Section 7.3 Properites of the Laplace Transform

Theorem 3：Translation in s

If the Laplace transform $L\{f\}(s)=F(s)$ exists for $s>\alpha$, then $L\{e^{at}f(t)\}(s)=F(s-a)$ for $s>\alpha+a$.

Theorem 4：Laplace Transform of the Derivative

Let $f(t)$ be continuous on $[0,\infty)$ and $f'(t)$ be piecewise continuous on $[0,\infty)$, with both of exponential order α. Then, for $s>\alpha$,

$L\{f'\}(s)=sL\{f\}(s)-f(0)$.

Theorem 5：Laplace Transform of Higher-Order Derivatives

Let $f(t)$, $f'(t),\ldots,f^{(n-1)}(t)$ be continuous on $[0,\infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0,\infty)$, with both of exponential order α. Then, for $s>\alpha$,

$L\{f^{(n)}\}(s)=s^nL\{f\}(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0)$.

Theorem 6：Derivatives of the Laplace Transform

Let $F(s)=L\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0,\infty)$, and of exponential order α. Then, for $s>\alpha$,

$$L\{t^nf(t)\}(s)=(-1)^n\frac{d^nF}{ds^n}(s).$$

◇ Determine the Laplace transform of the given function using Table 7.1 and the properties of the transform given in Table 7.2.

1. $t^2+e^t\sin 2t$

Sol.

$$\begin{aligned}
&L\{t^2+e^t\sin 2t\}(s)\\
&=L\{t^2\}(s)+L\{e^t\sin 2t\}(s)\\
&=\frac{2!}{s^3}+\frac{2}{(s-1)^2+2^2}\\
&=\frac{2}{s^3}+\frac{2}{(s-1)^2+4},\quad s>1
\end{aligned}$$

9. $e^{-t}t\sin 2t$

Sol.

$$\boxed{\text{Theorem 6：}\quad L\{t^nf(t)\}(s)=(-1)^n\frac{d^nF}{ds^n}(s)=(-1)^n\frac{d^n}{ds^n}[L\{f\}(s)]}$$

$$\begin{aligned}
&L\{e^{-t}t\sin 2t\}(s)\\
&=(-1)^1\frac{d}{ds}[L\{e^{-t}\sin 2t\}(s)]
\end{aligned}$$

$$= (-1)\frac{d}{ds}\left[\frac{2}{(s+1)^2+4}\right]$$
$$= -\frac{0-2\cdot 2(s+1)}{[(s+1)^2+4]^2}$$
$$= \frac{4(s+1)}{[(s+1)^2+4]^2}$$

21. Given that $L\{\cos bt\}(s) = s/(s^2+b^2)$, use the translation property to compute $L\{e^{at}\cos bt\}$.

<u>Sol.</u>

Theorem 3 : If $L\{f\}(s) = F(s)$, for $s > \alpha$, then $L\{e^{at} f(t)\}(s) = F(s-a)$

$$L\{\cos bt\}(s) = \frac{s}{s^2+b^2} = F(s)$$
$$L\{e^{at}\cos bt\}(s) = F(s-a) = \frac{s-a}{(s-a)^2+b^2}$$

25. Use formula (6) to help determine

(a) $L\{t\cos bt\}$

<u>Sol.</u>

$$L\{t\cos bt\}(s)$$
$$= (-1)^1\frac{d}{ds}[L\{\cos bt\}(s)]$$
$$= -\frac{d}{ds}\left[\frac{s}{s^2+b^2}\right]$$
$$= -\frac{(s^2+b^2)-s\cdot 2s}{(s^2+b^2)^2}$$
$$= \frac{s^2-b^2}{(s^2+b^2)^2}$$

(b) $L\{t^2\cos bt\}$

<u>Sol.</u>

$$L\{t^2\cos bt\}(s)$$
$$= (-1)^2\frac{d^2}{ds^2}[L\{\cos bt\}(s)]$$
$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+b^2}\right]$$
$$= \frac{d}{ds}\left[\frac{b^2-s^2}{(s^2+b^2)^2}\right]$$

$$= \frac{-2s(s^2+b^2)^2-(b^2-s^2)\cdot 4s(s^2+b^2)}{(s^2+b^2)^4}$$

$$= \frac{-2s(s^2+b^2)[(s^2+b^2)+2(b^2-s^2)]}{(s^2+b^2)^4}$$

$$= \frac{2s^3-6sb^2}{(s^2+b^2)^3}$$