Section 2.6 Substitutions and Transformations

Definition：Homogeneous Equation

If the right-hand side of the equation $\frac{dy}{dx} = f(x,y)$ can be expressed as a function of the ratio y/x alone, then we say the equation is **homogeneous**.

Method for Solving Homogeneous Equations

$$\frac{dy}{dx} = f(x,y) = G(y/x)$$

Let $z = \frac{y}{x}$

$$\Rightarrow y = xz$$

$$\Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx} = G(z)$$

$$\Rightarrow x\frac{dz}{dx} = G(z) - z \quad \text{(separable)}$$

$$\Rightarrow \int \frac{1}{G(z) - z}\,dz = \int \frac{1}{x}\,dx$$

Equations of the Form $dy/dx = G(ax+by)$

When the right-hand side of the equation $\frac{dy}{dx} = f(x,y)$ can be expressed as a function of the combination $ax+by$, where a and b are constants.

Method for Solving $dy/dx = G(ax+by)$

Let $z = ax + by$

$$\Rightarrow \frac{dz}{dx} = a + b\frac{dy}{dx} = a + bG(z) \quad \text{(separable)}$$

$$\Rightarrow \int \frac{1}{a + bG(z)}\,dz = \int dx$$

Definition：Bernoulli Equation

A first-order equation that can be written in the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ where $P(x)$ and $Q(x)$ are continuous on an interval (a,b) and n is a real number, is called a **Bernoulli Equation.**

Method for Solving Bernoulli Equations

$$\frac{dy}{dx}+P(x)y=Q(x)y^n,\ n\neq 0 \text{ or } 1$$

$$\Rightarrow y^{-n}\frac{dy}{dx}+P(x)y^{1-n}=Q(x)$$

Let $z=y^{1-n}$

$$\Rightarrow \frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dz}{dx}$$

$$\Rightarrow \frac{1}{1-n}\frac{dz}{dx}+P(x)z=Q(x)\quad (\text{ linear })$$

Equation with Linear Coefficients

Equations of the form $(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0$, where the a_i's, b_i's, and c_i's

Method for Solving Equation with Linear Coefficients

Type I. $c_1=c_2=0\Rightarrow$ homogeneous

Type II. $a_1b_2=a_2b_1\Rightarrow \dfrac{dy}{dx}=G(ax+by)$

Type III. $a_1b_2\neq a_2b_1\Rightarrow$ Let $\begin{cases} x=u+h \\ y=v+k \end{cases}$

$$\Rightarrow (a_1u+b_1v+a_1h+b_1k+c_1)du+(a_2u+b_2v+a_2h+b_2k+c_2)dv=0$$

To find (h,k) such that $\begin{cases} a_1h+b_1k+c_1=0 \\ a_2h+b_2k+c_2=0 \end{cases}$

$$\Rightarrow (a_1u+b_1v)du+(a_2u+b_2v)dv=0$$

$$\Rightarrow \frac{dv}{du}=\frac{-(a_1u+b_1v)}{a_2u+b_2v}=G(\frac{v}{u})\quad (\text{ homogenous })$$

◇ Use the method discussed under "Homogeneous Equations" to solve problems.

10. $(3x^2 - y^2)dx + (xy - x^3y^{-1})dy = 0$

Sol.

$$\frac{dy}{dx} = \frac{y^2 - 3x^2}{xy - x^3y^{-1}} = \frac{y^3 - 3x^2y}{xy^2 - x^3} = \frac{y}{x}\cdot\frac{y^2 - 3x^2}{y^2 - x^2} = \frac{y}{x}\cdot\frac{(y/x)^2 - 3}{(y/x)^2 - 1}$$

