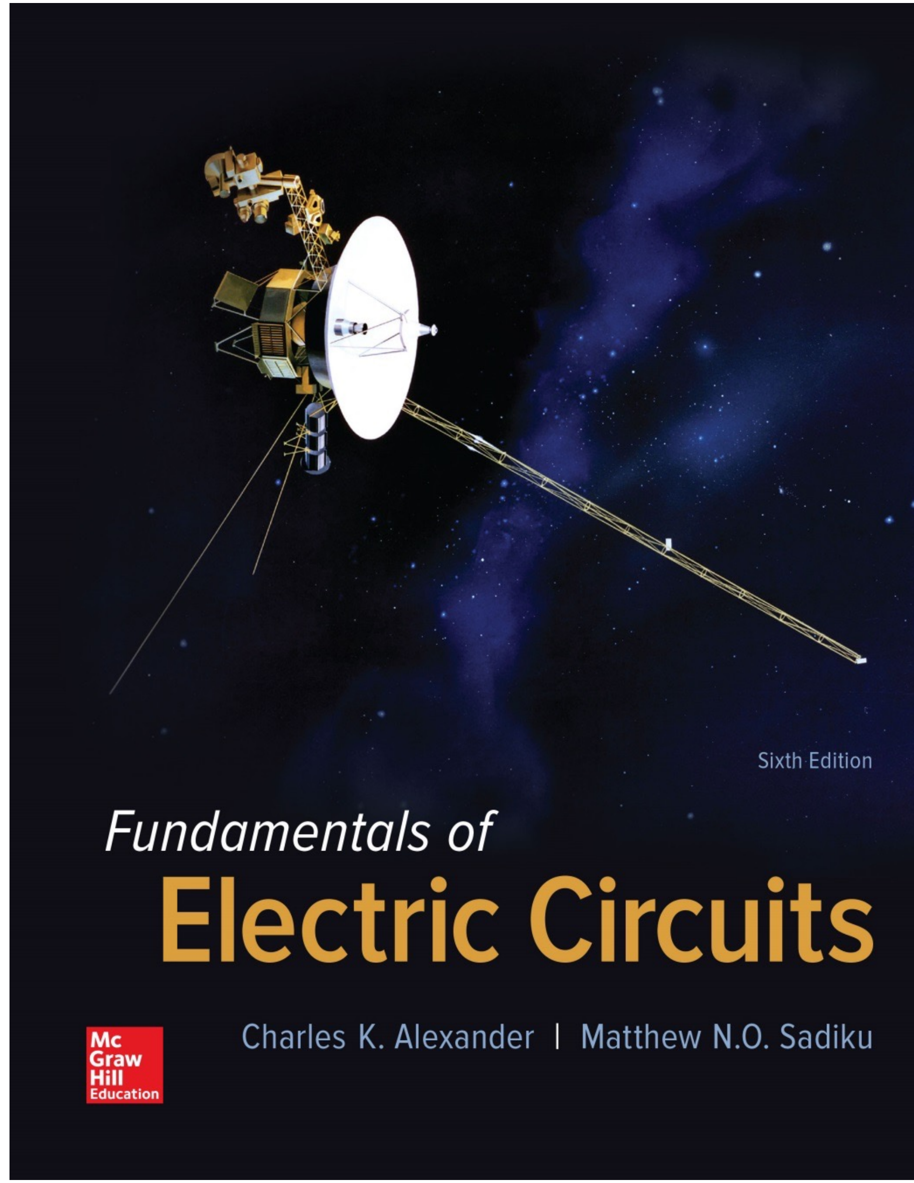
Fundamentals of
Electric Circuits
Chapter 4

Circuit Theorems



4.1 Introduction

- In this chapter, the concept of **superposition** will be introduced.
- **Source transformation** will also be covered.
- **Thevenin** and **Norton's** theorems will be covered.
- Examples of applications for these concepts will be presented.

4.2 Linearity Property

- Linearity in a circuit means that as current is changed, the voltage changes proportionally
- It also requires that the response of a circuit to a sum of sources will be the sum of the individual responses from each source separately
- A resistor satisfies both of these criteria

Linearity

current is increased by a constant k.

$$v = iR \longrightarrow kiR = kv$$

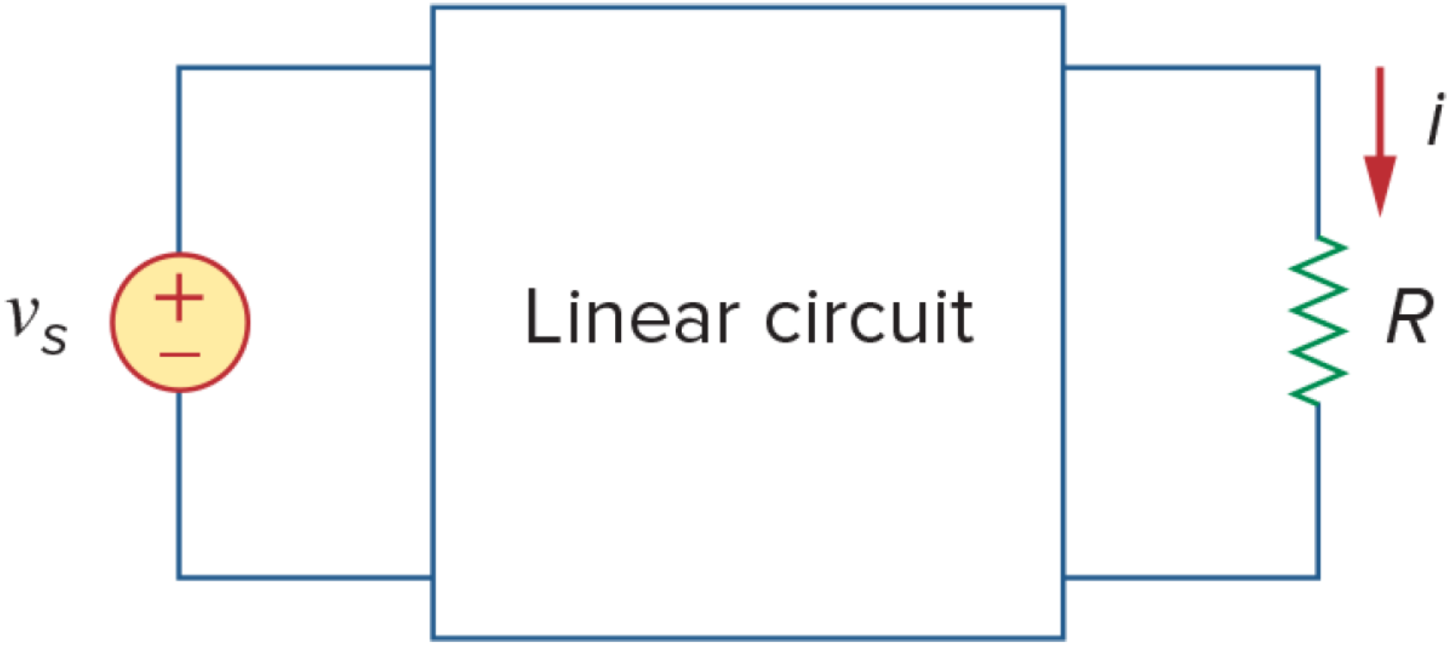
additivity property

$$\begin{aligned} v_1 &= i_1 R \\ v_2 &= i_2 R \end{aligned} \xrightarrow{(i_1 + i_2)} \begin{aligned} v &= (i_1 + i_2)R \\ &= i_1 R + i_2 R \\ &= v_1 + v_2 \end{aligned}$$

$$\begin{aligned} p_1 &= Ri_1^2 \\ p_2 &= Ri_2^2 \end{aligned} \qquad \begin{aligned} p_3 &= R(i_1 + i_2)^2 \\ &= Ri_1^2 + Ri_2^2 + 2Ri_1 i_2 \\ &\neq p_1 + p_2 \end{aligned}$$

The Power relationship is nonlinear.

A linear circuit



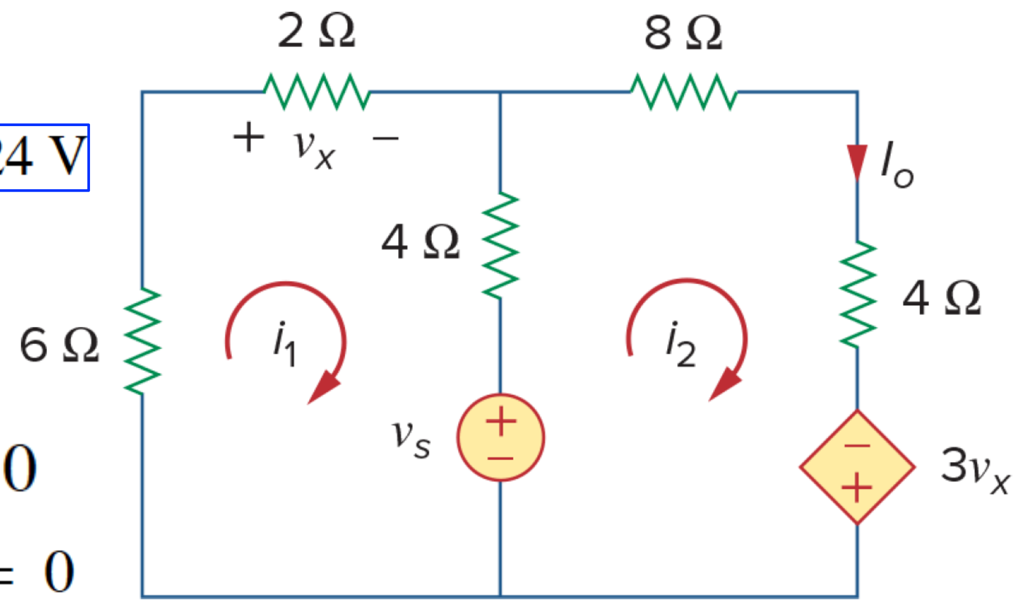
Suppose $v_s = 10$ V gives $i = 2$ A.

$v_s = 1$ V will give $i = 0.2$ A

$i = 1$ mA must be due to $v_s = 5$ mV

Example 4.1

$\boxed{\text{find } I_o \text{ when } v_s = 12 \text{ V and } v_s = 24 \text{ V}}$



KVL to the two loops

$$12i_1 - 4i_2 + v_s = 0$$

$$-4i_1 + 16i_2 - 3v_x - v_s = 0$$

$$\xrightarrow{v_x = 2i_1} -10i_1 + 16i_2 - v_s = 0$$

$$\Rightarrow i_1 = -6i_2$$

$$12i_1 - 4i_2 + v_s = 0 \Rightarrow i_2 = \frac{v_s}{76}$$

$$\boxed{v_s = 12 \text{ V}} \Rightarrow I_o = i_2 = \frac{12}{76} \text{A}$$

$$\boxed{v_s = 24 \text{ V}} \Rightarrow I_o = i_2 = \frac{24}{76} \text{A}$$

when the source value is doubled,

$\longrightarrow$ I_o doubles.

4.3 Superposition

- If there are two or more independent sources there are two ways to solve for the circuit parameters:
  - Nodal or Mesh analysis
  - Use superposition
- Superposition
  - voltage across (or current through) an element in a linear circuit
    = the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

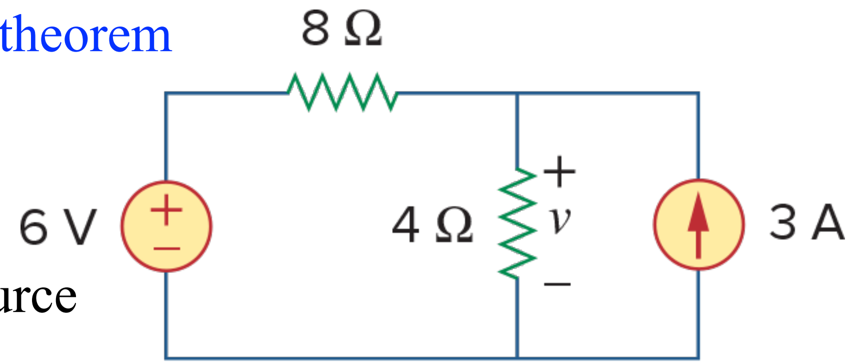
Applying Superposition

- Superposition
  - applying one independent source at a time
- Dependent sources are left alone
- The steps are:

1. Turn off all independent sources except one source.
   - Find the output (voltage or current) due to that active source using the techniques covered in Chapters 2 and 3.
2. Repeat step 1 for each of the other independent sources.
3. Find the total contribution by adding algebraically all the contributions due to the independent sources.

Example 4.3

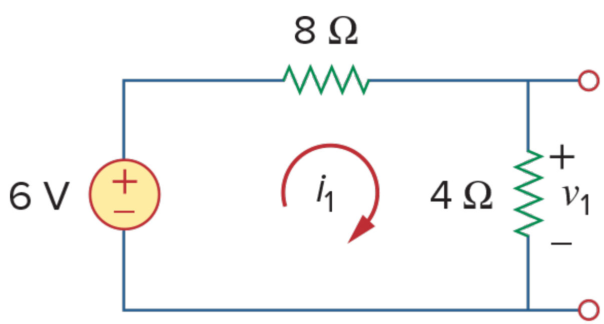
Use the superposition theorem to find v in the circuit



Since there are two sources:

$v = v_1 + v_2$ v_1 due to the 6-V voltage source

v_2 due to the 3-A current source

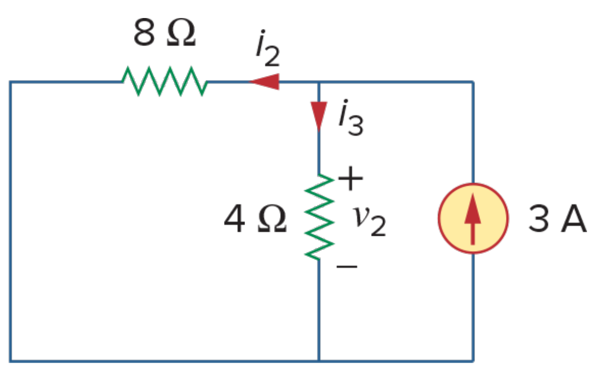


set the current source to zero

Applying KVL: $12i_1 - 6 = 0 \implies i_1 = 0.5 \text{ A}$

$\implies v_1 = 4i_1 = 2 \text{ V}$

or voltage division to get: $v_1 = \dfrac{4}{4+8}(6) = 2 \text{ V}$



set the voltage source to zero

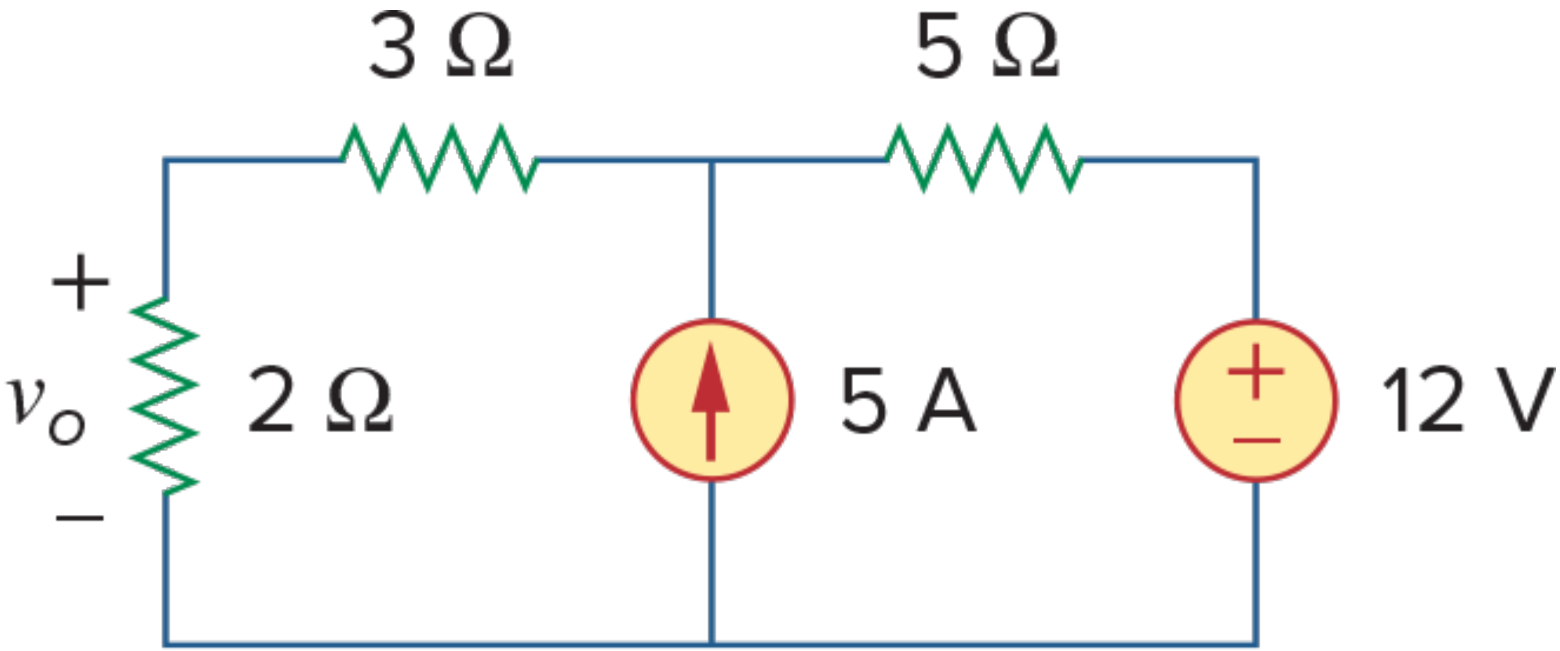
Using current division: $i_3 = \dfrac{8}{4+8}(3) = 2 \text{ A}$

