Section 4.7 Variable-Coefficient Equations

Definition：Cauchy-Euler, or Equidimensional, Equations

A linear second-order equation that can be expressed in the form $at^2y''(t)+bty'(t)+cy=g(t)$, where a, b and c are constants, is called a Cauchy-Euler, or Equidimensional, Equation.

Theorem 5：Existence and Uniqueness of Solutions

Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval (a,b) that contains the point t_0. Then, for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a,b) to the I.V.P. $y''(t)+p(t)y'(t)+q(t)y(t)=g(t)$ ； $y(t_0)=Y_0$, $y'(t_0)=Y_1$.

Characteristic Equation：

To solve a homogeneous Cauchy-Euler equation $at^2y''(t)+bty'(t)+cy=0$ ：

Substitute $y=t^r$, $y'=rt^{r-1}$, $y''=r(r-1)t^{r-2}$, we can obtain the Characteristic Equation is

$$ar^2+(b-a)r+c=0$$

$$\Rightarrow r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\Rightarrow\begin{cases}\text{相異實根}(r_1\neq r_2)\Rightarrow y(t)=c_1t^{r_1}+c_2t^{r_2}\\ \text{相同實根}(r_1=r_2)\Rightarrow y(t)=c_1t^{r_1}+c_2t^{r_1}\ln t\\ \text{共軛複根}(r=\alpha\pm\beta i)\Rightarrow y(t)=c_1t^{\alpha}\cos(\beta\ln t)+c_2t^{\alpha}\sin(\beta\ln t)\end{cases}$$

Theorem 8：Reduction of Order

Let $y_1(t)$ be a solution, not identically zero, to the homogeneous differential equation $y''(t)+p(t)y'(t)+q(t)y(t)=0$ in an interval I. Then, $y_2(t)=y_1(t)\int\frac{e^{-\int p(t)dt}}{y_1(t)^2}dt$ is a second, linearly independent solution.

◇ Use Theorem 5 to discuss the existence and uniqueness of a solution to the differential equation that satisfies the initial conditions $y(1)=Y_0$, $y'(1)=Y_1$, where Y_0 and Y_1 are real constants.

2. $(1+t^2)y''+ty'-y=\tan t$

Sol.

$$(1+t^2)y''+ty'-y=\tan t$$

$$\Rightarrow y''+\frac{t}{1+t^2}y'-\frac{1}{1+t^2}y=\frac{\tan t}{1+t^2}$$

$\because$ $\frac{t}{1+t^2}$, $\frac{1}{1+t^2}$, and $\frac{\tan t}{1+t^2}$ are continuous on $(-\frac{\pi}{2},\frac{\pi}{2})$ and $1\in(-\frac{\pi}{2},\frac{\pi}{2})$

$\therefore$ there exists a unique solution on $(-\frac{\pi}{2},\frac{\pi}{2})$ to the I.V.P. $(1+t^2)y''+ty'-y=\tan t$ ；

$y(1)=Y_0$, $y'(1)=Y_1$

◇ Determine whether Theorem5 applies. If it does, then discuss what conclusions can be drawn. If it does not, explain why.

8. $(1-t)y''+ty'-2y=\sin t$; $y(0)=1$, $y'(0)=1$

<u>Sol.</u>

$$(1-t)y''+ty'-2y=\sin t$$

$$\Rightarrow y''+\frac{t}{1-t}y'-\frac{2}{1-t}y=\frac{\sin t}{1-t}\text{, for } t\neq 1$$

$\because$ $\frac{t}{1-t}$, $\frac{2}{1-t}$, and $\frac{\sin t}{1-t}$ are continuous on $(-\infty,1)$ and $0\in(-\infty,1)$

$\therefore$ there exists a unique solution on $(-\infty,1)$ to the I.V.P. $(1-t)y''+ty'-2y=\sin t$; $y(0)=1$, $y'(0)=1$.

◇ Find a general solution to the given Cauchy-Euler equation for $t>0$.

15. $y''(t)-\frac{1}{t}y'(t)+\frac{5}{t^2}y(t)=0$

<u>Sol.</u>

$$y''-\frac{1}{t}y'+\frac{5}{t^2}y=0$$

$$\Rightarrow t^2y''-ty'+5y=0 \quad (a=1,b=-1,c=5)$$

$$\Rightarrow r^2+(-1-1)r+5=0$$

$$\Rightarrow r^2-2r+5=0$$

$$\Rightarrow r=\frac{2\pm\sqrt{4-20}}{2}$$

$$\Rightarrow r=1\pm 2i$$

$\therefore$ the general solution is $y(t)=c_1t\cos(2\ln t)+c_2t\sin(2\ln t)$

◇ Solve the given initial value problem for the Cauchy-Euler equation.

19. $t^2y''(t)-4ty'(t)+4y(t)=0$; $y(1)=-2$, $y'(1)=-11$

<u>Sol.</u>

$$t^2y''-4ty'+4y=0 \quad (a=1,b=-4,c=4)$$

$$\Rightarrow r^2+(-4-1)r+4=0$$

$$\Rightarrow r^2-5r+4=0$$

$$\Rightarrow (r-1)(r-4)=0$$

$$\Rightarrow r=1,4$$

$\therefore$ $y(t)=c_1t+c_2t^4$ and $y'(t)=c_1+4c_2t^3$

$\because$ $y(1)=-2$, $y'(1)=-11$

$$\Rightarrow\begin{cases}c_1+c_2=-2\\c_1+4c_2=-11\end{cases}\Rightarrow\begin{cases}c_1=1\\c_2=-3\end{cases}\qquad\therefore\ y(t)=t-3t^4$$

◇ Use variation of parameters to find a general solution to the differential equation given that the functions y_1 and y_2 are linearly independent solutions to the corresponding homogeneous equation for $t>0$. Remember to put the equation in standard form.

