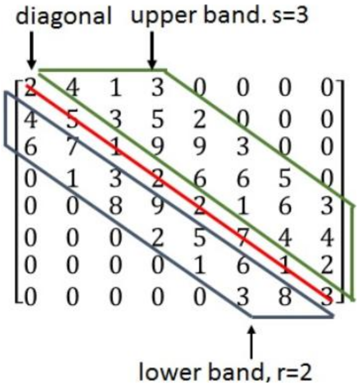
Programming Assignment 4: Banded LU-Decomposition

A kind of sparse matrix is banded matrix. If square matrix A is of size $n \times n$, a lower band element of bandwidth r is the element $a_{i,j}$ such that $0<i-j \le r$ and an upper band element of bandwidth s is the element $a_{i,j}$ such that $0<j-i \le s$. Only the elements on the diagonal, on the lower band, and on the upper band can be non-zero; all other elements are called off-band elements and they are all zeros. The following is an example of an 8×8 banded matrix with the lower bandwidth r of 2 and the upper bandwidth s of 3:



Banded LU-decomposition is to factorize the $n \times n$ banded matrix A into two $n \times n$ triangular matrices, L and U such that $A=L \times U$. If A is of lower bandwidth r and upper bandwidth s, then L is a lower banded triangular matrix of lower bandwidth r and U is an upper banded triangular matrix of upper bandwidth s. The upper bandwidth of L and lower bandwidth of U can be viewed as 0. Let $A^{(k)}$ be the $(n-k) \times (n-k)$ sub-matrix of A after removing the first k rows and the first k columns, i.e.,

$$A^{(0)} = A,$$

$$A^{(k)} = \begin{bmatrix} a_{k,k} & a_{k,k+1} & \cdots & 0 & 0 \\ a_{k+1,k} & a_{k+1,k+1} & a_{k+1,k+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & a_{n-2,n-2} & a_{n-2,n-1} \\ 0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} \end{bmatrix}, k \le n-1.$$

Note that $A^{(k)}$ is also of lower bandwidth r and upper bandwidth s. Starting from $A^{(0)}$, matrices L and U are generated by computing, given $A^{(k)}$, $0 \le k \le n-1$, sub-matrices $A^{(k+1)}$ as the following steps:

1. Compute elements of the k-th row of matrix U: $u_{k,j} = a_{k,j}$, for $k \le j \le \min(n-1, k+s)$;
2. Compute elements of the k-th column of matrix L: $l_{i,k} = a_{i,k}/a_{k,k}$, for $k \le i \le \min(n-1, k+r)$; (note that, $l_{k,k}==1$)
3. Compute elements of submatrix $A^{(k+1)}$: $a_{i,j} = a_{i,j} - l_{i,k} \times u_{k,j}$, for $k+1 \le i \le \min(n-1, k+r)$ and $\max(k+1, i-r) \le j \le \min(n-1, \min(i+s, k+s))$.

Write a C program to input an integer n of banded square matrix size for matrices A, L, and U, and input integer r as the lower bandwidth and integer s as the upper bandwidth. Randomly generate non-banded elements $a_{i,j}$ for matrix A. Then, compute LU-decomposition $A=L \times U$ to generate matrices L and U. For the input matrix, keep a copy $A1$, and check whether $A1=L \times U$ to verify correctness of the program. Output matrices A, L, and U. In this assignment, you must submit two files:

the source code of the solution **assgn4_DXXXXXXX.c** (80%) and the assignment report **assgn4_DXXXXXXX.pdf** (20%), where DXXXXXXX is your student ID. In the assignment report, you must explain correctness of the bandwidth conditions for matrices of L, U, and A, in steps 1, 2, and 3. Programming assignment 4 is due by **11:59 pm, Monday, November 21**. Submit your solution and the report to **iLearn2**.

Example of program execution

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D:\>lu_decomposition_banded
Enter matrix size n: 10

Enter the lower bandwidth and the of matrix A, (r, s): 4 5

Matrix A:
  0.6642   0.1594   0.2151   0.2872   0.1274   0.7966
  0.6534   0.1045   0.9909   0.0619   0.5018   0.7341   0.9972
  0.1179   0.2752   0.5189   0.0744   0.9486   0.7577   0.1089   0.2002
  0.1360   0.4037   0.5627   0.1243   0.5270   0.3616   0.3888   0.1429   0.7257
  0.5778   0.7899   0.0098   0.1494   0.9745   0.7751   0.2228   0.1989   0.2426   0.2319
           0.1092   0.0057   0.0080   0.7960   0.7350   0.4135   0.9140   0.3314   0.5517
                    0.7642   0.4405   0.0730   0.6363   0.2695   0.7081   0.9252   0.3324
                             0.1372   0.8285   0.5191   0.5476   0.7438   0.6026   0.0601
                                      0.9894   0.8948   0.6649   0.5405   0.7831   0.2062
                                               0.7849   0.5625   0.3058   0.1418   0.1470

Matrix L:
  1.0000
  0.9837   1.0000
  0.1775  -4.7202   1.0000
  0.2048  -7.0938   1.4539   1.0000
  0.8699 -12.4500   2.2901  26.3468   1.0000
          -2.0876   0.3925   2.7065   0.1331   1.0000
                    0.1837 -32.0451  -1.2784  -0.7488   1.0000
                            -7.0060  -0.2319  -4.3309   3.6823   1.0000
                                      0.0512   0.1101   0.8925   1.8320   1.0000
                                               7.6565  -4.1829   1.6119   2.1431   1.0000

Matrix U:
  0.6642   0.1594   0.2151   0.2872   0.1274   0.7966
          -0.0523   0.7793  -0.2206   0.3765  -0.0495   0.9972
                    4.1592  -1.0180   2.7030   0.3824   4.8159   0.2002
                            -0.0196  -0.7583  -0.7090   0.4611  -0.1482   0.7257
                                     19.3383  17.2687 -10.5397   3.6440 -18.8773   0.2319
                                               0.1025   0.7593   0.7515   0.8791   0.5208
                                                        1.2555   1.1447   0.7058   1.0189
                                                                -0.4096   2.5183  -1.3821
                                                                         -3.5913   1.7596
                                                                                  -1.1222

The LU-decomposition program is correct.

D:\>
微軟注音 半 :
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