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CONNECTEDNESS OF THE HILBERT SCHEME

by Robin HARTSHORNE

CONTENTS

INTRODUCTION (1)

In this paper we study the Hilbert scheme of subschemes of projective space, as defined by Grothendieck [FGA, p. 221-01 ff] (2). Let S be a noetherian prescheme, let n be an integer, and let $p \in \mathbf{Q}[z]$ be a polynomial. Then the Hilbert scheme

$$\mathrm{H}^p = Hilb^p(\mathbf{P}^n_{\mathrm{S}}/\mathrm{S})$$

parametrizes subschemes of projective n-space over S, which are flat over every point of S. Our main theorem states that if S is connected (e.g. $\mathrm{S} = \mathrm{Spec}\, k$, k a field), then H^p is connected. Furthermore, we determine for which polynomials p, H^p is non-empty.

It develops in the course of the proof that all the deformations performed are linear; that is, they can be carried out over the affine line. Thus we have proved more: H^p is linearly connected.

It also appears that the Hilbert scheme is never actually needed in the proof. Therefore we define the notion of a connected functor, and prove that the functor $\mathbf{Hilb}^p$ is connected. (A representable functor is connected $\Leftrightarrow$ the prescheme representing it is connected.)

Chapter 1 contains preliminary material. Chapters 2, 3, and 4 study some special subschemes of projective space called fans, their deformations, and some numerical characters of subschemes. Chapter 5 contains the main theorem and its proof.

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(2) Numbers or letters in brackets refer to the bibliography.

CHAPTER I

PRELIMINARIES

Projective space.

If A is a ring, and $\mathfrak{a}$ a homogeneous ideal in $R=A[x_0, x_1, \ldots, x_r]$, then $\widetilde{\mathfrak{a}}$ [EGA, ch. II, 2.5] is a quasi-coherent sheaf of ideals on $X=\mathbf{P}^r_A$, and so defines a closed subscheme which we will call $V(\mathfrak{a})$. Conversely, if $Y \subseteq X$ is a closed subscheme defined by a sheaf of ideals $\mathscr{I}_Y$, then $\Gamma_*(\mathscr{I}_Y)$ [EGA, ch. II, 2.6] is a homogeneous ideal of R, which we will call $I(Y)$. Thus we have a correspondence between homogeneous ideals in R and closed subschemes of X which, however, is not one-to-one. What one can say is this: for any closed subscheme Y of X, $V(I(Y))=Y$; for any homogeneous ideal $\mathfrak{a}$ in R, $\mathfrak{a} \subseteq I(V(\mathfrak{a}))$, and there is equality if and only if no associated prime ideal of $\mathfrak{a}$ contains the " irrelevant " prime ideal $R_+=(x_0, x_1, \ldots, x_r)$. If Y is a closed subscheme of X, we will speak of $I(Y)$ as *the* ideal of Y.

Hilbert polynomials.

We recall the definition and elementary properties from [EGA, ch. III, § 2.5]. Let k be a field, let X be a projective scheme over k, and let F be a coherent sheaf on X. For each $n \in \mathbf{Z}$, define

$$\chi_n(F) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, F(n)).$$

Then there is a polynomial $p(z) \in \mathbf{Q}[z]$, called the *Hilbert polynomial* of F, such that $p(n)=\chi_n(F)$ for all $n \in \mathbf{Z}$. It is a polynomial with positive leading coefficient. Its degree is equal to the dimension of the support of F. It is zero if and only if F is zero.

If
$$0 \to F' \to F \to F'' \to 0$$
is an exact sequence of coherent sheaves on X, the Hilbert polynomials add: $p(F)=p(F')+p(F'')$. Thus the Hilbert polynomial is actually a function on the Grothendieck group $K(X)$ of coherent sheaves on X (see [2, § 4]).

If F is a coherent sheaf on X, and $k \subseteq k'$ is a base field extension, then the Hilbert polynomial of the extended sheaf $F_{k'}$ on $X_{k'}$ is equal to the Hilbert polynomial of F.

If F is a coherent sheaf on X with Hilbert polynomial $p(z)$, then for all large enough $n \in \mathbf{Z}$,

$$p(n) = \dim_k H^0(X, F(n)),$$

by Serre's theorem [EGA, ch. III, thm. 2.2.1]. If M is any graded module over $R=k[x_0, x_1, \ldots, x_r]$ such that $\widetilde{M}=F$, then for all large enough $n \in \mathbf{Z}$,

$$p(n)=\dim_k M_n.$$

Definition. — *If* $p_1(z)$, $p_2(z)$ *are polynomials in* $\mathbf{Q}[z]$, *we say* $p_1(z) \leq p_2(z)$ *if for all large enough* $n \in \mathbf{Z}$, $p_1(n) \geq p_2(n)$.

Lemma **(1.1)**. — *Let* $f: X \rightarrow Y$ *be a projective morphism, with* Y *locally noetherian, let* F *be a coherent sheaf on* X, *and let* y *be a point of* Y. *Then there is an* $n_0 \in \mathbf{Z}$, *such that for* $n \geq n_0$,

$$f_*(F(n) \otimes \boldsymbol{k}(y))=f_*(F(n)) \otimes \boldsymbol{k}(y).$$

Proof. — If F is flat over Y, the result follows from [EGA, III, 7.9.9] and Serre's theorem [EGA, III, 2.2.1]. For the general case, we may assume Y affine. By embedding X in a projective space $\mathbf{P}_Y^n$, we reduce to the case $X=\mathbf{P}_Y^n$. Then there are coherent sheaves L_0, L_1, on X, flat over Y, and an exact sequence

$$L_1 \rightarrow L_0 \rightarrow F \rightarrow 0.$$

Applying Serre's theorem to this sequence and to the exact sequence

$$L_1 \otimes \boldsymbol{k}(y) \rightarrow L_0 \otimes \boldsymbol{k}(y) \rightarrow F \otimes \boldsymbol{k}(y) \rightarrow 0,$$

we find that for large enough n, the sequences

$$f_*(L_1(n)) \rightarrow f_*(L_0(n)) \rightarrow f_*(F(n)) \rightarrow 0$$

and

$$f_*(L_1(n) \otimes \boldsymbol{k}(y)) \rightarrow f_*(L_0(n)) \otimes \boldsymbol{k}(y)) \rightarrow f_*(F(n) \otimes \boldsymbol{k}(y)) \rightarrow 0$$

are exact. Now applying the result to L_0 and L_1 (which are flat over Y), and using the five-lemma, we complete the proof.

Theorem **(1.2)**. — *Let* $f: X \rightarrow Y$ *be a projective morphism, with* Y *locally noetherian, and let* F *be a coherent sheaf on* X. *If* F *is flat over* Y, *then the function*

$$y \rightsquigarrow p(F_y)$$

which associates to each point $y \in Y$ *the Hilbert polynomial of the restriction of* F *to the fibre of* X *at* y, *is a locally constant function on* Y. *The converse is true if* Y *is integral.*

Proof. — The first statement is [EGA, III, 7.9.11]. For the converse, we reduce to the case where Y is the spectrum of a local noetherian domain A, with residue field k and quotient field K. We wish to show that F is flat over A$\Leftrightarrow$for all large enough $n \in \mathbf{Z}$,

$$\dim_k H^0(X_k, F_k(n))=\dim_K H^0(X_K, F_K(n)).$$

By the Lemma, this condition is equivalent to the condition that for all large enough $n \in \mathbf{Z}$,

$$\dim_k M_n \otimes k=\dim_K M_n \otimes K,$$

where $$M_n = H^0(X, F(n)).$$

But since A is a local noetherian domain and M_n is of finite type, this is equivalent to saying that M_n is free over A for all large enough $n \in \mathbf{Z}$ [3, ch. II, § 3, no. 2, Prop. 7]. This is equivalent to saying F is flat over Y, by [EGA, III, 7.9.14].

Flatness over a non-singular curve.

Proposition (**1.3**). — *Let* $f: X \to Y$ *be a morphism of locally noetherian preschemes, where* Y *is a non-singular curve* (i.e. *a non-singular noetherian scheme of dimension one*). *Let* F *be a quasi-coherent sheaf on* X. *Then* F *is flat over* Y *if and only if no associated prime cycle of* F *on* X *lies over a closed point of* Y.

Proof. — Since the question is local on X and Y, we reduce immediately to the case where Y is the spectrum of a discrete valuation ring A, X is the spectrum of a ring B, and $F = \widetilde{M}$, where M is a B-module. Since A is a discrete valuation ring, M is flat$\Leftrightarrow$it is torsion-free [3, ch. I, Prop. 3, p. 29], and for that it is sufficient to check that a generator t of the maximal ideal of A is not a zero-divisor in M. For that, it is necessary and sufficient that the image of t in B be contained in no associated prime of M [3, ch. IV, Cor. 2, p. 132]. But for a prime ideal $\mathfrak{p}$ of B to contain the image of t is the same as for its restriction to A to be the maximal ideal, i.e. for $\mathfrak{p}$ to lie over the closed point.

Proposition (**1.4**). — *Let* $f: X \to Y$ *be a morphism of locally noetherian preschemes, where* Y *is a non-singular curve. Let* U *be a dense open subset of* Y, *and let* Z *be a closed subprescheme of* $f^{-1}(U)$, *flat over* U. *Then there exists a unique closed subprescheme* $\overline{Z}$ *of* X, *flat over* Y, *whose intersection with* $f^{-1}(U)$ *is* Z. *Moreover,* $\overline{Z}$ *depends functorially on* X, Y, f, U, Z *(meaning that if* X′, Y′, f', U′, Z′ *is another such quintuple, and there are compatible maps* $X \to X'$, $Y \to Y'$, etc., *then there is a unique map* $\overline{Z} \to \overline{Z}'$ *compatible with the other maps).*

Proof (compare [FGA, Lemma 3.7, p. 221-16]). — Let $\mathscr{I}_Z$ be the sheaf of ideals of Z on $f^{-1}(U)$. Let $\overline{\mathscr{I}}$ be the largest subsheaf of $\mathcal{O}_X$ whose restriction to $f^{-1}(U)$ is $\mathscr{I}_Z$ [EGA, ch. I, 9.4.2]. Then $\overline{\mathscr{I}}$ defines a closed subprescheme $\overline{Z}$ of X. Reducing to the case where Y is the spectrum of a discrete valuation ring, one sees easily that $\overline{Z}$ is unique, and has the desired properties. Its construction is clearly functorial.

Linear Connectedness.

Definition. — *Let* X *be a prescheme over a field* k, *and let* x, x' *be points of* X. *We say* x *specializes linearly to* x' *(written* $x \rightsquigarrow x'$*), if there exists an extension field* k_1 *of* k, *and a morphism* $f: \operatorname{Spec} k_1[t]_{(t)} \to X$, *which sends the generic point to* x *and the special point to* x'. *We say that two points* x, x' *of* X *can be connected by a sequence of linear specializations if there is a sequence of points*

$$x = x_1, x_2, \ldots, x_n = x'$$

of X *such that for each* i, *either* $x_i \rightsquigarrow x_{i+1}$ *or* $x_{i+1} \rightsquigarrow x_i$ *is a linear specialization.*

Definition. — *A prescheme* X *over* k *is linearly connected if any two points can be connected by a sequence of linear specializations.*

Remark. — If $f:X\rightarrow Y$ is a surjective morphism of preschemes over k, and if X is linearly connected, then so is Y. More generally, a necessary and sufficient condition that Y be linearly connected is that whenever y, y' are points of Y, then there exist linearly connected preschemes $X_1, \ldots, X_s$ over k, and k-morphisms $f_i: X_i \rightarrow Y$ such that $y \in f_1(X_1)$, $y' \in f_s(X_s)$, and for each i, $f_i(X_i) \cap f_{i+1}(X_{i+1}) \neq \emptyset$.

Definition. — *A rational curve over a field* k *is a one-dimensional integral scheme of finite type over* k, *whose function field is a pure transcendental extension of* k.

Definition. — *Let* X *be a prescheme over* k. *Two points* x_1 *and* x_2 *of* X *are said to be connected by a rational curve if there exists an extension field* k_1 *of* k, *a rational curve* Y *over* k_1, *a morphism* $f: Y\rightarrow X$, *and points* $y_1, y_2 \in Y$, *rational over* k_1, *such that* $f(y_1)=x_1$ *and* $f(y_2)=x_2$. *We say that points* x *and* x' *of* X *can be connected by a sequence of rational curves if there is a sequence*

$$x=x_1, x_2, \ldots, x_n=x'$$

of points of X *such that for each* i, x_i *and* x_{i+1} *can be connected by a rational curve.*

Lemma (**1.5**). — *A rational curve* X *over a field* k *is linearly connected.*

Proof. — It will be sufficient to show that if x is the generic point of X, and x' is any closed point, then $x \rightsquigarrow x'$ is a linear specialization. In the first place, by making a finite base field extension and taking an irreducible component of the lifted curve, we reduce to the case where x' is rational over k. Passing to the normalization of X, we reduce to the case where X is non-singular [EGA, ch. II, 7.4.5]. But a non-singular rational curve over k is locally isomorphic to $\mathbf{P}^1_k$ [EGA, ch. II, 7.5.16], and so the local ring of any rational point is isomorphic to $k[t]_{(t)}$, and we are done.

Proposition (**1.6**). — *Let* X *be a prescheme over* k. *If any two points of* X *can be connected by a sequence of rational curves, then* X *is linearly connected. The converse is true if* X *is of finite type over* k.

Proof. — The first statement follows from the Lemma and the remark above. For the converse, suppose X is of finite type over k. It is sufficient to prove that if $x \rightsquigarrow x'$ is a linear specialization in X, then x and x' can be connected by a rational curve. Moreover, for that, we can assume that X is affine, say $X = \operatorname{Spec} B$, where $B = k[u_1, \ldots, u_n]$ is an algebra of finite type over k.

Since $x \rightsquigarrow x'$ is a linear specialization, there is an extension field k_1 of k, and a morphism

$$f: \operatorname{Spec} k_1[t]_{(t)} \rightarrow X$$

which sends the generic point to x and the special point to x'. Let $A = k_1[t]_{(t)}$, and denote also by f the corresponding homomorphism of rings $f: B\rightarrow A$. Let

$A_0 = k_1[t, f(u_1), \ldots, f(u_n)]$. Then Spec A_0 is a rational curve over k_1, and there is a morphism Spec $A_0 \to X$ which sends the generic point to x and a closed point, rational over k_1, to x'.

Thus, replacing X by Spec A_0, we reduce to the case where X is a rational curve over k, x is the generic point, and x' is a closed point, rational over k. Let $k_1 = k(t)$, where t is an indeterminate, and let $Y = X_{k_1}$. Then Y is a rational curve over k_1, since k_1 is a regular field extension of k [9, ch. III], and there are closed points y, y' of Y, rational over k_1, which map to x, x', respectively [EGA, ch. I, 3.4.9]. Thus x and x' can be joined by a rational curve.

Proposition (**1.7**). — *Let k be a field. Then any open subset of $\mathbf{P}^r_k$ is linearly connected. The scheme* Spec $k[x_1, \ldots, x_r]_{\mathfrak{m}}$ *is linearly connected (where $x_1, \ldots, x_r$ are indeterminates, and $\mathfrak{m}$ is the maximal ideal $(x_1, \ldots, x_r)$).*

Proof. — Using the remark above, it is sufficient to prove the latter assertion. So let $B = k[x_1, \ldots, x_r]_{\mathfrak{m}}$. It will be sufficient to show that any point in Spec B can be connected by a sequence of linear specializations to the closed point $\mathfrak{m}$.

Let $\mathfrak{p} \subseteq B$ be any prime ideal. Let $k_1 = k(\mathfrak{p})$, and let $f : B \to k_1$ be the canonical homomorphism. Let t be an indeterminate, and define a homomorphism

$$g : k[x_1, \ldots, x_r] \to k_1[t]$$

by $g(x_i) = tf(x_i)$. This homomorphism extends by localization to homomorphisms

$$g_0 : B \to k_1[t]_{(t)}$$

and

$$g_1 : B \to k_1[t]_{(t-1)}.$$

Now if we let $\mathfrak{q}$ be the ideal generated by the homogeneous rational functions in $\mathfrak{p}$, then $\mathfrak{q}$ is prime, and g_0, g_1 describe linear specializations $\mathfrak{q} \rightsquigarrow \mathfrak{m}$ and $\mathfrak{q} \rightsquigarrow \mathfrak{p}$, respectively. Thus $\mathfrak{p}$ can be connected to $\mathfrak{m}$ by a sequence of two linear specializations.

Definition. — *A prescheme* X *over a field k is geometrically connected if for every extension field k' of k, $X_{k'}$ is connected.*

Definition. — *A morphism $f : X \to Y$ of preschemes is connected if*

a) *the fibres of f are geometrically connected, and*

b) *f is universally submersive.* (A morphism is said to be *submersive* if it is surjective, and the image space has the quotient topology. It is *universally submersive* if it is submersive, and remains so after any base extension.)

Remarks. — 1. This definition is stronger than the one given in [EGA, IV, 4.5.5].

2. For a morphism $f : X \to Y$ of preschemes to be universally submersive, it is sufficient that any one of the three following conditions hold:

1) f has a section;

2) f is surjective and proper;

3) f is surjective, flat, and quasi-compact.

