Section 2.2　Separable Equations

Definition： Separable Equation

If the right-hand side of the equation $\frac{dy}{dx}=f(x,y)$ can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y, then the differential equation is called **separable**.

Method for Solving Separable Equations

To solve the equation $\frac{dy}{dx}=g(x)p(y)$

1. To obtain $\frac{1}{p(y)}dy=g(x)dx$
2. Integrate both sides： $\int\frac{1}{p(y)}dy=\int g(x)dx \Rightarrow P(y)=G(x)+C$

◇ Determine whether the given differential equation is separable.

5. $(xy^2+3y^2)dy-2xdx=0$

Sol.

$$(xy^2+3y^2)dy-2xdx=0$$

$$\Rightarrow (xy^2+3y^2)dy=2xdx$$

$$\Rightarrow \frac{dy}{dx}=\frac{2x}{xy^2+3y^2)}=\frac{2x}{x+3}\cdot\frac{1}{y^2}$$

Hence, the equation is separable.

◇ Solve the equation.

11. $\frac{dy}{dx}=\frac{\sec^2 y}{1+x^2}$

Sol.

$$\frac{dy}{dx}=\frac{\sec^2 y}{1+x^2}$$

$$\Rightarrow \frac{1}{\sec^2 y}dy=\frac{1}{1+x^2}dx$$

$$\Rightarrow \int\frac{1}{\sec^2 y}dy=\int\frac{1}{1+x^2}dx$$

Way1：

$$\int\frac{1}{\sec^2 y}dy$$

$$=\int\cos^2 ydy$$

$$=\frac{1}{2}\int(\cos 2y+1)dy$$

$$=\frac{1}{2}(\frac{1}{2}\sin 2y+y)+C=\frac{\sin 2y}{4}+\frac{y}{2}+C$$

＊ $\cos 2x=2\cos^2 x-1$

$$\Rightarrow \cos^2 x=\frac{1}{2}(\cos 2x+1)$$

Way2：分部積分 $\begin{pmatrix} u=\cos y & dv=\cos ydy \\ du=-\sin ydy & v=\sin y \end{pmatrix}$

$$\Rightarrow \frac{\sin 2y}{4}+\frac{y}{2}=\tan^{-1}x+C$$

15. $y^{-1}dy+ye^{\cos x}\sin xdx=0$

Sol.

$$y^{-1}dy+ye^{\cos x}\sin xdx=0$$

$$\Rightarrow y^{-2}dy=-e^{\cos x}\sin xdx$$

$$\Rightarrow \int y^{-2}dy=\int -e^{\cos x}\sin xdx$$

$$\Rightarrow \int y^{-2}dy=\int e^{\cos x}d(\cos x)$$

$$\Rightarrow -\frac{1}{y}=e^{\cos x}+C_1$$

$$\Rightarrow y=\frac{1}{C-e^{\cos x}}, \text{ where } \; C=-C_1 \; \text{(explicit solution)}$$

15.

$$\int -e^{\cos x}\sin xdx \quad \begin{pmatrix} u=\cos x \\ du=-\sin xdx \end{pmatrix}$$

$$=\int e^u du$$

$$=e^u+C_1$$

$$=e^{\cos x}+C_1$$

◇ Solve the initial value problem.

19. $\frac{dy}{dx}=2\sqrt{y+1}\cos x, \quad y(\pi)=0$

Sol.

$$\frac{dy}{dx}=2\sqrt{y+1}\cos x$$

$$\Rightarrow \frac{1}{2}(y+1)^{\frac{-1}{2}}dy=\cos xdx$$

$$\Rightarrow \frac{1}{2}\int (y+1)^{\frac{-1}{2}}dy=\int \cos xdx$$

$$\Rightarrow (y+1)^{\frac{1}{2}}=\sin x+C$$

$$\because \; y(\pi)=0 \Rightarrow (0+1)^{\frac{1}{2}}=\sin\pi+C$$

$$\Rightarrow C=1^{\frac{1}{2}}-0=1$$

$$\Rightarrow (y+1)^{\frac{1}{2}}=\sin x+1$$

$$\Rightarrow y=(\sin x+1)^2-1$$

19.

$$\frac{1}{2}\int (y+1)^{\frac{-1}{2}}dy \quad \begin{pmatrix} u=y+1 \\ du=dy \end{pmatrix}$$

$$=\frac{1}{2}\int u^{\frac{-1}{2}}du$$

$$=\frac{1}{2}(2u^{\frac{1}{2}})+C$$

$$=(y+1)^{\frac{1}{2}}+C$$

26. $\sqrt{y}dx+(1+x)dy=0,\quad y(0)=1$

Sol.

$$\sqrt{y}dx+(1+x)dy=0$$

$$\Rightarrow y^{\frac{1}{2}}dx=-(1+x)dy$$

$$\Rightarrow -(1+x)^{-1}dx=y^{\frac{-1}{2}}dy$$

$$\Rightarrow -\int(1+x)^{-1}dx=\int y^{\frac{-1}{2}}dy$$

$$\Rightarrow -\ln\left|1+x\right|=2y^{\frac{1}{2}}+C$$

$\Rightarrow -\ln(1+x)=2y^{\frac{1}{2}}+C$ (since at initial point, $x=0\Rightarrow x+1>0$)

$\because\ y(0)=1\Rightarrow -\ln 1=2+C\Rightarrow C=-2$

$\therefore\ 2y^{\frac{1}{2}}=-\ln|1+x|-(-2)\Rightarrow y^{\frac{1}{2}}=\dfrac{2-\ln|1+x|}{2}\Rightarrow y=\dfrac{[2-\ln(1+x)]^2}{4}$

29. **Uniqueness Questions.** In Chapter 1 we indicated that in applications most *initial value problems* will have a unique solution. In fact, the existence of unique solutions was so important that we stated an existence and uniqueness theorem, Theorem1, page12. The method for separable equations can give us a solution, but it may not give us all the solutions (also see problem30). To illustrate this, consider the equation $\dfrac{dy}{dx}=y^{1/3}$.

