Section 6.3 Undetermined Coefficients and the Annihilator Method

Definition：Annihilator

A linear differential operator A is said to annihilate a function f if

(2) $A[f](x)=0$,

for all x. That is, A annihilates f if f is a solution to the homogenous linear differential equation (2) on $(-\infty,\infty)$.

☆ x^m 之消去元為 D^{m+1}；

e^{rx} 之消去元為 $(D-r)$；

$e^{\alpha x}\cos\beta x$ or $e^{\alpha x}\sin\beta x$ 之消去元為 $(D-\alpha)^2+\beta^2$ 。

◇ Use the method of undetermined coefficients to determine the form of a particular solution for the given equation.

3. $y'''+3y''-4y=e^{-2x}$ ($m=0$, $r=-2\to s=2$)

Sol.

$$r^3+3r^2-4=0$$
$$\Rightarrow (r-1)(r^2+4r+4)=0$$
$$\Rightarrow (r-1)(r+2)^2=0$$
$$\Rightarrow r=1,-2,-2$$

$\therefore$ $y_h=c_1e^x+c_2e^{-2x}+c_3xe^{-2x}$

$\therefore$ The form of a particular solution is $y_p=Ax^2e^{-2x}$.

◇ Find a general solution to the given equation.

9. $y'''-3y''+3y'-y=e^x$ ($m=0$, $r=1\to s=3$)

Sol.

$$r^3-3r^2+3r-1=0$$
$$\Rightarrow (r-1)(r^2-2r+1)=0$$
$$\Rightarrow (r-1)(r-1)^2=0$$
$$\Rightarrow r=1,\ 1,\ 1$$

$\therefore$ $y_h=(c_1+c_2x+c_3x^2)e^x$

Let $y_p=Ax^3e^x$

$$y_p'=A(3x^2e^x+x^3e^x)=A(3x^2+x^3)e^x$$
$$y_p''=A[(6x+3x^2)e^x+(3x^2+x^3)e^x]=A(6x+6x^2+x^3)e^x$$

$$y_p''' = A[(6+12x+3x^2)e^x + (6x+6x^2+x^3)e^x] = A(6+18x+9x^2+x^3)e^x$$

$$\Rightarrow [A(6+18x+9x^2+x^3) - 3A(6x+6x^2+x^3) + 3A(3x^2+x^3) - Ax^3]e^x = e^x$$
$$\Rightarrow 6A = 1$$

$$\Rightarrow A = \frac{1}{6}$$

$$\therefore \quad y(x) = y_h + y_p = (c_1 + c_2 x + c_3 x^2)e^x + \frac{1}{6}x^3 e^x$$

◇ Find a differential operator that annihilates the given function.

11. $x^4 - x^2 + 11$

Sol.

$$A = D^5$$

15. $e^{2x} - 6e^x$

Sol.

e^{2x} 之消去元為 $(D-2)$

$6e^x$ 之消去元為 $(D-1)$

$\therefore \quad A = (D-2)(D-1)$

19. $xe^{-2x} + xe^{-5x}\sin 3x$

Sol.

xe^{-2x} 之消去元為 $(D+2)^2$

$xe^{-5x}\sin 3x$ 之消去元為 $[(D+5)^2 + 3^2]^2$

$\therefore \quad A = (D+2)^2[(D+5)^2 + 3^2]^2$

◇ Use the annihilator method to determine the form of a particular solution for the given equation.

23. $y'' - 5y' + 6y = e^{3x} - x^2$

Sol.

〔way 1〕

$e^{3x} - x^2$ 之消去元為 $D^3(D-3)$

$$D^3(D-3)(D^2-5D+6)[y] = D^3(D-3)(e^{3x} - x^2) = 0$$
$$\Rightarrow D^3(D-3)(D^2-5D+6)[y] = 0$$
$$\Rightarrow D^3(D-3)[(D-3)(D-2)][y] = 0$$
$$\Rightarrow D^3(D-3)^2(D-2) = 0$$

$$\therefore \quad y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{3x} + c_5 x e^{3x} + c_6 e^{2x}$$

$$(D^2 - 5D + 6)[y] = [(D-3)(D-2)][y] = 0$$

$\therefore \quad y_h = c_4 e^{3x} + c_6 e^{2x}$

$\therefore \quad y_p = y(x) - y_h = c_1 + c_2 x + c_3 x^2 + c_5 x e^{3x}$

〔way 2〕

$r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0 \Rightarrow r = 2, 3$

$\therefore y_h = c_1 e^{2x} + c_2 e^{3x}$

$e^{3x} - x^2$ 之消去元為 $D^3(D-3)$

$D^3(D-3)(D^2-5D+6)[y] = D^3(D-3)(e^{3x} - x^2) = 0$

$\Rightarrow D^3(D-3)(D^2-5D+6)[y] = 0$

$\Rightarrow D^3(D-3)[(D-3)(D-2)][y] = 0$

$\Rightarrow D^3(D-3)^2(D-2) = 0$

$\therefore \quad y(x) = c_1 e^{2x} + c_2 e^{3x} + c_3 + c_4 x + c_5 x^2 + c_6 x e^{3x}$

$\therefore \quad y_p = c_3 + c_4 x + c_5 x^2 + c_6 x e^{3x}$

25. $y'' - 6y' + 9y = \sin 2x + x$

Sol.

