Complex analysis

Based on lectures by Richard Borcherds
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These notes are based on Richard Borcherds's YouTube series on complex analysis[1].

[1] See https://www.youtube.com/playlist?list=PL8yHsr3EFj537_iYA5QrvwhvMlpkJ1yGN.

1 Introduction

Many things in complex analysis are similar to things in real analysis: we have the usual arithmetic operations, exponentials, trigonometric functions, differentiation, integration, limits, series, etc.

There are notable differences between real and complex analysis:

– We can write trigonometric functions in terms of exponentials:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2};$$

– If a complex function is differentiable, it is infinitely differentiable.

In real analysis, we can interpret

$$\int_0^1 f(x)\,dx$$

in only one way: there is only one path from 0 to 1. But there are infinitely many in complex analysis. Cauchy's theorem resolves this problem: the integral is almost independent of the path from 0 to 1.

One doesn't learn how to evaluate integrals like

$$\int_0^\infty \frac{\sin x}{x}\,dx = \pi/2$$

or sums like

$$\sum_n \frac{1}{n^2} = \pi^2/6$$

in introductory calculus courses. (The antiderivative of $(\sin x)/x$ does not have a closed form.) We will use complex integration to compute integrals and sums like these.

Further, any complex differentiable function on (for instance) $(0,1)$ has a unique continuation on any larger open connected set. For example, Riemann's ζ function

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

converges only if $\Re(s) > 1$. However, ζ has an analytic continuation to the plane, and only considering this continuation can one state the Riemann hypothesis.

Recall the construction of the Mandelbrot set: Fix a complex number c. Let $z_0 = 0$, and consider the sequence $\{z_0, z_1 = z_0^2 + c, z_2 = z_1^2 + c, \ldots\}$. If this sequence is bounded, then c

is in the Mandelbrot set. Despite having such a simple definition, the Mandelbrot set is incredibly intricate. The study of things like the Mandelbrot set is complex dynamics.

"If you want to check whether a textbook on complex analysis is good or not, there's a very simple test: you check to see if it has a section on the gamma function and a section on elliptic functions. If it doesn't have these sections, then the author doesn't really understand complex analysis. [Reading such a textbook] is like reading a book on music by somebody who is tone deaf." Dr. Borcherds recommends something like Complex Analysis by Lars Ahlfors.

2 Arithmetic

Operations on complex numbers are defined in the obvious way:

- $(a+ib)+(c+id)=(a+c)+i(b+d)$;
- $(a+ib)-(c+id)=(a-c)+i(b-d)$;
- $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$.

One can do cumbersome computations to check distributivity, associativity, etc. However, one sees that $\mathbf{C}=\mathbf{R}[i]/(i^2+1)$ is a ring.

Recall that complex conjugation (given by $\overline{a+ib}=a-ib$) preserves all properties of the complex numbers. (In fact, $z\mapsto\overline{z}$ is an automorphism of $\mathbf{C}$.) One notices that

$$z^{-1}=\frac{\overline{z}}{|z|^2}.$$

"You must never use Ξ as a complex variable!" Here's why:

$$\frac{\Xi}{\overline{\Xi}}.$$

"[This] wins the prize for the most awful mathematical notation of all time."

Another useful notion is the norm: $|z|=\sqrt{z\overline{z}}$. In particular, we have $|zw|=|z|\cdot|w|$ and $|z-w|\leq|z|+|w|$.

One makes $\mathbf{C}$ a metric space by defining the distance between z and w by $|z-w|$.

Problem 2.1. What integers can be written as a sum of two squares?

