Section 2.4 Exact Equations

Definition：Exact Differential Form

The differential form $M(x,y)dx+N(x,y)dy$ is said to be **exact** in a rectangle R if there is a function $F(x,y)$ such that $\frac{\partial F}{\partial x}(x,y)=M(x,y)$ and $\frac{\partial F}{\partial y}(x,y)=N(x,y)$ for all (x,y) in R.

That is, the total differential of $F(x,y)$ satisfies $dF(x,y)=M(x,y)dx+N(x,y)dy$.

If $M(x,y)dx+N(x,y)dy$ is an exact differential form, then the equation $M(x,y)dx+N(x,y)dy=0$ is called an **exact equation**.

Test for Exactness

$M(x,y)dx+N(x,y)dy=0$ is called an exact equation $\Leftrightarrow \frac{\partial M}{\partial y}(x,y)=\frac{\partial N}{\partial x}(x,y)$.

Method for Solving Exact Equations

1. $M(x,y)dx+N(x,y)dy=0$ is exact.
2. To determine $F(x,y)$：
   i. $F(x,y)=\int M(x,y)dx+g(y)$ or
   ii. $F(x,y)=\int N(x,y)dy+g(x)$
3. To determine $g(y)$ or $g(x)$
   i. $\frac{\partial F}{\partial y}=f(x,y)+g'(y)=N(x,y)\Rightarrow g'(y)=?\Rightarrow g(y)=?$
   ii. $\frac{\partial F}{\partial x}=f(x,y)+g'(x)=M(x,y)\Rightarrow g'(x)=?\Rightarrow g(x)=?$
4. To write the solution：$F(x,y)=C$

◇ Classify the equation as separable, linear, exact, or none of these. Notice that some equations may have more than one classification.

6. $y^2dx+(2xy+\cos y)dy=0$

Sol.

(1) The equation is not separable.

(2) $\because\ y^2dx+(2xy+\cos y)dy=0\Rightarrow \frac{dx}{dy}+\underbrace{\frac{2}{y}}_{P(y)}x=\underbrace{-\frac{\cos y}{y^2}}_{Q(y)}$ $\therefore$ The equation is linear.

(3) $\because\ \frac{\partial}{\partial y}[y^2]=2y=\frac{\partial}{\partial x}[2xy+\cos y]$ $\therefore$ The equation is exact.

◇ Determine whether the equation is exact. If it is, then solve it.

15. $\cos\theta dr-(r\sin\theta-e^{\theta})d\theta=0$

Sol.

$$\cos\theta dr-(r\sin\theta-e^{\theta})d\theta=0\ \Rightarrow \cos\theta dr+(-r\sin\theta+e^{\theta})d\theta=0$$

$\because\ \frac{\partial}{\partial\theta}[\cos\theta]=-\sin\theta$ $\quad\therefore$ it's an exact equation.

Let $F(r,\theta)=\int\cos\theta dr+g(\theta)=r\cos\theta+g(\theta)$

$\because\ \frac{\partial F}{\partial\theta}=-r\sin\theta+g'(\theta)=-r\sin\theta+e^{\theta}\ \Rightarrow g'(\theta)=e^{\theta}\Rightarrow g(\theta)=e^{\theta}$

$\therefore\ F(r,\theta)=r\cos\theta+e^{\theta}=C$ is a solution.

19. $\left(2x+\frac{y}{1+x^2y^2}\right)dx+\left(\frac{x}{1+x^2y^2}-2y\right)dy=0$

Sol.

$\because\ \frac{\partial}{\partial y}\left[2x+\frac{y}{1+x^2y^2}\right]=\frac{(1+x^2y^2)-y(2x^2y)}{(1+x^2y^2)^2}=\frac{1-x^2y^2}{(1+x^2y^2)^2}$ and

$\frac{\partial}{\partial x}\left[\frac{x}{1+x^2y^2}-2y\right]=\frac{(1+x^2y^2)-x(2xy^2)}{(1+x^2y^2)^2}=\frac{1-x^2y^2}{(1+x^2y^2)^2}$

$\therefore$ it's an exact equation.

