Section 4.5 The Superposition Principle and Undetermined Coefficients Revisited

Theorem： Superposition Principle

Let y_1 be a solution to the differential equation $ay''+by'+cy=f_1(t)$, and let y_2 be a solution to $ay''+by'+cy=f_2(t)$. Then for any constants c_1 and c_2, the function $c_1y_1+c_2y_2$ is a solution to the differential equation $ay''+by'+cy=f_1(t)+f_2(t)$

◇ Decide whether the method of undermined coefficients together with superposition can be applied to find a particular solution of the given equation. Do not solve the equation.

10. $3y''+2y'+8y=t^2+4t-t^2e^t\sin t$

Sol.

可利用未定係數法分別求出$3y''+2y'+8y=t^2+4t$的特解y_{p_1}與

$3y''+2y'+8y=t^2e^t\sin t$的特解y_{p_2}，再根據疊加原理(superposition principle)可得知

$3y''+2y'+8y=t^2+4t-t^2e^t\sin t$的特解為$y_{p_1}-y_{p_2}$。

11. $y''-6y'-4y=4\sin 3t-e^{3t}t^2+1/t$

Sol.

$y''-6y'-4y=4\sin 3t$和$y''-6y'-4y=-e^{3t}t^2$的特解皆可用未定係數法求得，但
$y''-6y'-4y=1/t$中的$1/t$不屬未定係數法可用之形式，故無法用此法求出特解。

14. $y''-2y'+3y=\cosh t+\sin^3 t$

Sol.

$$y''-2y'+3y=\cosh t=\frac{e^t}{2}+\frac{e^{-t}}{2}$$

$$y''-2y'+3y=\sin^3 t=-\frac{1}{12}\sin(3t)+\frac{1}{12}\cos(3t)-\frac{3}{4}\sin t-\frac{3}{4}\cos t$$

可利用未定係數法分別求出的$y''-2y'+3y=\dfrac{e^t}{2}$特解$y_{p_1}$與$y''-2y'+3y=\dfrac{e^{-t}}{2}$的特解$y_{p_2}$，

再根據疊加原理(superposition principle)可得知$y''-2y'+3y=\cosh$的特解為$y_{p_1}+y_{p_2}$。

同理，分別求出的$y''-2y'+3y=-\dfrac{1}{12}\sin(3t)+\dfrac{1}{12}\cos(3t)$特解$y_{p_1}$與

$y''-2y'+3y=-\dfrac{3}{4}\sin t-\dfrac{3}{4}\cos t$的特解$y_{p_2}$，得知$y''-2y'+3y=\sin^3 t$的特解為$y_{p_1}+y_{p_2}$。

◇ Find a **general solution(通解 or 一般解)** to the differential equation.

20. $y''(\theta)+4y(\theta)=\sin\theta-\cos\theta$

<u>Sol.</u>

$r^2+4=0 \Rightarrow r=\pm 2i$

$\therefore\ \ y_h=c_1\cos 2\theta+c_2\sin 2\theta$

$\because\ \ r=i$ is not a root of $r^2+4=0$

$\therefore$ take $s=0$

Let $y_p=A\sin\theta-B\cos\theta$

$\Rightarrow y_p'=A\cos\theta+B\sin\theta$

$y_p''=-A\sin\theta+B\cos\theta$

$\Rightarrow(-A+4A)\sin\theta+(B-4B)\cos\theta=\sin\theta-\cos\theta$

$$\Rightarrow\begin{cases}-3A=1\\-3B=-1\end{cases}$$

$$\Rightarrow\begin{cases}A=\dfrac{1}{3}\\B=\dfrac{1}{3}\end{cases}$$

$$\therefore y_p=\frac{1}{3}\sin\theta-\frac{1}{3}\cos\theta$$

Hence, the general solution is $y(\theta)=y_h+y_p=c_1\cos 2\theta+c_2\sin 2\theta+\frac{1}{3}\sin\theta-\frac{1}{3}\cos\theta$

◇ Find the solution to the initial value problem.

29. $y''(\theta)-y(\theta)=\sin\theta-e^{2\theta}$; $y(0)=1$, $y'(0)=-1$.

