An Exploration of Time Decay Dominance in an Iterative Hedging Strategy

A Quantitative Analysis

Options trading strategies face the fundamental tension between time decay and potential gains from favorable price movements, especially amongst retail strategies. The following is a mathematical framework explaining why empirical option trading backtests will show profitability for single-day strategies but consistent losses for longer holding periods.

1 Setup

Let:

$$
\begin{aligned}
S_0 &= \text{Initial stock price} && (1)\\
K &= \text{Strike price} && (2)\\
T &= \text{Time to expiration (days)} && (3)\\
r &= \text{Risk-free rate (annual)} && (4)\\
\sigma &= \text{Volatility (annual)} && (5)\\
P_0 &= \text{Initial option premium} && (6)\\
f &= \text{Daily trading fee} && (7)\\
n &= \text{Number of days held} && (8)
\end{aligned}
$$

The initial call option premium is given by the Black-Scholes formula:

$$P_0 = S_0 \cdot N(d_1) - K \cdot e^{-rT/365} \cdot N(d_2) \tag{9}$$

where:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)(T/365)}{\sigma\sqrt{T/365}} \tag{10}$$

$$d_2 = d_1 - \sigma\sqrt{T/365} \tag{11}$$

where $N(\cdot)$ is the CDF of the normal distribution function. Now, assume an underlying stock follows geometric Brownian motion, as was assumed in the iterative strategy:

$$S_n = S_0 \cdot \exp\left[\left(r - \frac{\sigma^2}{2}\right)\frac{n}{365} + \sigma\sqrt{\frac{n}{365}} \cdot Z\right] \tag{12}$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable. The option value after holding for n days is:

$$P_n = S_n \cdot N(d_1^n) - K \cdot e^{-r(T-n)/365} \cdot N(d_2^n) \tag{13}$$

where:

$$d_1^n = \frac{\ln(S_n/K) + (r + \sigma^2/2)((T-n)/365)}{\sigma\sqrt{(T-n)/365}} \tag{14}$$

$$d_2^n = d_1^n - \sigma\sqrt{(T-n)/365} \tag{15}$$

2 Main Results

Theorem 1 (Time Decay Dominance)**.** *For a long options strategy with daily trading fees, there exists a critical time period* T^* *such that:*

- $\mathbb{E}[P^*(n)] > 0$ *when* $n < T^*$
- $\mathbb{E}[P^*(n)] < 0$ *when* $n > T^*$

where $P^*(n)$ *represents the profit function after* n *days.*

The total profit after holding for n days is:

$$P^*(n) = P_n - P_0 - n \cdot f \tag{16}$$

Let the time decay of an option be:

$$\Theta = -\frac{\partial P}{\partial T} = -\left[S_0 \cdot \phi(d_1) \cdot \frac{\sigma}{2\sqrt{T/365}} + r \cdot K \cdot e^{-rT/365} \cdot N(d_2)\right] \tag{17}$$

Representing how much an option value decreases per day; where $\phi(\cdot)$ is the standard normal probability density function. $\phi(d_1)$ is the standard normal probability density function, where the first term is the effect of volatility decay, and the second term is representative of the interest rate discounting. Therefore, in order to get daily theta decay:

$$\Theta_{\text{daily}} = \frac{|\Theta|}{365} \approx \frac{P_0 \cdot \sqrt{2\pi/(T/365)} \cdot e^{-d_1^2/2}}{365} \tag{18}$$

Is an appropriate approximation; showing that daily theta decay is *roughly* proportionate to the initial option premium, a factor that increases as time to expiration decreases, and an exponential decay factor based on how far the option is from being an ATM option. Let Γ be the convexity of the option, which provides potential upside from price movements:

$$\Gamma = \frac{\partial^2 P}{\partial S^2} = \frac{\phi(d_1)}{S_0\sigma\sqrt{T/365}} \tag{19}$$

Gamma essentially represents the option's ability to benefit from large price movements. The potential upside from the price movement is as follows:

$$\mathbb{E}[\text{Gamma Benefit}] = \frac{1}{2} \cdot \Gamma \cdot \sigma^2 \cdot S_0^2 \cdot \frac{1}{365} \tag{20}$$

This showcases that the benefit is proportional to the squared volatility, scaled by gamma and the square of a stock price, and converted to a daily basis by dividing by 365.

3 Proof of Main Theorem

We will first analyze two cases to establish the threshold seen in the prior section.

Case 1: Short-term holding $(n = 1)$

The expected profit for a one-day hold is:

$$\mathbb{E}[\Pi(1)] = \mathbb{E}[P_1] - P_0 - f \tag{21}$$

For small time intervals, we will use Taylor expansion; this is because the stock price follows Brownian motion, the normal CDF functions depend on a random stock price, and taking the expectation would ergo be impractical, and we should instead linearized the nonlinear:

$$\mathbb{E}[P_1] \approx P_0 - \Theta_{\text{daily}} + \frac{1}{2} \cdot \Gamma \cdot \sigma^2 \cdot S_0^2 \cdot \frac{1}{365} \tag{22}$$

This allows us to separate certain loss from expected gain and maintain reasonable accuracy. Therefore:

$$\mathbb{E}[\Pi(1)] \approx \frac{1}{2} \cdot \Gamma \cdot \sigma^2 \cdot S_0^2 \cdot \frac{1}{365} - \Theta_{\text{daily}} - f \tag{23}$$

For sufficiently high volatility, this expression can be positive.