One notices that the set of these integers is closed under multiplication: $(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2$. We have $a^2+b^2=|a+ib|^2$ and $c^2+d^2=|c+id|^2$. Then

$$\begin{aligned}(a^2+b^2)(c^2+d^2)&=|(a+ib)(c+id)|^2\\&=|(ac-bd)+i(ad+bc)|^2\\&=(ac-bd)^2+(ad+bc)^2.\end{aligned}$$

For example, $5 = 1^2 + 2^2$ and $13 = 2^2 + 3^2$. We have $5 \cdot 13 = 65 = 8^2 + 1^2 = 4^2 + 7^2$. Why are there two ways? Well, $|1 + 2i|^2 = 5$ and $|2 + 3i|^2 = 13$, but we also have $|2 - 3i|^2 = 13$. So $(1 + 2i)(2 + 3i) = -4 + 7i$ gives one solution, and $(1 + 2i)(2 - 3i) = 8 + i$ gives another. (Also, squaring any Gaussian integer generates a Pythagorean triple.)

Hamilton came up with the quaternions—an extension of the complex numbers—wherein $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. They are not commutative. If $z = a + bi + cj + dk$, we define $\overline{z} = a - bi - cj - dk$, and we find that $z\overline{z} = a^2 + b^2 + c^2 + d^2$. Moreover, just like in the complex case, we let

$$(a + bi + cj + dk)^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}.$$

Sums of three squares are not closed under multiplication, but sums of four squares are. (We show this as we did for the sum of two squares case.)

3 exp, log, sin, cos

We define $\exp z = e^z$ in the expected way:

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_k \frac{z^k}{k!}.$$

This is absolutely convergent. We can show that $\exp(z_1 + z_2) = (\exp z_1)(\exp z_2)$ just as one does for the real case (by expanding and rearranging) since it is absolutely convergent.

If $z = \alpha + i\beta$, then $\exp z = (\exp \alpha)(\exp i\beta)$. We know what $\exp \alpha$ is since α is real, but what about $\exp i\beta$? Let's try it:

$$\begin{aligned}
\exp i\beta &= 1 + i\beta + \frac{i^2\beta^2}{2!} + \frac{i^3\beta^3}{3!} + \frac{i^4\beta^4}{4!} + \frac{i^5\beta^5}{5!} + \cdots \\
&= 1 + i\beta - \frac{\beta^2}{2!} - i\frac{\beta^3}{3!} + \frac{\beta^4}{4!} + i\frac{\beta^5}{5!} + \cdots \\
&= 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \cdots + i\beta - i\frac{\beta^3}{3!} + i\frac{\beta^5}{5!} + \cdots \\
&= \cos \beta + i \sin \beta.
\end{aligned}$$

One sees that $|\exp z| = \exp \Re z$. The map $\exp : \mathbf{C} \longrightarrow \mathbf{C}^*$ is a surjective group homomorphism. Further, $\ker \exp = 2\pi i\mathbf{Z}$.

Suppose we want to solve $\exp w = z$. Write $z = r(\cos\theta + i \sin\theta)$. Since $\arg z$ is defined up to multiples of 2π, $\log z$ is defined up to multiples of $2\pi i$, since $\log z = \log |z| + i \arg z$.

What about $z_1^{z_2}$? That is just $\exp(z_2 \log z_1)$, but $\log z_1$ is ambiguous. It is well defined if $z_1 > 0$ is real and we take $\log z_1$ real. It is also okay if $n = z_2$ is an integer, since $\exp(n2\pi i) = 1$.

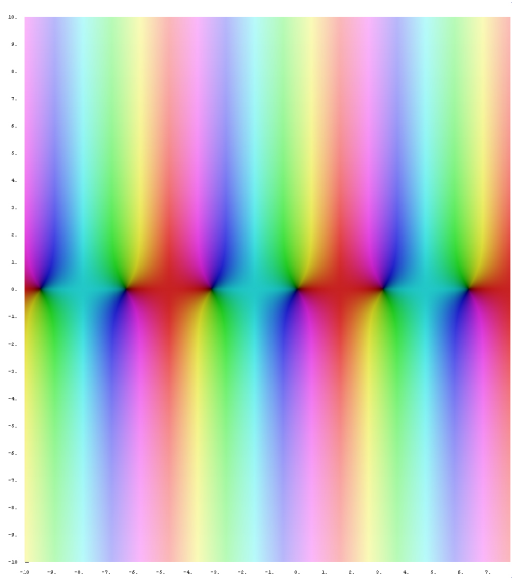

Figure 1: A domain colouring of $\sin z$ where brightness indicates $|\sin z|$ and hue indicates $\arg(\sin z)$.

