## Section 7.5　Solving Initial Value Problems

**Method of Laplace Transforms**

To solve an initial value problem：

(i) Take the Laplace transform of both sides of the equation.

(ii) Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.

(iii) Determine the inverse Laplace transform of the solution by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

◇ Solve the given initial value problem using the method of Laplace transforms.

7. $y''-7y'+10y=9\cos t+7\sin t$ ； $y(0)=5$, $y'(0)=-4$.

Sol.

$$L\{y''-7y'+10y\}=L\{9\cos t+7\sin t\}$$

$$\Rightarrow L\{y''\}-7L\{y'\}+10L\{y\}=\frac{9s}{s^2+1}+\frac{7}{s^2+1}$$

Let $L\{y\}=Y(s)$

$$\Rightarrow\left[s^2Y-sy(0)-y'(0)\right]-7\left[sY-y(0)\right]+10Y=\frac{9s}{s^2+1}+\frac{7}{s^2+1}$$

$$\Rightarrow(s^2Y-5s+4)-7(sY-5)+10Y=\frac{9s+7}{s^2+1}$$

$$\Rightarrow(s^2-7s+10)Y-(5s-39)=\frac{9s+7}{s^2+1}$$

$$\Rightarrow Y=\frac{5s^3-39s^2+14s-32}{(s^2-7s+10)(s^2+1)}=\frac{5s^3-39s^2+14s-32}{(s-5)(s-2)(s^2+1)}=\frac{A}{s-5}+\frac{B}{s-2}+\frac{Cs+D}{s^2+1}$$

$$\Rightarrow 5s^3-39s^2+14s-32=A(s-2)(s^2+1)+B(s-5)(s^2+1)+(Cs+D)(s-5)(s-2)$$

Let $s=2\Rightarrow-120=-15B\Rightarrow B=8$

Let $s=5\Rightarrow-312=78A\Rightarrow A=-4$

Let $s=0\Rightarrow-32=-2A-5B+10D\Rightarrow10D=-32-8+40=0\Rightarrow D=0$

Let $s=1\Rightarrow-52=-2A-8B+4C\Rightarrow4C=-52-8+64=4\Rightarrow C=1$

$$\therefore\ Y=\frac{-4}{s-5}+\frac{8}{s-2}+\frac{s}{s^2+1}$$

$$\therefore\ y(t)=L^{-1}\left\{\frac{-4}{s-5}+\frac{8}{s-2}+\frac{s}{s^2+1}\right\}=-4e^{5t}+8e^{2t}+\cos t$$

13. $y''-y'-2y=-8\cos t-2\sin t$ ； $y(\pi/2)=1$, $y'(\pi/2)=0$

Sol.

Let $w(t)=y(t+\pi/2)\Rightarrow w'(t)=y'(t+\pi/2),\ w''(t)=y''(t+\pi/2)$ and
$w(0)=y(0+\pi/2)=1,\ \ w'(0)=y'(0+\pi/2)=0$

故原 I.V.P $y''(t+\pi/2)-y'(t+\pi/2)-2y(t+\pi/2)=-8\cos(t+\pi/2)-2\sin(t+\pi/2)$；
$y(\pi/2)=1, y'(\pi/2)=0$

可轉換成 $w''-w'-2w=8\sin t-2\cos t$； $w(0)=1,\ \ w'(0)=0$

$$L\{w''-w'-2w\}=L\{8\sin t-2\cos t\}$$
$$\Rightarrow L\{w''\}-L\{w'\}-2L\{w\}=\frac{8}{s^2+1}-\frac{2s}{s^2+1}$$
$$\Rightarrow\left[s^2W-sw(0)-w'(0)\right]-\left[sW-w(0)\right]-2W=\frac{8-2s}{s^2+1}$$
$$\Rightarrow(s^2W-s)-(sW-1)-2W=\frac{8-2s}{s^2+1}$$
$$\Rightarrow(s^2-s-2)W+(-s+1)=\frac{8-2s}{s^2+1}$$
$$\Rightarrow W=\frac{s^3-s^2-s+7}{(s^2-s-2)(s^2+1)}=\frac{s^3-s^2-s+7}{(s+1)(s-2)(s^2+1)}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{Cs+D}{s^2+1}$$
$$\Rightarrow s^3-s^2-s+7=A(s-2)(s^2+1)+B(s+1)(s^2+1)+(Cs+D)(s+1)(s-2)$$

$$\boxed{\begin{array}{l}\cos(t+\pi/2)=-\sin t\\ \sin(t+\pi/2)=\cos t\\ \cos(t-\pi/2)=\sin t\\ \sin(t-\pi/2)=-\cos t\end{array}}$$

Let $s=2\Rightarrow 9=15B\Rightarrow B=\dfrac{3}{5}$

Let $s=-1\Rightarrow 6=-6A\Rightarrow A=-1$

Let $s=0\Rightarrow 7=-2A+B-2D\Rightarrow 2D=2+\dfrac{3}{5}-7=\dfrac{-22}{5}\Rightarrow D=\dfrac{-11}{5}$

Let $s=1\Rightarrow 6=-2A+4B-2(C+D)\Rightarrow 2C=2+\dfrac{12}{5}+\dfrac{22}{5}-6=\dfrac{14}{5}\Rightarrow C=\dfrac{7}{5}$

$$\therefore\ W=\frac{-1}{s+1}+\frac{3}{5(s-2)}+\frac{7s}{5(s^2+1)}-\frac{11}{5(s^2+1)}$$
$$\therefore\ w(t)=L^{-1}\left\{\frac{-1}{s+1}+\frac{3}{5(s-2)}+\frac{7s}{5(s^2+1)}-\frac{11}{5(s^2+1)}\right\}=-e^{-t}+\frac{3}{5}e^{2t}+\frac{7}{5}\cos t-\frac{11}{5}\sin t$$
$$y(t)=w(t-\pi/2)=-e^{-(t-\pi/2)}+\frac{3}{5}e^{2(t-\pi/2)}+\frac{7}{5}\cos(t-\pi/2)-\frac{11}{5}\sin(t-\pi/2)$$
$$=-e^{\pi/2-t}+\frac{3}{5}e^{2t-\pi}+\frac{7}{5}\sin t+\frac{11}{5}\cos t$$

◇ Solve the given third-order initial value problem for $y(t)$ using the method of Laplace transforms.