Let $z = \frac{y}{x} \Rightarrow y = xz \Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx} = z\cdot\frac{z^2-3}{z^2-1} \Rightarrow x\frac{dz}{dx} = \frac{z^3 - 3z}{z^2-1} - z = \frac{-2z}{z^2-1} = \frac{2z}{1-z^2}$

$$\Rightarrow \int \frac{1-z^2}{2z}dz = \int \frac{1}{x}dx$$

$$\Rightarrow \frac{1}{2}\int\left(\frac{1}{z} - z\right)dz = \int\frac{1}{x}dx$$

$$\Rightarrow \frac{1}{2}(\ln|z| - \frac{1}{2}z^2) = \ln|x| + C_1$$

$$\Rightarrow 2\ln|z| - z^2 = 4\ln|x| + C$$

$$\Rightarrow \ln\left(\frac{z^2}{x^4}\right) - z^2 = C$$

$$\Rightarrow \ln\left(\frac{y^2}{x^6}\right) - \frac{y^2}{x^2} = C$$

13. $\frac{dx}{dt} = \frac{x^2 + t\sqrt{t^2+x^2}}{tx}$

Sol.

$$\frac{dx}{dt} = \frac{x}{t} + \frac{\sqrt{t^2+x^2}}{x} = \frac{x}{t} + \frac{\sqrt{1+(\frac{x}{t})^2}}{x/t}$$

Let $z = \frac{x}{t} \Rightarrow x = tz \Rightarrow \frac{dx}{dt} = z + t\frac{dz}{dt} = z + \frac{\sqrt{1+z^2}}{z} \Rightarrow t\frac{dz}{dt} = \frac{\sqrt{1+z^2}}{z}$

$$\Rightarrow \int\frac{z}{(1+z^2)^{1/2}}dz = \int\frac{1}{t}dt$$

$$\Rightarrow (1+z^2)^{1/2} = \ln|t| + C$$

$$\Rightarrow \left(1 + \frac{x^2}{t^2}\right)^{1/2} = \ln|t| + C$$

$$\boxed{\begin{aligned} &\int\frac{z}{(1+z^2)^{1/2}}dz \quad \begin{pmatrix} u = 1+z^2 \\ du = 2zdz \end{pmatrix} \\ &= \frac{1}{2}\int u^{\frac{-1}{2}}du \\ &= u^{\frac{1}{2}} + C \\ &= (1+z^2)^{\frac{1}{2}} + C \end{aligned}}$$

15. $\frac{dy}{dx} = \frac{x^2 - y^2}{3xy}$

Sol.

$$\frac{dy}{dx}=\frac{1-\left(\frac{y}{x}\right)^2}{\frac{3y}{x}}$$

Let $z=\frac{y}{x}\Rightarrow y=xz\Rightarrow\frac{dy}{dz}=z+x\frac{dz}{dx}=\frac{1-z^2}{3z}\Rightarrow x\frac{dz}{dx}=\frac{1-z^2}{3z}-z=\frac{1-4z^2}{3z}$

$$\Rightarrow 3\int\frac{z}{1-4z^2}dz=\int\frac{1}{x}dx$$

$$\Rightarrow\frac{-3}{8}\ln|1-4z^2|=\ln|x|+C_1$$

$$\Rightarrow -3\ln|1-4z^2|=8\ln|x|+C_2,\quad \text{where } C_2=8C_1$$

$$\Rightarrow |1-4z^2|^{-3}=Cx^8,\quad \text{where } C=e^{C_2}$$

$$\Rightarrow\left|1-\frac{4y^2}{x^2}\right|^{-3}=Cx^8$$

$$\int\frac{z}{1-4z^2}dz\quad\begin{pmatrix}u=1-4z^2\\ du=-8zdz\end{pmatrix}$$
$$=\frac{-1}{8}\int\frac{1}{u}du$$
$$=\frac{-1}{8}\ln|u|+C$$
$$=\frac{-1}{8}\ln|1-4z^2|+C$$

◇ Use the method discussed under "Equations of the Form $dy/dx=G(ax+by)$" to solve problems.

17. $\frac{dy}{dx}=\sqrt{x+y}-1$

Sol.