We have $\exp iz = \cos z + i \sin z$, hence $\exp(-iz) = \cos z - i \sin z$. So

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

In the same way,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

One can derive any trigonometric identity using these definitions. The differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

has a solution $y = e^{\lambda x}$. For instance, if

$$\frac{d^2y}{dx} + 2\frac{dy}{dx} + 2y = 0,$$

we have

$$\lambda^2 + 2\lambda + 2 = 0,$$

so $\lambda = -1 \pm i$. So the solutions to this differential equation are $y = e^{-(1+i)x}$ and $y = e^{-(1-i)x}$, (one may write these in terms of cos and sin).

Complex numbers simplify Fourier series. If $f(x) = f(2\pi + x)$, we can write

$$\begin{aligned} f(x) &= \sum_{n>0} a_n \sin nx + \sum_{n>0} b_n \cos nx \\ &= \sum_{n\in\mathbf{Z}} c_n e^{inx},\ c_n \in \mathbf{C}. \end{aligned}$$

The function

$$\tan z := \frac{\sin z}{\cos z}$$

is defined in the obvious way. On the upper half plane, $\tan z \approx i$ and on the lower half plane, $\tan z \approx -i$. Near the real axis $\tan z$ oscillates like it does in the real case. (In real analysis one thinks of sin and cos bounded and tan being "wild," and in complex analysis it is approximately the opposite.)

Exercise 3.1. Express $\arccos z$ using log and $\sqrt{\ }$.

Notice that

$$\begin{aligned} \sin(iz) &= i \sinh z; \\ \cos(iz) &= \cosh z; \\ \tan(iz) &= i \tanh z. \end{aligned}$$

4 Holomorphic functions

A function $f : \mathbf{R} \longrightarrow \mathbf{R}$ is called differentiable at $x_0 \in \mathbf{R}$ is f is approximately linear there: that is, $f(x) = f(x_0) + a(x - x_0) + \text{small error}$. ("Small error" means less than any nonzero linear function, or error$/(x - x_0)$ goes to 0 as x goes to x_0.) Equivalently, f is differentiable at x_0 if a certain limit exists.

Suppose $w = u + iv$ is a function of $z = x + iy$ (where u and v are functions of x and y). We write u and v as a vector that we want to be approximately linear:

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \text{error}.$$

We can approximate w as a linear function. There's nothing really complex going on yet. Above is differentiability of real functions in two variables.

Now we seek a complex derivative. The function w is differentiable if

$$w(z) = w(z_0) + A(z - z_0) + \text{small error}.$$

Here, A is a complex number. Hence, for w to be differentiable as a complex function (from the real two-variable case above), we must also have

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} = \begin{pmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{pmatrix}.$$

That is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are called the *Cauchy–Riemann equations*. We also have the expected limit definition:

$$\lim_{z \to z_0} \frac{w(z) - w(z_0)}{z - z_0}.$$

Definition 4.1. Suppose w is a complex function of $z \in U \subset \mathbf{C}$ where U is an open set. The function w is called *holomorphic* if it has a complex derivative everywhere. (That is, it is continuous with real derivatives satisfying the Cauchy–Riemann equations.)

Remark 4.2. Sometimes you will see "analytic" instead of "holomorphic." A function is *analytic* if it has a power series expansion at each point. Complex functions are holomorphic if and only if they are analytic.

