

Chapter 3 Methods of Analysis

Overview

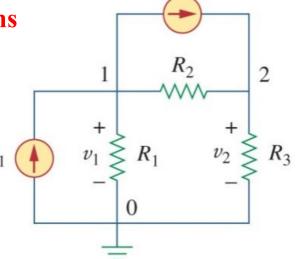
- With Ohm's and Kirchhoff's law established, they may now be applied to circuit analysis.
- Two techniques will be presented in this chapter:
 - Nodal analysis, which is based on Kirchhoff current law (KCL)
 - Mesh analysis, which is based on Kirchhoff voltage law (KVL)
- Any linear circuit can be analyzed using these two techniques.
- The analysis will result in a set of simultaneous equations which may be solved by Cramer's rule or computationally (using MATLAB for example)
- Computational circuit analysis using PSpice will also be introduced here.

3.2 Nodal Analysis

- If instead of focusing on the voltages of the circuit elements, one looks at the voltages at the nodes of the circuit, the number of simultaneous equations to solve for can be reduced.
- Given a circuit with n nodes, without voltage sources, the nodal analysis is accomplished via three steps:
 - 1. Select a node as the reference node. Assign voltages $V_1, V_2, ..., V_n$ to the remaining n-1 nodes, voltages are relative to the reference node.
 - 2. Apply KCL to each of the n-1 non-reference nodes. Use Ohm's law to express the branch currents in terms of node voltages I_2
 - 3. Solve the resulting *n*-1 simultaneous equations to obtain the unknown node voltages.
- The reference, or datum, node is commonly referred to as the ground since its voltage is by default zero.

Current flows from a higher potential to a lower potential in a resistor.

$$i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R}$$



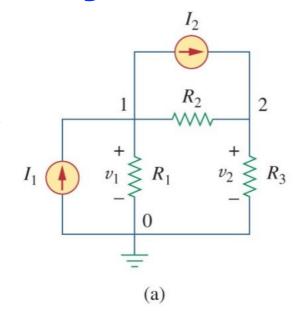
Applying Nodal Analysis

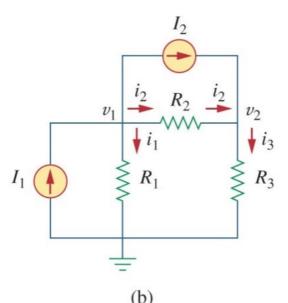
- Let's apply nodal analysis to this circuit to see how it works.
- This circuit has a node that is designed as ground. We will use that as the reference node (node 0)
- The remaining two nodes are designed 1 and 2 and assigned voltages v₁ and v₂.
- Now apply KCL to each node:
- At node 1

$$I_1 = I_2 + i_1 + i_2$$

At node 2

$$I_2 + i_2 = i_3$$



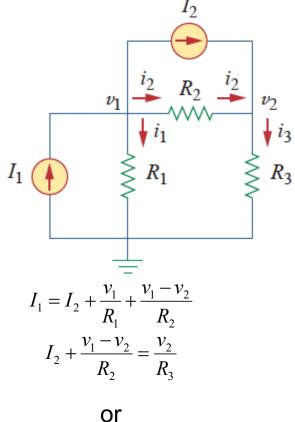


Apply Nodal Analysis II

- We can now use OHM's law to express the unknown currents i_1 , i_2 , and i_3 in terms of node voltages.
- In doing so, keep in mind that current flows from high potential to low
- From this we get:

$$i_1 = \frac{v_1 - 0}{R_1}$$
 or $i_1 = G_1 v_1$
 $i_2 = \frac{v_1 - v_2}{R_2}$ or $i_2 = G_2 (v_1 - v_2)$
 $i_3 = \frac{v_2 - 0}{R_3}$ or $i_3 = G_3 v_2$

$$I_1 = I_2 + i_1 + i_2$$
 $I_2 + i_2 = i_3$
Substituting back into the node equations



$$I_1 = I_2 + G_1 v_1 + G_2 (v_1 - v_2)$$
$$I_2 + G_2 (v_1 - v_2) = G_3 v_2$$

The last step is to solve the system of equations: $v_1 = ?$ and $v_2 = ?$

Example 3.1 Calculate the node voltages in the circuit

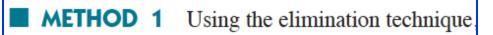
At node 1, applying KCL and Ohm's law gives

$$i_1 = i_2 + i_3 \qquad \Rightarrow \qquad 5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2}$$