Proposition (**1.8**). — *Let* $f: X \to Y$ *be a connected morphism of preschemes, and let* $Y' \to Y$ *be a base extension. If* Y' *is connected* (resp. *geometrically connected*), *then so is* $X' = X \times_Y Y'$.

Proof. — Since the conditions *a)* and *b)* are stable under base extension, we are reduced to proving that if Y is connected and the fibers are connected then X is connected. This follows immediately from *b)*.

We leave the reader the definition of a *geometrically linearly connected* prescheme, the definition of a *linearly connected* morphism of preschemes, and the statement and proof of a proposition about linearly connected morphisms, analogous to Proposition 1.8.

Connected Functors.

This section is a variation on the theme " anything you can do with preschemes, you can do with the functors they represent ". Here is the situation: If X is a prescheme, we define (see [EGA, ch. 0_{III}, § 8] and [FGA, p. 195-01 ff]) a contravariant functor h_X from the category of preschemes, (***Prsch***), to the category of sets, (***Sets***), by setting

$$h_X(Y) = \mathrm{Hom}(Y, X)$$

for each prescheme Y. Then $h: X \rightsquigarrow h_X$

is a covariant functor from (***Prsch***) to the category (***Fun***) of contravariant functors from (***Prsch***) to (***Sets***). h is a fully faithful functor, that is, it gives en equivalence of (***Prsch***) with a full subcategory of (***Fun***). A functor $F \in \mathrm{Ob}(\boldsymbol{Fun})$ is called *representable* if it lies in this full subcategory, i.e. if there exists a prescheme X and an isomorphism of functors $\xi: h_X \xrightarrow{\sim} F$. In that case the pair (X, ξ), or simply X, is said to *represent* F.

The general philosophy is that definitions, theorems, and their proofs can be extended from (***Prsch***) to the larger category (***Fun***). In this case, we take four ways of defining a connected prescheme, and show how each can be extended to functors. The four definitions are: X is connected if

1. X is not the union of two disjoint non-empty open subsets.
2. Any morphism of X into a disjoint sum of two preschemes has its image in one or the other.
3. Any two points in X can be connected by a sequence of images in X of connected preschemes.
4. Any two points in X can be joined by a sequence of generizations and specializations in X.

The usual proof shows that the four definitions, applied to functors, are equivalent (at least for the category of locally noetherian preschemes). The details are mostly a matter of notation, so we will give only indications of proof.

If F is a representable functor, we may denote by $|F|$ the prescheme which represents it. If F is a functor, a *prescheme over* F is a pair (X, ξ) (sometimes written simply X), where X is a prescheme, and $\xi \in F(X)$. A *morphism* $(X, \xi) \to (Y, \eta)$ of preschemes over F is a morphism $X \to Y$ such that the map $F(Y) \to F(X)$ sends η to ξ.

We recall [8] that a morphism $u : F \to G$ of functors is called *relatively representable* (resp. *relatively representable by open immersions*) if for every prescheme X over G, the fibred product $X \times_G F$ is representable (resp. representable by an open subprescheme of X). (Note: Since the sets F(X) and $\mathrm{Hom}(h_X, F)$ are canonically identified, to give a prescheme X over F is the same as to give a morphism of functors $h_X \to F$. So if X is a prescheme over G, what we really mean by $X \times_G F$ is $h_X \times_G F$.) If G is representable, then the morphism $u : F \to G$ is relatively representable if and only if F is representable.

Definition. — *A family* $\{f_i : F_i \to F\}$ *of morphisms of functors in* (***Fun***) *is collectively surjective if for every field* K *over* F (*meaning of course,* " *prescheme* Spec K *over* F ") *at least one of the fibred products* $F_i \times_F K$ *is non-empty.* (*We say that a functor* F *is empty if* $F(Y) = \varnothing$ *for every* Y.)

If the functors F_i, F are all representable, then a family $\{f_i : F_i \to F\}$ of morphisms is collectively surjective if and only if $|F| = \bigcup f_i(|F_i|)$.

Definition. — *Let* F_1, F_2 *be functors in* Ob(***Fun***). *We define their disjoint sum,* $F_1 \amalg F_2$ *to be the functor*

$$Y \rightsquigarrow \{(\alpha, \eta_1, \eta_2) \mid \alpha \textit{ is a representation } Y = Y_1 \amalg Y_2 \textit{ of } Y$$

$$\textit{as a disjoint sum of subpreschemes, and } \eta_1 \in F_1(Y_1),\ \eta_2 \in F_2(Y_2)\}.$$

Z Note that $F_1 \amalg F_2$ is in general not the direct sum in the categoric sense of F_1 and F_2.

If $F = F_1 \amalg F_2$, then F is representable if and only if F_1 and F_2 are both representable, and in that case $|F| = |F_1| \amalg |F_2|$.

Proposition (**1.9**). — *Let* $F \in \mathrm{Ob}(\textbf{\textit{Fun}})$ *be a functor restricted to the category of locally noetherian preschemes. Then the following conditions are equivalent.*

(i) *Whenever* $u_1 : U_1 \to F$ *and* $u_2 : U_2 \to F$ *are morphisms of functors, relatively representable by open immersions, such that* $U_1 \times_F U_2 = \varnothing$, *and* u_1, u_2 *are collectively surjective, then either* $U_1 = \varnothing$ *or* $U_2 = \varnothing$.

(ii) *Whenever* $f : F \to G_1 \amalg G_2$ *is a morphism of* F *into a disjoint sum of functors* G_1, G_2, *then* f *factors through one of the canonical inclusions* $G_1 \to G_1 \amalg G_2$ *or* $G_2 \to G_1 \amalg G_2$.

(iii) *Whenever* X, X′ *are non-empty connected preschemes over* F, *there exists a sequence*

$$X = X_1, X_2, \ldots, X_n = X'$$

of non-empty connected preschemes over F, *such that for each i, there is a morphism either* $X_i \to X_{i+1}$ *or* $X_{i+1} \to X_i$ *of preschemes over* F.

(iv) *Whenever* X, X′ *are spectra of fields over* F, *there exists a sequence*

$$X = X_1, X_2, \ldots, X_n = X'$$

of preschemes X_i *over* F *and morphisms* $X_i \to X_{i+1}$ *or* $X_{i+1} \to X_i$ *over* F *for each i, such that*

a) *each* X_i *is the spectrum of a local domain and*

b) *each morphism* $X_i \to X_{i+1}$ *or* $X_{i+1} \to X_i$ *comes from either a field extension, or the map of a local domain onto its residue field, or the map of a local domain into its quotient field.*

Proof (in outline). — (i) $\Rightarrow$ (ii) Given $f: \mathrm{F} \to \mathrm{G}_1 \amalg \mathrm{G}_2$ as in (ii), define a functor U_1 as the fibred product $\mathrm{F} \times_{\mathrm{G}_1 \amalg \mathrm{G}_2} \mathrm{G}_1$. Define U_2 similarly, and apply (i).

(ii) $\Rightarrow$ (iii) Given (X, ξ) a non-empty connected prescheme over F, define a functor G_1 as follows. If Y is connected,

$$\mathrm{G}_1(\mathrm{Y}) = \{(\mathrm{Y}, \eta) \mid \eta \in \mathrm{F}(\mathrm{Y}), \text{ and } (\mathrm{Y}, \eta) \text{ can be joined to } (\mathrm{X}, \xi) \text{ as in condition (iii)}\}.$$

Otherwise, $\mathrm{G}_1(\mathrm{Y}) = \prod \mathrm{G}_1(\mathrm{Y}_i)$, where $\mathrm{Y} = \amalg \mathrm{Y}_i$ is the decomposition of Y into its connected components. (Note that the connected components of a locally noetherian prescheme are open, and hence the prescheme is the sum of its connected components.) Define G_2 similarly, by taking those (Y, η) which cannot be joined to (X, ξ). Apply (ii) and deduce that any (Y, η), whose Y is connected, can be joined to (X, ξ).

(iii) $\Rightarrow$ (iv) Use the fact that any two points in a locally noetherian connected prescheme can be joined by a sequence of generizations and specializations.

(iv) $\Rightarrow$ (iii) obvious.

(iii) $\Rightarrow$ (i) Given U_1 and U_2 as in condition (i), suppose that for some connected prescheme Y over F, $\mathrm{U}_1(\mathrm{Y}) \neq \emptyset$. Then use (iii) to show that for every connected prescheme Z over F, $\mathrm{U}_2(\mathrm{Z}) = \emptyset$, and hence $\mathrm{U}_2 = \emptyset$.

Definition. — A functor $\mathrm{F} \in \mathrm{Ob}(\boldsymbol{Fun})$ *satisfying the conditions* (i), (ii), (iii) *of Proposition* 1.9 *is said to be connected.*

Note that if F is representable, then F is connected if and only if the prescheme representing it is connected.

Proposition **(1.10)**. — *Let* $\mathrm{F} \in \mathrm{Ob}(\boldsymbol{Fun})$ *be a functor defined on preschemes over a field k. Then the following conditions are equivalent:*

(iii lin) *Whenever* X, X′ *are linearly connected preschemes over* F, *there exists a sequence*

$$\mathrm{X} = \mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n = \mathrm{X}'$$

of non-empty linearly connected preschemes over F, *such that for each i, there is a morphism either* $\mathrm{X}_i \to \mathrm{X}_{i+1}$ *or* $\mathrm{X}_{i+1} \to \mathrm{X}_i$ *of preschemes over* F.

(iv lin) *Whenever* X, X′ *are spectra of fields over* F, *there exists a sequence*

$$\mathrm{X} = \mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n = \mathrm{X}'$$

of preschemes X_i *over* F, *and morphisms* $\mathrm{X}_i \to \mathrm{X}_{i+1}$ *or* $\mathrm{X}_{i+1} \to \mathrm{X}_i$ *over* F *for each i, such that*

a) *each* X_i *is the spectrum of a field or of a discrete valuation ring of the form* $k_1[t]_{(t)}$, *where* k_1 *is an extension field of k, and*

b) *each morphism* $\mathrm{X}_i \to \mathrm{X}_{i+1}$ *or* $\mathrm{X}_{i+1} \to \mathrm{X}_i$ *comes from a field extension, or the map of a discrete valuation ring to its residue field or its quotient field.*

Proof. — The proof is similar to the proof of Proposition 1.9, and is left to the reader.

Definition. — A functor $\mathrm{F} \in \mathrm{Ob}(\boldsymbol{Fun})$ *satisfying the equivalent conditions of Proposition* 1.10 *is said to be linearly connected.*

Note that a linearly connected functor is connected. Note also that if F is a representable functor, then F is linearly connected if and only if the prescheme representing it is linearly connected.

We leave to the reader the definition of a *geometrically connected functor* (resp. *geometrically linearly connected functor*) on $(\boldsymbol{Prsch}/k)$; the definition of a *connected morphism of functors* (resp. *linearly connected morphism of functors*); and the statement and proof of a proposition analogous to Proposition 1.8 for these types of morphisms of functors.

Hilbert Schemes.

Let $f:X\to S$ be a morphism of preschemes. For each prescheme S' over S we define (see [FGA, p. 221-17])

$$\mathbf{Hilb}_{X/S}(S')$$

to be the set of closed subpreschemes of $X'=X\times_S S'$, flat over S'. If S'' is another prescheme over S, and $S''\to S'$ is a morphism over S, then we define a map

$$\mathbf{Hilb}_{X/S}(S')\to\mathbf{Hilb}_{X/S}(S'')$$

by

$$Z\rightsquigarrow Z\times_{S'}S''.$$

Note that flatness is preserved under base entension. Thus $\mathbf{Hilb}_{X/S}$ appears as a contravariant functor from the category $(\boldsymbol{Prsch}/S)$ of preschemes over S to $(\boldsymbol{Sets})$.

Now suppose that X is projective over S locally noetherian, and let $p=p(z)$ be a polynomial in $\mathbf{Q}[z]$. We define

$$\mathbf{Hilb}^p_{X/S}(S')$$

to be the subset of $\mathbf{Hilb}_{X/S}(S')$ consisting of those subpreschemes Z of X', flat over S', such that for every $s'\in S'$, the fibre $Z_{s'}$ of Z over s' has Hilbert polynomial p. Since the Hilbert polynomial is stable under base field extension, this defines a subfunctor $\mathbf{Hilb}^p_{X/S}$ of $\mathbf{Hilb}_{X/S}$. By virtue of Theorem 1.2, we see that

$$\mathbf{Hilb}_{X/S}=\coprod_{p\in\mathbf{Q}[z]}\mathbf{Hilb}^p_{X/S},$$

using the notation of the previous section, and so the functor $\mathbf{Hilb}_{X/S}$ is representable if and only if the functors $\mathbf{Hilb}^p_{X/S}$ are all representable, and in that case the prescheme representing the former is the disjoint sum of the preschemes representing the latter.

The larger part of the séminaire Bourbaki, exposé 221 [FGA, p. 221-1 to p. 221-28] is devoted to the proof of the following theorem.

Theorem* **(1.11) (Grothendieck) [FGA, thm. 3.2, p. 221-12]. — *Let* $f:X\to S$ *be a projective morphism of preschemes, with* S *noetherian. Then for each polynomial* $p\in\mathbf{Q}[z]$, *the functor* $\mathbf{Hilb}^p_{X/S}$ *is representable by a prescheme* $Hilb^p(X/S)$, *projective over* S. *Hence the functor* $\mathbf{Hilb}_{X/S}$ *is also representable, and is represented by the disjoint sum of the preschemes* $Hilb^p(X/X)$.

We call the prescheme $Hilb^p(X/S)$ the *Hilbert scheme of* X *over* S *with Hilbert polynomial* p.

Note that since $H = Hilb^p(X/S)$ represents a functor, it comes equipped with a canonical element $\theta \in \mathbf{Hilb}^p_{X/S}(H)$, i.e. a canonical subprescheme Z of $X_H = X \times_S H$, flat over H. Restating the definition of the Hilbert scheme, we see that the pair (H, Z) is characterized by the following universal property: Whenever S′ is a prescheme over S, and Y is a closed subprescheme of $X_{S'} = X \times_S S'$, flat over S′, and with the Hilbert polynomials of all the fibres $Y_{s'}$ equal to p, then there exists a unique S-morphism $g : S' \to H$, such that $Y = Z \times_H S'$.$_*$

Remark. — Since all our proofs are independent of the existence of the Hilbert scheme, we will surround by asterisks $^* \dots _*$ every passage which dependes on their existence.

Specialization of Subpreschemes.

The consideration of connected preschemes over the functor $\mathbf{Hilb}_{X/S}$ leads us to make the following definitions.

If X is a prescheme over S, k a field, and $\operatorname{Spec} k \to S$ a morphism, we call the product $X_k = X \times_S \operatorname{Spec} k$ a *generalized fibre* of X over S.

Definition. — Let X *be a prescheme over* S, *and let* $Z_1 \subseteq X_{k_1}$ *and* $Z_2 \subseteq X_{k_2}$ *be closed subpreschemes of generalized fibres of* X *over* S. *We say* Z_1 *specializes to* Z_2 *(written* $Z_1 \rightsquigarrow Z_2$*) if either*

a) Z_1 *is obtained from* Z_2 *by a base field extension* $k_2 \subseteq k_1$, *or*

b) *there exists a local domain* A, *with quotient field* k_1 *and residue field* k_2, *a morphism* $\operatorname{Spec} A \to S$, *and a closed subprescheme* Z *of* $X_A = X \times_S \operatorname{Spec} A$, *flat over* A, *whose fibre over the generic point of* Spec A *is* Z_1, *and whose fibre over the closed point of* Spec A *is* Z_2.

If moreover S *is a prescheme over a field* k, *we say* Z_1 *specializes linearly to* Z_2 *if either*

a) *as above, or*

b) *as above but with the additional requirement that* Spec A *be linearly connected.*

Definition. — Let X *be a prescheme over* S. *A connected sequence of specializations in* X *is a sequence*

$$Z_1, Z_2, \dots, Z_n,$$

where for each i, Z_i *is a closed subprescheme of a generalized fibre* X_{k_i} *of* X *over* S, *and where for each* i, *either* Z_i *specializes to* Z_{i+1}, *or vice versa. Similarly, if* S *is a prescheme over a field* k, *one defines a connected sequence of linear specializations in* X.

Examples. — 1. Let X be a prescheme over S, let Y be a connected prescheme over S, and let Z be a closed subprescheme of $X \times_S Y$, flat over Y. If y', y'' are any two points of Y, and Z′, Z″ are the projections on X of the fibres of Z over y', y'', respectively, then Z′ and Z″ can be joined by a connected sequence of specializations

$$Z' = Z_1, Z_2, \dots, Z_n = Z''$$

in X. Indeed, one need only join y' to y'' by a sequence of generizations and specializations in Y, and take for the Z_i the projections on X of the fibres of Z over the intervening points of Y.

Similarly, if Y is linearly connected, Z' can be joined to Z'' by a connected sequence of linear specializations in X.

2. Let X be a prescheme over a field k, and let G be a connected group prescheme over k, acting on X. Let Z be a closed subprescheme of X, let g be a point of G with values in a field k' over k, and let $Z'=Z^g$ be the image in $X_{k'}$ of $Z_{k'}$ under the action of g. Then Z and Z' can be joined by a connected sequence of specializations in X.