Let $z=x+y\Rightarrow y=z-x\Rightarrow\frac{dy}{dx}=\frac{dz}{dx}-1=\sqrt{z}-1\Rightarrow\frac{dz}{dx}=\sqrt{z}$

$$\Rightarrow\int\frac{1}{\sqrt{z}}dz=\int dx$$

$$\Rightarrow 2z^{1/2}=x+C$$

$$\Rightarrow 2(x+y)^{1/2}=x+C$$

$$\Rightarrow (x+y)^{1/2}=\frac{x+C}{2}$$

$$\Rightarrow x+y=\frac{(x+C)^2}{4}$$

$\Rightarrow y=\frac{(x+C)^2}{4}-x$ and $y=-x$ are solutions.

19. $\frac{dy}{dx}=(x-y+5)^2$

Sol.

Let $z = x - y \Rightarrow y = x - z \Rightarrow \dfrac{dy}{dx} = 1 - \dfrac{dz}{dx} = (z+5)^2 \Rightarrow \dfrac{dz}{dx} = 1 - (z+5)^2$

$$\Rightarrow \int \frac{1}{1-(z+5)^2}\,dz = \int dx$$

$$\Rightarrow \int \frac{1}{-z^2-10z-24}\,dz = \int dx$$

$$= \int \frac{1}{z^2+10z+24}\,dz = -\int dx$$

$$= \frac{1}{2}\int \left(\frac{1}{z+4} - \frac{1}{z+6}\right)dz = -x + C_1$$

$$\Rightarrow \frac{1}{2}[\ln|z+4| - \ln|z+6|] = -x + C_1$$

$$\Rightarrow \ln\left|\frac{z+4}{z+6}\right| = -2x + C_2$$

$$\Rightarrow \left|\frac{z+4}{z+6}\right| = Ce^{-2x}$$

$$\Rightarrow \left|\frac{x-y+4}{x-y+6}\right| = Ce^{-2x}$$

◇ Use the method discussed under "Bernoulli Equations" to solve problems.

23. $\dfrac{dy}{dx} = \dfrac{2y}{x} - x^2y^2$

Sol.

$$\frac{dy}{dx} - \frac{2}{x}y = -x^2y^2$$

$$\Rightarrow y^{-2}\frac{dy}{dx} - \frac{2}{x}y^{-1} = -x^2$$

Let $z = y^{-1}$

$$\Rightarrow \frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$$

$$\Rightarrow y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$$

$$\Rightarrow -\frac{dz}{dx} - \frac{2}{x}z = -x^2$$

$$\Rightarrow \frac{dz}{dx} + \frac{2}{x}z = x^2$$

Let $\mu(x) = e^{2\int \frac{1}{x}dx} = e^{2\ln|x|} = x^2$

$$\Rightarrow x^2\frac{dz}{dx} + 2xz = x^4$$

$$\Rightarrow \frac{d}{dx}[x^2z] = x^4$$

$$\Rightarrow x^2z = \int x^4dx = \frac{1}{5}x^5 + C_1$$

$$\Rightarrow z = \frac{x^3}{5} + \frac{C_1}{x^2}$$

$$\Rightarrow y^{-1} = \frac{x^5+C}{5x^2}$$

$\Rightarrow y = \dfrac{5x^2}{x^5+C}$ and $y \equiv 0$ are the solutions

25. $\dfrac{dx}{dt} + tx^3 + \dfrac{x}{t} = 0$

Sol.

$$\frac{dx}{dt}+\frac{x}{t}=-tx^3$$

$$\Rightarrow x^{-3}\frac{dx}{dt}+\frac{1}{t}x^{-2}=-t$$

Let $z=x^{-2}$

$$\Rightarrow \frac{dz}{dt}=-2x^{-3}\frac{dx}{dt}$$

$$\Rightarrow x^{-3}\frac{dx}{dt}=\frac{-1}{2}\frac{dz}{dt}$$

$$\Rightarrow \frac{-1}{2}\frac{dz}{dt}+\frac{1}{t}z=-t$$

$$\Rightarrow \frac{dz}{dt}+(-\frac{2}{t})z=2t$$

Let $\mu(t)=e^{-2\int\frac{1}{t}dt}=e^{-2\ln|t|}=\frac{1}{t^2}$

$$\Rightarrow \frac{1}{t^2}\frac{dz}{dt}+(-\frac{2}{t^3})z=\frac{2}{t}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{t^2}z\right]=\frac{2}{t}$$

