Stochastic processes and SDE's

Giacomo Babudri

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1 Introduction

In probability theory and related fields, ***a stochastic or random process*** is a mathematical object usually defined as a sequence of random variables, where the index of the sequence often has the interpretation of time. Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner.

In the general case, a stochastic process is characterized by a collection of random variables indexed by time. For a discrete-time stochastic process, these random variables might be denoted as $X_0, X_1, X_2, \ldots$, representing the state of the system at different time points. For continuous-time processes, the random variables are indexed by a continuous parameter, often time itself.

A ***stochastic differential equation (SDE)*** is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. The general form of a one-dimensional stochastic differential equation is given by:

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

Here:

- X_t is the state of the system at time t.
- $\mu(t, X_t)$ represents the deterministic drift term, describing the system's average or expected behavior.
- $\sigma(t, X_t)$ is the stochastic diffusion term, representing the random variations.
- dW_t represents the random component (Brownian Motion).

2 Simulation of the main processes

For the simulation was done preliminarily an infrastructure, in html, and then added javsascript, based on the stochastic process to represent. The Html can be found on this link: https://jackbab0.github.io/Statistics/Homework7.html.

2.1 Random Walk

A Random Walk is a mathematical model describing a path that consists of a series of random steps. In its simplest form, each step is independent and has equal probability in either direction. The resulting path exhibits a random, unpredictable pattern.

Widely used in various fields, including finance for modeling stock prices, physics for particle motion, and biology for modeling random processes.

Simulation

```
function simulateRandomWalk(numSteps, timeDelta) {
    const steps = [0];

    for (let i = 1; i < numSteps; i++) {
        const step = steps[i - 1] + (Math.random() > 0.5 ? 1 : -1) * timeDelta;
        steps.push(step);
    }

    return steps;
}
```

Figure 1: Code of the simulation

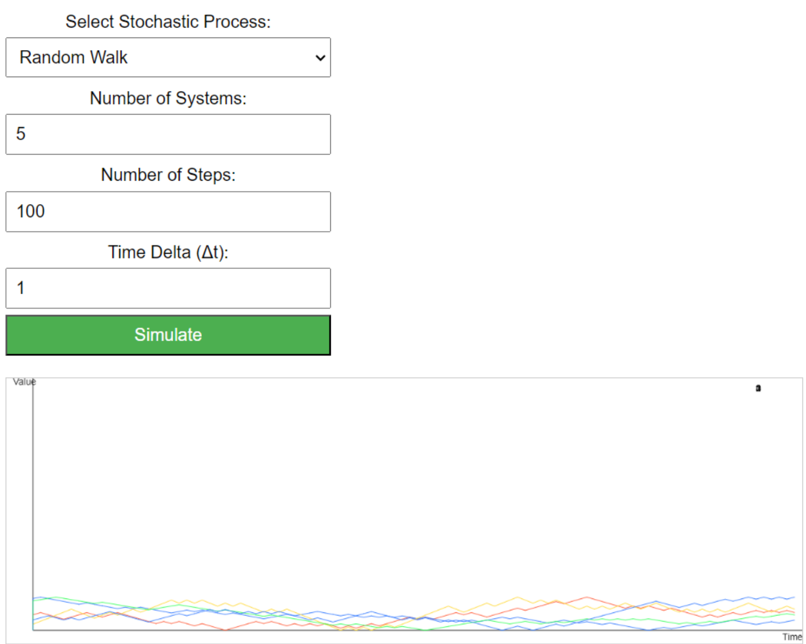


Figure 2: Simulation

2.2 Arithmetic Brownian Motion

Arithmetic Brownian Motion, also known as Brownian Motion, is a stochastic process characterized by normally distributed and independent changes. It includes a constant drift term (μ) representing the average rate of change and a stochastic term (σdW_t) capturing random fluctuations. The formula for Arithmetic Brownian Motion is given by:

$$dX_t = \mu\, dt + \sigma\, dW_t$$

Used in various fields to model random and independent movements, such as in physics for particle motion and in finance for modeling asset prices.

Simulation

```
function simulateArithmeticBrownian(numSteps, timeDelta, drift, volatility) {
    const steps = [0];

    for (let i = 1; i < numSteps; i++) {
        const randomIncrement = drift * timeDelta + volatility * Math.sqrt(timeDelta) * (2 * Math.random() - 1);
        const step = steps[i - 1] + randomIncrement;
        steps.push(step);
    }

    return steps;
}
```

