Section 4.4 Nonhomogeneous Equations：The Method of Undetermined Coefficients

Definition：Nonhomogeneous Linear Equations
Linear second-order constant-coefficient differential equation：

$$ay'' + by' + cy = f(t) \quad (a \neq 0) \quad \text{with} \quad f(t) \neq 0,$$

we called the equation is **nonhomogeneous**.

Method of Undetermined Coefficients
To find a particular solution to the differential equation $ay'' + by' + cy = Ct^m e^{rt}$, use the form

(14) $y_p(t) = t^s (A_m t^m + \cdots + A_1 t + A_0) e^{rt}$

with
(i) $s = 0$，if r is not a root of $ar^2 + br + c = 0$
(ii) $s = 1$，if r is a simple root of $ar^2 + br + c = 0$
(iii) $s = 2$，if r is a double root of $ar^2 + br + c = 0$

To find a particular solution to the differential equation $ay'' + by' + cy = Ct^m e^{\alpha t} \cos \beta t$ or $ay'' + by' + cy = Ct^m e^{\alpha t} \sin \beta t$, use the form

(15) $y_p(t) = t^s (A_m t^m + \cdots + A_1 t + A_0) e^{\alpha t} \cos \beta t + t^s (B_m t^m + \cdots + B_1 t + B_0) e^{\alpha t} \sin \beta t$

with
(iv) $s = 0$，if r is not a root of $ar^2 + br + c = 0$
(v) $s = 1$，if r is a root of $ar^2 + br + c = 0$

◇ Find a particular solution to the differential equation.

10. $y'' + 3y = -9 \qquad (m = 0, r = 0)$

Sol.

$r^2 + 3 = 0 \Rightarrow r = \pm\sqrt{3}i$

$\because$ $r = 0$ is not a root of $r^2 + 3 = 0$

$\therefore$ take $s = 0$

Let $y_p = A_0$

$\Rightarrow y_p' = 0$ and $y_p'' = 0$

$\Rightarrow 0 + 3A = -9$

$\Rightarrow A = -3$

$\therefore$ $y_p = -3$

15. $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y=xe^x \qquad (m=1, r=1)$

Sol.

$r^2-5r+6=(r-2)(r-3)=0 \Rightarrow r=2,3$

$\because$ $r=1$ is not a root of $r^2-5r+6=0$

$\therefore$ take $s=0$

Let $y_p=(A_1x+A_0)e^x$

$$\Rightarrow y_p'=A_1e^x+(A_1x+A_0)e^x=(A_1x+A_1+A_0)e^x$$
$$y_p''=A_1e^x+(A_1x+A_1+A_0)e^x=(A_1x+2A_1+A_0)e^x$$
$$\Rightarrow (A_1x+2A_1+A_0)e^x-5(A_1x+A_1+A_0)e^x+6(A_1x+A_0)e^x=xe^x$$
$$\Rightarrow (A_1-5A_1+6A_1)xe^x+(2A_1+A_0-5A_1-5A_0+6A_0)e^x=xe^x$$
$$\Rightarrow \begin{cases} 2A_1=1 \\ -3A_1+2A_0=0 \end{cases}$$
$$\Rightarrow \begin{cases} A_1=\dfrac{1}{2} \\ A_0=\dfrac{3}{4} \end{cases}$$

$$\therefore y_p=(\frac{1}{2}x+\frac{3}{4})e^x$$

17. $y''-2y'+y=8e^t \qquad (m=0, r=1)$

Sol.

$r^2-2r+1=(r-1)^2=0 \Rightarrow r=1$ （重根）

$\because$ $r=1$ is a double root of $r^2-2r+1=0$

$\therefore$ take $s=2$

Let $y_p=At^2e^t$

$$\Rightarrow y_p'=A(2te^t+t^2e^t)=(2At+At^2)e^t$$
$$y_p''=(2A+2At)e^t+(2At+At^2)e^t=(2A+4At+At^2)e^t$$
$$\Rightarrow (2A+4At+At^2)e^t-2(2At+At^2)e^t+At^2e^t=8e^t$$
$$\Rightarrow 2Ae^t=8e^t$$
$$\Rightarrow A=4$$

$$\therefore \ y_p=4t^2e^t$$

21. $x''(t)-4x'(t)+4x(t)=te^{2t} \qquad (m=1, r=2)$

Sol.

$r^2-4r+4=(r-2)^2=0 \Rightarrow r=2$ （重根）

$\because$ $r=2$ is a double root of $r^2-4r+4=0$

$\therefore$ take $s=2$

Let $x_p=t^2(A_1t+A_0)e^{2t}=(A_1t^3+A_0t^2)e^{2t}$

$$\Rightarrow x_p'=(3A_1t^2+2A_0t)e^{2t}+2(A_1t^3+A_0t^2)e^{2t}=(3A_1t^2+2A_0t+2A_1t^3+2A_0t^2)e^{2t}$$

$$x_p''=(6A_1t+2A_0+6A_1t^2+4A_0t)e^{2t}+2(3A_1t^2+2A_0t+2A_1t^3+2A_0t^2)e^{2t}$$

$$\Rightarrow (6A_1t+2A_0+6A_1t^2+4A_0t)e^{2t}+2(3A_1t^2+2A_0t+2A_1t^3+2A_0t^2)e^{2t}$$
$$-4(3A_1t^2+2A_0t+2A_1t^3+2A_0t^2)e^{2t}+4(A_1t^3+A_0t^2)e^{2t}=te^{2t}$$

$$\Rightarrow (6A_1t+2A_0)e^{2t}=te^{2t}$$

$$\Rightarrow \begin{cases} A_1=\dfrac{1}{6} \\ A_0=0 \end{cases}$$

$\therefore$ $x_p=t^2(A_1t+A_0)e^{2t}=\dfrac{1}{6}t^3e^{2t}$

23. $y''(\theta)-7y'(\theta)=\theta^2 \qquad (m=2, r=0)$

Sol.

