Section 2.3　Linear Equations

Definition： Linear first-order equation

✧ General form： $a_1(x)\dfrac{dy}{dx}+a_0(x)y=b(x)$, where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x not on y.

✧ Standard form： $\dfrac{dy}{dx}+P(x)y=Q(x)$.

Method for Solving Linear Equations

1. Write the equation in the standard form： $\dfrac{dy}{dx}+P(x)y=Q(x)$
2. Calculate the integrating factor $\mu(x)$ by the formula： $\mu(x)=\exp[\int P(x)dx]$.
3. $\mu(x)\times$standard form and the left-hand side is just $\dfrac{d}{dx}[\mu(x)y]$：

$$\underbrace{\mu(x)\frac{dy}{dx}+\mu(x)P(x)y}_{\frac{d}{dx}[\mu(x)y]}=\mu(x)Q(x)$$

4. Integrate the last equation and solve for y.

◇ Obtain the general solution to the equation.

10. $\dfrac{dr}{d\theta}+r\tan\theta=\sec\theta$

Sol.

Let $\mu(\theta)=e^{\int\tan\theta d\theta}=e^{-\ln|\cos\theta|}=\dfrac{1}{|\cos\theta|}=|\sec\theta|$

$$\begin{aligned}
\text{原式}\times\mu(\theta) &\Rightarrow |\sec\theta|\frac{dr}{d\theta}+r|\sec\theta|\tan\theta=|\sec\theta|\sec\theta\\
&\Rightarrow \sec\theta\frac{dr}{d\theta}+r\sec\theta\tan\theta=\sec^2\theta\\
&\Rightarrow \frac{d}{d\theta}[r\sec\theta]=\sec^2\theta\\
&\Rightarrow r\sec\theta=\int\sec^2\theta d\theta\\
&\qquad\qquad=\tan\theta+C\\
&\Rightarrow r=\sin\theta+C\cdot\cos\theta
\end{aligned}$$

$$\begin{aligned}
&\int\tan\theta d\theta\\
&=\int\frac{\sin\theta}{\cos\theta}d\theta\\
&\begin{pmatrix}u=\cos\theta\\ du=-\sin\theta d\theta\end{pmatrix}\\
&=-\int\frac{1}{u}du\\
&=-\ln|u|+C\\
&=-\ln|\cos\theta|+C
\end{aligned}$$

14. $x\dfrac{dy}{dx}+3y+3x^2=\dfrac{\sin x}{x}$

Sol.

$$x\frac{dy}{dx}+3y+3x^2=\frac{\sin x}{x}$$

$$\Rightarrow \frac{dy}{dx}+\frac{3}{x}y=\frac{\sin x}{x^2}-3x \quad \text{-- (1)}$$

Let $\mu(x)=e^{3\int\frac{1}{x}dx}=e^{3\ln|x|}=|x|^3$

$$(1)\times\mu(x)\Rightarrow x^3\frac{dy}{dx}+x^3\cdot\frac{3}{x}y=x^3(\frac{\sin x}{x^2}-3x)$$

$$\Rightarrow x^3\frac{dy}{dx}+3x^2y=x\sin x-3x^4$$

$$\Rightarrow \frac{d}{dx}[x^3y]=x\sin x-3x^4$$

$$\Rightarrow x^3y=\int(x\sin x-3x^4)dx$$

$$=-x\cos x+\sin x-\frac{3}{5}x^5+C$$

$$\Rightarrow y=-\frac{\cos x}{x^2}+\frac{\sin x}{x^3}-\frac{3}{5}x^2+\frac{C}{x^3}$$

◇ Solve the initial value problem.