$\implies v_2 = 4i_3 = 8 \text{ V}$

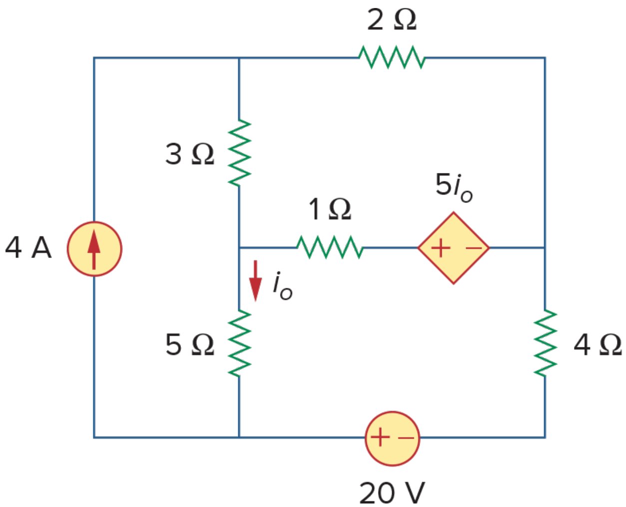
$$\boxed{v = v_1 + v_2 = 2 + 8 = 10 \text{ V}}$$

Practice Problem 4.3

Using the superposition theorem, find v_1 in the circuit

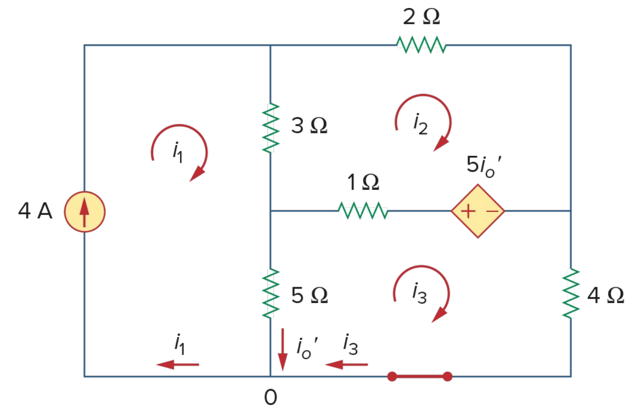


Example 4.4 Find i_o using superposition



$$i_o = i'_o + i''_o$$

i'_o due to the 4-A current source



$\boxed{\text{For loop 1,}} \Longrightarrow i_1 = 4\text{ A}$

$\boxed{\text{For loop 2,}}$

$$-3i_1 + 6i_2 - 1i_3 - 5i'_o = 0$$

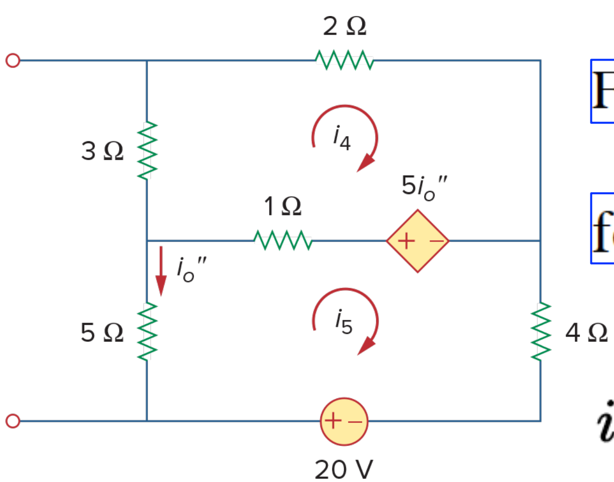
$\boxed{\text{For loop 3,}}$

$$-5i_1 - 1i_2 + 10i_3 + 5i'_o = 0$$

$\boxed{\text{node 0,}} \Longrightarrow i_3 = i_1 - i'_o = 4 - i'_o$

$$\Longrightarrow \begin{matrix} 3i_2 - 2i'_o = 8 \\ i_2 + 5i'_o = 20 \end{matrix} \Longrightarrow \boxed{i'_o = \frac{52}{17}\text{ A}}$$

$$i_o = i'_o + i''_o$$
$$= -\frac{8}{17} = -0.4706\text{ A}$$



$\boxed{\text{For loop 4,}}$

$$6i_4 - i_5 - 5i''_o = 0$$

$\boxed{\text{for loop 5,}}$ Be careful!

$$-i_4 + 10i_5 - 20 + 5i''_o = 0$$

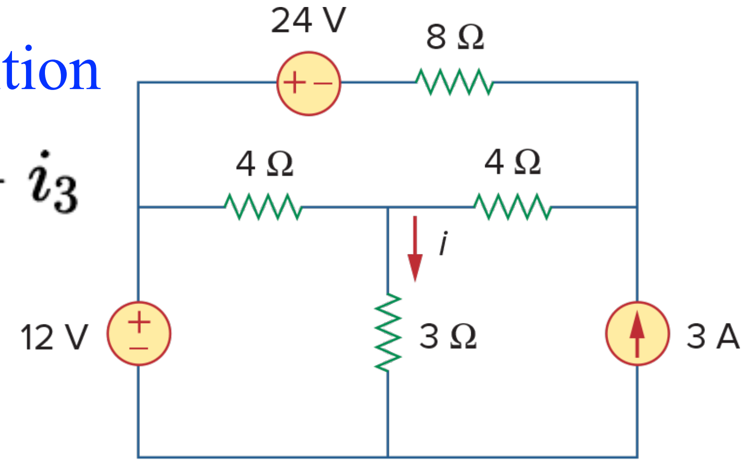
$$i_5 = -i''_o \Longrightarrow \begin{matrix} 6i_4 - 4i''_o = 0 \\ i_4 + 5i''_o = -20 \end{matrix} \Longrightarrow \boxed{i''_o = -\frac{60}{17}\text{ A}}$$

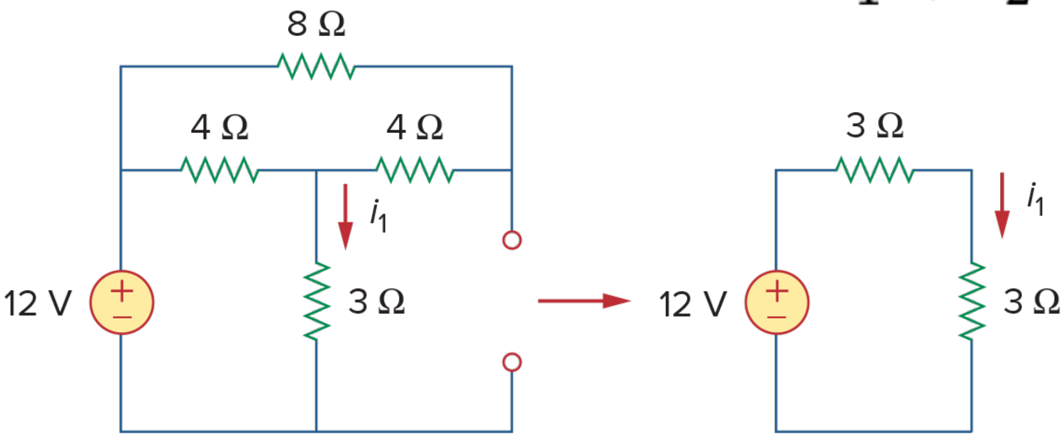
i''_o due to the 20-V voltage source

Example 4.5

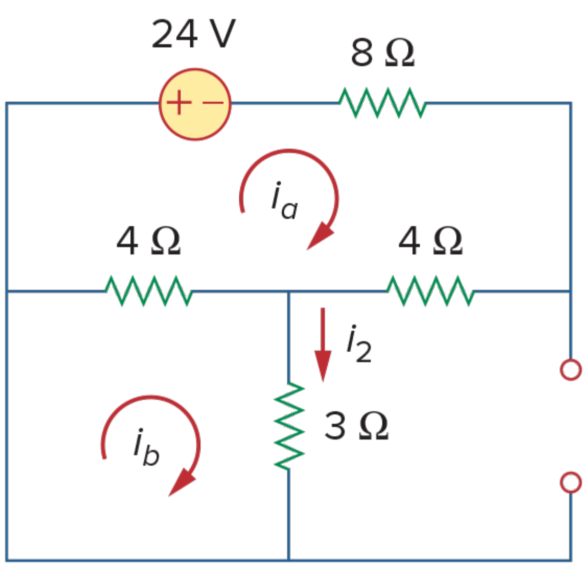
Find i using superposition

$$i = i_1 + i_2 + i_3$$





$$\Longrightarrow \boxed{i_1 = \frac{12}{6} = 2\ \text{A}}$$



mesh analysis:

For Loop i_a $\Longrightarrow 16i_a - 4i_b + 24 = 0$

$$\Longrightarrow 4i_a - i_b = -6$$

For Loop i_b $\Longrightarrow 7i_b - 4i_a = 0$

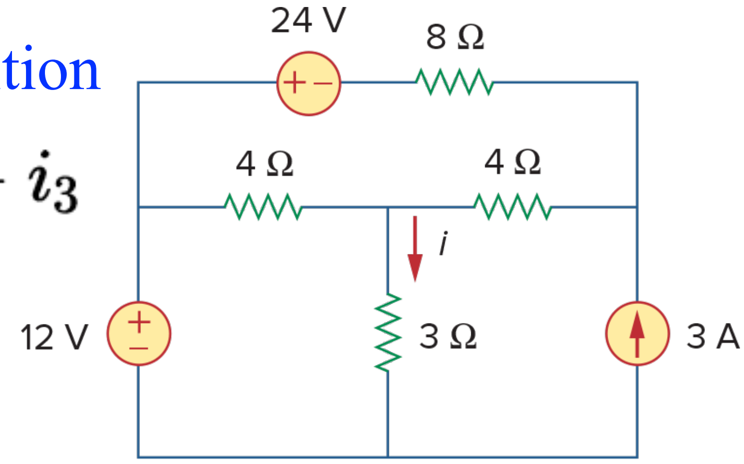
$$\Longrightarrow i_a = \frac{7}{4} i_b$$

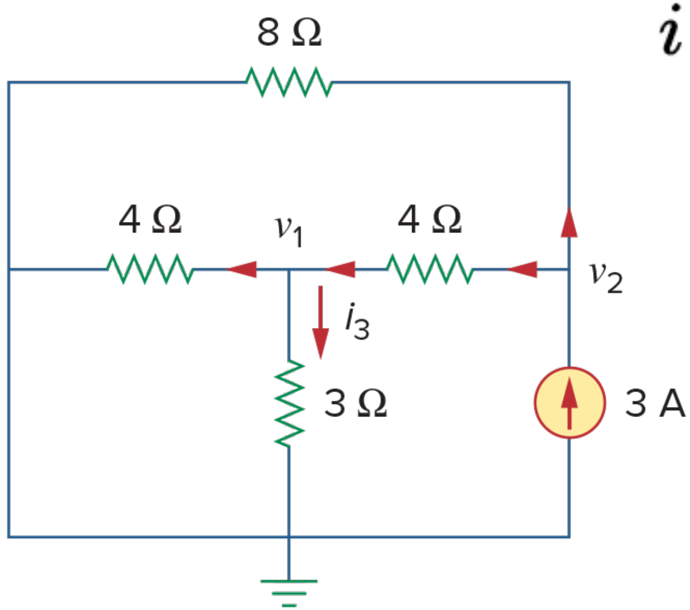
$$\Longrightarrow \boxed{i_2 = i_b = -1}$$

Example 4.5

Find i using superposition

$$i = i_1 + i_2 + i_3$$





$$i = i_1 + i_2 + i_3$$

$$\boxed{= 2 - 1 + 1 = 2 \text{ A}}$$

nodal analysis:

For Node v_2

$$3 = \frac{v_2}{8} + \frac{v_2 - v_1}{4} \implies 24 = 3v_2 - 2v_1$$

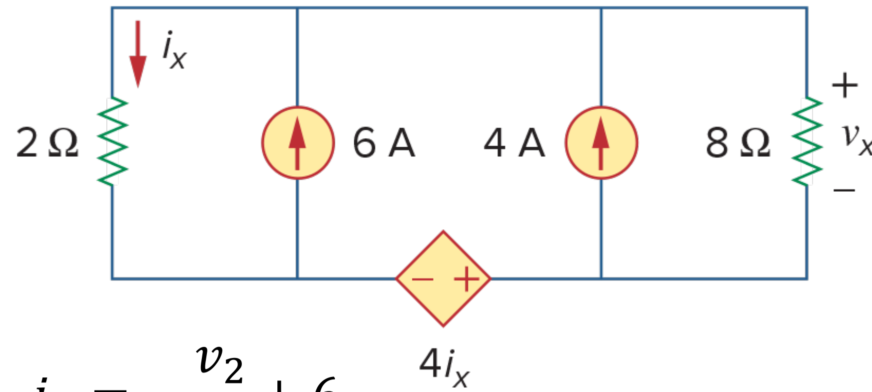
For Node v_1

$$\frac{v_2 - v_1}{4} = \frac{v_1}{4} + \frac{v_1}{3} \implies v_2 = \frac{10}{3} v_1$$

$$\implies v_1 = 3$$

$$\implies \boxed{i_3 = \frac{v_1}{3} = 1 \text{ A}}$$

Prob. 4.19. Use superposition to solve for v_x



v_1: $i_x + \frac{v_1}{8} = 4$

$$i_x = \frac{v_1 - (-4i_x)}{2} \Rightarrow i_x = -\frac{v_1}{2}$$

$$\Rightarrow -\frac{v_1}{2} + \frac{v_1}{8} = 4$$

$$\Rightarrow v_1 = -\frac{32}{3} = -10.6667(V)$$

$$\boxed{\begin{aligned} v_x &= v_1 + v_2 \\ &= -10.6667 - 16 \\ &= -26.6667 \end{aligned}}$$

How do you check the coherence?

$$2 \cdot i_x = v_x + 4 \cdot i_x$$

$$\Rightarrow i_x = -\frac{v_x}{2} = 13.3333(V)$$

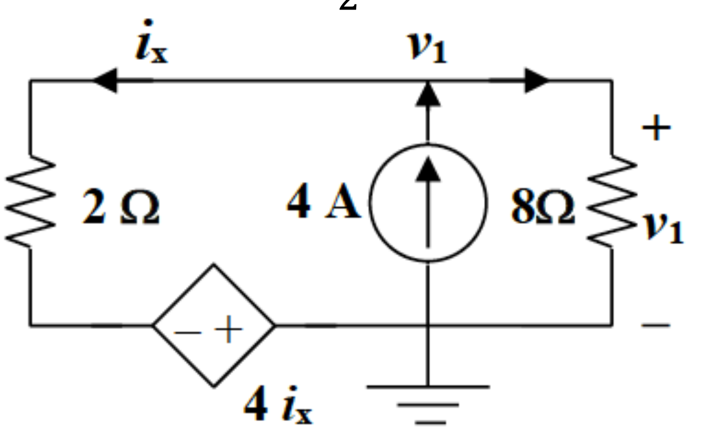
v_2: $i_x = -\frac{v_2}{8} + 6$

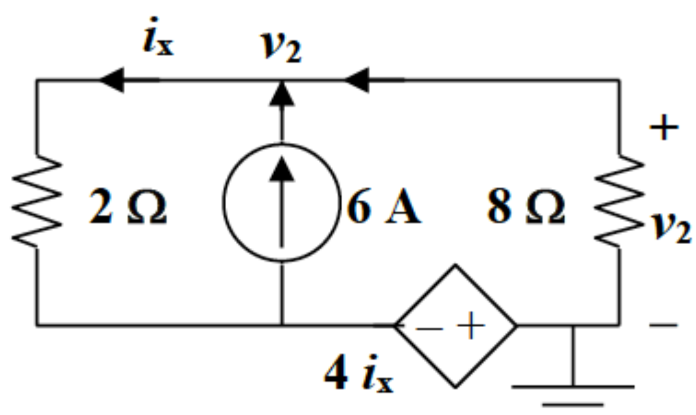
$$i_x = \frac{v_2 - (-4i_x)}{2} \Rightarrow i_x = -\frac{v_2}{2}$$

$$\Rightarrow -\frac{v_2}{2} = -\frac{v_2}{8} + 6$$

$$\Rightarrow v_2 = -\frac{48}{3} = -16(V)$$

KCL: $i_x + \frac{v_x}{8} = 6 + 4$



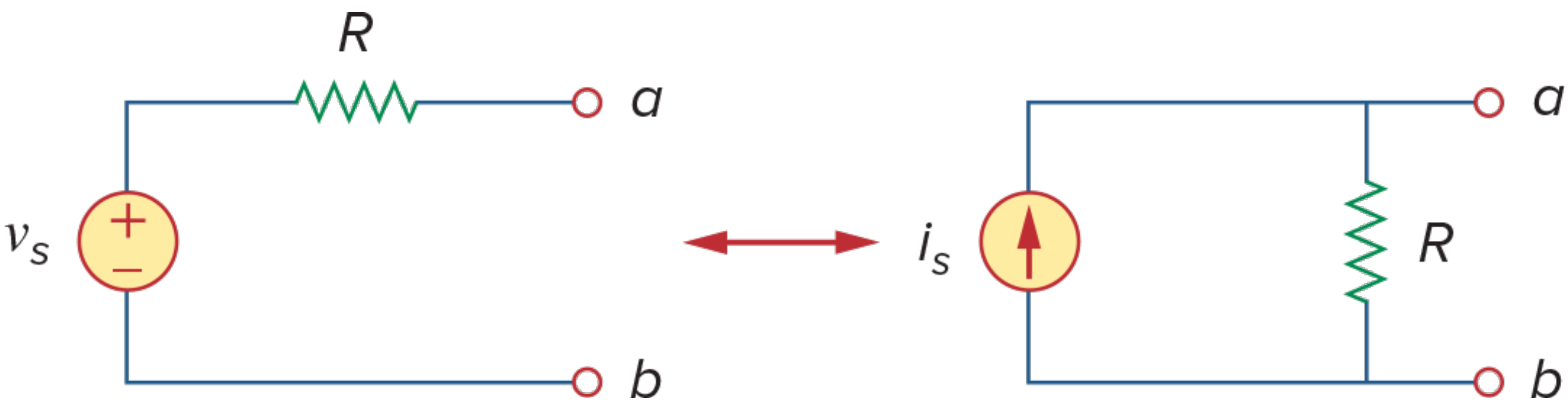


4.4 Source Transformation

- Much like the Δ–Y transformation, it is possible to transform a source from one form to another
- This can be useful for simplifying circuits
- The principle behind all of these transformations is equivalence