<u>Sol.</u>

$r^2-1=0 \Rightarrow r=1,-1$

$\therefore\ \ y_h=c_1e^{\theta}+c_2e^{-\theta}$

(i) $\sin\theta$ $(m=0, r_1=i)$
(ii) $e^{2\theta}$ $(m=0, r_2=2)$

$\because\ \ r_1=i$ and $r_2=2$ are not a root of $r^2-r=0$

$\therefore$ take $s=0$

Let $y_{p_1}=A\cos\theta+B\sin\theta$

$$\Rightarrow y'_{p_1} = -A\sin\theta + B\cos\theta$$

$$y''_{p_1} = -A\cos\theta - B\sin\theta$$

$$\Rightarrow (-A - A)\cos\theta + (-B - B)\sin\theta = \sin\theta$$

$$\Rightarrow \begin{cases} A = 0 \\ B = \dfrac{-1}{2} \end{cases}$$

$$\therefore \quad y_{p_1} = \frac{-1}{2}\sin\theta$$

Let $y_{p_2} = Ce^{2\theta}$

$$\Rightarrow y'_{p_2} = 2Ce^{2\theta}$$

$$y''_{p_2} = 4Ce^{2\theta}$$

$$\Rightarrow 3C = -1$$

$$\Rightarrow C = \frac{-1}{3}$$

$$\therefore \quad y_{p_2} = \frac{-1}{3}e^{2\theta}$$

Hence $y(\theta) = c_1 e^{\theta} + c_2 e^{-\theta} - \dfrac{1}{2}\sin\theta - \dfrac{1}{3}e^{2\theta}$

$$\Rightarrow y'(\theta) = c_1 e^{\theta} - c_2 e^{-\theta} - \frac{1}{2}\cos\theta - \frac{2}{3}e^{2\theta}$$

$$\because \quad y(0) = 1, \quad y'(0) = -1$$

$$\Rightarrow \begin{cases} c_1 + c_2 - \dfrac{1}{3} = 1 \\ c_1 - c_2 - \dfrac{1}{2} - \dfrac{2}{3} = -1 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 + c_2 = \dfrac{4}{3} \\ c_1 - c_2 = \dfrac{1}{6} \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \dfrac{3}{4} \\ c_2 = \dfrac{7}{12} \end{cases}$$

$$\therefore \quad y(\theta) = \frac{3}{4}e^{\theta} + \frac{7}{12}e^{-\theta} - \frac{1}{2}\sin\theta - \frac{1}{3}e^{2\theta}$$

◇ Find a **particular solution** （特解）to the given higher-order equation.

39. $y''' + y'' - 2y = te^t + 1$

<u>Sol.</u>

$$r^3 + r^2 - 2 = 0 \Rightarrow (r-1)(r^2+2r+2) = 0 \Rightarrow r = 1,\ -1 \pm i$$

(i) te^t $(m = 1, r = 1 \to \text{take} \quad s = 1)$

Let $y_{p_1} = t(At+B)e^t = (At^2+Bt)e^t$

$$\Rightarrow y'_{p_1} = (2At+B)e^t + (At^2+Bt)e^t = (At^2+2At+Bt+B)e^t$$

$$y''_{p_1} = (2At+2A+B)e^t + (At^2+2At+Bt+B)e^t = (At^2+4At+Bt+2A+2B)e^t$$

$$y'''_{p_1} = (2At+4A+B)e^t + (At^2+4At+Bt+2A+2B)e^t = (At^2+6At+Bt+6A+3B)e^t$$

$$\Rightarrow [(At^2+6At+Bt+6A+3B)+(At^2+4At+Bt+2A+2B)-2(At^2+Bt)]e^t = te^t$$

$$\Rightarrow (10At+8A+2B)e^t = te^t$$

$$\Rightarrow \begin{cases} 10A = 1 \\ 8A+5B = 0 \end{cases} \Rightarrow \begin{cases} A = \dfrac{1}{10} \\ B = \dfrac{-4}{25} \end{cases}$$

$$\therefore \ y_{p_1} = (\frac{1}{10}t^2 - \frac{4}{25}t)e^t$$

(ii) 1 $(m = 0, r = 0 \to \text{take} \quad s = 0)$

Let $y_{p_2} = C$

$$\Rightarrow y'_{p_2} = y''_{p_2} = y'''_{p_2} = 0$$

$$\Rightarrow -2C = 1$$

$$\Rightarrow C = \frac{-1}{2}$$

$$\therefore \ y_{p_2} = \frac{-1}{2}$$

Hence, $y_p = \dfrac{1}{10}t^2e^t - \dfrac{4}{25}te^t - \dfrac{1}{2}$.