Game Theory and Logic

Jack Beecher

March 13, 2025

Abstract

This paper is an expository mathematics paper exploring an overlap of game theory and logic. Specifically, it uses the Ehrenfeucht-Fraïssé game to conclude different results about the theory of discrete linear orderings without endpoints.

1 Background

To understand the coming discussion, it is important to understand the description of the game from the statement of the problem [3]:

Suppose we have a game G with two players named Alice and Bob respectively. A strategy for Bob is a function τ such that if Alice's first n moves are $c_1, \ldots, c_n$ then Bob's nth move will be $\tau(c_1, \ldots, c_n)$. We say that Bob uses the strategy τ if the play of the game looks like:

$$\text{Alice: } c_1; \quad \text{Bob: } \tau(c_1); \quad \text{Alice: } c_2; \quad \text{Bob: } \tau(c_1, c_2); \ \ldots$$

We say that τ is a winning strategy for Bob if for any sequence of plays $c_1, c_2, \ldots$ that Alice makes, Bob will win by following τ. We define winning strategies for Alice analogously.

Next, let $\mathfrak{L}$ be a language with no function symbols. Suppose we have two structures $\mathfrak{A}$ and $\mathfrak{B}$ for $\mathfrak{L}$ with $|\mathfrak{A}| \cap |\mathfrak{B}| = \emptyset$. If $A \subseteq |\mathfrak{A}|$ and $B \subseteq |\mathfrak{B}|$ and $f : A \to B$, we say that f is a partial embedding if the function

$$f \cup \{(c_{\mathfrak{A}}, c_{\mathfrak{B}}) : c \text{ is a constant in } \mathfrak{L}\}$$

is a bijection preserving all relations of $\mathfrak{L}$. We will define an infinite two-player game $G_\omega(\mathfrak{A}, \mathfrak{B})$ between two players called Alice and Bob. A play of the game will consist of a (countably) infinite number of stages. Together they will build a partial embedding f from $\mathfrak{A}$ to $\mathfrak{B}$. At the ith stage, Alice moves first and either plays $a_i \in |\mathfrak{A}|$, challenging Bob to put a_i into the domain of f, or $b_i \in |\mathfrak{B}|$, challenging Bob to put b_i into the range of f. If Alice plays a_i then Bob must play $b_i \in |\mathfrak{B}|$, whereas if Alice plays b_i then Bob must play $a_i \in |\mathfrak{A}|$. Bob wins the game if $f = \{(a_i, b_i) : i = 1, 2, \ldots\}$ is the graph of a partial embedding.

Also, note that in this paper, whenever we refer to a linear ordering, it is inherently a strict linear ordering.

Accompanying the background for the game, we will also commonly reference these below definitions and theorems when needed. Feel free to skip forward to the introduction for now and come back as needed while reading.

1.1 Definitions

Definition 1. (As stated in the structures hand out in class) If $\mathfrak{A}$ and $\mathfrak{B}$ are two linear orderings, a function $f : |\mathfrak{A}| \to |\mathfrak{B}|$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ if and only if:

1. The function f is a bijection.

2. For all $x, w \in |\mathfrak{A}|$, we have that

$$x <_{\mathfrak{A}} w \iff f(x) <_{\mathfrak{B}} f(w)$$

If such an isomorphism f exists, we say $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

Definition 2. We define the lexicographic ordering $\prec$ as follows: $(x, m) \prec (y, n)$ if and only if $x < y$ or $x = y$ and $m < n$, where $<$ is defined in the standard sense.

Definition 3. Definition of Complete Theory from Enderton's textbook [1, p. 156]:
A theory T is said to be complete iff for every sentence σ, either $\sigma \in T$ or $(\neg\sigma) \in T$.

1.2 Theorems

Theorem 1. The lexicographic ordering $(L \times \mathbb{Z}, \prec)$ is a discrete linear ordering without endpoints.