Source Transformation

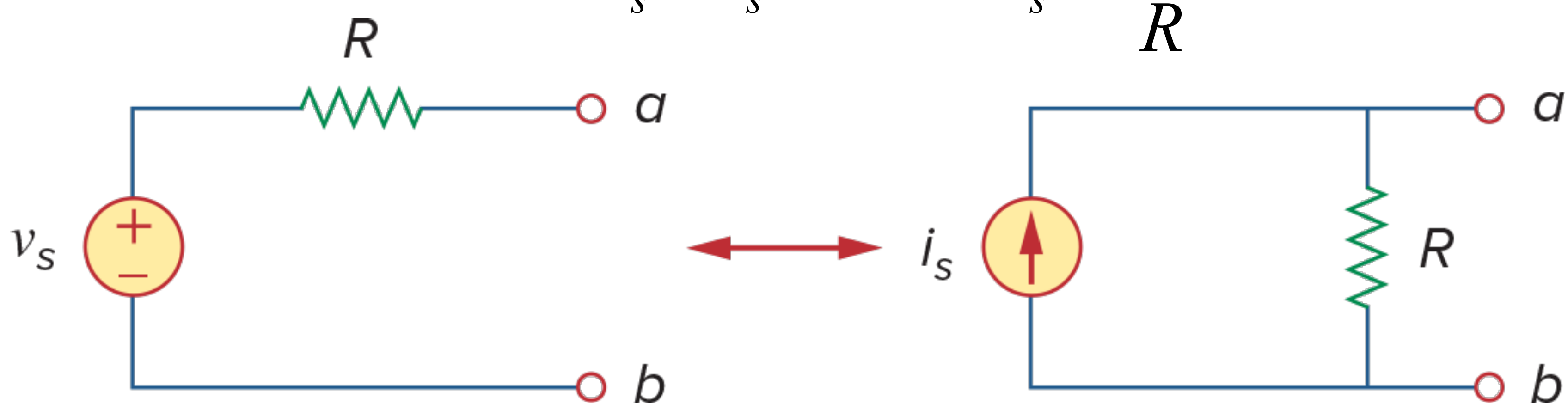
- A source transformation is the process of
  - replacing a voltage source v_s in series with a resistor R
  - by a current source i_s in parallel with a resistor R
  - or vice versa.



Terminal Equivalency

- These transformations work because the two sources have equivalent behavior at their terminals
- If the sources are turned off the resistance at the terminals are both R
- If the terminals are short circuited, the currents need to be the same
- From this we get the following requirement:

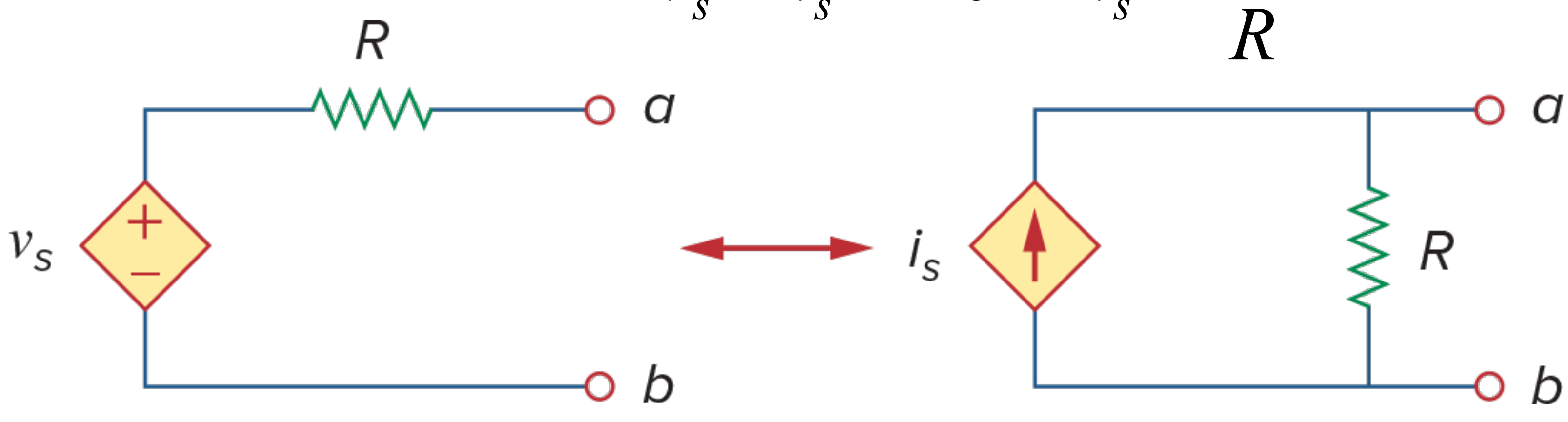
$$v_s = i_s R \quad \text{or} \quad i_s = \frac{v_s}{R}$$



Dependent Sources

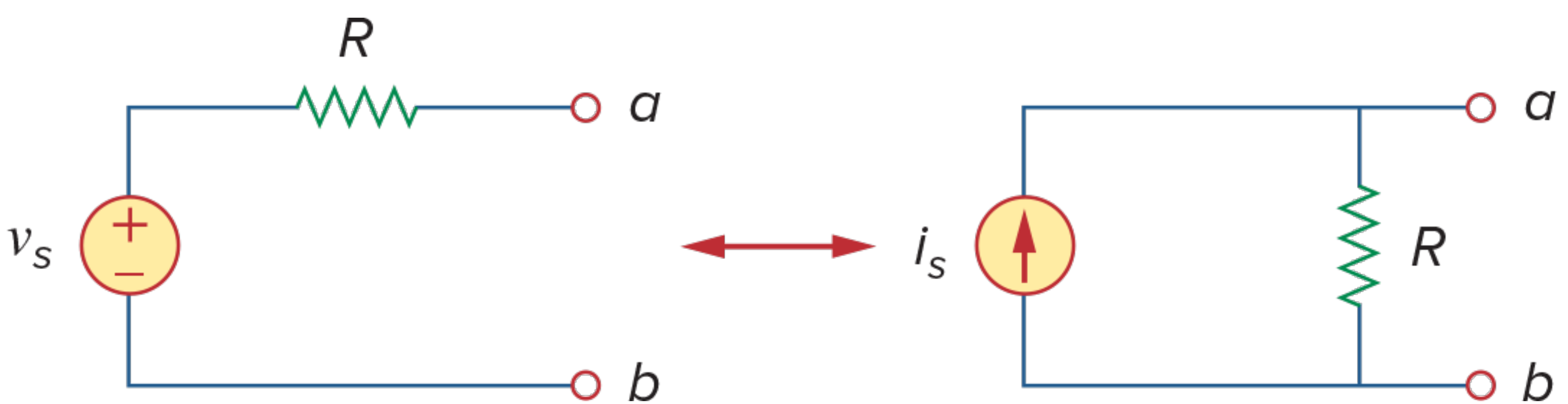
- **Source transformation** also applies to dependent sources
- But, the **dependent variable** must be handled carefully
- The same relationship between the voltage and current holds here:

$$v_s = i_s R \quad \text{or} \quad i_s = \frac{v_s}{R}$$



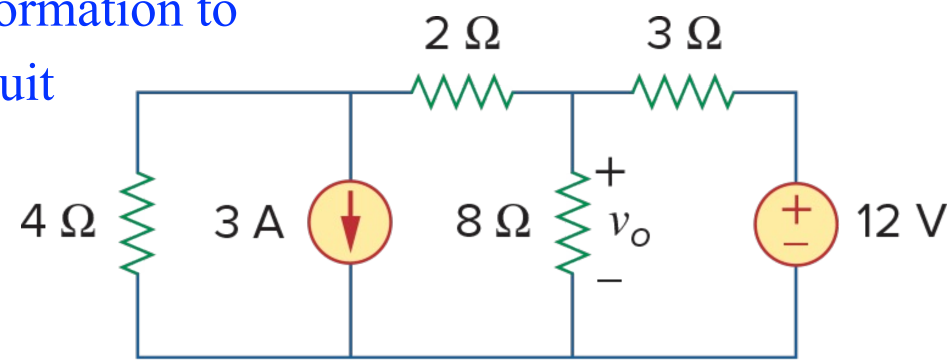
Source transformation rules

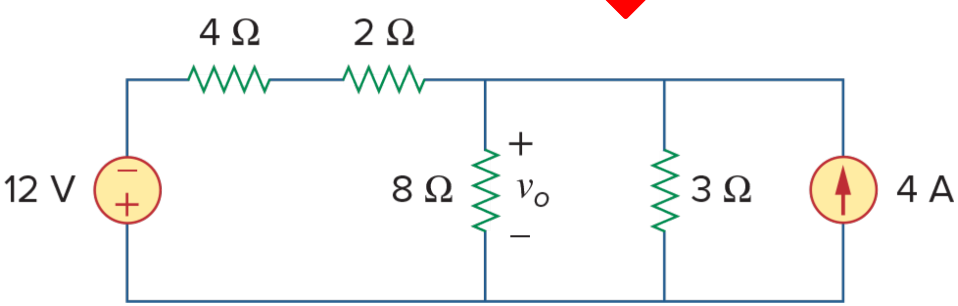
- Note that the arrow of the current source is directed towards the positive terminal of the voltage source
- Source transformation is not possible when R=0 for an ideal voltage source
- For a realistic source, R≠0
- For an ideal current source, R=∞ also prevents the use of source transformation

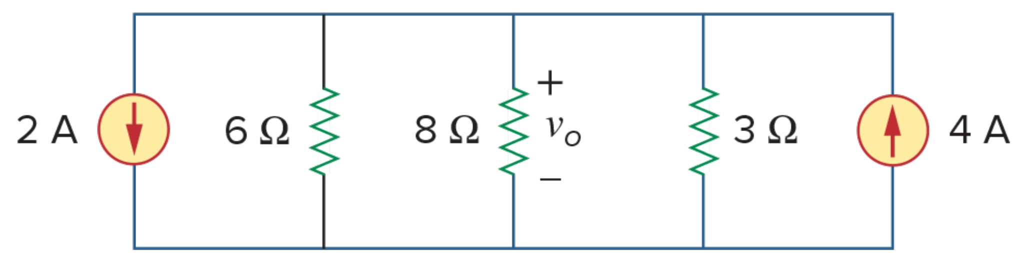


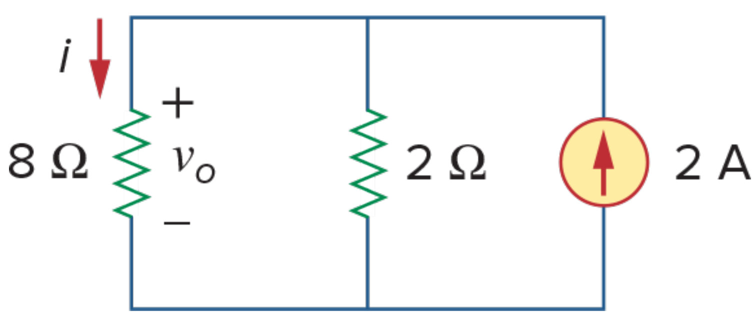
Example 4.6

Use source transformation to find v_O in the circuit







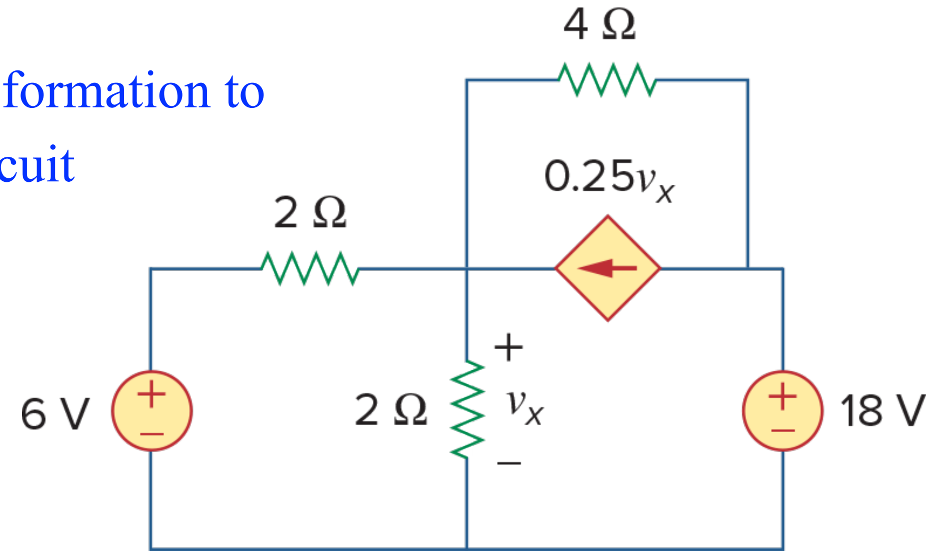


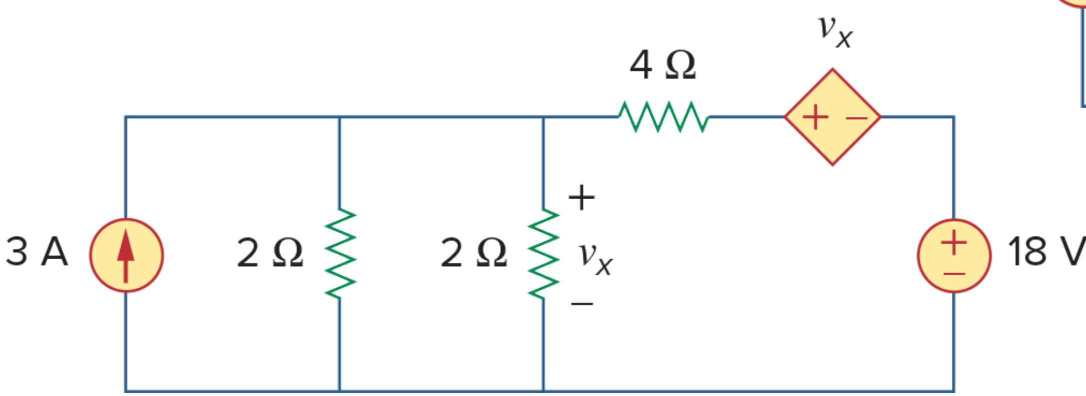
$$i = \frac{2}{2+8}(2) = 0.4 \text{ A}$$

$$v_o = 8i = 8(0.4) = 3.2 \text{ V}$$

Example 4.7

Use source transformation to find v_x in the circuit





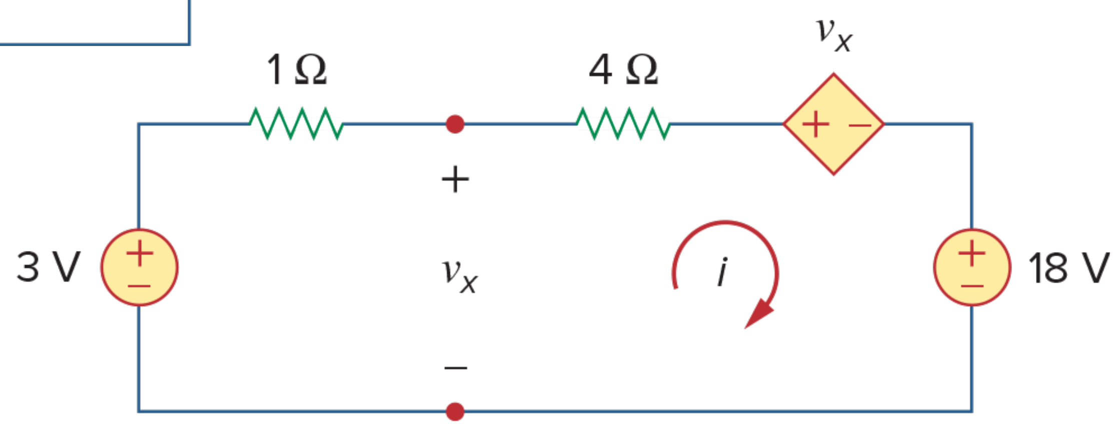
Applying KVL

$$-3 + 5i + v_x + 18 = 0$$

$$v_x = 3 - i$$

$$\Rightarrow 15 + 5i + 3 - i = 0$$

$$i = -4.5\text{ A}$$



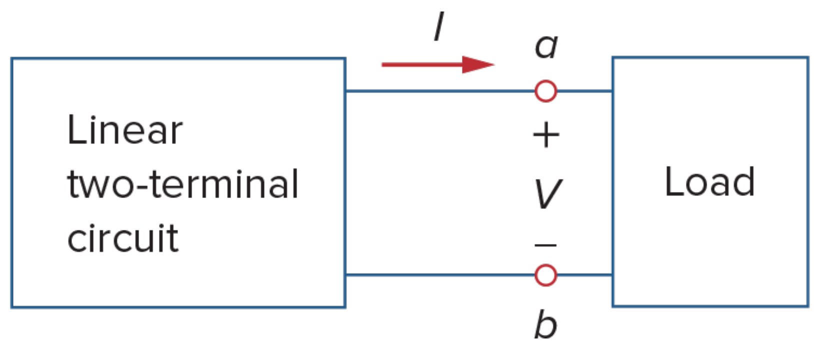
$$\Rightarrow \boxed{v_x = 3 - i = 7.5\text{ V}}$$