28. $y'''+y''+3y'-5y=16e^{-t}$； $y(0)=0,\ \ y'(0)=2,\ \ y''(0)=-4$.

Sol.

$$L\{y''' + y'' + 3y' - 5y\} = L\{16e^{-t}\}$$

$$\Rightarrow L\{y'''\} + L\{y''\} + 3L\{y'\} - 5L\{y\} = \frac{16}{s+1}$$

$$\Rightarrow \left[s^3Y - s^2y(0) - sy'(0) - y''(0)\right] + \left[s^2Y - sy(0) - y'(0)\right] + 3\left[sY - y(0)\right] - 5Y = \frac{16}{s+1}$$

$$\Rightarrow (s^3Y - 2s + 4) + (s^2Y - 2) + 3sY - 5Y = \frac{16}{s+1}$$

$$\Rightarrow (s^3 + s^2 + 3s - 5)Y + (-2s + 2) = \frac{16}{s+1}$$

$$\Rightarrow Y = \frac{2s^2 + 14}{(s^3 + s^2 + 3s - 5)(s+1)} = \frac{2s^2 + 14}{(s-1)[(s+1)^2 + 2^2](s+1)} = \frac{A}{s-1} + \frac{B(s+1) + 2C}{(s+1)^2 + 2^2} + \frac{D}{s+1}$$

$$\Rightarrow 2s^2 + 14 = A[(s+1)^2 + 2^2](s+1) + [B(s+1) + 2C](s-1)(s+1) + D(s-1)[(s+1)^2 + 2^2]$$

Let $s = -1 \Rightarrow 16 = -8D \Rightarrow D = -2$

Let $s = 1 \Rightarrow 16 = 16A \Rightarrow A = 1$

Let $s = 0 \Rightarrow 14 = 5A - (B + 2C) - 5D \Rightarrow B + 2C = 5 + 10 - 14 = 1$

Let $s = 2 \Rightarrow 22 = 39A + 3(3B + 2C) + 13D \Rightarrow 3B + 2C = 3$

$$\Rightarrow \begin{cases} B + 2C = 1 \\ 3B + 2C = 3 \end{cases} \Rightarrow B = 1, \quad C = 0$$

$$\therefore \quad Y = \frac{1}{s-1} + \frac{s+1}{(s+1)^2 + 2^2} - \frac{2}{s+1}$$

$$\therefore \quad y(t) = L^{-1}\left\{\frac{1}{s-1} + \frac{s+1}{(s+1)^2 + 2^2} - \frac{2}{s+1}\right\} = e^t + e^{-t}\cos 2t - 2e^{-t}$$

◇ Find solutions to the given initial value problem.

35. $y'' + 3ty' - 6y = 1$ ; $y(0) = 0$, $y'(0) = 0$.

Sol.

Let $Y(s) = L\{y\}(s)$

$$L\{y'' + 3ty' - 6y\} = L\{1\}$$

$$\Rightarrow L\{y''\} + 3L\{ty'\} - 6L\{y\} = \frac{1}{s}$$

$$\Rightarrow \left[s^2Y - sy(0) - y'(0)\right] + 3\cdot(-1)\frac{d}{ds}\left[sY - y(0)\right] - 6Y = \frac{1}{s}$$

$$\Rightarrow s^2Y + 3\cdot(-1)\frac{d}{ds}[sY] - 6Y = \frac{1}{s}$$

$$\Rightarrow s^2Y - 3(Y + sY') - 6Y = \frac{1}{s}$$

$$\Rightarrow (s^2 - 9)Y - 3sY' = \frac{1}{s}$$

$$\overset{\div(-3s)}{\Rightarrow} Y' + \left(\frac{3}{s} - \frac{s}{3}\right)Y = \frac{-1}{3s^2}$$

Let $\mu(s) = e^{\int(\frac{3}{s}-\frac{s}{3})ds} = e^{3\ln|s| - \frac{s^2}{6}} = s^3 e^{\frac{-s^2}{6}}$

$$\Rightarrow s^3 e^{\frac{-s^2}{6}} Y' + s^3 e^{\frac{-s^2}{6}}\left(\frac{3}{s} - \frac{s}{3}\right)Y = \frac{-s}{3} e^{\frac{-s^2}{6}}$$

$$\Rightarrow \frac{d}{ds}\left(s^3 e^{\frac{-s^2}{6}} Y\right) = \frac{-1}{3} s e^{\frac{-s^2}{6}}$$

$$\Rightarrow s^3 e^{\frac{-s^2}{6}} Y = \int \frac{-1}{3} s e^{\frac{-s^2}{6}} ds = e^{\frac{-s^2}{6}}$$

$$\Rightarrow Y = \frac{1}{s^3}$$

$$\therefore \quad y(t) = L^{-1}\left\{\frac{1}{s^3}\right\} = L^{-1}\left\{\frac{1}{2}\cdot\frac{2!}{s^{2+1}}\right\} = \frac{1}{2}t^2$$