# Section 7.4 Inverse Laplace Transform

**Definition：Inverse Laplace Transform**
Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0,\infty)$ and satisfies $L\{f\}=F$, then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f=L^{-1}\{F\}$.

**Theorem：Linearity of the Inverse Transform**
Assume that Let $L^{-1}\{F\}$, $L^{-1}\{F_1\}$, and $L^{-1}\{F_2\}$ exist and continuous on $[0,\infty)$ and let $c$ be any constant. Then
(i) $L^{-1}\{F_1+F_2\}=L^{-1}\{F_1\}+L^{-1}\{F_2\}$
(ii) $L^{-1}\{cF\}=cL^{-1}\{F\}$.

◇ Determine the inverse Laplace transform of the given function.

3. $\dfrac{s+1}{s^2+2s+10}$

Sol.

$$\frac{s+1}{s^2+2s+10}=\frac{s+1}{(s+1)^2+3^2}$$

$$L^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\}(t)=e^{-t}\cos 3t$$

◇ Determine the partial fraction expansions for the given rational function.

13. $\dfrac{-2s^2-3s-2}{s(s+1)^2}$

Sol.

$$\frac{-2s^2-3s-2}{s(s+1)^2}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+1)^2}$$

$$\Rightarrow \frac{-2s^2-3s-2}{s(s+1)^2}=\frac{A(s+1)^2+Bs(s+1)+Cs}{s(s+1)^2}$$

$$\Rightarrow -2s^2-3s-2=A(s+1)^2+Bs(s+1)+Cs$$

Let $s=-1 \Rightarrow -2+3-2=-C \Rightarrow C=1$
Let $s=0 \Rightarrow -2=A$
Let $s=1 \Rightarrow -7=-2\cdot 2^2+2B+1 \Rightarrow B=0$

$$\therefore \frac{-2s^2-3s-2}{s(s+1)^2}=\frac{-2}{s}+\frac{1}{(s+1)^2}$$

19. $\dfrac{1}{(s-3)(s^2+2s+2)}$

<u>Sol.</u>

$$\frac{1}{(s-3)(s^2+2s+2)}=\frac{1}{(s-3)[(s+1)^2+1^2]}=\frac{A}{s-3}+\frac{B(s+1)+C}{(s+1)^2+1^2}$$

$$\Rightarrow\frac{1}{(s-3)(s^2+2s+2)}=\frac{A[(s+1)^2+1^2]+B(s+1)(s-3)+C(s-3)}{(s-3)[(s+1)^2+1^2]}$$

$$\Rightarrow 1=A[(s+1)^2+1^2]+B(s+1)(s-3)+C(s-3)$$

Let $s=3\Rightarrow 1=17A\Rightarrow A=\dfrac{1}{17}$

Let $s=-1\Rightarrow 1=A-4C\Rightarrow 4C=\dfrac{1}{17}-1=\dfrac{-16}{17}\Rightarrow C=\dfrac{-4}{17}$

Let $s=0\Rightarrow 1=2A-3B-3C\Rightarrow 3B=\dfrac{2}{17}+\dfrac{12}{17}-1=\dfrac{-3}{17}\Rightarrow B=\dfrac{-1}{17}$

$$\therefore\ \frac{1}{(s-3)(s^2+2s+2)}=\frac{1}{17}\left[\frac{1}{s-3}-\frac{s+1}{(s+1)^2+1^2}-\frac{4}{(s+1)^2+1^2}\right]$$

◇ Theorem 6 in Section 7.3 can be expressed in terms of the inverse Laplace transform as

$L^{-1}\left\{\dfrac{d^nF}{ds^n}\right\}(t)=(-t)^n f(t)$, where $f=L^{-1}\{F\}$. Use this equation to compute $L^{-1}\{F\}$.

35. $F(s)=\ln\left(\dfrac{s^2+9}{s^2+1}\right)$

<u>Sol.</u>

$$F(s)=\ln\left(\frac{s^2+9}{s^2+1}\right)=\ln(s^2+9)-\ln(s^2+1)$$

$$\frac{dF}{ds}=\frac{2s}{s^2+9}-\frac{2s}{s^2+1}=2\left(\frac{s}{s^2+9}-\frac{s}{s^2+1}\right)$$

$$L^{-1}\left\{\ln\left(\frac{s^2+9}{s^2+1}\right)\right\}(t)$$

$$=L^{-1}\left\{2\left(\frac{s}{s^2+9}-\frac{s}{s^2+1}\right)\right\}(t)$$

$$=2\cdot L^{-1}\left\{\frac{s}{s^2+9}-\frac{s}{s^2+1}\right\}(t)$$

$$=2(\cos 3t-\cos t)=(-t)f(t)$$

$$\Rightarrow f(t)=\frac{2(\cos 3t-\cos t)}{-t}=\frac{2(\cos t-\cos 3t)}{t}\qquad\therefore\ L^{-1}\{F(s)\}(t)=\frac{2(\cos t-\cos 3t)}{t}$$