There is another way of getting the Cauchy–Riemann equations. We define

$$\partial/\partial z := \frac{1}{2}(\partial/\partial x - i\partial/\partial y);$$
$$\partial/\partial \overline{z} := \frac{1}{2}(\partial/\partial x + i\partial/\partial y);$$

these are called *Wirtinger derivatives*. These are chosen such that

$$(\partial/\partial z)z = 1;\ (\partial/\partial z)\overline{z} = 0;\ (\partial/\partial \overline{z})z = 0;\ (\partial/\partial \overline{z})\overline{z} = 1.$$

Hence the Cauchy–Riemann equations are equivalent to

$$\frac{\partial w}{\partial \overline{z}} = 0.$$

Informally, "holomorphic" means "depends on z but not on $\overline{z}$."

Example 4.3 (Holomorphic functions)**.**

- The functions 1 and z are holomorphic (since they are linear).

- Suppose f and g are holomorphic on $U \subset \mathbf{C}$. Then $f+g$, $f-g$, fg, f/g, and $f \circ g$ are holomorphic.
- If a power series $a_0 + a_1 z + a_2 z^2 + \cdots$ converges for $|z| < R$, then it is holomorphic and its derivative is $a_1 + 2a_2 z + \cdots$.
- Hence sin, cos, tan, exp, and log are holomorphic when defined.
- If f is holomorphic, then its derivative is.

Example 4.4 (Non-holomorphic functions)**.**

- $\Re z$;
- $\Im z$;
- $|z|$;
- $|z|^2$;
- $\overline{z}$.

What polynomials in x and y are holomorphic? Equivalently, what polynomials in $z = x + iy$ and $\overline{z} = x - iy$ are holomorphic? When is $\sum a_{mn} z^m \overline{z}^n$ holomorphic? We need

$$\partial/\partial\overline{z} = \sum a_{mn} z^m n \overline{z}^{n-1} = 0.$$

So $a_{mn} n = 0$ for all m and n, hence $a_{mn} = 0$ if $n > 0$. So $f(x,y) = \sum a_{m0} z^m$ is a polynomial in z.

5 Harmonic functions

Suppose $w = u + iv$ is a complex function of $z = x + iy$. Given u of x and y, can we find a holomorphic function w such that $u = \Re w$? In general, no. If a complex function is holomorphic, it satisfies the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These imply that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 y}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}.$$

In particular,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This is the Laplace equation, and functions that satisfy it are called harmonic functions. It has a physical interpretation: it gives the steady state heat of some planar region. (Equivalently one writes $\nabla u = 0$.)

We use this to find solutions to the Laplace equation by taking holomorphic function and taking its real or imaginary part.

Example 5.1. Find all harmonic polynomials in x and y. The polynomials that result from taking the real and imaginary parts of powers of $z = x + iy$ are harmonic, and they happen to be a basis for harmonic polynomials.

Example 5.2. The function

$$ze^z = (x + iy)e^x(\cos y + i \sin y)$$

is holomorphic. Hence the real part of this is $e^x(x \cos y - y \sin y)$. This is harmonic.

Is any harmonic function u the real part of some holomorphic function $w = u + iv$? It is useful to take u as a function on an open set $U \subset \mathbf{C}$. The answer depends on U.

We are trying to solve the Cauchy–Riemann equations. They determine v up to a constsnt.

Problem 5.3. Given f and g functions of x and y, can we solve $\partial v/\partial x = f$ and $\partial v/\partial y = g$?

We need $\partial f/\partial y = \partial f/\partial x$ because both sides equal $\partial^2 v/\partial x \partial y$. Sometimes we can solve for v. Suppose U is a rectangle containing the origin. Put $v(0, 0) = 0$. We see that

$$v(x, 0) = \int_0^x f(x, 0)\, dx$$

and

$$v(x, y) = v(x, 0) + \int_0^y g(x, y)\, dy.$$

Clearly $\partial v/\partial y = g$. Is $\partial v/\partial x = f$? It is true on the x axis by definition. We see

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial}{\partial x} v(x, 0) + \int_0^y \frac{\partial g}{\partial x}(x, y)\, dy \\ &= f(x, 0) + \int_0^y \frac{\partial f}{\partial y}\, dy \\ &= f(x, y). \end{aligned}$$

So $\partial v/\partial x = f$.

