4.4 二次曲面的化简与分类

1. 解：$\lambda^3-3\lambda-2=0$.

$\lambda_1=2,\ \lambda_2=\lambda_3=-1$.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt3}&\frac{1}{\sqrt2}&-\frac{1}{\sqrt6}\\\frac{1}{\sqrt3}&-\frac{1}{\sqrt2}&-\frac{1}{\sqrt6}\\\frac{1}{\sqrt3}&0&\frac{2}{\sqrt6}\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}+\begin{pmatrix}0\\0\\0\end{pmatrix}$$

化简方程得 $y^2+z^2-2x^2=9$：单叶双曲面

2. 解：$\lambda^3-\lambda^2-\lambda+1=0$.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&0\\-\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}+\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

化简得 $-x^2+y^2+z^2+1=0$：双叶双曲面.

3. 解：$\lambda^3-2\lambda^2-3\lambda=0$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt3}&-\frac{1}{\sqrt2}&\frac{1}{\sqrt6}\\\frac{1}{\sqrt3}&0&-\frac{2}{\sqrt6}\\\frac{1}{\sqrt3}&\frac{1}{\sqrt2}&\frac{1}{\sqrt6}\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}+\cancel{\begin{pmatrix}\frac32\\0\\-1\end{pmatrix}}\ \cancel{\begin{pmatrix}\frac18\\-\frac14\\-\frac98\end{pmatrix}}\ \begin{pmatrix}0\\-\frac32\\-\frac12\end{pmatrix}$$

化简得 $3z^2-y^2-2=0$：双曲柱面

(15) $\lambda^3-18\lambda^2+99\lambda-162=0$. $\lambda_{1,2,3}=3,6,9$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad ①$$

易知其为 $\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+I_4/I_3=0$ 形式的简化方程，故不必代入①式化简立即可得 $3x^2+6y^2+9z^2-\frac{8}{27}=0$

(16) $\lambda^3+6\lambda^2-216\lambda=0$, $\lambda_1=0$, $\lambda_2=12$, $\lambda_3=-18$

(17) $\lambda^3-7\lambda^2+10\lambda=0$. $\lambda_1=0$, $\lambda_2=2$, $\lambda_3=5$

18) ~~$2x^2+1$~~

$\lambda^3-12\lambda^2+21\lambda-10=0$. $\lambda_1=1$, $\lambda_2=1$, $\lambda_3=10$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

得 $x^2+y^2+10z^2-1=0$. ∴椭球面.

19) $\lambda^3-7\lambda^2+14\lambda-6=0$. $\lambda_1=3$, $\lambda_2=2+\sqrt{2}$, $\lambda_3=2-\sqrt{2}$.

简化前的过渡方程略. 因为 $\lambda_1\lambda_2\lambda_3\neq0$. 知化简方程必为椭圆抛物面. 知为

$3x^2+(2+\sqrt{2})y^2+(2-\sqrt{2})z^2-11/2=0$

2. 证明: 特征方程为 $\lambda^3-\frac{1}{4}\lambda=0$. $\lambda_1=0$, $\lambda_2=\frac{1}{2}$, $\lambda_3=-\frac{1}{2}$.

又其中心为不存在. 知为(II)类方程

$\frac{1}{2}x^2-\frac{1}{2}y^2+d=0$ 的形式.

不论 d 或正或负 都有方程为双曲抛物面.

3. 解：取 $\begin{pmatrix} \vec{e_1'} \\ \vec{e_2'} \\ \vec{e_3'} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \vec{e_1} \\ \vec{e_2} \\ \vec{e_3} \end{pmatrix}$ 和三个平面交点 $(0,-\frac{1}{2},\frac{1}{2})$ 为原点，得

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

即

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y+\frac{1}{2} \\ z-\frac{1}{2} \end{pmatrix}.$$

得 $O'(0,0,\frac{1}{\sqrt{2}})$. $A'(\frac{1}{\sqrt{3}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{2}})$ $B'(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}},0)$.

代入二次方程，得 $-2x^2+y^2-z^2+\frac{1}{2}=0$.

其余各类型方程不存在.