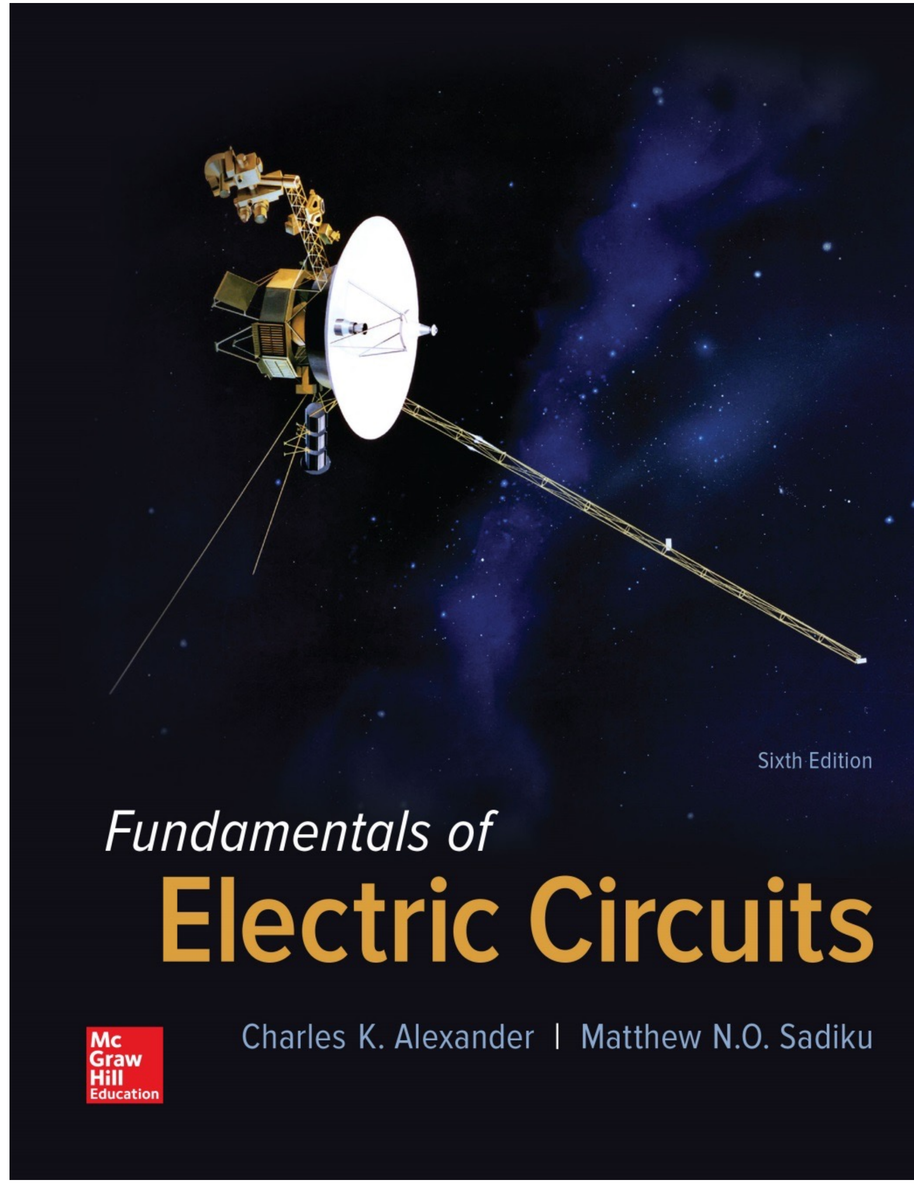
Fundamentals of Electric Circuits
Chapter 14

Frequency Response



- This chapter will introduce the idea of the **transfer function**:
  - a means of describing the relationship between the **input** and **output** of a circuit.
- **Bode plots** and their utility
  - in describing the **frequency response** of a circuit will also be introduced.
- The concept of **resonance** as applied to LRC circuits will be covered as well
- Finally, **frequency filters** will be discussed.

The **frequency response** of a circuit is the variation in its behavior with change in signal frequency.

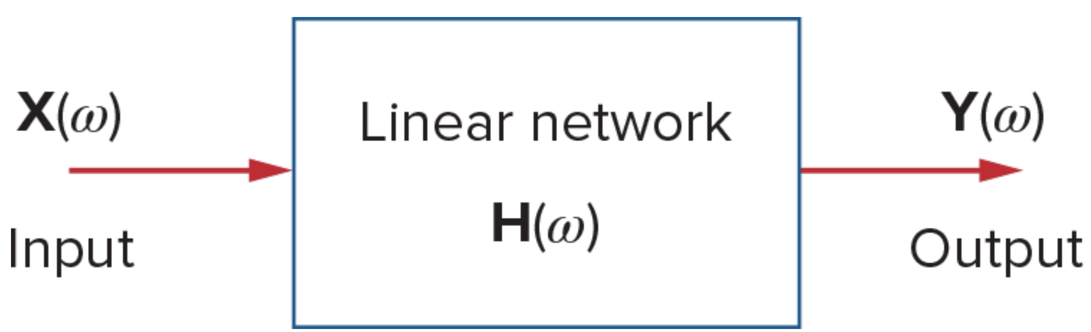
- **Frequency response is the variation in a circuit's behavior with change in signal frequency.**
- **This is significant for applications involving filters.**
- **Filters play critical roles in blocking or passing specific frequencies or ranges of frequencies.**
- **Without them, it would be impossible to have multiple channels of data in radio communications.**

14.2 Transfer Function

- One useful way to analyze the frequency response of a circuit is the concept of the transfer function $H(\omega)$.
- It is the frequency dependent ratio of a forced function $Y(\omega)$ to the forcing function $X(\omega)$.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$\mathbf{X}(\omega)$ and $\mathbf{Y}(\omega)$ denote the input and output phasors of a network



Transfer Function

- There are four possible input/output combinations:

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} \tag{14.2a}$$

$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)} \tag{14.2b}$$

$$\mathbf{H}(\omega) = \text{Transfer Impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)} \tag{14.2c}$$

$$\mathbf{H}(\omega) = \text{Transfer Admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)} \tag{14.2d}$$

Zeros and Poles

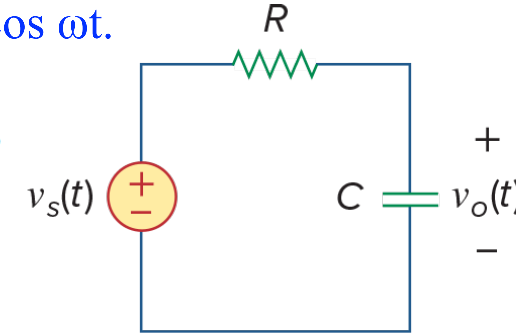
- To obtain H(ω), we first convert to frequency domain equivalent components in the circuit.
- H(ω) can be expressed as the ratio of numerator N(ω) and denominator D(ω) polynomials.

$$H(\omega) = \frac{N(\omega)}{D(\omega)}$$

- Zeros
  - where the transfer function goes to zero.
- Poles
  - where it goes to infinity.
- They can be related to the roots of N(ω) and D(ω)

Example 14.1

For the RC circuit, obtain the transfer function V_O/V_S and its frequency response. Let $v_s = V_m \cos \omega t$.



Convert to the frequency domain

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

obtain the magnitude and phase of $\mathbf{H}(\omega)$

$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

$$\phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

where $\omega_0 = 1/RC$

Plot H and ϕ for $0 < \omega < \infty$

TABLE 14.1

For Example 14.1.

ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	−84°
1	0.71	−45°	20	0.05	−87°
2	0.45	−63°	100	0.01	−89°
3	0.32	−72°	∞	0	−90°

Multiplication:

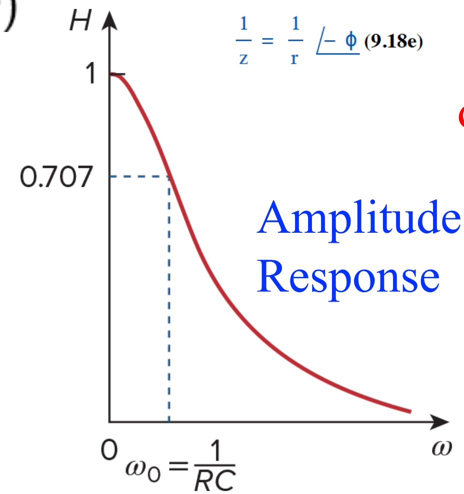
$$z_1 z_2 = r_1 r_2 \underline{/\phi_1 + \phi_2} \quad (9.18c)$$

Division:

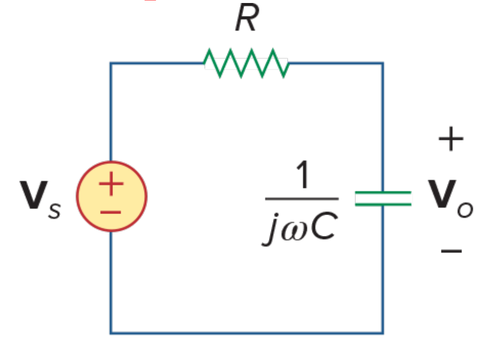
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \underline{/\phi_1 - \phi_2} \quad (9.18d)$$

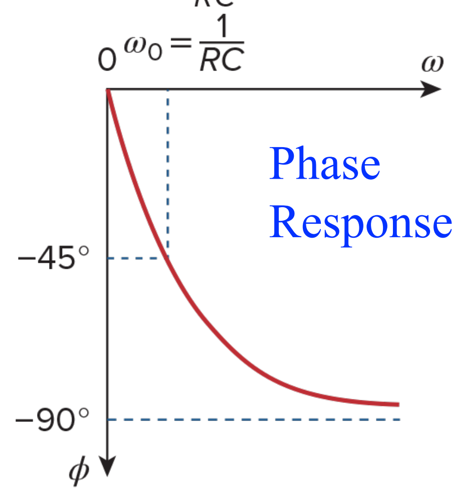
Reciprocal:

$$\frac{1}{z} = \frac{1}{r} \underline{/-\phi} \quad (9.18e)$$



Convert to the frequency domain equivalent circuit





Low-pass Filters

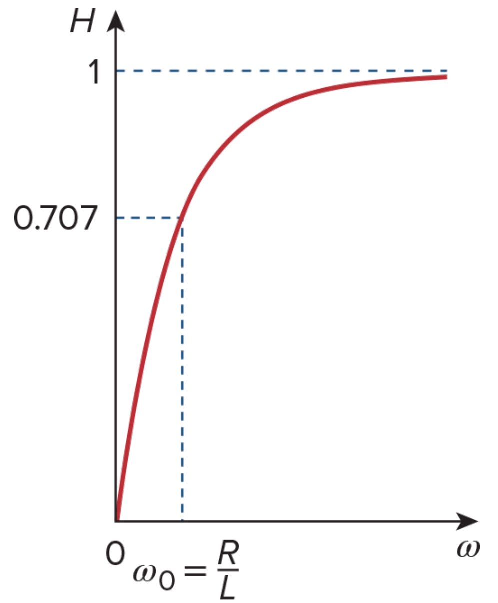
Practice Problem 14.1

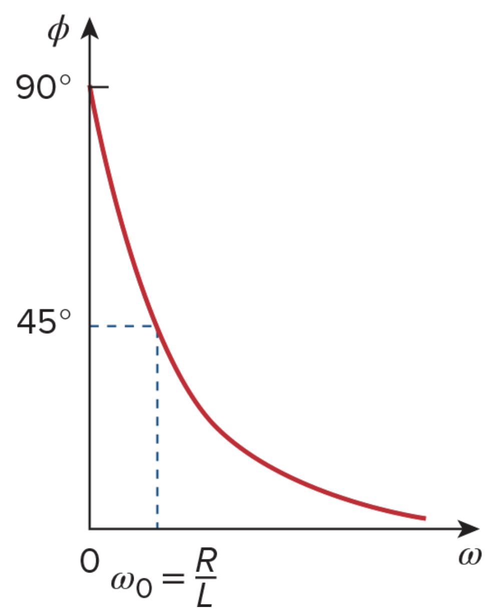
For the RC circuit, obtain the transfer function V_O/V_S and its frequency response. Let $v_s = V_m \cos \omega t$.

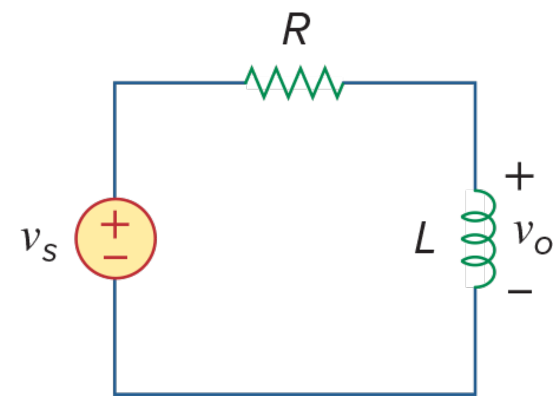
$$H(\omega) = \frac{V_O}{V_S} = \frac{j\omega L}{R + j\omega L}$$

$$= \frac{1}{1 + \frac{R}{j\omega L}}$$

$$= \frac{1}{1 - j\frac{R/L}{\omega}}$$

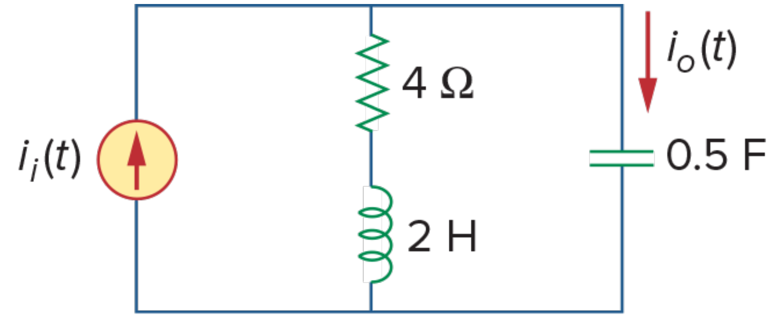






Example 14.2

For the circuit, calculate the gain $I_o(\omega)/I_i(\omega)$ and its poles and zeros.