Suppose we are given u on a rectangle and suppose u is harmonic. Then we can find $w = u + iv$ with w holomorphic. We need to solve $\partial v/\partial y = \partial u/\partial x = g$ and $\partial v/\partial y = -\partial u/\partial y = f$. We need $\partial g/\partial x = \partial f/\partial y$, and that follows because u is harmonic.

What about other regions U? Suppose $U = \mathbf{C} \setminus \{0\}$. Take $u = \log r$. The function u is harmonic (it is more-or-less the fundamental solution of the Laplace equation) except at $x = y = 0$. Further, $u = \Re(\log z)$, and we take $v = \Im(\log z) = \arg z$. However, $\log z$ is not defined on all of U since $\arg z$ is not defined. (If we take a disk and "extend" it into a bigger region by taking disks that overlap in a circular formation around some point, we get a problem when the last disk overlaps the original one: Is $\arg z$ near 0 or 2π? This is the problem of being defined up to a constant.) If a region U has "holes," we cannot extend a harmonic function u to a holomorphic function w.

If U is simply connected, then any harmonic function u is the real part of a holomorphic function w where $\Im w$ is unique up to a constant.

In differential geometry, one might write

$$\partial v/\partial x = f,\ \partial v/\partial y = g$$

as a 1-form

$$dv = (\partial v/\partial x)dx + (\partial v/\partial y)dy.$$

Does there exist a v such that $dv = \omega = f\,dx + g\,dy$? We need $d\omega = 0$, or $df/dy = dg/dx$. We saw that $\omega = dv$ for some v if and only if $dv = 0$ on a simply connected region U.

Definition 5.4. The *de Rham cohomology group* of an open region U of the complex numbers $H^1(U)$ is the set of closed 1-forms ω ($d\omega = 0$) modulo the set of exact forms ω ($\omega = dv$ for some v).

We have shown that U is simply connected implies $H^1(U) = 0$.

6 Integration

We cannot really integrate a complex function f from α to β since there are many paths between them. So, we define an integral over a path C from α to β.

If α and β are real and we consider the path along the real line, we write $f = g + ih$ for real functions g and h, and then we have

$$\int_\alpha^\beta f(z)\,dz = \int_\alpha^\beta g(x)\,dx + i\int_\alpha^\beta h(x)\,dx.$$

Recall that an integral is a "sum."

A path C from α to β is given by a function $\rho : [r, s] \longrightarrow \mathbf{C}$ where $\rho(r) = \alpha$ and $\rho(s) = \beta$. Now suppose that ρ has a continuous derivative and we consider

$$\int_C f(z)\, dz.$$

We write $z = \rho(x)$ and $dz = \rho'(x)\, dx$ (the proof of this is similar to that in the real case). Hence

$$\int_C f(z)\, dz = \int_r^s f(\rho(x))\rho'(x)\, dx$$

This is an alternative definition (to the "limit of a sum" one).

Now write $f = u + iv$ and $z = x + iy$. We can write a complex integral as a path integral:

$$\begin{aligned} \int_C f(z)\, dz &= \int_C (u + iv)\,(dx + idy) \\ &= \int_C u\, dx - v\, dy + i \int_C v\, dx + u\, dy. \end{aligned}$$

Notice that

$$\int_C f(z)\, dz$$

is almost independent of the parameterization ρ since the summands of the integral are independent of the parameterization. However, we need to pay attention to the direction of the path (that is, if $\rho(r) = \beta$ and $\rho(s) = \alpha$). In this case, a sign changes.