37. $ty''-(t+1)y'+y=t^2$; $y_1=e^t$, $y_2=t+1$

Sol.

$$y_h=c_1e^t+c_2(t+1)$$

Let $y_p=v_1e^t+v_2(t+1)$

$$\Rightarrow\begin{cases}v_1'e^t+v_2'(t+1)=0\\ v_1'e^t+v_2'=\dfrac{t^2}{t}=t\end{cases}$$

$$\Rightarrow\begin{cases}v_2't=-t\\ v_1'=\dfrac{-v_2'(t+1)}{e^t}\end{cases}$$

$$\Rightarrow\begin{cases}v_2'=-1\\ v_1'=\dfrac{(t+1)}{e^t}=e^{-t}(t+1)\end{cases}$$

$$\Rightarrow\begin{cases}v_2=-t\\ v_1=-te^{-t}-2e^{-t}\end{cases}$$

$$\int e^{-t}(t+1)dt\quad\begin{pmatrix}u=t+1 & dv=e^{-t}\\ du=dt & v=-e^{-t}\end{pmatrix}$$
$$=-e^{-t}(t+1)+\int e^{-t}dt$$
$$=-e^{-t}(t+1)+(-e^{-t})+C$$
$$=-te^{-t}-2e^{-t}+C$$

$$\therefore\ y_p=[-te^{-t}-2e^{-t}]e^t-t(t+1)=-t-2-t^2-t=-t^2-2t-2$$

$$\therefore\ y(t)=y_h+y_p=c_1e^t+c_2(t+1)-t^2-2t-2$$

39. $ty''+(5t-1)y'-5y=t^2e^{-5t}$; $y_1=5t-1$, $y_2=e^{-5t}$

Sol.

$$y_h=c_1(5t-1)+c_2e^{-5t}$$

Let $y_p=v_1(5t-1)+v_2e^{-5t}$

$$\Rightarrow\begin{cases}v_1'(5t-1)+v_2'e^{-5t}=0\quad(\times5)\\ 5v_1'-5v_2'e^{-5t}=\dfrac{t^2e^{-5t}}{t}=te^{-5t}\end{cases}$$

$$\Rightarrow\begin{cases}5v_1'(5t-1)+5v_2'e^{-5t}=0\\ 5v_1'-5v_2'e^{-5t}=te^{-5t}\end{cases}$$

$$\Rightarrow\begin{cases}5v_1'(5t-1+1)=te^{-5t}\\ v_2'=-v_1'e^{5t}(5t-1)\end{cases}$$

$$\Rightarrow \begin{cases} 25v_1't = te^{-5t} \\ v_2' = -v_1'e^{5t}(5t-1) \end{cases}$$

$$\Rightarrow \begin{cases} v_1' = \dfrac{1}{25}e^{-5t} \\ v_2' = \dfrac{-1}{25}(5t-1) \end{cases}$$

$$\Rightarrow \begin{cases} v_1 = \dfrac{1}{25}\displaystyle\int e^{-5t}dt = \dfrac{1}{25}\cdot\dfrac{-1}{5}e^{-5t} = \dfrac{-1}{125}e^{-5t} \\ v_2 = \dfrac{-1}{25}\left(\dfrac{5}{2}t^2 - t\right) = \dfrac{-1}{10}t^2 + \dfrac{1}{25}t \end{cases}$$

$$\therefore\ y_p = \frac{-1}{125}e^{-5t}(5t-1) + \left(\frac{-1}{10}t^2 + \frac{1}{25}t\right)e^{-5t} = \frac{1}{125}e^{-5t} - \frac{1}{10}t^2e^{-5t}$$

$$\therefore\ y(t) = c_1(5t-1) + c_2e^{-5t} + \frac{1}{125}e^{-5t} - \frac{1}{10}t^2e^{-5t}$$

◇ A differential equation and a non-trivial solution f are given. Find a second linearly independent solution using reduction of order.

47. $tx'' - (t+1)x' + x = 0$, $t > 0$, $f(t) = e^t$

Sol.

$$tx'' - (t+1)x' + x = 0$$

$$\Rightarrow x'' - \frac{t+1}{t}x' + \frac{1}{t}x = 0$$

$$p(t) = -\frac{t+1}{t} = -1 - \frac{1}{t} = -(1+\frac{1}{t})$$

$$-\int p(t)dt = \int (1+\frac{1}{t})dt = t + \ln t$$

$$\int te^{-t}dt \quad \begin{pmatrix} u = t & dv = e^{-t} \\ du = dt & v = -e^{-t} \end{pmatrix}$$

$$= -te^{-t} + \int e^{-t}dt$$

$$= -te^{-t} + (-e^{-t}) + C$$

$$= -te^{-t} - e^{-t} + C$$

$$\therefore\ f_2(t) = e^t \cdot \int \frac{e^{t+\ln t}}{e^{2t}}dt = e^t \cdot \int \frac{te^t}{e^{2t}}dt = e^t \cdot \int te^{-t}dt = e^t \cdot (-te^{-t} - e^{-t}) = -t - 1$$