Section 7.2 Definition of the Laplace Transform

Definition 1：Laplace Transform

Let $f(t)$ be a function on $[0,\infty)$. The **Laplace transform** of f is the function F defined by the integral $F(s) := \int_0^\infty e^{-st} f(t)dt$.

The domain of $F(s)$ is all the values of s for which the integral in (1) exists. The Laplace transform of f is denote by both F and $L\{f\}$.

Theorem：Linearity of the Transform

Let f, f_1, and f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then, for $s > \alpha$,

(i) $L\{f_1 + f_2\} = L\{f_1\} + L\{f_2\}$

(ii) $L\{cf\} = cL\{f\}$

Definition 3：Exponential Order α

A function $f(t)$ is said to be of **exponential order** α if there exist positive constants T and M such that $|f(t)| \le Me^{\alpha t}$, for all $t \ge T$

◇ Use Definition 1 to determine the Laplace transform of the given function.

7. $e^{2t}\cos 3t$

Sol.

Let $f(t) = e^{2t}\cos 3t$

$$F(s) = \int_0^\infty e^{-st}\cdot e^{2t}\cos 3t dt$$

$$= \lim_{N\to\infty}\int_0^N e^{(2-s)t}\cos 3t dt$$

$$= \lim_{N\to\infty}\left\{\frac{e^{(2-s)t}}{9+(2-s)^2}\left[3\sin 3t + (2-s)\cos 3t\right]_0^N\right\} = \frac{s-2}{(s-2)^2+9},\quad s>2$$

$$* \int e^{(2-s)t}\cos 3t dt \quad \begin{pmatrix} u = e^{(2-s)t} & dv = \cos 3t dt \\ du = (2-s)e^{(2-s)t} & v = \frac{1}{3}\sin 3t \end{pmatrix}$$

$$= \frac{1}{3}e^{(2-s)t}\sin 3t - \frac{(2-s)}{3}\int e^{(2-s)t}\sin 3t dt \quad \begin{pmatrix} u = e^{(2-s)t} & dv = \sin 3t dt \\ du = (2-s)e^{(2-s)t} & v = \frac{-1}{3}\cos 3t \end{pmatrix}$$

$$= \frac{1}{3}e^{(2-s)t}\sin 3t - \frac{(2-s)}{3}\left(\frac{-1}{3}e^{(2-s)t}\cos 3t + \frac{(2-s)}{3}\int e^{(2-s)t}\cos 3t dt\right)$$

$$= \frac{1}{3}e^{(2-s)t}\sin 3t + \frac{(2-s)}{9}e^{(2-s)t}\cos 3t - \frac{(2-s)^2}{9}\int e^{(2-s)t}\cos 3t dt$$

$$\Rightarrow \int e^{(2-s)t}\cos 3t dt = \frac{e^{(2-s)t}}{9+(2-s)^2}[3\sin 3t + (2-s)\cos 3t]$$

11. $f(t)=\begin{cases}\sin t & , \quad 0<t<\pi \\ 0 & , \quad \pi<t\end{cases}$

Sol.

$$
\begin{aligned}
F(s) &= \int_0^{\pi} e^{-st}\cdot \sin t dt + \int_{\pi}^{\infty} e^{-st}\cdot 0 dt \\
&= \int_0^{\pi} e^{-st}\cdot \sin t dt \\
&= \left[\frac{-e^{-st}}{1+s^2}(\cos t + s\sin t)\right]_0^{\pi} \\
&= \frac{e^{-s\pi}}{1+s^2} - \left(\frac{-1}{1+s^2}\right) \\
&= \frac{e^{-s\pi}+1}{1+s^2}
\end{aligned}
$$

◇ Use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.

19. $L\{t^4e^{5t} - e^t\cos\sqrt{7}t\}$

Sol.

$$
\begin{aligned}
& L\{t^4e^{5t} - e^t\cos\sqrt{7}t\} \\
&= L\{t^4e^{5t}\} - L\{e^t\cos\sqrt{7}t\} \\
&= \frac{4!}{(s-5)^{4+1}} - \frac{s-1}{(s-1)^2+(\sqrt{7})^2} \\
&= \frac{4!}{(s-5)^5} - \frac{s-1}{(s-1)^2+7}, \quad s>5
\end{aligned}
$$