Section 6.1　Basic Theory of Linear Differential Equations

Definition：Linear Differential Equation of order n

Form：

(1)　$a_n(x)y^n(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_0(x)y(x)=b(x)$,

Where $a_0(x), a_1(x), \ldots, a_n(x)$ and $b(x)$ depend only on x, not y.

I. $a_0, a_1, \ldots, a_n$ are all constants, (1) is constant coefficients；otherwise it is variable coefficients.

II. If $b(x)=0$, (1) is called homogeneous； otherwise it is nonhomogeneous.

Standard form：

(2)　$y^n(x)+p_1(x)y^{(n-1)}(x)+\cdots+p_n(x)y(x)=g(x)$

Theorem 1：Existence and Uniqueness

Suppose $p_1(x), \ldots, p_n(x)$, and $g(x)$ are each continuous on an interval (a,b) that contains the point x_0. Then, for any choice of the initial values γ_0, $\gamma_1, \ldots, \gamma_{n-1}$, there exists a unique solution $y(x)$ on the whole interval (a,b) to the I.V.P.

$$y^n(x)+p_1(x)y^{(n-1)}(x)+\cdots+p_n(x)y(x)=g(x),\quad y(x_0)=\gamma_0,\ y'(x_0)=\gamma_1, \ldots,\ y^{(n-1)}(x_0)=\gamma_{n-1}.$$

Definition 1：Wronskian

Let $f_1, \ldots, f_n$ be any n functions that are $(n-1)$ times differentiable. The function

$$W[f_1,\ldots,f_n](x):=\begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the Wronskian of $f_1, \ldots, f_n$.

Definition 2：Linear Dependence of Functions

The m functions f_1, $f_2, \ldots, f_m$ are said to be **linearly dependent on an interval** I if at least one of them can be expressed as a linear combination of the others on I；equivalently, they are linearly dependent if there exist constants c_1, $c_2, \ldots, c_m$, not all zero, such that $c_1f_1(x)+c_2f_2(x)+\cdots+c_mf_m(x)=0$ for all x in I. Otherwise, they are said to be **linearly independent on** I.

Theorem 3：Linear Dependence and the Wronskian

If y_1, $y_2,\ldots,y_n$ are n solutions to $y^n + p_1 y^{(n-1)} + \cdots + p_n y = 0$ on (a,b), with p_1, $p_2,\ldots,p_n$ continuous on (a,b), then the following statements are equivalent：

(i) y_1, $y_2,\ldots,y_n$ are L.D. on (a,b).

(ii) $\exists x_0 \in (a,b)$, $W[y_1,\ldots,y_n](x_0) = 0$

(iii) $\forall x \in (a,b)$, $W[y_1,\ldots,y_n](x) = 0$

These statements are also equivalent：

(iv) y_1, $y_2,\ldots,y_n$ are L.I. on (a,b).

(v) $\exists x_0 \in (a,b)$, $W[y_1,\ldots,y_n](x_0) \neq 0$

(vi) $\forall x \in (a,b)$, $W[y_1,\ldots,y_n](x) \neq 0$

Whenever (iv), (v), or (vi) is met, $\{y_1, y_2,\ldots,y_n\}$ is called a fundamental solution set for $y^n + p_1 y^{(n-1)} + \cdots + p_n y = 0$ on (a,b).

◇ Determine the largest interval (a,b) for which Theorem 1 guarantees the existence of a unique solution on (a,b) to the given initial value problem.

5. $x\sqrt{x+1}y''' - y' + xy = 0$ ； $y(1/2) = y'(1/2) = -1$, $y''(1/2) = 1$

Sol.

$$x\sqrt{x+1}y''' - y' + xy = 0$$

$$\Rightarrow y''' - \frac{1}{x\sqrt{x+1}} y' + \frac{x}{x\sqrt{x+1}} y = 0$$

(1) $p_1(x) = 0$ is continuous on $(-\infty,\infty)$

(2) $p_2(x) = \dfrac{-1}{x\sqrt{x+1}}$ is continuous on $(-1,0)\cup(0,\infty)$

(3) $p_3(x) = \dfrac{x}{x\sqrt{x+1}}$ is continuous on $(-1,\infty)$

(4) $g(x) = 0$ is continuous on $(-\infty,\infty)$

$\Rightarrow p_1(x), p_2(x), p_3(x)$, and $g(x)$ are continuous on $(-1,0)\cup(0,\infty)$ and $x_0 = \dfrac{1}{2} \in (0,\infty)$

$\therefore$ The largest interval is $(0,\infty)$.

◇ Determine whether the given functions are linearly dependent or linearly independent on the specified interval. Justify your decisions.

8. $\{x^2, x^2 - 1, 5\}$ on $(-\infty,\infty)$

Sol.

