Section 4.2 Homogeneous Linear Equations：The General Solution

Definition：Homogeneous Linear Equations

Linear second-order constant-coefficient differential equation：

(1) $ay'' + by' + cy = f(t) \quad (a \neq 0)$

with the special case where the function $f(t)$ is zero：

(2) $ay'' + by' + cy = 0$

Equation (2) is called the **homogeneous** form of equation (1).

Auxiliary Equation：

Substitute $y = e^{rt}$, $y' = re^{rt}$, $y'' = r^2 e^{rt}$ into (2), we obtain

$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$

$\Rightarrow (ar^2 + br + c)e^{rt} = 0 \quad (\because e^{rt}$ is never zero)

$\Rightarrow (ar^2 + br + c) = 0$ is called the **auxiliary equation**.

$$\Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \begin{cases} \text{相同實根}(r_1 = r_2) \Rightarrow y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t} \\ \text{相異實根}(r_1 \neq r_2) \Rightarrow y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \\ \text{共軛複根}(r = \alpha \pm \beta i) \Rightarrow y(t) = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t \end{cases}$$

Theorem：Existence and Uniqueness：Homogeneous Case

For any real number $a(\neq 0)$, b, c, t_0, Y_0, and Y_1, there exists a unique solution to the initial value problem

(10) $ay'' + by' + cy = 0$, $y(t_0) = Y_0$, $y'(t_0) = Y_1$.

The solution is valid for all t in $(-\infty, \infty)$.

Definition：Linear Independent of Two Functions

(1) $y_1(t)$ and $y_2(t)$ is said to be **linearly independent** on the interval $I \Leftrightarrow y_1 \neq ky_2$ on I.

(2) $y_1(t)$ and $y_2(t)$ is said to be **linearly dependent** on the interval $I \Leftrightarrow y_1 = ky_2$ on I.

Theorem：Representation of Solution to Initial Value Problem

If $y_1(t)$ and $y_2(t)$ are two solution to the differential equation (2) that are linearly independent on $(-\infty, \infty)$, then unique constants c_1 and c_2 can always be found so that $c_1 y_1(t) + c_2 y_2(t)$ satisfies the initial value problem (10) on $(-\infty, \infty)$.

Lemma：A Condition for Linear Dependence of Solutions

For any real number $a(\neq 0)$, b, and c, if $y_1(t)$ and $y_2(t)$ are two solution to the differential equation (2) on $(-\infty, \infty)$ and if the equality

(11) $y_1(\tau) y_2'(\tau) - y_1'(\tau) y_2(\tau) = 0$

Hold at any point τ, then y_1 and y_2 are linearly dependent on $(-\infty, \infty)$.

Wronskian：

$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$ is called the **Wronskian** of y_1 and y_2.

(1) $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = 0 \Rightarrow y_1$ and y_2 are L.D.

(2) $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \neq 0 \Rightarrow y_1$ and y_2 are L.I.

◇ Find a general solution to the given differential equation.

6. $y'' + 8y' + 16y = 0$

Sol.

Consider the auxiliary equation $r^2 + 8r + 16 = 0$

$\Rightarrow (r+4)^2 = 0 \quad \Rightarrow r = -4$ （重根）

Hence, the general solution is $y(t) = C_1 e^{-4t} + C_2 t e^{-4t}$.

11. $4w'' + 20w' + 25w = 0$

Sol.

$4r^2 + 20r + 25 = 0$

$\Rightarrow (2r+5)^2 = 0 \Rightarrow r = \dfrac{-5}{2}$ （重根）

$\therefore \quad y(t) = C_1 e^{\frac{-5}{2}t} + C_2 t e^{\frac{-5}{2}t}$.

◇ Solve the given initial value problem.

13. $y'' + 2y' - 8y = 0$ ； $y(0) = 3$, $y'(0) = -12$

Sol.

$r^2 + 2r - 8 = 0$

$\Rightarrow (r+4)(r-2) = 0$

$\Rightarrow r = -4, 2$

$\therefore \quad y(t) = C_1 e^{-4t} + C_2 e^{2t}$

$\Rightarrow y'(t) = -4C_1 e^{-4t} + 2C_2 e^{2t}$

$\because \quad y(0) = 3$, $y'(0) = -12$

$$\Rightarrow \begin{cases} C_1 + C_2 = 3 \\ -4C_1 + 2C_2 = -12 \end{cases} \Rightarrow \begin{cases} C_1 = 3 \\ C_2 = 0 \end{cases}$$

$\therefore \quad y(t) = 3e^{-4t}$

18. $y'' - 6y' + 9y = 0$ ； $y(0) = 2$, $y'(0) = 25/3$

Sol.

$$r^2 - 6r + 9 = 0$$
$$\Rightarrow (r-3)^2 = 0$$
$$\Rightarrow r = 3 \text{（重根）}$$
$$\therefore \quad y(t) = C_1 e^{3t} + C_2 te^{3t}$$
$$\Rightarrow y'(t) = 3C_1 e^{3t} + C_2 (e^{3t} + 3te^{3t})$$
$$\because \quad y(0) = 2, \quad y'(0) = 25/3$$
$$\Rightarrow \begin{cases} C_1 = 2 \\ 3C_1 + C_2 = \dfrac{25}{3} \end{cases} \Rightarrow \begin{cases} C_1 = 2 \\ C_2 = \dfrac{7}{3} \end{cases}$$
$$\therefore y(t) = 2e^{3t} + \frac{7}{3} te^{3t}$$

21. **First-Order Constant-Coefficient Equations.**

(a) Substituting $y = e^{rt}$, find the auxiliary equation for the first-order linear equation $ay' + by = 0$, where a and b are constants with $a \neq 0$.