Let $F(x,y)=\int(2x+\frac{y}{1+x^2y^2})dx+g(y)=x^2+\tan^{-1}(xy)+g(y)$

$\because\ \frac{\partial F}{\partial y}=\frac{x}{1+x^2y^2}+g'(y)=\frac{x}{1+x^2y^2}-2y\ \Rightarrow g'(y)=-2y\Rightarrow g(y)=-y^2$

$\therefore F(x,y)=x^2+\tan^{-1}(xy)-y^2=C$ is a solution.

◇ Solve the initial value problem.

23. $(e^ty+te^ty)dt+(te^t+2)dy=0,\ \ y(0)=-1$

Sol.

$\because\ \frac{\partial}{\partial y}[e^ty+te^ty]=e^t+te^t$ and $\frac{\partial}{\partial t}[te^t+2]=e^t+te^t$ $\quad\therefore$ it's an exact equation.

Let $F(t,y)=\int(te^t+2)dy+g(t)=te^ty+2y+g(t)$

$\because\ \frac{\partial F}{\partial t}=y(e^t+te^t)+g'(t)=e^ty+te^ty\Rightarrow g'(t)=0\Rightarrow g(t)=C_1$ or take $g(t)=0$

$\therefore\ F(t,y)=te^ty+2y=C$

$\because \quad y(0) = -1 \Rightarrow 0 \cdot (-1) - 2 = C \Rightarrow C = -2$

$\therefore \quad F(t, y) = te^t y + 2y = -2$ is the solution of the IVP.

25. $(y^2 \sin x)dx + (\frac{1}{x} - \frac{y}{x})dy = 0$, $y(\pi) = 1$

Sol.

$\because \quad \frac{\partial}{\partial y}[y^2 \sin x] = 2y \sin x \neq \frac{\partial}{\partial x}\left[\frac{1}{x} - \frac{y}{x}\right] = -\frac{1}{x^2} + \frac{y}{x^2}$ $\quad \therefore$ it's not an exact equation.

(另解)

$$(y^2 \sin x)dx + (\frac{1}{x} - \frac{y}{x})dy = 0$$

$$\Rightarrow y^2 \sin x dx + \frac{1-y}{x} dy = 0$$

$$\Rightarrow x \sin x dx + \frac{1-y}{y^2} dy = 0$$

$$\Rightarrow x \sin x dx = \frac{y-1}{y^2} dy$$

$$\Rightarrow \int x \sin x dx = \int \frac{y-1}{y^2} dy$$

$$\Rightarrow \int x \sin x dx = \int (\frac{1}{y} - \frac{1}{y^2}) dy$$

$$\Rightarrow -x \cos x + \sin x = \ln|y| + \frac{1}{y} + C$$

$$\int x \sin x dx$$

$$\begin{pmatrix} u = x & dv = \sin x dx \\ du = dx & v = -\cos x \end{pmatrix}$$

$\because \quad y(\pi) = 1 \Rightarrow -\pi \cos \pi + \sin \pi = \ln 1 + 1 + C \Rightarrow \pi = 1 + C \Rightarrow C = \pi - 1$

$\therefore -x \cos x + \sin x = \ln y + \frac{1}{y} + \pi - 1$ is the solution of the IVP.

($\ln|y| = \ln y$ since the initial point, $y > 0$)

29. Consider the equation

$$(y^2 + 2xy)dx - x^2 dy = 0.$$

(a) Show that this equation is not exact.

Sol.

$\because \quad \frac{\partial}{\partial y}[y^2 + 2xy] = 2y + 2x \neq \frac{\partial}{\partial x}\left[-x^2\right] = -2x$ $\quad \therefore$ it's not an exact equation.

(b) Show that multiplying both sides of the equation by y^{-2} yields a new equation that is exact.