Similarly, if G is linearly connected (as for example $\mathbf{G}_a$, $\mathbf{G}_m$, $\mathbf{GL}(n)$), then Z can be joined to Z' by a connected sequence of linear specializations in X.

Proposition **(1.12)**. — *Let* X *be projective over* S *locally noetherian, and let* $p \in \mathbf{Q}[z]$ *be a polynomial. Then the functor* $\mathbf{Hilb}^p_{X/S}$ *is connected* (resp. *linearly connected*) *if and only if whenever* $Z' \subseteq X_{k'}$ *and* $Z'' \subseteq X_{k''}$ *are closed subpreschemes of generalized fibres of* X, *with Hilbert polynomial* p, *then there exists a connected sequence of specializations* (resp. *linear specializations*)

$$Z' = Z_1, Z_2, \ldots, Z_n = Z''$$

in X.

Proof. — This follows immediately from the definition of the functor $\mathbf{Hilb}^p_{X/S}$, and from the criteria (iii), (iv), of Proposition 1.9 (resp. criteria (iii lin), (iv lin) of Proposition 1.10).

CHAPTER 2

THE INTEGERS n_i

In this chapter we associate with every coherent sheaf F on $\mathbf{P}^r_k$ integers $n_r(\mathrm{F}), n_{r-1}(\mathrm{F}), \ldots, n_0(\mathrm{F})$. The integer $n_i(\mathrm{F})$ measures the sections of F whose support is of dimension i. (Recall that the support of a section s of an abelian sheaf F on a topological space X is the set of $x \in \mathrm{X}$ such that the image s_x of s in the stalk F_x of F at x is non-zero. The support is always a closed subset of X.) If $\mathrm{F}=\mathcal{O}_\mathrm{Z}$, where $\mathrm{Z} \subseteq \mathbf{P}^r_k$ is an integral subscheme of dimension q, then $n_i(\mathrm{F})=0$ for $i \neq q$, and $n_q(\mathrm{F})$ is the degree of Z.

The Operations $\mathbf{F}^i$, $\mathbf{F}_i$.

Definition. — Let X *be a locally noetherian prescheme, and* F *a coherent sheaf on* X. *Define* $\mathrm{R}^i(\mathrm{F})$ *to be the subsheaf of* F *whose sections over an open set* U *are those sections of* F *over* U *whose support has codimension* $\geqslant i$. *Define* F^i *to be* $\mathrm{F}/\mathrm{R}^i(\mathrm{F})$.

Remarks. — 1. Recall that for Y a closed subset of X, the *codimension* of Y in X is the minimum, taken over pairs Y_i, X_i where Y_i is an irreducible component of Y, X_i is an irreducible component of X, and $\mathrm{Y}_i \subseteq \mathrm{X}_i$, of $\mathrm{codim}(\mathrm{Y}_i, \mathrm{X}_i)$. To see that $\mathrm{R}^i(\mathrm{F})$ is a subsheaf of F, one has only to remark:

a) when a section s of F is multiplied by a section f of $\mathcal{O}_\mathrm{X}$, then $\mathrm{Supp}\, fs \subseteq \mathrm{Supp}\, s$, and

b) if $\mathrm{U}' \subseteq \mathrm{U}$ are open sets, and Y is a closed subset of U, then

$$\mathrm{codim}(\mathrm{Y} \cap \mathrm{U}', \mathrm{U}') \geqslant \mathrm{codim}(\mathrm{Y}, \mathrm{U}).$$

2. R^i is a left exact functor.

3. The associated prime cycles of F^i are just those associated prime cycles of F whose codimension is $<i$, and $\mathrm{R}^i(\mathrm{F})$ is the smallest subsheaf of F such that every associated prime cycle of the quotient has codimension $<i$ [see EGA, IV, § 3.1 for associated prime cycles].

4. We say that F^i is obtained by " throwing away components of codimension $\geqslant i$ ". In fact, if $\mathrm{F}=\mathcal{O}_\mathrm{Z}$, where $\mathrm{Z} \subseteq \mathrm{X}$ is a subscheme of X without embedded components, this is literally the case.

5. If $j \leqslant i$, then $(\mathrm{F}^i)^j = (\mathrm{F}^j)^i = \mathrm{F}^j$.

Definition. — If $\mathrm{X}=\mathbf{P}^r_k$ *is a projective space over a field* k, *and* F *is a coherent sheaf on* X, *we define* $\mathrm{R}_i(\mathrm{F})$ *to be* $\mathrm{R}^{r-i}(\mathrm{F})$ *and* F_i *to be* $\mathrm{F}/\mathrm{R}_i(\mathrm{F})=\mathrm{F}^{r-i}$. *Similarly if* Z *is a closed* *subscheme of* X, *then for any* i, $\mathcal{O}_Z^i$ *is the structure sheaf of a closed subscheme of* Z *which we denote by* Z^i *or* Z_{r-i}.

Proposition **(2.1)**. — *The operation* $F \rightsquigarrow F^i$ *is a functor from coherent sheaves on* X *to coherent sheaves on* X. *It takes injections into injections, and surjections into surjections, but is not semi-exact.*

Proof. — The statements are mostly obvious. Observe that if $f: F \to G$ is a homomorphism of sheaves, then f maps $R^i(F)$ into $R^i(G)$, since the codimension of a section can at most increase. Furthermore, if $F \subseteq G$, then $R^i(F) = F \cap R^i(G)$. Thus the functor $F \rightsquigarrow F^i$ takes injections into injections.

To see that the functor is not semi-exact, take any exact sequence of the form

$$0 \to \mathcal{I}_{Z'} \to \mathcal{O}_Z \to \mathcal{O}_{Z'} \to 0$$

where Z is an irreducible subscheme of X and $Z' \subseteq Z$ is a subscheme of Z of lower dimension.

Proposition **(2.2)**. — *Let* X *be a projective space* $\mathbf{P}_k^r$, *and let* F *be a coherent sheaf on* X. *Let* K *be an extension field of* k, *and let* X_K, F_K *be obtained by base extension. Then for each* i,

$$R^i(F_K) = R^i(F) \otimes K \quad \textit{and} \quad (F_K)^i = F^i \otimes K.$$

In other words, formation of $R^i(F)$ and F^i commutes with base field extension.

Proof. — In the first place, it is sufficient to prove the first of these relations, since the second follows. In the second place, we can assume that X is affine, since the statement is local on X. Thus we may assume that $X = \operatorname{Spec} k[x_1, \ldots, x_n]$.

Now base field extension is an exact functor, so the inclusion $R^i(F) \subseteq F$ gives an inclusion $R^i(F) \otimes K \subseteq F_K$. In fact, it is clear that

$$R^i(F) \otimes K \subseteq R^i(F_K).$$

To show that they are equal, by the third remark above it will be sufficient to show that all associated primes of the quotient $F_K/R^i(F) \otimes K = (F^i)_K$ have codimension $< i$. By [3, ch. IV, thm. 2, p. 154], we have

$$\operatorname{Ass}(F^i)_K = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(F^i)} \operatorname{Ass}(K[x_1, \ldots, x_n]/\mathfrak{p}'),$$

where, if $\mathfrak{p} \subseteq k[x_1, \ldots, x_n]$ is a prime ideal, $\mathfrak{p}'$ is its extension to $K[x_1, \ldots, x_n]$. But by [ZS, vol. II, thm. 36, p. 244], each associated prime of $\mathfrak{p}'$ has the same dimension as $\mathfrak{p}$ and hence the same codimension. Thus every associated prime of $(F^i)_K$ has codimension $< i$. q.e.d.

Proposition **(2.3)**. — *Let* $f: X \to Y$ *be a morphism of finite type, where* X *is irreducible,* Y *is noetherian and integral, and* Y *is universally catenary.* (Recall [EGA, IV, 5.6] that a prescheme Y is said to be universally catenary if for every $y \in Y$ and every $r \geq 0$, the polynomial ring $\mathcal{O}_Y[T_1, \ldots, T_r]$ satisfies the chain condition for prime ideals. In particular, if Y is regular, or of finite type over a field, then Y is universally catenary.)

Let F *be a coherent sheaf on* X. *Then there is a non-empty open subset* $V \subseteq Y$ *such that for all* $y \in V$, *and for all* i,

$$R^i(F_y) = R^i(F) \otimes \boldsymbol{k}(y)$$

and
$$(F_y)^i = F^i \otimes \boldsymbol{k}(y),$$

where $F_y = F \otimes \boldsymbol{k}(y)$ *is the fibre of* F *at* y. *In other words, formation of* $R^i(F)$ *and* F^i *commutes with passage to the fibres of a morphism of finite type, almost always.*

Proof. — I claim there is a non-empty open subset V of Y with the following properties:

(i) $\mathcal{O}_X$ is flat over V; for each j, F^j is flat over V, and for each associated prime cycle Z of $F_V = F|f^{-1}(V)$, $\mathcal{O}_Z$ is flat over V. (We give Z the reduced subscheme structure.)

(ii) If $Z \in \operatorname{Ass} F_V$ or if $Z = X$, then $f: Z \to V$ is surjective.

(iii) If $Z \in \operatorname{Ass} F_V$ or if $Z = X$, then for each $y \in V$, all the irreducible components of the fibre Z_y have the same dimension.

(iv) If $Z \in \operatorname{Ass} F_V$, then for each $y \in V$, the fibre Z_y has no embedded components.

(v) For each sheaf $E = F^j$ or $R^j(F)$, and for each $y \in V$, we have

$$\operatorname{Ass}(E_y) = \bigcup_{Z \in \operatorname{Ass}(E_V)} \operatorname{Ass}(Z_y).$$

Indeed, since Y is irreducible, any finite intersection of non-empty open sets is non-empty. Hence it is sufficient to consider each property and each j (resp. Z, E) separately, and find a V which works in that case. Condition (i) is possible to satisfy by the Theorem of Generic Flatness [EGA, IV, 6.9.1]. It implies [EGA, IV, 3.3.2] that each $Z \in \operatorname{Ass} F_V$ or $Z = X$ dominates Y. Hence to satisfy condition (ii) it is sufficient to replace V by a non-empty open subset V′ contained in the image $f(Z)$. Condition (iii) follows from (ii) and [EGA, IV, 9.5.6]. Conditions (iv) and (v) follow from [EGA, IV, 9.8.3].

Now any V satisfying the conditions (i)-(v) will do for the Proposition. First we establish

(vi) If $Z \in \operatorname{Ass} F_V$ is of codimension i, then for every $y \in V$, every associated prime cycle of the fibre Z_y is of codimension i in X_y.

Indeed, using (i), (ii) [EGA, IV, 6.1.4], and the fact that X is catenary, we deduce that Z_y has codimension i in X_y. But by (iv), Z_y has no embedded components, and by (iii) all the irreducible components of Z_y have the same dimension. For the same reason, all the irreducible components of X_y have the same dimension, and hence, since X_y is catenary, all the irreducible components of Z_y have the same codimension i.

Now we proceed as in the proof of Proposition 2.2, restricting our attention to V. In the first place, since F^i is flat over V, we have, for any $y \in V$, an inclusion

$$R^i(F) \otimes \boldsymbol{k}(y) \subseteq F_y.$$

Furthermore by (v) and (vi), every associated prime cycle of $R_i(F)\otimes \boldsymbol{k}(y)$ is of codimension $\geqslant i$, so

$$R^i(F)\otimes \boldsymbol{k}(y)\subseteq R^i(F_y).$$

To show they are equal, it will be sufficient, by Remark 3 above, to show that all the associated prime cycles of the quotient $F_y/R^i(F)\otimes \boldsymbol{k}(y)=(F^i)_y$ have codimension $<i$. This again follows from (v) and (vi), and the fact that $\operatorname{Ass} F^i \subseteq \operatorname{Ass} F$. This gives the first statement of the proposition, and the second follows immediately.

Remark. — For the rest of this chapter we will be concerned mostly with sheaves on projective space $X=\mathbf{P}^r_k$ over a field k. Since for any closed subscheme Z of X,

$$\dim Z+\operatorname{codim} Z=r,$$

we will permit ourselves to use interchangeably the notations $R_i(F)=R^{r-i}(F)$, $F_i=F^{r-i}$, and to use the preceeding results translated into the language of $R_i(F)$, F_i.

The integers $n_i(F)$.

Definition. — Let X *be a projective space* $\mathbf{P}^r_k$ *over a field* k, *and let* F *be a coherent sheaf on* X. *For each* i *we define*

$$n_i(F)=(i\,!)\ (\text{coefficient of } z^i \text{ in the Hilbert polynomial of } R_i(F)=R^{r-i}(F)).$$

Remarks. — 1. The integers $n_i(F)$ are all non-negative. They are zero for $i<0$ and $i>\dim \operatorname{Supp}(F)$.

2. We refer to the $(r+1)$-tuple of integers $(n_r(F), n_{r-1}(F), \ldots, n_0(F))$ as $n_*(F)$. If n_*, m_* are two $(r+1)$-tuples of integers, we say $n_*\geq m_*$ in the *pointwise ordering* if $n_i\geq m_i$ for each i. We say $n_*\geq m_*$ in the *lexicographic ordering* if $n_r>m_r$, or if $n_r=m_r$ and $n_{r-1}>m_{r-1}$, etc. Unless otherwise specified, $n_*\geq m_*$ will always mean in the pointwise ordering.

3. If X is a subscheme of $\mathbf{P}^r_k$, we set $n_i(X)=n_i(\mathcal{O}_X)$ for each i.

Example (**2.5**). — Let X be a reduced subscheme of $\mathbf{P}^r_k$, all of whose irreducible components are linear subspaces of $\mathbf{P}^r_k$. Then for each i, $n_i(X)$ is the number of components of X of dimension i.

Proof. — One reduces easily to the case where X is irreducible of dimension i. Then $X\cong\mathbf{P}^i_k$, whose Hilbert polynomial [see FAC, Prop. 3, p. 275] is

$$(1/i\,!)(z+1)\ldots(z+i).$$

Thus $n_i(X)=1$, as required.

Proposition (**2.6**). — *Let* $0\to F'\to F\to F''\to 0$ *be an exact sequence of coherent sheaves on* $X=\mathbf{P}^r_k$. *Then for each* i,

$$n_i(F')\leq n_i(F)\leq n_i(F')+n_i(F'').$$

Proof. — This follows immediately from the exact sequence

$$0 \to R_i(F') \to R_i(F) \to R_i(F''),$$

and the properties of Hilbert polynomials.

Lemma **(2.7)**. — *Let* $F \to G \to 0$ *be a surjection of coherent sheaves on* X, *where* F *is a sheaf whose associated primes are all of dimension i. Then* $n_i(F) \geq n_i(G)$, *and there is equality if and only if* $F \to G$ *is an isomorphism.*

Proof. — Let K be the kernel. Then we have an exact sequence

$$0 \to K \to F \to G \to 0,$$

and so, letting p denote the Hilbert polynomial,

$$p(F) = p(K) + p(G).$$

It follows immediately that $n_i(F) \geq n_i(G)$, since these are all polynomials of degree $\leq i$. Furthermore, if $n_i(F) = n_i(G)$, then $p(K)$ is a polynomial of degree $< i$. But $\operatorname{Ass} K \subseteq \operatorname{Ass} F$, so all the associated primes of K are of dimension i, and so K must be 0, showing $F \to G$ is an isomorphism.

Proposition **(2.8)**. — *Let* $F \to G \to 0$ *be a surjection of coherent sheaves on* X. *Then* $n_*(F) \geq n_*(G)$ *in the lexicographic ordering and there is equality if and only if* $F \to G$ *is an isomorphism.*

Proof. — It will suffice to prove that the following statement is true for each i: If $n_j(F) = n_j(G)$ for all $j > i$, then $n_i(F) \geq n_i(G)$, and there is equality if and only if $F_{i-1} \to G_{i-1}$ is an isomorphism. We proceed by descending induction on i, the case $i = r+1$ being trivial ($r = \dim X$).

Suppose then that $n_j(F) = n_j(G)$ for all $j > i$. We can assume by the induction hypothesis that $F_i \to G_i$ is an isomorphism. Thus we have an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R_i(F)/R_{i-1}(F) \arrow[r] \arrow[d, "\alpha"] & F_{i-1} \arrow[r] \arrow[d, "\beta"] & F_i \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & R_i(G)/R_{i-1}(G) \arrow[r] & G_{i-1} \arrow[r] \arrow[d] & G_i \arrow[r] & 0. \\
& & 0 & &
\end{tikzcd}$$

A diagram chase shows that α is surjective. Moreover, all the associated primes of $R_i(F)/R_{i-1}(F)$ are of dimension i, so we can apply the Lemma to α. Thus $n_i(F) \geq n_i(G)$, and we have equality if and only if α is an isomorphism. But by the 5-Lemma, α is an isomorphism if and only if β is. q.e.d.

Behavior under Base Change.

Proposition (**2.9**). — *Let* X *be a projective space* $\mathbf{P}^r_k$, *and let* F *be a coherent sheaf on* X. *Let* K *be an extension field of* k, *and let* X_K, F_K *be obtained by base extension. Then* $n_*(F_K) = n_*(F)$.

Proof. — This is an immediate consequence of Proposition 2.2 and the fact the Hilbert polynomial is preserved under base field extension.

