Section 6.2 Homogeneous Linear Equations with Constant Coefficients

Homogeneous linear *n*th-order differential equation：

(1) $a_n(x)y^n(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_0(x)y(x)=0$,

where $a_n, a_{n-1}, \ldots, a_0$ are real constants.

If $y_1,\ldots,y_n$ are the solutions and they are L.I., we can express a general solution to (1) in the form

(2) $y(x)=C_1y_1+\cdots+C_ny_n$, where $C_1,\ldots,C_n$ as arbitrary constants.

Let $L[y]:=a_ny^n+a_{n-1}y^{(n-1)}+\cdots+a_0y\overset{(1)}{\Rightarrow}L[y](x)=0$.

For $y=e^{rx}$, we find **auxiliary equation**

(6) $P(r)=a_nr^n+a_{n-1}r^{(n-1)}+\cdots+a_0=0$

Since $r_1,\ldots,r_n$ are the zeros of the auxiliary polynomial $P(r)$ can be factored as

(9) $P(r)=a_n(r-r_1)\cdots(r-r_n)$.

Consequently, the operator $L[y]=a_ny^n+a_{n-1}y^{(n-1)}+\cdots+a_0y$ can be expressed in terms of the differentiation operator D as the following composition：

(10) $L=P(D)=a_n(D-r_1)\cdots(D-r_n)$.

◇ Find a general solution for the differential equation with x as the independent variable.

3. $6z'''+7z''-z'-2z=0$

Sol.

Let $z=e^{rx}$，輔助方程式為

$$6r^3+7r^2-r-2=0$$

$$\Rightarrow (r+1)(6r^2+r-2)=0$$

$$\Rightarrow (r+1)(3r+2)(2r-1)=0$$

$$\Rightarrow r=-1,-\frac{2}{3},\frac{1}{2}$$

$$\therefore\ z(x)=c_1e^{-x}+c_2e^{-\frac{2}{3}x}+c_3e^{\frac{1}{2}x}$$

9. $u'''-9u''+27u'-27u=0$

Sol.

Let $z=e^{rx}$，輔助方程式為

$$r^3-9r^2+27r-27=0$$

$$\Rightarrow (r-3)^3=0$$

$\Rightarrow r = 3, 3, 3$

$\therefore \quad u(x) = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}$

◇ Find a general solution to the given homogeneous equation.

17. $(D+4)(D-3)(D+2)^3(D^2+4D+5)^2 \cdot D^5[y] = 0$

Sol.

Let $y = e^{rx}$，輔助方程式為

$(r+4)(r-3)(r+2)^3(r^2+4r+5)^2 \cdot r^5 = 0$

$\Rightarrow r_1 = -4$， $r_2 = 3$， $r_3 = r_4 = r_5 = -2$， $r_6 = r_7 = -2+i$， $r_8 = r_9 = -2-i$

$r_{10} = r_{11} = r_{12} = r_{13} = r_{14} = 0$

Hence,

$$y(t) = c_1 e^{-4x} + c_2 e^{3x} + (c_3 + c_4 x + c_5 x^2)e^{-2x} + (c_6 + c_7 x)e^{-2x}\cos x + (c_8 + c_9 x)e^{-2x}\sin x$$
$$+ c_{10} + c_{11}x + c_{12}x^2 + c_{13}x^3 + c_{14}x^4$$

18. $(D-1)^3(D-2)(D^2+D+1)\cdot(D^2+6D+10)^3[y] = 0$

Sol.

Let $y = e^{rx}$，輔助方程式為

$(r-1)^3(r-2)(r^2+r+1)\cdot(r^2+6r+10)^3 = 0$

$\Rightarrow r_1 = r_2 = r_3 = 1$， $r_4 = 2$， $r_5 = \dfrac{-1+\sqrt{3}i}{2}$， $r_6 = \dfrac{-1-\sqrt{3}i}{2}$， $r_7 = r_8 = r_9 = -3+i$，

$r_{10} = r_{11} = r_{12} = -3-i$

Hence,

$$y(t) = (c_1 + c_2 x + c_3 x^2)e^{x} + c_4 e^{2x} + c_5 e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}x}{2} + c_6 e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}x}{2}$$
$$+ (c_7 + c_8 x + c_9 x^2)e^{-3x}\cos x + (c_{10} + c_{11}x + c_{12}x^2)e^{-3x}\sin x$$

◇ Solve the given initial value problem.

21. $y''' - 4y'' + 7y' - 6y = 0$； $y(0) = 1$， $y'(0) = 0$， $y''(0) = 0$

Sol.

