Section 6.4 Variation of Parameters

Introduction：Variation of Parameters for high-order differential equation

Consider the nth-order differential equation

$$a_n(x)y^n(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_0(x)y(x)=g(x)$$

and we obtain $y_1,y_2,\ldots,y_n$ are solutions for $a_n(x)y^n(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_0(x)y(x)=0$.

Then we find a homogeneous solution is $y_h=c_1y_1+c_2y_2+\cdots+c_ny_n$.

In the method of variation of parameters, we let $y_p=v_1y_1+v_2y_2+\cdots+v_ny_n$

and determine the functions $v_1,v_2,\ldots,v_n$.

Method of Variation of Parameters

To solve $y_p(t)=v_1y_1+v_2y_2+\cdots+v_ny_n$

1. $W[y_1,y_2,\ldots,y_n](x)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$

2. $W_k(x)=(-1)^{n-k}W[y_1,y_2,\ldots,y_{k-1},y_{k+1},\ldots y_n]$

3. $v_k(x)=\int\dfrac{g(x)W_k(x)}{W[y_1,y_2,\ldots,y_n](x)}dx$

◇ Use the method of variation of parameters to determine a particular solution to the given equation.

3. $z'''+3z''-4z=e^{2x}$

<u>Sol.</u>

$$r^3+3r^2-4=0\Rightarrow(r-1)(r+2)^2=0\Rightarrow r=1,-2,-2$$

$$\therefore\ y_h=c_1e^x+c_2e^{-2x}+c_3xe^{-2x}$$

Let $y_p=v_1e^x+v_2e^{-2x}+v_3xe^{-2x}$

$$W[e^x,e^{-2x},xe^{-2x}](x)=\begin{vmatrix} e^x & e^{-2x} & xe^{-2x} \\ e^x & -2e^{-2x} & e^{-2x}-2xe^{-2x} \\ e^x & 4e^{-2x} & -4e^{-2x}+4xe^{-2x} \end{vmatrix}=9e^{-3x}$$

$$W_1[e^{-2x}, xe^{-2x}](x) = (-1)^{3-1}\begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x}-2xe^{-2x} \end{vmatrix} = e^{-4x}$$

$$W_2[e^{x}, xe^{-2x}](x) = (-1)^{3-2}\begin{vmatrix} e^{x} & xe^{-2x} \\ e^{x} & e^{-2x}-2xe^{-2x} \end{vmatrix} = -e^{-x}+3xe^{-x}$$

$$W_3[e^{x}, e^{-2x}](x) = (-1)^{3-3}\begin{vmatrix} e^{x} & e^{-2x} \\ e^{x} & -2e^{-2x} \end{vmatrix} = -3e^{-x}$$

$$v_1(x) = \int \frac{e^{2x}\cdot e^{-4x}}{9e^{-3x}}dx = \frac{1}{9}e^{x}+d_1$$

$$v_2(x) = \int \frac{e^{2x}\cdot(-e^{-x}+3xe^{-x})}{9e^{-3x}}dx = \frac{1}{9}\int(-e^{4x}+3xe^{4x})dx = \frac{1}{12}xe^{4x}-\frac{7}{144}e^{4x}+d_2$$

$$v_3(x) = \int \frac{e^{2x}\cdot(-3e^{-x})}{9e^{-3x}}dx = -\frac{1}{12}e^{4x}+d_3$$

$$y_p = (\frac{1}{9}e^{x}+d_1)\cdot e^{x} + (\frac{1}{12}xe^{4x}-\frac{7}{144}e^{4x}+d_2)\cdot e^{-2x} - (\frac{1}{12}e^{4x}+d_3)\cdot xe^{-2x}$$

let $d_1 = d_2 = d_3 = 0$

$$\therefore\ y_p = \frac{1}{9}e^{2x}+\frac{1}{12}xe^{2x}-\frac{7}{144}e^{2x}-\frac{1}{12}xe^{2x} = \frac{1}{16}e^{2x}$$

