Section 4.6 Variation of Parameters

Introduction：Variation of Parameters

Consider the $ay''+by'+cy=g(t)$ and let $y_1(t)$ and $y_2(t)$ be two L.I. solutions for $ay''+by'+cy=0$. Then we know that a general solution is $y_h(t)=c_1y_1(t)+c_2y_2(t)$.

Let $y_p(t)=v_1(t)y_1(t)+v_2(t)y_2(t)$

$$\Rightarrow y_p'(t)=\underbrace{(v_1'y_1+v_2'y_2)}_{set\ 0}+(v_1y_1'+v_2y_2')$$
$$\Rightarrow y_p''(t)=(v_1'y_1'+v_2'y_2')+(v_1y_1''+v_2y_2'')$$
$$\Rightarrow a(v_1'y_1'+v_2'y_2')+a(v_1y_1''+v_2y_2'')+b(v_1y_1'+v_2y_2')+c(v_1y_1+v_2y_2)=g$$
$$\Rightarrow a(v_1'y_1'+v_2'y_2')+v_1\underbrace{(ay_1''+by_1'+cy_1)}_{=0}+v_2\underbrace{(ay_2''+by_2'+cy_2)}_{=0}=g$$
$$\Rightarrow a(v_1'y_1'+v_2'y_2')=g \quad or$$
$$\Rightarrow (v_1'y_1'+v_2'y_2')=\frac{g}{a}$$

Method of Variation of Parameters

To determine a particular solution to $ay''+by'+cy=g$:

1. Find two L.I. solutions $\{y_1(t),y_2(t)\}$ to the $ay''+by'+cy=0$ and take

   $y_p(t)=v_1(t)y_1(t)+v_2(t)y_2(t)$.

2. Determine $v_1(t)$ and $v_2(t)$ by solving the system $\begin{cases} y_1v_1'+y_2v_2'=0 \\ y_1'v_1'+y_2'v_2'=\dfrac{g}{a}\end{cases}$ for $v_1'(t)$ and $v_2'(t)$

   and integrating.

3. Substitute $v_1(t)$ and $v_2(t)$ into the expression for $y_p(t)$ to obtain a particular solution.

◇ Find a general solution to the differential equation using the method of variation of parameters.

2. $y''+4y=\tan 2t$

Sol.

$$r^2+4=0 \Rightarrow r=\pm 2i$$
$$\therefore\ y_h(t)=c_1\cos 2t+c_2\sin 2t$$

Let $y_p(t)=v_1\cos 2t+v_2\sin 2t$

$$\Rightarrow \begin{cases} v_1'\cos 2t+v_2'\sin 2t=0 & (\times 2\sin 2t) \\ -2v_1'\sin 2t+2v_2'\cos 2t=\underbrace{\tan 2t}_{=\frac{\sin 2t}{\cos 2t}} & (\times \cos 2t) \end{cases}$$

$$\Rightarrow \begin{cases} 2v_1'\sin 2t\cos 2t + 2v_2'\sin^2 2t = 0 \\ -2v_1'\sin 2t\cos 2t + 2v_2'\cos^2 2t = \sin 2t \end{cases}$$

$$\boxed{\begin{aligned} \int \sec at\,dt &= \frac{1}{a}\ln|\sec at + \tan at| + C \\ \int \csc at\,dt &= \frac{-1}{a}\ln|\csc at + \cot at| + C \end{aligned}}$$

$$\Rightarrow \begin{cases} 2v_2' = \sin 2t \\ v_1' = -v_2' \cdot \dfrac{\sin 2t}{\cos 2t} \end{cases}$$

$$\Rightarrow \begin{cases} v_2' = \dfrac{1}{2}\sin 2t \\ v_1' = -\dfrac{\sin^2 2t}{2\cos 2t} = -\dfrac{(1-\cos^2 2t)}{2\cos 2t} = -\dfrac{1}{2}\left(\dfrac{1}{\cos 2t} - \cos 2t\right) = -\dfrac{1}{2}(\sec 2t - \cos 2t) \end{cases}$$

$$\Rightarrow \begin{cases} v_2 = \dfrac{1}{2}\displaystyle\int \sin 2t\,dt = -\dfrac{1}{4}\cos 2t \\ v_1 = -\dfrac{1}{2}\displaystyle\int(\sec 2t - \cos 2t)dt = -\dfrac{1}{2}\left(\dfrac{1}{2}\ln|\sec 2t + \tan 2t| - \dfrac{1}{2}\sin 2t\right) \end{cases}$$

$$\Rightarrow y_p = \left(-\frac{1}{4}\ln|\sec 2t + \tan 2t| + \frac{1}{4}\sin 2t\right)\cos 2t - \frac{1}{4}\cos 2t \sin 2t$$

$$= -\frac{1}{4}\ln|\sec 2t + \tan 2t|\cos 2t$$

$$\therefore\ \ y(t) = y_h + y_p = c_1\cos 2t + c_2\sin 2t - \frac{1}{4}\ln|\sec 2t + \tan 2t|\cos 2t$$