$(\sin 2x + x)$ 之消去元為 $D^2(D^2 + 2^2)$

$D^2(D^2+4)(D^2-6D+9)[y] = D^2(D^2+4)(\sin 2x + x) = 0$

$\Rightarrow D^2(D^2+4)(D^2-6D+9)[y] = 0$

$\Rightarrow D^2(D^2+4)(D-3)^2[y] = 0$

$\therefore \quad y(x) = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x + c_5 e^{3x} + c_6 x e^{3x}$ and

$y_h = c_5 e^{3x} + c_6 x e^{3x}$

$\therefore \quad y_p = y - y_h = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x$

27. $y'' + 2y' + 2y = e^{-x} \cos x + x^2$

Sol.

$(e^{-x} \cos x + x^2)$ 之消去元為 $D^3[(D+1)^2 + 1^2]$

$D^3[(D+1)^2+1](D^2+2D+2)[y] = D^3[(D+1)^2+1](e^{-x}\cos x + x^2) = 0$

$\Rightarrow D^3[(D+1)^2+1][(D+1)^2+1][y] = 0$

$\Rightarrow D^3[(D+1)^2+1]^2[y] = 0$

$\therefore \quad y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}\cos x + c_5 e^{-x} \sin x + c_6 x e^{-x} \cos x + c_7 x e^{-x} \sin x$ and

$y_h = c_4 e^{-x}\cos x + c_5 e^{-x}\sin x$

$\therefore \quad y_p = c_1 + c_2 x + c_3 x^2 + c_6 x e^{-x}\cos x + c_7 x e^{-x}\sin x$

34. Use the annihilator method to show that if $a_0 \neq 0$ in equation (4) and $f(x)$ has the form

(17) $f(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, then $y_p(x) = B_m x^m + B_{m-1}x^{m-1} + \cdots + B_1 x + B_0$

is the form of a particular solution to equation (4).

Sol.

(4) $a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = f(x)$

$f(x)$之消去元為 D^{m+1}

$D^{m+1}(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)[y] = D^{m+1}[f(x)] = 0$

$\Rightarrow D^{m+1}(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)[y] = 0$

Let $y = e^{rx}$

$\Rightarrow r^{m+1}(a_n r^n + a_{n-1}r^{n-1} + \cdots + a_0) = 0$

$\Rightarrow r = 0$ $(m+1)$個, $a_n r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0$

$\therefore \quad y(x) = B_0 + B_1 x + \cdots + B_{m-1}x^{m-1} + B_m x^m + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ and

$y_h = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

$\therefore \quad y_p = B_0 + B_1 x + \cdots + B_{m-1}x^{m-1} + B_m x^m$

35. Use the annihilator method to show that if $a_0 = 0$ and $a_1 \neq 0$ in (4) and $f(x)$ has the form given in (17), then equation (4) has a particular solution of the form

$y_p(x) = x\{B_m x^m + B_{m-1}x^{m-1} + \cdots + B_1 x + B_0\}$.

Sol.

(4) $a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1 y'(x) = f(x)$

$f(x)$之消去元為 D^{m+1}

$D^{m+1}(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D)[y] = D^{m+1}[f(x)] = 0$

$\Rightarrow D^{m+1}(a_n D^n + a_{n-1}D^{n-1} + \cdots + a_1 D)[y] = 0$

Let $y = e^{rx}$

$\Rightarrow r^{m+1}(a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r) = 0$

$\Rightarrow r^{m+1} \cdot r(a_n r^{n-1} + a_{n-1}r^{n-2} + \cdots + a_1) = 0$

$\Rightarrow r = 0$ $(m+1)$個, $r = 0$, $a_n r^{n-1} + a_{n-1}r^{n-2} + \cdots + a_1 = 0$

$\therefore$ $y(x) = c_0 + c_1 y_1 + c_2 y_2 + \cdots + c_{n-1}y_{n-1} + x(B_0 + B_1 x + \cdots + B_{m-1}x^{m-1} + B_m x^m)$ and

$y_h = c_0 + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

$\therefore$ $y_p = x(B_0 + B_1 x + \cdots + B_{m-1}x^{m-1} + B_m x^m)$