$$\Rightarrow \qquad 3v_1 - v_2 = 20 \qquad 2\Omega \lessgtr \qquad 6\Omega$$

At node 2, we do the same thing and get

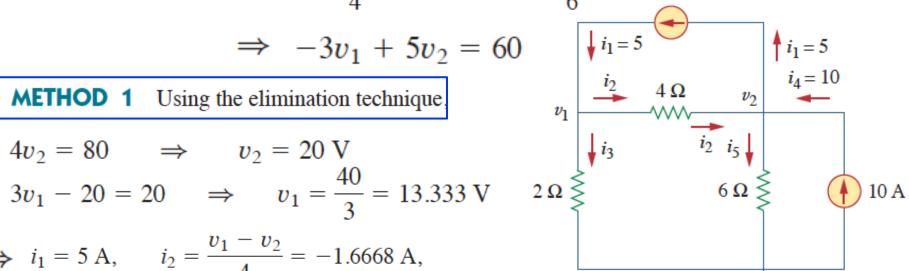
$$i_2 + i_4 = i_1 + i_5$$
 $\Rightarrow \frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$ 5A



$$4v_2 = 80 \qquad \Rightarrow \qquad v_2 = 20 \text{ V}$$

$$\implies i_1 = 5 \text{ A}, \qquad i_2 = \frac{v_1 - v_2}{4} = -1.6668 \text{ A},$$

$$i_3 = \frac{v_1}{2} = 6.666 \text{ A}$$
 $i_4 = 10 \text{ A}$, $i_5 = \frac{v_2}{6} = 3.333 \text{ A}$



5 A

METHOD 2 To use Cramer's rule,

$$3v_1 - v_2 = 20$$
$$-3v_1 + 5v_2 = 60$$

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

$$i_1 = 5 \text{ A}, \qquad i_2 = \frac{v_1 - v_2}{4} = -1.6668 \text{ A},$$

$$\Delta = \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = 15 - 3 = 12$$

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{\Delta} = \frac{100 + 60}{12} = 13.333 \text{ V}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{\Delta} = \frac{180 + 60}{12} = 20 \text{ V}$$

Example 3.2

Determine the voltages at the nodes

At node 1,

$$3 = i_1 + i_x \implies$$

$$3 = \frac{v_1 - v_3}{4} + \frac{v_1 - v_2}{2}$$

$$\Rightarrow$$

$$3v_1 - 2v_2 - v_3 = 12$$

At node 2,

$$i_x = i_2 + i_3 \implies$$

$$\frac{v_1 - v_2}{2} = \frac{v_2 - v_3}{8} + \frac{v_2 - 0}{4}$$

$$\Rightarrow$$

$$-4v_1 + 7v_2 - v_3 = 0$$

At node 3,

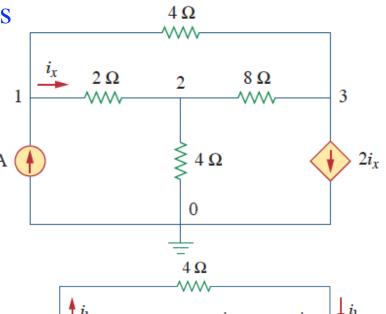
$$i_1 + i_2 = 2i_x \implies$$

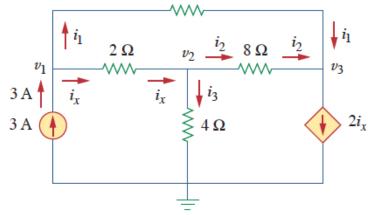
$$i_1 + i_2 = 2i_x$$
 \Rightarrow $\frac{v_1 - v_3}{4} + \frac{v_2 - v_3}{8} = \frac{2(v_1 - v_2)}{2}$

$$\Rightarrow$$

$$2v_1 - 3v_2 + v_3 = 0$$

$$2+3 \Rightarrow v_1 = 2v_2$$





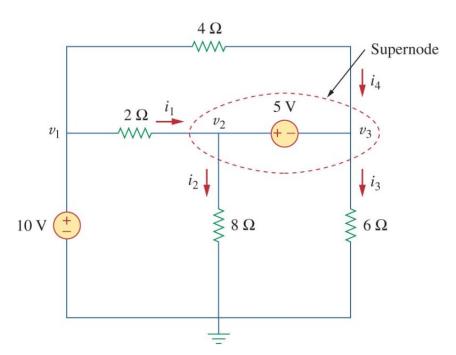
$$v_2 = 2.4 \text{ V}$$

 $v_1 = 4.8 \text{ V}$ $\Rightarrow v_3 = -2.4 \text{ V}$

3.3

Nodal Analysis with Voltage Sources

- Depending on what nodes the source is connected to, the approach varies
- Between the reference node and a non-reference mode:
 - Set the voltage at the nonreference node to the voltage of the source
 - In the example circuit $v_1=10V$
- Between two non-reference nodes
 - The two nodes form a supernode.