4. $y'' - 2y' + y = t^{-1}e^t$

<u>Sol.</u>

$r^2 - 2r + 1 = 0 \Rightarrow r = 1$（重根）

$\therefore\ \ y_h(t) = c_1e^t + c_2te^t$

Let $y_p(t) = v_1e^t + v_2te^t$

$$\Rightarrow \begin{cases} v_1'e^t + v_2'te^t = 0 \\ v_1'e^t + v_2'(e^t + te^t) = t^{-1}e^t \end{cases}$$

$$\Rightarrow \begin{cases} v_2'e^t = t^{-1}e^t \\ v_1' = -v_2't \end{cases}$$

$$\Rightarrow \begin{cases} v_2' = t^{-1} \\ v_1' = -1 \end{cases}$$

$$\Rightarrow \begin{cases} v_2 = \ln|t| \\ v_1 = -t \end{cases}$$

$$\Rightarrow y_p = -te^t + te^t\ln|t|$$

$$\therefore\ \ y(t) = c_1e^t + c_2te^t - te^t + te^t\ln|t|$$

8. $y'' + 4y = \csc^2(2t)$

Sol.

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

$$\therefore \quad y_h(t) = c_1 \cos 2t + c_2 \sin 2t$$

Let $y_p(t) = v_1 \cos 2t + v_2 \sin 2t$

$$\Rightarrow \begin{cases} v_1' \cos 2t + v_2' \sin 2t = 0 & (\times 2\cos 2t) \\ -2v_1' \sin 2t + 2v_2' \cos 2t = \underbrace{\csc^2 2t}_{=\frac{1}{\sin^2 2t}} & (\times \sin 2t) \end{cases}$$

$$\Rightarrow \begin{cases} 2v_1' \cos^2 2t + 2v_2' \sin 2t \cos 2t = 0 \\ -2v_1' \sin^2 2t + 2v_2' \sin 2t \cos 2t = \dfrac{1}{\sin 2t} \end{cases}$$

$$\Rightarrow \begin{cases} 2v_1' = -\dfrac{1}{\sin 2t} = -\csc 2t \\ v_2' = -v_1' \cdot \dfrac{\cos 2t}{\sin 2t} \end{cases}$$

$$\Rightarrow \begin{cases} v_1' = -\dfrac{1}{2}\csc 2t \\ v_2' = \dfrac{1}{2}\dfrac{\cos 2t}{\sin^2 2t} \end{cases}$$

$$\boxed{\begin{aligned} &\int \frac{\cos 2t}{\sin^2 2t} dt \quad \begin{pmatrix} u = \sin 2t \\ du = 2\cos 2t dt \end{pmatrix} \\ &= \frac{1}{2}\int u^{-2} du \\ &= -\frac{1}{2}u^{-1} + C \\ &= -\frac{1}{2}\frac{1}{\sin 2t} + C \end{aligned}}$$

$$\Rightarrow \begin{cases} v_1 = -\dfrac{1}{2}\displaystyle\int \csc 2t dt = -\dfrac{1}{2}\left[-\dfrac{1}{2}\ln|\csc 2t + \cot 2t|\right] = \dfrac{1}{4}\ln|\csc 2t + \cot 2t| \\ v_2 = \dfrac{1}{2}\displaystyle\int \dfrac{\cos 2t}{\sin^2 2t} dt = \dfrac{1}{2}\left(-\dfrac{1}{2}\dfrac{1}{\sin 2t}\right) = -\dfrac{1}{4}\dfrac{1}{\sin 2t} \end{cases}$$

$$\Rightarrow y_p = \frac{1}{4}\cos 2t \ln|\csc 2t + \cot 2t| - \frac{1}{4}$$

$$\therefore \quad y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4}\cos 2t \ln|\csc 2t - \cot 2t| - \frac{1}{4}$$

◇ Find a general solution to the differential equation.