Figure 3: Code of the simulation

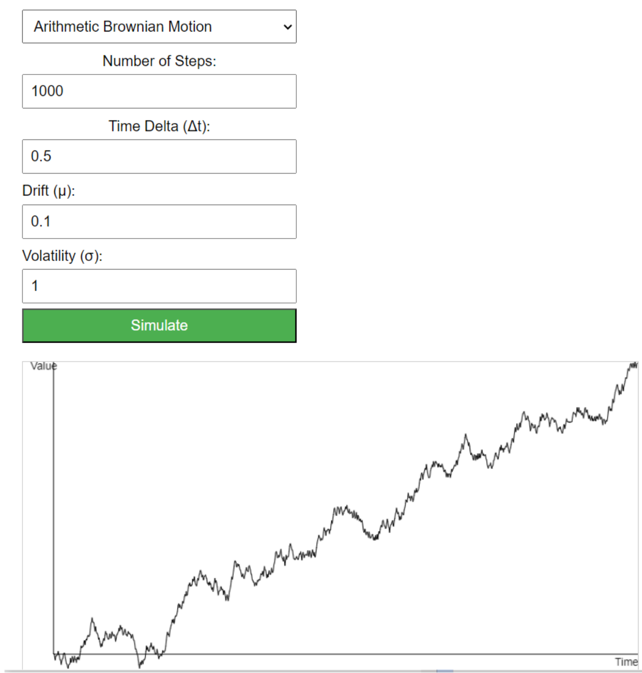


Figure 4: Simulation

2.3 Geometric Brownian Motion

Geometric Brownian Motion is an extension of Arithmetic Brownian Motion, commonly applied in finance, especially in the Black–Scholes option pricing model. It introduces a multiplicative term, resulting in a log-normal distribution for the variable. The formula for Geometric Brownian Motion is given by:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

Predominantly used in finance for modeling the dynamic behavior of financial instruments, particularly stock prices and option pricing.

Simulation

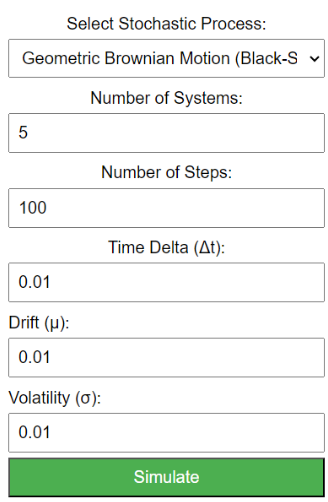
```
function simulateGeometricBrownian(numSteps, timeDelta, drift, volatility) {
    const steps = [1];

    for (let i = 1; i < numSteps; i++) {
        const randomIncrement = drift * timeDelta + volatility * Math.sqrt(timeDelta) * normalRandom();
        const step = steps[i - 1] * Math.exp(randomIncrement);
        steps.push(step);
    }

    return steps;
}
```

```
function normalRandom() {
    let u = 0, v = 0;
    while (u === 0) u = Math.random();
    while (v === 0) v = Math.random();
    return Math.sqrt(-2.0 * Math.log(u)) * Math.cos(2.0 * Math.PI * v);
}
```

Figure 5: Code of the simulation



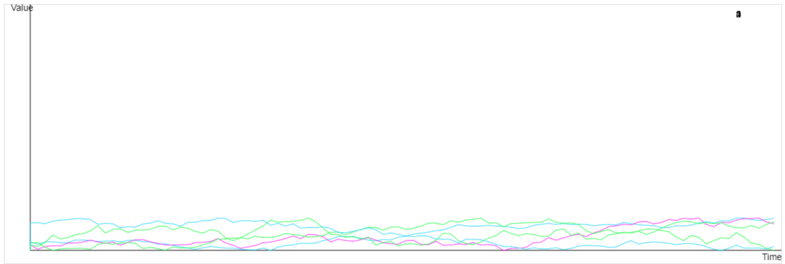


Figure 6: Simulation

2.4 Ornstein–Uhlenbeck Process (Mean-Reverting Process)

The Ornstein–Uhlenbeck process is a mean-reverting stochastic process used to model systems that tend to return to a certain mean or equilibrium value over time. It includes a drift term $(\theta(\mu - X_t))$, where θ is the speed of mean reversion, and a stochastic term $(\sigma \, dW_t)$. The formula for the Ornstein–Uhlenbeck process is given by:

$$dX_t = \theta(\mu - X_t) \, dt + \sigma \, dW_t$$

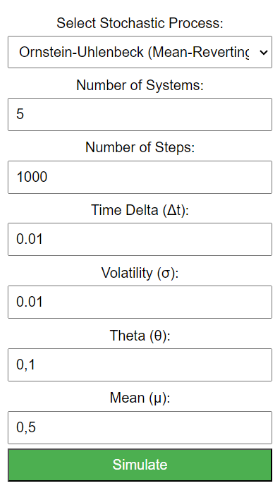
Commonly applied in finance for modeling interest rates and in biology for population dynamics.

