Section 1.2 Solutions and Initial Value Problems

Definition： Explicit Solution

A function $\phi(x)$ that when substituted for y in equation $F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0$ or

$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1} y}{dx^{n-1}}\right) = 0$ satisfies the equation for all x in the interval I is called an

explicit solution to the equation on I.

Definition： Implicit Solution

A relation $G(x, y) = 0$ is said to be an **implicit solution** to equation $F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0$

on the interval I if it defines one or more explicit solutions on I.

Theorem： Existence and Uniqueness of Solution

Given the initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, assume that f and $\frac{\partial f}{\partial y}$ are continuous functions in a rectangle $R = \{(x, y) : a < x < b, c < y < d\}$ that contains the point (x_0, y_0). Then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.

18. Let $c > 0$. Show that the function $\phi(x) = (c^2 - x^2)^{-1}$ is a solution to the initial value problem $dy/dx = 2xy^2$, $y(0) = 1/c^2$, on the interval $-c < x < c$. Note that this solution becomes unbounded as x approaches $\pm c$. Thus, the solution exists on the interval $(-\delta, \delta)$ with $\delta = c$, but not for larger δ. This illustrates that in Theorem 1 the existence interval can be quite small (if c is small) or quite large (if c is large).
Notice also that there is no clue from the equation $dy/dx = 2xy^2$ itself, or from the initial value, that the solution will "blow up" at $x = \pm c$.

Sol.

Let $y(x) = \phi(x) = (c^2 - x^2)^{-1}$

$$\Rightarrow \frac{dy}{dx} = \phi'(x) = -(c^2 - x^2)^{-2} \cdot (-2x) = 2x(c^2 - x^2)^{-2} = 2xy^2 \quad \text{and} \quad y(0) = \phi(0) = (c^2 - 0)^{-1} = \frac{1}{c^2}$$

$\therefore$ $\phi(x) = (c^2 - x^2)^{-1}$ is a solution of the initial value problem on the interval $-c < x < c$.

19. Show that the equation $(dy/dx)^2 + y^2 + 4 = 0$ has no (real-valued) solution.

Sol.

$\because$ $(dy/dx)^2 + y^2 \ge 0$, $\forall x$

$\Rightarrow (dy/dx)^2 + y^2 + 4 \ne 0$, $\forall x$

$\therefore$ the equation $(dy/dx)^2 + y^2 + 4 = 0$ has no (real-valued) solution.

29. (a) For the initial value problem (12) of Example 9, show that $\phi_1(x) \equiv 0$ and $\phi_2(x) = (x-2)^3$ are solutions. Hence, this initial value problem has multiple solutions.

Sol.

Example 9

$$\frac{dy}{dx} = 3y^{2/3}, \quad y(2) = 0 \qquad (12)$$

(i) Let $y = \phi_1(x) \equiv 0$,

$$\Rightarrow \frac{dy}{dx} = 0 = 3y^{2/3} \text{ and } y(2) = 0$$

$\therefore$ $\phi_1(x)$ is a solution

(ii) Let $y = \phi_2(x) = (x-2)^3$

$$\Rightarrow \frac{dy}{dx} = 3(x-2)^2 = 3y^{2/3} \text{ and } y(2) = (2-2)^3 = 0$$

$\therefore$ $\phi_2(x) = (x-2)^3$ is also a solution (Therefore the IVP has multiple solutions.)