(a) Use the method of separation of variables to show that $y=\left(\dfrac{2x}{3}+C\right)^{3/2}$ is a solution.

Sol.

$$\frac{dy}{dx}=y^{\frac{1}{3}}$$

$$\Rightarrow y^{-\frac{1}{3}}dy=dx$$

$$\Rightarrow \int y^{-\frac{1}{3}}dy=\int dx$$

$$\Rightarrow \frac{3}{2}y^{\frac{2}{3}}=x+C_1$$

$$\Rightarrow y^{\frac{2}{3}}=\frac{2}{3}x+\frac{2}{3}C_1$$

$\Rightarrow y=(\frac{2}{3}x+C)^{\frac{3}{2}}$, where $C=\frac{2}{3}C_1$.

(b) Show that the initial value problem $\frac{dy}{dx}=y^{1/3}$ with $y(0)=0$ is satisfied for $C=0$ by $y=(2x/3)^{3/2}$ for $x\geq 0$.

<u>Sol.</u>

From (a), $y=(\frac{2x}{3}+C)^{\frac{3}{2}}$ is a general solution of $y'=y^{\frac{1}{3}}$.

For $C=0$, $y=(\frac{2}{3}x)^{\frac{3}{2}}$ is also a solution of $y'=y^{\frac{1}{3}}$ and $y(0)=(\frac{2}{3}\cdot 0)^{\frac{3}{2}}=0$

Hence, it satisfies the I.V.P. $y'=y^{\frac{1}{3}}$, $y(0)=0$

(c) Now show that the constant function $y\equiv 0$ also satisfies the initial value problem given in part (b). Hence, this initial value problem does not have a unique solution.

<u>Sol.</u>

Clearly, $y\equiv 0$ satisfies the I.V.P. $y'=y^{\frac{1}{3}}$, $y(0)=0$. Hence, the I.V.P. has multiple solutions.

(d) Finally , show that the conditions of Theorem 1 on page 12 are not satisfied.
(The solution $y\equiv 0$ was lost because of the division by zero in the separation process.)

<u>Sol.</u>

$f(x,y)=y^{\frac{1}{3}}\Rightarrow\frac{\partial}{\partial y}[y^{\frac{1}{3}}]=\frac{1}{3}y^{-\frac{2}{3}}=\frac{1}{3y^{\frac{2}{3}}}, y\neq 0$, $\frac{\partial f}{\partial y}$ is not continuous in any rectangle contains $(0,0)$. Hence, the IVP has no unique solution.

(i.e. IVP 可能有解，但有兩個以上的解，或根本沒有解)

30. As stated in this section, the separation of equation (2) on page 40 requires division by $p(y)=0$, and this may disguise the fact that the roots of the equation $p(y)=0$ are actually constant solutions to the differential equation.

(a) To explore this further, separate the equation $\frac{dy}{dx}=(x-3)(y+1)^{2/3}$ to derive the solution,

$y = -1 + (x^2/6 - x + C)^3$.

Sol.

$$\frac{dy}{dx} = (x-3)(y+1)^{\frac{2}{3}}$$

$$\Rightarrow (y+1)^{\frac{-2}{3}} dy = (x-3)dx$$

$$\Rightarrow \int (y+1)^{\frac{-2}{3}} dy = \int (x-3)dx$$

$$\Rightarrow 3(y+1)^{\frac{1}{3}} = \frac{1}{2}(x-3)^2 + C_1$$

$$\Rightarrow (y+1)^{\frac{1}{3}} = \frac{1}{6}(x-3)^2 + \frac{1}{3}C_1$$

$$= \frac{x^2}{6} - x + C, \text{ where } C = \frac{3}{2} + \frac{1}{3}C_1$$

$$\Rightarrow y = -1 + (\frac{x^2}{6} - x + C)^3$$

(b) Show that $y \equiv -1$ satisfies the original equation $\frac{dy}{dx} = (x-3)(y+1)^{2/3}$

Sol.

For $y \equiv -1$

$\Rightarrow y' = 0$ and $(x-3)(y+1)^{\frac{2}{3}} = 0$

$\Rightarrow y \equiv -1$ is a solution of the original equation.

(c) Show that there is no choice of the constant C that will make the solution in part (a) yield the solution $y \equiv -1$. Thus, we lost the solution $y \equiv -1$ when we divided by $(y+1)^{2/3}$.

Sol.

$\because \frac{x^2}{6} - x + C \neq 0$ for any constant C

$\Rightarrow$ There is no choice of C such that $y = -1 + (\frac{x^2}{6} - x + C)^3 = -1$.

(表示這題在使用變數分離法解題時，會遺漏了$y \equiv -1$這個解)