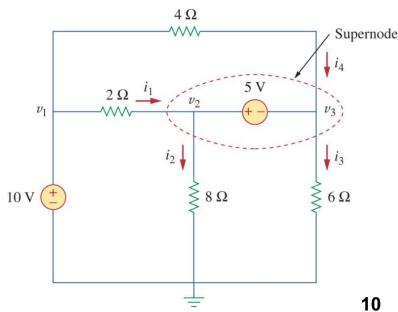
Supernode

- A supernode is formed by enclosing a voltage source (dependent or independent) connected between two non-reference nodes and any elements connected in parallel with it.
- Why?
 - Nodal analysis requires applying KCL
 - The current through the voltage source cannot be known in advance (Ohm's law does not apply)
 - By lumping the nodes together, the current balance can still be described
- In the example circuit node 2 and 3 form a supernode
- The current balance would be:

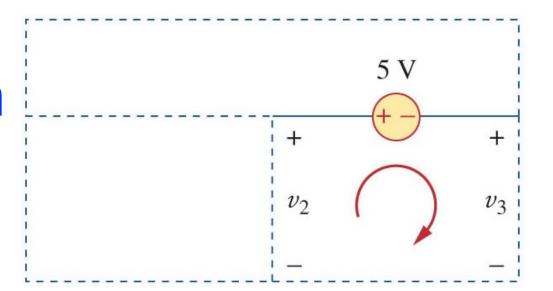
$$i_1 + i_4 = i_2 + i_3$$

Or this can be expressed as:

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6}$$



Analysis with a supernode



- In order to apply KVL to the supernode in the example, the circuit is redrawn as shown.
- Going around this loop in the clockwise direction gives:

$$-v_2 + 5 + v_3 = 0 \implies v_2 - v_3 = 5$$

- Note the following properties of a supernode:
 - 1. The voltage source inside the supernode provides a constraint equation needed to solve for the node voltages
 - 2. A supernode has no voltage of its own
 - 3. A supernode requires the application of both KCL and KVL

Example 3.3 Find the node voltages

Applying KCL to the supernode

$$\Rightarrow$$
 2 = $i_1 + i_2 + 7$

$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7 \qquad \Rightarrow \qquad 8 = 2v_1 + v_2 + 28$$

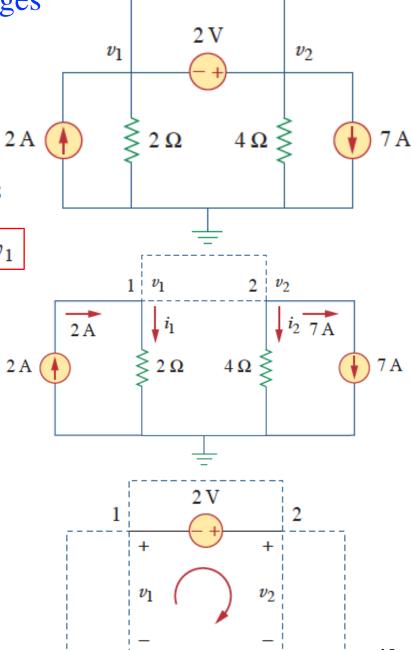
$$\Rightarrow$$
 $v_2 = -20 - 2v_1$

apply KVL

$$-v_1 - 2 + v_2 = 0 \implies v_2 = v_1 + 2$$
 2A

$$\Rightarrow v_1 = -7.333 \text{ V}$$

 $v_2 = v_1 + 2 = -5.333 \text{ V}$



 10Ω

Example 3.4 Find the node voltages in the circuit

Figure 3.12

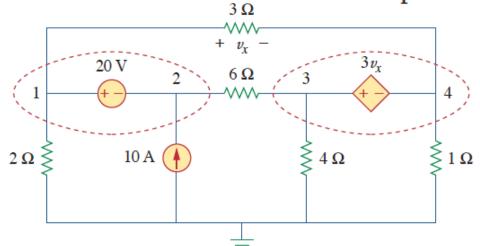
For Example 3.4.

At supernode 1-2,

$$i_3 + 10 = i_1 + i_2$$

$$\Rightarrow \frac{v_3 - v_2}{6} + 10 = \frac{v_1 - v_4}{3} + \frac{v_1}{2}$$

$$\Rightarrow 5v_1 + v_2 - v_3 - 2v_4 = 60$$

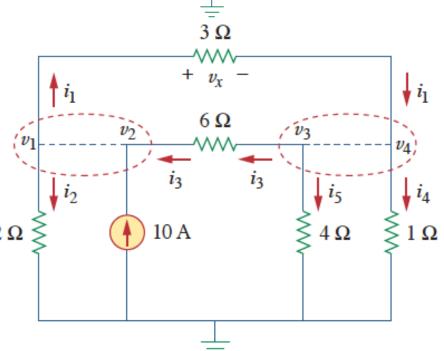


At supernode 3-4,

$$i_1 = i_3 + i_4 + i_5$$

$$\Rightarrow \frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + \frac{v_4}{1} + \frac{v_3}{4}$$