Sol.

Let $y = e^{rt}$
$$\Rightarrow y' = re^{rt}$$
$$\Rightarrow are^{rt} + be^{rt} = 0$$
$$\Rightarrow (ar + b)e^{rt} = 0$$
$\therefore$ $ar + b$ is the auxiliary equation for $ay' + by = 0$

(b) Use the result of part(a) to find the general solution.

Sol.

$$ar + b = 0 \Rightarrow r = \frac{-b}{a}$$
$$\therefore \quad y(t) = Ce^{\frac{-b}{a}t}$$

◇ Use Definition：Linear Independent of Two Functions to determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0,1)$.

31. $y_1(t) = \tan^2 t - \sec^2 t$, $y_2(t) \equiv 3$

Sol.

$\because$ $\tan^2 t - \sec^2 t = -1 \Rightarrow -3y_1 = y_2$

$\therefore$ y_1 and y_2 are linearly dependent.

34. **Wronskian.** For any two differentiable functions y_1 and y_2, the function

(18) $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

is called the Wronskian of y_1 and y_2. This function plays a crucial role on proof of Theorem2.

(a) Show that $W[y_1, y_2]$ can be conveniently expressed as the 2×2 determinant

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

Sol.

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) = W[y_1, y_2](t)$$

(b) Let $y_1(t), y_2(t)$ be a pair of solutions to the homogeneous equation $ay'' + by' + cy = 0$ (with $a \neq 0$) on an open interval I. Prove that $y_1(t)$ and $y_2(t)$ are linearly independent on I if and only if their Wronskian is never zero on I. [*Hint*: This is just a reformulation of Lemma.]

Sol.

$(\Rightarrow)$

From Lemma 1 (p.172),

$y_1y_2' - y_1'y_2 = 0 \Rightarrow y_1$ and y_2 are L.D. on I.

Hence, y_1 and y_2 are L.I. on $I \Rightarrow y_1y_2' - y_1'y_2 \neq 0$

$(\Leftarrow)$

Assume that y_1 and y_2 are L.D. on I

$\Rightarrow \exists C$, such that $y_1 = Cy_2$ on I

$\Rightarrow y_1' = Cy_2'$

$\Rightarrow \; y_1y_2' - y_1'y_2 = Cy_2y_2' - Cy_2'y_2 = 0 \quad \rightarrow\leftarrow$

Hence, y_1 and y_2 are L.I. on I.

35. **Linear Dependence of Three Functions.** For each of the following, determine whether the given three functions are linearly dependent or linearly independent on $(-\infty, \infty)$:

(a) $y_1(t) = 1$, $y_2(t) = t$, $y_3(t) = t^2$.

Sol.

Consider $C_1y_1 + C_2y_2 + C_3y_3 = 0$

$\Rightarrow C_1 + C_2t + C_3t^2 = 0$

$\Rightarrow C_1 = C_2 = C_3 = 0$

Hence, y_1, y_2 and y_3 are L.I.

(b) $y_1(t) = -3$, $y_2(t) = 5\sin^2 t$, $y_3(t) = \cos^2 t$.

Sol.

Consider $C_1 y_1 + C_2 y_2 + C_3 y_3 = 0$

$$\Rightarrow -3C_1 + 5C_2 \sin^2 t + C_3 \cos^2 t = 0$$

$$\Rightarrow \begin{cases} C_1 = \dfrac{-5}{3} \\ C_2 = 1 \\ C_3 = 5 \end{cases} \quad \text{satisfies the equation.}$$

Hence, y_1, y_2 and y_3 are L.D.

(c) $y_1(t) = e^t$, $y_2(t) = te^t$, $y_3(t) = t^2 e^t$.

Sol.

Consider $C_1 y_1 + C_2 y_2 + C_3 y_3 = 0$

$$\Rightarrow C_1 e^t + C_2 te^t + C_3 t^2 e^t = 0$$
$$\Rightarrow (C_1 + C_2 t + C_3 t^2) e^t = 0$$
$$\Rightarrow C_1 + C_2 t + C_3 t^2 = 0$$
$$\Rightarrow C_1 = C_2 = C_3 = 0$$

Hence, y_1, y_2 and y_3 are L.I.

(d) $y_1(t) = e^t$, $y_2(t) = e^{-t}$, $y_3(t) = \cosh t$.

Sol.

Consider $C_1 y_1 + C_2 y_2 + C_3 y_3 = 0$

$$\Rightarrow C_1 e^t + C_2 e^{-t} + C_3 \cosh t = 0$$
$$\Rightarrow C_1 e^t + C_2 e^{-t} + C_3 \cdot \frac{e^t + e^{-t}}{2} = 0$$
$$\Rightarrow (C_1 + \frac{C_3}{2}) e^t + (C_2 + \frac{C_3}{2}) e^{-t} = 0$$
$$\Rightarrow \begin{cases} C_1 + \dfrac{C_3}{2} = 0 \\ C_2 + \dfrac{C_3}{2} = 0 \end{cases}$$
$$\Rightarrow \begin{cases} C_1 = C_2 = -1 \\ C_3 = 2 \end{cases} \quad \text{satisfies the equation.}$$

Hence, y_1, y_2 and y_3 are L.D.