# ECON-899: Problem Set 7 Solutions

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This problem set is based on Section 3 of Michaelides and Ng (2000) used to assess the properties of a Simulated Methods of Moments (SMM) estimator, as well as indirect inference and efficient method of moments estimators. A simple statement of the three estimators is given in Section 2 of that paper. As stated on p. 237, "With the SMM, the practitioner only needs to specify the empirical moments and is the easiest to implement". While that section provides an example estimating the $\ell = 1$ parameter of an MA(1), which was presented in class, here we will consider estimating $\ell = 2$ parameters of an AR(1).

Suppose the true data generating process for a series $\{x_t\}_{t=1}^T$ is given by the following AR(1) model:

$$x_t = \rho_0 x_{t-1} + \varepsilon_t, \tag{1}$$

where $\varepsilon_t \sim \mathcal{N}(0, \sigma_0^2)$, $\rho_0 = 0.50$, $\sigma_0 = 1$, $x_0 = 0$ and $T = 200$. Let $b_0 = (\rho_0, \sigma_0^2)$. We will take the model generation process to be

$$y_t(b) = \rho y_{t-1}(b) + e_t, \quad e_t \sim^{i.i.d.} \mathcal{N}(0, \sigma^2), \tag{2}$$

where $b = (\rho, \sigma^2)$.

## Tasks

1. Derive the following asymptotic moments associated with $m_3(x)$: mean, variance, first order autocorrelation. Furthermore, compute $\nabla_b g(b_0)$. Which moments are informative for estimating $b$?

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**Answer:** First, notice that we can iterate backwards $x_t$ in order to get the following function of initial conditions $x_0$ and the error terms $\{\varepsilon_t\}_{t=0}^T$

$$x_t = \rho_0^t x_0 + \sum_{j=0}^{t-1} \rho_0^j \varepsilon_{t-j},$$

such that, by taking $t \to +\infty$, we have that

$$x_t = \sum_{j=0}^{\infty} \rho_0^j \varepsilon_{t-j}$$

That being said, we have that the mean of the AR(1) process is given by

$$\mathbb{E}[x_t] = \mathbb{E}\left[\sum_{j=0}^{\infty} \rho_0^j \varepsilon_{t-j}\right] = \sum_{j=0}^{\infty} \rho_0^j \mathbb{E}[\varepsilon_{t-j}] = 0.$$

Regarding the variance, notice that

$$x_t^2 = \left(\sum_{j=0}^{\infty} \rho_0^j \varepsilon_{t-j}\right)^2 = \sum_{j=0}^{\infty} \rho_0^{2j} \varepsilon_{t-j} + 2\sum_{i<j} \rho_0^i \rho_0^j \varepsilon_{t-i}\varepsilon_{t-j},$$

such that

$$\begin{aligned} Var(x_t) = \mathbb{E}[x_t^2] &= \mathbb{E}\left[\sum_{j=0}^{\infty} \rho_0^{2j} \varepsilon_{t-j} + 2\sum_{i<j} \rho_0^i \rho_0^j \varepsilon_{t-i}\varepsilon_{t-j}\right] \\ &= \sum_{j=0}^{\infty} \rho_0^{2j} \mathbb{E}[\varepsilon_{t-j}^2] = \sigma_0^2 \sum_{j=0}^{\infty} \rho_0^{2j} = \frac{\sigma_0^2}{1-\rho_0^2}. \end{aligned}$$

Finally, for the autocovariance of order 1, we have that

$$\begin{aligned} Cov(x_t, x_{t-1}) = \mathbb{E}[x_t x_{t-1}] = \mathbb{E}\left[\left(\sum_{j=0}^{\infty} \rho_0^j \varepsilon_{t-j}\right)\left(\sum_{j=0}^{\infty} \rho_0^j \varepsilon_{t-1-j}\right)\right] &= \sum_{j=0}^{\infty} \rho_0^{2j+1} \mathbb{E}[\varepsilon_{t-j}^2] \\ &= \rho_0 \sigma_0^2 \sum_{j=0}^{\infty} \rho_0^{2j} = \frac{\rho_0 \sigma_0^2}{1-\rho^2}, \end{aligned}$$