$$\Rightarrow |4v_1 + 2v_2 - 5v_3 - 16v_4 = 0|$$



Example 3.4 Find the node voltages in the circuit

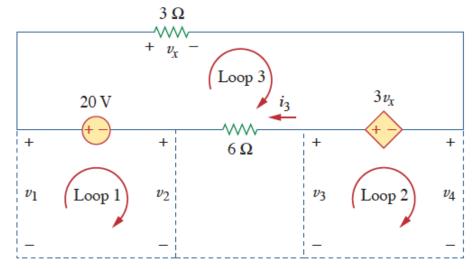
$$5v_1 + v_2 - v_3 - 2v_4 = 60$$

$$4v_1 + 2v_2 - 5v_3 - 16v_4 = 0$$
2

apply KVL to the branches

For loop 1,
$$-v_1 + 20 + v_2 = 0$$

 $\Rightarrow v_1 - v_2 = 20$



For loop 2,
$$-v_3 + 3v_x + v_4 = 0$$

But
$$v_x = v_1 - v_4 \implies 3v_1 - v_3 - 2v_4 = 0$$

For loop 3,
$$v_x - 3v_x + 6i_3 - 20 = 0$$

But
$$6i_3 = v_3 - v_2$$
 and $v_x = v_1 - v_4$. $\Rightarrow -2v_1 - v_2 + v_3 + 2v_4 = 20$

$$v_2 = v_1 - 20.$$

$$3 \Rightarrow 3v_1 - v_3 - 2v_4 = 0$$

$$1 \Rightarrow 6v_1 - v_3 - 2v_4 = 80 \Rightarrow$$

$$2 \Rightarrow 6v_1 - 5v_3 - 16v_4 = 40$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & -2 \\ 6 & -1 & -2 \\ 6 & -5 & -16 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 40 \end{bmatrix}$$

Practice Problem 3.4

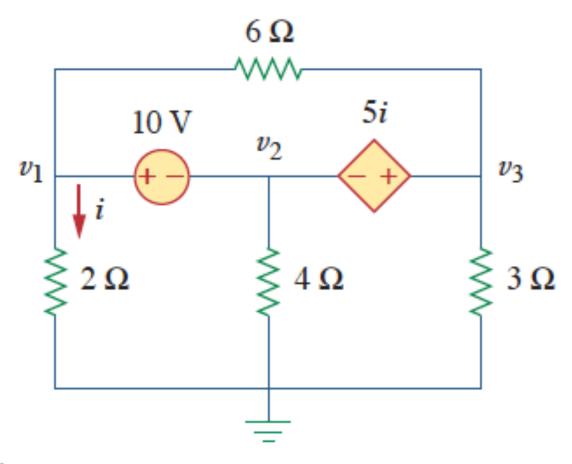
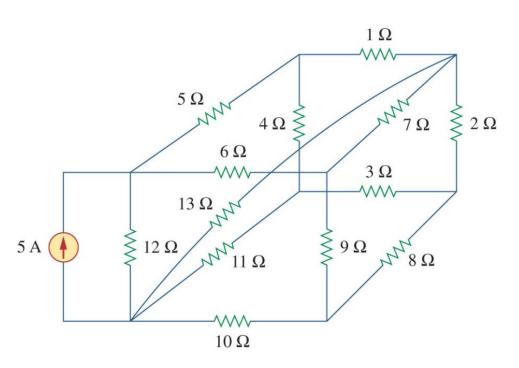


Figure 3.14

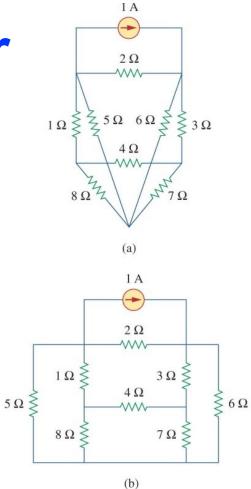
3.4 Mesh Analysis

- Another general procedure for analyzing circuits is to use the mesh currents as the circuit variables.
- Recall:
 - A loop is a closed path with no node passed more than once
 - A mesh is a loop that does not contain any other loop within it
- Mesh analysis uses KVL to find unknown currents
- Mesh analysis is limited in one aspect:
 - It can only apply to circuits that can be rendered planar.
- A planar circuit can be drawn such that there are no crossing branches.