Proof. To show this, notice that for any (l_i, x), $(l_i, x-1)$ is the immediate predecessor, $(l_i, x+1)$ is the immediate successor, and they both always exist. $\square$

Theorem 2. Any countably infinite discrete linear order without endpoints that does not contain elements infinitely far apart is isomorphic to $(\mathbb{Z}, <)$.

Proof. Trivially, just index the countably infinite order with the integers and map them to the integers by the indexing. Then, $<$ holds and the mapping is bijective. $\square$

Theorem 3. Let L be a finite language without function symbols and let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures. Then $\mathfrak{A} \equiv \mathfrak{B}$ if and only if Bob has a winning strategy in $G_n(\mathfrak{A}, \mathfrak{B})$ for all n. (This theorem was given to us to use in question 4 without proof).

Theorem 4. Corollary 26I b) from Enderton's textbook [1, p. 157]:
A complete axiomatizable theory (in a reasonable language) is decidable.

Theorem 5. Corollary 26F a) from Enderton's textbook [1, p. 154]:
Let Σ be a set of sentences in a countable language. If Σ has some infinite model, then Σ has models of every infinite cardinality.

2 Introduction

To eventually draw conclusions about the theory of discrete linear orderings without endpoints using game theory and the game described above, we build simply. In the coming sections, we will walk from what a discrete linear ordering without endpoints is, to how the game described above can help us draw conclusions, and eventually, to a discussion of results. In order to help things remain clear, we start here with a background of the game without so much technical emphasis.

The game described above is called the Ehrenfeucht-Fraïssé game. In simple terms, the game consists of two players, a spoiler and a duplicator. The game also consists of two "boards." The spoilers goal is to make a move on one board that the duplicator is unable to replicate on the other. In a mathematical context, the two boards are two domains of separate models. The inability to replicate a move is the inability to preserve the ordering across the boards.

If Alice is the spoiler, Alice wins if she can trap Bob with an unduplicatable move. For example, Alice plays between two previous moves on board 1 and on board 2 those previous elements don't have space in between them. If such a round occurs, Bob loses the game and the game stops.

On the other hand, for Bob to win, he must be able to duplicate for countably infinite rounds of the game. He is declared the winner if it is theoretically provable that his strategy avoids all of Alice's traps.

Eventually, we will get to a more technical analysis of this game where the two boards are different variations of discrete linear orders without endpoints. First, however, we start by clearly defining the game boards we will play on.

3 Linear orderings without endpoints

Working in a language $\mathfrak{L}$ with the two-place predicate $<$, we define precisely what it means for a model to be a dense linear ordering without endpoints. We then will define precisely what it for a model to be a discrete linear ordering without endpoints as well. We define both as it is easier to see the game play out for the first time on dense boards. Once we have a feel for the game, we will then turn our attention to playing on discrete boards.

3.1 Define Γ (axioms for dense linear ordering without endpoints)

To understand how to play on a dense linearly ordered game board without endpoints, we define an axiom set Γ that will act as the rules.

Linear Ordering: Note again, that whenever we say linear ordering in this paper, we are referring to a strict linear ordering. A strict linear ordering has the following qualities: transitive, antisymmetric, irreflexive. Thus, we add the following three axioms below to Γ:

$$\forall x\ \forall y\ \forall z\ ((x < y \wedge y < z) \rightarrow x < z) \tag{1}$$

$$\forall x \; \forall y \; ((x < y \rightarrow y \not< x)) \tag{2}$$

$$\forall x \; (x \not< x) \tag{3}$$

Without Endpoints: A linear ordering not having endpoints means our game boards stretch off into infinity. There is no end of the board in either direction. To represent that, we include these two axioms below in our set Γ:

$$\forall x \; \exists y \; (x < y) \tag{4}$$

$$\forall x \; \exists z \; (z < x) \tag{5}$$

Dense: Our game boards are dense if between any two spots on the board, there is always another open spot. The axiom below represents this:

$$\forall x \; \forall y \; (x < y \rightarrow \exists z \; (x < z \wedge z < y)) \tag{6}$$

Therefore, if we have a model of Γ, it is a dense linear order without endpoints. Also, if we have a structure that doesn't model Γ, it is not a dense linear order without endpoints as there is a statement above not true in our structure. Therefore, we have a model for Γ if and only if it is a dense linear order without endpoints.