which implies that the first order autocorrelation is given by

$$Corr(x_t, x_{t-1}) = \frac{Cov(x_t, x_{t-1})}{\sqrt{Var(x_t)}\sqrt{Var(x_{t-1})}} = \rho_0,$$

thus, we have that

$$m_3 = \begin{bmatrix} 0 \\ \frac{\sigma_0^2}{1-\rho_0^2} \\ \frac{\rho_0 \sigma_0^2}{1-\rho_0^2} \end{bmatrix},$$

which implies that the Jacobian matrix is given by

$$\nabla_b g(b_0) = \begin{bmatrix} 0 & 0 \\ \frac{2\rho_0 \sigma_0^2}{(1-\rho_0^2)^2} & \frac{2\sigma_0}{1-\rho_0^2} \\ \frac{\sigma_0^2(1+\rho_0^2)}{(1-\rho_0^2)^2} & \frac{2\rho_0 \sigma_0}{1-\rho_0^2} \end{bmatrix}.$$

The variance and the autocovariance are informative moments to estimate $b = (\rho, \sigma^2)$.

2. Simulate a series of "true" data of length $T = 200$ using (1). We will use this to compute $M_T(x)$.

   **Answer:** Figure 1 displays the simulated series of the "true" data.

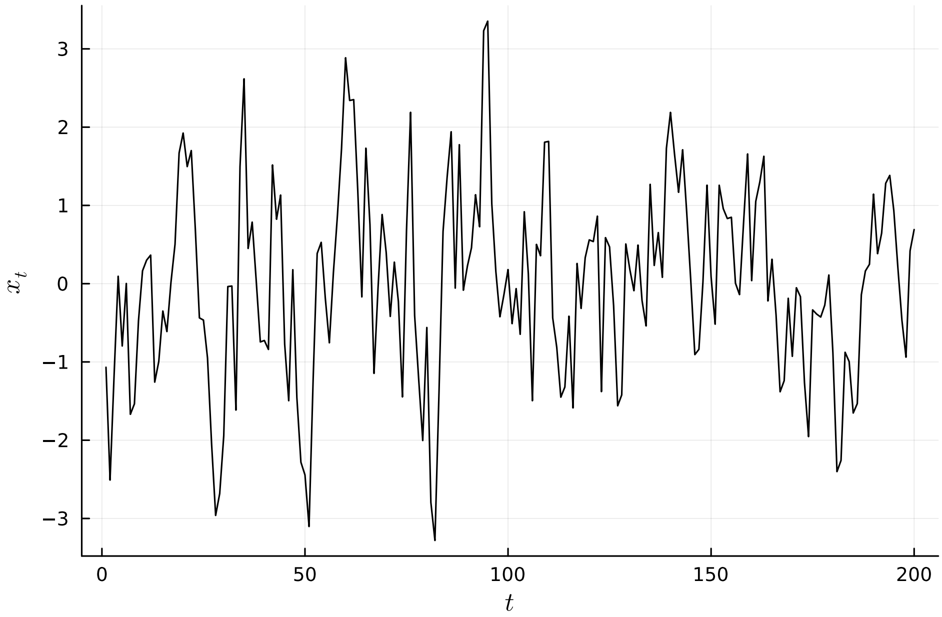



Figure 1: Simulated data

3. Set $H = 10$ and simulate $H$ vectors of length $T = 200$ random variables $e_t$ from $\mathcal{N}(0, 1)$. We will use this to compute $M_{TH}(y(b))$. Store these vectors. You will use the same vector of random variables throughout the entire exercise. Since this exercise requires you to estimate $\sigma^2$, you want to change the variance of $e_t$ during the estimation. You can simply use $\sigma e_t$ when the variance is $\sigma^2$.

4. We will start by estimating the $\ell = 2$ vector $b$ for the just identified case where $m_2$ uses mean and variance. Given what you found in part (1), do you think there will be a problem? Of course, in general we would not know whether this case would be a problem, so hopefully the standard error of the estimate of $b$ as well as the $J$ test will tell us something. Let's see.

