Section 4.3 Auxiliary Equations with Complex Roots

Review：Euler's Formula

(1) $e^{i\theta} = \cos\theta + i\sin\theta$

If $r = \alpha \pm \beta i$ is the solution of $ar^2 + br + c = 0$, we can find $y(t) = C_1 e^{(\alpha+\beta i)t} + C_2 e^{(\alpha-\beta i)t}$.

(1) is used in $e^{(\alpha+\beta i)t}$, we find $e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$.

Complex Conjugate Roots：

If the auxiliary equation has complex conjugate roots $\alpha \pm \beta i$, then two linearly independent solutions to $ay'' + by' + c = 0$ are $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$, and a general solution is

$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$, where c_1 and c_2 are arbitrary constants.

◇ The auxiliary equation for the given differential equation has complex roots. Find a general solution.

6. $w'' + 4w' + 6w = 0$

Sol.

$$r^2 + 4r + 6 = 0$$

$$\Rightarrow r = \frac{-4 \pm \sqrt{16-24}}{2}$$

$$\Rightarrow r = -2 \pm \sqrt{2}i \quad (\alpha = -2, \beta = \sqrt{2})$$

$$\therefore \quad y(t) = c_1 e^{-2t}\cos\sqrt{2}t + c_2 e^{-2t}\sin\sqrt{2}t$$

◇ Solve the given initial value problem.

27. $y''' - 4y'' + 7y' - 6y = 0$ ； $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$

Sol.

$$r^3 - 4r^2 + 7r - 6 = 0$$

$$\Rightarrow (r-2)(r^2 - 2r + 3) = 0$$

$$\Rightarrow r = 2,\ \frac{2 \pm \sqrt{4-12}}{2}$$

$$\Rightarrow r = 2,\ 1 \pm \sqrt{2}i \quad (\alpha = 1, \beta = \sqrt{2})$$

$$\therefore \quad y(t) = c_1 e^{2t} + c_2 e^{t}\cos\sqrt{2}t + c_3 e^{t}\sin\sqrt{2}t$$

$$\Rightarrow y'(t) = 2c_1e^{2t} + c_2(e^t \cos\sqrt{2}t - \sqrt{2}e^t \sin\sqrt{2}t) + c_3(e^t \sin\sqrt{2}t + \sqrt{2}e^t \cos\sqrt{2}t)$$
$$= 2c_1e^{2t} + (c_2 + \sqrt{2}c_3)e^t \cos\sqrt{2}t + (c_3 - \sqrt{2}c_2)e^t \sin\sqrt{2}t$$

and

$$y''(t) = 4c_1e^{2t} + (c_2 + \sqrt{2}c_3)(e^t \cos\sqrt{2}t - \sqrt{2}e^t \sin\sqrt{2}t) + (c_3 - \sqrt{2}c_2)(e^t \sin\sqrt{2}t + \sqrt{2}e^t \cos\sqrt{2}t)$$
$$= 4c_1e^{2t} + [(c_2 + \sqrt{2}c_3) + \sqrt{2}(c_3 - \sqrt{2}c_2)]e^t \cos\sqrt{2}t + [(c_3 - \sqrt{2}c_2) - \sqrt{2}(c_2 + \sqrt{2}c_3)]e^t \sin\sqrt{2}t$$

$\because \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$

$$\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ 2c_1 + c_2 + \sqrt{2}c_3 = 0 \\ 4c_1 - c_2 + 2\sqrt{2}c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \\ c_3 = -\sqrt{2} \end{cases}$$

$\therefore \quad y(t) = e^{2t} - \sqrt{2}e^t \sin\sqrt{2}t$

29. Find a general solution to the following higher-order equations.

(a) $y''' - y'' + y' + 3y = 0$

<u>Sol.</u>

$$r^3 - r^2 + r + 3 = 0$$

$$\Rightarrow (r+1)(r^2 - 2r + 3)$$

$$\Rightarrow r = -1, \ \frac{2 \pm \sqrt{4-12}}{2}$$

$$\Rightarrow r = -1, \ 1 \pm \sqrt{2}i \qquad (\alpha = 1, \beta = \sqrt{2})$$

$\therefore \quad y(t) = c_1e^{-t} + c_2e^t \cos\sqrt{2}t + c_3e^t \sin\sqrt{2}t$

(b) $y''' + 2y'' + 5y' - 26y = 0$

<u>Sol.</u>

$$r^3 + 2r^2 + 5r - 26 = 0$$

$$\Rightarrow (r-2)(r^2 + 4r + 13)$$

$$\Rightarrow r = 2, \ \frac{-4 \pm \sqrt{16-52}}{2}$$

$$\Rightarrow r = 2, \ -2 \pm 3i \qquad (\alpha = -2, \beta = 3)$$

$\therefore \quad y(t) = c_1e^{2t} + c_2e^{-2t} \cos 3t + c_3e^{-2t} \sin 3t$

(c) $y^{iv} + 13y'' + 36y = 0$

<u>Sol.</u>

$$r^4 + 13r^2 + 36 = 0$$

$\Rightarrow (r^2 + 4)(r^2 + 9)$

$\Rightarrow r = \pm 2i,\ \pm 3i \quad (\alpha_1 = 0, \beta_1 = 2\ ; \alpha_2 = 0, \beta_2 = 3)$

$\therefore \quad y(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 \cos 3t + c_4 \sin 3t$

37. The auxiliary equations for the following differential equations have repeated complex roots. Adapt the "repeated root" procedure of Section 4.2 to find their general solutions：

(a) $y^{iv} + 2y'' + y = 0$

<u>Sol.</u>

$$r^4 + 2r^2 + 1 = 0$$

$\Rightarrow (r^2 + 1)^2 = 0$

$\Rightarrow r = \pm i$ (重根)

$\therefore \quad y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$

(b) $y^{iv} + 4y''' + 12y'' + 16y' + 16y = 0$. [*Hint*：The auxiliary equation is $(r^2 + 2r + 4)^2 = 0$]

<u>Sol.</u>

$$r^4 + 4r^3 + 12r^2 + 16r + 16 = 0$$

$\Rightarrow (r^2 + 2r + 4)^2 = 0$

$\Rightarrow r = \dfrac{-2 \pm \sqrt{4 - 16}}{2}$

$\Rightarrow r = -1 \pm \sqrt{3} i$ (重根)

$\therefore \quad y(t) = c_1 e^{-t} \cos \sqrt{3} t + c_2 e^{-t} \sin \sqrt{3} t + c_3 t e^{-t} \cos \sqrt{3} t + c_4 t e^{-t} \sin \sqrt{3} t$