$r^3 - 4r^2 + 7r - 6 = 0$

$\Rightarrow (r-2)(r^2 - 2r + 3) = 0$

$\Rightarrow r = 2, 1 \pm \sqrt{2}i$

$\therefore \quad y(x) = c_1 e^{2x} + c_2 e^{x}\cos\sqrt{2}x + c_3 e^{x}\sin\sqrt{2}x$

$$y'(x)=2c_1e^{2x}+c_2(e^x\cos\sqrt{2}x-\sqrt{2}e^x\sin\sqrt{2}x)+c_3(e^x\sin\sqrt{2}x+\sqrt{2}e^x\cos\sqrt{2}x)$$
$$=2c_1e^{2x}+(c_2+\sqrt{2}c_3)e^x\cos\sqrt{2}x+(c_3-\sqrt{2}c_2)e^x\sin\sqrt{2}x$$
$$y''(x)=4c_1e^{2x}+(c_2+\sqrt{2}c_3)(e^x\cos\sqrt{2}x-\sqrt{2}e^x\sin\sqrt{2}x)$$
$$+(c_3-\sqrt{2}c_2)(e^x\sin\sqrt{2}x+\sqrt{2}e^x\cos\sqrt{2}x)$$
$$=4c_1e^{2x}+(c_2+\sqrt{2}c_3+\sqrt{2}c_3-2c_2)e^x\cos\sqrt{2}x+(c_3-\sqrt{2}c_2+c_3-\sqrt{2}c_2)e^x\sin\sqrt{2}x$$
$$=4c_1e^{2x}+(-c_2+2\sqrt{2}c_3)e^x\cos\sqrt{2}x+(-2\sqrt{2}c_2+2c_3)e^x\sin\sqrt{2}x$$

$\because\ y(0)=1,\ y'(0)=0,\ y''(0)=0$

$$\Rightarrow\begin{cases}c_1+c_2=1\\2c_1+c_2+\sqrt{2}c_3=0\\4c_1-c_2+2\sqrt{2}c_3=0\end{cases}\Rightarrow\begin{cases}c_1=1\\c_2=0\\c_3=-\sqrt{2}\end{cases}$$

$\therefore\ y(x)=e^{2x}-\sqrt{2}e^x\sin\sqrt{2}x$

25. Show that the m functions $e^{rx},\ xe^{rx},\dots,\ x^{m-1}e^{rx}$ are linearly independent on $(-\infty,\infty)$. [*Hint*: Show that these functions are linearly independent if and only if $1,\ x,\dots,x^{m-1}$ are linearly independent.]

<u>Sol.</u>

Assume $c_0,\ c_1,\dots,c_{m-1}$ are constants for which

$$c_0e^{rx}+c_1xe^{rx}+\cdots+c_{m-1}x_{m-1}e^{rx}=0$$
$$\Leftrightarrow (c_0+c_1x+\cdots+c_{m-1}x_{m-1})e^{rx}=0$$
$$\Leftrightarrow c_0=c_1=\cdots=c_{m-1}=0$$

$\therefore\ e^{rx},\ xe^{rx},\dots,x^{m-1}e^{rx}$ are L.I.

30. (a) Derive the form $y(x)=A_1e^x+A_2e^{-x}+A_3\cos x+A_4\sin x$ for the general solution to the equation $y^{(4)}=y$, from the observation that the fourth roots of unity are 1, -1, i, and $-i$.

<u>Sol.</u>

Let $y=e^{rx}$，輔助方程式為

$$r^4-1=0$$
$$\Rightarrow (r-1)(r+1)(r^2+1)$$
$$\Rightarrow r=\pm1,\ \pm i$$

$\therefore \quad y(x) = A_1 e^x + A_2 e^{-x} + A_3 \cos x + A_4 \sin x$

(b) Derive the form $y(x) = A_1 e^x + A_2 e^{-x/2} \cos(\sqrt{3}x/2) + A_3 e^{-x/2} \sin x(\sqrt{3}x/2)$ for the general solution to the equation $y^{(3)} = y$, from the observation that the cube roots of unity are 1, $e^{i2\pi/3}$, and $e^{-i2\pi/3}$.

<u>Sol.</u>

Let $y = e^{rx}$，輔助方程式為

$$r^3 - 1 = 0$$
$$\Rightarrow (r-1)(r^2 + r + 1) = 0$$
$$\Rightarrow r = 1,\ \frac{-1 \pm \sqrt{3}i}{2}$$

$\therefore \quad y(x) = A_1 e^x + A_2 e^{-x/2} \cos(\sqrt{3}x/2) + A_3 e^{-x/2} \sin x(\sqrt{3}x/2)$

31. **Higher-Order Cauchy-Euler Equations.** A differential equation that can be expressed in the form $a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_0 y(x) = 0$ where a_n, $a_{n-1}, \ldots, a_0$ are constants, is called a homogeneous **Cauchy-Euler** equation. (The second-order case is discussed in Section 4.7.) Use the substitution $y = x^r$ to help determine a fundamental solution set for the following Cauchy-Euler equations：

(a) $x^3 y''' + x^2 y'' - 2xy' + 2y = 0$, $x > 0$.