   (a) Set $\mathbb{W} = I$ and graph in three dimensions the objective function

   $$J_{TH}(b) \equiv \left[M_T(x) - M_{TH}(y(b))\right]' \mathbb{W} \left[M_T(b) - M_{TH}(y(b))\right],$$

   over $\rho \in [0.35, 0.65]$ and $\sigma \in [0.8, 1.2]$. Obtain an estimate of $b$ by using $\mathbb{W} = I$ in

   $$\hat{b}_{TH} = \arg\min_b J_{TH}(b), \tag{3}$$

   using `fminsearch`. Report $\hat{b}^1_{TH}$.

   (b) Set $i(T) = 4$. Obtain an estimate of $\mathbb{W}^*$. Using $\hat{\mathbb{W}}^*_{TH} = \hat{S}^{-1}_{TH}$ in (3), obtain an estimate of $\hat{b}^2_{TH}$. Report $\hat{b}^2_{TH}$.

(c) To obtain standard errors, compute numerically $\nabla_b g_T(\hat{b}^2_{TH})$ defined as

$$\nabla_b g_T(\hat{b}^2_{TH}) = \begin{bmatrix} \frac{\partial g_T(\hat{b}^2_{TH})}{\partial \rho} & \frac{\partial g_T(\hat{b}^2_{TH})}{\partial \sigma} \end{bmatrix}.$$

Report the values of $\nabla_b g_T(\hat{b}^2_{TH})$. Next, obtain the $\ell \times \ell$ variance-covariance matrix of $\hat{b}^2_{TH}$ as follows

$$\frac{1}{T}\left[\nabla_b g_T(\hat{b}^2_{TH})' \hat{S}^{-1}_{TH} \nabla_b g_T(\hat{b}^2_{TH})\right]^{-1}.$$

Finally, compute the standard errors defined as

$$\sqrt{diag\left(\frac{1}{T}\left[\nabla_b g_T(\hat{b}^2_{TH})' \hat{S}^{-1}_{TH} \nabla_b g_T(\hat{b}^2_{TH})\right]^{-1}\right)}.$$

How can we use the information on $\nabla_b g_T(\hat{b}^2_{TH})$ to think about local identification?

(d) Since we are in the just identified case, the $J$ test should be zero (on a computer this may be not be exact). However, given the identification issues in this particular case where we use mean and variance, the $J$ test may not be zero. Compute the value of the $J$ test:

$$T\frac{H}{1+H} \times J_{TH}(\hat{b}^2_{TH}) \to \chi^2,$$

noting that in this just identified case $n - \ell = 0$ degrees of freedom recognizing that there really is not distribution.

5. Next we estimating the $\ell = 2$ vector $b$ for the just identified case where $m_2$ uses the variance and autocorrelation. Given what you found in part (1), do you now think there will be a problem? If not, hopefully the standard error of the estimate of $b$ as well as the $J$ test will tell us something. Let's see. For this case, perform steps (a)-(d) above.

6. Next, we will consider the overidentified case where $m_3$ uses the mean, variance and autocorrelation. Let's see. For this case, perform steps (a)-(d) above. Furthermore, bootstrap the the finite sample distribution of the estimators using the following algorithm:

   (a) Draw $\varepsilon_t$ and $e^h_t$ from $\mathcal{N}(0,1)$ for $t = 1, 2, \ldots, T$ and $h = 1, 2, \ldots, H$. Compute $\left(\hat{b}^1_{TH}, \hat{b}^2_{TH}\right)$ as described.