Assume c_1, c_2, and c_3 are constants for which $c_1x^2+c_2(x^2-1)+5c_3=0$

Set $x=0, 1, \text{and} -1$

$$\Rightarrow \begin{cases} -c_2+5c_3=0 \\ c_1+5c_3=0 \\ c_1+5c_3=0 \end{cases} \Rightarrow 5c_3=c_2=-c_1 \Rightarrow \begin{cases} c_3=1 \\ c_2=5 \\ c_1=-5 \end{cases}$$

$\Rightarrow \{x^2, x^2-1, 5\}$ are L.D. on $(-\infty,\infty)$.

12. $\{\cos 2x, \cos^2 x, \sin^2 x\}$ on $(-\infty,\infty)$

Sol.

Assume c_1, c_2, and c_3 are constants for which $c_1\cos 2x+c_2\cos^2 x+c_3\sin^2 x=0$

Set $x=0, \dfrac{\pi}{2}, \text{and } \pi$

$$\Rightarrow \begin{cases} c_1+c_2=0 \\ -c_1+c_3=0 \\ c_1+c_2=0 \end{cases} \Rightarrow c_1=-c_2=c_3 \Rightarrow \begin{cases} c_1=1 \\ c_2=-1 \\ c_3=1 \end{cases}$$

$\Rightarrow \{\cos 2x, \cos^2 x, \sin^2 x\}$ are L.D. on $(-\infty,\infty)$.

13. $\{x, x^2, x^3, x^4\}$ on $(-\infty,\infty)$

Sol.

Assume c_1, c_2, c_3, and c_4 are constants for which $c_1x+c_2x^2+c_3x^3+c_4x^4=0$

Set $x=1, -1, 2, \text{and} -2$

$$\Rightarrow \begin{cases} c_1+c_2+c_3+c_4=0 \\ -c_1+c_2-c_3+c_4=0 \\ 2c_1+4c_2+8c_3+16c_4=0 \\ -2c_1+4c_2-8c_3+16c_4=0 \end{cases} \Rightarrow c_1=c_2=c_3=c_4=0$$

$\Rightarrow \{x, x^2, x^3, x^4\}$ are L.I on $(-\infty,\infty)$.

◇ Using the Wronskian, verify that the given functions form a fundamental solution set for the given differential equation and find a general solution.

16. $y'''-y''+4y'-4y=0$; $\{e^x, \cos 2x, \sin 2x\}$

Sol.

$$W[e^x,\cos 2x,\sin 2x]=\begin{vmatrix} e^x & \cos 2x & \sin 2x \\ e^x & -2\sin 2x & 2\cos 2x \\ e^x & -4\cos 2x & -4\sin 2x \end{vmatrix}$$
$$=8e^x\sin^2 2x+2e^x\cos^2 2x-4e^x\sin 2x\cos 2x-(-2e^x\sin^2 2x-8e^x\cos^2 2x-4e^x\sin 2x\cos 2x)$$
$$=10e^x(\sin^2 2x+\cos^2 2x)$$
$$=10e^x\neq 0$$

By Theorem3, $\{e^x,\cos 2x,\sin 2x\}$ is a fundamental solution set and hence the general solution is $y(x)=c_1e^x+c_2\cos 2x+c_3\sin 2x$

◇ A particular solution and a fundamental solution set are given for a nonhomogeneous equation and its corresponding homogeneous equation.

(a) Find a general solution to the nonhomogeneous equations.

(b) Find the solution that satisfies the specified initial conditions.

19. $y'''+y''+3y'-5y=2+6x-5x^2$; $y(0)=-1,\ y'(0)=1,\ y''(0)=-3$; $y_p=x^2$;
$\{e^x,e^{-x}\cos 2x,e^{-x}\sin 2x\}$

Sol.

(a) $y(x)=c_1e^x+c_2e^{-x}\cos 2x+c_3e^{-x}\sin 2x+x^2$

(b)

$$y'(x)=c_1e^x+c_2(-e^{-x}\cos 2x-2e^{-x}\sin 2x)+c_3(-e^{-x}\sin 2x+2e^{-x}\cos 2x)+2x$$
$$=c_1e^x+(-c_2+2c_3)e^{-x}\cos 2x+(-2c_2-c_3)e^{-x}\sin 2x+2x$$
$$y''(x)=c_1e^x+(-c_2+2c_3)(-e^{-x}\cos 2x-2e^{-x}\sin 2x)+(-2c_2-c_3)(-e^{-x}\sin 2x+2e^{-x}\cos 2x)+2$$
$$=c_1e^x+(c_2-2c_3-4c_2-2c_3)e^{-x}\cos 2x+(2c_2-4c_3+2c_2+c_3)e^{-x}\sin 2x+2$$
$$=c_1e^x+(-3c_2-4c_3)e^{-x}\cos 2x+(4c_2-3c_3)e^{-x}\sin 2x+2$$

$\because$ $y(0)=-1,\quad y'(0)=1,\quad y''(0)=-3$

$$\Rightarrow\begin{cases} c_1+c_2=-1 \\ c_1-c_2+2c_3=1 \\ c_1-3c_2-4c_3+2=-3 \end{cases}\quad\Rightarrow\begin{cases} c_1=-1 \\ c_2=0 \\ c_3=1 \end{cases}$$

$\therefore$ $y(x)=-e^x+e^{-x}\sin 2x+x^2$

25. Prove that L defined in (7) is a linear operator by verifying that properties (9) and (10) hold for any $n-$times differentiable functions $y,y_1,\ldots,y_m$ on (a,b).