Case 2: Long-term holding $(n \to T)$

As n increases, the cumulative effects dominate. We will now need to derive such an approximation in order to better understand what occurs to expected profit with n approaching infinity. Instead of treating theta as a time function, lets model it as an exponentially accelerating decay rate; as it is a fundamental truth that theta decay accelerations as expiration approaches. Therefore, if we assume $\Theta_{\text{daily}}(t) \approx \frac{P_0}{\tau} e^{t/\tau}$ where τ is a characteristic time scale.

A natural choice is $\tau = \frac{\sqrt{T}}{\sqrt{2\pi}}$, as it is dimensionally correct with the $\sqrt{2\pi}$ factor being retrieved from black scholes, giving:

$$\text{Cumulative Loss} = \int_0^n \frac{P_0}{\tau} e^{t/\tau}\, dt$$

$$= P_0 \left[e^{t/\tau}\right]_0^n = P_0 \left(e^{n/\tau} - 1\right)$$

With $\tau = \frac{\sqrt{T}}{\sqrt{2\pi}}$:

$$\text{Cumulative Loss} = P_0 \left(e^{\sqrt{2\pi n}/\sqrt{T}} - 1\right)$$

Rearranging to match:

$$\text{Cumulative Theta Loss} \approx P_0 \left(1 - e^{-\sqrt{2\pi n}/\sqrt{T}}\right)$$

The sign flip occurs because we're measuring loss (positive quantity) rather than the negative theta decay. Additionally:

$$\text{Cumulative Fees} = n \cdot f$$

Which is self-explanatory as it is a linear, daily fee.
The expected profit becomes:

$$\mathbb{E}[\Pi(n)] = \mathbb{E}[P_n] - P_0 - n \cdot f \tag{24}$$

As n increases:

1. Theta decay compounds exponentially
2. Fees accumulate linearly
3. Gamma benefits diminish due to reduced time to expiration

Critical Threshold
The critical time T^* where expected profit equals zero is approximately:

$$T^* = \frac{\sqrt{2\pi} \cdot P_0}{\Theta_{\text{daily}} + f} \tag{25}$$

This means that for $n > T^*$, the cumulative theta decay and fees exceed any reasonable expected gains from favorable price movements.

4 Numerical Example

Consider the following parameters:

$$S_0 = \$100 \tag{26}$$
$$K = \$105 \text{ (5\% OTM call)} \tag{27}$$
$$\sigma = 30\% \text{ (annual volatility)} \tag{28}$$
$$r = 3\% \tag{29}$$
$$f = \$1.00 \text{ (daily fee)} \tag{30}$$
$$T = 30 \text{ days} \tag{31}$$

Calculations:

$$P_0 \approx \$2.50 \text{ (Black-Scholes premium)} \tag{32}$$
$$\Theta_{\text{daily}} \approx \$0.08 \text{ per day} \tag{33}$$
$$\text{Total daily cost} = \$0.08 + \$1.00 = \$1.08 \tag{34}$$

Critical threshold:

$$T^* \approx \frac{\sqrt{2\pi} \cdot 2.50}{1.08} \approx 5.8 \text{ days} \tag{35}$$

5 Implications and Conclusions

Corollary 1. *The mathematical structure of options pricing creates an inherent bias toward short-term strategies for long option positions when considering iterative strategies, regardless of directional market knowledge.*

Key Insights:

1. **Certainty vs. Uncertainty:** Theta decay and fees represent certain losses, while profitable price movements are probabilistic.
2. **Exponential vs. Linear Costs:** Time decay accelerates as expiration approaches, while fees accumulate linearly.
3. **Diminishing Returns:** Longer holding periods provide diminishing marginal benefits from gamma convexity.

Practical Applications:
While some approximation in my analysis may lead to slight inaccuracies, this analysis explains why empirical backtests of the aforementioned strategy consistently show:

- Positive returns for 1-day option strategies
- Negative returns for extended holding periods
- Poor performance even with advance knowledge of market events

The mathematical framework demonstrates that successful options trading requires either:

1. Very short holding periods to minimize theta exposure (oversimplification).
2. Selling options at varying periods, dependent on real-world decisions, rather than buying them to benefit from time decay and selling at a predetermined date that an iterative strategy would entail.
3. Sophisticated dynamic hedging strategies beyond simple long positions, or a larger amount of money where decay would matter much less with cheaper options.

6 References

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3), 637-654.

Hull, J. C. (2017). *Options, Futures, and Other Derivatives* (10th ed.). Pearson.