Theorem (**2.10**). — *Let* $X = \mathbf{P}^r_Y$, *and let* $f : X \to Y$ *be the projection, where* Y *is an integral noetherian universally catenary prescheme. Let* F *be a coherent sheaf on* X. *Let* η *be the generic point of* Y, *and let* y *be any other point of* Y. *Then each of the following conditions implies the next:*

(i) $n_*(F_y) = n_*(F_\eta)$.

(ii) F *is flat over* Y *at all points* $x \in f^{-1}(y)$.

(iii) $n_*(F_y) \geq n_*(F_\eta)$ *in the pointwise ordering.*

(iv) $n_*(F_y) \geq n_*(F_\eta)$ *in the lexicographic ordering.*

Furthermore, (iv) *is satisfied for all* $y \in Y$, *and* (i), (ii), (iii) *are satisfied for all* y *in a non-empty open subset* V *of* Y.

Proof. — If $Y' = \operatorname{Spec} A$ is the spectrum of a discrete valuation ring, and $g : Y' \to Y$ is a morphism sending the closed point y' onto y, and the generic point η' onto η, then, by Proposition 2.9, the conditions (i), (iii), (iv) are equivalent to the corresponding conditions (i)′, (iii)′, (iv)′ for the extended situation X′, F′, y', η' over Y′. Furthermore, by the Valuative Criterion of Flatness [EGA, IV, 11.8.1] condition (ii) is equivalent to saying that for all such base extensions $g : Y' \to Y$, (ii′) holds. Such base extensions exist by [EGA, ch. II, 7.1.9, p. 141]. Thus to prove the implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) and the truth of (iv), we are reduced to the case where $Y = \operatorname{Spec} A$ is the spectrum of a discrete valuation ring, with closed point y and generic point η.

(iii) ⇒ (iv) is obvious.

To prove (ii) ⇒ (iii), let F be flat over Y. Then $R^{r-i}(F)$ and F^{r-i} are also flat over Y, since over a discrete valuation ring, flatness depends only on the set of associated primes (see Proposition 1.3). Thus by Theorem 1.2, $R^{r-i}(F) \otimes \boldsymbol{k}(\eta)$ and $R^{r-i}(F) \otimes \boldsymbol{k}(y)$ have the same Hilbert polynomial. The first of these is equal to $R_i(F_\eta)$ (follows e.g. from Proposition 2.3). The second is a subsheaf of F_y, since F^{r-i} is flat over Y. Moreover, since its Hilbert polynomial, being equal to that of $R_i(F_\eta)$, is of degree $< i$, it has support of dimension $\leq i$. Hence

$$R^{r-i}(F) \otimes \boldsymbol{k}(y) \subseteq R_i(F_y).$$

and so $n_i(F_y) \geq n_i(F_\eta)$.

To prove (i) ⇒ (ii) and the truth of (iv), let F be arbitrary. Let $T \subseteq F$ be the torsion subsheaf of F, and let $F' = F/T$. Then F′ is flat over Y, $F'_\eta = F_\eta$, and there is an exact sequence

$$0 \to T_y \to F_y \to F'_y \to 0.$$

Applying Proposition 2.8, we find that $n_*(F_y) \geq n_*(F'_y)$ in the lexicographic ordering. On the other hand, by the implication (ii) $\Rightarrow$ (iii) above, $n_*(F'_y) \geq n_*(F'_\eta) = n_*(F_\eta)$ in the pointwise ordering. Thus $n_*(F_y) \geq n_*(F_\eta)$ in the lexicographic ordering, which establishes (iv). If there is equality, then by Proposition 2.8, $F_y \to F'_y$ is an isomorphism, i.e. $T_y = 0$. But then by Nakayama's Lemma $T = 0$ and $F = F'$ is flat over Y. Thus (i) $\Rightarrow$ (ii).

For the last statement of the theorem, choose a non-empty open subset $V \subseteq Y$ such that

1) each $R^j(F)$ is flat over V, and
2) for all $y \in V$, $R^j(F) \otimes \boldsymbol{k}(y) = R^j(F_y)$.

This is possible by the Theorem of Generic Flatness [EGA, IV, 6.9.1] and by Proposition 2.3. Now if $y \in V$, condition (i), hence also conditions (ii) and (iii) are satisfied.

Remark. — One can show by easy examples that the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) of the theorem are all strict.

Corollary **(2.11)**. — *Let* $f : X \to Y$, F *be as in the theorem, and assume that* F *is flat over* Y. *Then, for any* $(r+1)$*-tuple of integers* m_*, *the set*

$$E = \{y \in Y \mid n_*(F_y) \leq m_*\}$$

is open, and the set

$$G = \{y \in Y \mid n_*(F_y) > m_*\}$$

is closed. The function $y \rightsquigarrow n_*(F_y)$ *is upper semi-continuous on* Y.

Proof. — It is sufficient to prove the first statement. Using the criterion for openness given in [EGA, 0_{III}, 9.2.6, p. 16], we must show that whenever $Z \subseteq Y$ is a closed irreducible subset meeting E, then E contains a non-empty open subset of Z. Give Z the reduced induced structure, and make the base change $Z \to Y$. Then the result follows from the theorem applied to $f_Z : X_Z \to Z$ and the sheaf F_Z.

CHAPTER 3

FANS IN PROJECTIVE SPACE

In this chapter we study subschemes of projective space of a special type, called *fans*. We will deal mostly with a fixed projective space $\mathbf{P}^r_k$, and a fixed homogeneous coordinate system $x_0, x_1, \ldots, x_r$.

Definition. — *A fan* X *in* $\mathbf{P}^r_k$ *is a subscheme whose ideal* $\mathfrak{a}$ *can be written as an intersection of prime ideals* $\mathfrak{p}$ *of the form*

$$\mathfrak{p} = (x_1 - a_1 x_0, x_2 - a_2 x_0, \ldots, x_q - a_q x_0)$$

for various q, *and various* $a_1, \ldots, a_q \in k$. *A tight fan* X *is a fan whose ideal* $\mathfrak{a}$ *can be written as an intersection of prime ideals of the form*

$$\mathfrak{p} = (x_1, x_2, \ldots, x_{q-1}, x_q - a_q x_0)$$

for various q *and various* $a_q \in k$.

Remarks. — 1. A fan is a reduced subscheme of $\mathbf{P}^r_k$, all of whose irreducible components are linear subspaces. Moreover, for each q, all of the q-dimensional components of the fan contain a common $(q-1)$-dimensional linear subspace (hence the name "fan"). A tight fan has the additional property that, for each q, all of its q-dimensional components are contained in a common $(q+1)$-dimensional linear subspace (which is not a component of the fan, of course).

2. If X is a fan, then for each i, $n_i(\mathrm{X})$ is the number of i-dimensional linear subspaces which are components of X. (See Example 2.5.)

Lemma (**3.1**). — *Let* X *be a tight fan in* $\mathbf{P}^r_k$, *and let its ideal* $\mathfrak{a}$ *be written as an irredundant intersection of prime ideals* $\mathfrak{a} = \bigcap \mathfrak{p}_{ij}$, $i = 1, \ldots, r$; $j = 1, \ldots, t_i$, *where*

$$\mathfrak{p}_{ij} = (x_1, x_2, \ldots, x_{i-1}, x_i - a_{ij} x_0),$$

with $a_{ij} \in k$. *Let* s *be the largest index* i *for which* $t_i \neq 0$. *Then*

1) $a_{ij} \neq 0$ *for* $i = 1, \ldots, s-1$ *and* $j = 1, \ldots, t_i$, *and*

2) $a_{ij_1} \neq a_{ij_2}$ *for* $i = 1, \ldots, s$; $j_1, j_2 = 1, \ldots, t_i$; $j_1 < j_2$. *Furthermore,* $\mathfrak{a}$ *can be written*

$$\mathfrak{a} = (x_1\pi_1, x_2\pi_1\pi_2, \ldots, x_{s-1}\pi_1 \cdots \pi_{s-1}, \pi_1\pi_2 \cdots \pi_s)$$

where for each i,

$$\pi_i = \prod_{j=1}^{t_i} (x_i - a_{ij} x_0).$$

Proof. — The first statement is obvious, and the second follows from the first, together with some easy calculations in polynomial rings.

Proposition (**3.2**). — *Let* X_1, X_2 *be two tight fans in* $\mathbf{P}_k^r$. *Then the following conditions are equivalent:*

(i) $n_*(X_1)=n_*(X_2)$.

(ii) X_1 *and* X_2 *have the same Hilbert polynomial.*

(iii) *There exists a subscheme* X_3 *of* $\mathbf{P}_K^r$, *for suitable* K, *and linear specializations*

$$\begin{tikzcd} & X_3 \arrow[dl, squiggly] \arrow[dr, squiggly] & \\ X_1 & & X_2 \end{tikzcd}$$

Proof. — (i) $\Rightarrow$ (ii) We will show that the Hilbert polynomial of a tight fan X is determined by $n_*(X)$. Let X be given, and let its ideal $\mathfrak{a}$ be written as an irredundant intersection of prime ideals $\mathfrak{a}=\bigcap \mathfrak{p}_{ij}$, $i=1, \ldots, r; j=1, \ldots, t_i$, as in the Lemma. Note that for each $i=1, \ldots, r$, $t_i=n_{r-i}(X)$, so we need only show that the Hilbert polynomial of X is determined by the integers t_i.

Case 1. — Suppose there is an integer $1 \leq s \leq r$ such that $t_i=0$ for $i \neq s$. Then, using the notation of the Lemma,

$$\mathfrak{a}=(x_1, x_2, \ldots, x_{s-1}, \pi_s).$$

Now π_s is a homogeneous polynomial of degree t_s, so X is a hypersurface of degree t_s in the projective $(r-s+1)$-space defined by $x_1=\ldots=x_{s-1}=0$, and as such its Hilbert polynomial is determined. In fact, one can write down its Hilbert polynomial explicitly (see Corollary below).

Case 2. — In general, let s be the largest index i for which $t_i \neq 0$. Write $\mathfrak{a}=\mathfrak{b} \cap \mathfrak{c}$, where

$$\mathfrak{b}=\bigcap_{i<s} \mathfrak{p}_{ij} \quad \text{and} \quad \mathfrak{c}=\bigcap \mathfrak{p}_{sj}.$$

Proceeding by induction on s, and using Case 1, we can assume that the Hilbert polynomials of the subschemes defined by $\mathfrak{b}$ and $\mathfrak{c}$ have already been determined, and depend only on the t_i. Because of the exact sequence

$$0 \rightarrow R/\mathfrak{a} \rightarrow R/\mathfrak{b} \oplus R/\mathfrak{c} \rightarrow R/\mathfrak{b}+\mathfrak{c} \rightarrow 0$$

(where $R=k[x_0, \ldots, x_r]$) and the additivity of Hilbert polynomials, to show that the Hilbert polynomial of X depends only on the t_i, it will be sufficient to show that the Hilbert polynomial of the subscheme defined by the ideal $\mathfrak{b}+\mathfrak{c}$ depends only on the t. By the Lemma,

$$\mathfrak{b}=(x_1\pi_1, x_2\pi_1\pi_2, \ldots, x_{s-2}\pi_1 \cdots \pi_{s-2}, \pi_1 \cdots \pi_{s-1})$$

and

$$\mathfrak{c}=(x_1, \ldots, x_{s-1}, \pi_s).$$

Thus

$$\mathfrak{b}+\mathfrak{c}=(x_1, x_2, \ldots, x_{s-1}, \pi_1 \cdots \pi_{s-1}, \pi_s).$$

But $\pi_1 \cdots \pi_{s-1}$ can be replaced by $Ax_0^{t_1+\cdots+t_{s-1}}$, where $A = \prod_{i<s} a_{ij}$ is a non-zero element of k (by the Lemma, part 1). Thus

$$\mathfrak{b}+\mathfrak{c}=(x_1, x_2, \ldots, x_{s-1}, x_0^{t_1+\cdots+t_{s-1}}, \pi_s).$$

The subscheme defined by $\mathfrak{b}+\mathfrak{c}$ is therefore a complete intersection of hypersurfaces of degrees $t_1+\cdots+t_{s-1}$ and t_s in the projective $(r-s+1)$-space defined by $x_1=\cdots=x_{s-1}=0$, and as such its Hilbert polynomial is determined by the t_i. In general, if Y is a complete intersection of hypersurfaces of degrees m, n in projective space $X=\mathbf{P}_k^r$, then the Hilbert polynomial of Y can be calculated from the exact sequence

$$0 \to \mathcal{O}_X(-m-n) \to \mathcal{O}_X(-m) \oplus \mathcal{O}_X(-n) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

and the fact that the Hilbert polynomial of $\mathcal{O}_X(n)$ is $\binom{z+n+r}{r}$.

(ii) $\Rightarrow$ (i) We show that the Hilbert polynomial $f(z)$ of a tight fan X of dimension s determines $n_*(X)$. In the first place, $n_s(X)$ is determined as $s!$ times the leading coefficient of $f(z)$. Suppose inductively that $n_s, n_{s-1}, \ldots, n_{i+1}$ have been determined. Then (using the notation of Chapter 2), X_i is a tight fan with $n_*(X_i)=(n_s, n_{s-1}, \ldots, n_{i+1}, 0, \ldots, 0)$. By the implication (i) $\Rightarrow$ (ii) above, its Hilbert polynomial $g(z)$ is determined. Therefore the Hilbert polynomial $f(z)-g(z)$ of $R_i(\mathcal{O}_X)$ is also determined, and so also $n_i(X)$. By induction we see that $n_*(X)$ is determined.

(i) + (ii) $\Rightarrow$ (iii) Let X_1, X_2 be tight fans in $\mathbf{P}_k^r$ with $n_*(X_1)=n_*(X_2)$ and with the same Hilbert polynomials. Let their ideals $\mathfrak{a}^{(1)}$ and $\mathfrak{a}^{(2)}$ be given as in the Lemma as intersections of prime ideals determined by constants $a_{ij}^{(1)}$ (resp. $a_{ij}^{(2)}$), $i=1, \ldots, r$; $j=1, \ldots, t_i$. (Note that the t_i are the same for X_1 and X_2.) Let s be the largest index i for which $t_i \neq 0$.

Take indeterminates u_{ij} over k, $i=1, \ldots, s$; $j=1, \ldots, t_i$, and let $Y=\operatorname{Spec} k[u_{ij}]$.

Let $$\mathfrak{p}_{ij}=(x_1, x_2, \ldots, x_{i-1}, x_i-u_{ij}x_0),$$

and let $\mathfrak{a}=\bigcap \mathfrak{p}_{ij}$. Let X be the closed subscheme of $\mathbf{P}_Y^r$ defined by $\mathfrak{a}$. Then

$$\mathfrak{a}=(x_1\pi_1, x_2\pi_1\pi_2, \ldots, x_{s-1}\pi_1\cdots\pi_{s-1}, \pi_1\cdots\pi_s)$$

where for each i, $$\pi_i=\prod_{j=1}^{t_i}(x_i-u_{ij}x_0).$$

Thus the fibre of X at the point $y_1 \in Y$ given by $u_{ij}=a_{ij}^{(1)}$ is X_1, and the fibre at the point $y_2 \in Y$ given by $u_{ij}=a_{ij}^{(2)}$ is X_2. Let X_3 be the fibre of X at the generic point of Y. Then X_1, X_2, X_3 are all tight fans with the same n_*, hence the same Hilbert polynomials, by (i) $\Rightarrow$ (ii) above. Therefore X is flat over Y at the points above y_1, y_2, by Theorem 1.2, and so there are linear specializations as required.

(iii) $\Rightarrow$ (ii) is obvious. q.e.d.

Remarks. — 1. The proof (i) $\Rightarrow$ (ii) above gives at the same time a slightly stronger result, namely that if X_1, X_2 are tight fans with $n_*(X_1)=n_*(X_2)$, then they have the same Hilbert function. (The *Hilbert function* of a subscheme X of $\mathbf{P}^r_k$ is the function $f(n)=\dim H^0(X, \mathcal{O}_X(n))$.)

2. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) above are both false for loose fans, as can be seen from the following examples (a *loose fan* is a fan which is not tight):

	n_0	n_1	Hilb. poly.
a) Three lines in a plane, meeting at a point.......	0	3	$3z$
b) Three lines in three-space, meeting at a point but not lying in a plane..........................	0	3	$3z+1$
c) Same as *a)*, plus a point......................	1	3	$3z+1$.

Corollary (**3.3**). — *The Hilbert polynomial of a tight fan* X *in* $\mathbf{P}^r_k$, *with* $n_*(X)=(n_{r-1}, \ldots, n_0)$ *is*

$$f(z)=\sum_{t=0}^{r-1} g(n_t+\cdots+n_{r-1}, t),$$

where for any $n, t\in\mathbf{Z}$, $t\geq 0$,

$$g(n,t)=\binom{z+t}{t+1}-\binom{z+t-n}{t+1}.$$

Proof. — For any n, r, define $h(n,r)$ to be the Hilbert polynomial of a hypersurface of degree n in projective r-space. For integers m, n, r, define $c(m,n,r)$ be to the Hilbert polynomial of a complete intersection of hypersurfaces of degrees m and n in projective r-space. In other words (by an easy calculation)

$$h(n,r)=\binom{z+r}{r}-\binom{z+r-n}{r}$$

and

$$c(m,n,r)=\binom{z+r}{r}-\binom{z+r-n}{r}-\binom{z+r-m}{r}+\binom{z+r-m-n}{r}.$$

Using the proof (i) $\Rightarrow$ (ii) of the proposition as a guide, one can show that the Hilbert polynomial of X is

$$f(z)=\sum_{t=0}^{r-1} h(n_t, t+1)-\sum_{t=0}^{r-2} c(n_t, n_{t+1}+\cdots+n_{r-1}, t+1).$$

Now a little juggling of the binomial coefficients gives the result of the Corollary.