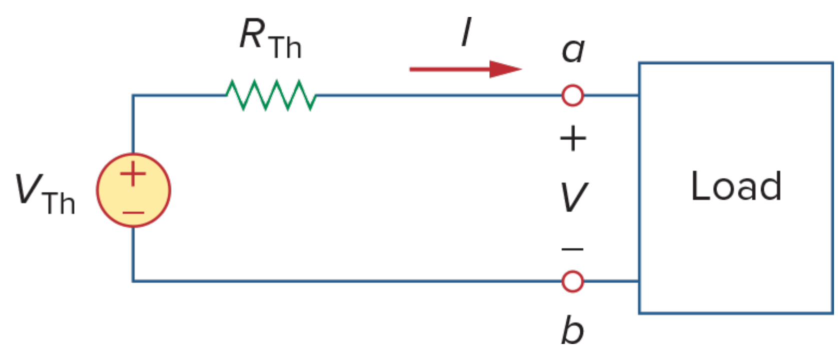
4.5 Thevenin's Theorem

- In many circuits, one element will be variable
- An example of this is mains power; many different appliances may be plugged into the outlet, each presenting a different resistance
- This variable element is called the load
- Ordinarily one would have to reanalyze the circuit for each change in the load

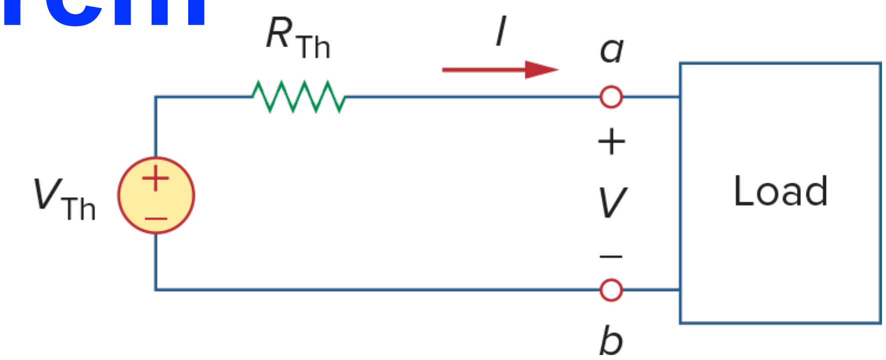
Thevenin's Theorem

- **Thevenin's theorem states that a linear two terminal circuit may be replaced with a voltage source + resistor**
- **The voltage source's value is equal to the open circuit voltage at the terminals**
- **The resistance is equal to the resistance measured at the terminals when the independent sources are turned off.**

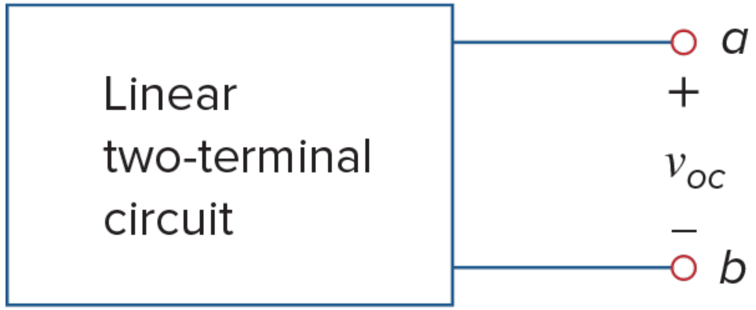




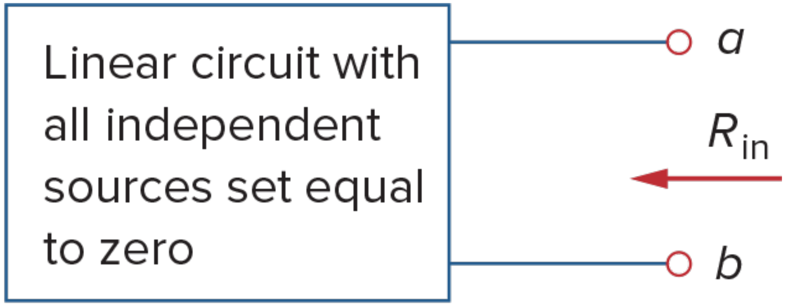
Thevenin’s Theorem



- **There are two cases to consider when finding the equivalent resistance**
- **Case 1: no dependent sources**
  - ✓ the resistance may be found by simply turning off all the sources

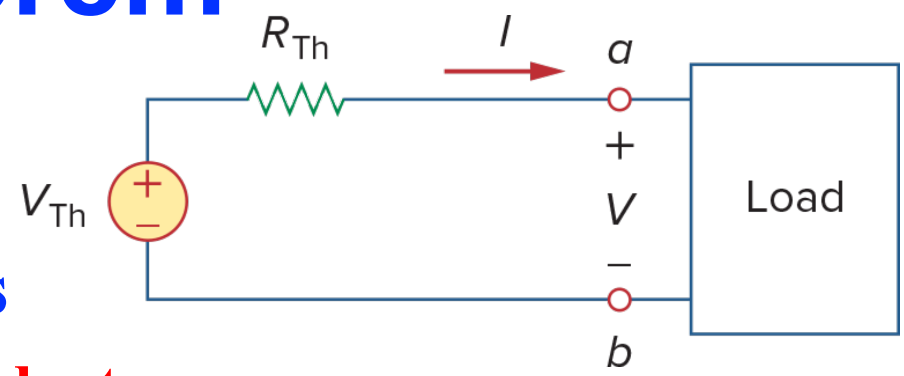


$V_{Th} = v_{oc}$

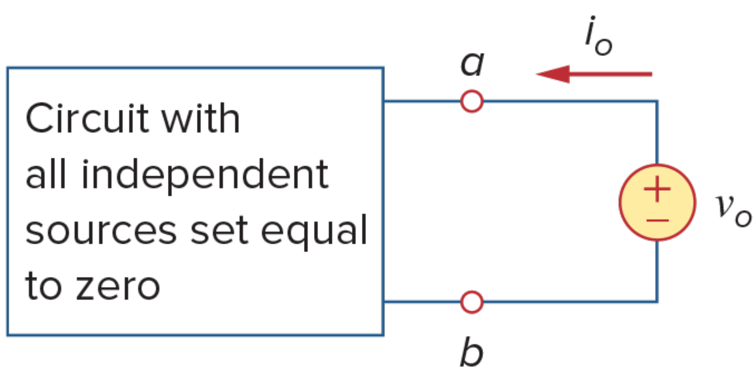


$R_{Th} = R_{in}$

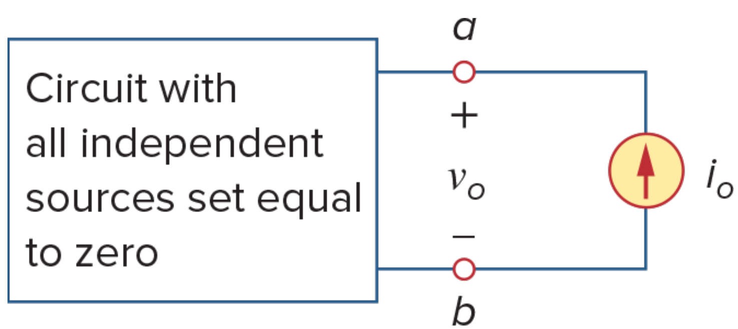
Thevenin’s Theorem



- **Case 2: dependent sources**
  - ✓ **still turn off all the independent sources.**
  - ✓ **apply a voltage v_0 (or current i_0) to the terminals**
  - ✓ **determine the current i_0 (voltage v_0).**



$$R_{Th} = \frac{v_o}{i_o}$$



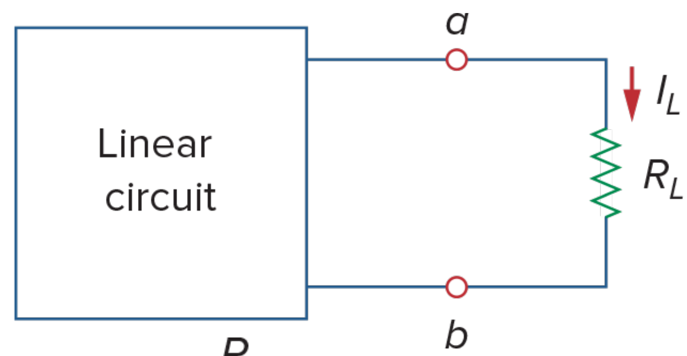
$$R_{Th} = \frac{v_o}{i_o}$$

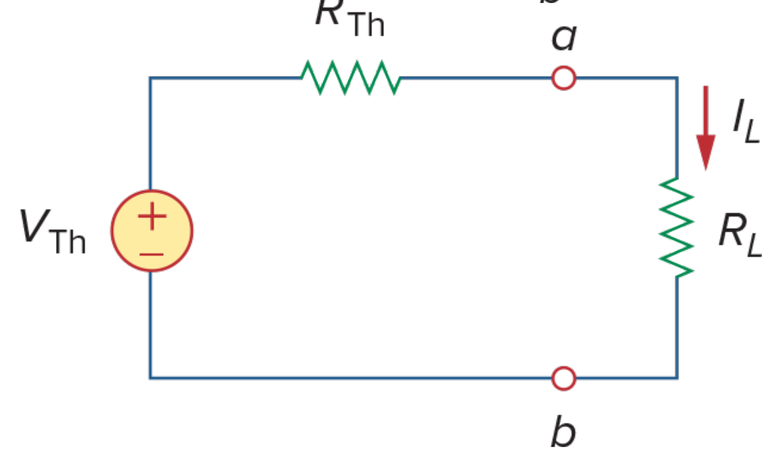
Thevenin's Theorem

- Thevenin's theorem is very powerful in circuit analysis.
- It allows one to **simplify a circuit**
- A large circuit may be replaced by
  - a **single independent voltage source**
  - and a **single resistor**.
- The **equivalent circuit** behaves externally exactly the same as the **original circuit**.

Negative Resistance?

- It is possible for the result of this analysis to end up with a negative resistance.
- This implies the circuit is supplying power
- This is reasonable with dependent sources
- Note that in the end, the Thevenin equivalent makes working with variable loads much easier.
- Load current can be calculated with a voltage source and two series resistors
- Load voltages use the voltage divider rule.



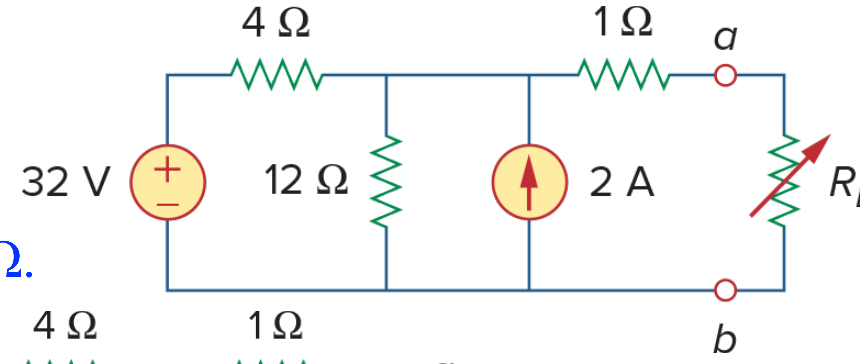


$$I_L = \frac{V_{Th}}{R_{Th} + R_L}$$

$$V_L = R_L I_L = \frac{R_L}{R_{Th} + R_L} V_{Th}$$

Example 4.8

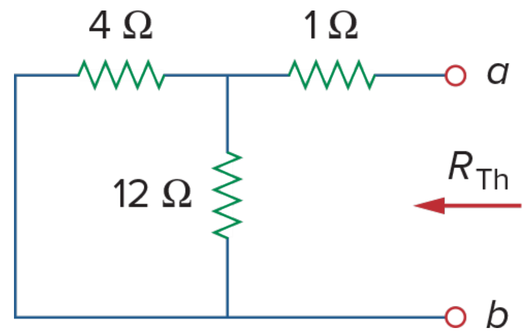
Find the Thevenin equivalent circuit of the circuit to the left of the terminals a-b.
Find the current through R_L = 6, 16, and 36 Ω.



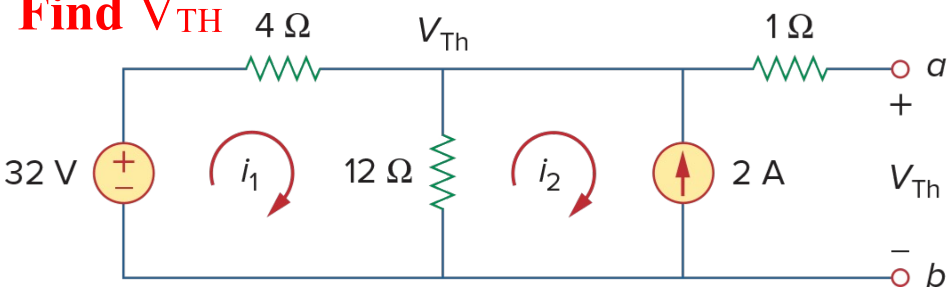
Find R_{TH}

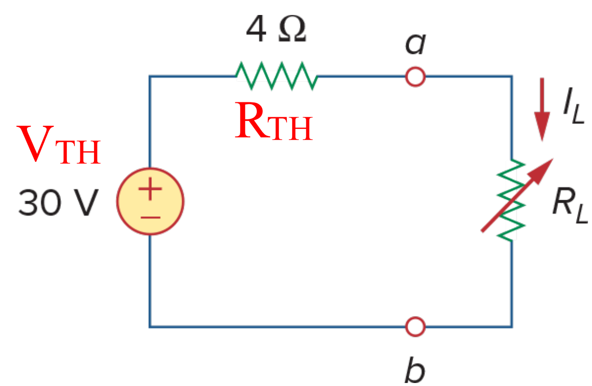
turning off the 32-V voltage source (short circuit) and the 2-A current source (open circuit).

$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$



Find V_{TH}





mesh analysis

$$-32 + 4i_1 + 12(i_1 - i_2) = 0$$

$$i_2 = -2\ \text{A}$$

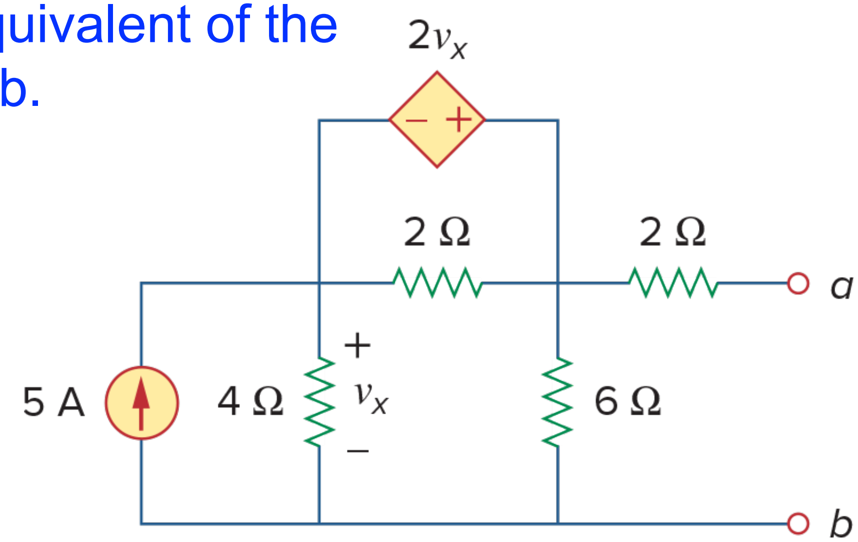
$$\Longrightarrow\ i_1 = 0.5\ \text{A}$$

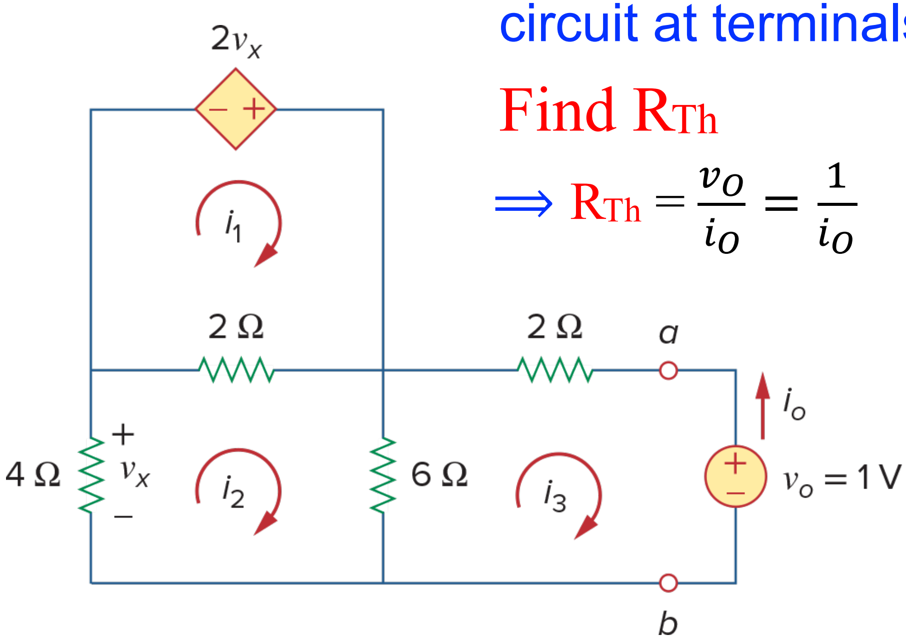
$$\Longrightarrow\ V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30\ \text{V}$$

$$\boxed{I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L}}$$

Example 4.9

Find the Thevenin equivalent of the circuit at terminals a-b.