By current division

$$\mathbf{I}_o(\omega) = \frac{4 + j2\omega}{4 + j2\omega + 1/j0.5\omega}\mathbf{I}_i(\omega)$$

$$\frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)} = \frac{j0.5\omega(4 + j2\omega)}{1 + j2\omega + (j\omega)^2} = \frac{s(s+2)}{s^2 + 2s + 1} \Leftarrow \; s = j\omega$$

The zeros

$$s(s+2) = 0 \Rightarrow \; z_1 = 0, z_2 = -2$$

The poles

$$s^2 + 2s + 1 = (s+1)^2 = 0 \Rightarrow \; p = -1$$

repeated pole
(or double pole)

14.3 †The Decibel Scale

- Bode plots
- These plots are based on logarithmic scales.
- The transfer function
  - as an expression of gain.
- Gain expressed in log form
  - typically expressed in bels
  - or more commonly decibels (1/10 of a bel)

1. $\log P_1 P_2 = \log P_1 + \log P_2$
2. $\log P_1 / P_2 = \log P_1 - \log P_2$
3. $\log P^n = n \log P$
4. $\log 1 = 0$

power gain G

$$G_{dB} = 10 \log_{10} \frac{P_2}{P_1}$$

reason why logarithms are greatly used

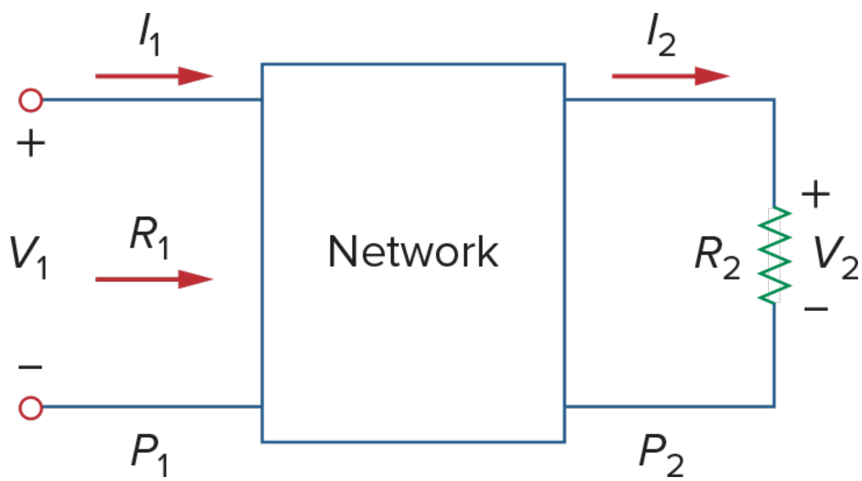
$$G_{dB} = 10 \log_{10} 2 \simeq 3 \text{ dB}$$

$$G_{dB} = 10 \log_{10} 0.5 \simeq -3 \text{ dB}$$

The logarithm of the reciprocal of a quantity is simply negative the logarithm of that quantity.

The gain G can be expressed in terms of voltage or current ratio

If P_1 is the input power,
P_2 is the output (load) power,
R_1 is the input resistance,
R_2 is the load resistance.



$$G_{dB} = 10 \log_{10} \frac{P_2}{P_1} = 10 \log_{10} \frac{V_2^2/R_2}{V_1^2/R_1}$$

$$= 10 \log_{10}\left(\frac{V_2}{V_1}\right)^2 + 10 \log_{10} \frac{R_1}{R_2}$$

$$G_{dB} = 20 \log_{10} \frac{V_2}{V_1} - 10 \log_{10} \frac{R_2}{R_1}$$

For the case when $R_2 = R_1$

when comparing voltage levels

$$G_{dB} = 20 \log_{10} \frac{V_2}{V_1}$$

when comparing current levels

$$G_{dB} = 20 \log_{10} \frac{I_2}{I_1}$$

14.4 Bode Plots

Bode plots are semilog plots of the magnitude (in decibels) and phase (in degrees) of a transfer function versus frequency.

- One problem with the transfer function is that it needs to cover a large range in frequency.
- Plotting the frequency response on a semilog plot makes the task easier
  - where the x axis is plotted in log form
- ⇒ Bode plots
- Bode plots either show
  - magnitude (in decibels)
  - phase (in degrees)

  as a function of frequency.

$$\mathbf{H} = H\angle\phi = He^{j\phi}$$

$$\ln \mathbf{H} = \ln H + \ln e^{j\phi} = \ln H + j\phi$$

$$H_{\text{dB}} = 20 \log_{10} H$$

Standard Form

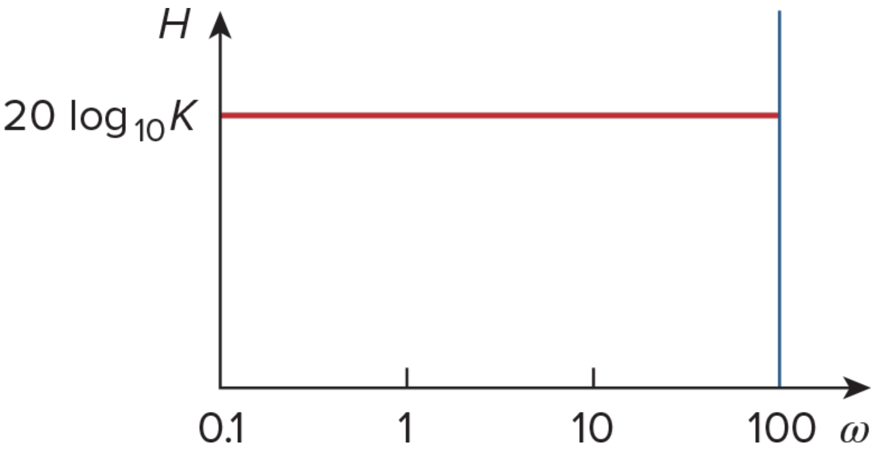
- The transfer function:

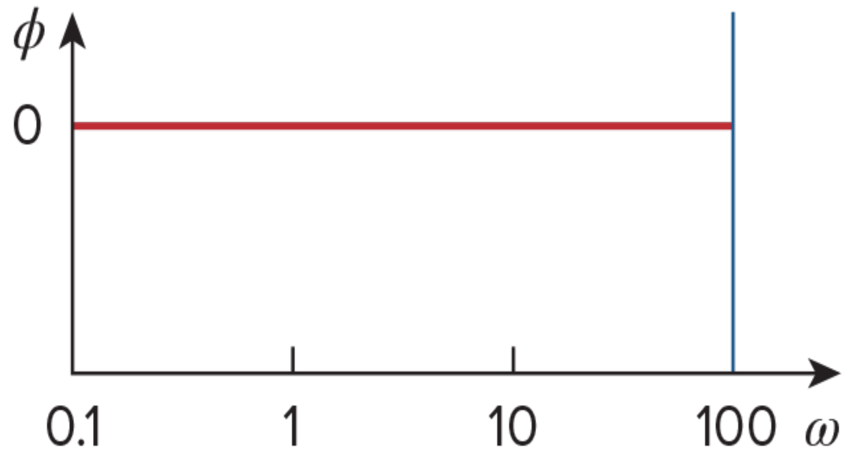
$$H(\omega)=\frac{K(j\omega)^{\pm 1}(1+j\omega/z_1)\left[1+j2\zeta_1\omega/\omega_k+(j\omega/\omega_k)^2\right]\cdots}{(1+j\omega/p_1)\left[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2\right]\cdots}$$

- This standard form may include the following seven factors in various combinations:
  - A gain K
  - A pole $(j\omega)^{-1}$
    or a zero $(j\omega)$
  - A simple pole $1/(1+j\omega/p_1)$
    or a simple zero $(1+j\omega/z_1)$
  - A quadratic pole $1/[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2]$
    or zero $[1+j2\zeta_1\omega/\omega_n+(j\omega/\omega_k)^2]$

Bode Plots

- In a bode plot, each of these factors is plotted separately and then added graphically.
- Gain, K: the magnitude is $20\log_{10}K$ and the phase is 0°. Both are constant with frequency.