17. $\dfrac{dy}{dx}-\dfrac{y}{x}=xe^x$, $y(1)=e-1$

Sol.

$$\frac{dy}{dx}-\frac{y}{x}=xe^x\Rightarrow\frac{dy}{dx}+(-\frac{1}{x})y=xe^x$$

Let $\mu(x)=e^{-\int\frac{1}{x}dx}=e^{-\ln|x|}=\dfrac{1}{|x|}$

$$原式\times\mu(x)\Rightarrow\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}=e^x$$

$$\Rightarrow\frac{d}{dx}[\frac{1}{x}y]=e^x$$

$$\Rightarrow\frac{1}{x}y=\int e^x dx$$

$$=e^x+C$$

$$\Rightarrow y=xe^x+Cx$$

$\because$ $y(1)=e-1\Rightarrow e-1=e+C\Rightarrow C=-1$ $\quad\therefore$ $y=xe^x-x$

22. $\sin x \dfrac{dy}{dx} + y\cos x = x\sin x$, $y(\dfrac{\pi}{2}) = 2$

Sol.

$$\sin x \frac{dy}{dx} + y\cos x = x\sin x$$

$$\Rightarrow \frac{dy}{dx} + \frac{\cos x}{\sin x} y = x \quad \text{-- (1)}$$

Let $\mu(x) = e^{\int \frac{\cos x}{\sin x} dx} = e^{\ln|\sin x|} = |\sin x|$

$$(1) \times \mu(x) \Rightarrow \sin x \frac{dy}{dx} + y\cos x = x\sin x$$

$$\Rightarrow \frac{d}{dx}[y\sin x] = x\sin x$$

$$\Rightarrow y\sin x = \int x\sin x dx$$

$$= -x\cos x + \sin x + C$$

$$\Rightarrow y = -x\cot x + 1 + \frac{C}{\sin x}$$

$\because \; y(\dfrac{\pi}{2}) = 2 \Rightarrow 2 = -\dfrac{\pi}{2}\cot\dfrac{\pi}{2} + 1 + \dfrac{C}{\sin\frac{\pi}{2}} \Rightarrow 2 = 0 + 1 + C \Rightarrow C = 1$

$\therefore \; y = -x\cot x + 1 + \dfrac{1}{\sin x} = -x\cot x + \csc x + 1$

22.

$\int \dfrac{\cos x}{\sin x} dx \quad \begin{pmatrix} u = \sin x \\ du = \cos x dx \end{pmatrix}$

$= \int \dfrac{1}{u} du$

$= \ln|u| + C$

$= \ln|\sin x| + C$

$\int x\sin x dx \quad \begin{pmatrix} u = x & dv = \sin x dx \\ du = dx & v = -\cos x \end{pmatrix}$

$= -x\cos x + \int \cos x dx$

$= -x\cos x + \sin x + C$

28. **Constant Multiples of Solutions.**

(a) Show that $y = e^{-x}$ is a solution of the linear equation

(16) $\dfrac{dy}{dx} + y = 0$

and $y = x^{-1}$ is a solution of the nonlinear equation

(17) $\dfrac{dy}{dx} + y^2 = 0$

Sol.

(i) For $y = e^{-x}$ and $\dfrac{dy}{dx} = -e^{-x} \Rightarrow \dfrac{dy}{dx} + y = -e^{-x} + e^{-x} = 0$

Hence, $y = e^{-x}$ is a solution of $\dfrac{dy}{dx} + y = 0$.

(ii) For $y = x^{-1}$ and $\dfrac{dy}{dx} = -x^{-2} \Rightarrow \dfrac{dy}{dx} + y^2 = -x^{-2} + x^{-2} = 0$

Hence, $y = x^{-1}$ is a solution of $\dfrac{dy}{dx} + y^2 = 0$.

(b) Show that for any constant C, the function Ce^{-x} is a solution of equation (16), while Cx^{-1} is a solution of equation (17) only when $C=0$ or 1.

Sol.

(i) Let $y=Ce^{-x} \Rightarrow \frac{dy}{dx}=-Ce^{-x} \Rightarrow \frac{dy}{dx}+y=-Ce^{-x}+Ce^{-x}=0$

Hence, $y=Ce^{-x}$ is a solution of $\frac{dy}{dx}+y=0$ for ant constant C.