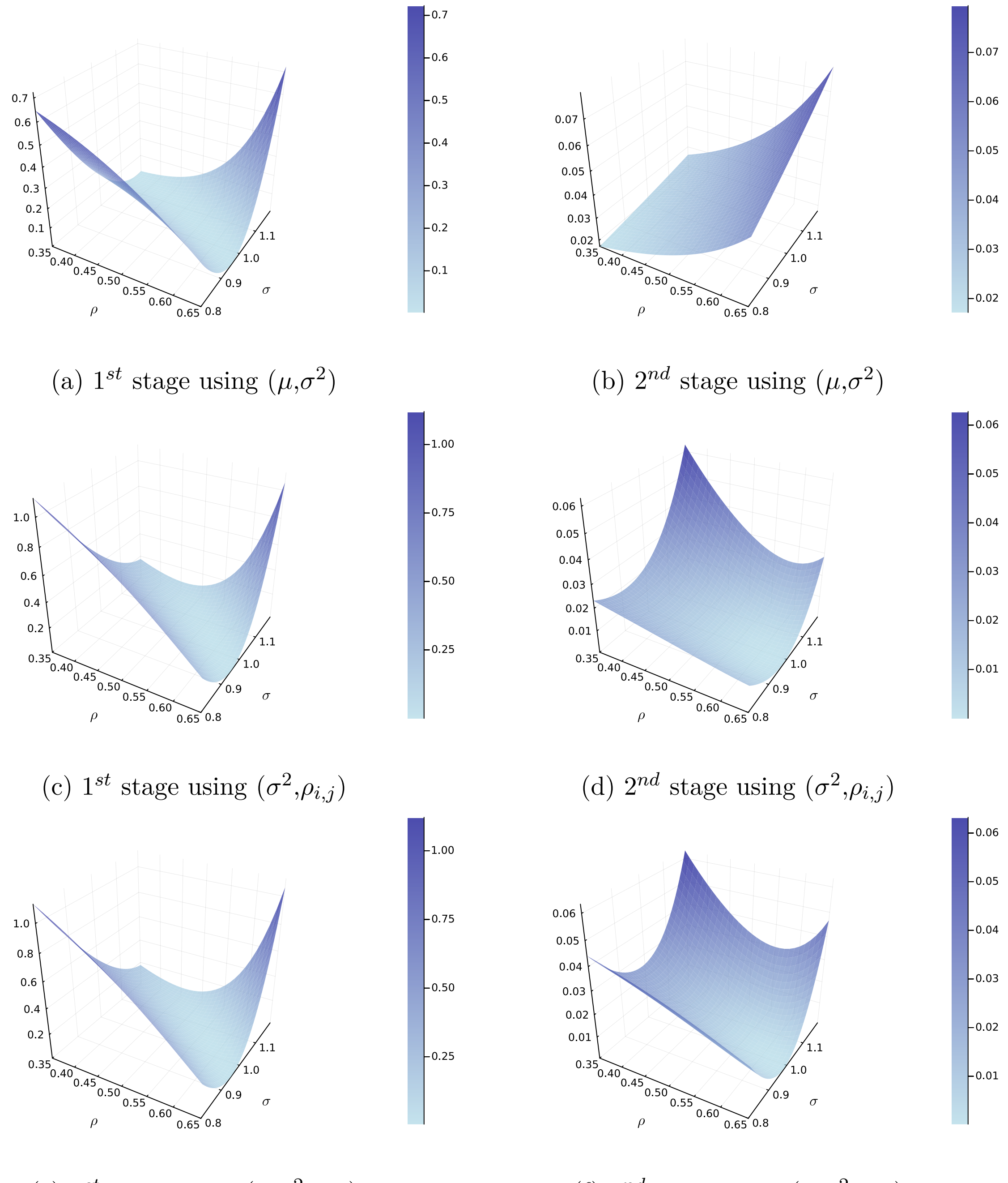
   (b) Repeat (e) using another seed.

   Every time you do step (i), the seed needs to change. Otherwise you will keep getting the same estimators.

   **Answer:** Table 1 report our estimates for the parameter vector $b = (\rho, \sigma)$ for each stage (i.e., 1 and 2) using the moments: (i) mean and variance, (ii) variance and autocovariance, and (iii) mean, variance and auto-covariance. As mentioned before when deriving the asymptotic moments for the AR(1) process, the mean is not informative to estimate the parameter $b$. We can see that in the first row of Table 1 where we have that $\hat{\rho}_i$ for $i \in \{1, 2\}$ is far apart from the corresponding true value of 0.50. We observe the same pattern with $\hat{\sigma}_i$ and, also, we can see that the standard errors are large when using the mean and variance. Our estimates get better when we use the (ii) variance and the autocovariance and (iii) mean, variance and autocovariance; however, only when using all three moments, we cannot reject the null-hypothesis, at standard levels, of Sargan's that the over-identifying restrictions are valid.