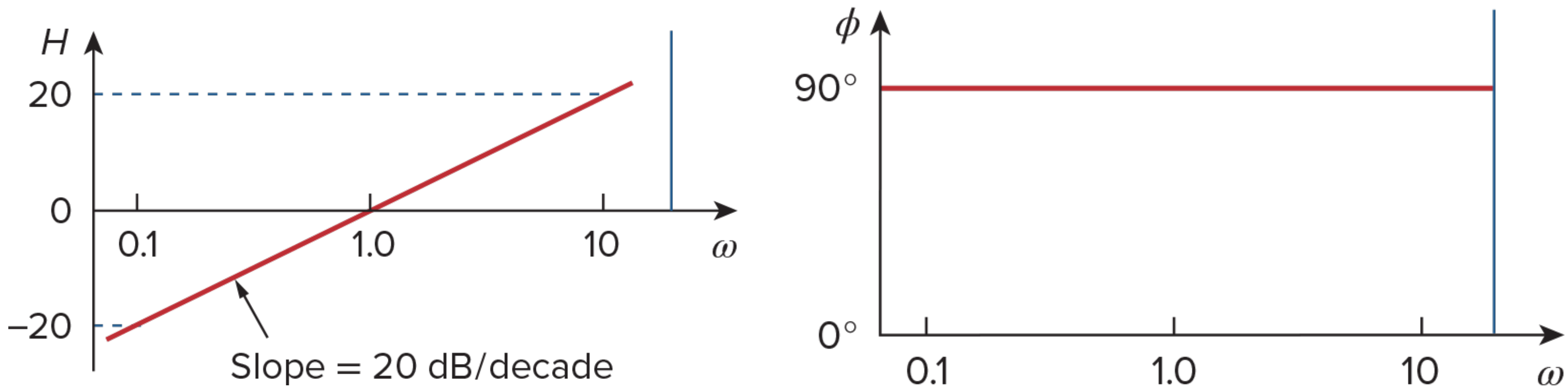




Bode Plots

$$H(\omega)=\frac{K(j\omega)^{\pm1}(1+j\omega/z_1)\left[1+j2\zeta_1\omega/\omega_k+(j\omega/\omega_k)^2\right]\cdots}{(1+j\omega/p_1)\left[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2\right]\cdots}$$

- Pole/zero at the origin:
  - For the zero (jω), the slope in magnitude is 20 dB/decade and the phase is 90°.
  - For the pole $(j\omega)^{-1}$ the slope in magnitude is -20 dB/decade and the phase is -90°



Bode Plots

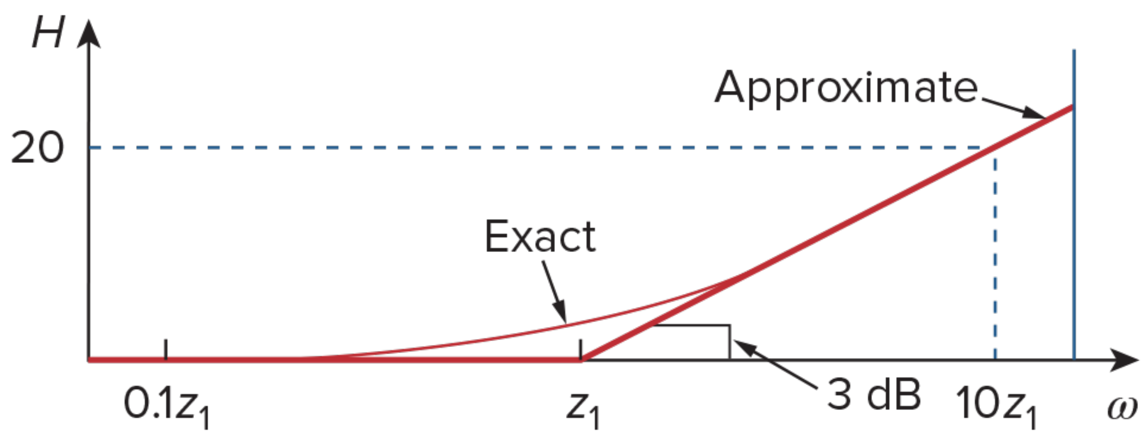
$$H(\omega)=\frac{K(j\omega)^{\pm 1}(1+j\omega/z_1)\left[1+j2\zeta_1\omega/\omega_k+(j\omega/\omega_k)^2\right]\cdots}{(1+j\omega/p_1)\left[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2\right]\cdots}$$

- Simple zero
  - the magnitude is $20\log_{10}|1+j\omega/z_1|$
  - the phase is $\tan^{-1}\omega/z_1$

    where:

$$H_{dB}=20\log_{10}\left|1+\frac{j\omega}{z_1}\right| \Rightarrow \underset{as\ \omega\to\infty}{20\log_{10}\frac{\omega}{z_1}}$$

- The pole is similar, except the corner frequency is at $\omega=p_1$, the magnitude has a negative slope
- approximated as a flat line and sloped line that intersect at $\omega=z_1$ (corner or break frequency)

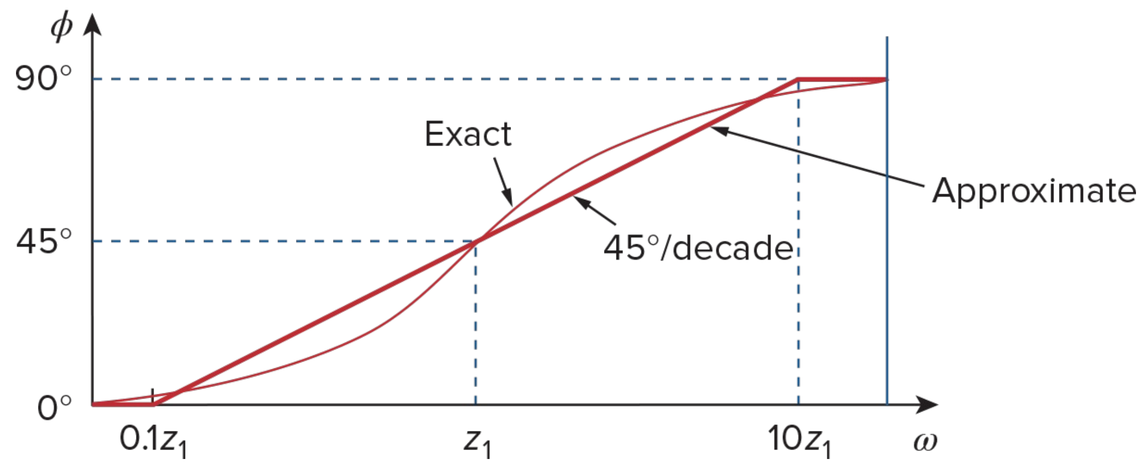


Bode Plots

$$H(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)\left[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2\right]\cdots}{(1 + j\omega/p_1)\left[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2\right]\cdots}$$

- The phase can be plotted as a series straight lines

$$\phi = \tan^{-1}\left(\frac{\omega}{z_1}\right) = \begin{cases} 0, & \omega = 0 \\ 45^\circ, & \omega = z_1 \\ 90^\circ, & \omega \to \infty \end{cases}$$

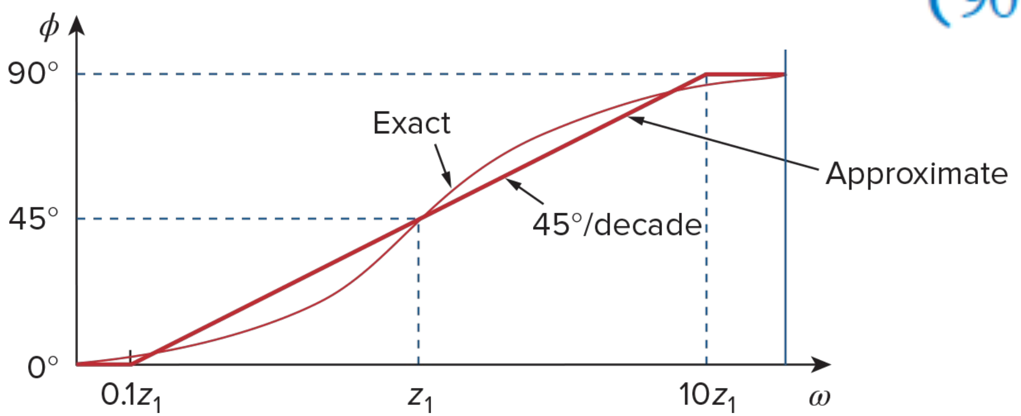


Bode Plots

$$H(\omega)=\frac{K(j\omega)^{\pm1}(1+j\omega/z_1)\left[1+j2\zeta_1\omega/\omega_k+(j\omega/\omega_k)^2\right]\cdots}{(1+j\omega/p_1)\left[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2\right]\cdots}$$