(ii) Let $y=Cx^{-1} \Rightarrow \frac{dy}{dx}=-Cx^{-2} \Rightarrow \frac{dy}{dx}+y^2=-Cx^{-2}+C^2x^{-2}=x^{-2}(C^2-C)=0$

If $(C^2-C)=0 \Rightarrow C=0$ or 1

Hence, $y=Cx^{-1}$ is a solution of $\frac{dy}{dx}+y^2=0$ only when $C=0$ or 1.

(c) Show that for any linear equation of the form $\frac{dy}{dx}+P(x)y=0$, if $\hat{y}(x)$ is a solution, then for any constant C the function $C\hat{y}(x)$ is also a solution.

Sol.

Substitution of $C\hat{y}(x)$ for y in $\frac{dy}{dx}+P(x)y=0$

$\Rightarrow \frac{d}{dx}[C\hat{y}]+P\cdot C\hat{y}=0 \Rightarrow C\hat{y}'+P\cdot C\hat{y}=0 \Rightarrow C(\hat{y}'+P\hat{y})=0$

$\because$ $\hat{y}$ is a solution of $\frac{dy}{dx}+P(x)y=0$

$\Rightarrow \hat{y}'+P\hat{y}=0$

$\Rightarrow C(\hat{y}'+P\hat{y})=0$ for any constant C

$\therefore$ $C\hat{y}(x)$ is also a solution of $\frac{dy}{dx}+P(x)y=0$.

31. **Discontinuous Coefficients.** As we will see in Chapter 3, occasions arise when the coefficient $P(x)$ in a linear equation fails to be continuous because of jump discontinuities. Fortunately, we may still obtain a "reasonable" solution. For example, consider the initial value problem $\frac{dy}{dx}+P(x)y=x$, $y(0)=1$, where $P(x):=\begin{cases}1 & , \quad 0\le x\le 2\\ 3 & , \quad x>2\end{cases}$

(a) Find the general solution for $0\le x\le 2$.

Sol.

For $0\le x\le 2$, $P(x)=1$

$$\frac{dy}{dx}+y=x$$

Let $\mu(x)=e^{\int 1dx}=e^x$

$$\Rightarrow e^x \frac{dy}{dx} + e^x y = xe^x$$

$$\Rightarrow \frac{d}{dx}[e^x y] = xe^x$$

$$\Rightarrow e^x y = \int xe^x dx$$

$$= xe^x - e^x + C$$

$$\Rightarrow y = x - 1 + \frac{C}{e^x}$$

(b) Choose the constant in the solution of part (a) so that the initial condition is satisfied.

Sol.

$\because \quad y(0) = 1 \Rightarrow 1 = 0 - 1 + C \Rightarrow C = 2$

$\therefore \quad y = x - 1 + \dfrac{2}{e^x}$

(c) Find the general solution for $x > 2$.

Sol.

For $x > 2$, $P(x) = 3$

$$\frac{dy}{dx} + 3y = x$$

Let $\mu(x) = e^{\int 3dx} = e^{3x}$

$$\Rightarrow e^{3x} \frac{dy}{dx} + 3e^{3x} y = xe^{3x}$$

$$\Rightarrow \frac{d}{dx}[e^{3x} y] = xe^{3x}$$

$$\Rightarrow e^{3x} y = \int xe^{3x} dx$$

$$= \frac{1}{3} xe^{3x} - \frac{1}{9} e^{3x} + C$$

$$\Rightarrow y = \frac{x}{3} - \frac{1}{9} + \frac{C}{e^{3x}}$$

(d) Now choose the constant in the general solution from part (c) so that the solution from part (b) and the solution from part (c) agree at $x = 2$. By patching the two solutions together, we can obtain a continuous function that satisfies the differential equation except at $x = 2$, where its derivative is undefined.