A common representation of a dense linear order without endpoints is the reals with the standard $<$. Therefore, we can imagine our game boards as two number lines. We will come back to this mental image when we play our first game. Before that, we define the second type of board we will eventually play on.

3.2 Define Σ (axioms for discrete linear ordering without endpoints)

For Σ, our rule set for discrete boards, we use axioms 1-3 (linear ordering) and 4-5 (without endpoints) from Γ but replace axiom 6 (dense) with two new axioms (discrete) to represent a discrete linear order without endpoints.

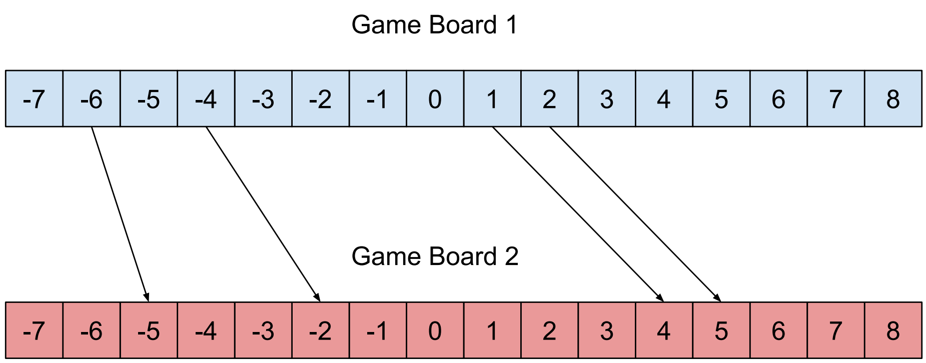
Discrete: For a linear order to be discrete, every element has both an immediate successor and predecessor. Or in terms of our game board, there is a square both directly ahead and behind every other square. We can represent that by adding the two axioms below to Σ to replace axiom 6 from Γ:

$$\forall x\ \exists y\ (x < y \wedge \forall z\ \neg(x < z \wedge z < y)) \tag{6}$$

$$\forall x\ \exists y\ (y < x \wedge \forall z\ \neg(y < z \wedge z < x)) \tag{7}$$

Thus, axioms 1 through 5 from Γ and the two axioms above compose our set of axioms Σ. We reach the same conclusion that a structure models those axioms if and only if it is a discrete linear order without endpoints.

A common representation of this is the integers with $<$. Therefore, we can imagine our two game boards as two infinitely long rows of squares indexed by the integers.



The arrows in this image are the game state. If at any point, the arrows cross, Bob loses as the board is no longer a partial embedding.

Now, that we have an idea of the game boards we will play on, we bridge into a discussion of our game.

4 Playing the game on dense boards

The goal of this section is the show that for any two countable boards (structures $\mathfrak{A}$ and $\mathfrak{B}$), Bob has a winning strategy for the game ($G_\omega(\mathfrak{A}, \mathfrak{B})$) if and only if $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

First, let's imagine we are Bob. How does winning the game imply the boards are isomorphic? By the definition, a winning strategy means we can win every game no matter what set of moves Alice plays. Thus, we can create a partial embedding from $\mathfrak{A}$ to $\mathfrak{B}$ for any combination of Alice's moves.