Planar vs Nonplanar



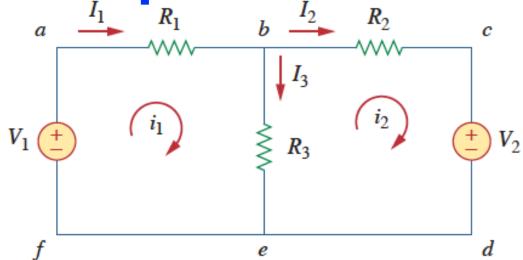
The figure on the left is a nonplanar circuit: The branch with the 13Ω resistor prevents the circuit from being drawn without crossing branches



The figure on the right is a planar circuit: It can be redrawn to avoid crossing branches

Mesh Analysis Steps

Mesh analysis follows these steps:

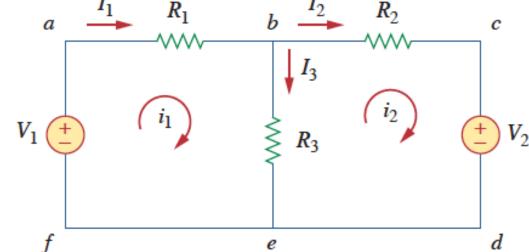


- 1. Assign mesh currents i_1 , i_2 , ... i_n to the n meshes
- 2. Apply KVL to each of the *n* mesh currents.
- 3. Solve the resulting *n* simultaneous equations to get the mesh currents

Mesh Analysis

Example ^a

 The above circuit has two paths that are meshes: (abefa and bcdeb)



- The outer loop (abcdefa) is a loop, but not a mesh
- First, mesh currents i₁ and i₂ are assigned to the two meshes.
- Applying KVL to the meshes: $-V_1 + R_1i_1 + R_3(i_1 i_2) = 0$ $R_2i_2 + V_2 + R_3(i_2 i_1) = 0$

$$(R_1 + R_3)i_1 - R_3i_2 = V_1$$
 $-R_3i_1 + (R_2 + R_3)i_2 = -V_2$

Example 3.5

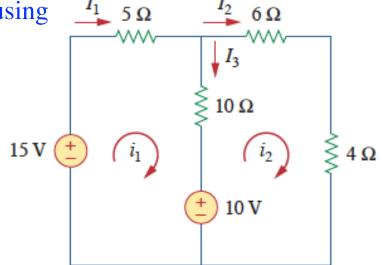
Find the branch currents and using I_1 5Ω I_2 6Ω mesh analysis.

For mesh 1,
$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

$$\Rightarrow 3i_1 - 2i_2 = 1$$

For mesh 2,
$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0$$

$$\Rightarrow i_1 = 2i_2 - 1$$



METHOD 1 Using the substitution method,

$$6i_2 - 3 - 2i_2 = 1 \implies i_2 = 1 \text{ A}$$

 $i_1 = 2i_2 - 1 = 2 - 1 = 1 \text{ A}.$

$$I_3 = i_1 - i_2 = 0$$

Figure 3.18

$$\Delta = \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = 6 - 2 = 4$$

$$\Delta_1 = \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} = 2 + 2 = 4,$$

$$\Delta_2 = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = 3 + 1 = 4$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$i_1 = \frac{\Delta_1}{\Lambda} = 1 \text{ A},$$

$$i_2 = \frac{\Delta_2}{\Delta} = 1 \text{ A}$$

$$I_3 = i_1 - i_2 = 0$$

Example 3.6 Use mesh analysis to find the current I_0 $\frac{l_1}{l_2}$ $\frac{l_2}{l_3}$

For mesh 1,
$$-24 + 10(i_1 - i_2) + 12(i_1 - i_3) = 0$$

$$\Rightarrow 11i_1 - 5i_2 - 6i_3 = 12$$

For mesh 2,
$$24i_2 + 4(i_2 - i_3) + 10(i_2 - i_1) = 0$$

$$\Rightarrow -5i_1 + 19i_2 - 2i_3 = 0$$

For mesh 3,
$$4I_o + 12(i_3 - i_1) + 4(i_3 - i_2) = 0$$

 $I_o = i_1 - i_2$,

$$\Rightarrow 4(i_1 - i_2) + 12(i_3 - i_1) + 4(i_3 - i_2) = 0$$

$$\Rightarrow -i_1 - i_2 + 2i_3 = 0$$

$$\begin{bmatrix} 11 & -5 & -6 \\ -5 & 19 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} \implies \begin{cases} \Delta = 192 \\ \Delta_1 = 432 \\ \Delta_2 = 144 \\ \Delta_3 = 288 \end{cases} \qquad i_1 = \frac{\Delta_1}{\Delta} = \frac{432}{192} = 2.25 \text{ A},$$