Find R_{Th}

$$\Longrightarrow R_{Th} = \frac{v_O}{i_O} = \frac{1}{i_O}$$

mesh analysis

Loop 1

$$-2v_x + 2(i_1 - i_2) = 0 \Longrightarrow v_x = i_1 - i_2$$

$$-4i_2 = v_x = i_1 - i_2 \Longrightarrow i_1 = -3i_2$$

Loop 2

$$4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) = 0$$

Loop 3

$$6(i_3 - i_2) + 2i_3 + 1 = 0$$

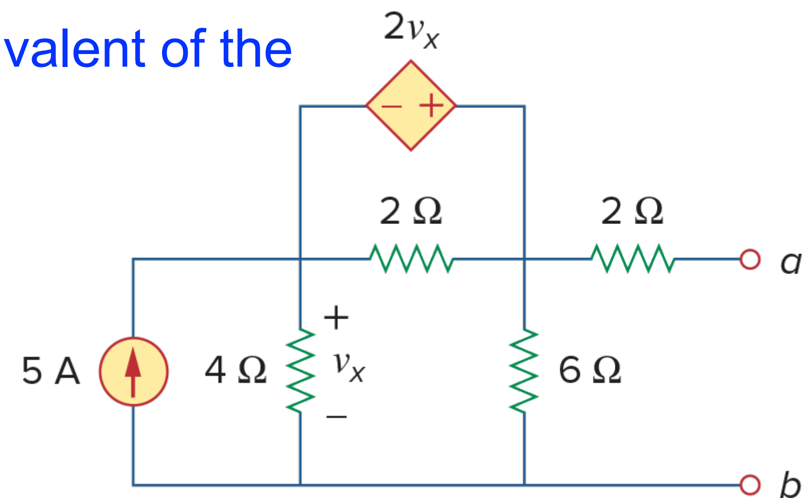
$$\boxed{\Longrightarrow i_3 = -\frac{1}{6}A}$$

$$\boxed{\Longrightarrow i_O = -i_3 = \frac{1}{6}A}$$

$$\boxed{R_{Th} = \frac{1}{i_O} = 6\Omega}$$

Example 4.9

Find the Thevenin equivalent of the circuit at terminals a-b.



Find V_{Th}

Loop 1

$$\Longrightarrow \quad i_1 = 5$$

Loop 2

$$\Longrightarrow \quad -2v_x + 2(i_3 - i_2) = 0$$

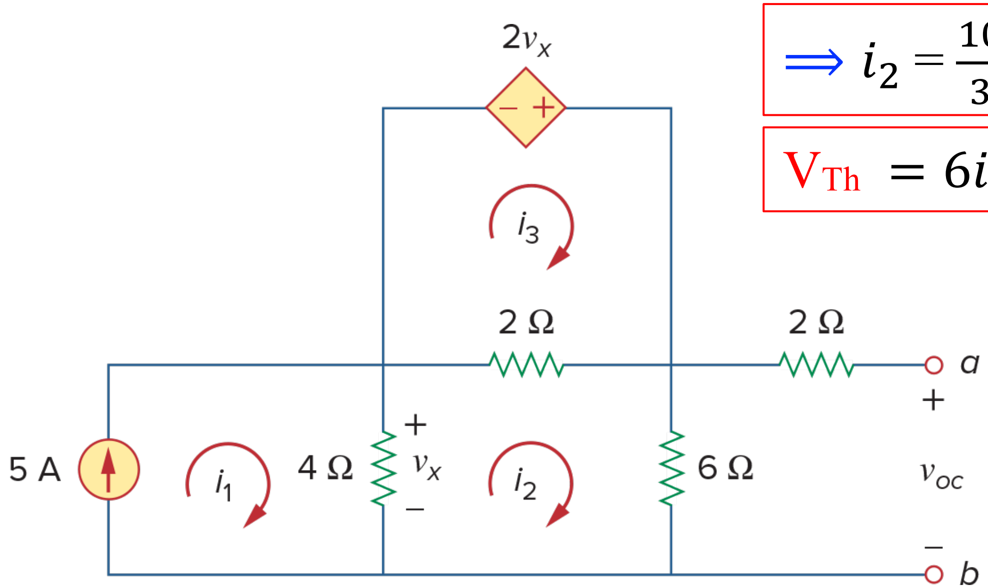
$$\Longrightarrow \quad v_x = i_3 - i_2$$

$$4(i_1 - i_2) = v_x$$

Loop 3

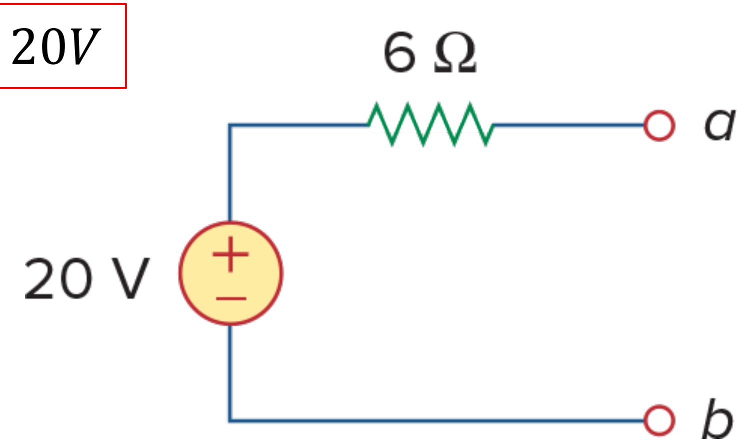
$$\Longrightarrow 4(i_2 - i_1) + 2(i_2 - i_3) + 6i_2 = 0$$

$$\Longrightarrow 12i_2 - 4i_1 - 2i_3 = 0$$

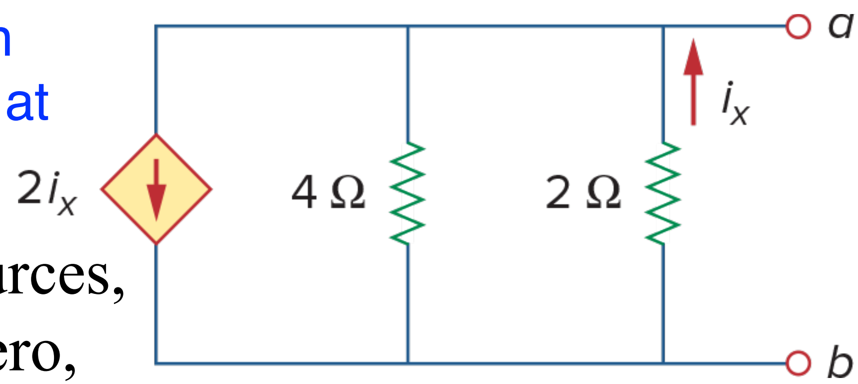


$$\boxed{\Longrightarrow i_2 = \frac{10}{3} A}$$

$$\boxed{V_{Th} = 6i_2 = 20V}$$



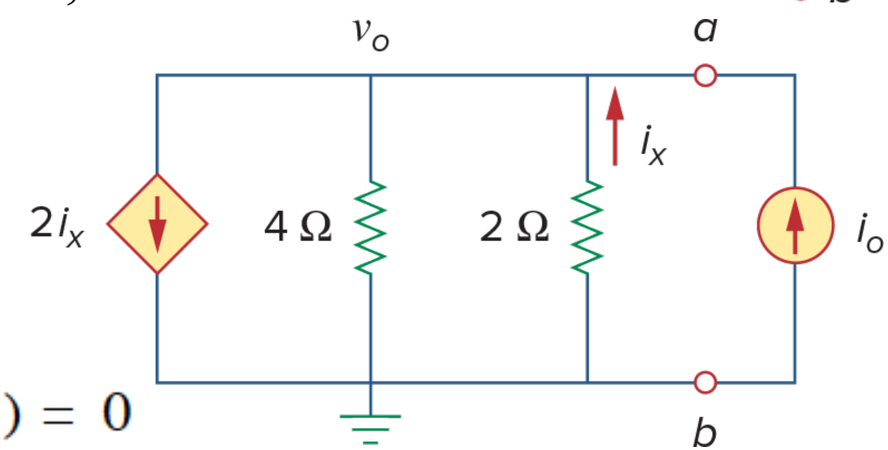
Example 4.10 Determine the Thevenin equivalent of the circuit at terminals a-b.



- When you have no independent sources, the value for V_{Th} will be equal to zero, so you will only have to find R_{Th}.

Find R_{Th}

Assuming $i_o = 1$ A



Nodal Analysis:

$$2i_x + (v_o - 0)/4 + (v_o - 0)/2 + (-1) = 0$$

$$i_x = (0 - v_o)/2 = -v_o/2$$

$$\Longrightarrow\ 2(-v_o/2) + (v_o - 0)/4 + (v_o - 0)/2 + (-1) = 0$$

$$\Longrightarrow\ v_o = -4\text{ V}$$

$$\boxed{R_{Th} = v_o/1 = -4\Omega}$$

The negative value of the resistance tells us that, according to the passive sign convention, the circuit is supplying power.

Example 4.10

Providing a load to the original circuit

Mesh Analysis:

mesh 1:

$$8i_x + 4i_1 + 2(i_1 - i_2) = 0$$

$$i_x = i_2 - i_1$$

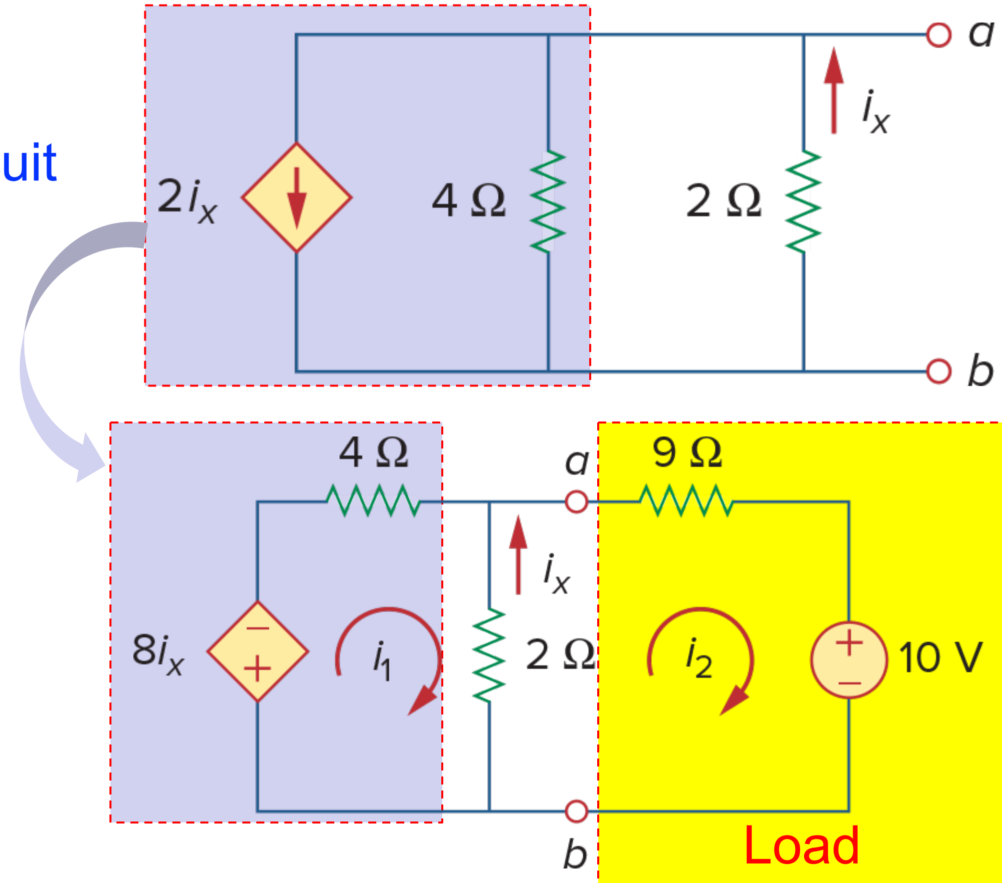
$$\Rightarrow -2i_1 + 6i_2 = 0$$

mesh 2:

$$2(i_2 - i_1) + 9i_2 + 10 = 0$$

$$\Rightarrow -2i_1 + 11i_2 = -10$$

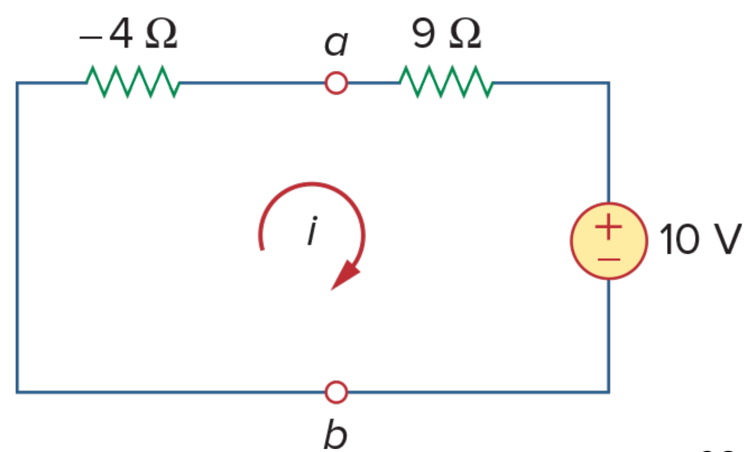
$\Rightarrow \boldsymbol{i_2} = -2$ (A) load current



Providing the load to the Thevenin equivalent circuit

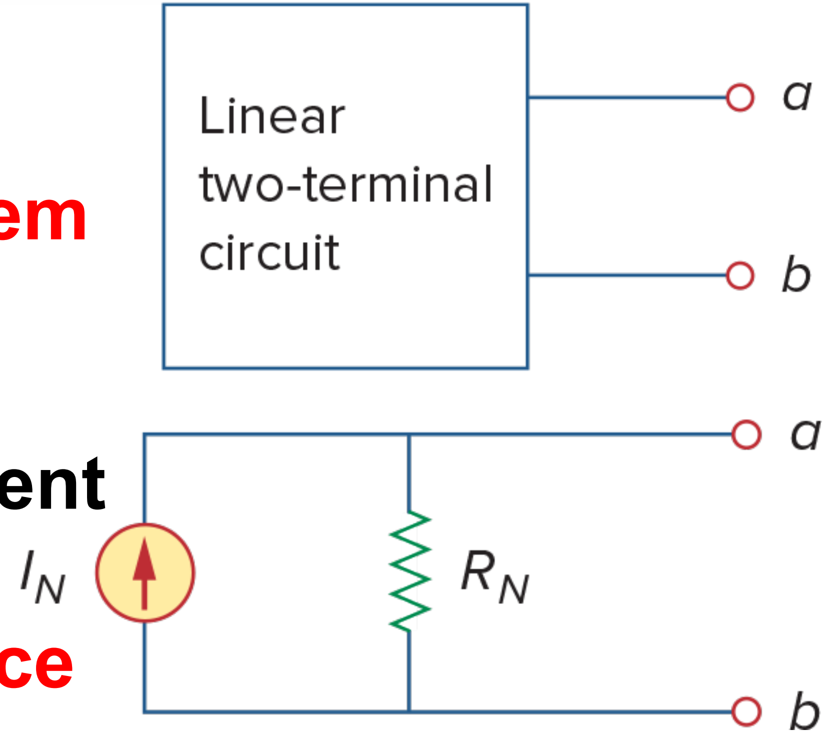
$$-4i + 9i + 10 = 0$$

$\Rightarrow \boldsymbol{i} = -2$ (A) load current



4.6 Norton's Theorem

- Similar to Thevenin's theorem, **Norton's theorem** states that a linear two terminal circuit may be replaced with an equivalent circuit containing:
  a resistor + **current source**

