An asymptotic analysis of preconditioned linear systems arising from discretisations of the transport equation within a diffusive regime.

Jack Blake, Ivan Graham and Alastair Spence

Mathematical Sciences,
Bath University, Bath, UK, BA2 7AY

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The problem

The (mono-energetic, steady-state) neutron transport equation in 5D with an isotropic source, $q(\mathbf{r})$, is given by

$$\Omega \cdot \nabla \psi(\mathbf{r}, \Omega) + \sigma_T(\mathbf{r}) \psi(\mathbf{r}, \Omega) = \frac{\sigma_S(\mathbf{r})}{4\pi} \int_{\mathbb{S}^2} \psi(\mathbf{r}, \Omega') \, \mathrm{d}\Omega' + q(\mathbf{r})$$

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with $\mathbf{r} \in V$ and $\Omega \in \mathbb{S}^2$. The argument $\psi(\mathbf{r}, \Omega)$ is called the neutron flux. Boundary conditions

$$\psi(\mathbf{r}, \Omega) = 0, \quad \text{when} \quad n(\mathbf{r}) \cdot \Omega < 0, \quad \mathbf{r} \in \delta V.$$

Note: σ_T, σ_S and σ_A are called cross-sections. They are all strictly positive and satisfy $\sigma_T = \sigma_S + \sigma_A$

Operator notation and source iteration

Introduce

$$\mathcal{T}(\cdot) \equiv \Omega \cdot \nabla(\cdot) + \sigma_T(\mathbf{r})(\cdot) \quad \text{and define} \quad \phi(\mathbf{r}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \psi(\mathbf{r}, \Omega) \, \mathrm{d}\Omega$$

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Source iteration is defined as follows

$$\begin{aligned} \mathcal{T}\psi^{(k+1)} &= \sigma_S \phi^{(k)} + q \\ \phi^{(k+1)} &= \tfrac{1}{4\pi}\int_{\mathbb{S}^2} \psi^{(k+1)}\, \mathrm{d}\Omega \end{aligned}$$

If the cross-sections (σ_T, σ_S and σ_A) are constant, this basic iterative method is known to converge since $\sigma_S/\sigma_T < 1$.

Source iteration with spatially-dependent cross-sections

A new result for continuous cross-sections:

Theorem

$$\left\| \phi(\mathbf{r}) - \phi^{(k+1)}(\mathbf{r}) \right\|_{L^2(V)} \leq \left\| \frac{\sigma_S(\mathbf{r})}{\sigma_T(\mathbf{r})} \right\|_{L^\infty(V)} \left\| \phi(\mathbf{r}) - \phi^{(k)}(\mathbf{r}) \right\|_{L^2(V)}$$

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By their definitions, $\sigma_S/\sigma_T < 1$ for all $\mathbf{r} \in V$, therefore

$$\left\|\phi - \phi^{(k)}\right\|_{L^2(V)} \to 0, \quad \text{as} \quad k \to \infty,$$

so source iteration converges with spatially dependent cross-sections. Discrete 1D versions of this result are given in both Ashby et. al, 1995, and Bihari and Brown, 2009, for different discretisations.

Limitations of source iteration

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We will explain one method of overcoming this limitation in a setting of 1 spatial and 1 angular dimension, where the transport equation is given by

$$\mu\frac{\partial}{\partial x}\psi(x,\mu) + \sigma_T(x)\psi(x,\mu) = \sigma_S(x)\phi(x) + q(x)$$

where $x \in [0,1]$ and $\mu \in [-1,1]$, and with boundary conditions

$$\begin{aligned}\psi(0,\mu) &= 0, \quad \text{when } \mu > 0,\\ \psi(1,\mu) &= 0, \quad \text{when } \mu < 0.\end{aligned}$$

The scalar flux, ϕ, is defined by

$$\phi(x) \equiv \frac{1}{2}\int_{[-1,1]} \psi(x,\mu)\,\mathrm{d}\mu.$$

Asymptotic Diffusion Approximation

Propose that the expansion

$$\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \dots$$

holds away from spatial boundaries, ϵ small. Assume $\sigma_T = \hat{\sigma}_T/\epsilon$, $\sigma_A = \epsilon\hat{\sigma}_A$ and $q = \epsilon Q$. Matching powers of ϵ (see Habetler and Matkowsky, 1975), yields the following diffusion equation

$$\frac{-1}{3}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sigma_T}\frac{\mathrm{d}}{\mathrm{d}x}\Theta(x)\right) + \sigma_A\Theta(x) = q(x),$$

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Here, $\phi(x) = \Theta(x) + \mathcal{O}(\epsilon^2)$, and a matched boundary layer analysis results in boundary conditions

$$\begin{aligned}\Theta(0) - \tfrac{\lambda}{\sigma_T}\tfrac{\mathrm{d}}{\mathrm{d}x}\Theta(0) = 0,\\ \Theta(1) + \tfrac{\lambda}{\sigma_T}\tfrac{\mathrm{d}}{\mathrm{d}x}\Theta(1) = 0,\end{aligned}$$

λ a known constant.