- The phase can be plotted as a series straight lines
- From $\omega=0$ to $\omega \le z_1/10$, we let $\phi=0$
- At $\omega=z_1$ we let $\phi=45°$
- For $\omega \ge 10z_1$, we let $\phi= 90°$

$$\phi=\tan^{-1}\left(\frac{\omega}{z_1}\right)=\begin{cases}0, & \omega=0\\ 45°, & \omega=z_1\\ 90°, & \omega\to\infty\end{cases}$$

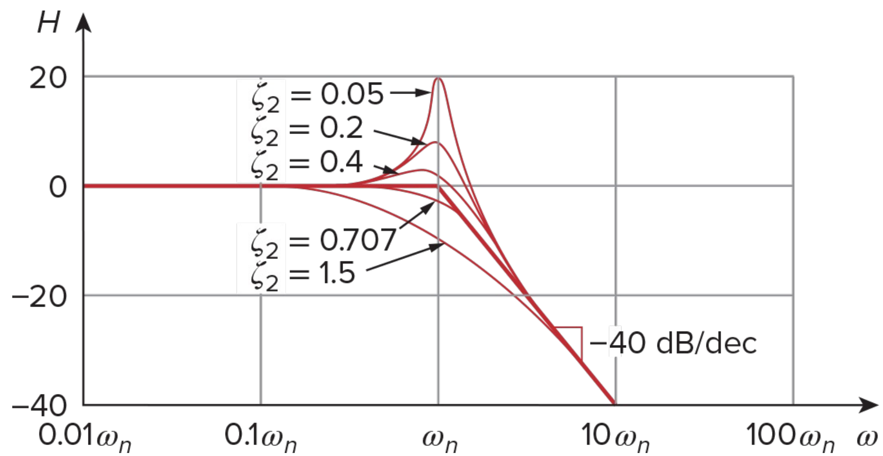


Bode Plots Quadratic pole/zero

- Quadratic pole:
  - The magnitude of the quadratic pole $1/[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2]$
    - $-20\log_{10}[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2]$

$$H_{dB} \underset{\text{as } \omega\to 0}{\Rightarrow} 0 \quad \underset{\text{as } \omega\to\infty}{\Rightarrow} -40\log_{10}\frac{\omega}{\omega_n}$$

  - The magnitude plot will be two lines:
    - slope zero for $\omega<\omega_n$
    - slope -40dB/decade, with ω_n as the corner frequency



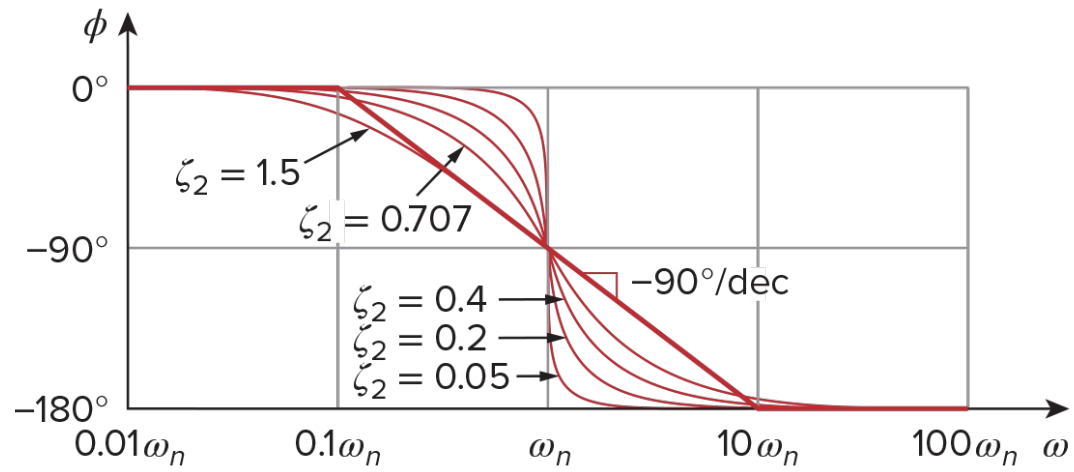
- **Quadratic zero, the plots are inverted.**

Bode Plots Quadratic pole/zero

- The phase can be expressed as:

$$\phi = -\tan^{-1}\frac{2\zeta_2\omega/\omega_n}{1-\omega^2/\omega_n^{\,2}} = \begin{cases} 0 & \omega = 0 \\ -90^\circ & \omega = \omega_n \\ -180^\circ & \omega \to \infty \end{cases}$$

  - This will be a straight line with slope of -90°/decade starting at $\omega_n/10$ and ending at 10 ω_n.



Bode Plots

TABLE 14.3

Summary of Bode straight-line magnitude and phase plots.

Factor — **Magnitude** — **Phase**

K

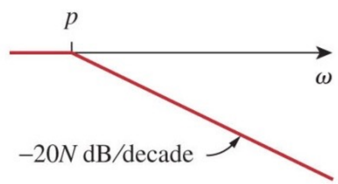
$(j\omega)^N$

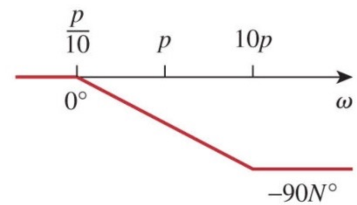
$\dfrac{1}{(j\omega)^N}$

$\left(1 + \dfrac{j\omega}{z}\right)^N$

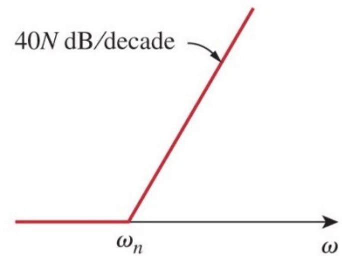
Bode Plots

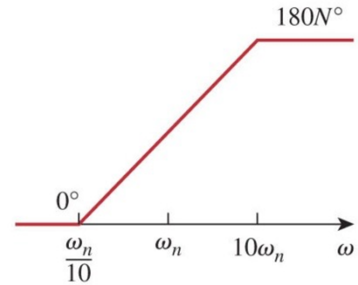
$$\frac{1}{(1 + j\omega/p)^N}$$



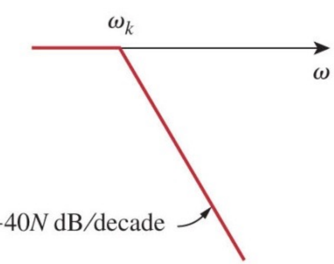


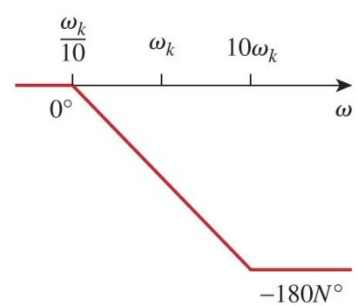
$$\left[1 + \frac{2j\omega\zeta}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right]^N$$





