Ito Integration ans Calculus, Concept and Didactical Simulations

Giacomo Babudri

January 2024

1 Introduction

The principles of Ito integration and calculus stand as fundamental elements in stochastic calculus, a specialized branch of mathematics that extends the traditional calculus to handle stochastic processes. Stochastic calculus holds significant importance in the realm of mathematical finance, offering a robust framework for the modeling and analysis of financial instruments in dynamic and uncertain environments.

1.1 Stochastic Differential Equation (SDE)

In the realm of Ito calculus, the foundation lies in the concept of stochastic differential equations (SDEs). An SDE is an equation that combines deterministic differentials and stochastic differentials. A typical representation of an SDE takes the form:

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

where:

- X_t is the stochastic process,
- $\mu(t, X_t)$ is the drift term,
- $\sigma(t, X_t)$ is the diffusion term,
- dt signifies the deterministic component,
- dW_t is the stochastic differential associated with a Wiener process.

1.2 Ito Integral

The Ito integral serves as a stochastic extension of the Riemann-Stieltjes integral. This integral plays a crucial role in solving stochastic differential equations and denotes the accumulation of a stochastic process over time. The Ito integral of a process Y_t with respect to a Wiener process W_t is denoted as:

$$\int_0^t Y_s \, dW_s$$

1.3 Ito's Lemma

Itô's Lemma is a fundamental result in stochastic calculus, serving as the stochastic analog of the chain rule. It is employed to find the differential of a time-dependent function of a stochastic process. In the context of a stochastic differential equation (SDE) of the form:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t$$

where B_t is a Wiener process, Itô's Lemma provides a formula for the differential $df(t, X_t)$ of a differentiable function $f(t, x)$ as:

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu(t, X_t) \frac{\partial f}{\partial x} + \frac{\sigma^2(t, X_t)}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma(t, X_t) \frac{\partial f}{\partial x} dB_t$$

This formula is crucial in mathematical finance, particularly in deriving equations for option values, including the well-known Black–Scholes equation.

2 simulation

2.1 Code

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters of the GBM process
mu = 0.1  # drift rate
sigma = 0.2  # volatility
T = 1.0  # final time
N = 1000  # number of steps

# Generate a Geometric Brownian Motion (GBM)
t = np.linspace(0, T, N+1)
dt = T / N
W = np.random.randn(N+1) * np.sqrt(dt).cumsum()

# Calculate the GBM process
X = np.exp((mu - 0.5 * sigma**2) * t + sigma * W)

# Calculate the approximate Ito integral
Ito_integral = np.cumsum(X[:-1] * (np.diff(t) + sigma * X[:-1] * np.diff(W)))
```

```
# Visualize the results
plt.figure(figsize=(10, 6))

# Plot the GBM process
plt.subplot(2, 1, 1)
plt.plot(t, X, label='GBM')
plt.title('Geometric Brownian Motion')
plt.xlabel('Time')
plt.ylabel('X(t)')
plt.legend()

# Plot the Ito integral
plt.subplot(2, 1, 2)
plt.plot(t[:-1], Ito_integral, label="Ito Integral")
plt.title("Ito Integral of GBM")
plt.xlabel('Time')
plt.ylabel('Ito Integral')
plt.legend()

plt.tight_layout()
plt.show()
```

2.2 Output

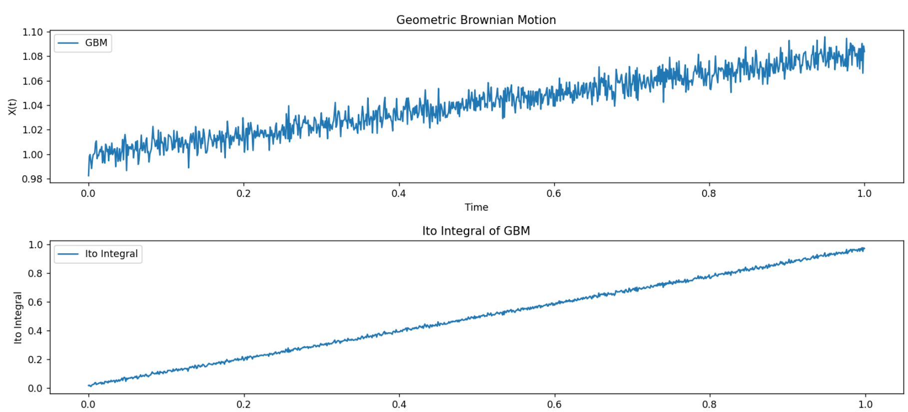

Figure 1: Simulation's output