Some properties:

–

$$\int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz = \int_{C_1 \cup C_2} f(z)\, dz;$$

–

$$\int_C f(z) + g(z)\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz;$$

– If M is an upper bound for $|f|$ and $|C|$ is the length of C, then

$$\left| \int_C f(z)\, dz \right| \leq M\, |C|\, .$$

Example 6.1. Compute

$$\int_C z^n \, dz$$

where C is a path from 1 to 1. If C is the trivial path (the parameterization $\rho = 1$), then (clearly)

$$\int_C z^n \, dz = 0.$$

Instead, if C is given by

$$\rho : [0, 2\pi] \longrightarrow \mathbf{C} : \alpha \longmapsto e^{i\alpha},$$

then

$$\begin{aligned} \int_C z^n \, dz &= \int_0^{2\pi} e^{inx} i e^{ix} \, dx \\ &= i \int_0^{2\pi} e^{i(n+1)x} \, dx \\ &= \begin{cases} 2\pi i & n = -1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So the integral does depend on the path.

7 Cauchy's theorem

Theorem 7.1 (Cauchy)**.** *Suppose C_1 and C_2 are homotopic paths and w is holomorphic. Then*

$$\int_{C_1} w(z) \, dz = \int_{C_2} w(z) \, dz.$$

Recall that "homotopic" means you can slide one path into the other while keeping the endpoints fixed.

Theorem 7.2 (Cauchy)**.** *Suppose C is homotopic to a constant curve and w is holomorphic (in C). Then*

$$\int_C w(z) \, dz = 0.$$

One can see that these are equivalent. We will prove a weaker version, and we will use Green's theorem.

Theorem 7.3 (Green).

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_D \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\,dx\,dy$$

where D is the interior of C.

Hence

$$\begin{aligned}\int_C w(z)\,dz &= \int_C (u+iv)(dx+i\,dy)\\ &= \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx\\ &= \iint_D \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\,dx\,dy + i\iint_D \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\,dx\,dy\\ &= 0.\end{aligned}$$

1. In a simply connected region, a holomorphic function w has an antiderivative. We might define it as

   $$\int_\alpha^z w(z)\,dz,$$

   but is this well defined? There are many paths from α to z. Since the region is simply connected, an two paths from α to z are homotopic, so the integrals are the same.

2. What is

   $$\int_C \frac{z^2+3}{z^4-2z+3}\,dz$$

   where C is the circle of radius two? We cannot just apply Cauchy's theorem, since z^4-2z+3 has roots inside C. (In particular it is not holomorphic.) Instead, let C be the circle of radius $R>2$. We see that

   $$\begin{aligned}\int_C \frac{z^2+3}{z^4-2z+3}\,dz &\le 2\pi R\cdot(2/R^2)\\ &\le 4\pi/R\end{aligned}$$

   for all real $R>2$. Hence the integral is 0.

8 Cauchy's integral formula

If f is holomorphic in U, then

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz$$

where C goes around w once counterclockwise.

Proof. Write

$$f(z) = \text{constant} + g(z)$$

where $g(0) = 0$. If f is constant we need to show that

$$\int_C 1/z\, dz = 2\pi i$$

If we let C be a circle of radius r and $z = re^{i\theta}$ then

$$\begin{aligned}\int_C 1/z\, dz &= \int_0^{2\pi} r^{-1}e^{-i\theta} rie^{i\theta}\, d\theta \\ &= 2\pi i.\end{aligned}$$

If g vanishes at 0, we want to show that

$$\int_C \frac{g(z)}{z}\, dz = 0.$$

Now we let C be a circle of small radius r. Then if ε is an upper bound for g

$$\begin{aligned}\int_C \frac{g(z)}{z}\, dz &\leq |C| \sup |g(z)/z| \\ &\leq 2\pi r(\varepsilon/r) \\ &= 2\pi\varepsilon.\end{aligned}$$

□

1. Given a holomorphic function on a contour C, f is determined inside C.
2. Suppose f is defined on a disk U. Then

$$|f(w)| \leq \sup_{z\in\partial U} f(z).$$

3. We know that

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\, dz.$$

   Then taking the nth derivative

$$f^{(n)}(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-w)^{n+1}} n!\, dz.$$

   Differentiating under the integral sign is valid because C has finite length and $f(z)/(z-w)$ has continuous derivatives near C. So the nth derivative of f exists! That is, if f is differentiable (in an open region) then it is infinitely differentiable.

