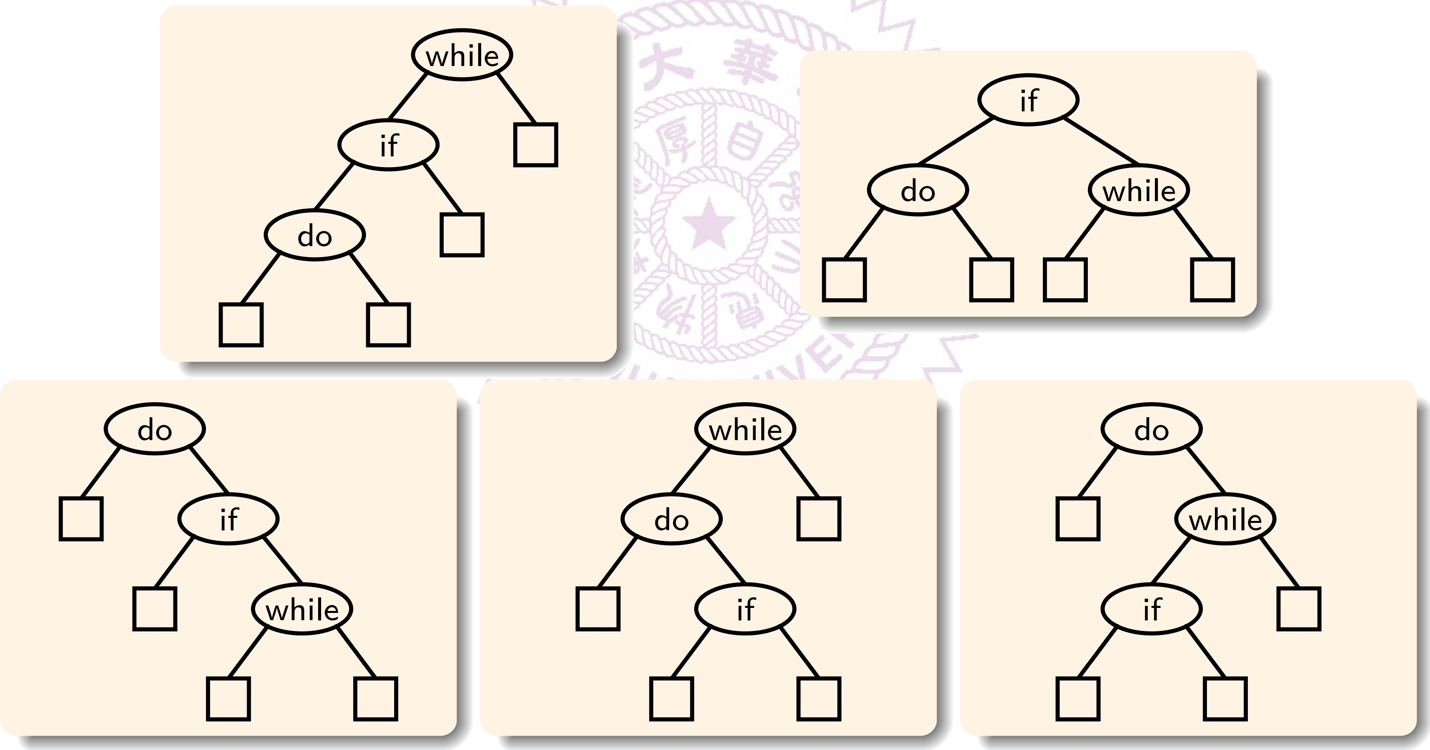
$$\begin{bmatrix} 0 & 4 & 11 \\ 6 & 0 & 2 \\ 3 & \infty & 0 \end{bmatrix} \quad D^{(0)}. \qquad \begin{bmatrix} 0 & 4 & 11 \\ 6 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix} \quad D^{(1)}.$$

$$\begin{bmatrix} 0 & 4 & 6 \\ 6 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix} \quad D^{(2)}. \qquad \begin{bmatrix} 0 & 4 & 6 \\ 5 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix} \quad D^{(3)}.$$

- The minimum cost between all vertices, i and j, is given by $D^{(3)}[i, j]$, $1 \leq i, j \leq 3$.
- To print out the shortest paths for each pair of vertices, the intermediate vertex k on line 10 should be memorized to another matrix $P[1 : n, 1 : n]$.
- Using matrix P the shortest paths can be printed out.
- Correctness of the algorithm can be found in textbooks, [Horowitz], pp. 284-287, and [Cormen], pp. 693-695.