$$i_2 = \frac{\Delta_2}{\Delta} = \frac{144}{192} = 0.75 \text{ A},$$

$$\Delta_3 = 288$$

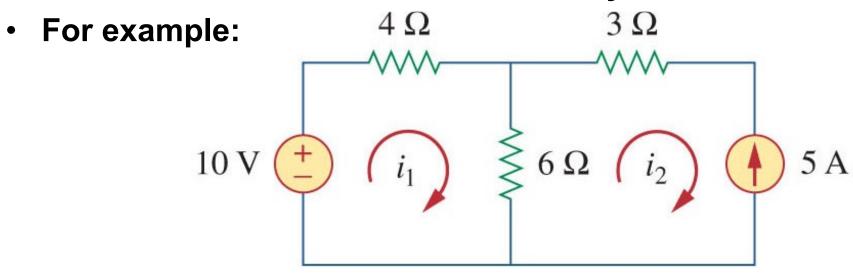
$$i_1 = \frac{\Delta_1}{\Delta} = \frac{432}{192} = 2.25 \text{ A},$$
 $i_2 = \frac{\Delta_2}{\Delta} = \frac{144}{192} = 0.75 \text{ A},$
 $i_3 = \frac{\Delta_3}{\Delta} = \frac{288}{192} = 1.5 \text{ A}$

Figure 3.20

$$I_o = i_1 - i_2 = 1.5 \text{ A}.$$

3.5 Mesh Analysis with Current Sources

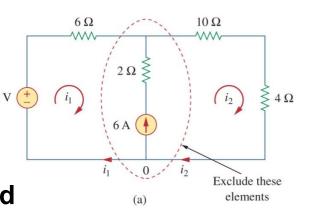
- The presence of a current source makes the mesh analysis simpler in that it reduces the number of equations.
- If the current source is located on only one mesh, the current for that mesh is defined by the source.

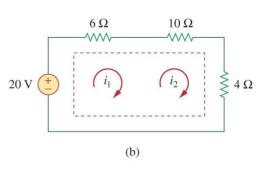


Here, the current i₂ is equal to -5A

Supermesh

Similar to the case of nodal analysis where a voltage source shared

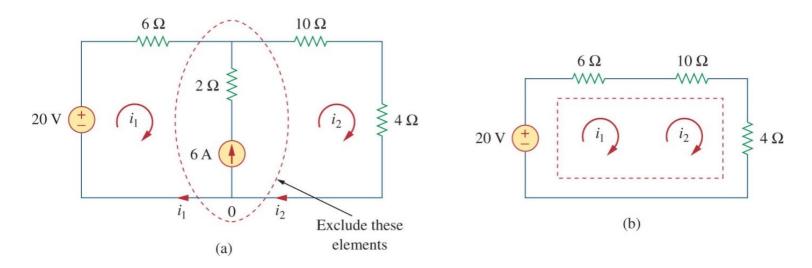




two non-reference nodes, current sources (dependent or independent) that are shared by more than one mesh need special treatment

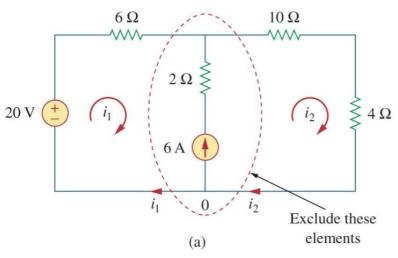
- The two meshes must be joined together, resulting in a supermesh.
- The supermesh is constructed by merging the two meshes and excluding the shared source and any elements in series with it
- A supermesh is required because mesh analysis uses KVL
- But the voltage across a current source cannot be known in advance.
- Intersecting supermeshes in a circuit must be combined to for a larger supermesh.

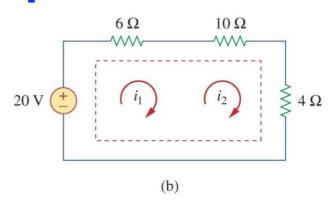
Creating a Supermesh



- In this example, a 6A current course is shared between mesh 1 and 2.
- The supermesh is formed by merging the two meshes.
- The current source and the 2Ω resistor in series with it are removed.

Supermesh Example





Apply KVL to the supermesh

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$
 or $6i_1 + 14i_2 = 20$

 We next apply KCL to the node in the branch where the two meshes intersect.

$$i_2 = i_1 + 6$$

Solving these two equations we get:

$$i_1 = -3.2A$$
 $i_2 = 2.8A$

Note that the supermesh required using both KVL and KCL

Example 3.7 find i_1 to i_4 using mesh analysis.

