The Idea Behind Krylov Methods [1]

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What are they?

Do they make sense?

Can they be practical?

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A matrix, A, is diagonalisable if there are matrices S and Λ such that $A = S\Lambda S^{-1}$, where Λ is diagonal.

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"a Krylov method solves $A\mathbf{x} = \mathbf{b}$ by repeatedly performing matrix-vector multiplications involving A" [1]

What is a Krylov space?

Krylov space of dimension k:

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Residual: $\mathbf{r}_k \equiv \mathbf{b} - A\mathbf{x}_k$

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General Krylov:

number of iterations required depends upon eigenvalues

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The future of the talk...

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...BOARD...

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Summarising Theorem:

Theorem ([1] p6)

If the minimal polynomial of the nonsingular matrix A has degree m, then the solution to $A\mathbf{x} = \mathbf{b}$ lies in the space $\mathcal{K}_m(A, \mathbf{b})$.

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How could this be better?

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Ammended Theorem:

Theorem ([1] p6 (adapted))

If matrix A is nonsingular and diagonalisable with d distinct eigenvalues, then the solution to $A\mathbf{x} = \mathbf{b}$ lies in the space $\mathcal{K}_d\left(A, \mathbf{b}\right)$.

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- Realistic expectation: PA has a few 'clusters' of eigenvalues

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Take nonsingular A with one cluster of eigenvalues, radius ρ, centred at 1. I.e. $|1 - \lambda_i| < \rho, \forall i = 1, \ldots, d$.

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Now let the eigenvalues of A satisfy:

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$$\leq\; C\rho^{k-1}$$

Numerical Example

Take

$$A = \begin{pmatrix} 1 & -\rho & & & \\ -\rho & 1 & -\rho & & \\ & -\rho & 1 & -\rho & \\ & & \ddots & \ddots & \ddots \end{pmatrix}^{N \times N}$$

which has eigenvalues

$$\lambda_i = 2\cos\left(\frac{i\pi}{N+1}\right)$$

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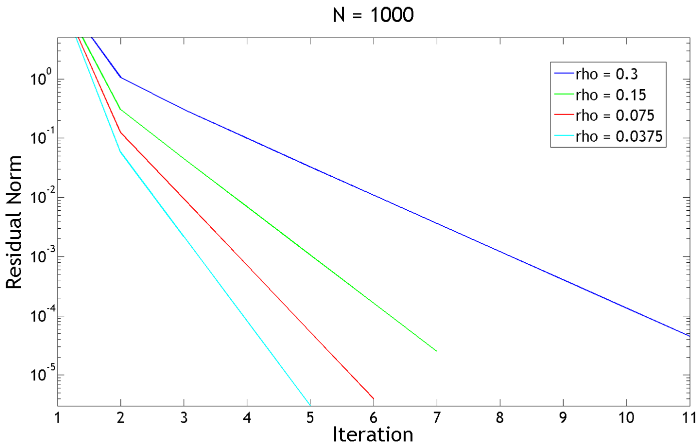

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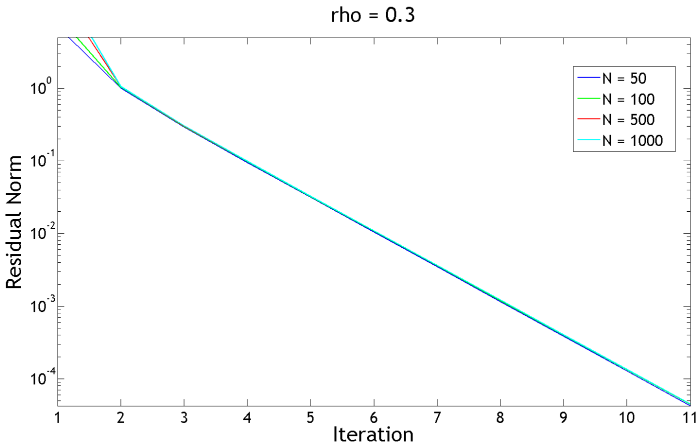

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