Diffusion Synthetic Acceleration (DSA)

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Synthetic acceleration methods were first suggested by Kopp in 1963. DSA is such a method.
2-step process:

1. Do one step of source iteration,
2. use the diffusion approximation to estimate the error in step 1.

Limitations of DSA

DSA improves upon or maintains the convergence of plain source iteration for all values of σ_S/σ_T

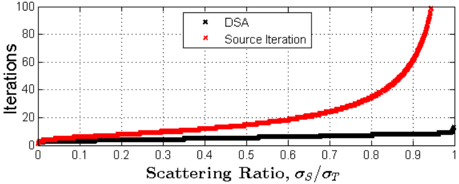

Figure: Iterations to converge to a tolerance of 10^{-6}

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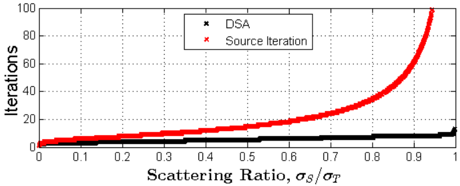

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In the presence of discontinuities in the cross-sections, multidimensional DSA suffers degraded effectiveness (Azmy, 1998, Warsa et. al., 2004).

Warsa et. al., 2004: Preconditioned krylov approach.

A model problem

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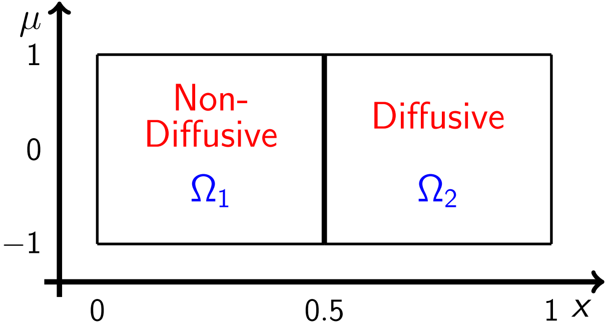
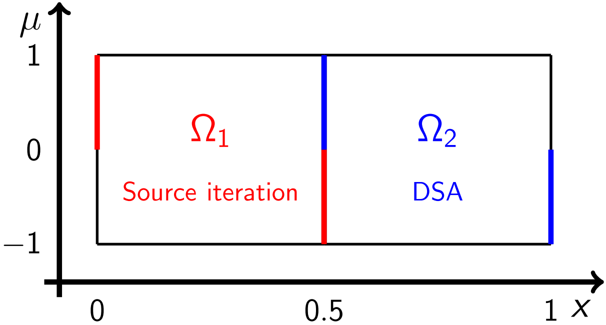
We pose the following problem.

$$\frac{\sigma_S(x)}{\sigma_T(x)} = \begin{cases} \epsilon^2, & x \in [0,\ 0.5], \\ 1-\epsilon^2, & x \in [0.5,\ 1]. \end{cases}$$

We will solve on Ω_1 using source iteration, and on Ω_2 using DSA.



How our algorithm is organised



- Iterate on Ω_1,
- Use estimate on Ω_1 to impose boundary condition for Ω_2,
- Iterate on Ω_2,
- Use estimate on Ω_2 to impose boundary condition for Ω_1.

Difficulty: DSA boundary conditions

Results

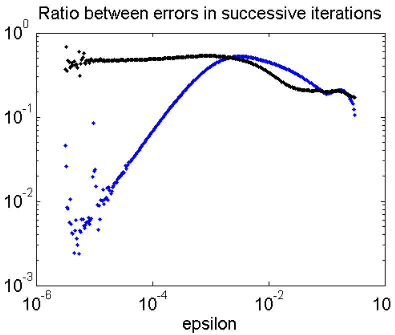


Figure: Ratio between successive errors during iteration

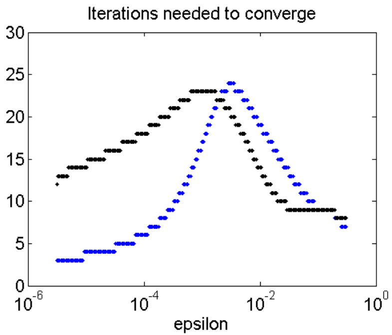


Figure: Number of iterations needed to converge

- **BLACK**: DSA applied over the whole spatial domain,
- **BLUE**: Source iteration on Ω_1, DSA on Ω_2