$$\Rightarrow \frac{1}{t^2}z=2\int\frac{1}{t}dt=2\ln|t|+C$$

$$\Rightarrow z=t^2\ln t^2+Ct^2$$

$$\Rightarrow x^{-2}=t^2\ln t^2+Ct^2$$

$$\Rightarrow x^2=(t^2\ln t^2+Ct^2)^{-1}$$

$\Rightarrow x=\pm(t^2\ln t^2+Ct^2)^{\frac{-1}{2}}$ and $x\equiv 0$ are the solutions

27. $\frac{dr}{d\theta}=\frac{r^2+2r\theta}{\theta^2}$

Sol.

$$\frac{dr}{d\theta}=\frac{r^2}{\theta^2}+\frac{2r}{\theta}$$

$$\Rightarrow \frac{dr}{d\theta}-\frac{2}{\theta}r=\frac{r^2}{\theta^2}$$

$$\Rightarrow r^{-2}\frac{dr}{d\theta}-\frac{2}{\theta}r^{-1}=\frac{1}{\theta^2}$$

Let $z=r^{-1}$

$$\Rightarrow \frac{dz}{d\theta}=-r^{-2}\frac{dr}{d\theta}$$

$$\Rightarrow r^{-2}\frac{dr}{d\theta}=-\frac{dz}{d\theta}$$

$$\Rightarrow -\frac{dz}{d\theta}-\frac{2}{\theta}z=\frac{1}{\theta^2}$$

$$\Rightarrow \frac{dz}{d\theta}+\frac{2}{\theta}z=-\frac{1}{\theta^2}$$

Let $\mu(\theta)=e^{2\int\frac{1}{\theta}d\theta}=e^{2\ln|\theta|}=\theta^2$

$$\Rightarrow \theta^2\frac{dz}{d\theta}+2\theta z=-1$$

$$\Rightarrow \frac{d}{d\theta}[\theta^2 z]=-1$$

$$\Rightarrow \theta^2 z=-\int d\theta=-\theta+C$$

$$\Rightarrow z=\frac{-\theta+C}{\theta^2}$$

$$\Rightarrow r^{-1}=\frac{-\theta+C}{\theta^2}$$

$\Rightarrow r=\frac{\theta^2}{-\theta+C}$ and $r\equiv 0$ are the solutions

◇ Use the method discussed under "Equations with Linear Coefficients" to solve problems.

29. $(-3x+y-1)dx+(x+y+3)dy=0$

Sol.

$$\text{Let } \begin{cases} x = u + h \\ y = v + k \end{cases} \Rightarrow \begin{cases} dx = du \\ dy = dv \end{cases}$$

$$(-3u + v - 3h + k - 1)du + (u + v + h + k + 3)dv = 0$$

$$\Rightarrow \begin{cases} -3h + k - 1 = 0 \\ h + k + 3 = 0 \end{cases} \Rightarrow \begin{cases} h = -1 \\ k = -2 \end{cases}$$

$$\Rightarrow (-3u + v)du + (u + v)dv = 0$$

$$\Rightarrow \frac{dv}{du} = \frac{3u - v}{u + v} = \frac{3 - \frac{v}{u}}{1 + \frac{v}{u}}$$

$$\text{Let } \quad z = \frac{v}{u}$$

$$\Rightarrow v = uz$$

$$\Rightarrow \frac{dv}{du} = z + u\frac{dz}{du} = \frac{3 - z}{1 + z} \Rightarrow u\frac{dz}{du} = \frac{3 - z}{1 + z} - z = \frac{3 - 2z - z^2}{1 + z}$$