Optimal Binary Search Tree

- Possible binary search trees for three identifiers
  - Successful searches terminate at an internal node, shown in ellipse
  - Unsuccessful searches terminate at an external node, shown in square
  - n internal nodes and $n + 1$ external nodes



Optimal Binary Search Tree — *cost*

- For each identifier, a_i, at $level(a_i)$ in the tree, each successful search needs $level(a_i)$ comparisons.
- Note that for n identifiers there are $n+1$ possible unsuccessful searches.
  - Name these unsuccessful events, E_j, $0 \le j \le n$.
  - For each unsuccessful search E_i at $level(E_i)$ of the binary tree, there are $level(E_i) - 1$ comparisons.
- Let p_i be the probability of searching for identifier a_i and q_i be the probability of searching for E_i.

$$\sum_{i=1}^{n} p_i + \sum_{j=0}^{n} q_i = 1. \tag{6.2.5}$$

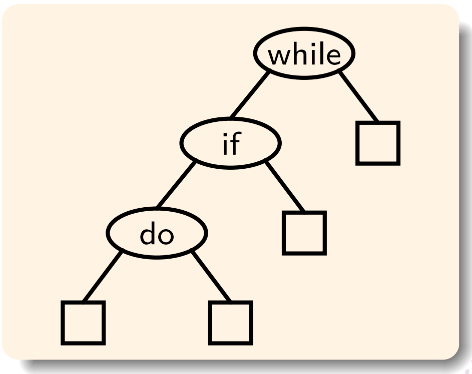
- The cost of the binary search tree is the expected value of the number of comparisons

$$cost(t) = \sum_{i=1}^{n} p_i \times level(a_i) + \sum_{j=0}^{n} q_i \times (level(E_i) - 1). \tag{6.2.6}$$

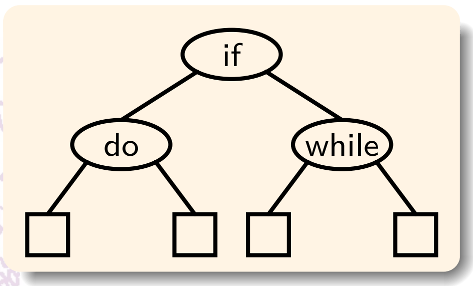
- The optimal binary search tree is the binary tree such that the cost of the tree is minimum.

