Section 2.5　Special Integrating Factors

Definition：Integrating Factor
If the equation $M(x,y)dx+N(x,y)dy=0$ is not exact,
but the equation $\mu(x,y)M(x,y)dx+\mu(x,y)N(x,y)dy=0$ is exact,
then $\mu(x,y)$ is called an **integrating factor** of $M(x,y)dx+N(x,y)dy=0$.

【Theorem】Special Integrating Factors

(1) $\mu(x)=\exp[\int \frac{\partial M/\partial y-\partial N/\partial x}{N}dx]$

(2) $\mu(y)=\exp[\int \frac{\partial N/\partial x-\partial M/\partial y}{M}dy]$

◇ Identify the equation as separable, linear, exact, or having an integrating factor that is a function of either x alone or y alone.

6. $(2y^2x-y)dx+xdy=0$

Sol.

(1) The equation is not separable.

(2) The equation is not linear.

$$\frac{dy}{dx}=\frac{y-2y^2x}{x}=\frac{1}{x}y-2y^2 \Rightarrow \frac{dy}{dx}-\underbrace{\frac{1}{x}}_{P(x)}y=\underbrace{-2y^2}_{Q(y)}$$

(3) The equation is not exact.

$$\frac{\partial M}{\partial y}=4yx-1\neq\frac{\partial N}{\partial x}=1$$

(4) Integrating factor depending on y alone.

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{(4yx-1)-1}{x}=\frac{4yx-2}{x}=4y-\frac{2}{x}$$

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{1-(4yx-1)}{2y^2x-y}=\frac{2-4yx}{y(2xy-1)}=\frac{2(1-2yx)}{y(2xy-1)}=-\frac{2}{y}\qquad (y \text{ alone})$$

◇ Solve the equation.

10. $(2y^2+2y+4x^2)dx+(2xy+x)dy=0$

Sol.

Let $M=2y^2+2y+4x^2$, $N=2xy+x$

$$\frac{\partial M}{\partial y}=4y+2\neq\frac{\partial N}{\partial x}=2y+1 \quad \text{(not exact)}$$

Consider $$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{(4y+2)-(2y+1)}{2xy+x}=\frac{2y+1}{x(2y+1)}=\frac{1}{x} \quad (x \text{ alone})$$

Let $\mu(x)=e^{\int\frac{1}{x}dx}=e^{\ln|x|}=|x|$

原式 $\times\mu(x)\Rightarrow(2xy^2+2xy+4x^3)dx+(2x^2y+x^2)dy=0$

Let $F(x,y)=\int(2x^2y+x^2)dy+g(x)=x^2y^2+x^2y+g(x)$

$\because \frac{\partial F}{\partial x}=2xy^2+2xy+g'(x)=2xy^2+2xy+4x^3\Rightarrow g'(x)=4x^3\Rightarrow g(x)=x^4$

$\therefore F(x,y)=x^2y^2+x^2y+x^4=C$

11. $(y^2+2xy)dx-x^2dy=0$

Sol.

Let $M=y^2+2xy$, $N=-x^2$

$$\frac{\partial M}{\partial y}=2y+2x\neq\frac{\partial N}{\partial x}=-2x \quad \text{(not exact)}$$

Consider $$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{(-2x)-(2y+2x)}{y^2+2xy}=\frac{-4x-2y}{y^2+2xy}=\frac{-2(2x+y)}{y(y+2x)}=\frac{-2}{y} \quad (y \text{ alone})$$

(note: $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{(2y+2x)-(-2x)}{-x^2}=\frac{2y+4x}{-x^2}=\frac{-2y}{x^2}-\frac{4}{x}$ is not a function of just x, it's a function of x and y)

Let $\mu(y)=e^{-2\int\frac{1}{y}dx}=e^{-2\ln|y|}=\frac{1}{y^2}$

$\Rightarrow(1+2xy^{-1})dx+(-x^2y^{-2})dy=0$

Let $F(x,y)=\int(1+2xy^{-1})dx+g(y)=x+x^2y^{-1}+g(y)$

$\because \frac{\partial F}{\partial y}=-x^2y^{-2}+g'(y)=-x^2y^{-2}\Rightarrow g'(y)=0\Rightarrow$ take $g(y)=0$

$\therefore F(x,y)=x+x^2y^{-1}=C$ and $y\equiv0$ are solutions.

◇ Find an integrating factor of the form $x^n y^m$ and solve the equation.

13. $(2y^2-6xy)dx+(3xy-4x^2)dy=0$

Sol.

$$\text{原式}\times x^n y^m \Rightarrow (2x^n y^{m+2}-6x^{n+1}y^{m+1})dx+(3x^{n+1}y^{m+1}-4x^{n+2}y^m)dy=0$$

$$\frac{\partial}{\partial y}[2x^n y^{m+2}-6x^{n+1}y^{m+1}]=2(m+2)x^n y^{m+1}-6(m+1)x^{n+1}y^m$$

$$\frac{\partial}{\partial x}[3x^{n+1}y^{m+1}-4x^{n+2}y^m]=3(n+1)x^n y^{m+1}-4(n+2)x^{n+1}y^m$$

$$\Rightarrow \begin{cases}2(m+2)=3(n+1)\\6(m+1)=4(n+2)\end{cases}\Rightarrow\begin{cases}2m+4=3n+3\\6m+6=4n+8\end{cases}\Rightarrow\begin{cases}2m-3n=-1\\6m-4n=2\end{cases}\Rightarrow\begin{cases}m=1\\n=1\end{cases}$$

Hence, the integrating factor is xy

$$(2xy^3-6x^2y^2)dx+(3x^2y^2-4x^3y)dy=0$$

Let $F(x,y)=\int(2xy^3-6x^2y^2)dx+g(y)=x^2y^3-2x^3y^2+g(y)$

$\because$ $\dfrac{\partial F}{\partial y}=3x^2y^2-4x^3y+g'(y)=3x^2y^2-4x^3y \Rightarrow$ take $g(y)=0$

$\therefore$ $F(x,y)=x^2y^3-2x^3y^2=C$ is solution.