5. $y''' + y' = \tan x,\ \ 0 < x < \pi/2$

Sol.

$$r^3 + r = 0 \Rightarrow r(r^2+1) = 0 \Rightarrow r = 0,\ \pm i$$

$$\therefore\ y_h = c_1 + c_2\cos x + c_3\sin x$$

Let $y_p = v_1 + v_2\cos x + v_3\sin x$

$$W[1,\cos x,\sin x](x) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1$$

$$W_1[\cos x,\sin x](x) = (-1)^{3-1}\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$W_2[1,\sin x](x) = (-1)^{3-2}\begin{vmatrix} 1 & \sin x \\ 0 & \cos x \end{vmatrix} = -\cos x$$

$$W_3[1,\cos x](x) = (-1)^{3-3}\begin{vmatrix} 1 & \cos x \\ 0 & -\sin x \end{vmatrix} = -\sin x$$

$$v_1(x) = \int \frac{\tan x\cdot 1}{1}dx = -\ln|\cos x| + d_1$$

$$v_2(x) = \int \frac{\tan x\cdot(-\cos x)}{1}dx = -\int \sin x dx = \cos x + d_2$$

$$v_3(x)=\int\frac{\tan x\cdot(-\sin x)}{1}dx=\int\frac{\cos^2 x-1}{\cos x}dx=\int(\cos x-\sec x)dx=\sin x-\ln|\sec x+\tan x|+d_3$$

$$y_p=-\ln|\cos x|+d_1+\cos^2 x+d_2\cos x+(\sin x-\ln|\sec x+\tan x|)\sin x+d_3\sin x$$

let $d_1=d_2=d_3=0$ and

$\because\ 0<x<\pi/2$

$$\boxed{-\ln|\cos x|=-\ln(\cos x)=\ln(\cos x)^{-1}=\ln(\sec x)}$$

$\therefore\ y_p=\ln(\sec x)+1-(\sin x)\ln(\sec x+\tan x)$

7. Find a general solution to the Cauchy-Euler equation $x^3y'''-3x^2y''+6xy'-6y=x^{-1},\ \ x>0$, given that $\{x,x^2,x^3\}$ is a fundamental solution set for the corresponding homogeneous equation.

<u>Sol.</u>

$$x^3y'''-3x^2y''+6xy'-6y=x^{-1}\Rightarrow y'''-3x^{-1}y''+6x^{-2}y'-6x^{-3}y=\underbrace{x^{-4}}_{g(x)}$$

$\because\ \{x,x^2,x^3\}$ is a fundamental solution set for the corresponding homogeneous equation

$\therefore\ y_h=c_1x+c_2x^2+c_3x^3$

Let $y_p=v_1x+v_2x^2+v_3x^3$

$$W[x,x^2,x^3](x)=\begin{vmatrix}x & x^2 & x^3\\ 1 & 2x & 3x^2\\ 0 & 2 & 6x\end{vmatrix}=2x^3$$

$$W_1[x^2,x^3](x)=(-1)^{3-1}\begin{vmatrix}x^2 & x^3\\ 2x & 3x^2\end{vmatrix}=x^4$$

$$W_2[x,x^3](x)=(-1)^{3-2}\begin{vmatrix}x & x^3\\ 1 & 3x^2\end{vmatrix}=-2x^3$$

$$W_3[x,x^2](x)=(-1)^{3-3}\begin{vmatrix}x & x^2\\ 1 & 2x\end{vmatrix}=x^2$$

$$v_1(x)=\int\frac{x^{-4}\cdot x^4}{2x^3}dx=-\frac{1}{4}x^{-2}+d_1$$

$$v_2(x)=\int\frac{x^{-4}\cdot(-2x^3)}{2x^3}dx=\frac{1}{3}x^{-3}+d_2$$

$$v_3(x)=\int\frac{x^{-4}\cdot x^2}{2x^3}dx=-\frac{1}{8}x^{-4}+d_3$$

$$y_p=(-\frac{1}{4}x^{-2}+d_1)x+(\frac{1}{3}x^{-3}+d_2)x^2+(-\frac{1}{8}x^{-4}+d_3)x^3$$