Optimal Binary Search Tree — Example

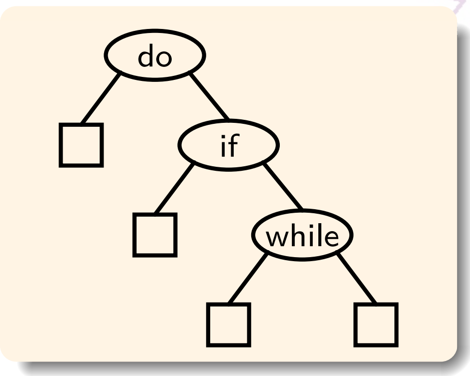
- Suppose $p_i = q_i = 1/7$ then



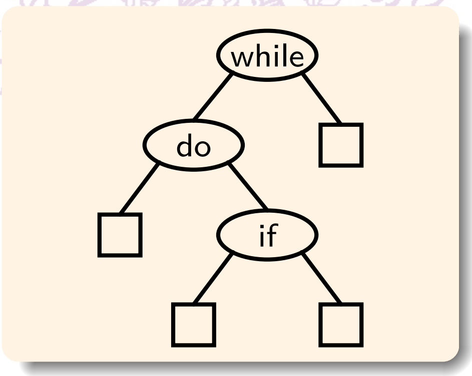
$cost = 15/7$



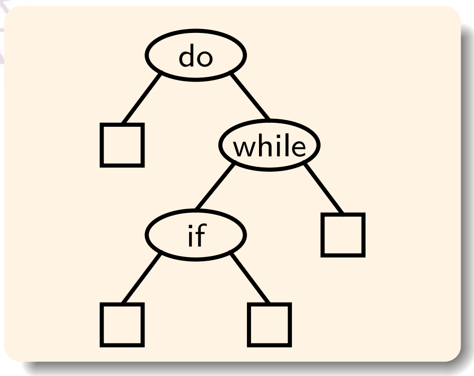
$cost = 13/7$, optimal



$cost = 15/7$



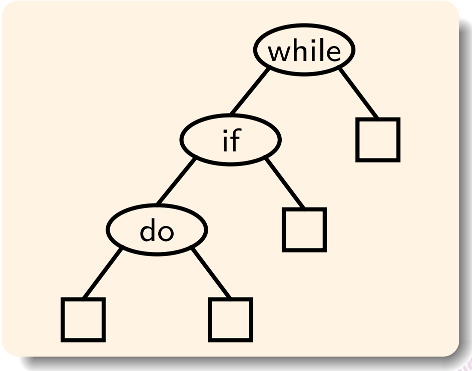
$cost = 15/7$



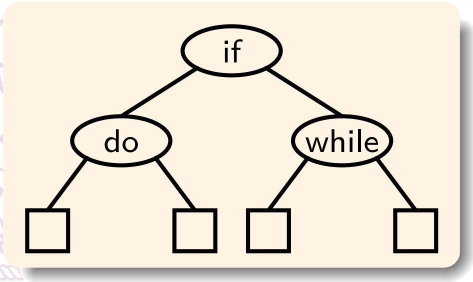
$cost = 15/7$

Optimal Binary Search Tree — Example II

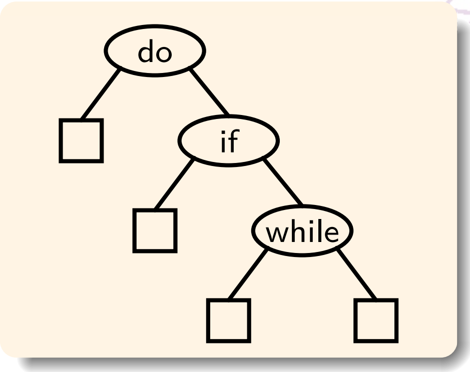
- Suppose $p_1 = 0.5$(do), $p_2 = 0.1$(if), $p_3 = 0.05$(while), $q_0 = 0.15$, $q_1 = 0.1$, $q_2 = 0.05$, $q_3 = 0.05$, then



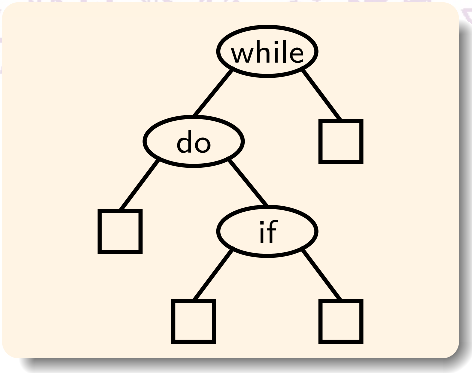
$cost = 2.65$



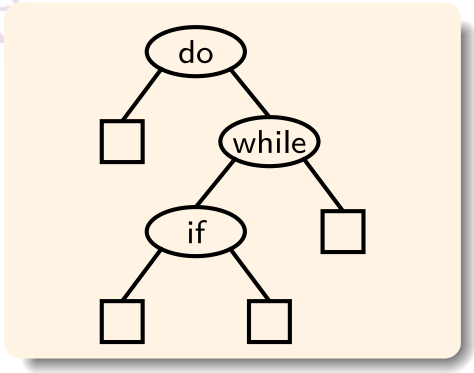
$cost = 1.9$



$cost = 1.5$, optimal



$cost = 2.15$



$cost = 1.6$

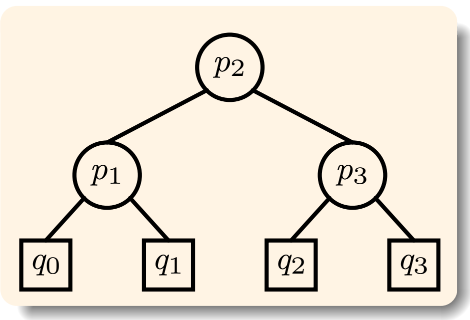
Optimal Binary Search Tree — Properties

- Given internal nodes $\{a_1, a_2, \cdots, a_n\}$ with probabilities $\{p_1, p_2, \cdots, p_n\}$ and the external nodes with probabilities $\{q_0, q_1, \cdots, q_n\}$.
- If a_k is the root of a binary search tree, then its left subtree consists of internal nodes $\{a_1, a_2, \cdots, a_{k-1}\}$ and external nodes $\{q_0, q_1, \cdots, q_{k-1}\}$.
- The right subtree consists of internal nodes $\{a_{k+1}, \cdots, a_n\}$ and external nodes $\{q_k, \cdots, q_n\}$.
- Let the cost of the left subtree be c_l and the cost of the right subtree be c_r, then the cost of the tree with a_k as the root is

$$c(a_k) = c_l + c_r + w(1, n) \tag{6.2.7}$$

where

$$w(1, n) = \sum_{i=1}^{n} p_i + \sum_{i=0}^{n} q_i. \tag{6.2.8}$$



- Example

$$c_l = p_1 + q_0 + q_1$$
$$c_r = p_3 + q_2 + q_3$$
$$\begin{aligned} c(p_2) &= p_2 + 2(p_1 + q_0 + q_1) + 2(p_3 + q_2 + q_3) \\ &= c_l + c_r + p_1 + p_2 + p_3 + q_0 + q_1 + q_2 + q_3 \end{aligned}$$