To construct our isomorphism, analyze the game in which Alice's strategy is to play every element of $|\mathfrak{A}|$. Since we can win that game, there is a partial embedding from every element of $|\mathfrak{A}|$ to $|\mathfrak{B}|$. This partial embedding is also onto and 1-1 given the countable nature of our two structures and game. Therefore, our partial embedding is a full isomorphism from $|\mathfrak{A}|$ to $|\mathfrak{B}|$, which implies $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

Now, let's flip to the other direction. We, as Bob, know the two boards we are playing on are isomorphic, and we need a strategy to win every time. Well, since we win if the game boards duplicate each other at the end of the game, and we know the isomorphism is a duplicating function, we play by the isomorphism. This means that whenever Alice plays on game board A, we play the mapping of that move by the isomorphism on game board B. Similarly, if she plays on game board B, we play on game board A by the inverse mapping of the isomorphism. Formally, we can denote our strategy as follows (where f is the isomorphism from $|\mathfrak{A}|$ to $|\mathfrak{B}|$):

$$\tau(c_1, \ldots, c_i) = \begin{cases} f(c_i) & \text{if } c_i \in |\mathfrak{A}| \\ f^{-1}(c_i) & \text{if } c_i \in |\mathfrak{B}| \end{cases}$$

Thus, by following strategy τ, we create a partial embedding from $\mathfrak{A}$ to $\mathfrak{B}$ and win the game. We conclude that Bob has a winning strategy in a game across countable boards if and only if the two boards are isomorphic.

4.1 Showing two game boards are isomorphic

Using this notion, we can show that two countably dense game boards are isomorphic. All we need to do is come up with a strategy so that no matter how Alice plays, we (as Bob) can duplicate her move on the other dense board.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be these two countable dense linear orderings without endpoints in our $\mathfrak{L}$. As we previously mentioned, it is useful to imagine these two game boards as two number lines of the reals. We will show we have a winning strategy on these boards inductively.

On move 1, independent of where Alice plays, we can play wherever to "duplicate" her board.

This duplication is straightforward because there is only one element and no notion of comparison yet.

However, let's explore how we will make a move in a general case, say at move n if the boards are still duplicated at move n - 1. Thinking ahead to try to guess at what Alice may do to stump us, we realize she only ever has two general types of moves.

1. She can play beyond (either greater or less than) any previous move on either board

2. She can play between any two previous moves on either board

Understanding this, we realize we can't lose. If she plays beyond any previous move, we can as well since the boards don't have endpoints. If she plays between two previous moves, we can follow suit as the boards are dense.

Therefore, for countably many rounds, we have a duplicating move to counter Alice. She can't win! This means that the two countable boards we played on were isomorphic. Or, formally, $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ for any $\mathfrak{A}$ and $\mathfrak{B}$ that are dense linear orderings without endpoints.

Now that we've played the game (and won) with a simple example, we can test ourselves by playing the game on more difficult boards: ones that are discrete and not dense.

5 Playing the game on discrete boards

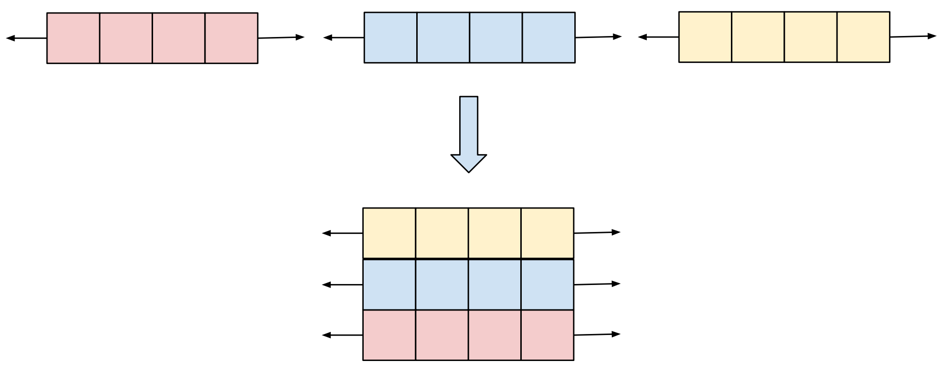
To build our strategy for how we as Bob will eventually play against Alice on discrete boards, we first explore the subtle differences between different discrete linear orderings without endpoints.