| Moments | $\hat{\rho}_1$ | s.e.$(\hat{\rho}_1)$ | $\hat{\sigma}_1$ | s.e.$(\hat{\sigma}_1)$ | $\hat{\rho}_2$ | s.e.$(\hat{\rho}_2)$ | $\hat{\sigma}_2$ | s.e.$(\hat{\sigma}_2)$ | **J**-statistic | $p$-value |
|---|---|---|---|---|---|---|---|---|---|---|
| (i) | -0.97 | 0.93 | 0.3 | 4.7 | -0.99 | 0.13 | 0.19 | 1.51 | 0.00 | 0.00 |
| (ii) | 0.62 | 0.05 | 0.99 | 0.07 | 0.62 | 0.09 | 0.99 | 0.11 | 0.00 | 0.00 |
| (iii) | 0.62 | 0.05 | 0.99 | 0.07 | 0.63 | 0.09 | 1.00 | 0.09 | 0.02 | 0.12 |

Table 1: Estimates



(a) $1^{st}$ stage using $(\mu,\sigma^2)$

(b) $2^{nd}$ stage using $(\mu,\sigma^2)$

(c) $1^{st}$ stage using $(\sigma^2,\rho_{i,j})$

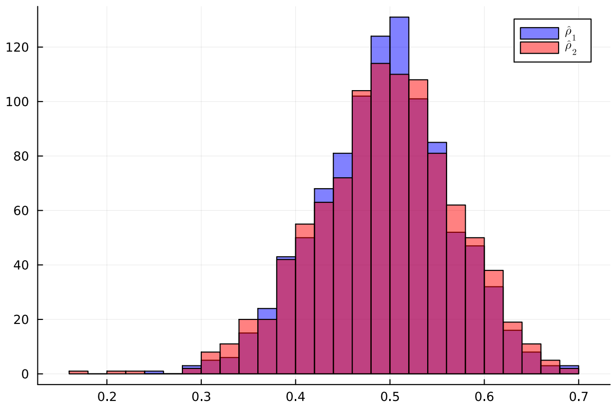
(d) $2^{nd}$ stage using $(\sigma^2,\rho_{i,j})$

(e) $1^{st}$ stage using $(\mu,\sigma^2,\rho_{i,j})$

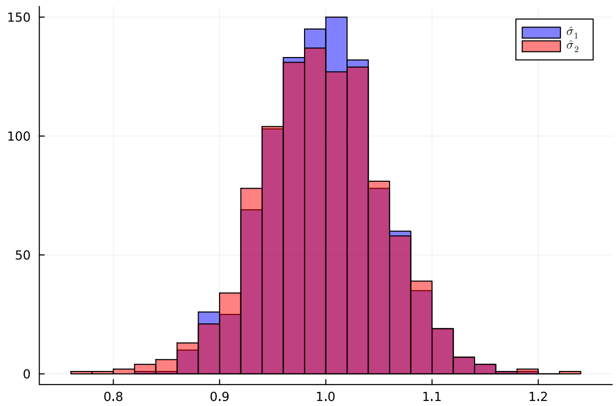
(f) $2^{nd}$ stage using $(\mu,\sigma^2,\rho_{i,j})$

Figure 2: Objective functions

Figure 2 display the corresponding surface plots for the objective function for both stages for each the identified and over-identified cases. Finally, Figure 3 displays the bootstrap of the finite sample distribution of the estimators $\hat{\rho}_i$ and $\hat{\sigma}_i$ using both stages $i \in \{1, 2\}$.




(a) Histogram $\hat{\rho}$




(b) Histogram $\hat{\sigma}$

Figure 3: Empirical distribution

# References


A. Michaelides and S. Ng. Estimating the Rational Expectations Model of Speculative Storage: A Monte Carlo Comparison of Three Simulation Estimators. *Journal of Econometrics*, 96 (2):231–266, 2000.