let $d_1=d_2=d_3=0$

$$\therefore\ y_p=\frac{-x^{-1}}{4}+\frac{x^{-1}}{3}-\frac{x^{-1}}{8}=\frac{-x^{-1}}{24}$$

$$\therefore \quad y(x) = c_1 x + c_2 x^2 + c_3 x^3 - \frac{x^{-1}}{24}$$

11. Find a general solution to the Cauchy-Euler equation $x^3 y''' - 3xy' + 3y = x^4 \cos x$, $x > 0$.

Sol.

Let $y = x^r \Rightarrow y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$, and $y''' = r(r-1)(r-2)x^{r-3}$

$$\Rightarrow x^3 \cdot r(r-1)(r-2)x^{r-3} - 3x \cdot rx^{r-1} + 3x^r = 0$$
$$\Rightarrow [r(r-1)(r-2) - 3r + 3]x^r = 0$$
$$\Rightarrow r(r-1)(r-2) - 3r + 3 = 0$$
$$\Rightarrow r(r-1)(r-2) - 3(r-1) = 0$$
$$\Rightarrow (r-1)[r(r-2) - 3] = 0$$
$$\Rightarrow (r-1)(r^2 - 2r - 3) = 0$$
$$\Rightarrow r = 1,\ -1,\ 3$$

$$\therefore \quad y_h = c_1 x + c_2 x^{-1} + c_3 x^3$$

$$x^3 y''' - 3xy' + 3y = x^4 \cos x \Rightarrow y''' - 3x^{-2} y' + 3x^{-3} y = x \cos x$$

Let $y_p = v_1 x + v_2 x^{-1} + v_3 x^3$

$$W[x, x^{-1}, x^3](x) = \begin{vmatrix} x & x^{-1} & x^3 \\ 1 & -x^{-2} & 3x^2 \\ 0 & 2x^{-3} & 6x \end{vmatrix} = -16$$

$$W_1[x^{-1}, x^3](x) = (-1)^{3-1} \begin{vmatrix} x^{-1} & x^3 \\ -x^{-2} & 3x^2 \end{vmatrix} = 4x$$

$$W_2[x, x^3](x) = (-1)^{3-2} \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = -2x^3$$

$$W_3[x, x^{-1}](x) = (-1)^{3-3} \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1}$$

$$v_1(x) = \int \frac{x \cos x \cdot 4x}{-16} dx = -\frac{1}{4} \int x^2 \cos x dx = -\frac{1}{4}(x^2 \sin x + 2x \cos x - 2\sin x) + d_1$$

$$v_2(x) = \int \frac{x \cos x \cdot (-2x^3)}{-16} dx$$
$$= \frac{1}{8} \int x^4 \cos x dx$$
$$= \frac{1}{8}(x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24\sin x) + d_2$$
$$= \frac{1}{8} x^4 \sin x + \frac{1}{2} x^3 \cos x - \frac{3}{2} x^2 \sin x - 3x \cos x + 3\sin x + d_2$$

$$v_3(x) = \int \frac{x\cos x \cdot (-2x^{-1})}{-16} dx = \frac{1}{8}\int \cos x dx = \frac{1}{8}\sin x + d_3$$

$$y_p = (-\frac{1}{4}x^2 \sin x - \frac{1}{2}x\cos x + \frac{1}{2}\sin x + d_1)x$$
$$+ (\frac{1}{8}x^4 \sin x + \frac{1}{2}x^3 \cos x - \frac{3}{2}x^2 \sin x - 3x\cos x + 3\sin x + d_2)x^{-1} + (\frac{1}{8}\sin x + d_3)x^3$$

let $d_1 = d_2 = d_3 = 0$

$$y_p = -x\sin x - 3\cos x + 3x^{-1}\sin x$$

$$\therefore \quad y(x) = c_1 x + c_2 x^{-1} + c_3 x^3 - x\sin x - 3\cos x + 3x^{-1}\sin x$$