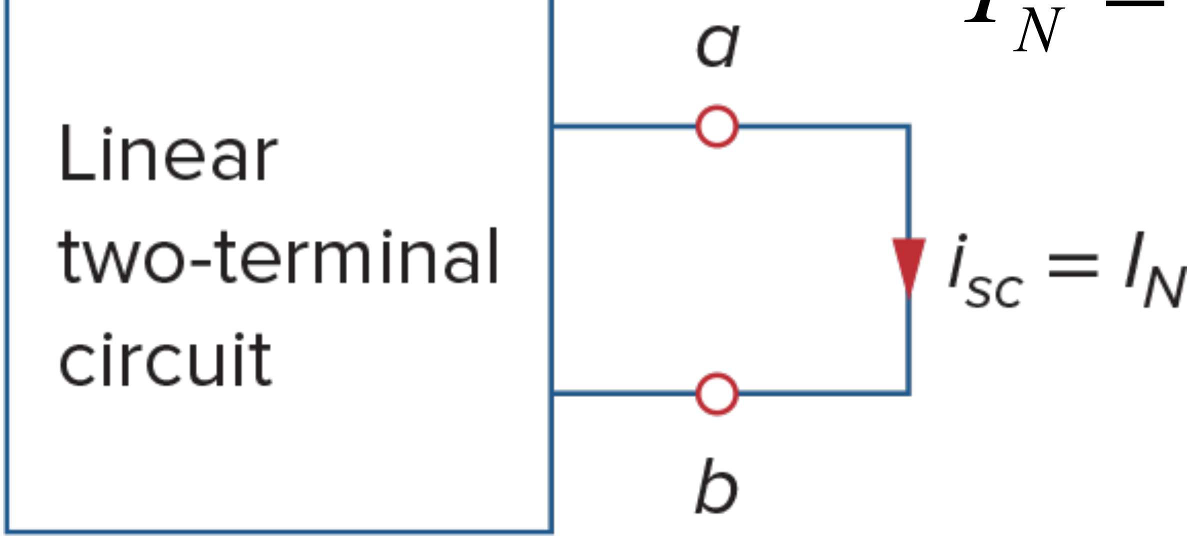


- **The Norton resistance** will be exactly the same as **the Thevenin**

$$\boxed{R_N = R_{Th}}$$

Norton's Theorem

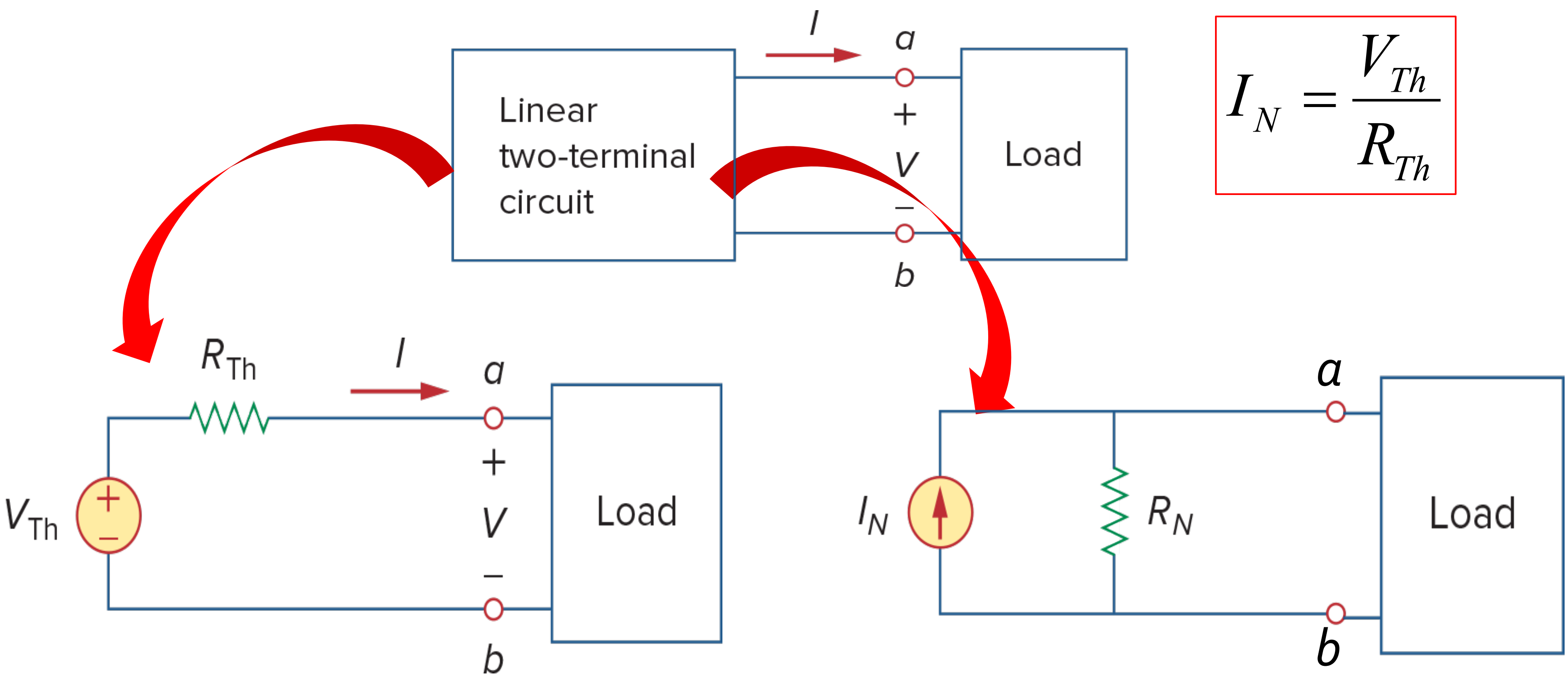
- The **Norton current I_N** is found by:
  - **short circuiting** the circuit's terminals and measuring the resulting **current**



$$I_N = i_{sc}$$

Norton vs. Thevenin

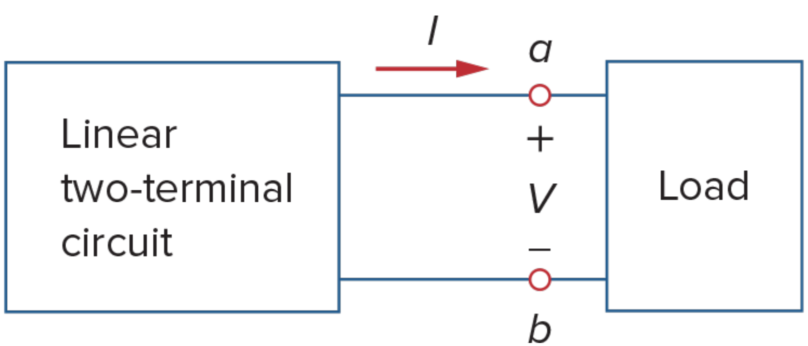
- These two equivalent circuits can be related to each other
- The **Norton current** and **Thevenin voltage** are related to each other as follows:

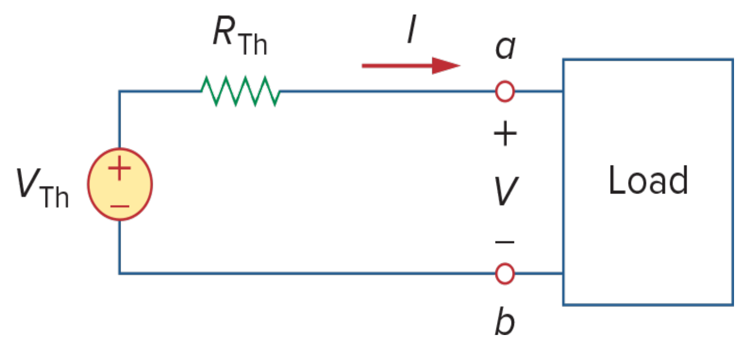


$$\boxed{I_N = \frac{V_{Th}}{R_{Th}}}$$

Norton vs. Thevenin

- **With V_{TH}, I_N, and ($R_{TH}=R_N$) related, finding the Thevenin or Norton equivalent circuit requires:**
  - **The open-circuit voltage across terminals *a* and *b*.**
  - **The short-circuit current at terminals *a* and *b*.**
  - **The equivalent or input resistance at terminals *a* and *b* *when* all independent sources are turned off.**

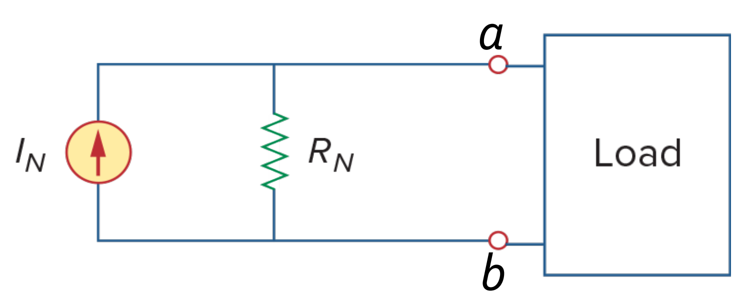




$$V_{Th} = v_{oc}$$

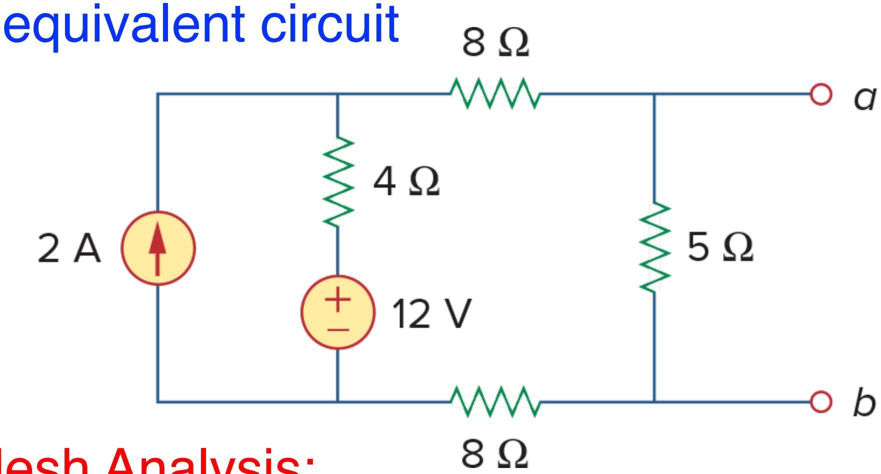
$$I_N = i_{sc}$$

$$R_{Th} = \frac{v_{oc}}{i_{sc}} = R_N$$



Example 4.11

Find the Norton equivalent circuit at terminals a-b.



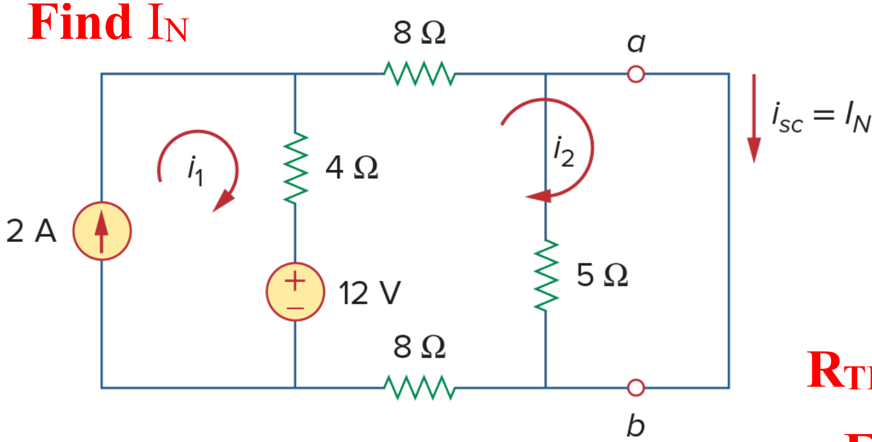
Norton equivalent

Find R_N

Set all the independent sources equal to zero.

$$R_N = 5 \parallel (8+4+8) = 5 \parallel 20 = \frac{20 \times 5}{25} = 4\ \Omega$$

Find I_N

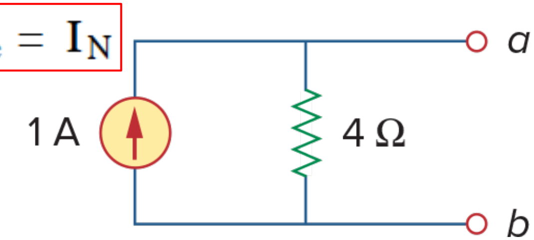


Mesh Analysis:

Mesh 1: $i_1 = 2\text{ A}$

Mesh 2: $20i_2 - 4i_1 - 12 = 0$

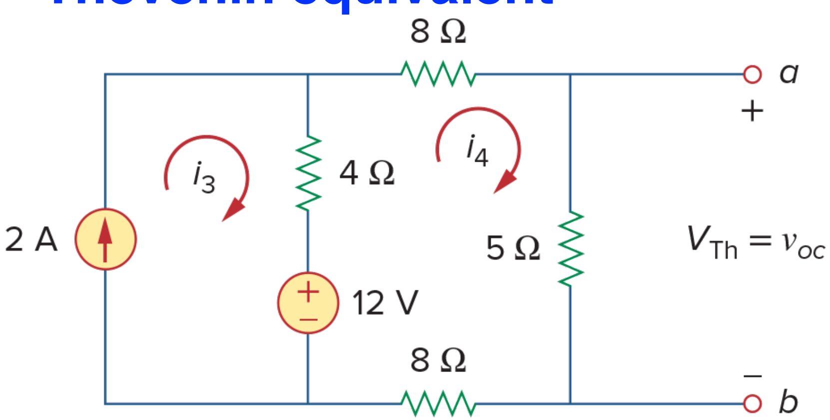
$\Rightarrow$ $\boxed{i_2 = 1\text{ A} = i_{sc} = I_N}$



$R_{TH} = R_N$

Thevenin equivalent

Find V_{TH}



Mesh 3: $i_3 = 2\text{ A}$

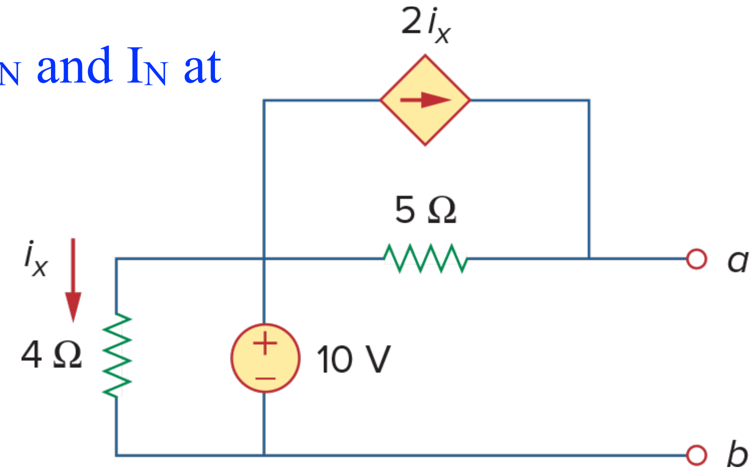
Mesh 4: $25i_4 - 4i_3 - 12 = 0$

$\Rightarrow$ $i_4 = 0.8\text{ A}$

$$\boxed{v_{oc} = V_{Th} = 5i_4 = 4\text{ V}}$$

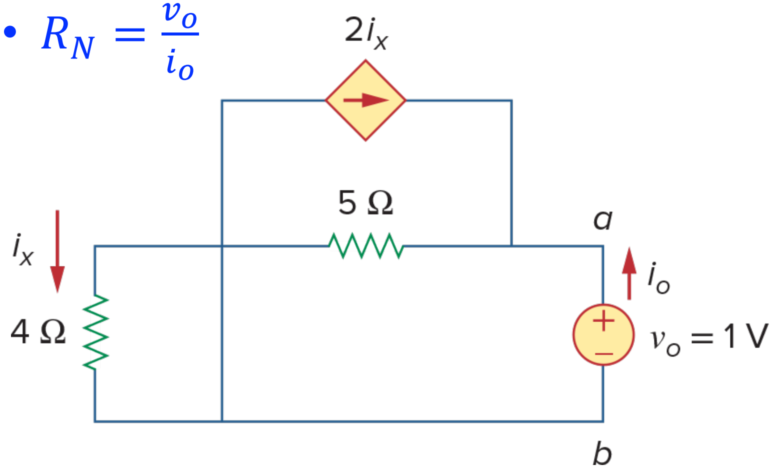
$$\boxed{I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1\text{ A}}$$

Example 4.12 Using Norton's theorem, find R_N and I_N at terminals ab.



Find R_N

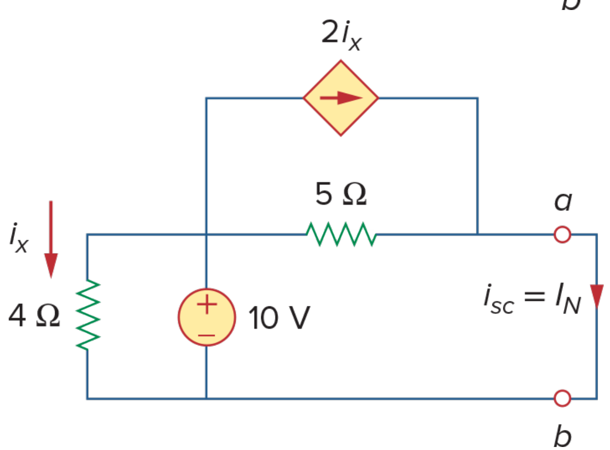
- set the independent voltage source equal to zero
- connect a voltage source of $v_o = 1$ V
  (or any unspecified voltage vo) to the terminals
- $R_N = \frac{v_o}{i_o}$



$$R_N = \frac{v_o}{i_o} = \frac{1}{0.2} = 5\ \Omega$$

Find I_N

- short-circuit terminals a and b and find the current $i$sc



$$i_x = \frac{10}{4} = 2.5\ \text{A}$$

KCL at node a gives:

$$i_{sc} = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7\ \text{A}$$

$$\Longrightarrow\ I_N = 7\ \text{A}$$

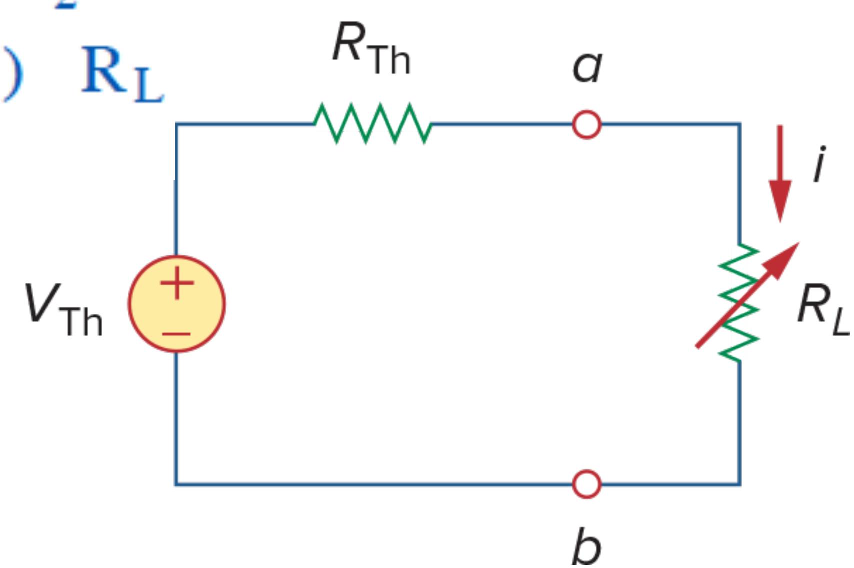
4.8 Maximum Power Transfer

- **In many applications, a circuit is designed to power a load**
- **Among those applications there are many cases where we wish to maximize the power transferred to the load**
- **Unlike an ideal source, internal resistance will restrict the conditions where maximum power is transferred.**