Remark. — Following Nagata [10], let us define a *numerical polynomial* to be a polynomial $f(z)\in\mathbf{Q}[z]$ which takes integer values for all large enough integers. (For example, binomial coefficients and Hilbert polynomials are numerical polynomials.) Then, as in the proof of (20.8) (*loc. cit.*, p. 69), one can show that any numerical polynomial $f(z)$ of degree s can be written uniquely in the form

$$f(z)=\sum_{k=0}^{s} g(m_k, k),$$

with $m_k\in\mathbf{Z}$. Thus the Corollary states that a necessary and sufficient condition for a numerical polynomial to be the Hilbert polynomial of a tight fan is that when expressed in this form, $m_0 \geq m_1 \geq \cdots \geq m_s \geq 0$. (For there exist tight fans with arbitrary $n_* \geq 0$). We will see later that this is also a necessary and sufficient condition that $f(z)$ be the Hilbert polynomial of some subscheme of projective space.

Corollary* **(3.4). — *The points of* $Hilb(\mathbf{P}_k^r)$ *corresponding to tight fans with given* n_* (resp. *given Hilbert polynomial*) *form a constructible subset, irreducible in the induced topology, and linearly connected.*

Proof. — This follows from the proposition, the definition of the Hilbert scheme, and Chevalley's theorem (see [4], exposé 7, theorem 3, p. 7-09).$_*$

Now we study the relationship between loose fans and tight fans.

Definition. — *Let* X *be a fan in* $\mathbf{P}_k^r$, *defined by an ideal* $\mathfrak{a}$ *which is represented as an irredundant intersection of prime ideals* $\mathfrak{a} = \bigcap \mathfrak{p}$, *where each prime* $\mathfrak{p}$ *is of the form*

$$\mathfrak{p} = (x_1 - a_1 x_0, x_2 - a_2 x_0, \ldots, x_q - a_q x_0)$$

for various q *and various* $a_1, \ldots, a_q \in k$. *Define* $p(X)$ *to be the largest integer* $p \leq r$ *such that for every prime* $\mathfrak{p}$ *of the above form occurring in* $\mathfrak{a}$,

1) *if* $q \leq p$, *then* $a_1 = a_2 = \cdots = a_{q-1} = 0$, *and*
2) *if* $q > p$, *then* $a_1 = a_2 = \cdots = a_{p-1} = 0$.

Notice that $p(X) \geq 1$, and $p(X) = r$ if and only if X is a tight fan.

Lemma **(3.5)**. — *Let* X *be a fan in* $\mathbf{P}_k^r$, *and let* p *be an integer* $\leqslant p(X)$. *Then the Hilbert polynomial of* X *is determined by* $n_*(X)$ *and the Hilbert polynomial of the union* X′ *of those components of* X *of codimension* $> p$.

Proof. — By induction on p. If $p = 1$, then the ideal $\mathfrak{a}$ of X can be written as an intersection

$$\mathfrak{a} = (f_1) \cap \cdots \cap (f_s) \cap \mathfrak{q}$$

where each f_i is of the form $f_i = x_1 - a_i x_0$ with $a_i \in k$, and where $\mathfrak{q}$ is the ideal of X′. By induction on s, it is sufficient to show that if $\mathfrak{b}$ is any ideal in $R = k[x_0, \ldots, x_r]$, and f is a linear form not contained in any associated prime ideal of $\mathfrak{b}$, then the Hilbert polynomial of $R/(\mathfrak{b} \cap (f))$ is determined by that of $R/\mathfrak{b}$. As in the proof of Proposition 3.2 ist is sufficient to show that the Hilbert polynomial of $R/(\mathfrak{b} + (f))$ is determined by that of $R/\mathfrak{b}$, since the Hilbert polynomial of $R/(f)$ is independent of f. Now we consider the exact sequence

$$0 \to R/\mathfrak{b} \xrightarrow{f} R/\mathfrak{b} \to R/(\mathfrak{b} + (f)) \to 0$$

where the first map, multiplication by f, is a map of graded modules of degree 1. Therefore the Hilbert polynomial of $R/(\mathfrak{b} + (f))$ is the first difference function of that of $R/\mathfrak{b}$, and so is determined by it.

In case $p > 1$, we write the ideal $\mathfrak{a}$ of X as an irredundant intersection of prime ideals $\mathfrak{a} = \bigcap \mathfrak{p} \bigcap \mathfrak{q} \bigcap \mathfrak{r}$, where each $\mathfrak{p}$ is of the form

$$\mathfrak{p} = (x_1, x_2, \ldots, x_{q-1}, x_q - a_q x_0), \qquad q < p$$

each $\mathfrak{q}$ is of the form

$$\mathfrak{q}=(x_1, x_2, \ldots, x_{p-1}, x_p-a_p x_0),$$

and each $\mathfrak{r}$ is of the form

$$\mathfrak{r}=(x_1, x_2, \ldots, x_{p-1}, x_p-a_p x_0, \ldots, x_q-a_q x_0), \qquad q>p.$$

Applying the induction hypothesis to the same X, but with $p-1$ for the integer p, we find that the Hilbert polynomial of X is determined by $n_*(X)$ and the Hilbert polynomial of X″, the fan defined by the ideal $\bigcap \mathfrak{q} \bigcap \mathfrak{r}$. To calculate this latter polynomial, we can pass to the quotient ring $\overline{R}=R/(x_1, \ldots, x_{p-1})$, and apply the case $p=1$ above to see that the Hilbert polynomial of X″ is determined by the number of $\mathfrak{q}$'s and the Hilbert polynomial of X′, which is the fan defined by the ideal $\bigcap \mathfrak{r}$.

Proposition **(3.6)**. — *Let* X_0 *be a fan in* $\mathbf{P}^r_k$. *Then there are subschemes* X_1, X_2, X_3 *of* $\mathbf{P}^r$ *(over suitable fields) and linear specializations*

$$\begin{tikzcd} & & X_2 \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] & \\ & X_1 \arrow[dl, rightsquigarrow] & & X_3 \\ X_0 & & & \end{tikzcd}$$

such that either

a) X_3 *is a tight fan, or*

b) X_3 *is a fan with* $p(X_3)>p(X_0)$, *or*

c) X_3 *is a subscheme with* $n_*(X_3)>n_*(X_0)$.

Proof. — If X_0 is a tight fan, there is nothing to prove, so let us assume that X_0 is not a tight fan, i.e. $p=p(X_0)<r$. Write the ideal $\mathfrak{a}$ of X_0 as an irredundant intersection of prime ideals $\mathfrak{a}=\bigcap \mathfrak{p} \bigcap \mathfrak{q}$, where each $\mathfrak{p}$ is of the form

$$\mathfrak{p}=(x_1, x_2, \ldots, x_{q-1}, x_q-a_q x_0), \qquad q\leq p,$$

and each $\mathfrak{q}$ is of the form

$$\mathfrak{q}=(x_1, x_2, \ldots, x_{p-1}, x_p-a_p x_0, x_{p+1}-a_{p+1}x_0, \ldots, x_q-a_q x_0),$$

$q>p$, and the $a_i \in k$.

For the first step, let λ, μ be indeterminates over k, and perform the automorphism of projective space given by

$$\begin{cases} x'_p=x_p-\lambda x_0 \\ x'_{p+1}=x_{p+1}-\mu x_p \\ x'_i=x_i & \text{for } i\neq p, p+1. \end{cases}$$

Let $k_1=k(\lambda, \mu)$, and let $X_1 \subseteq \mathbf{P}^r_{k_1}$, be the result of applying this automorphism to X_0. Then $X_1 \rightsquigarrow X_0$ is a linear specialization. Moreover, one sees easily that X_1 is a fan with $p(X_1)=p(X_0)$; $n_*(X_1)=n_*(X_0)$. Writing the ideal $\mathfrak{a}$ of X_1 as $\bigcap \mathfrak{p} \bigcap \mathfrak{q}$ as above (with the $a_i \in k_1$ now), and defining for each $\mathfrak{q}$ a new ideal

$$\mathfrak{q}'=(x_1, \ldots, x_{p-1}, x_p, x_{p+1}-a_{p+1}x_0, \ldots, x_q-a_q x_0),$$

we see also by an easy calculation that the intersection $\mathfrak{a}'=\bigcap \mathfrak{p} \bigcap \mathfrak{q}'$ is irredundant. (It was for this purpose that the indeterminates λ, μ were introduced: without them the intersection $\mathfrak{a}'$ might fail to be irredundant.) So we define the fan X' to be the one given by the ideal $\mathfrak{a}'$, and observe that $p(X')>p(X_1)$, and $n_*(X')=n_*(X_1)$. We will now attempt to deform X_1 into X'.

Let t be an indeterminate over k_1, let $A=k_1[t]$, and define a subscheme $X \subseteq \mathbf{P}^r_A$ by the ideal $\mathfrak{a}''=\bigcap \mathfrak{p} \bigcap \mathfrak{q}''$, where for each $\mathfrak{q}$ above,

$$\mathfrak{q}''=(x_1, x_2, \ldots, x_{p-1}, x_p-ta_p x_0, x_{p+1}-a_{p+1}x_0, \ldots, x_q-a_q x_0).$$

Then X is flat over A since each associated prime lies over the generic point (Proposition 1.3). I claim that the fibre of X at the point $t=1$ is X_1 (and let the reader beware of setting $t=1$ in each ideal and then taking the intersection: specialization does not commute with intersection of ideals!). Indeed, letting X_2 be the fibre of X at the generic point of A, X_1 and X_2 have the same Hilbert polynomial. Indeed, by the Lemma, it is sufficient to show that the fans defined by $\bigcap \mathfrak{q}$ and $\bigcap \mathfrak{q}''$ have the same Hilbert polynomial, and this is true because they differ by an automorphism of the projective space. But X_1 is a closed subscheme of the fibre $(X)_{t=1}$, and X is flat over A, so by Theorem 1.2, X_1 and $(X)_{t=1}$ have the same Hilbert polynomial, and therefore $X_1=(X)_{t=1}$. (In general, if Y is a closed subscheme of X, and has the same Hilbert polynomial as X, then $Y=X$. For in the exact sequence $0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$, $\mathscr{I}_Y$ will have Hilbert polynomial 0, hence is 0 by Serre's theorem.) Thus $X_2 \rightsquigarrow X_1$ is a linear specialization.

Finally, let X_3 be the fibre of X at the point $t=0$. Then X' is a closed subscheme of X_3, by construction. By Theorem 2.10, $n_*(X_3) \geq n_*(X_2)$. So either $n_*(X_3)>n_*(X_2)=n_*(X_0)$, in which case condition *c)* is satisfied, or $n_*(X_3)=n_*(X_2)$. In the latter case, $n_*(X_3)=n_*(X')$ and so by Proposition 2.8, $X_3=X'$, and condition *b)* is satisfied.

Corollary **(3.7)**. — *Let* X_0 *be a fan in* $\mathbf{P}^r_k$. *Then there is a connected sequence of linear specializations* $X_0, X_1, \ldots, X_s$ *in* $\mathbf{P}^r$, *such that either*

a) X_s *is a tight fan, or*

b) X_s *is a subscheme with* $n_*(X_s)>n_*(X_0)$.

Definition. — *Let* Z_1 *and* Z_2 *be closed subpreschemes of a prescheme* X, *given by sheaves of ideals* $\mathscr{I}_1$ *and* $\mathscr{I}_2$, *respectively. We define closed subpreschemes* $Z_1 \cap Z_2$ *and* $Z_1 \cup Z_2$ *by the sheaves of ideals* $\mathscr{I}_1+\mathscr{I}_2$ *and* $\mathscr{I}_1 \cap \mathscr{I}_2$, *respectively.*

Note that taking intersections of closed subpreschemes is compatible with base extension, but that taking unions is not in general.

Lemma **(3.8)**. — *Let $f: X \to Y$ be a morphism of finite type with Y locally noetherian and integral, and let Z_1, Z_2 be closed subpreschemes of X. Then there is a dense open subset $V \subseteq Y$ such that if $Y' \to Y$ is a morphism which factors through V then*

$$(Z_1 \cup Z_2)' = Z_1' \cup Z_2'$$

where $'$ denotes base extension to Y'.

Proof. — Whenever Z_1 and Z_2 are closed subpreschemes of a prescheme X, there is an exact sequence

$$0 \to \mathcal{O}_{Z_1 \cup Z_2} \to \mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z_2} \to \mathcal{O}_{Z_1 \cap Z_2} \to 0.$$

Since intersection of subpreschemes is compatible with base extension, we need only take V such that $\mathcal{O}_{Z_1 \cap Z_2} | f^{-1}(V)$ is flat over V (by the Theorem of Generic Flatness), and apply the exact sequence on X and on X'.

Proposition **(3.9)**. — *Let integers $r > 0$ and $m_{r-1}, \ldots, m_0 \geq 0$ be given, and let $m_* = (m_{r-1}, \ldots, m_0)$. Then the set of polynomials $p \in \mathbf{Q}[z]$ such that there exists a field k and a fan $X \subseteq \mathbf{P}_k^r$ with Hilbert polynomial p and with $n_*(X) = m_*$, is finite.*

Proof. — We will construct a noetherian scheme T over $\mathbf{Z}$ and a closed subscheme W of $\mathbf{P}_T^r$, flat over T, such that whenever k is a field and $X \subseteq \mathbf{P}_k^r$ a fan with $n_*(X) = m_*$, then X arises from W by a base extension $\operatorname{Spec} k \to T$. This will prove the proposition, for it shows there can be no more different Hilbert polynomials than there are connected components of T.

Given $m_* = (m_{r-1}, \ldots, m_0)$. Take indeterminates t_{ij}^q over $\mathbf{Q}$ for $q = 1, \ldots, r$; $i = 1, \ldots, q$; and $j = 1, \ldots, m_{r-q}$. Let

$$A = \mathbf{Z}[t_{ij}^q],$$

let

$$\mathfrak{p}_{jq} = (x_1 - t_{1j}^q x_0, \ldots, x_q - t_{qj}^q x_0)$$

for each j, q, and let

$$\mathfrak{a} = \bigcap \mathfrak{p}_{jq}.$$

Then $\mathfrak{a}$ defines a closed subscheme Z of $\mathbf{P}_A^r$ which is the union of the closed subschemes Z_{jq} defined by the prime ideals $\mathfrak{p}_{jq}$. Now clearly whenever k is a field and $X \subseteq \mathbf{P}_k^r$ a fan with $n_*(X) = m_*$ there is a morphism $\operatorname{Spec} k \to \operatorname{Spec} A$ such that

$$X = \bigcup (Z_{jq} \otimes_A k).$$

Thus in the pair $(\operatorname{Spec} A, Z)$ we have what we want except for two things: Z may not be flat over $\operatorname{Spec} A$, and the expression $Z = \bigcup Z_{jq}$ may not survive base extension.

However, by using the Theorem of Generic Flatness [EGA, IV, 6.9.1] and the above lemma repeatedly we can find a noetherian scheme T, which is the disjoint union of a finite number of locally closed subschemes of $\operatorname{Spec} A$ such that the morphism $T \to \operatorname{Spec} A$ is bijective and such that if

$$W_{jq} = Z_{jq} \times_A T$$

and

$$W = \bigcup W_{jq},$$

then W is flat over T, and for any morphism $T' \to T$,

$$W \times_T T' = \bigcup (W_{jq} \times_T T').$$

This pair T, W is the one we want.

Corollary **(3.10)**. — *Let an integer $r > 0$ and a polynomial $p \in \mathbf{Q}[z]$ be given. Then the set of r-tuples $m_* = (m_{r-1}, \ldots, m_0)$ of integers such that there exists a field k and a fan $X \subseteq \mathbf{P}^r_k$ with Hilbert polynomial p and with $n_*(X) = m_*$, is finite.*

Proof. — We may assume degree $p < r$. Then for any such m_*, m_{r-1} is already determined by the coefficient of z^{r-1} in p. Suppose inductively we have shown that $m_{r-1}, \ldots, m_{i+1}$ can have only finitely many values. Then we will show that m_i can have only finitely many values.

Now m_i is determined by the coefficient of z^i in the Hilbert polynomial of $R_i(\mathcal{O}_X)$ (using the notation of Chapter 2). But from the exact sequence

$$0 \to R_i(\mathcal{O}_X) \to \mathcal{O}_X \to \mathcal{O}_{X_i} \to 0$$

it follows that

$$p(R_i(\mathcal{O}_X)) = p(\mathcal{O}_X) - p(\mathcal{O}_{X_i}),$$

where p denotes the Hilbert polynomial. We are supposing that $p(X) = p$. Moreover, since X is a fan with $n_*(X) = m_*$, X_i is a fan with

$$n_*(X_i) = (m_{r-1}, \ldots, m_{i+1}, 0, \ldots, 0).$$

Now there are only finitely many values of $m_{r-1}, \ldots, m_{i+1}$ by our induction hypothesis, and by Proposition 3.9 there are only finitely many possible polynomials $p(X_i)$ corresponding to each choice of $m_{r-1}, \ldots, m_{i+1}$. Hence there are only finitely many possibilities for m_i. q.e.d.