$$\frac{1}{[1 + 2j\omega\zeta/\omega_k + (j\omega/\omega_k)^2]^N}$$





Example 14.3

Construct the Bode plots for the transfer function

$$\mathbf{H}(\omega) = \frac{200 j\omega}{(j\omega + 2)(j\omega + 10)}$$

Converting H(ω) in the standard form

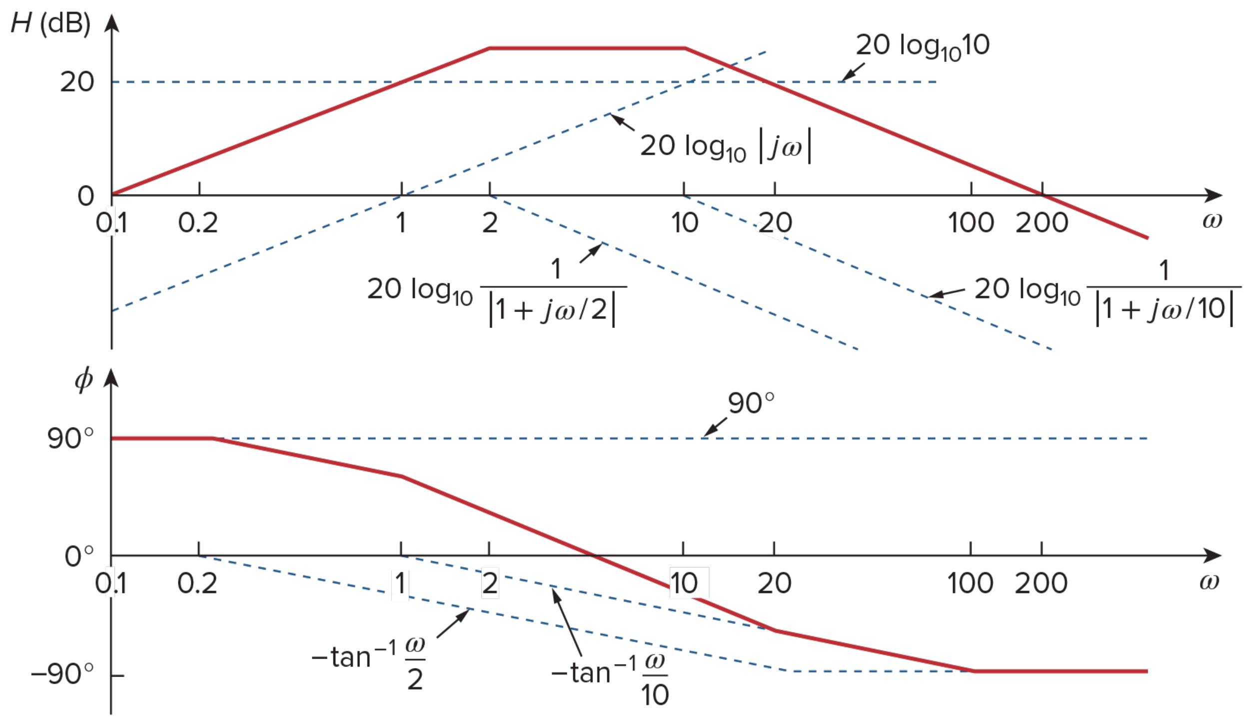
$$\mathbf{H}(\omega) = \frac{10 j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

$$= \frac{10|j\omega|}{|1 + j\omega/2||1 + j\omega/10|} \angle 90^\circ - \tan^{-1}\omega/2 - \tan^{-1}\omega/10$$

$$H_{\text{dB}} = 20\log_{10} 10 + 20\log_{10}|j\omega| - 20\log_{10}\left|1 + \frac{j\omega}{2}\right| - 20\log_{10}\left|1 + \frac{j\omega}{10}\right|$$

$$\phi = 90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10}$$

two corner frequencies at ω=2, 10 rad/s



Example 14.4

Construct the Bode plots for the transfer function

$$\mathbf{H}(\omega) = \frac{j\omega + 10}{j\omega(j\omega + 5)^2}$$

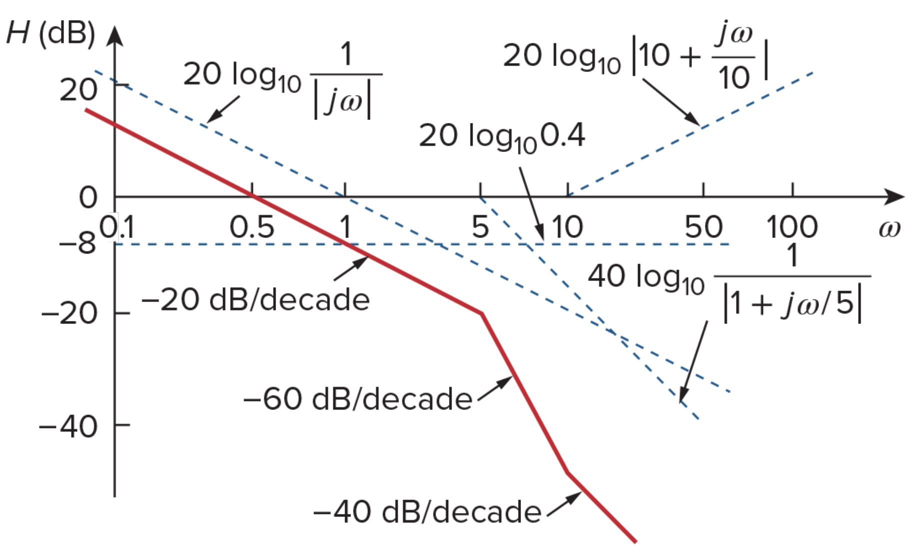
Converting H(ω) in the standard form

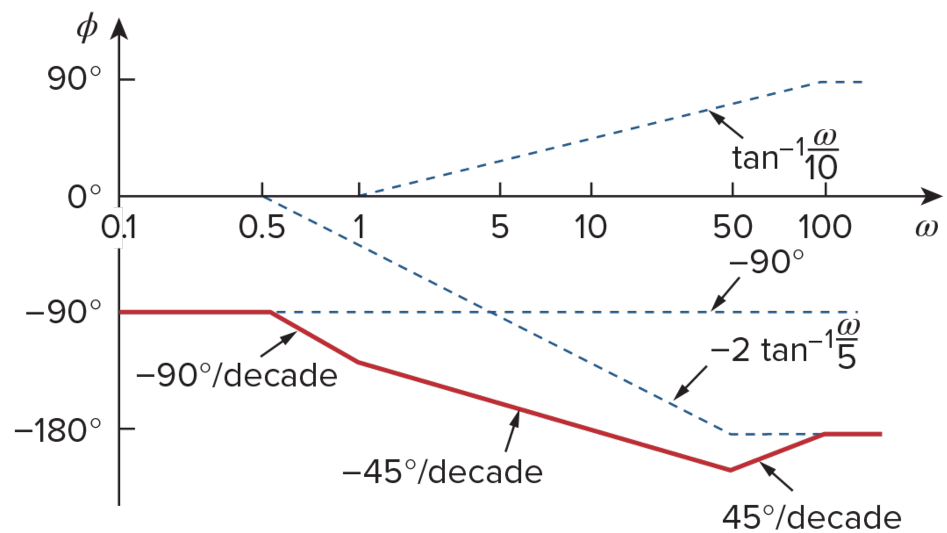
$$\mathbf{H}(\omega) = \frac{0.4(1 + j\omega/10)}{j\omega(1 + j\omega/5)^2}$$

$$H_{\mathrm{dB}} = 20\log_{10}0.4 + 20\log_{10}\left|1 + \frac{j\omega}{10}\right| - 20\log_{10}|j\omega| - 40\log_{10}\left|1 + \frac{j\omega}{5}\right|$$

$$\phi = 0^\circ + \tan^{-1}\frac{\omega}{10} - 90^\circ - 2\tan^{-1}\frac{\omega}{5}$$

two corner frequencies at ω=5, 10 rad/s



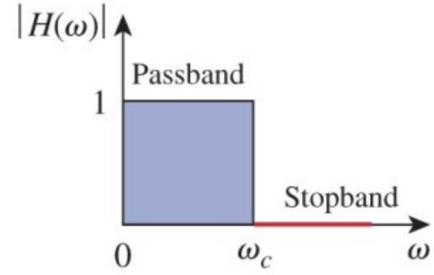


14.7 Passive Filters

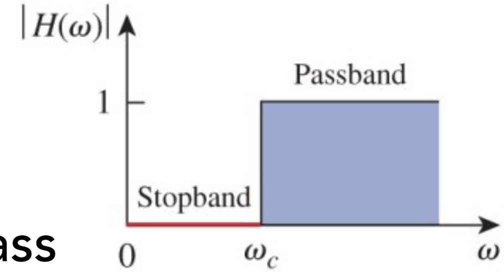
- A filter is a circuit that is designed to
  - pass signals with desired frequencies
  - reject or attenuate others.
- Passive filter consists only of passive elements:
  - R, L, and C.
- They are very important circuits in that many technological advances would not have been possible without the development of filters.

