

CS513 HW3 Sample Solutions

Notice: you can use any known NP-hard problems for the polynomial reduction.

1. In a BoxTiling problem, there is a large Box R (in 3D) with width W , length L and height H , and several small boxes r_1, r_2, \dots, r_n where box r_i has width w_i , length ℓ_i and height h_i . The question is whether there is a subset of the small boxes that can be put inside the large box R with no gaps or overlaps. Show that BoxTiling is NP-hard.

Solution: Reduction from SubsetSum (given numbers x_1, x_2, \dots, x_n , is there is a subset of numbers that add up to $X = \sum_i x_i/2$). We create boxes with width 1, length 1 and height x_i . And make R to be width 1, length 1 and height X . If we answer yes to SubsetSum then we answer yes to BoxTiling.

2. Adam wants to ask if there is a simple path (i.e., not repeating vertices) in a graph G that goes through at least $1/3$ of the vertices. Show that this problem is NP-hard.

Solution: Reduce from Hamiltonian Path problem. Given a graph G of n vertices, create $2n$ additional singleton vertices that are not connected to any other vertices. Call this new graph G' . The existence of a hamiltonian path in G is equivalent to the existence of a path in G' that visits $1/3$ vertices.

3. Consider a directed graph $G = (V, E)$, a number k , and a set of paths P_1, P_2, \dots, P_m of G , is it possible to select at least k of the paths such that no two of the selected paths share any vertices? Show that this problem is NP-hard.

Solution: We reduce from 3D matching to this problem. Given Y_1, Y_2, Y_3 each of n vertices and 3-tuples $U = (u, v, w)$ with $u \in Y_1$, $v \in Y_2$, and $w \in Y_3$, we ask whether there are n tuples of U such that each vertex of Y_1, Y_2, Y_3 appears exactly once. Now we formulate the graph in our problem. The vertices are $Y_1 \cup Y_2 \cup Y_3$ and the edges are the tuples in U . We take $k = n$. If 3D matching answers YES, then there is a subset of k paths that are disjoint. The opposite direction is also true.

4. Given a finite set U of size n and a collection A_1, A_2, \dots, A_m of subsets of U . You are also given numbers c_1, c_2, \dots, c_m . The question is that, does there exist a subset $X \subseteq U$ such that the cardinality of $X \cap A_i$ is equal to c_i ? Prove that this problem is NP-complete.

(Hint: you may use some variants of 3SAT: https://en.wikipedia.org/wiki/Boolean_satisfiability_problem#3-satisfiability)

Solution: Reduce from one-in-three 3-SAT, i.e., exactly one literal in each clause is TRUE. Given such an instance, we take the elements in U to be all the literals $\{x_i, \bar{x}_i\}$, and each set A_i contains the three literals in the i th clause. And fix all $c_i = 1$. Now suppose the one-in-three SAT is satisfiable, then there is a set X of size n (with the TRUE literals) such that for each A_i the intersection is exactly one. The other direction is similar.