Maximum Power Transfer

- **We can use the Thevenin equivalent circuit for finding the maximum power in a linear circuit**
- **We will assume that the load resistance can be varied**
- **Looking at the equivalent circuit with load included, the power transferred is:**

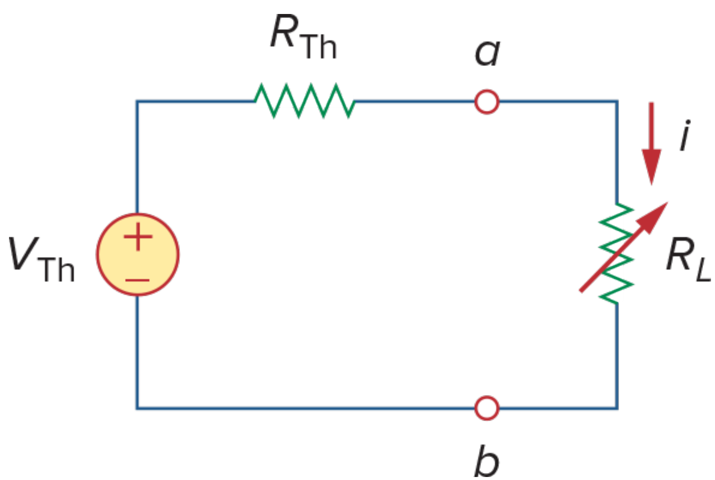
$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$

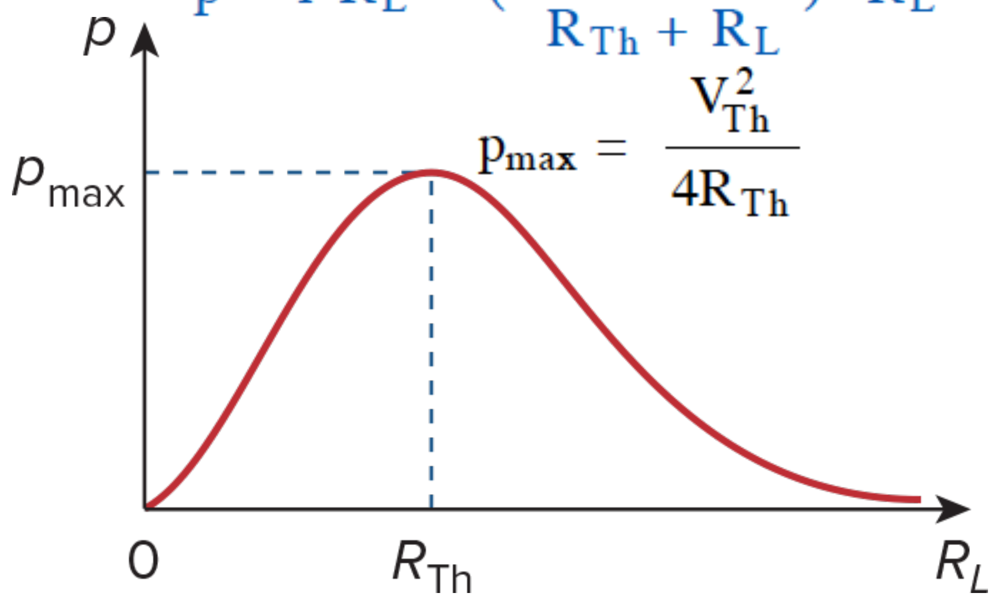


Maximum Power Transfer

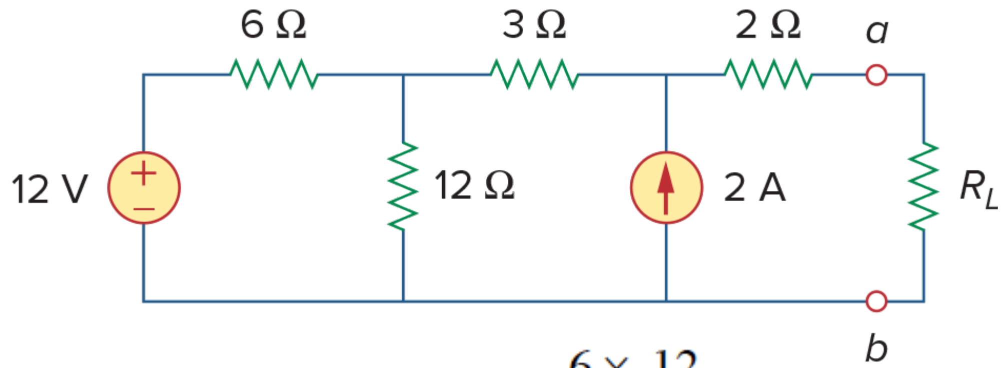
- For a given circuit, V_{TH} and R_{TH} are fixed.
- By varying the load resistance R_L,
  ⇒ the power delivered to the load varies as
- as $R_L \rightarrow 0$ and ∞
  ⇒ the power transferred $\rightarrow 0$
- The **maximum power** transferred → when $R_L = R_{TH}$

$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$



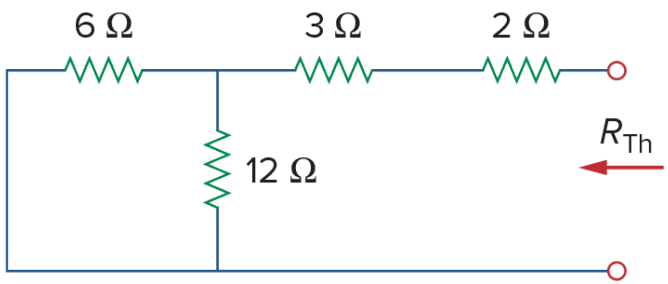


Example 4.13 Find the value of R_L for maximum power transfer in the circuit. Find the maximum power.



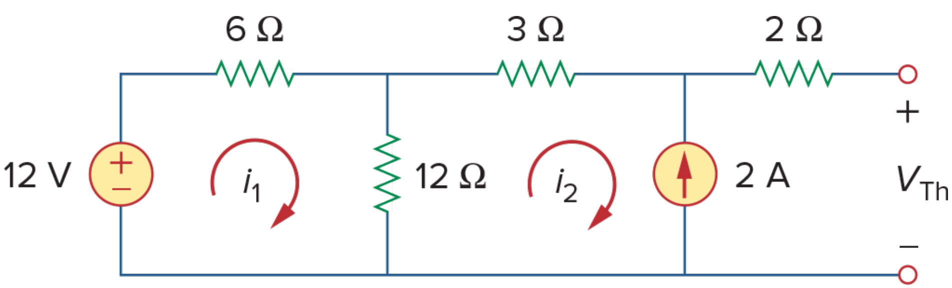
Thevenin equivalent

Find R_{TH}



$$R_{Th} = 2 + 3 + 6 \parallel 12 = 5 + \frac{6 \times 12}{18} = 9\ \Omega$$

Find V_{TH}



Mesh Analysis:

Mesh 1: $-12 + 18i_1 - 12i_2 = 0$

Mesh 2: $i_2 = -2\ A$

$\Rightarrow i_1 = -2/3.$

KVL:

$-12 + 6i_1 + 3i_2 + 2(0) + V_{Th} = 0$

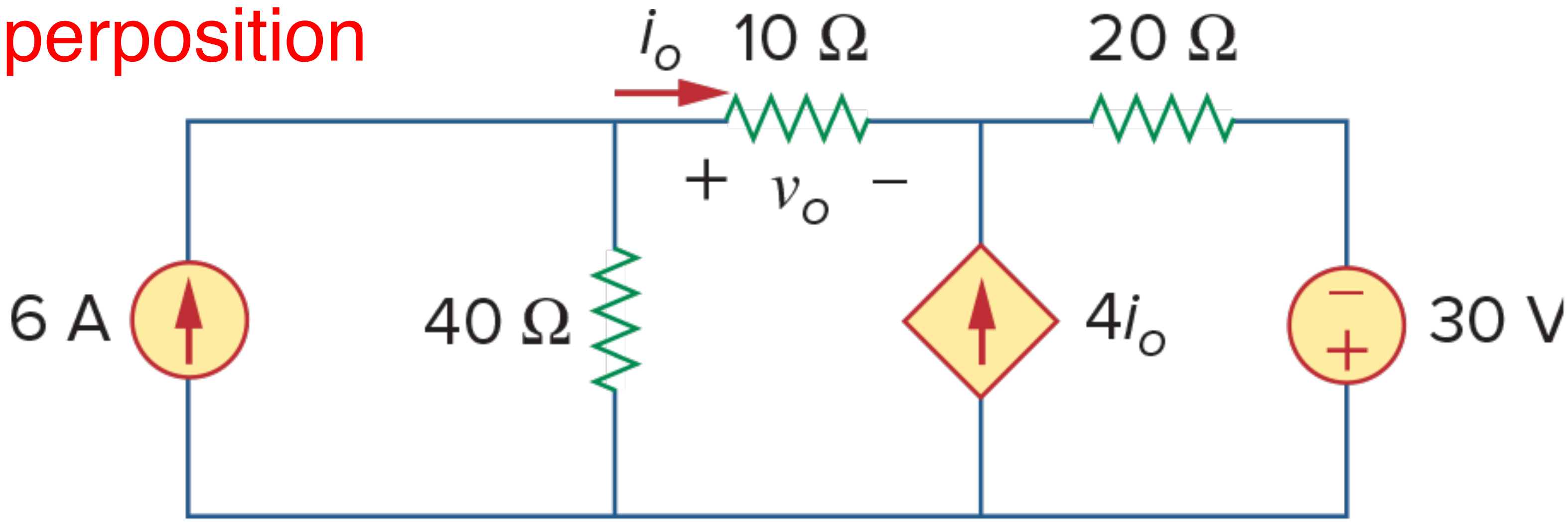
$\Rightarrow V_{Th} = 22\ V$

maximum power transfer:

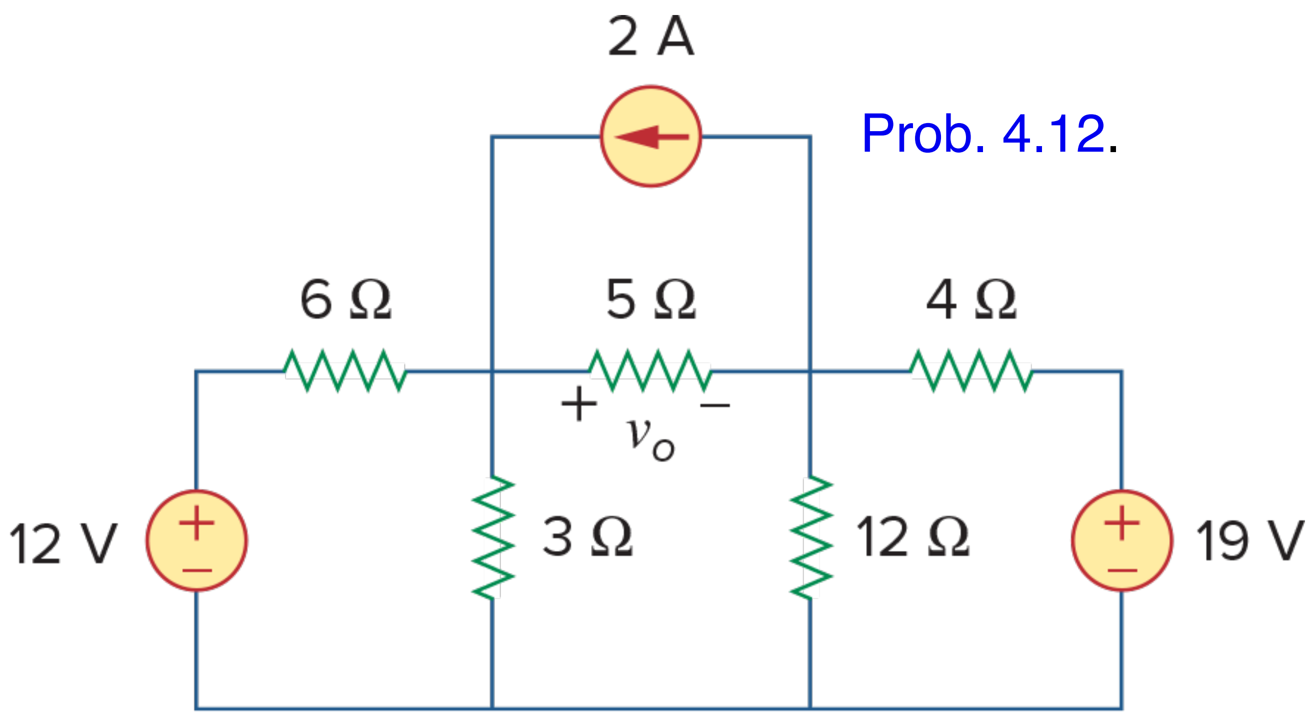
$$\boxed{R_L = R_{Th} = 9\ \Omega}$$

$$\boxed{p_{max} = \frac{V_{Th}^2}{4R_L} = \frac{22^2}{4 \times 9} = 13.44\ W}$$

Superposition



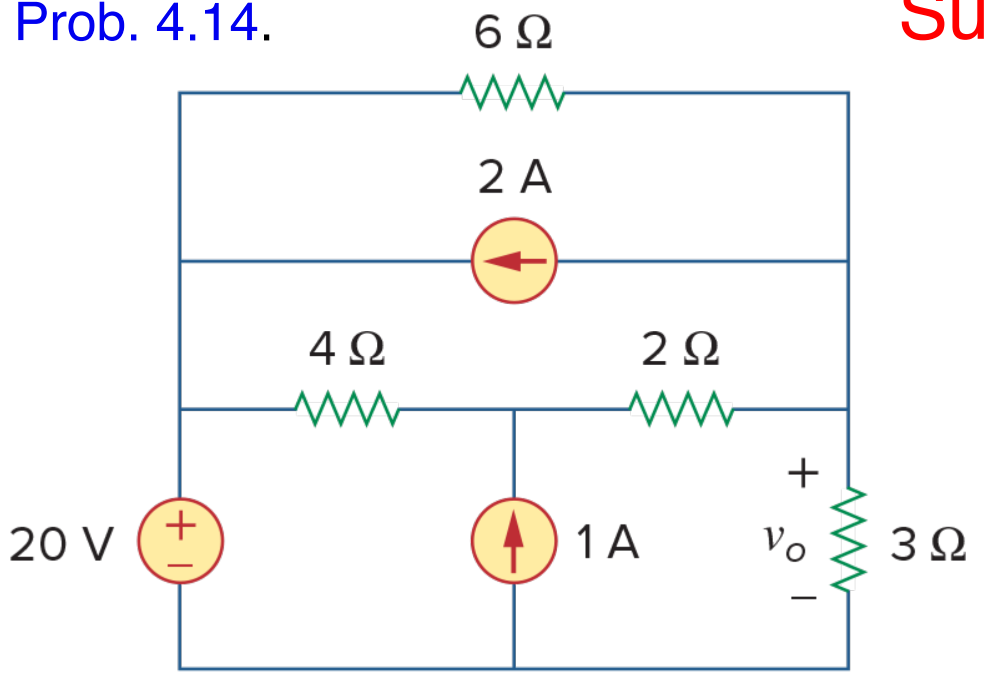
Prob. 4.11.



Prob. 4.12.

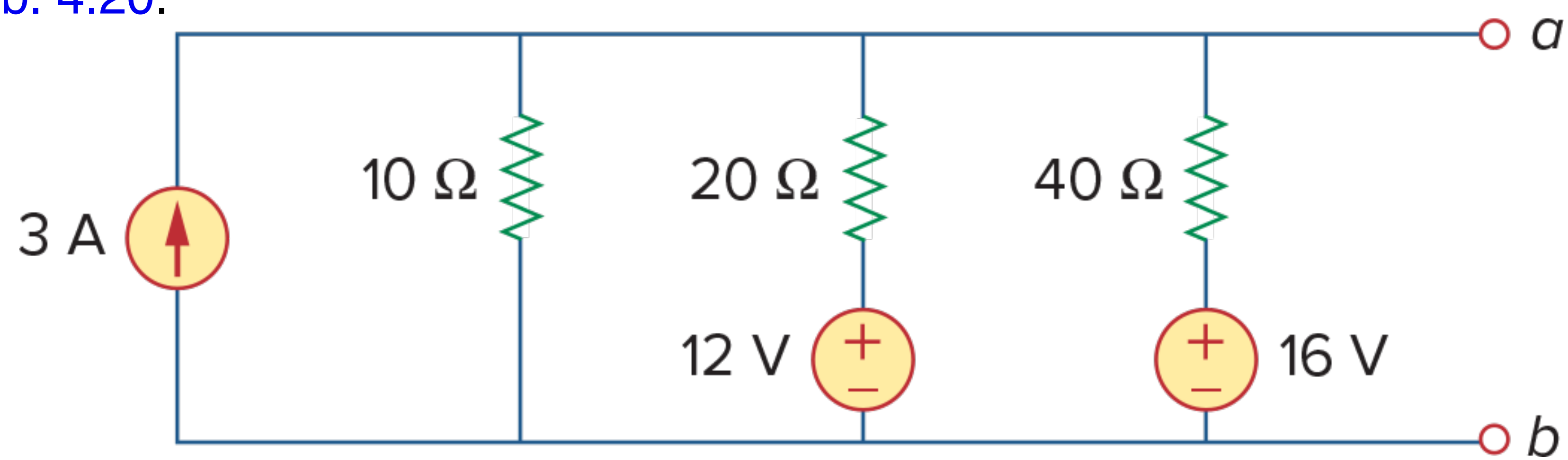
Prob. 4.14.