Passive Filters

- Low-pass Filter
  - passes only low frequencies and blocks high frequencies.
- High-pass Filter
  - does the opposite of lowpass
- Band-pass Filter
  - only allows a range of frequencies to pass through.
- Band-stop Filter
  - does the opposite of bandpass



(a)



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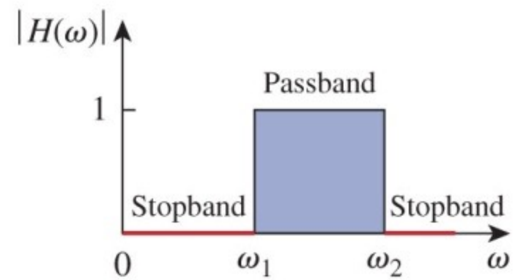

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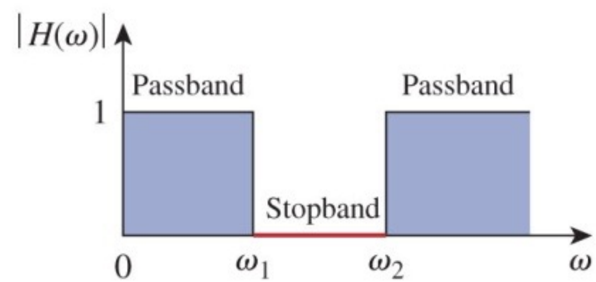


(d)

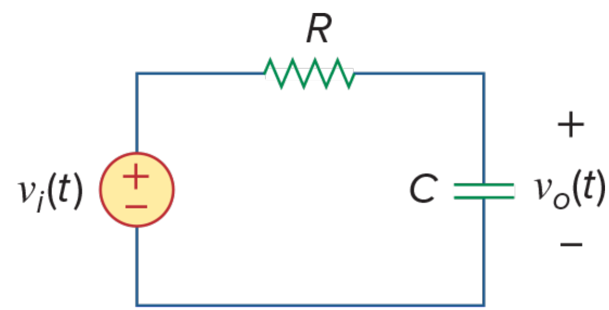
TABLE 14.5

Summary of the characteristics of ideal filters.

Type of Filter	$H(0)$	$H(\infty)$	$H(\omega_c)$ or $H(\omega_0)$
Lowpass	1	0	$1/\sqrt{2}$
Highpass	0	1	$1/\sqrt{2}$
Bandpass	0	0	1
Bandstop	1	1	0

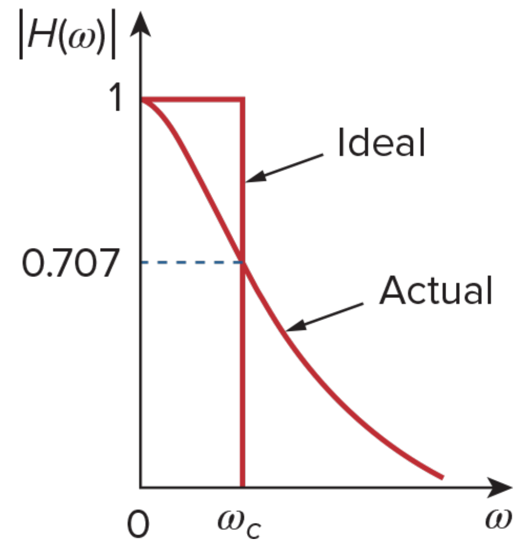
ω_c is the cutoff frequency for lowpass and highpass filters; ω_0 is the center frequency for bandpass and bandstop filters.

14.7.1 Low-Pass Filter



$$H(\omega) = \frac{V_o}{V_i} = \frac{1/j\omega C}{R + 1/j\omega C} \Rightarrow H(\omega) = \frac{1}{1 + j\omega RC}$$

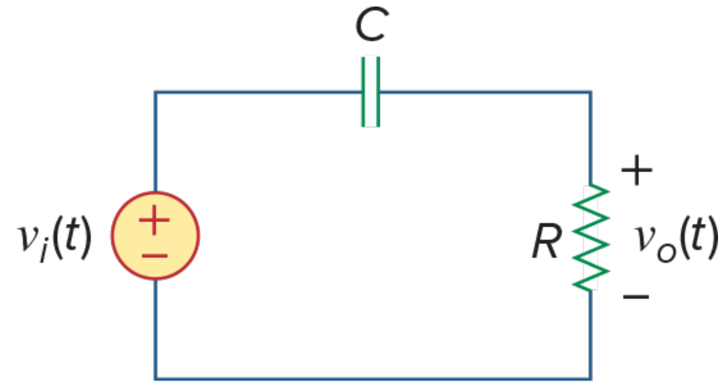
$$H(\omega_c) = \frac{1}{\sqrt{1 + \omega_c^2 R^2 C^2}} = \frac{1}{\sqrt{2}} \Rightarrow \omega_c = \frac{1}{RC}$$



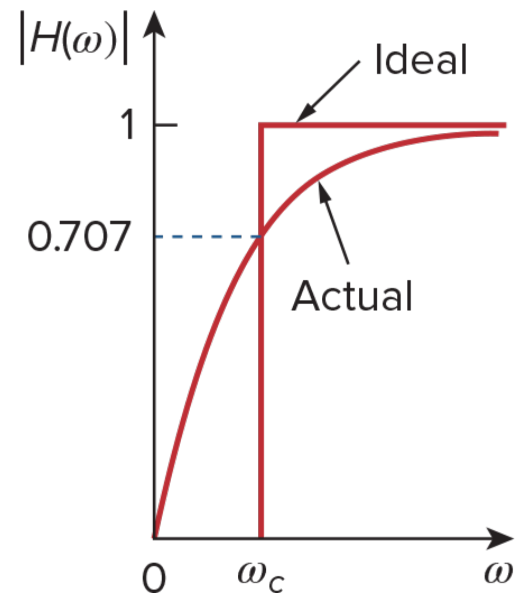
- Low-pass filter
  - when the output of a RC circuit is taken off the capacitor.
- ω_C
  - The half power frequency
  - cutoff frequency
  - or 3dB frequency.
- The filter is designed to pass from DC up to ω_C

14.7.2 High-Pass Filter

$$H(\omega) = \frac{V_o}{V_i} = \frac{R}{R + 1/j\omega C} \Rightarrow H(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$