CHAPTER 4

DISTRACTIONS

Remarks on Monomial Ideals.

Let B be a ring, and $S = B[x_1, \ldots, x_r]$ a polynomial ring over B. An ideal $\mathfrak{b} \subseteq S$ will be called a *monomial ideal* if, whenever a polynomial

$$f = \Sigma b_{i_1, \ldots, i_r} x_1^{i_1} \ldots x_r^{i_r}$$

is in $\mathfrak{b}$, then each monomial

$$m = b_{i_1, \ldots, i_r} x_1^{i_1} \ldots x_r^{i_r}$$

of f is also in $\mathfrak{b}$.

We can introduce an *r-fold grading* in the ring S by setting the *r-degree* of a monomial m as above equal to the r-tuple $(i_1, \ldots, i_r)$. Then S is the direct sum of the subgroups $S_{i_1, \ldots, i_r}$ of elements of r-degree $(i_1, \ldots, i_r)$. A monomial ideal if then an ideal which is r-homogeneous for this r-fold grading of S. The definition and theorems of [ZS, vol. II, ch. VII, § 2] extend immediately to the case of multiply graded rings, so we have the following results for monomial ideals in S:

1) An ideal $\mathfrak{b} \subseteq S$ is a monomial ideal if and only if it has a basis consisting of monomials in the x_i with coefficients in B [*loc. cit.*, thm. 7, p. 151].

2) If $\mathfrak{b}$, $\mathfrak{c}$ are monomial ideals in S, then $\mathfrak{b}+\mathfrak{c}$, $\mathfrak{b} \cap \mathfrak{c}$, $\mathfrak{b}\mathfrak{c}$, $\mathfrak{b} : \mathfrak{c}$ and $\sqrt{\mathfrak{b}}$ are monomial ideals [*loc. cit.*, thm. 8, p. 152].

3) If $\mathfrak{b} \subseteq S$ is a monomial ideal, admitting a representation as an intersection of (not necessarily monomial) primary ideals, then the isolated components of $\mathfrak{b}$ and the associated primes of $\mathfrak{b}$ are monomial ideals [*loc. cit.*, Corollary to thm. 9, p. 154].

4) Let $\mathfrak{b}$ be an ideal in S. If $\mathfrak{b}$ is a monomial ideal, then for every polynomial $f(x_1, \ldots, x_r) \in \mathfrak{b}$, and for every r-tuple $t_1, \ldots, t_r$ of elements of B, we have $(t_1 x_1, \ldots, t_r x_r) \in \mathfrak{b}$. The converse is true if B is an infinite field [*loc. cit.*, thm. 10, p. 155].

Finally note that the prime monomial ideals are just the ideals of the form

$$\mathfrak{p} + (x_{i_1}, \ldots, x_{i_s})$$

where $\mathfrak{p}$ is a prime ideal of B.

Remarks on Change of Polynomial Ring.

In this chapter we will be dealing with polynomial rings, and localizations of polynomial rings, in which the number of indeterminates is variable. To avoid semantic difficulties, we make some preliminary definitions.

Let k be a field, and let $\{z_i\}$, $i=1, 2, \ldots$ be infinitely many independent indeterminates over k. For any finite set $\alpha=(i_1, \ldots, i_s)$ of indices, let

$$\overline{R}_\alpha = k[z_{i_1}, \ldots, z_{i_s}].$$

If α, β are two finite sets of indices, and if $\mathfrak{a}\subseteq\overline{R}_\alpha$ and $\mathfrak{b}\subseteq\overline{R}_\beta$ are ideals, we say that the pairs $(\alpha, \mathfrak{a})$ and $(\beta, \mathfrak{b})$ are *equivalent* if the extensions $\mathfrak{a}'$, $\mathfrak{b}'$ of $\mathfrak{a}$ and $\mathfrak{b}$ to the ring $\overline{R}_{\alpha\cup\beta}$ are the same. This is clearly an equivalence relation. We define a **-ideal* (with respect to the field k and the indeterminates $\{z_i\}$) to be an equivalence class of pairs $(\alpha, \mathfrak{a})$ as above.

The justification for this terminology lies in the fact that all the usual operations on ideals pass to these equivalence classes. In particular, we can talk about the sum, intersection, product, and quotient of two *-ideals, and about the inclusion of one *-ideal in another. We can talk about prime *-ideals, primary *-ideals, and primary representations of *-ideals, with the usual uniqueness theorems. Finally we can speak of a sequence of elements being prime to a *-ideal.

Indeed, all of these concepts are invariant under the operation of passing to a polynomial ring over a ring (e.g. if $(\alpha, \mathfrak{a}_1)$ is equivalent to $(\beta, \mathfrak{b}_1)$ and $(\alpha, \mathfrak{a}_2)$ is equivalent to $(\beta, \mathfrak{b}_2)$, then $(\alpha, \mathfrak{a}_1+\mathfrak{a}_2)$ is equivalent to $(\beta, \mathfrak{b}_1+\mathfrak{b}_2)$). The statements about operations on ideals are obvious. For the primary representation, see [3, ch. IV, Prop. 11 and ex. 3, pp. 157, 158]. The statement about prime sequences comes from [EGA, 0_{IV}, 15.1.14], and the fact that a polynomial ring over a ring is a faithfully flat extension [3, ch. I, Prop. 9, p. 51]. See also [10, (6.13), ..., (6.17), pp. 17, 18].

Having made these remarks, we will abuse language and write simply " ideal " instead of " *-ideal ". We will let $\overline{R}$ denote indifferently any one of the rings $\overline{R}_\alpha$, and we will confuse a *-ideal $\{(\alpha, \mathfrak{a})\}$ with its representative $\mathfrak{a}$ in $\overline{R}_\alpha$.

We make similar conventions in another situation. Let k be a field, let $x_0, \ldots, x_r$ be indeterminates, and let $\{t_i\}$, $i=1, 2, \ldots$ be infinitely many independent indeterminates over $k(x_0, \ldots, x_r)$. For any finite set $\alpha=(i_1, \ldots, i_s)$ of indices, let $A_\alpha=k[t_{i_1}, \ldots, t_{i_s}]_\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal $(t_{i_1}, \ldots, t_{i_s})$. Let $R'_\alpha=A_\alpha[x_0, \ldots, x_r]$. Then we define equivalence of pairs $(\alpha, \mathfrak{a})$, where $\mathfrak{a}$ is an ideal in R'_α, as above, and define *-ideals in this context. Again all the ideal operations listed above make sense. (Compare [3, ch. IV, Prop. 11 and ex. 1, pp. 157, 158] ; [ZS, vol. I, ch. IV, § 10, p. 223 ff.], and note that if $\alpha\subseteq\beta$, R'_β is faithfully flat over R'_α.)

Again by abuse of language we will drop *'s and α's.

Canonical Distractions.

For the rest of this chapter we will use the following notation (with the conventions listed above).

k is a field.

$R = k[x_0, \dots, x_r]$ is a polynomial ring.

$\{t_{ij}\}$, $i = 1, \dots, r$; $j = 1, 2, \dots$ are infinitely many independent indeterminates over $k(x_0, \dots, x_r)$.

$A = k[t_{ij}]_{\mathfrak{m}}$ where $\mathfrak{m}$ is the maximal ideal (t_{ij}).

We will also denote by $\mathfrak{m}$ the maximal ideal of A.

$R' = A[x_0, \dots, x_r]$.

Let $\mathfrak{a} \subseteq R$ be an ideal generated by monomials in $x_1, \dots, x_r$. Note that $\mathfrak{a}$ has a unique minimal basis of monomials. It consists of those monomials

$$x_1^{s_1} \dots x_r^{s_r}$$

in $\mathfrak{a}$ for which the r-tuple $(s_1, \dots, s_r)$ is minimal in the partial ordering given by

$$(s_1, \dots, s_r) \leq (s'_1, \dots, s'_r) \Leftrightarrow s_k \leq s'_k \qquad \text{for each } k.$$

Definition. — *Let* $\mathfrak{a} \subseteq R$ *be an ideal generated by monomials in* $x_1, \dots, x_r$. *Its canonical distraction is the ideal* $\mathfrak{a}' \subseteq R'$ *generated by the expressions*

$$\prod_{j=1}^{s_1} (x_1 - t_{1j}x_0) \cdot \prod_{j=1}^{s_2} (x_2 - t_{2j}x_0) \cdot \ \dots \ \cdot \prod_{j=1}^{s_r} (x_r - t_{rj}x_0)$$

as $x_1^{s_1} \dots x_r^{s_r}$ *ranges over the unique minimal monomial basis of* $\mathfrak{a}$.

The rest of this chapter will be devoted to discussing various properties of the canonical distraction of an ideal.

Proposition (**4.1**). — *Let* $\mathfrak{b}$ *be an ideal in* R' *generated by monomials in expressions of the form* $(x_k - t_{ij}x_0)$, $k = 1, \dots, r$. *Then the following conditions are equivalent:*

(i) $R'/\mathfrak{b}$ *is flat over* A.

(ii) $\{t_{ij}\}$ *is a prime sequence in* $R'/\mathfrak{b}$.

(iii) x_0 *is prime to* $\mathfrak{b}$.

Proof. — (i) $\Rightarrow$ (ii). This is a special case of [EGA, 0_{IV}, 15.1.14]. For clearly $\{t_{ij}\}$ is a prime sequence in A.

(ii) $\Rightarrow$ (iii). Note that $R'/(\mathfrak{b} + (t_{ij})) \cong R/\mathfrak{b}_0$ where $\mathfrak{b}_0$ is an ideal generated by monomials in $x_1, \dots, x_r$. Thus x_0 is prime to $(\mathfrak{b} + (t_{ij}))$ and so $\{t_{ij}, x_0\}$ is a prime sequence in $R'/\mathfrak{b}$. Now $R'/\mathfrak{b}$ is a graded ring with $(R'/\mathfrak{b})_0 \cong A$, which is a local ring; the t_{ij} are in $\mathfrak{m}$, and x_0 is homogeneous of positive degree. Hence the property of being a prime sequence in $R'/\mathfrak{b}$ is independent of the order [1, Cor. 2.9, p. 633], and so x_0 is prime to $\mathfrak{b}$.

(iii) $\Rightarrow$ (i). Since x_0 is prime to $\mathfrak{b}$, we have an exact sequence

$$0 \to R'/\mathfrak{b} \xrightarrow{x_0} R'/\mathfrak{b} \to R'/(\mathfrak{b} + (x_0)) \to 0.$$

Now the ideal $\mathfrak{b}+(x_0)$ can be generated by monic monomials in $x_0, \ldots, x_r$, so $R'/(\mathfrak{b}+(x_0))$ is free, hence flat over A. Hence, for any A-module N we have an exact sequence

$$0 \to \mathrm{Tor}_1^A(R'/\mathfrak{b}, N) \xrightarrow{x_0} \mathrm{Tor}_1^A(R'/\mathfrak{b}, N) \to 0.$$

Since the Tor is an R'-module graded in positive degrees, and since x_0 is homogeneous of degree $+1$, we have $\mathrm{Tor}_1^A(R'/\mathfrak{b}, N)=0$. Thus $R'/\mathfrak{b}$ is flat over A.

Theorem (**4.2**). — *Let* $\mathfrak{a} \subseteq R$ *be an ideal generated by monomials in* $x_1, \ldots, x_r$, *and let* $\mathfrak{a}'$ *be its canonical distraction. Then* $R'/\mathfrak{a}'$ *is flat over* A, *and* $R'/\mathfrak{a}'+\mathfrak{m} \cong R/\mathfrak{a}$.

Proof. — The second statement is obvious from the definition of $\mathfrak{a}'$. For the first, we use criterion (ii) of the previous proposition. We may assume that only as many t_{ij} occur in R' as we need to express $\mathfrak{a}'$. Write them in lexicographic order. It will be sufficient to show that for each $k=1, \ldots, r$, if $R''=R'/(t_{ij})_{i<k}$ and $\mathfrak{a}''=(\mathfrak{a}'+(t_{ij})_{i<k})/(t_{ij})_{i<k}$, then $t_{k1}, \ldots, t_{ks_k}$ is a prime sequence in $R''/\mathfrak{a}''$ (where s_k is the number of t_{ij} with $i=k$). From the definition of $\mathfrak{a}'$, it follows that $\mathfrak{a}''$ can be written as

$$\mathfrak{a}''=\mathfrak{b}_0+(x_k-t_{k1}x_0)\mathfrak{b}_1+\cdots+(x_k-t_{k1}x_0)\cdots(x_k-t_{ks_k})\mathfrak{b}_{s_k},$$

where the $\mathfrak{b}_q$ are ideals generated by the x_i for $i \neq k$, and the t_{ij} for $i>k$. Let B be the ring $A_\alpha[x_0, \ldots, \hat{x}_k, \ldots, x_r]$, where $\alpha=\{(i,j) \mid i>k \text{ and } j=1, \ldots, s_i\}$. Then R'' is a localization of the polynomial ring $B[x_k, t_{k1}, \ldots, t_{ks_k}]$. Since prime sequences are preserved under localization, we have only to prove the following technical Lemma.

Lemma (**4.3**). — *Let* B *be a ring. Let* $\mathfrak{b}_0, \ldots, \mathfrak{b}_s$ *be ideals in* B, *and let* u *be an element of* B. *Let* $x, t_1, \ldots, t_s$ *be indeterminates, and let* $C=B[x, t_1, \ldots, t_s]$. *Let* $\mathfrak{a}$ *be the ideal*

$$\mathfrak{a}=\mathfrak{b}_0+(x-ut_1)\mathfrak{b}_1+\cdots+(x-ut_1)\ \ldots\ (x-ut_s)\mathfrak{b}_s$$

in C. *Then* $(t_1, \ldots, t_s)$ *is a prime sequence in* $C/\mathfrak{a}$.

Proof. — We must show for each $q=1, \ldots, s$ that t_q is prime to the ideal

$$\mathfrak{a}_q=\mathfrak{b}_0+x\mathfrak{b}_1+\cdots+x^{q-1}\mathfrak{b}_{q-1}+x^{q-1}(x-ut_q)\mathfrak{b}_q+\cdots+x^{q-1}(x-ut_q)\cdots(x-ut_s)\mathfrak{b}_s,$$

in $C_q=B[x, t_q, \ldots, t_s]$. To simplify notation, write t for t_q, and let

$$\mathfrak{c}=\mathfrak{b}_0+x\mathfrak{b}_1+\cdots+x^{q-1}\mathfrak{b}_q$$

$$\mathfrak{d}=x^{q-1}\mathfrak{b}_q+x^{q-1}(x-ut_{q+1})\mathfrak{b}_{q+1}+\cdots+x^{q-1}(x-ut_{q+1})\cdots(x-ut_s)\mathfrak{b}_s,$$

so that

$$\mathfrak{a}_q=\mathfrak{c}+(x-ut)\mathfrak{d}.$$

1) Note that for any $a \in C_q$, if $x^n a \in \mathfrak{c}$, for some n, then $x^{q-1}a \in \mathfrak{c}$. This follows from the fact that the ideal $\mathfrak{c}$ is homogeneous for the grading in C_q defined by x, and is generated by things in degrees $\leq q-1$.

2) Note that $\mathfrak{d} \subseteq (x^{q-1})$.

3) Now we show by a direct argument that t is prime to the ideal $\mathfrak{a}_q$. Let $a \in C_q$, and suppose that $ta \in \mathfrak{a}_q$. Then we can write

$$ta = c + (x - ut)d$$

where $c \in \mathfrak{c}$ and $d \in \mathfrak{d}$. Furthermore, since $C_q = C_{q+1}[t]$, and $\mathfrak{c}$, $\mathfrak{d}$ are extensions of ideals in C_{q+1}, we can write

$$c = c_0 + c_1 t, \qquad d = d_0 + d_1 t,$$

where $c_i \in \mathfrak{c}$, $d_i \in \mathfrak{d}$, and $c_0, d_0 \in C_{q+1}$. Thus

$$ta = c_0 + c_1 t + (x - ut)(d_0 + d_1 t) \qquad (*).$$

Reducing mod t, we find that $c_0 + xd_0 = 0$, which implies that $xd_0 \in \mathfrak{c}$. By 2) above, $d_0 = x^{q-1}d_0'$, so we have $x^q d_0' \in \mathfrak{c}$. By 1) above it follows that $x^{q-1}d_0' \in \mathfrak{c}$, or $d_0 \in \mathfrak{c}$. Now, cancelling $c_0 + xd_0$ from (*), and dividing by t, we have

$$a = (c_1 - ud_0) + (x - ut)d_1$$

which is in $\mathfrak{c} + (x - ut)\mathfrak{d} = \mathfrak{a}_q$. q.e.d.

An Auxiliary construction.

We introduce some further notation.

$\{z_{ij}\}$, $i = 1, \ldots, r$; $j = 1, 2, \ldots$ will be infinitely many independent indeterminates.

$\overline{R} = k[z_{ij}]$ will be a polynomial ring in finitely many of the z_{ij} (using the conventions listed above).