Superposition



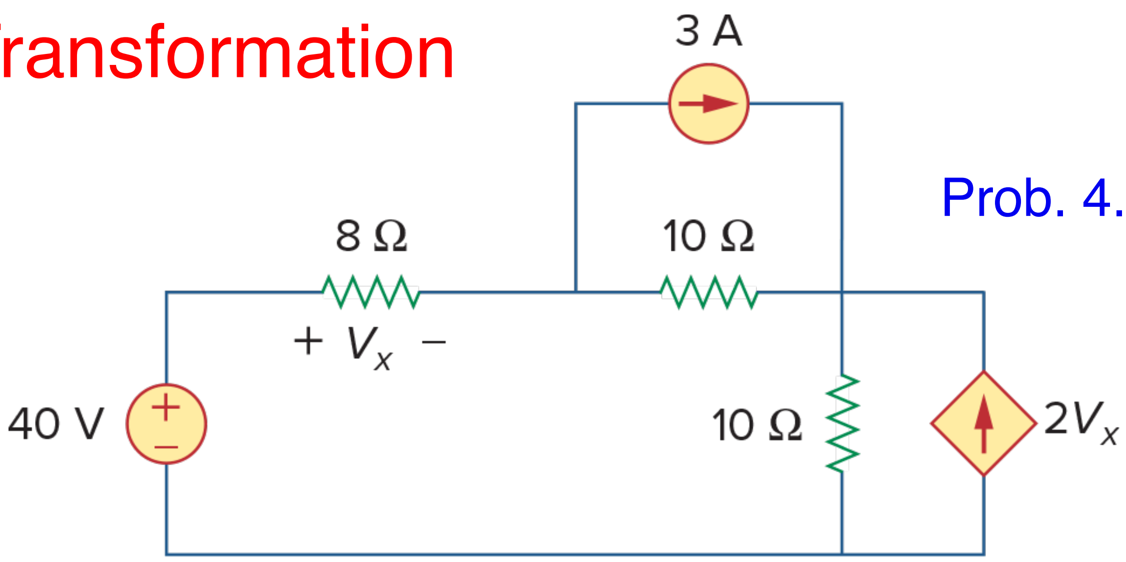
Source Transformation

Prob. 4.20.

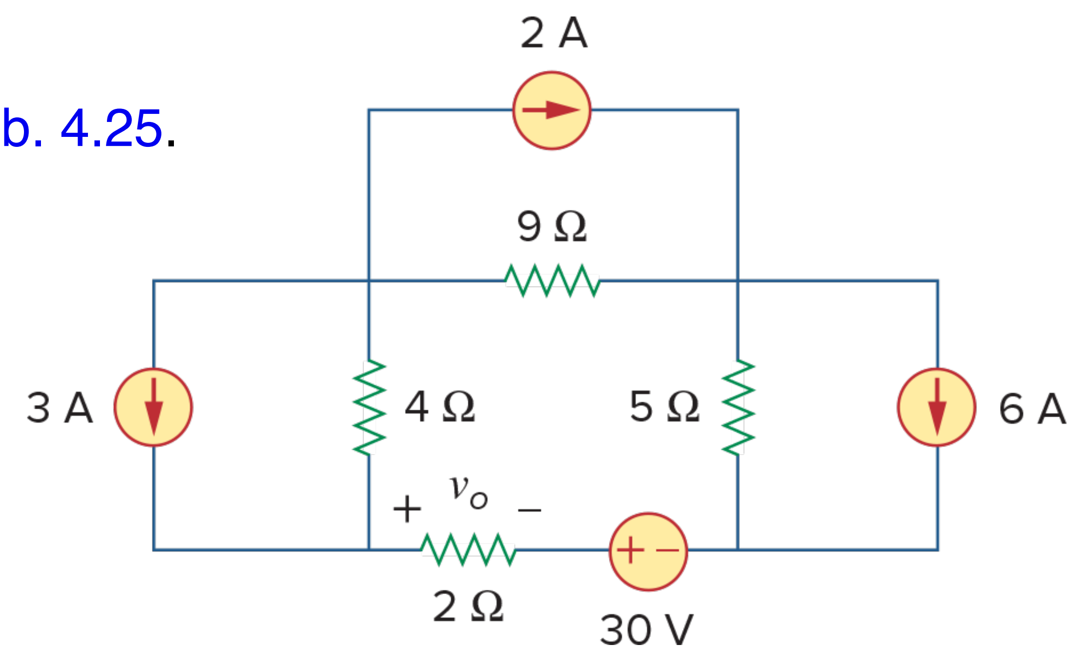


Source Transformation

Prob. 4.24.

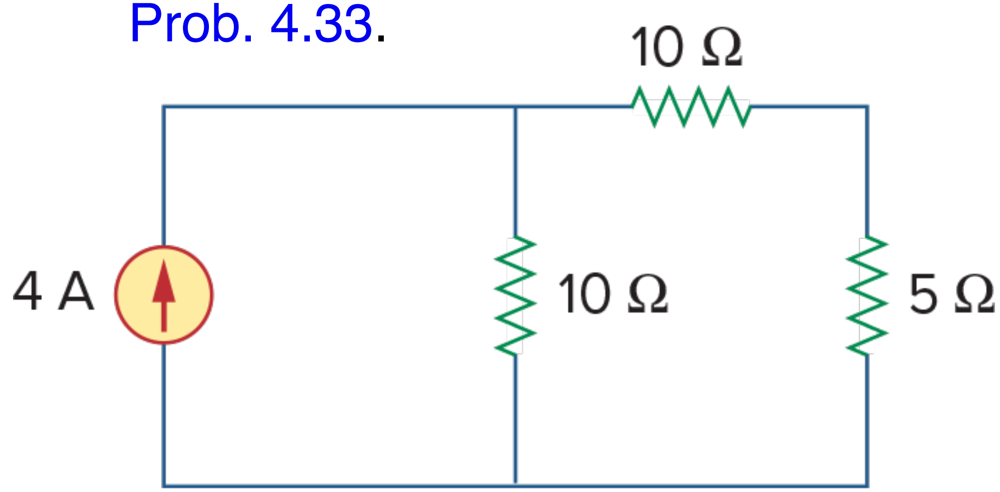


Prob. 4.25.

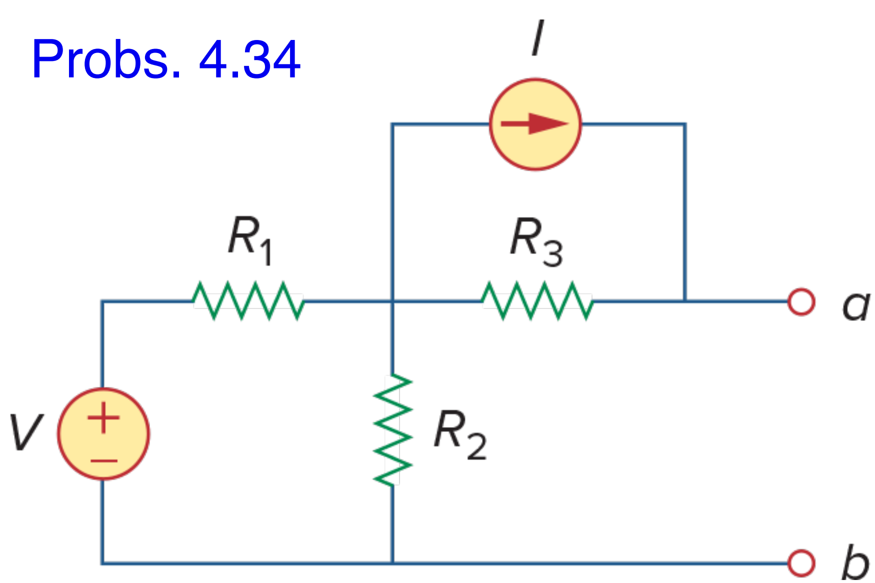


Thevenin's & Norton's

Prob. 4.33.

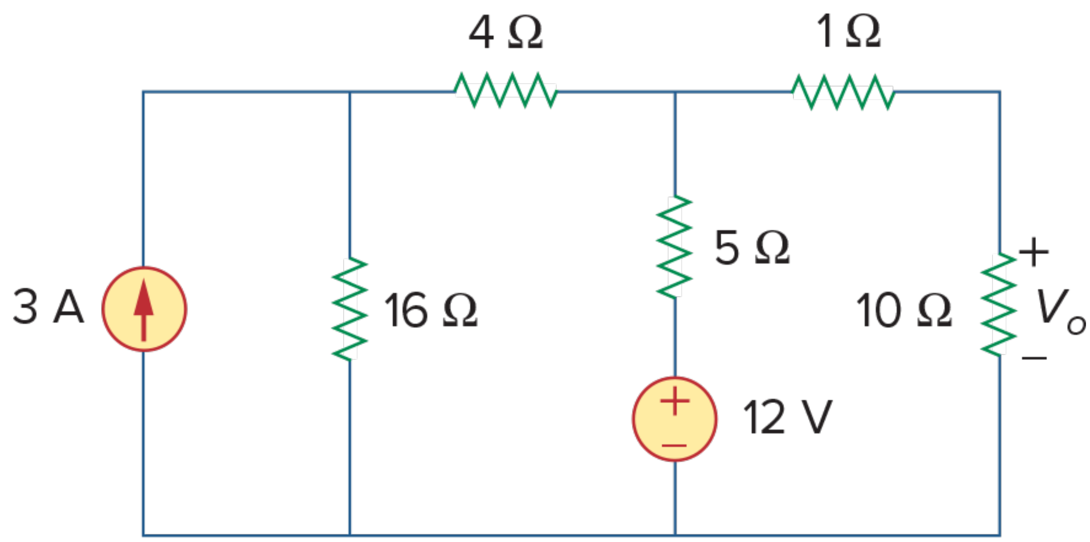


Probs. 4.34

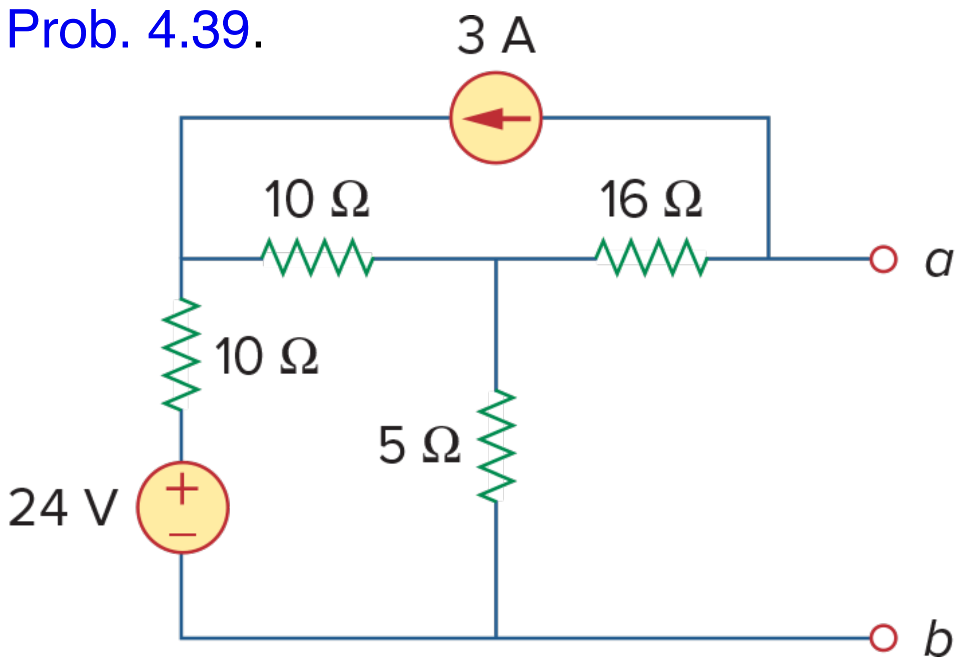


Thevenin's & Norton's

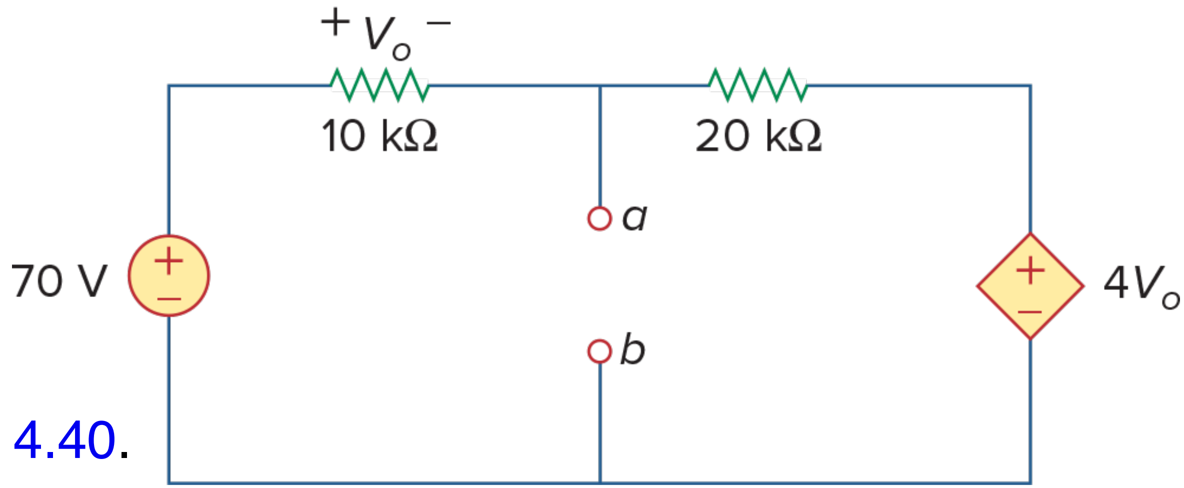
Prob. 4.38. Apply Thevenin's theorem to find Vo



Prob. 4.39.

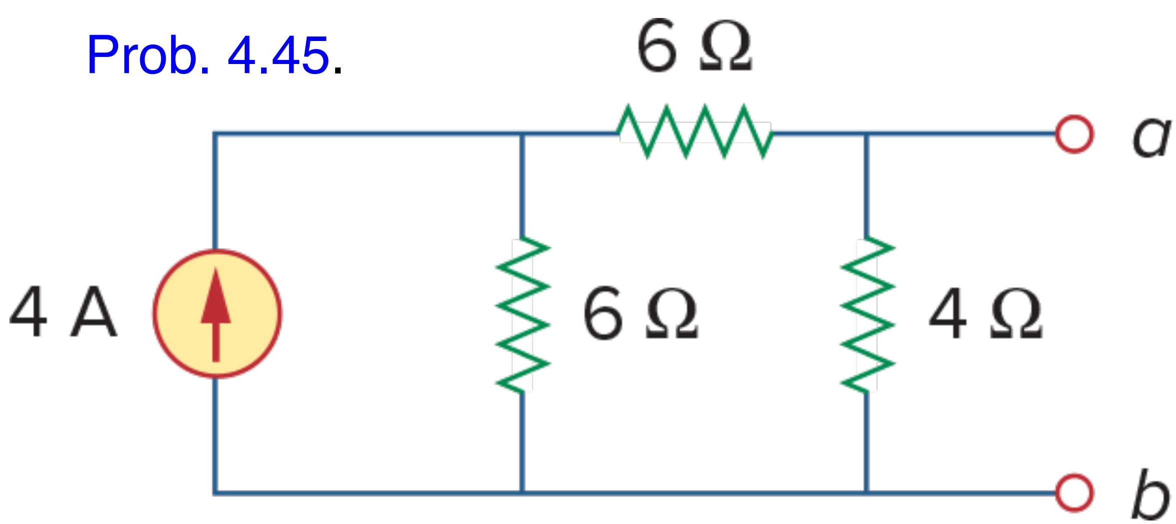


Thevenin's & Norton's



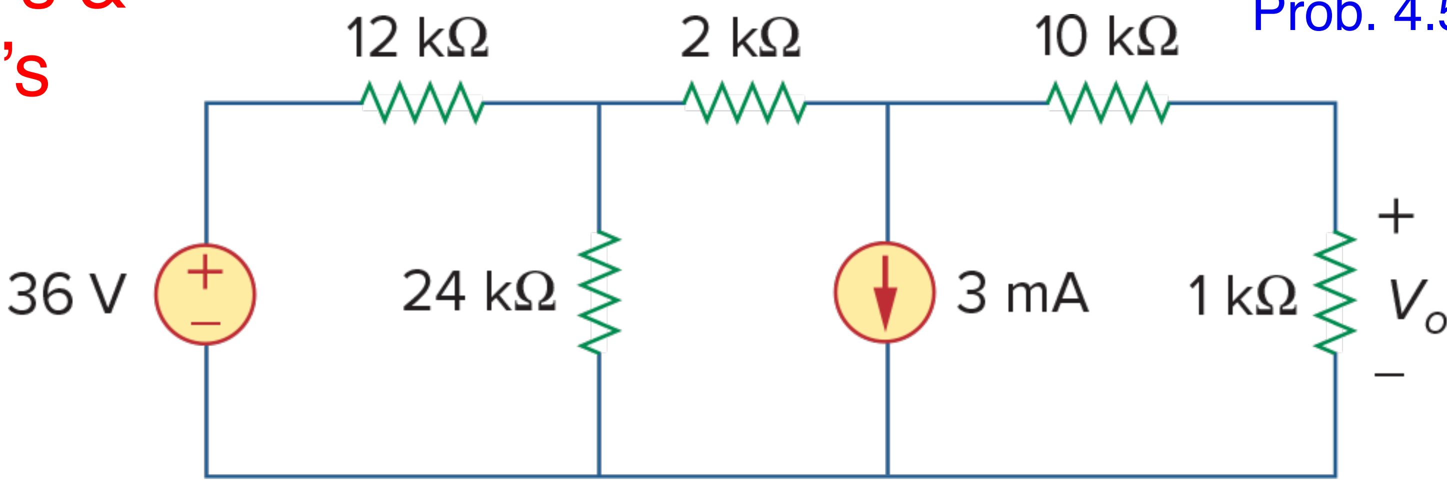
Prob. 4.40.

Prob. 4.45.



Thevenin's & Norton's

Prob. 4.56.



Probs. 4.57