(7) $L[y]:=\dfrac{d^ny}{dx^n}+p_1\dfrac{d^{n-1}y}{dx^{n-1}}+\cdots+p_ny=(D^n+p_1D^{n-1}+\cdots+p_n)[y]$

(9) $L[y_1+y_2+\cdots+y_m]=L[y_1]+L[y_2]+\cdots+L[y_m]$

(10) $L[cy] = cL[y]$ (c any constant)

Sol.

I.

$$L[y_1 + y_2] = \frac{d^n}{dx^n}(y_1 + y_2) + p_1 \frac{d^{n-1}}{dx^{n-1}}(y_1 + y_2) + \cdots + p_n(y_1 + y_2)$$

$$= \frac{d^n}{dx^n} y_1 + \frac{d^n}{dx^n} y_2 + p_1 \frac{d^{n-1}}{dx^{n-1}} y_1 + p_1 \frac{d^{n-1}}{dx^{n-1}} y_1 + \cdots + p_n y_1 + p_n y_2$$

$$= \left(\frac{d^n}{dx^n} y_1 + p_1 \frac{d^{n-1}}{dx^{n-1}} y_1 + \cdots + p_n y_1 \right) + \left(\frac{d^n}{dx^n} y_2 + p_1 \frac{d^{n-1}}{dx^{n-1}} y_2 + \cdots + p_n y_2 \right)$$

$$= L[y_1] + L[y_2]$$

II.

$$L[cy] = \frac{d^n}{dx^n}(cy) + p_1 \frac{d^{n-1}}{dx^{n-1}}(cy) + \cdots + p_n(cy)$$

$$= c \cdot \frac{d^n}{dx^n} y + c \cdot p_1 \frac{d^{n-1}}{dx^{n-1}} y + \cdots + c \cdot p_n y$$

$$= c\left(\frac{d^n}{dx^n} y + p_1 \frac{d^{n-1}}{dx^{n-1}} y + \cdots + p_n y \right)$$

$$= cL[y]$$

27. Show that the set of functions $\{1, x, x^2, \ldots, x^n\}$, where n is a positive integer, is linearly independent on every open interval (a,b). [*Hint*: Use the fact that a polynomial of degree at most n has no more than n zeros unless it is identically zero.]

Sol.

Assume c_0, c_1, $c_2, \ldots, c_n$ are constants for

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} + c_n x^n = 0$$

$$f'(x) = c_1 + 2c_2 x + \cdots + (n-1)c_{n-1} x^{n-2} + nc_n x^{n-1} = 0$$

$$f''(x) = 2c_2 + \cdots + (n-1)(n-2)c_{n-1} x^{n-3} + n(n-1)c_n x^{n-2} = 0$$

$$\vdots$$

$$f^{(n-1)}(x) = (n-1)!c_{n-1} + n!c_n x = 0$$

$$f^{(n)}(x) = n!c_n = 0$$

$\Rightarrow c_n = 0$ and by backward substitution , $c_{n-1} = c_{n-2} = \cdots = c_1 = c_0 = 0$

$\therefore$ $\{1, x, x^2, \ldots, x^n\}$ is L.I.

28. The set of functions $\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx\}$, where n is a positive integer, is linearly independent on every interval (a,b). Prove this in the special case $n = 2$ and $(a,b) = (-\infty, \infty)$.

Sol.

For $n = 2$, $\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx\} = \{1, \cos x, \sin x, \cos 2x, \sin 2x\}$

Assume c_1, c_2, c_3, c_4 and c_5 are constants for which

$$c_1 + c_2 \cos x + c_3 \sin x + c_4 \cos 2x + c_5 \sin 2x = 0$$

Set $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and $\frac{\pi}{4}$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_4 = 0 \\ c_1 + c_3 - c_4 = 0 \\ c_1 - c_2 + c_4 = 0 \\ c_1 - c_3 - c_4 = 0 \\ c_1 + \frac{\sqrt{2}}{2} c_2 + \frac{\sqrt{2}}{2} c_3 + c_5 = 0 \end{cases} \qquad \Rightarrow c_1 = c_2 = c_3 = c_4 = c_5 = 0$$

$\Rightarrow \{1, \cos x, \sin x, \cos 2x, \sin 2x\}$ is L.I.