We start by proving that every game board which models Σ is isomorphic to a game board of the form $(L \times \mathbb{Z}, \prec)$, where L is a linear order and $\prec$ is the lexicographic order given in Definition 2.

Let $\mathfrak{A}$ model Σ. We will show $\mathfrak{A}$ is isomorphic to a model of the form $(L \times \mathbb{Z}, \prec)$, where L is a linear order and $\prec$ is the lexicographic order. We construct an equivalence relation for our discrete linear ordering as follows:

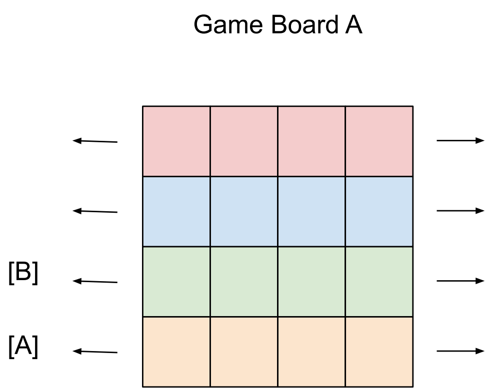
$$a \equiv b \text{ if and only if there are finitely many elements between a and b.}$$

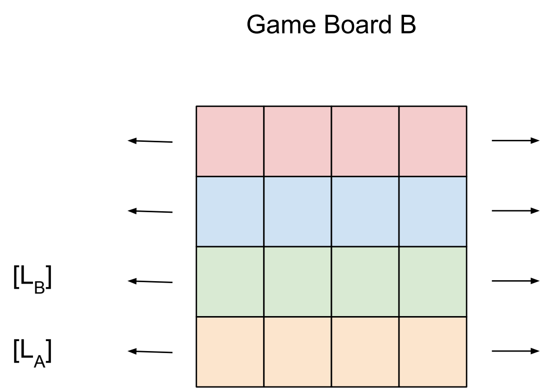
In a less technical sense, this means if two spots on a game board are unreachable from each other, they exist on different sections of the board. For a good mental image, take our long row of sequential game squares (possible moves), and separate any spots unreachable from each other by putting them on their own row. So instead of one long row, we have a NxM game grid where N is the number of separate chunks containing members infinitely far apart. Also, notice each row M is still its own discrete linear ordering without endpoints.



By that equivalence relation and the corresponding mental image, we define the notion of an equivalence chunk. We say the number of equivalence chunks in a discrete linear ordering is equal to the number of equivalence classes (N) there are. Above we claimed that each row M (equivalence chunk) is its own discrete linear ordering without endpoints. To formalize this notion, imagine it wasn't. Then there would be a value in the chunk without an immediate successor or predecessor, which implies that $\mathfrak{A}$ does not model Σ, or that our game board is not a discrete linear ordering without endpoints.

Understanding this representation of any discrete linear ordering, let $\mathfrak{B}$ be our model $(L \times \mathbb{Z}, \prec)$ such that L is ordered by $<$ and $|L| = N$. We now show $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$. By our construction of $\mathfrak{B}$ and the definition of $\prec$, $\mathfrak{B}$ has the same number of equivalence chunks as $\mathfrak{A}$. That is, the two game boards have the same number of rows (N). To show this, notice that $a \equiv b$ if and only if $a = (l_i, x)$ and $b = (l_i, y)$ where l_i is any element taken from the set L and x and y are any integers. Thus, there are $|L| = N$ chunks. Notationally, index the equivalence classes by their ordering so that [a] corresponds to l_a for all equivalence classes. Meaning if $a \not\equiv b$ and $a < b$, $l_a < l_b$.