$$\Rightarrow \int \frac{1 + z}{3 - 2z - z^2}dz = \int \frac{1}{u}du$$

$$\Rightarrow \frac{-1}{2}\ln|3 - 2z - z^2| = \ln|u| + C_1$$

$$\Rightarrow \ln|3 - 2z - z^2| = -2\ln|u| + C_2$$

$$\Rightarrow |3 - 2z - z^2| = Cu^{-2}$$

$$\Rightarrow \left|3 - \frac{2v}{u} - (\frac{v}{u})^2\right| = Cu^{-2}$$

$$\Rightarrow \left|3 - \frac{2(y + 2)}{x + 1} - (\frac{y + 2}{x + 1})^2\right| = C(x + 1)^{-2}$$

$$\int \frac{1 + z}{3 - 2z - z^2}dz \quad \begin{pmatrix} u = 3 - 2z - z^2 \\ du = (-2 - 2z)dz \\ \quad = -2(1 + z)dz \end{pmatrix}$$

$$= \frac{-1}{2}\int \frac{1}{u}du$$

$$= \frac{-1}{2}\ln|u| + C$$

$$= \frac{-1}{2}\ln|3 - 2z - z^2| + C$$

31. $(2x - y)dx + (4x + y - 3)dy = 0$

Sol.

$$\text{Let } \begin{cases} x = u + h \\ y = v + k \end{cases} \Rightarrow \begin{cases} dx = du \\ dy = dv \end{cases}$$

$$(2u - v + 2h - k)du + (4u + v + 4h + k - 3)dv = 0$$

$$\Rightarrow \begin{cases} 2h - k = 0 \\ 4h + k - 3 = 0 \end{cases} \Rightarrow \begin{cases} h = \frac{1}{2} \\ k = 1 \end{cases}$$

$$\Rightarrow (2u - v)du + (4u + v)dv = 0$$

$$\Rightarrow \frac{dv}{du} = 4\frac{-2u + v}{u + v} = \frac{-2 + \frac{v}{u}}{4 + \frac{v}{u}}$$

Let $z = \frac{v}{u}$

$\Rightarrow v = uz$

$\Rightarrow \frac{dv}{du} = z + u\frac{dz}{du} = \frac{-2+z}{4+z} \Rightarrow u\frac{dz}{du} = \frac{-2+z}{4+z} - z = \frac{-z^2-3z-2}{4+z}$

$\Rightarrow \int \frac{4+z}{z^2+3z+2}dz = -\int \frac{1}{u}du$

$\Rightarrow \int \left(\frac{3}{z+1} - \frac{2}{z+2}\right)dz = -\int \frac{1}{u}du$

$\Rightarrow 3\ln|z+1| - 2\ln|z+2| = -\ln|u| + C_1$

$\Rightarrow \frac{|z+1|^3}{|z+2|^2} = Cu^{-1}$

$\Rightarrow \frac{\left|\frac{v}{u}+1\right|^3}{\left|\frac{v}{u}+2\right|^2} = Cu^{-1}$

$\Rightarrow \frac{\left|\frac{y-1}{x-1/2}+1\right|^3}{\left(\frac{y-1}{x-1/2}+2\right)^2} = C(x-1/2)^{-1}$

45. **Coupled Equations.** In analyzing coupled equations of the form $\frac{dy}{dt} = ax + by$, $\frac{dx}{dt} = \alpha x + \beta y$, where $a, b, \alpha,$ and β are constants, we may wish to determine the relationship between x and y rather than the individual solutions $x(t)$, $y(t)$. For this purpose, divide the first equation by the second to obtain

(17) $\frac{dy}{dx} = \frac{ax+by}{\alpha x + \beta y}$.

This new equation is homogeneous, so we can solve it via the substitution $v = y/x$. We refer to the solution of (17) as integral curves. Determine the integral curves for the system

$\frac{dy}{dt} = -4x - y$, $\frac{dx}{dt} = 2x - y$.

Sol.