Applying KVL to the larger supermesh,

$$2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0$$

$$\Rightarrow i_1 + 3i_2 + 6i_3 - 4i_4 = 0$$

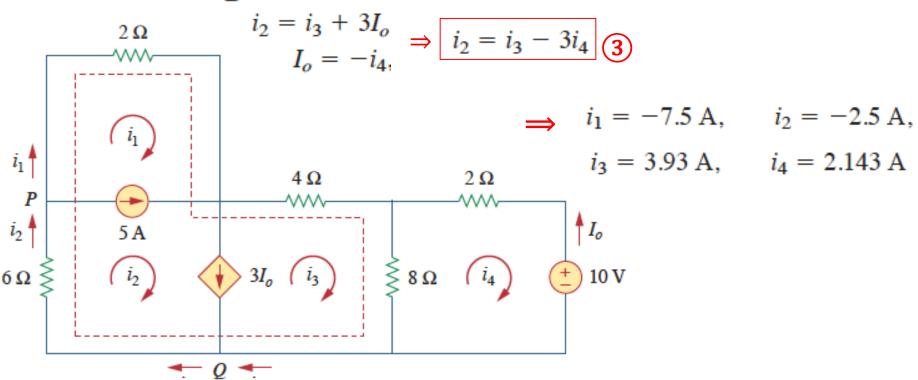
KCL to node *P*:
$$i_2 = i_1 + 5$$
 2

Applying KVL in mesh 4,

$$2i_4 + 8(i_4 - i_3) + 10 = 0$$

$$\Rightarrow 5i_4 - 4i_3 = -5$$

KCL to node Q:



3.38 Apply mesh analysis to the circuit in Fig. 3.84 and

obtain I_o .

We need 4 independent equations.

From Mesh 1:

$$i_1 = -5 \text{ A}$$

From Mesh 2:

$$1 \cdot (\mathbf{i}_2 - \mathbf{i}_1) + 2 \cdot (\mathbf{i}_2 - \mathbf{i}_4) + 22.5 + 4 \cdot \mathbf{i}_2 = 0$$

$$\Rightarrow 7 \cdot \mathbf{i}_2 - \mathbf{i}_4 = -27.5$$

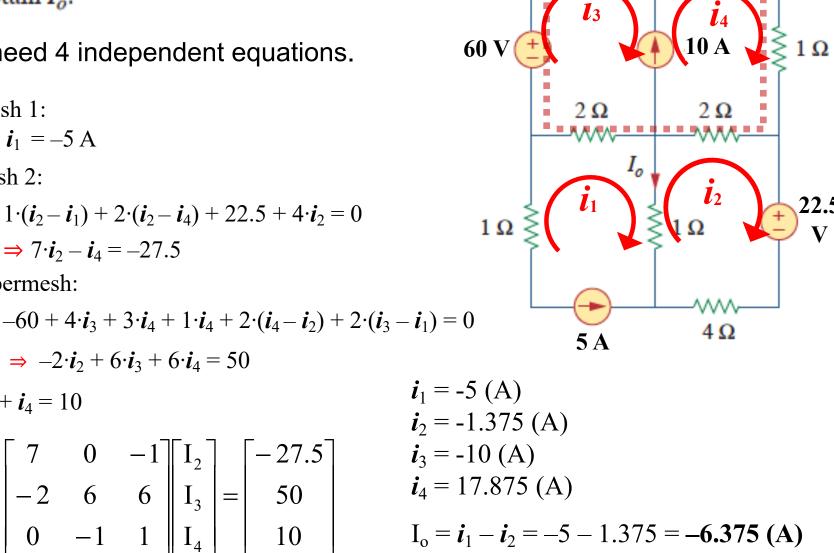
From Supermesh:

$$-60 + 4 \cdot \mathbf{i}_3 + 3 \cdot \mathbf{i}_4 + 1 \cdot \mathbf{i}_4 + 2 \cdot (\mathbf{i}_4 - \mathbf{i}_2) + 2 \cdot (\mathbf{i}_3 - \mathbf{i}_1) = 0$$

$$\Rightarrow -2 \cdot \mathbf{i}_2 + 6 \cdot \mathbf{i}_3 + 6 \cdot \mathbf{i}_4 = 50$$

And: $-i_3 + i_4 = 10$

$$\begin{bmatrix} 7 & 0 & -1 \\ -2 & 6 & 6 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} -27.5 \\ 50 \\ 10 \end{bmatrix}$$



 3Ω

3.6

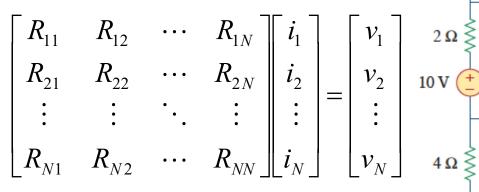
[†]Nodal and Mesh Analyses by Inspection

Mesh Analysis

There is a similarly fast way to construct a matrix for solving a circuit by mesh analysis