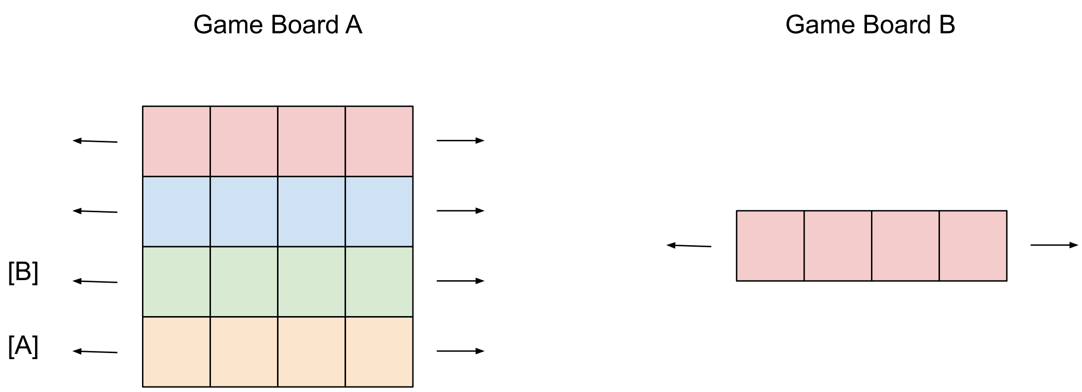
Now that we know we have a bijective mapping from equivalence chunks in $\mathfrak{A}$ to equivalence chunks in $\mathfrak{B}$, we can create our isomorphism $f : |\mathfrak{A}| \to |\mathfrak{B}|$. To do this, notice that since every countably infinite equivalence chunk is itself a discrete linear order without endpoints, there exists an isomorphism f_e from each chunk to the integers by Theorem 2. Define $f : \mathbb{Z} \to (L \times \mathbb{Z})$ as

$$f(a) = (l_a, f_e(a))$$

To show f is an isomorphism, we show that f is bijective and that $a < b$ if and only if $f(a) \prec f(b)$. It is clearly a bijection as the chunks and f_e are both bijective. To show $a < b \Leftrightarrow f(a) \prec f(b)$, note that either $a \not\equiv b$ or $a \equiv b$. If $a \not\equiv b$, $a < b \Leftrightarrow l_a < l_b \Leftrightarrow f(a) \prec f(b)$. If $a \equiv b$, $f(a) \prec f(b) \Leftrightarrow f_e(a) < f_e(b) \Leftrightarrow a < b$. Thus, f is an isomorphism. Since f is an isomorphism, every model of Σ is isomorphic to one of the form $(L \times \mathbb{Z}, \prec)$.

Now that we have a mental idea of the discrete game board in our heads from walking through a complicated example. We can play our game on a discrete board.

Our goal in this game is to show that any model $(L \times \mathbb{Z}, \prec)$ of Σ is elementary equivalent to $(\mathbb{Z}, <)$. Or that any gameboard of the lexicographic grid form described above can be duplicated to a game board of the standard row of boxes we described earlier. Based on the picture below, you may be skeptical that this is possible.



5.1 Start of discrete game

Let $\mathfrak{B}$ be our model $(\mathbb{Z}, <)$ and $\mathfrak{A}$ be our model $(L \times \mathbb{Z}, \prec)$. By Theorem 3, to show $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$, we show Bob has a winning strategy in $G_\omega(\mathfrak{A}, \mathfrak{B})$ for all n. Again, we will act as if we are Bob trying to beat Alice in a game. Without loss of generality, to show we have a winning strategy, all we need to do is show that we have a winning strategy if Alice plays exclusively in $|\mathfrak{A}|$. For if Alice makes moves in $|\mathfrak{B}|$, our reply is trivial as we can constrain our move to a singular l_i and treat $\mathfrak{A}$ as if it were just $(\{1\} \times \mathbb{Z}, \prec)$, which is isomorphic to $\mathfrak{B}$. Since the boards are isomorphic and countable, we know we have a winning strategy as we showed above. So assuming Alice is an intelligent opponent trying to win, we show we have a winning strategy for Alice playing exclusively in $|\mathfrak{A}|$. We first define a distance function d for $x = (a, i),\ y = (b, j) \in L \times \mathbb{Z}$ as follows:

$$d(x, y) = \begin{cases} |i - j| & \text{if } a = b \\ \infty & \text{if } a \neq b \end{cases}$$

This function states that if two spaces on the board are unable to be reached in a countable number of successive spaces, they are infinitely far apart and we can treat them as separate chunks. With that definition of distance in mind, we define our strategy τ for move i as follows (where f is the game state, or partial embedding, from all prior moves).