Sol.

$$y^{-2} \cdot (y^2 + 2xy)dx - y^{-2} \cdot x^2 dy = 0 \Rightarrow (1 + 2xy^{-1})dx + (-x^2 y^{-2})dy = 0$$

$\because \quad \frac{\partial}{\partial y}[2xy^{-1}] = -2xy^{-2} = \frac{\partial}{\partial x}\left[-x^2y^{-2}\right] = -2xy^{-2}$

$\therefore$ it's an exact equation.

(c) Use the solution of the resulting exact equation to solve the original equation.

Sol.

Let $F(x,y) = \int (1+2xy^{-1})dx + g(y) = x + x^2y^{-1} + g(y)$

$\because \quad \frac{\partial F}{\partial y} = -x^2y^{-2} + g'(y) = -x^2y^{-2} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$ or take $g(y) = 0$

$\therefore \quad F(x,y) = x + x^2y^{-1} = C$ is a solution of the original equation.

(d) Were any solutions lost in the process?

Sol.

Yes, $y \equiv 0$ is also a solution.

(會造成這問題是因為在題目原本的方程式中，y 並無特殊限制，但在(b)小題中，乘上 y^{-2} 時，便同時給了一個 $y \neq 0$ 之限制)

30. Consider the equation

$(5x^2y + 6x^3y^2 + 4xy^2)dx + (2x^3 + 3x^4y + 3x^2y)dy = 0$.

(a) Show that the equation is not exact.

Sol.

$\because \quad \frac{\partial}{\partial y}[5x^2y + 6x^3y^2 + 4xy^2] = 5x^2 + 12x^3y + 8xy \neq \frac{\partial}{\partial x}\left[2x^3 + 3x^4y + 3x^2y\right] = 6x^2 + 12x^3y + 6xy$

$\therefore$ it's not an exact equation.

(b) Multiply the equation by $x^n y^m$ and determine values for n and m that make the resulting equation exact.

Sol.

原式$\times x^n y^m$

$\Rightarrow (5x^{2+n}y^{1+m} + 6x^{3+n}y^{2+m} + 4x^{1+n}y^{2+m})dx + (2x^{3+n}y^m + 3x^{4+n}y^{1+m} + 3x^{2+n}y^{1+m})dy = 0$

$\frac{\partial}{\partial y}(5x^{2+n}y^{1+m} + 6x^{3+n}y^{2+m} + 4x^{1+n}y^{2+m}) = 5(1+m)x^{2+n}y^m + 6(2+m)x^{3+n}y^{1+m} + 4(2+m)x^{1+n}y^{1+m}$

$\frac{\partial}{\partial x}(2x^{3+n}y^m + 3x^{4+n}y^{1+m} + 3x^{2+n}y^{1+m}) = 2(3+n)x^{2+n}y^m + 3(4+n)x^{3+n}y^{1+m} + 3(2+n)x^{1+n}y^{1+m}$

$$\Rightarrow \begin{cases} 5+5m = 6+2n \\ 12+6m = 12+3n \\ 8+4m = 6+3n \end{cases} \Rightarrow \begin{cases} 5m-2n = 1 \\ 6m-3n = 0 \\ 4m-3n = -2 \end{cases} \Rightarrow \begin{cases} m = 1 \\ n = 2 \end{cases}$$

(c) Use the solution of the resulting exact equation to solve the original equation.

Sol.

$$(5x^4y^2+6x^5y^3+4x^3y^3)dx+(2x^5y+3x^6y^2+3x^4y^2)dy=0$$

Let $F(x,y)=\int(5x^4y^2+6x^5y^3+4x^3y^3)dx+g(y)=x^5y^2+x^6y^3+x^4y^3+g(y)$

$\because \dfrac{\partial F}{\partial y}=2x^5y+3x^6y^2+3x^4y^2+g'(y)=2x^5y+3x^6y^2+3x^4y^2$

$\Rightarrow g'(y)=0 \Rightarrow$ take $g(y)=0$

$\therefore F(x,y)=x^5y^2+x^6y^3+x^4y^3=C$ is a solution of the original equation.