If $\mathfrak{a}$ is an ideal in R generated by monomials in $x_1, \ldots, x_r$, we will denote by $\overline{\mathfrak{a}}$ the ideal in $\overline{R}$ generated by the expressions

$$\prod_{j=1}^{s_1} z_{1j} \prod_{j=1}^{s_2} z_{2j} \cdots \prod_{j=1}^{s_r} z_{rj}$$

as $x_1^{s_1} \cdots x_r^{s_r}$ ranges over the unique minimal monomial basis of $\mathfrak{a}$. Note that if $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ are two such ideals, then $\overline{\mathfrak{a}}_1 \subseteq \overline{\mathfrak{a}}_2$.

Proposition (**4.4**). — *Let* $\mathfrak{a} \subseteq R$ *be an ideal generated by monomials in* $x_1, \ldots, x_r$. *Then* $\overline{\mathfrak{a}}$ *is the irrendundant intersection of those prime ideals*

$$\mathfrak{p} = (z_{i_1 j_1}, \ldots, z_{i_s j_s})$$

such that

1) $i_1, \ldots, i_s$ *are all distinct.*

2) $\mathfrak{a} \subseteq (x_{i_1}^{j_1}, \ldots, x_{i_s}^{j_s})$.

3) $\mathfrak{a}$ *is not contained in any ideal generated by a proper subset of the* $x_{i_k}^{j_k}$.

Proof. — Since $\overline{\mathfrak{a}}$ is generated by monomials in which each variable occurs at most to the first power, $\overline{\mathfrak{a}}$ is a radical ideal. Hence it can be written as an irredundant intersection of prime ideals. Each such prime ideal is generated by some of the z_{ij} (see remarks on monomial ideals above), so can be written

$$\mathfrak{p}=(z_{i_1j_1}, \ldots, z_{i_sj_s}).$$

Whenever z_{ij} occurs in a monomial generator of $\overline{\mathfrak{a}}$, so does $z_{ij'}$ for any $j'<j$. Thus $i_1, \ldots, i_s$ are all distinct, for otherwise $\mathfrak{p}$ would not be minimal: if two z_{ij} have the same i, one could throw away the one with the larger j. Condition 2) is equivalent to saying $\overline{\mathfrak{a}}\subseteq\mathfrak{p}$, and condition 3) follows from the minimality of $\mathfrak{p}$. Thus every prime ideal in the irredundant representation of $\overline{\mathfrak{a}}$ satisfies 1), 2) and 3). Conversely, any prime ideal satisfying 1), 2) and 3) must occur, because by 2) it contains $\overline{\mathfrak{a}}$, and by 3) is minimal with that property.

Lemma **(4.5)**. — *Let* $\mathfrak{a}\subseteq\mathrm{R}$ *be an ideal generated by monomials in* $x_1, \ldots, x_r$. *Let it define the subscheme* $\mathrm{X}\subseteq\mathbf{P}_k^r$, *and let* $\mathfrak{a}_{(i)}$ *be the ideal of* $\mathrm{X}_i=\mathrm{X}^{r-i}$ (see definition in chapter 2). *Let* $t=r-i$, *and order the t-tuples of distinct indices* $1\leq k_1, \ldots, k_t\leq r$ *in any order, and let* μ *be the number of them. Set* $\mathfrak{a}_0=\mathfrak{a}$, *and define inductively for* $\nu=1, \ldots, \mu$,

$$\mathfrak{a}_\nu=\{y\in\mathrm{R}\mid(x_{k_1}, \ldots, x_{k_t})^n y\subseteq\mathfrak{a}_{\nu-1} \quad \text{for some } n\},$$

using the ν^{th} *t-tuple. Then*

$$\mathfrak{a}_0\subseteq\mathfrak{a}_1\subseteq\cdots\subseteq\mathfrak{a}_\mu,$$

and $\mathfrak{a}_\mu=\mathfrak{a}_{(i)}$.

Proof. — Indeed, let $\mathrm{X}(\nu)$ be the subscheme of $\mathbf{P}_k^r$ defined by $\mathfrak{a}_\nu$. Then $\mathrm{X}(\nu)$ is the closed subscheme of $\mathrm{X}(\nu-1)$ defined by the sheaf of ideals N_ν whose sections are those sections of $\mathcal{O}_{\mathrm{X}(\nu-1)}$ with support in the variety V_ν of the ideal $(x_{k_1}, \ldots, x_{k_t})$. Now since $\mathfrak{a}$ is generated by monomials in $x_1, \ldots, x_r$, every associated prime of $\mathfrak{a}$ of dimension $\leq i$ contains one of the ideals $(x_{k_1}, \ldots, x_{k_t})$, for $t=r-i$. Thus in passing from X to $\mathrm{X}(\mu)$ we kill all sections of $\mathcal{O}_\mathrm{X}$ with support of dimension $\leq i$, and only those, so $\mathrm{X}(\mu)=\mathrm{X}_i$.

Lemma **(4.6)**. — *Let* $\mathfrak{a}\subseteq\mathrm{R}$ *be an ideal generated by monomials in* $x_1, \ldots, x_r$. *Let* $1\leq k_1, \ldots, k_t\leq r$ *be a set of indices and lets*

$$\mathfrak{a}_1=\{y\in\mathrm{R}\mid(x_{k_1}, \ldots, x_{k_t})^n y\subseteq\mathfrak{a} \quad \text{for some } n\}.$$

Then $\overline{\mathfrak{a}}_1$ *is the intersection of those primes* $\mathfrak{p}=(z_{i_1j_1}, \ldots, z_{i_sj_s})$ *of* $\overline{\mathfrak{a}}$ *for which the set* $(k_1, \ldots, k_t)$ *is not contained in the set* $(i_1, \ldots, i_s)$.

Proof. — We use the criteria 1), 2), 3) of Proposition 4.4.

a) Let $\mathfrak{p}=(z_{i_1j_1}, \ldots, z_{i_sj_s})$ be a prime of $\overline{\mathfrak{a}}$, and suppose that

$$(k_1, \ldots, k_t)\nsubseteq(i_1, \ldots, i_s).$$

Then we show that $\mathfrak{p}$ is a prime of $\overline{\mathfrak{a}}_1$. Condition 1) is satisfied trivially. By our hypothesis, there is one of the k's, say k_1, which is different from all the i's. To verify condition 2), let $y\in\mathfrak{a}_1$. Then for some n, $x_{k_1}^n y\in\mathfrak{a}$. But since $\mathfrak{p}$ is a prime of $\overline{\mathfrak{a}}$,

$\mathfrak{a} \subseteq (x_{i_1}^{j_1}, \ldots, x_{i_s}^{j_s})$. Thus $x_{k_1}^n y \in (x_{i_1}^{j_1}, \ldots, x_{i_s}^{j_s})$, and so $y \in (x_{i_1}^{j_1}, \ldots, x_{i_s}^{j_s})$ which proves 2). Condition 3) is satisfied trivially, since $\mathfrak{a} \subseteq \mathfrak{a}_1$.

b) Let $\mathfrak{p} = (z_{i_1 j_1}, \ldots, z_{i_s j_s})$ be an ideal satisfying condition 3) for $\mathfrak{a}_1$ (which will be the case if $\mathfrak{p}$ is a prime of $\overline{\mathfrak{a}}$, since $\mathfrak{a} \subseteq \mathfrak{a}_1$), and suppose that $(k_1, \ldots, k_t) \subseteq (i_1, \ldots, i_s)$. Then we show that $\mathfrak{p}$ is not a prime of $\overline{\mathfrak{a}}_1$, by showing that condition 2) fails. By our hypothesis, $\mathfrak{a}_1$ is not contained in any ideal generated by a proper subset of the $x_{i_k}^{j_k}$. Hence, for each $l = 1, \ldots, s$, there is a monomial $y_l \in \mathfrak{a}_1$ such that y_l is not divisible by any $x_{i_k}^{j_k}$ with $k \neq l$. Write

$$y_l = x_{i_l}^{r_l} \cdot y_l'$$

where y_l' is not divisible by x_{i_l}. Let $y'' = \text{l.c.m.}\{y_l'\}$. Then y'' is not divisible by any $x_{i_k}^{j_k}$, so $y'' \notin (x_{i_1}^{j_1}, \ldots, x_{i_s}^{j_s})$. But for each l, $x_{i_l}^{r_l} \cdot y''$ is a multiple of y_l, hence is in $\mathfrak{a}_1$. Now since $(k_1, \ldots, k_t) \subseteq (i_1, \ldots, i_s)$, for n large enough,

$$(x_{k_1}, \ldots, x_{k_t})^n y'' \subseteq \mathfrak{a},$$

so $y'' \in \mathfrak{a}_1$. This shows that 2) fails for $\mathfrak{a}_1$, so $\mathfrak{p}$ is not a prime of $\overline{\mathfrak{a}}_1$.

c) It follows from *a)* and *b)* that if $\mathfrak{p}$ is a prime of $\overline{\mathfrak{a}}$, then $\mathfrak{p}$ is a prime of $\overline{\mathfrak{a}}_1$ if and only if $(k_1, \ldots, k_t) \not\subseteq (i_1, \ldots, i_s)$. Conversely, let $\mathfrak{p}_1 = (z_{i_1 j_1}, \ldots, z_{i_s j_s})$ be a prime of $\overline{\mathfrak{a}}_1$. Then by *b)* we know that $(k_1, \ldots, k_t) \not\subseteq (i_1, \ldots, i_s)$. On the other hand, since $\overline{\mathfrak{a}} \subseteq \overline{\mathfrak{a}}_1$, $\mathfrak{p}_1$ contains a prime $\mathfrak{p}$ of $\overline{\mathfrak{a}}$, for which it will also be true that $(k_1, \ldots, k_t) \not\subseteq (i_1, \ldots, i_s)$, since its i's are contained among those of $\mathfrak{p}_1$. Hence by *a)*, $\mathfrak{p}$ is a prime of $\overline{\mathfrak{a}}_1$, and $\mathfrak{p} = \mathfrak{p}_1$. q.e.d.

Proposition (**4.7**). — *Let $\mathfrak{a} \subseteq \mathrm{R}$ be an ideal generated by monomials in $x_1, \ldots, x_r$. Let $\mathrm{X} \subseteq \mathbf{P}_k^r$ be the scheme it defines in projective space, and let $\mathfrak{a}_{(i)}$ be the ideal of X_i. Then $\overline{\mathfrak{a}}_{(i)}$ is the intersection of those prime ideals of $\overline{\mathfrak{a}}$ having strictly fewer than $r-i$ generators.*

Proof. — The conclusion follows from Lemmas 4.5 and 4.6, because a prime $\mathfrak{p} = (z_{i_1 j_1}, \ldots, z_{i_s j_s})$ of $\overline{\mathfrak{a}}$ has strictly fewer than $r-i$ generators if and only if for every t-tuple $1 \leq k_1, \ldots, k_t \leq r$ of distinct indices, with $t = r-i$, $(k_1, \ldots, k_t) \not\subseteq (i_1, \ldots, i_s)$.

Definition. — *An ideal $\mathfrak{a} \subseteq \mathrm{R}$ generated by monomials in $x_1, \ldots, x_r$ is balanced if whenever f is a polynomial in $\mathfrak{a}$, and $i < j$ are indices, the polynomial f', obtained by replacing x_j by x_i, is also in $\mathfrak{a}$.*

Proposition (**4.8**). — *Let $\mathfrak{a} \subseteq \mathrm{R}$ be an ideal generated by monomials in $x_1, \ldots, x_r$. If $\mathfrak{a}$ is balanced, then the primes of $\overline{\mathfrak{a}}$ are all of the form*

$$\mathfrak{p} = (z_{1 j_1}, \ldots, z_{s j_s}).$$

Proof. — If $\mathfrak{p} = (z_{i_1 j_1}, \ldots, z_{i_s j_s})$ is a prime of $\overline{\mathfrak{a}}$ and $\mathfrak{a}$ is balanced, then it follows immediately from the conditions of Proposition 4.4 that $(i_1, \ldots, i_s) = (1, 2, \ldots, s)$, possibly in a different order.

Application to Canonical Distractions.

Theorem (4.9). — *Let* $\mathfrak{a} \subseteq \mathrm{R}$ *be an ideal generated by monomials in* $x_1, \ldots, x_r$, *and let* $\mathfrak{a}' \subseteq \mathrm{R}'$ *be its canonical distraction. Then*

1) $\mathfrak{a}'$ *is an intersection of prime ideals of the form*

$$\mathfrak{p} = (x_{i_1} - t_{i_1 j_1} x_0, \ldots, x_{i_s} - t_{i_s j_s} x_0), \qquad \textit{with } i_1 < \ldots < i_s$$

2) *If* $\mathrm{X} \subseteq \mathbf{P}_k^r$ *is the scheme in projective space defined by* $\mathfrak{a}$, *and if* $\mathfrak{a}_{(i)}$ *is the ideal of* X_i (see chapter 2) *then* $(\mathfrak{a}_{(i)})'$ *is the intersection of those prime ideals of* $\mathfrak{a}'$ *having strictly fewer than* $r-i$ *generators.*

3) *If* $\mathfrak{a}$ *is balanced, the prime ideals in* 1) *all have* $i_1, \ldots, i_s = 1, \ldots, s$.

Proof. — Consider the map $\varphi : \overline{\mathrm{R}} \to \mathrm{R}'$ defined by $\varphi(z_{ij}) = x_i - t_{ij} x_0$. Note that the fields $k(x_i, t_{ij})$ and $k(x_i, x_i - t_{ij} x_0)$ are the same, and are generated by the same number of elements. Since the x_i, t_{ij} are all independent indeterminates, it follows that the $x_i - t_{ij} x_0$ are algebraically independent [ZS, vol. I, ch. II, § 12, p. 95 ff]. Hence φ is an injection.

If $\mathfrak{a} \subseteq \mathrm{R}$ is an ideal generated by monomials in $x_1, \ldots, x_r$, then $\varphi(\overline{\mathfrak{a}})$ generates $\mathfrak{a}'$, and for each $\mathfrak{p} = (z_{i_1 j_1}, \ldots, z_{i_s j_s})$, $\varphi(\mathfrak{p})$ generates the prime ideal

$$(x_{i_1} - t_{i_1 j_1} x_0, \ldots, x_{i_s} - t_{i_s j_s} x_0).$$

Thus, using the results of Propositions 4.4, 4.7 and 4.8, we need only show that if $\overline{\mathfrak{a}} = \bigcap \mathfrak{p}$ is the prime representation of $\overline{\mathfrak{a}}$, then $\overline{\mathfrak{a}}\mathrm{R}' = \bigcap \mathfrak{p}\mathrm{R}'$.

Let $\mathrm{T} = k[x_i, x_i - t_{ij} x_0]$, and factor φ as follows:

$$\overline{\mathrm{R}} \to \mathrm{T} \to \mathrm{R}'.$$

Now $\mathrm{T} \cong \overline{\mathrm{R}}[x_0, \ldots, x_r]$, and passing to a polynomial ring preserves primary representations [3, ch. IV, Prop. 11, p. 157]. Hence

$$\overline{\mathfrak{a}}\mathrm{T} = \bigcap \mathfrak{p}\mathrm{T}.$$

Let S be the multiplicative system in T generated by x_0 and all expressions $x_0^q f(t_{ij})$, where $f(t_{ij})$ is a polynomial in the t_{ij} of degree $\leq q$, such that $f(0) \neq 0$. Then $\mathrm{T}_{\mathrm{S}} = \mathrm{R}'_{x_0}$, and we consider the ring inclusions

$$\mathrm{T} \to \mathrm{R}' \to \mathrm{R}'_{x_0}.$$

Since R'_{x_0} is a localization of T, we have

$$\overline{\mathfrak{a}}\mathrm{R}'_{x_0} = \bigcap \mathfrak{p}\mathrm{R}'_{x_0},$$

whence, by contraction to R′,

$$\overline{\mathfrak{a}}\mathrm{R}'_{x_0} \cap \mathrm{R}' = \bigcap (\mathfrak{p}\mathrm{R}'_{x_0} \cap \mathrm{R}').$$

But now observe that x_0 is prime to $\mathfrak{p}\mathrm{R}'$ (obvious since $i_1 < \ldots < i_s$), and x_0 is prime to $\overline{\mathfrak{a}}\mathrm{R}'$ (by Proposition 4.1 (iii) and Theorem 4.2)! Therefore

$$\overline{\mathfrak{a}}\mathrm{R}'_{x_0} \cap \mathrm{R}' = \overline{\mathfrak{a}}\mathrm{R}'$$

and

$$\mathfrak{p}\mathrm{R}'_{x_0} \cap \mathrm{R}' = \mathfrak{p}\mathrm{R}',$$

which completes the proof.

Geometrical Interpretation.

Theorem **(4.10)**. — *Let* $X \subseteq \mathbf{P}_k^r$ *be a subscheme whose ideal* $\mathfrak{a} \subseteq R$ *is generated by monomials in* $x_1, \ldots, x_r$, *and is balanced. Then there is a fan* $X'' \subseteq \mathbf{P}_K^r$, *for suitable* K, *and a linear specialization* $X'' \rightsquigarrow X$.