Simulation

```
function simulateOrnsteinUhlenbeck(numSteps, timeDelta,  volatility, theta, mean) {
    const steps = [mean];

    for (let i = 1; i < numSteps; i++) {
        const randomIncrement = theta * (mean - steps[i - 1]) * timeDelta + volatility * Math.sqrt(timeDelta) * normalRandom();
        const step = steps[i - 1] + randomIncrement;
        steps.push(step);
    }

    return steps;
}
```



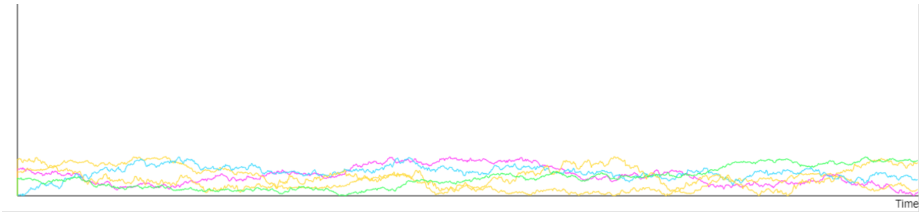


Figure 7: Simulation

Vasicek Model

The Vasicek model is a stochastic process used in interest rate modeling. It describes the evolution of interest rates over time and incorporates mean reversion. The process includes a deterministic drift term ($k(\mu - r_t)$), where k is the speed of mean reversion, μ is the long-term mean, and a stochastic term ($\sigma\, dW_t$). The formula for the Vasicek model is given by:

$$dr_t = k(\mu - r_t)\, dt + \sigma\, dW_t$$

Primarily employed in finance for modeling interest rate movements and assessing interest rate risk.

Simulation

```
function simulateVasicek(numSteps, timeDelta, interestRate, volatility, speed) {
    const steps = [speede];

    for (let i = 1; i < numSteps; i++) {
        const randomIncrement = speed * (interestRate - steps[i - 1]) * timeDelta + volatility * Math.sqrt(timeDelta) * normalRandom();
        const step = steps[i - 1] + randomIncrement;
        steps.push(step);
    }

  return steps;
}
```

Figure 8: Code of the simulation

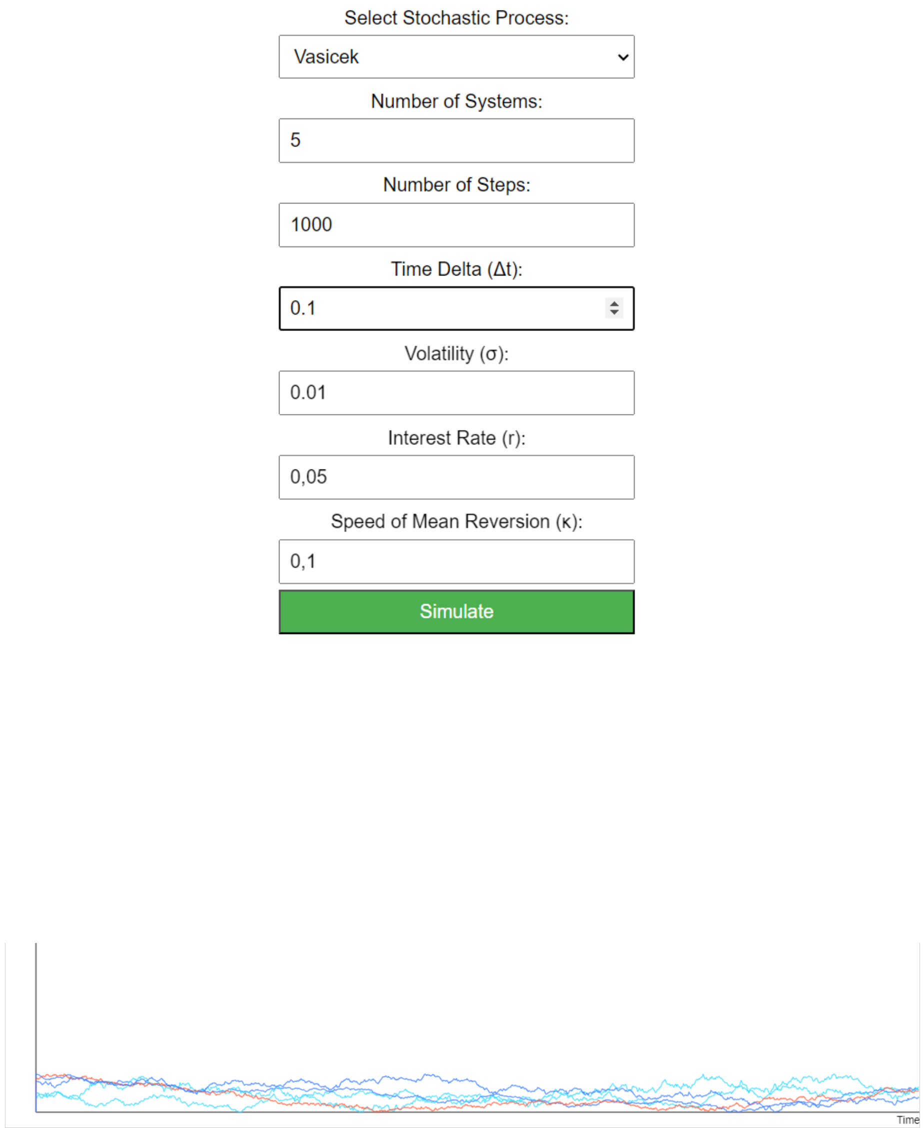


Figure 9: Simulation