Optimal Binary Search Tree — Recursive Algorithm

Algorithm 6.2.6. Recursive OBST

```
// Find the root r of the optimal binary search tree for nodes a_i to a_j.
// Input: range: i, j; probabilities: p[1 : n], q[0 : n]
// Output: cost, root r.
Algorithm OBSTr(i, j, p, q, r)
{
    if (i = j) then { // single vertex
        r := i;
        return q[i − 1] + q[i] + p[i];
    }
    cost := ∞;
    w := q[i − 1];
    for k := i to j do w := w + p[k] + q[k]; // calculate w(i, j)
    for k := i + 1 to j − 1 do { // try every vertex and find the minimum cost one
        cL := OBSTr(i, k − 1, p, q, rL); // find minimum cost left subtree
        cR := OBSTr(k + 1, j, p, q, rR); // find minimum cost right subtree
        if (cL + cR + w < cost) then {
            cost := cL + cR + w;
            r := k;
        }
    }
    return cost;
}
```


Optimal Binary Search Tree — Recursive Algorithm, II

- This algorithm finds the minimum-cost left subtree and right subtree and combines those two to form the minimum-cost binary search tree.
- The recursive algorithm is invoked by OBSTr$(1, n, p, q, r)$, where p is the array for the internal node probabilities, q is the array for the external nodes probabilities.
- It then finds the root r of the minimum-cost binary search tree.
  - The roots of the left and right subtrees should be found by calling OBSTr$(1, r-1, p, q, rL)$ and OBSTr$(r+1, n, p, q, rR)$ recursively.
- As most of the recursive function, the time complexity can be improved.

Optimal Binary Search Tree — Improved Algorithm

Algorithm 6.2.7. Optimal Binary Search Tree

```
// Find the matrix r. Each r[i, j] is the optimal root for a_i to a_j.
// Input: int n, probabilities: p[1 : n], q[0 : n]
// Output: r: optimal root matrix.
Algorithm OBST(n, p, q, r)
{
    for i := 0 to n − 1 do {
        w[i, i] := q[i] ;
        r[i, i] := 0 ;
        c[i, i] := 0 ;
        w[i, i + 1] := q[i] + p[i + 1] + q[i + 1] ; // one node trees
        r[i, i + 1] := i + 1 ;
        c[i, i + 1] := q[i] + p[i + 1] + q[i + 1] ;
    }
    w[n, n] := q[n] ;
    r[n, n] := 0 ;
    c[n, n] := 0 ;
    for m := 2 to n do { // Find optimal trees with m nodes
        for i := 0 to n − m do {
            j := i + m ;
            w[i, j] := w[i, j − 1] + p[j] + q[j] ;
            k := KnuthFind(c, r, i, j) ; // root with min cost of m-node tree
            r[i, j] := k ; // root for tree a_i to a_j
            c[i, j] := w[i, j] + c[i, k − 1] + c[k, j] ; // record min cost
        }
    } // When done, r[0, n] is the root, c[0, n] is the min cost
}
```


Optimal Binary Search Tree — KnuthFind

Algorithm 6.2.8. Knuth Find

```
// Find the min-cost root for tree a_i to a_j.
// Input: c[0 : n]: min cost, r[0 : n]: min cost root matrix
// Output: min cost root.
Algorithm KnuthFind(c, r, i, j)
{
    min := ∞ ;
    for m := r[i, j − 1] to r[i + 1, j] do {
        if ((c[i, m − 1] + c[m, j]) < min) then {
            min := c[i, m − 1] + c[m, j] ;
            l := m ;
        }
    }
    return l ;
}
```

- In the OBST Algorithm
  - $r[i, j]$ is the min-cost root for tree a_i to a_j
    - $p[i, j]$ is the probabilities of the internal nodes a_i to a_j
    - $q[i-1, j]$ is the probabilities of the external nodes
  - $c[i, j]$ is the cost of the optimal search tree
  - $w[i, j]$ is the sum of all the probabilities for internal and external nodes from a_i to a_j.

Optimal Binary Search Tree — OBST and Complexity

- After completion of the algorithm
  - The root of the optimal tree is given by $r[0, n]$
  - Let $k = r[0, n]$, then
  - The root of the left subtree is $r[0, k - 1]$
  - And the root of the right subtree is $r[k + 1, n]$
  - Repeating this process the entire tree can be built.
- Using `KnuthFind` function in `OBST` algorithm, the time complexity is $\mathcal{O}(n^2)$
  - Exercise
- And the complexity of using resulting $r[0, n]$ to build the optimal binary search tree is $\mathcal{O}(n)$

Summary

- Multistage graph problem
- Single-source shortest path
- All-pairs shortest paths
- Optimal binary search tree