Theorem 8.1 (Liouville). *If f is bounded and holomorphic in $\mathbf{C}$ then f is constant.*

Proof. Suppose C is a circle of large radius R. We know that

$$f'(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w)^2}\, dz.$$

Then

$$\begin{aligned} \left|f'(w)\right| &\leq (1/2\pi)(2\pi R)(\sup f)/R^2 \\ &\leq \frac{\sup f}{R}. \end{aligned}$$

Since f is bounded, $f'(w) = 0$ at all w. Hence f is constant. □

Exercise 8.2. Suppose $|f|$ is bounded by some $g \in \mathbf{C}[z]$. Show that f is a polynomial.

4. Any holomorphic function can be expanded as a power series. (This is not true for the real function

$$f(x) = \begin{cases} 0 & x = 0 \\ e^{-1/x^2} & \text{otherwise} \end{cases}$$

though it is infinitely differentiable.)

Suppose the Taylor series $\sum_k a_k z^k$ converges for some z_0. Then $a_n z_0^n$ is bounded, and $\sum_k a_k z^k$ converges if $|z| < |z_0|$. (The series converges if $|z| < R$ and diverges if $|z| > R$. Don't ask about what happens when $|z| = R$.)

Theorem 8.3 (Morera). *If*

$$\int_C f(z)\, dz = 0$$

then f is holomorphic.

Proof. Define

$$F(z) := \int_a^z f(z)\, dz.$$

This is well-defined. Then check that F has a derivative f. So F is holomorphic, and $F' = f$ is holomorphic. □

9 Analytic continuation

Suppose U is a connected open set and f is holomorphic on U. If you know what f is on any subregion of U, then f is determined uniquely on the rest of U. (What?!)

The point is that if f is holomorphic on U and f is not identically 0 then the roots of f are discrete in U (no limit points in U).

Suppose 0 is a limit point of the roots of f with $0 \in U$. Then $f = 0$ near 0. Otherwise,

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots ,$$

and we can assume that $a_n \neq 0$. So

$$f(z) = z^n (a_n + a_{n+1} z + \cdots),$$

and the parenthetical sum is continuous and nonzero at $z = 0$, so it is nonzero near $z = 0$. This contradicts that 0 is a limit point of the roots of f.

Now let V be the largest open subset of U with $f = 0$ on V. Then V is closed under taking limits by the above. And V is nonempty because we assumed that there was some limit point of roots of f, so V is closed and open. Therefore $V = \emptyset$ or $V = U$ since U is connected, but V is nonempty, so $V = U$. So f is identically 0.

Now suppose $f = g$ on some set S with a limit point in U. Then $f - g = 0$ on S, so $f - g = 0$ (otherwise we get a nonzero holomorphic function with roots having a limit point in U). So $f = g$. □

This is true for analytic real functions.

Suppose f is holomorphic on an open nonempty set V and $U \supseteq V$ is connected. Then there is at most one way to extend f to U. This "extension" is called the function's *analytic continuation*.

Example 9.1 (Gamma function). Recall

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} \, dt.$$

This is holomorphic for $\Re s > 0$. We can differentiate under the integral sign since $e^{-t}t^{s-1}$ has continuous derivative and is rapidly decreasing. Since

$$\Gamma(s) = \frac{\Gamma(s+1)}{s},$$

we can analytically continue Γ for $\Re s > -1$ ($s \neq 0$) and repeat this process such that $\Gamma(s)$ is holomorphic for all nonpositive integers s.

Example 9.2. Consider the function $\log z$ for $\Re z > 0$. You can continue log such that $\log(-1) = \pm \pi i$ depending on the region you choose.

