Preconditioning of Iterative Methods for the Transport Equation

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28th October, 2011

Neutron Transport Equation in Nuclear Fission

Diffusion Synthetic Acceleration (DSA)

An Example

Neutron Transport Equation

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The mono-energetic, steady-state, linear 3D Boltzmann transport equation:

$$\begin{aligned}\boldsymbol{\Omega} \cdot \nabla \Psi(\mathbf{r}, \boldsymbol{\Omega}) + \sigma(\mathbf{r}) \Psi(\mathbf{r}, \boldsymbol{\Omega}) = \\ & \frac{1}{4\pi} \sigma_s(\mathbf{r}) \int_{\mathbb{S}^2} \Psi(\mathbf{r}, \boldsymbol{\Omega}') \, \mathrm{d}\boldsymbol{\Omega}' \\ & + \frac{1}{4\pi} \nu(\mathbf{r}) \sigma_f(\mathbf{r}) \int_{\mathbb{S}^2} \Psi(\mathbf{r}, \boldsymbol{\Omega}') \, \mathrm{d}\boldsymbol{\Omega}' + \mathcal{Q}(\mathbf{r}, \boldsymbol{\Omega}).\end{aligned}$$

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In 1D:

$$\mu \frac{\partial \Psi}{\partial x} + \sigma \Psi = \frac{1}{2} \sigma_s \int_{-1}^{1} \Psi \, \mathrm{d}\mu' + \frac{1}{2} \nu \sigma_f \int_{-1}^{1} \Psi \, \mathrm{d}\mu' + \mathcal{Q}.$$

where σ, σ_s, σ_f and ν are functions of x, the spatial variable, and Ψ and $\mathcal{Q}$ are functions of both x and μ, the angular variable.

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Define:

- $\mathcal{T}(\cdot) \equiv \mu \frac{\partial(\cdot)}{\partial x} + \sigma(x)(\cdot)$,
- $\mathcal{S}(\cdot) \equiv \frac{1}{2}\sigma_s(x) \int_{-1}^{1} (\cdot)\, \mathrm{d}\mu'$,
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In operator form:

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In the previous operator form

$$(\mathcal{T} - \mathcal{S} - \gamma\mathcal{F})\Psi = \mathcal{Q}. \qquad \textit{Source Problem}$$

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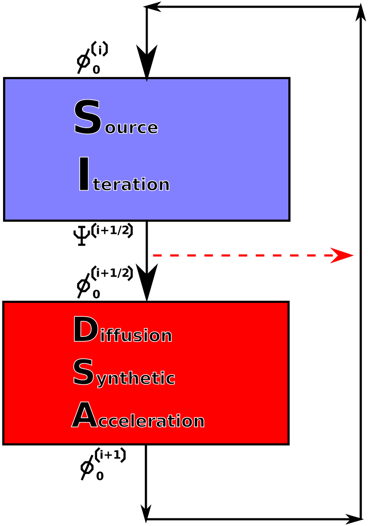
- Splitting methods ($\rightarrow$ *Source Iteration*)
- Synthetic Acceleration methods ($\rightarrow$ *Diffusion Synthetic Acceleration (DSA)*)

The Structure of a Diffusion Synthetic Acceleration Scheme

$$\phi_0^{(i)} \equiv \frac{1}{2}\int_{-1}^{1} \Psi^{(i)}\, d\mu'$$



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'Splitting' Applied to our problem of interest:

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Recall:

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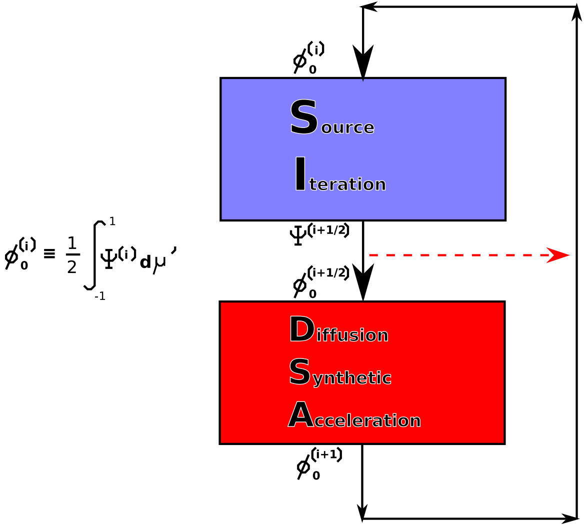

General Synthetic Acceleration

From the Source Iteration we have:

$$\begin{aligned} & \mathcal{T}\Psi^{(i+1/2)} &= \ & \mathcal{S}\Psi^{(i)} &+ \ & \mathcal{Q}. \\ \text{and} \quad & \mathcal{T}\Psi &= \ & \mathcal{S}\Psi &+ \ & \mathcal{Q}. \end{aligned}$$

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Subtracting we obtain an expression for a *linear correction* term

$$\begin{aligned} \mathcal{T}\left(\Psi - \Psi^{(i+1/2)}\right) &= \mathcal{S}\left(\Psi - \Psi^{(i)}\right), \\ &= \mathcal{S}\left(\Psi - \Psi^{(i+1/2)} \right) + \mathcal{S}\left(\Psi^{(i+1/2)} - \Psi^{(i)}\right), \end{aligned}$$