<u>Sol.</u>

Let $y = x^r \Rightarrow y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$, and $y''' = r(r-1)(r-2)x^{r-3}$

$$\Rightarrow x^3 \cdot r(r-1)(r-2)x^{r-3} + x^2 \cdot r(r-1)x^{r-2} - 2x \cdot rx^{r-1} + 2x^r = 0$$
$$\Rightarrow [r(r-1)(r-2) + r(r-1) - 2r + 2]x^r = 0$$
$$\Rightarrow r(r-1)(r-2) + r(r-1) - 2r + 2 = 0$$
$$\Rightarrow r(r-1)(r-2) + r(r-1) - 2(r-1) = 0$$
$$\Rightarrow (r-1)[r(r-2) + r - 2] = 0$$
$$\Rightarrow (r-1)(r^2 - r - 2) = 0$$
$$\Rightarrow r = 1,\ 2,\ -1$$

$\therefore \quad y_1 = x,\ y_2 = x^2,\ y_3 = x^{-1}$ are the solutions.

If $y_1 = x,\ y_2 = x^2,\ y_3 = x^{-1}$ are L.I., then $\{x, x^2, x^{-1}\}$ is the fundamental solution set.

(b) $x^4y^{(4)}+6x^3y'''+2x^2y''-4xy'+4y=0$, $x>0$.

Sol.

Let $y=x^r \Rightarrow y'=rx^{r-1}$, $y''=r(r-1)x^{r-2}$, $y'''=r(r-1)(r-2)x^{r-3}$, and

$y^{(4)}=r(r-1)(r-2)(r-3)x^{r-4}$

$$\Rightarrow x^4\cdot r(r-1)(r-2)(r-3)x^{r-4}+6x^3\cdot r(r-1)(r-2)x^{r-3}+2x^2\cdot r(r-1)x^{r-2}-4x\cdot rx^{r-1}+4x^r=0$$
$$\Rightarrow [r(r-1)(r-2)(r-3)+6r(r-1)(r-2)+2r(r-1)-4r+4]x^r=0$$
$$\Rightarrow r(r-1)(r-2)(r-3)+6r(r-1)(r-2)+2r(r-1)-4r+4=0$$
$$\Rightarrow r(r-1)(r-2)(r-3)+6r(r-1)(r-2)+2r(r-1)-4(r-1)=0$$
$$\Rightarrow (r-1)[r(r-2)(r-3)+6r(r-2)+2r-4]=0$$
$$\Rightarrow (r-1)(r-2)[r(r-3)+6r+2]=0$$
$$\Rightarrow (r-1)(r-2)(r^2+3r+2)=0$$
$$\Rightarrow r=1,\ 2,\ -1,\ -2$$

$\therefore$ $y_1=x,\ y_2=x^2,\ y_3=x^{-1},\ x^{-2}$ are solutions.

If $y_1=x, y_2=x^2, y_3=x^{-1}, x^{-2}$ are L.I., then $\{x, x^2, x^{-1}, x^{-2}\}$ is the fundamental solution set.

(c) $x^3y'''-2x^2y''+13xy'-13y=0$, $x>0$.

Sol.

$$\Rightarrow x^3\cdot r(r-1)(r-2)x^{r-3}-2x^2\cdot r(r-1)x^{r-2}+13x\cdot rx^{r-1}-13x^r=0$$
$$\Rightarrow [r(r-1)(r-2)-2r(r-1)+13r-13]x^r=0$$
$$\Rightarrow r(r-1)(r-2)-2r(r-1)+13r-13=0$$
$$\Rightarrow r(r-1)(r-2)-2r(r-1)+13(r-1)=0$$
$$\Rightarrow (r-1)[r(r-2)-2r+13]=0$$
$$\Rightarrow (r-1)(r^2-4r+13)=0$$
$$\Rightarrow r=1, 2\pm 3i$$

$\therefore$ $y_1=x,\ y_2=x^2\cos(3\ln x),\ y_3=x^2\sin(3\ln x)$ are solutions.

If $y_1=x,\ y_2=x^2\cos(3\ln x),\ y_3=x^2\sin(3\ln x)$ are L.I., then $\{x,\ x^2\cos(3\ln x), x^2\sin(3\ln x)\}$

is the fundamental solution set.