Proof. — Let $A = k[t_{ij}]_{\mathfrak{m}}$, and let $X' \subseteq \mathbf{P}_A^r$ be the scheme defined by the canonical distraction $\mathfrak{a}'$ of $\mathfrak{a}$. Then by Theorem 4.2, X′ is flat over A, and has closed fibre equal to X. By Theorem 4.9, parts 1) and 3), the general fibre $X'' \subseteq \mathbf{P}_K^r$ is a fan, where K is the quotient field of A.

Let $\mathfrak{a}_{(i)}$ be the ideal of X_i, and let $(X_i)' \subseteq \mathbf{P}_A^r$ be the subscheme defined by the canonical distraction $(\mathfrak{a}_{(i)})'$ of $\mathfrak{a}_{(i)}$. Then $(X_i)'$ is flat over A and has closed fibre X_i. Furthermore, by Theorem 4.9 part 2), the general fibre of $(X_i)'$ is $(X'')_i$! Thus for any i, X_i and $(X'')_i$ have the same Hilbert polynomial, and so $n_*(X) = n_*(X'')$.

CHAPTER 5

THE CONNECTEDNESS THEOREM

We will need the following simple Lemma from group theory.

Lemma (5.1). — *Let* G *be a (set-theoretic) group and let* G_1, H *be subgroups of* G. *Then the following conditions are equivalent:*

(i) G_1 *is a normal subgroup of* G, *and the canonical homomorphism* $\pi : G \to G/G_1$ *induces an isomorphism of* H *onto* G/G_1.

(ii) *There is a group homomorphism* $\theta : G \to H$ *such that*

a) θ *restricted to* H *is the identity on* H.

b) θ *maps* G_1 *to the unit element* $e \in H$.

c) *Fot all* $g \in G$, $\theta(g)^{-1} g \in G_1$.

Definition. — *If* G *is a group and* G_1, H *are subgroups satisfying the equivalent conditions of the Lemma, we say that* G *is the semi-direct product of* G_1 *and* H.

Let G be a group prescheme and let G_1, H be sub-group preschemes. We say that G is the *semi-direct product* of G_1 and H if for every prescheme S, the group G(S) is the semi-direct product of the subgroups $G_1(S)$ and H(S).

Remark. — This is to justify the process of defining maps involving group preschemes by " choosing elements ". For example, we define the inverse map of a group prescheme G into itself by saying " for any $g \in G$, send g into g^{-1} ". What this means is that for any prescheme S, we map the group G(S) into itself by $g \rightsquigarrow g^{-1}$. The map thus defined depends functorially on S, so defines a morphism of the functor h_G into itself, and therefore also a morphism of G into itself, which is the one we want.

More generally, this justifies talking as if the points of a group prescheme formed a group, as if a sub-group prescheme were a subgroup, and so on. For example, with this convention the fact that a group prescheme G is the semi-direct product of sub-group preschemes G_1 and H can be expressed by saying that there is a homomorphism of group preschemes $\theta : G \to H$ with properties *a)*, *b)* and *c)* of the Lemma.

Similarly, one can define subpreschemes by giving elements.

Definition. — *Let* $\alpha : G \times X \to X$ *be an action of a group prescheme* G *on a prescheme* X, *and let* Y *be a subprescheme of* X. *We say that* Y *is stable under the action of* G *if for every prescheme* S, *the subset* Hom(S, Y) *of* Hom(S, X) *is stable under the action of the (set-theoretic) group* Hom(S, G).

Proposition (5.2). — *Let* G *be a group prescheme over a field* k, *acting on a prescheme* X *over* k. *Let* G *be the semi-direct product of sub-group preschemes* G_1 *and* H, *where* H *is isomorphic to the additive group* $\mathbf{G}_a$ *or the multiplicative group* $\mathbf{G}_m$ *over* k. *Let* Y_0 *be a closed subprescheme of* X, *stable under the action of* G_1. *Then there are closed subpreschemes* Y_1 *of* X_K *and* Y_2 *of* X *(where* K *is a suitable extension field of* k*) and linear specializations*

$$\begin{tikzcd} & Y_1 \arrow[dl, squiggly] \arrow[dr] & \\ Y_0 & & Y_2 \end{tikzcd}$$

in X, *such that* Y_2 *is stable the action of* G *(the possibility* $Y_2=\emptyset$ *is not excluded)*.

Proof. — (This proof is an adaptation of the proof of the Borel fixed-point theorem [6, thm. 2, p. 206-05], and [5, thm. 3, p. 5-14].)

Since H is isomorphic to $\mathbf{G}_a$ or $\mathbf{G}_m$ we can embed H as an open subscheme of $\mathbf{P}^1=\mathbf{P}^1_k$ and extend the group action of H on itself to an action of H on $\mathbf{P}^1$, under which the point $\infty\in\mathbf{P}^1$ is stable. Since G is the semi-direct product of G_1 and H, there is a homomorphism of group preschemes $\theta : G\rightarrow H$ having the properties *a)*, *b)* and *c)* of the Lemma above. G acts via θ on $\mathbf{P}^1$; it also acts on X by hypothesis, so we can define the product action of G on $\mathbf{P}^1\times X$ by

$$g(u, x)=(\theta(g)u, gx)$$

for all $g\in G$, $u\in\mathbf{P}^1$, and $x\in X$.

Let Z be the closed subprescheme of $H\times X$ defined by

$$Z=\{(h, hy)\,|\,h\in H, y\in Y_0\}.$$

Then Z is flat over H (since it is isomorphic, over H, to $H\times Y_0$) and its fibre over the unit element $e\in H$ is Y_0. I claim that Z, as a locally closed subprescheme of $\mathbf{P}^1\times X$, is stable under the action of G defined above. Indeed, let $g\in G$ and $(h, hy)\in Z$ (using the conventions of the Remark above). Then

$$g(h, hy)=(\theta(g)h, ghy).$$

But using the properties *a)*, *b)* and *c)* of θ and the hypothesis that Y_0 is stable under the action of G_1, one finds that

$$\theta(g)h=\theta(gh)$$

and that there is a $y'\in Y_0$ such that

$$ghy=\theta(gh)y'.$$

Therefore $$g(h, hy) = (\theta(gh), \theta(gh)y')$$

is in Z and so Z is stable under the action of G.

We now apply Proposition 1.4 and let $\overline{Z}$ be the unique extension of Z to a closed subprescheme of $\mathbf{P}^1 \times X$, flat over $\mathbf{P}^1$. Then clearly $G \times \overline{Z}$ is the unique extension of $G \times Z$ to a closed subprescheme of $G \times \mathbf{P}^1 \times X$, flat over $\mathbf{P}^1$, and so by the functorial property of such extensions, $G \times \overline{Z}$ maps to $\overline{Z}$, i.e. $\overline{Z}$ is stable under the action of G.

Let Y_1 be the fibre of $\overline{Z}$ over the generic point of $\mathbf{P}^1$ and let Y_2 be the fibre of $\overline{Z}$ over the point $W \in \mathbf{P}^1$. Then by results of Chapter 1, there are linear specializations

$$\begin{tikzcd} & Y_1 \arrow[dl, rightsquigarrow] \arrow[dr] & \\ Y_0 & & Y_2. \end{tikzcd}$$

Furthermore, Y_2 is stable under the action of G since it is the fibred product over $\mathbf{P}$ of the stable subpreschemes $\overline{Z}$ and ∞.

Corollary **(5.3)**. — *Let k be a field and let Y_0 be a closed subscheme of $\mathbf{P}^r_k$. Then there is a connected sequence of linear specializations*

$$Y_0, Y_1, \ldots, Y_s$$

in $\mathbf{P}^r$, such that $Y_s \subseteq \mathbf{P}^r_k$ is stable under the action of the triangular group scheme $T(r+1)$ of matrices (a_{ij}), $0 \leq i, j \leq r$, with $a_{ij} = 0$ for $i < j$. Furthermore, $n_*(Y_s) \geq n_*(Y_0)$.

Proof. — It is well known that $T(r+1)$ is solvable, and has a composition series whose quotients are isomorphic to copies of $\mathbf{G}_a$ or $\mathbf{G}_m$. Moreover, one can find a composition series which is a semi-direct product at each step, so we need only apply the proposition to each step, starting at the bottom.

For the last statement, note that in the proposition if X is projective, then $n_*(Y_2) \geq n_*(Y_0)$. For Y_1 is obtained from Y_0 by an automorphism of X, so $n_*(Y_1) = n_*(Y_0)$, and $n_*(Y_2) \geq n_*(Y_1)$ by Theorem 2.10.

Proposition **(5.4)**. — *Let Y be a closed subscheme of $\mathbf{P}^r_k$ and let $\mathfrak{a} \subseteq k[x_0, x_1, \ldots, x_r]$ be the ideal of Y. If Y is stable under the action of $T(r+1)$ (where $T(r+1)$ acts on the coordinates of $\mathbf{P}^r$ in the order $x_1, \ldots, x_r, x_0$), then $\mathfrak{a}$ is generated by monomials in $x_1, \ldots, x_r$, and is balanced (see definition in Chapter 4).*

Proof. — Making a base field extension, one reduces to the case where k is an infinite field. If Y is stable under the action of $T(r+1)$, then $\mathfrak{a}$ must be stable under the action of $T(r+1)$ on $k[x_0, x_1, \ldots, x_r]$. This action is defined by

$$x_i \rightsquigarrow \sum_{j=1}^{r+1} a_{ij} x_j \qquad i = 1, \ldots, r+1,$$

where $(a_{ij}) \in \mathrm{T}(r+1)$ and where we have relabeled x_0 as x_{r+1}. We consider in particular matrices of the forms

$$\begin{pmatrix} t_1 & & & & & 0 \\ & t_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & t_{r+1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & \vdots & & & 0 \\ & 1 & \vdots & & & \\ & & \vdots & \cdot & & \\ & & \vdots & & \cdot & \\ \cdots & \cdots & 1 & \cdots & \cdots & \cdots \\ 0 & & \vdots & & \ddots & 1 \end{pmatrix}$$

where $t_i \in k$, $\Pi t_i \neq 0$, and where the second is an identity matrix with an extra 1 in the i^{th} column and the j^{th} row. Acting on $k[x_1, \ldots, x_{r+1}]$ with these matrices, we see that if $f(x_1, \ldots, x_{r+1})$ is a polynomial in $\mathfrak{a}$, then $\mathfrak{a}$ contains also

a) the polynomial obtained by replacing each x_i by $t_i x_i$ in f, and

b) the polynomial obtained by replacing x_j by $x_i + x_j$ in f,

for any $i < j$.

From *a)* and the fact that k is infinite it follows that $\mathfrak{a}$ is generated by monomials in $x_1, \ldots, x_{r+1}$ (see the remarks on monomial ideals in Chapter 4). From *b)* and the fact that $\mathfrak{a}$ is generated by monomials it follows that $\mathfrak{a}$ also contains the polynomial obtained by replacing x_j by x_i in f, for any $i < j$. Therefore one sees easily that the associated prime ideals of $\mathfrak{a}$ are all of the form $(x_1, \ldots, x_q)$ for various q. But $(x_1, \ldots, x_{r+1})$ cannot occur, since $\mathfrak{a}$ is *the* ideal of Y. Thus $\mathfrak{a}$ is generated by monomials in $x_1, \ldots, x_r$ and is balanced.

Recall from Chapter 3 the definition of a *numerical polynomial*, the notation

$$g(n, r) = \binom{z+r}{r+1} - \binom{z+r-n}{r+1}$$

for any $r, n \in \mathbf{Z}$, $r \geq 0$, and the fact that any numerical polynomial $p(z)$ can be written uniquely in the form

$$(*) \qquad p(z) = \sum_{t=0}^{\infty} g(m_t, t)$$

with $m_t \in \mathbf{Z}$.

Proposition (5.5). — *Let $p \in \mathbf{Q}[z]$ be a numerical polynomial of degree $< r$, whose expression in the form $(*)$ above has*

$$m_0 \geq m_1 \geq \cdots \geq m_{r-1} \geq 0.$$

Then there exists a proper subscheme X *of $\mathbf{P}^r_{\mathbf{Z}}$ flat over* $\mathbf{Z}$ *whose fibre at every point of* $\mathbf{Z}$ *has Hilbert polynomial p.*

Proof. — Let k be an infinite field. Then we can find a tight fan $\mathrm{X}'' \subseteq \mathbf{P}^r_k$ with

$$n_*(\mathrm{X}'') = (m_{r-1}, m_{r-2} - m_{r-1}, \ldots, m_0 - m_1)$$

since k is infinite and all the prescribed n_i are ≥ 0. Then by Corollary 3.3, X″ has Hilbert polynomial p. Applying Corollary 5.3, we can find a subscheme X′ of $\mathbf{P}^r_k$ also with Hilbert polynomial p (since specialization preserves Hilbert polynomials), and whose ideal in $k[x_0, \ldots, x_r]$ is generated by (monic) monomials in $x_1, \ldots, x_r$ (Proposition 5.4). Let $\mathfrak{a} \subseteq \mathbf{Z}[x_0, \ldots, x_r]$ be the ideal generated by the same monomials in $x_1, \ldots, x_r$ as the ideal of X'. Then $\mathfrak{a}$ defines a closed subscheme of $\mathbf{P}^r_{\mathbf{Z}}$, clearly flat over $\mathbf{Z}$, whose Hilbert polynomial at every point of $\mathbf{Z}$ is p, since the Hilbert polynomial of the quotient of a polynomial ring by an ideal generated by monomials is independent of the base field.

Theorem **(5.6)**. — *Let k be a field and let* X *be a closed subscheme of* $\mathbf{P}^r_k$. *Then there is a connected sequence of linear specializations*

$$X = X_1, X_2, \ldots, X_n = X'$$

in $\mathbf{P}^r$, *where* $X' \subseteq \mathbf{P}^r_{k'}$ *(k' a field containing k) is a tight fan.*

Proof. — Let X be given. Then by Corollary 5.3 there is a connected sequence of linear specializations joining X to a subscheme X_1 of $\mathbf{P}^r_k$, stable under the action of $T(r+1)$ (where we take $T(r+1)$ to act on the coordinates of $\mathbf{P}^r$ in the order $x_1, \ldots, x_r, x_0$). Furthermore $n_*(X_1) \geq n_*(X)$. By Proposition 5.4, the ideal of X_1 is generated by monomials in $x_1, \ldots, x_r$ and is balanced. Therefore by Theorem 4.10 there is a fan X_2 and a linear specialization $X_2 \rightsquigarrow X_1$, such that $n_*(X_2) = n_*(X_1)$. Then by Corollary 3.7 there is a connected sequence of linear specializations joining X_2 to a subscheme X_3 of $\mathbf{P}^r$ such that either X_3 is a tight fan (in which case we have finished) or X_3 is a subscheme with $n_*(X_3) > n_*(X_2)$, and hence $n_*(X_3) > n_*(X)$.

In the latter case we start over with X_3. Proceeding in this manner we must reach a tight fan after a finite number of steps, because by Corollary 3.10 the possible n_*'s of fans with a given Hilbert polynomial form a finite set.

Corollary **(5.7)**. — *Let k be a field, $r>0$ an integer and $p \in \mathbf{Q}[z]$ a numerical polynomial. Then a necessary and sufficient condition that p be the Hilbert polynomial of a proper closed subscheme of $\mathbf{P}^r_k$ is that when p is written in the form (*) above,*

$$m_0 \geq m_1 \geq \cdots \geq m_{r-1} \geq 0$$

and

$$m_r = m_{r+1} = \cdots = 0 \quad (^1).$$

Proof. — The necessity follows from the theorem and Corollary 3.3, and the sufficiency from Proposition 5.5 (make the base extension $\mathbf{Z} \to k$).

Theorem **(5.8)**. — *Let* S *be an arbitrary prescheme, $r>0$ an integer, and $p \in \mathbf{Q}[z]$ a numerical polynomial satisfying the condition of Corollary* 5.7. *Then*

$$f : \mathbf{Hilb}^p_{\mathbf{P}^r_S/S} \to S$$

is a linearly connected morphism of functors.

Proof. — Using Theorem 5.6, Proposition 3.2, and the criterion of Proposition 1.12, we see that for any field k, the functor

$$\mathbf{Hilb}^p_{\mathbf{P}^r_k/k}$$

(1) Professor J.-P. Serre has called my attention to the fact that this result was already known to Macaulay [15].

is linearly connected over k. Thus the fibres of f are geometrically linearly connected functors. Using Proposition 5.5 and making the base extension $S \to \operatorname{Spec} \mathbf{Z}$, we see that f has a section. (Note that to say $m_r = m_{r+1} = \cdots = 0$ is equivalent to saying degree $p < r$.)

Corollary* **(5.9) (Connectedness of the Hilbert Scheme). — *Let* S *be a connected* (resp. *geometrically connected*; resp. *linearly connected*; resp. *geometrically linearly connected*) *noetherian prescheme, let* $r > 0$ *be an integer, and* $p \in \mathbf{Q}[z]$ *a numerical polynomial. Then*

$$Hilb^p(\mathbf{P}^r_S/S)$$

is a connected (resp. *geometrically connected*, etc.) *prescheme, which is non-empty* $\Leftrightarrow S \neq \varnothing$ *and* p *satisfies the condition of Corollary* 5.7 *(or* $p = \sum_{t=0}^{r} g(1, t)$ *is the Hilbert polynomial of projective r-space itself).*

Proof. — Follows from the theorem and Proposition 1.8 and its analogues.*

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