It requires that all voltage sources within the circuit be independent

In general, for a circuit with N meshes, the mesh-current equations may be written as:



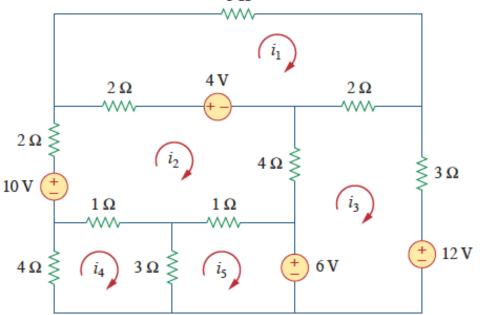


Figure 3.29

Each diagonal term on the resistance matrix is the sum of resistances in the mesh indicated by the matrix index

Mesh Analysis by Inspection II

- The off diagonal terms, R_{ik} are the negative of the sum of all resistances in common with meshes j and k with j≠k.
- The unknown mesh currents in the clockwise direction are denoted as ik
- The sum taken clockwise of all voltage sources in

mesh k are denoted as v_k . Voltage rises are treated

as positive.

 This matrix equation can be solved for the values of the unknown mesh currents

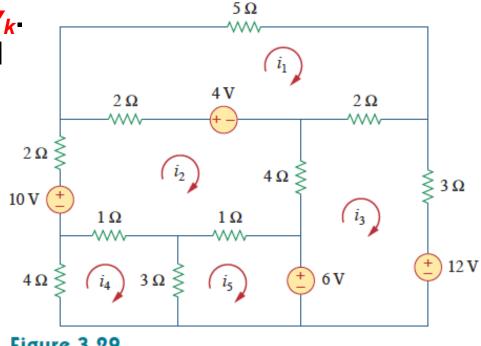


Figure 3.29

$$\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1N} \\ R_{21} & R_{22} & \cdots & R_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N1} & R_{N2} & \cdots & R_{NN} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}$$

diagonal

$$R_{11} = 5 + 2 + 2 = 9,$$
 $R_{22} = 2 + 4 + 1 + 1 + 2 = 1$
 $R_{33} = 2 + 3 + 4 = 9,$ $R_{44} = 1 + 3 + 4 = 8,$

2Ω 2Ω $2\Omega \lesssim$ 10 V 1Ω $R_{11} = 5 + 2 + 2 = 9$, $R_{22} = 2 + 4 + 1 + 1 + 2 = 10$, 12 V $R_{55} = 1 + 3 = 4$ Figure 3.29

off diagonal R_{jk}

$$R_{12} = -2$$
, $R_{13} = -2$, $R_{14} = 0 = R_{15}$, $R_{21} = -2$, $R_{23} = -4$, $R_{24} = -1$, $R_{25} = -1$, $R_{31} = -2$, $R_{32} = -4$, $R_{34} = 0 = R_{35}$, $R_{41} = 0$, $R_{42} = -1$, $R_{43} = 0$, $R_{45} = -3$, $R_{51} = 0$, $R_{52} = -1$, $R_{53} = 0$, $R_{54} = -3$

Voltage Vector

$$v_1 = 4$$
, $v_2 = 10 - 4 = 6$,
 $v_3 = -12 + 6 = -6$, $v_4 = 0$, $v_5 = -6$

Mesh Current Equations

 5Ω

$$\begin{bmatrix} 9 & -2 & -2 & 0 & 0 \\ -2 & 10 & -4 & -1 & -1 \\ -2 & -4 & 9 & 0 & 0 \\ 0 & -1 & 0 & 8 & -3 \\ 0 & -1 & 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -6 \\ 0 \\ -6 \end{bmatrix}$$

Selecting an Appropriate Approach

- In principle both the nodal analysis and mesh analysis are useful for any given circuit.
- What then determines if one is going to be more efficient for solving a circuit problem?
- There are two factors that dictate the best choice:
 - ✓ The nature of the particular network is the first factor
 - ✓ The second factor is the information required

Mesh analysis when...

- If the network contains:
 - √ Many series connected elements
 - √ Voltage sources
 - ✓ Supermeshes
 - ✓ A circuit with fewer meshes than nodes
- If branch or mesh currents are what is being solved for.
- Mesh analysis is the only suitable analysis for transistor circuits
- It is not appropriate for operational amplifiers because there is no direct way to obtain the voltage across an op-amp.

Nodal analysis if...

- If the network contains:
 - ✓ Many parallel connected elements
 - ✓ Current sources
 - ✓ Supernodes
 - ✓ Circuits with fewer nodes than meshes
- If node voltages are what are being solved for
- Non-planar circuits can only be solved using nodal analysis
- This format is easier to solve by computer