15. (a) Show that if $(\partial N/\partial x-\partial M/\partial y)/(xM-yN)$ depends only on the product xy, that is, $\dfrac{\partial N/\partial x-\partial M/\partial y}{xM-yN}=H(xy)$, then the equation $M(x,y)dx+N(x,y)dy=0$ has an integrating factor of the form $\mu(xy)$. Give the general formula for $\mu(xy)$.

Sol.

Let $\mu=\mu(xy)$ such that $\mu'=\mathrm{H}\mu$

$$\Rightarrow \mu'=\mu\cdot\left(\frac{\partial N/\partial x-\partial M/\partial y}{xM-yN}\right)$$

$$\Rightarrow \mu'xM-\mu'yN)=\mu(\partial N/\partial x-\partial M/\partial y)$$

$\because$ $\mu=\mu(xy)\Rightarrow \mu_x=\mu'\cdot y$ and $\mu_y=\mu'\cdot x$

$$\Rightarrow \mu_y M-\mu_x N=\mu(\partial N/\partial x-\partial M/\partial y)$$

$$\Rightarrow \mu_y M+\mu\cdot\partial M/\partial y=\mu\cdot\partial N/\partial x+\mu_x N$$

$$\Rightarrow \frac{\partial}{\partial y}[\mu M]=\frac{\partial}{\partial x}[\mu N]$$

$\Rightarrow \mu Mdx+\mu Ndy=0$ is exact

$\Rightarrow \mu=\mu(xy)$ is the integrating factor of $Mdx+Ndy=0$

$$\boxed{\because\ \mu'=\mathrm{H}\mu\Rightarrow\mu=\exp[\int Hd(xy)]=\exp[\int(\frac{\partial N/\partial x-\partial M/\partial y}{xM-yN})d(xy)]}$$

16. (a) Prove that $Mdx + Ndy = 0$ has an integrating factor that depends only on the sum $x + y$ if and only if the expression $\dfrac{\partial N/\partial x - \partial M/\partial y}{M - N}$ depends only on $x + y$.

Sol.

Let $\mu = \mu(x+y)$ be the integrating factor of $Mdx + Ndy = 0$

$\Leftrightarrow \mu Mdx + \mu Ndy = 0$ is exact.

$$\Leftrightarrow \frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N]$$

$$\Leftrightarrow \mu_y M + \mu \cdot \frac{\partial M}{\partial y} = \mu_x N + \mu \cdot \frac{\partial N}{\partial x}$$

$$\Leftrightarrow \mu' M + \mu \cdot \frac{\partial M}{\partial y} = \mu' N + \mu \cdot \frac{\partial N}{\partial x}$$

$$\Leftrightarrow \mu' = \mu \cdot \frac{\partial N/\partial x - \partial M/\partial y}{M - N}$$

$$\Leftrightarrow \frac{\mu'}{\mu} = \frac{\partial N/\partial x - \partial M/\partial y}{M - N}$$

$\because$ $\mu = \mu(x+y)$ depends only on $x + y \Rightarrow \mu'$ depends only on $x + y$ too

$\Leftrightarrow \dfrac{\mu'}{\mu} = \dfrac{\partial N/\partial x - \partial M/\partial y}{M - N}$ depends only on $x + y$.

(b) Use part (a) to solve the equation $(3 + y + xy)dx + (3 + x + xy)dy = 0$

Sol.

Let $M = 3 + y + xy$, $N = 3 + x + xy$

$$\frac{\partial N/\partial x - \partial M/\partial y}{M - N} = \frac{(1+y) - (1+x)}{(3+y+xy) - (3+x+xy)} = \frac{y - x}{y - x} = 1$$ is a function of $x + y$.

From part (a), the integrating factor $\mu = \mu(x+y)$ satisfies $\dfrac{\mu'}{\mu} = \dfrac{\partial N/\partial x - \partial M/\partial y}{M - N}$

Let $\dfrac{\partial N/\partial x - \partial M/\partial y}{M - N} = G(s)$, where $s = x + y$

$$\Rightarrow \mu'(s) = \mu(s)G(s) \Rightarrow \mu = \exp[\int G(s)ds] = \exp[\int ds] = e^{x+y}$$

原式$\times \mu \Rightarrow (3e^{x+y} + ye^{x+y} + xye^{x+y})dx + (3e^{x+y} + xe^{x+y} + xye^{x+y})dy = 0$

Let $F(x,y) = \int (3e^{x+y} + ye^{x+y} + xye^{x+y})dx + g(y)$

$$= 3e^{x+y} + ye^{x+y} + xye^{x+y} - ye^{x+y} + g(y)$$
$$= 3e^{x+y} + xye^{x+y} + g(y)$$

$\because$ $\dfrac{\partial F}{\partial y} = 3e^{x+y} + x(e^{x+y} + ye^{x+y}) + g'(y) = 3e^{x+y} + xe^{x+y} + xye^{x+y}$

$\Rightarrow g'(y) = 0 \Rightarrow$ take $g(y) = 0$

$\therefore$ $F(x,y) = 3e^{x+y} + xye^{x+y} = C$ is the solution of the equation.