Sol.

From part (b), $y = x - 1 + \dfrac{2}{e^x}$ for $0 \le x \le 2$.

For $x=2$, $y(2)=2-1+\frac{2}{e^2}=1+\frac{2}{e^2}$, and from part (c) $y=\frac{x}{3}-\frac{1}{9}+\frac{C}{e^{3x}}$ for $x>2$

$$\Rightarrow 1+\frac{2}{e^2}=\frac{2}{3}-\frac{1}{9}+\frac{C}{e^6}\Rightarrow 1+\frac{2}{e^2}=\frac{5}{9}+\frac{C}{e^6}\Rightarrow C=\frac{4}{9}e^6+2e^4$$

Hence, $y=\frac{x}{3}-\frac{1}{9}+e^{-3x}(\frac{4}{9}e^6+2e^4)$ for $x>2$.

(e) Sketch the graph of the solution from $x=0$ to $x=5$.

Sol.

33. **Singular Points.** Those values of x for which $P(x)$ in equation (4) is not defined are called **singular points** of the equation. For example, $x=0$ is a singular point of the equation $xy'+2y=3x$, since when the equation is written in the standard form, $y'+(2/x)y=3$, we see that $P(x)=2/x$ is not defined at $x=0$. On an interval containing a singular point, the questions of the existence and uniqueness of a solution are left unanswered, since Theorem 1 does not apply. To show the possible behavior of solutions near a singular point, consider the following equations.

(a) Show that $xy'+2y=3x$ has only one solution defined at $x=0$. Then show that the initial value problem for this equation with initial condition $y(0)=y_0$ has a unique solution when $y_0=0$ and no solution when $y_0\neq 0$.

Sol.

$$xy'+2y=3x$$

$$\Rightarrow y'+\frac{2}{x}y=3$$

Let $\mu(x)=e^{2\int\frac{1}{x}dx}=e^{2\ln|x|}=x^2$

$$\Rightarrow x^2y'+2xy=3x^2$$

$$\Rightarrow \frac{d}{dx}[x^2y]=3x^2$$

$$\Rightarrow x^2y=\int 3x^2dx$$

$$=x^3+C$$

$\Rightarrow y=x+\frac{C}{x^2}$ is defined at $x=0$ only when $C=0$

$\Rightarrow y=x$ is the only solution defined at $x=0$

Therefore, for the IVP $xy'+2y=3x$, $y(0)=y_0$

$\Rightarrow x=0$, $y=x$ is the only solution

$\Rightarrow y(0)=0$

Hence, the IVP has a unique solution $y=x$ when $y_0=0$, and has no solution when $y_0\neq 0$..

(b) Show that $xy' - 2y = 3x$ has an infinite number of solutions defined at $x = 0$. Then show that the initial value problem for this equation with initial condition $y(0) = 0$ has an infinite number of solutions.

Sol.

$$xy' - 2y = 3x$$

$$\Rightarrow y' - \frac{2}{x}y = 3$$

Let $\mu(x) = e^{-2\int \frac{1}{x}dx} = e^{-2\ln|x|} = x^{-2}$

$$\Rightarrow x^{-2}y' - 2x^{-3}y = 3x^{-2}$$

$$\Rightarrow \frac{d}{dx}[x^{-2}y] = 3x^{-2}$$

$$\Rightarrow x^{-2}y = \int 3x^{-2}dx$$

$$= -3x^{-1} + C$$

$\Rightarrow y = -3x + Cx^2$ is defined at $x = 0$ for any constant C.

Hence, the equation $xy' - 2y = 3x$ has an infinite number of solutions at $x = 0$.

$\because$ $y(0) = -3\cdot 0 + C\cdot 0 = 0$ for any constant C

$\therefore$ the IVP $xy' - 2y = 3x$, $y(0) = 0$ has an infinite number of solutions.