1. **Alice plays a move greater than all previous**

If Alice plays a_i such that $a_j \prec a_i \ \forall j < i$, we play $f(a_k) + 2^{n-i}$ where a_k was the previous greatest element played by Alice.

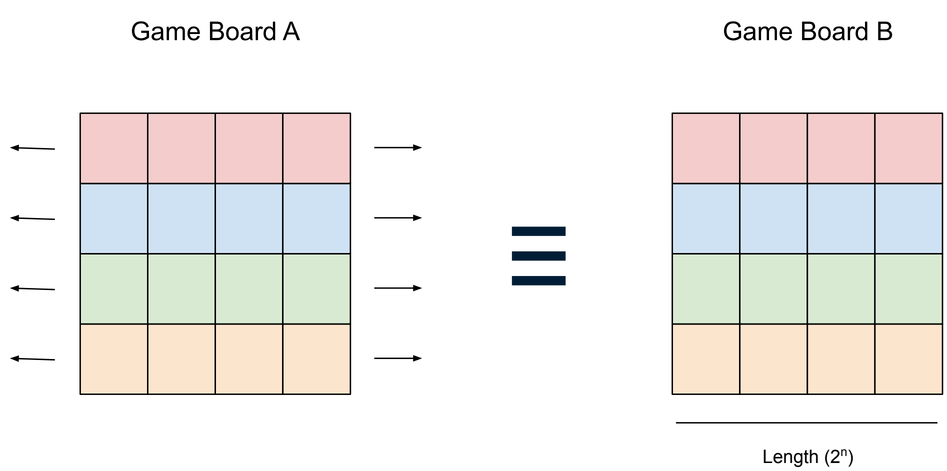
2. **Alice plays a move less than all previous**

   If Alice plays a_i such that $a_i \prec a_j \ \forall j < i$, we play $f(a_k) - 2^{n-i}$ where a_k was the previous smallest element played by Alice.

3. **Alice plays between two previous moves**

   If Alice plays a_i such that $a_k \prec a_i \prec a_j$, where a_k and a_j are the closest previous moves above and below a_i, we play in the middle by $\frac{f(a_k)+f(a_j)}{2}$.

To give some context as to why we play 2^{n-i} away for board-extending moves, it allows us to treat our countable row of sequential spaces as N rows of "long enough" spaces. We say long enough because playing 2^n spaces away in an n-move game gives us enough room to counter any of Alice's strategies. Even if Alice plays her best strategy, playing moves between two elements infinitely far apart, we can duplicate her for n moves.



With a formal argument, we now show this is a winning strategy. It is clear to see that if Alice plays above or below all previous elements, our strategy preserves relations and thus the game state remains a partial embedding as $f(a_k) - 2^{n-i} < f(a_k)$ and $f(a_k) + 2^{n-i} > f(a_k)$. So to try to beat our strategy, it would be best for Alice to play every move (after the initial two moves) between two others.

However, the whole point of our strategy is we can survive this. Even if Alice plays all of her moves between her first two elements which she can set an infinite distance apart, there will always be 2^{n-2} open spaces for us to make counter moves in $|\mathfrak{B}|$. Thus, we can continue to play our splitting the difference strategy for $\log_2(2^{n-2})+1=(n-2)+1=n-1$ more moves. But after move 2, there are only $n-2$ remaining rounds. Thus, our strategy works (and works by just one move to frustrate Alice even more). It is clear how this can be generalized to any mix of moves Alice may play. There will always be $n-j$ space between elements we must play between at move $i=j+1$, and we will always be able to make $\log_2(2^{n-j})+1=(n-j)+1$ more moves by splitting the difference. Since $(n-j)+1>n-j$, we can always beat Alice with this strategy.