15. $y'' + y = 3\sec t - t^2 + 1$

Sol.

$$y'' + y = 3\sec t - t^2 + 1 = 3\sec t - (t^2 - 1)$$

$$r^2 + 1 = 0 \Rightarrow r = \pm i$$

$$\therefore \quad y_h = c_1 \cos t + c_2 \sin t$$

(i) $3\sec t$

Let $y_{p_1} = v_1 \cos t + v_2 \sin t$

$$\Rightarrow \begin{cases} v_1' \cos t + v_2' \sin t = 0 & (\times \sin t) \\ -v_1' \sin t + v_2' \cos t = \underbrace{3\sec t}_{=\frac{3}{\cos t}} & (\times \cos t) \end{cases}$$

$$\Rightarrow \begin{cases} v_1' \sin t \cos t + v_2' \sin^2 t = 0 \\ -v_1' \sin t \cos t + v_2' \cos^2 t = 3 \end{cases}$$

$$\Rightarrow \begin{cases} v_2' = 3 \\ v_1' = -v_2' \cdot \dfrac{\sin t}{\cos t} = -3 \cdot \dfrac{\sin t}{\cos t} \end{cases}$$

$$\Rightarrow \begin{cases} v_2 = 3t \\ v_1 = 3\ln|\cos t| \end{cases}$$

$$\Rightarrow y_{p_1} = 3\cos t \ln|\cos t| + 3t\sin t$$

(ii) $t^2 - 1 \quad (m = 2, r = 0 \to s = 0)$

Let $y_{p_2} = At^2 + Bt + C$

$y_{p_2}' = 2At + B$

$y_{p_2}'' = 2A$

$$\Rightarrow 2A + At^2 + Bt + C = t^2 - 1$$

$$\Rightarrow \begin{cases} A = 1 \\ B = 0 \\ 2A + C = -1 \Rightarrow C = -3 \end{cases}$$

$\therefore \quad y_{p_2} = t^2 - 3$

$\therefore \quad y(t) = y_h + y_{p_1} - y_{p_2} = c_1 \cos t + c_2 \sin t + 3\cos t \ln|\cos t| + 3t\sin t - t^2 + 3$

17. $\dfrac{1}{2} y'' + 2y = \tan 2t - \dfrac{1}{2} e^t$

<u>Sol.</u>

$$\frac{1}{2} y'' + 2y = \tan 2t - \frac{1}{2} e^t$$

$$\Rightarrow y'' + 4y = 2\tan 2t - e^t$$

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

$\therefore \quad y_h = c_1 \cos 2t + c_2 \sin 2t$

(i) $2\tan 2t$

Let $y_{p_1} = v_1 \cos 2t + v_2 \sin 2t$

$$\Rightarrow \begin{cases} v_1' \cos 2t + v_2' \sin 2t = 0 & (\times 2\sin 2t) \\ -2v_1' \sin 2t + 2v_2' \cos 2t = \underbrace{2\tan 2t}_{=\frac{2\sin 2t}{\cos 2t}} & (\times \cos 2t) \end{cases}$$

$$\Rightarrow \begin{cases} 2v_1' \sin 2t \cos 2t + 2v_2' \sin^2 2t = 0 \\ -2v_1' \sin 2t \cos 2t + 2v_2' \cos^2 2t = 2\sin 2t \end{cases}$$

$$\Rightarrow \begin{cases} 2v_2' = 2\sin 2t \\ v_1' = -v_2' \cdot \dfrac{\sin 2t}{\cos 2t} \end{cases}$$

$$\Rightarrow \begin{cases} v_2' = \sin 2t \\ v_1' = -\dfrac{\sin^2 2t}{\cos 2t} = -\dfrac{(1-\cos^2 2t)}{\cos 2t} = \left(\cos 2t - \dfrac{1}{\cos 2t}\right) = \cos 2t - \sec 2t \end{cases}$$

$$\Rightarrow \begin{cases} v_2 = \int \sin 2t dt = -\dfrac{1}{2}\cos 2t \\ v_1 = \int (\cos 2t - \sec 2t) dt = \dfrac{1}{2}\sin 2t - \dfrac{1}{2}\ln|\sec 2t + \tan 2t| \end{cases}$$

$$\Rightarrow y_{p_1} = \left(\frac{1}{2}\sin 2t - \frac{1}{2}\ln|\sec 2t + \tan 2t|\right)\cos 2t - \frac{1}{2}\cos 2t \sin 2t$$

$$= -\frac{1}{2}\ln|\sec 2t + \tan 2t|\cos 2t$$

(ii) e^t $(m = 0, r = 1 \to s = 0)$

Let $y_{p_2} = Ae^t$

$y_{p_2}' = y_{p_2}'' = Ae^t$

$\Rightarrow Ae^t + 4Ae^t = e^t$

$\Rightarrow A = \dfrac{1}{5}$

$\therefore \quad y_{p_2} = \dfrac{1}{5}e^t$

$$\therefore \quad y(t) = y_h + y_{p_1} - y_{p_2} = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{2}\cos 2t \ln|\sec 2t + \tan 2t| - \frac{1}{5}e^t$$