$\frac{dy}{dx} = \frac{-4x-y}{2x-y} = \frac{-4-y/x}{2-y/x}$

Let $z = \frac{y}{x} \Rightarrow y = xz \Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx} = \frac{-4-z}{2-z} \Rightarrow x\frac{dz}{dx} = \frac{-4-z}{2-z} - z = \frac{z^2-3z-4}{2-z}$

$$\Rightarrow \int \frac{2-z}{z^2-3z-4}dz = \int \frac{1}{x}dx$$

$$\Rightarrow \frac{-1}{5}\int (\frac{2}{z-4}+\frac{3}{z+1})dz = \int \frac{1}{x}dx$$

$$\Rightarrow \frac{-1}{5}[2\ln|z-4|+3\ln|z+1|] = \ln|x|+C_1$$

$$\Rightarrow [2\ln|z-4|+3\ln|z+1|] = -5\ln|x|+C_2$$

$$\Rightarrow (z-4)^2\cdot|z+1|^3 = C|x|^{-5}$$

$$\Rightarrow (\frac{y}{x}-4)^2\cdot|\frac{y}{x}+1|^3 = C|x|^{-5} \quad \text{or} \quad (\frac{y}{x}-4)^4\cdot(\frac{y}{x}+1)^6 = Cx^{10}$$

47. **Riccati Equation.** An equation of the form

(18) $\quad \dfrac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$

is called a generalized Riccati equation.

(a) If one solution—say, $\mu(x)$ —of (18) is known, show that the substitution $y = u + 1/v$ reduces (18) to a linear equation in v.

Sol.

$$y = u + \frac{1}{v}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - v^{-2}\frac{dv}{dx} = P\cdot(u+\frac{1}{v})^2 - Q\cdot(u+\frac{1}{v}) + R = (Pu^2+Qu+R) + 2P\frac{u}{v} + Pv^{-2} + \frac{Q}{v}$$

$$= \frac{du}{dx} + (2Pu+Q)v^{-1} + Pv^{-2}$$

$$\Rightarrow \frac{du}{dx} - v^{-2}\frac{dv}{dx} = \frac{du}{dx} + (2Pu+Q)v^{-1} + Pv^{-2}$$

$$\Rightarrow -v^{-2}\frac{dv}{dx} - (2Pu+Q)v^{-1} = Pv^{-2}$$

$$\Rightarrow \frac{dv}{dx} + (2Pu+Q)v = -P \quad \text{is a linear equation.}$$

(b) Given that $u(x) = x$ is a solution to $\dfrac{dy}{dx} = x^3(y-x)^2 + \dfrac{y}{x}$, use the result of part (a) to find all the other solutions to this equation. (The particular solution $u(x) = x$ can be found by inspection or by using a Taylor series method; see Section 8.1.)

Sol.

$$\frac{dy}{dx} = x^3(y-x)^2 + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = x^3y^2 - 2x^4y + x^5 + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \underbrace{x^3}_{P(x)} y^2 + \underbrace{(\frac{1}{x} - 2x^4)}_{Q(x)} y + \underbrace{x^5}_{R(x)} \quad ..... \ (1)$$

From part (a), the substitution $y = u + \frac{1}{v}$ reduces (1) to a linear equation

$$\frac{dv}{dx} + (2Pu + Q)v = -P \Rightarrow \frac{dv}{dx} + (2x^3 \cdot x + \frac{1}{x} - 2x^4)v = -x^3 \Rightarrow \frac{dv}{dx} + \frac{v}{x} = -x^3 \quad ..... \ (2)$$

Let $\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|$

$$(2) \times \mu(x) \Rightarrow x\frac{dv}{dx} + v = -x^4$$

$$\Rightarrow \frac{d}{dx}[xv] = -x^4$$

$$\Rightarrow xv = -\frac{1}{5}x^5 + C_1$$

$$\Rightarrow v = -\frac{1}{5}x^4 + \frac{C_1}{x} = \frac{-x^5 + C}{5x}$$

$\because \quad y = x + \frac{1}{v}$

$$\Rightarrow v = \frac{1}{y - x} = \frac{-x^5 + C}{5x} \Rightarrow y - x = \frac{5x}{C - x^5} \Rightarrow y = \frac{5x}{C - x^5} + x$$