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$$\begin{aligned} (\mathcal{T} - \mathcal{S})(\Psi - \Psi^{(i+1/2)}) &= \mathcal{S}(\Psi^{(i+1/2)} - \Psi^{(i)}), \\ \Rightarrow \qquad \Psi &= \Psi^{(i+1/2)} + (\mathcal{T} - \mathcal{S})^{-1}\mathcal{S}(\Psi^{(i+1/2)} - \Psi^{(i)}). \end{aligned}$$

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DSA is a Synthetic Acceleration method that uses the P_1 diffusion approximation as the choice for M:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3\sigma}\frac{\mathrm{d}\phi_0}{\mathrm{d}x}\right) + \sigma_c\phi_0 = q_0 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{q_1}{\sigma}\right).$$

The P_1 Diffusion Approximation

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Legendre polynomials:

$$P_0(\mu) = 1$$
$$P_n(\mu) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}\mu^n}\left(\mu^2 - 1\right)^n, \quad n = 1, 2, \ldots.$$

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Orthogonality:

$$\int_{-1}^{1} P_n\left(\mu\right) P_{\hat{n}}\left(\mu\right)\,\mathrm{d}\mu = \frac{2\delta_{n,\hat{n}}}{2n+1}.$$

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Once nomalised they from a complete orthonormal sequence, so we can write the neutron flux as an expansion:

$$\Psi(x,\mu) = \sum_{n=0}^{\infty} (2n+1)\,\phi_n(x)\,P_n(\mu),$$

where $$\phi_n(x) \equiv \frac{1}{2}\int_{-1}^{1} \Psi(x,\mu')\,P_n(\mu')\ \mathrm{d}\mu'.$$

The P_1 Diffusion Approximation

The P_N approximation in slab geometry for the Transport equation, consisting of $N+1$ coupled differential equations is given by

$$\frac{n}{2n+1}\frac{\mathrm{d}\phi_{n-1}}{\mathrm{d}x}(x)+\frac{n+1}{2n+1}\frac{\mathrm{d}\phi_{n+1}}{\mathrm{d}x}(x)+\left(\sigma(x)-\sigma_{s,n}(x)\right)\phi_n(x)=q_n \qquad \text{for} \quad n=0,\ldots,N-1,$$

and

$$\frac{N}{2N+1}\frac{\mathrm{d}\phi_{N-1}}{\mathrm{d}x}(x)+\left(\sigma-\sigma_{s,N}\right)\phi_N \;=\; q_N.$$

where $q_n \equiv \frac{1}{2}\int_{-1}^{1} P_n(\mu)\, q(x,\mu)\, \mathrm{d}\mu$
and $\sigma_{s,n} \equiv 2\pi\int_{-1}^{1}\sigma_s(x,\hat{\mu})P_n(\hat{\mu})\,\mathrm{d}\hat{\mu}$

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To move from P_N to P_1 set $N = 1$, to obtain:

$$\begin{aligned} \frac{\mathrm{d}\phi_1}{\mathrm{d}x} \;+\; (\sigma - \sigma_{s,0})\,\phi_0 &= q_0, \\ \frac{1}{3}\frac{\mathrm{d}\phi_0}{\mathrm{d}x} \;+\; (\sigma - \sigma_{s,1})\,\phi_1 &= q_1. \end{aligned}$$

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Combining these and rearranging yields:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3\sigma}\frac{\mathrm{d}\phi_0}{\mathrm{d}x}\right) \;+\; \sigma_c\phi_0 \;=\; q_0 \;-\; \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{q_1}{\sigma}\right).$$

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Note: ϕ_0 depends only upon x, the spatial variable.

Diffusion Synthetic Acceleration

Linear correction:

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DSA finds an approximation for the integral of this error:

$$\begin{aligned} F^{(i+1)}(x) &\approx \int_{-1}^{1} \Psi(x,\mu') - \Psi^{(i+1/2)}(x,\mu')\,\mathrm{d}\mu', \\ &= 2\phi_0(x) - 2\phi_0^{(i+1/2)}(x), \end{aligned}$$

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hence

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Example Problem

[A. Greenbaum *Iterative methods for solving linear systems*]

water	water	iron	water
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$\sigma_t = 3.3333$ $\sigma_s = 3.3136$ $f = 1$	$\sigma_t = 3.3333$ $\sigma_s = 3.3136$ $f = 0$	$\sigma_t = 1.3333$ $\sigma_s = 1.1077$ $f = 0$	$\sigma_t = 3.3333$ $\sigma_s = 3.3136$ $f = 0$

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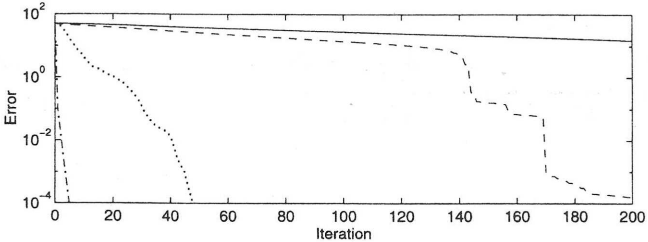

(a)

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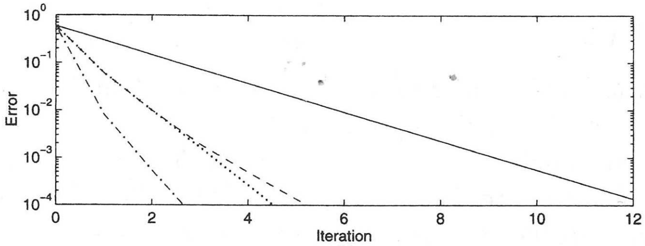

(b)

$\mathcal{T}$hank You!