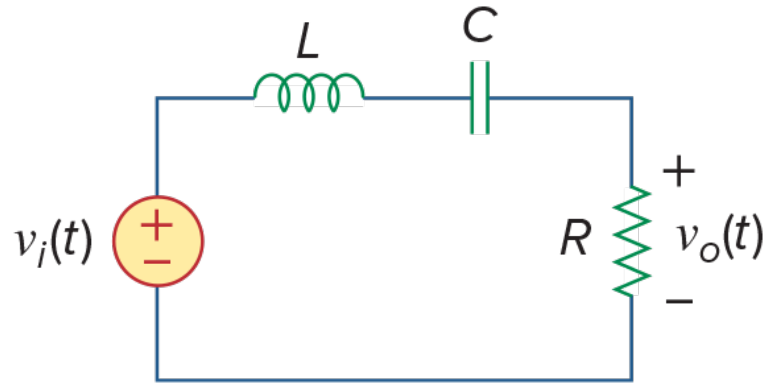
$$\Rightarrow \omega_c = \frac{1}{RC}$$



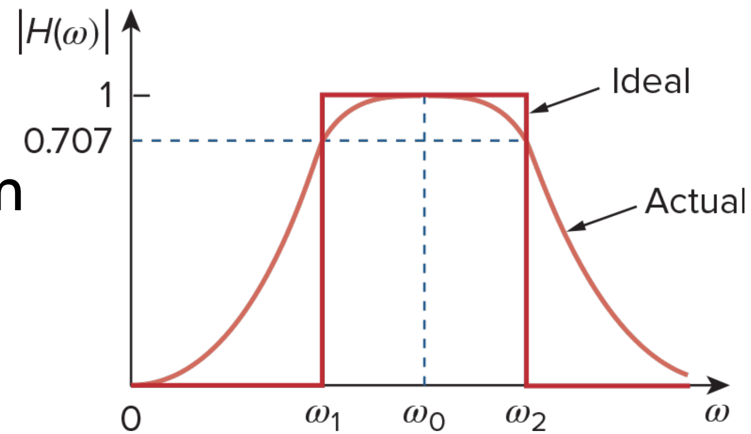
- High-pass filter
  - when the output taken off the resistor.
- ω_C
  - The half power frequency
  - cutoff frequency
  - or 3dB frequency.
- The cutoff frequency will be the same as the lowpass filter.
- The difference being that the frequencies passed go from ω_C to infinity.

14.7.3 Band-Pass Filter

$$H(\omega) = \frac{V_o}{V_i} = \frac{R}{R + j(\omega L - 1/\omega C)} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$



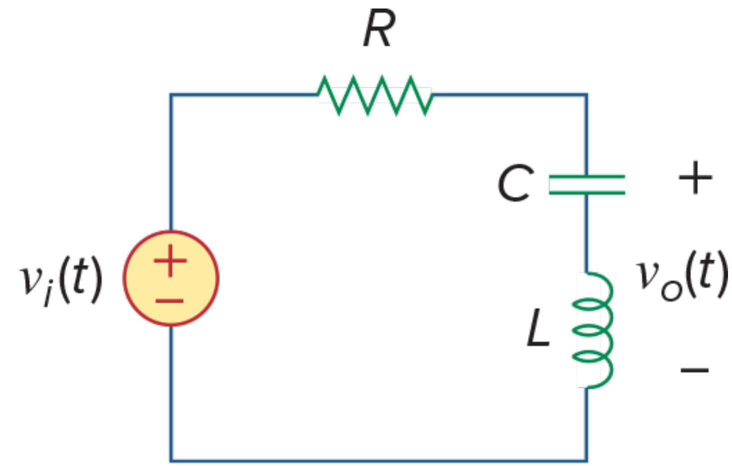
- Band-pass filter
  - The RLC series resonant circuit
  - when the output is taken off the resistor.
- ω_0
  - The center frequency



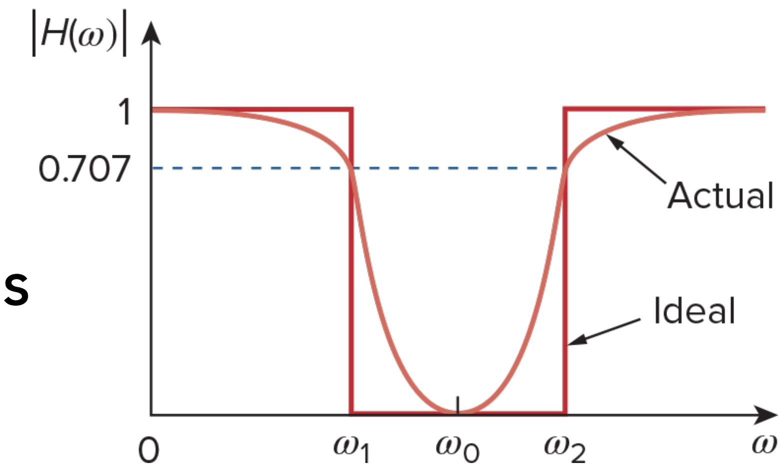
- The filter will pass frequencies from ω_1 to ω_2.
  - passes a band of frequencies
    - $\omega_1 < \omega < \omega_2$ centered on ω_0
- It can also be made by feeding the output from a low-pass to a high-pass filter.

14.7.4 Band-Stop Filter

$$H(\omega) = \frac{V_o}{V_i} = \frac{j(\omega L - 1/\omega C)}{R + j(\omega L - 1/\omega C)} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$



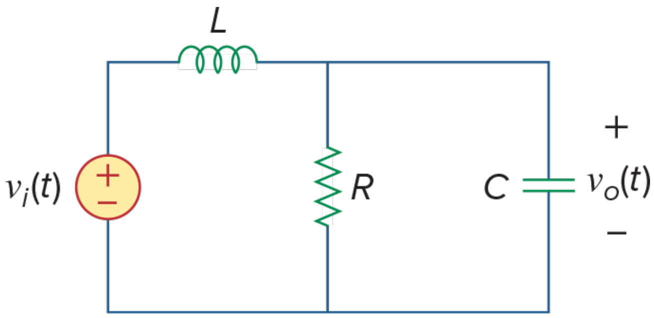
- Band-Stop (Band-Reject) filter
  - The RLC series resonant circuit
  - the output is taken off the taken off the LC series combination.
- The range of blocked frequencies
  - the same as the range of passed frequencies for the band-pass filter.



- ω_0 – center frequency
- The filter will reject frequencies from ω_1 to ω_2.
  - rejects a band of frequencies
    - $\omega_1 < \omega < \omega_2$ centered on ω_0

Example 14.10

Determine what type of filter. Calculate the corner or cutoff frequency. Take R = 2 kΩ, L = 2 H, and C = 2 μF.



The transfer function

$$\mathbf{H}(s) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{R \parallel 1/sC}{sL + R \parallel 1/sC}, \qquad s = j\omega \qquad R \left\| \frac{1}{sC} \right. = \frac{R/sC}{R + 1/sC} = \frac{R}{1 + sRC}$$

$$\Rightarrow \quad \mathbf{H}(s) = \frac{R/(1 + sRC)}{sL + R/(1 + sRC)} = \frac{R}{s^2RLC + sL + R}, \qquad s = j\omega$$

$$\mathbf{H}(\omega) = \frac{R}{-\omega^2RLC + j\omega L + R}$$

H (0) = 1 & H (∞) = 0 ⇒ second-order low-pass filter

$$H = \frac{R}{\sqrt{(R - \omega^2RLC)^2 + \omega^2L^2}}$$

The corner frequency is the same as the half-power frequency, that is, where H is reduced by a factor of $1/\sqrt{2}$.

dc value of H (ω) is 1

$$\Rightarrow \quad H^2 = \frac{1}{2} = \frac{R^2}{(R - \omega_c^2RLC)^2 + \omega_c^2L^2}$$

$$\Rightarrow \quad 2 = (1 - \omega_c^2LC)^2 + \left(\frac{\omega_c L}{R}\right)^2$$

$$2 = (1 - \omega_c^2\, 4 \times 10^{-6})^2 + (\omega_c\, 10^{-3})^2$$

$$2 = (1 - 4\omega_c^2)^2 + \omega_c^2$$

$$16\omega_c^4 - 7\omega_c^2 - 1 = 0$$

$$\omega_c^2 = 0.5509 \text{ and } -0.1134$$

$$\omega_c = 0.742 \text{ krad/s} = 742 \text{ rad/s}$$