Power method

$Av = \lambda v$ . . , take $x^{(0)}$, set $y := Ax^{(0)}$

then $x^{(1)} = \dfrac{y}{\|y\|}$, etc..

converges to eigenvector corresponding to largest eigenvalue of A.

Inverse iteration.

Power method using A^{-1} :

$$(A^{-1})v = \left(\frac{1}{\lambda}\right)v.$$

take $x^{(0)}$, solve $Ay = x^{(0)}$

then $x^{(1)} = \dfrac{y}{\|y\|}$, etc..

converges to eigenvector corresponding to evalue of A closest to zero.

Shifted inverse iteration

$$(A - \sigma I)^{-1} v = \left(\frac{1}{\lambda - \sigma}\right)v$$

take: $x^{(0)}$. solve $(A - \sigma I)y = x^{(0)}$

then $x^{(1)} = \dfrac{y}{\|y\|}$, etc..

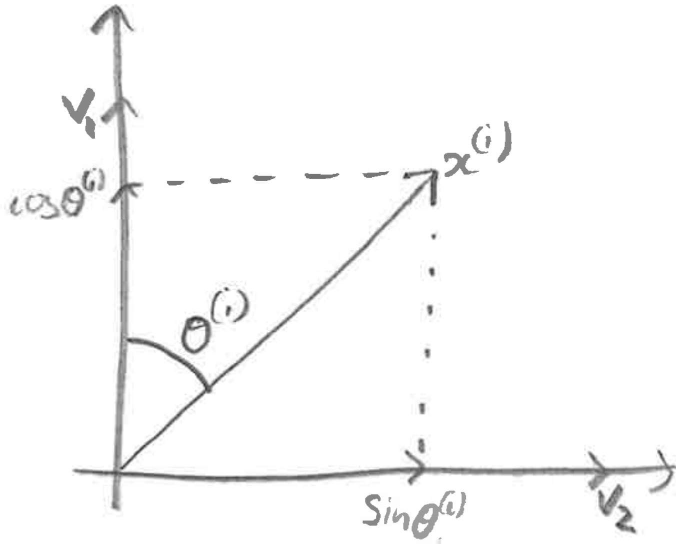
converges to eigenvector corresponding to eigenvalue closest to σ.

1i

let A have evectors $v_1, v_2,$
with evalues $\lambda_1, \lambda_2,$

choose a specific known RHS:

$$(A - s^{(i)} I) y = \cos\theta^{(i)} v_1 + \sin\theta^{(i)} v_2$$
$$= x^{(i)}$$



$$\| x^{(i)} - \cos\theta^{(i)} v_1 \| = |\sin\theta^{(i)}| \, \|v_2\|$$
$$= \sin\theta^{(i)}$$

$$y = \frac{\cos\theta^{(i)}}{\lambda_1 - s^{(i)}} v_1 + \frac{\sin\theta^{(i)}}{\lambda_2 - s^{(i)}} v_2$$

& $x^{(i+1)} = \dfrac{y}{\|y\|}$

so: $$\left.\begin{aligned} \cos\theta^{(i+1)} &= \frac{\cos\theta^{(i)}}{\|y\|(\lambda_1 - s^{(i)})} \\ \sin\theta^{(i+1)} &= \frac{\sin\theta^{(i)}}{\|y\|(\lambda_2 - s^{(i)})} \end{aligned}\right\} \Rightarrow \tan\theta^{(i+1)} = \frac{\lambda_1 - s^{(i)}}{\lambda_2 - s^{(i)}} \tan\theta^{(i)} \quad (*)$$

so if shift $s^{(i)}$ is fixed
then $(*) \Rightarrow \tan\theta^{(i+1)} \le C_1 \tan\theta^{(i)}$
i.e. linear convergence, provided $x^{(i)}$, (the RHS) is still updated.

if RHS $x^{(i)}$ is fixed
then $(*) \Rightarrow \tan\theta^{(i+1)} \le \dfrac{\lambda_1 - s^{(i)}}{\lambda_2 - s^{(i)}} C_2$
i.e. linear convergence, provided $s^{(i)}$ (the shift) is still updated.

~~$A,$~~ Krylov Subspace

$$A y_0 = b \quad , \quad r_0 = b - Ay_0$$

$$K_m(A, r_0) = \text{span}\{r_0, Ar_0, \dots, A^{m-1} r_0\}$$

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example method: <u>Projection method.</u>

solve:

$$Ay = b.$$

seek an approximate solution y_m from the subspace $y_0 + K_m$, st y_m satisfies:

$$b - Ay_m \perp L_m$$

where L_m is "some other" subspace.

Different choices for L_m leads to a family of Krylov methods.

2 main choices for L_m:
1) $L_m = K_m$
2) $L_m = AK_m$.

Choice 1: ~~Projection method~~ <u>GALERKIN - KRYLOV</u> (GK)

given arbitrary initial guess, y_0,
choose $y_m \in y_0 + K_m$ st: $b - Ay_m \perp K_m$

i.e. $V_m^T (b - Ay_m) = 0$, where V_m is the matrix whose columns form a basis of K_m.

Suppose we took $y_0 = 0$.

then: $\mathcal{K}_m(A, r_0) \equiv \mathcal{K}_m(A, b) = \text{span}\{b, Ab, \ldots, A^{m-1}b\}$

with this, we choose: $y_m \in \mathcal{K}_m$ s.t. $b - Ay_m \perp \mathcal{K}_m$

let V_m be the matrix whose columns are the spanning vectors of $\mathcal{K}_m$

then:

$$V^T(b - Ay_m) = 0.$$

Since the condition '$b - Ay_m \perp \mathcal{K}_m$' is known as the Galerkin condition, this method is called the 'Galerkin-Krylov' method.

Fixed iterations

Inverse iteration: given arbitrary $x^{(0)}$, with $\|x^{0}\| = 1$:

1. choose $\sigma^{(i)}$ & $\tau^{(i)}$
2. solve: $(A - \sigma^{(i)} I)\hat{y} = x^{(i)}$ inexactly, to tolerance $\tau^{(i)}$ — inner iteration
3. $x^{(i+1)} = \dfrac{\hat{y}}{\|\hat{y}\|}$
4. test for convergence.

(steps 1–4: outer iteration)

Suppose for the inner iteration we use GK.
This will work fine!
However... suppose we use GK <u>but</u>

1) use a fixed right hand side, $x^{(0)}$
2) use initial inner guess, $y_0 = 0$
3) use <u>exactly</u> m iterations, regardless of tolerance.
(for the sake of clarity, specify $m < n$
however equality does not invalidate any of the following discussion, it just renders it ultimately pointless...)