$r^2-7r=r(r-7)=0 \Rightarrow r=0,7$

$\because$ $r=0$ is a simple root of $r^2-7r=0$

$\therefore$ take $s=1$

Let $y_p=\theta(A_2\theta^2+A_1\theta+A_0)=A_2\theta^3+A_1\theta^2+A_0\theta$

$$\Rightarrow y_p'=3A_2\theta^2+2A_1\theta+A_0$$

$$y_p''=6A_2\theta+2A_1$$

$$\Rightarrow 6A_2\theta+2A_1-7(3A_2\theta^2+2A_1\theta+A_0)=\theta^2$$

$$\Rightarrow \begin{cases} 6A_2-14A_1=0 \\ 2A_1-7A_0=0 \\ -21A_2=1 \end{cases}$$

$$\Rightarrow \begin{cases} A_2=\dfrac{-1}{21} \\ A_1=\dfrac{-1}{49} \\ A_0=\dfrac{-2}{343} \end{cases}$$

$\therefore$ $y_p=\dfrac{-1}{21}\theta^3-\dfrac{1}{49}\theta^2-\dfrac{2}{343}\theta$

25. $y'' + 2y' + 4y = 111e^{2t}\cos 3t \qquad (m = 0, r = 2 + 3i)$

Sol.

$$r^2 + 2r + 4 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i$$

$\because\ r = 2 + 3i$ is not a root of $r^2 + 2r + 4 = 0$

$\therefore$ take $s = 0$

Let $y_p = Ae^{2t}\cos 3t + Be^{2t}\sin 3t$

$$\Rightarrow y_p' = A(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + B(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$$
$$= (2A + 3B)e^{2t}\cos 3t + (-3A + 2B)e^{2t}\sin 3t$$
$$y_p'' = (2A + 3B)(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + (-3A + 2B)(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$$
$$= (-5A + 12B)e^{2t}\cos 3t + (-12A - 5B)e^{2t}\sin 3t$$
$$\Rightarrow (-5A + 12B + 4A + 6B + 4A)e^{2t}\cos 3t + (-12A - 5B - 6A + 4B + 4B)e^{2t}\sin 3t = 111e^{2t}\cos 3t$$
$$\Rightarrow \begin{cases} 3A + 18B = 111 \\ -18A + 3B = 0 \end{cases}$$
$$\Rightarrow \begin{cases} A = 1 \\ B = 6 \end{cases}$$

$\therefore\ y_p = e^{2t}\cos 3t + 6e^{2t}\sin 3t$

◇ Use the method of undetermined coefficients to find a particular solution to the given higher-order equation.

33. $y''' - y'' + y = \sin t \qquad (m = 0, r = i)$

Sol.

Let $r_0 = i$ 代入 $r^3 - r^2 + 1 = 0 \Rightarrow -i + 1 + 1 = -i + 2 \neq 0$

$\because\ r_0 = i$ is not a root of $r^3 - r^2 + 1 = 0$

$\therefore$ take $s = 0$

Let $y_p = A\cos t + B\sin t$

$$\Rightarrow y_p' = -A\sin t + B\cos t$$
$$y_p'' = -A\cos t - B\sin t$$
$$y_p''' = A\sin t - B\cos t$$
$$\Rightarrow (-B + A + A)\cos t + (A + B + B)\sin t = \sin t$$
$$\Rightarrow \begin{cases} 2A - B = 0 \\ A + 2B = 1 \end{cases}$$

$$\Rightarrow \begin{cases} A = \dfrac{1}{5} \\ B = \dfrac{2}{5} \end{cases}$$

$$\therefore \quad y_p = \frac{1}{5}\cos t + \frac{2}{5}\sin t$$

35. $y''' + y'' - 2y = te^t \qquad (m = 1, r = 1)$

<u>Sol.</u>

$r^3 + r^2 - 2 = (r-1)(r^2 + 2r + 2) = 0 \Rightarrow r = 1,\ -1 \pm i$

$\because \quad r = 1$ is a simple root of $r^3 + r^2 - 2 = 0$

$\therefore$ take $s = 1$

Let $y_p = t(A_1 t + A_0)e^t = (A_1 t^2 + A_0 t)e^t$

$$\Rightarrow y_p' = (2A_1 t + A_0)e^t + (A_1 t^2 + A_0 t)e^t = (A_1 t^2 + 2A_1 t + A_0 t + A_0)e^t$$

$$y_p'' = (2A_1 t + 2A_1 + A_0)e^t + (A_1 t^2 + 2A_1 t + A_0 t + A_0)e^t = (A_1 t^2 + 4A_1 t + A_0 t + 2A_1 + 2A_0)e^t$$

$$y_p''' = (2A_1 t + 4A_1 + A_0)e^t + (A_1 t^2 + 4A_1 t + A_0 t + 2A_1 + 2A_0)e^t = (A_1 t^2 + 6A_1 t + A_0 t + 6A_1 + 3A_0)e^t$$

$$\Rightarrow (A_1 t^2 + 6A_1 t + A_0 t + 6A_1 + 3A_0 + A_1 t^2 + 4A_1 t + A_0 t + 2A_1 + 2A_0 - 2A_1 t^2 - 2A_0 t)e^t = te^t$$

$$\Rightarrow \begin{cases} 10A_1 = 1 \\ 8A_1 + 5A_0 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = \dfrac{1}{10} \\ A_0 = \dfrac{-4}{25} \end{cases}$$

$$\therefore \quad y_p = \left(\frac{1}{10}t^2 - \frac{4}{25}t\right)e^t$$