Since we have a winning strategy in $G_\omega(\mathfrak{A},\mathfrak{B})$, by Theorem 3, $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$. Thus, any discrete linear ordering without endpoints of the form $(L\times\mathbb{Z},\prec)$ is elementary equivalent to $(\mathbb{Z},<)$.

Perhaps, this result is quite surprising, or at least it was to me. We can mirror all moves by Alice on a board of uncountably infinite size onto a board of countably infinite size for countably many moves. This is incredible. We are playing two-dimensional chess against an opponent that has all the moves of an infinite-dimensional game.

6 Results from winning the discrete game

Besides our sense of pride for beating Alice, what else does the above victory uncover? Quite a lot.

Above we showed all models of Σ are isomorphic to a model of the lexicographic form $(L\times\mathbb{Z},\prec)$. This means all models of Σ are elementarily equivalent to such a lexicographic model. Therefore, since we just showed *all* of these lexicographic forms are elementarily equivalent to $(\mathbb{Z},<)$, we know all discrete linear orderings without endpoints are elementarily equivalent to each other. Therefore, $ThMod(\Sigma)=Th(\mathfrak{A})$ where $\mathfrak{A}$ is any model of Σ.

This allows us to prove that $C_n(\Sigma)$ is complete in a straightforward way. Since $C_n(\Sigma)=ThMod(\Sigma)=Th(\mathfrak{A})$, and the theory of any one model is complete, $C_n(\Sigma)$ is complete. To restate what this means, we know that for all sentences σ either $\sigma\in C_n(\Sigma)$ or $\neg\sigma\in C_n(\Sigma)$.

Furthermore, since we know we can axiomatize (with Σ) a theory for discrete linear orderings without endpoints and such a theory is complete, $C_n(\Sigma)$ is decidable. This follows from theorem 4, which states any complete axiomatizable theory is decidable.

We can also show that Σ is $\aleph_0$-categorical. By the definition of $\aleph_0$-categorical, we just need to show any two countably infinite models of the Σ are isomorphic. Well, we know that any two countable models of Σ are isomorphic to some countable lexicographic model. And we also know that we (Bob) have a winning strategy for a game between any countable model $(L \times \mathbb{Z}, \prec)$ and the model $(\mathbb{Z}, <)$. Thus, we know that all countable models of Σ are isomorphic to each other, which implies Σ is countably categorical.

6.1 More results

Continuing to reap the benefits of our win against Alice, there are a couple more conclusions about the theory of discrete linear orderings without endpoints that follow simply.

1. Σ has a model of every infinite cardinality

This is because we know $(\mathbb{Z}, <)$ is an infinite model of Σ. By Theorem 5, if Σ has some infinite model, then Σ has models of every infinite cardinality.

2. Σ has no finite models

By contradiction, assume there is a finite model. Since it is finite, there exists a greatest and least element, which contradicts the notion of without endpoints.

3. Σ is k-categorical

Any two models of k cardinality have k equivalence chunks. Thus, they are all isomorphic to any lexicographic structure of the form $(L \times \mathbb{Z}, \prec)$ where $|L| = k$. Therefore, all our models of k cardinality are isomorphic to each other.

7 Closing Comments

To recap, in this paper, we used a game theoretic approach to the Ehrenfeucht-Fraïssé game to prove that (among other things) the theory of a discrete linear ordering without endpoints is both complete and decidable.

References

[1] Enderton, Herbert B. *A Mathematical Introduction to Logic*. Second edition. Harcourt/Academic Press, 2001.

[2] "Ehrenfeucht–Fraïssé game." *Wikipedia, The Free Encyclopedia*, Wikipedia Foundation.

[3] "Final Paper Assignment" Course Materials, Math 69, Dartmouth College.