Example 9.3 (Riemann zeta function). Recall

$$\zeta(s) := \sum_k 1/k^s.$$

But ζ converges for $\Re s > 1$. We will show that ζ is holomorphic next section, so for now we will take that for granted. But $\zeta(s) - (2/2^s)\zeta(s)$ converges for $\Re s > 0$ to a holomorphic function (which we will show next time), so we can continue ζ as follows:

$$\frac{1}{1-2/2^s}\left(\zeta(s) - \frac{2}{2^s}\zeta(s)\right)$$

10 Locally uniform convergence

Suppose $f_1, f_2, \ldots$ are holomorphic on U. Does $\sum_i f_i$ converge and is it holomorphic? We want to know whether the sequence of partial sums (g_n) of this series converges.

There are some useful notions of convergence.

1. We say $g_n(z) \to g(z)$ *pointwise* if

   $$\lim_{n\to\infty} g_n(z) = g(z)$$

   for all z. This is too weak.

2. We say $g_n \to g$ uniformly if g_n tends to g roughly at the same rate at all points. This is too strong.

3. We can have $g_n \to g$ uniformly on compact sets, and this is just right.

Consider pointwise convergence. We would like

$$\int \lim g_n = \lim \int g_n.$$

We can construct a Dirac delta–like sequence of functions such that $\lim_{n\to\infty} g_n(z) = 0$ for all z and the integral is 1. And the limit of continuous functions need not be continuous. This makes pointwise convergence problematic.

Uniform convergence is more like having

$$\lim_{n\to\infty} \sup_z (g_n(z) - g(z)) = 0$$

for all z.

Definition 10.1 (Uniform convergence)**.** A sequence of functions (g_n) *converges uniformly* to a function g if for all $\varepsilon > 0$ there exists a natural number N such that for all z and $n \geq N$

$$|g_n(z) - g(z)| < \varepsilon.$$

A sequence of functions converges pointwise to g if for all $\varepsilon > 0$ and all z there exists a natural number N such that for all $n \geq N$ we have $|g_n(z) - g(z)| < \varepsilon$.

Remark 10.2. If $g_n \to g$ uniformly then

$$\int_a^b g_n(z)\, dz \longrightarrow \int_a^b g(z)\, dz.$$

(This is a finite interval.)

Remark 10.3. If $g_n \to g$ uniformly and g_n is holomorphic then g is holomorphic.

Proof. Suppose g_n is holomorphic. Then

$$\int_C g_n(z)\, dz = 0$$

by Cauchy's theorem. This implies that

$$\int_C g(z)\, dz = 0$$

by uniform convergence. Hence g is holomorphic by Morera's theorem. □

Warning 10.4. In real analysis, the uniform limit of analytic functions does not need to be analytic.

Problem 10.5. The series

$$\sum_k z^k = \frac{1}{1-z}$$

is not uniform on $|z| < 1$. The error

$$\left|z^n + z^{n+1} + z^{n+1} + \cdots\right| = \frac{z^n}{1-z}$$

is rather large for $z \approx 1$. We want an upper bound for the error that does not depend on z. Take $|z| < 1/2$. Then

$$\left|\frac{z^n}{1-z}\right| \leq (1/2)^{n-1}$$

tends to 0 independent of z. We can do this for 0.9, 0.99, etc. We want $\lim g_n$ to be holomorphic (locally). So we just need $g_n \to g$ uniformly on some neighbourhood of each point. We call this *local uniform convergence*. The example above is locally uniform convergent. Again, it is uniform convergence on compact subsets of U. (That local uniform convergence is equivalent to uniform convergence on compact subsets comes from $\mathbf{C}$ being locally compact.)

Suppose the power series $a_0 + a_1 z + a_2 z^2 + \cdots$ converges for $|z| < R$. We want to show that it is holomorphic for $|z| < R$. Suppose $r < s < R$. We will show that convergence is uniform for $|z| < r$. We know that $a_n s^n$ is bounded, so $a_n \le M/s^n$ for some fixed M. So $|a_n r^n| \le M(r/s)^n$. And $r/s < 1$, so

$$\begin{aligned} \left|a_n z^n + a_{n+1} z^{n+1} + \cdots\right| &\le M((r/s)^n + (r/s)^{n+1} + \cdots) \\ &\le M \frac{(r/s)^n}{1 - r/s} \end{aligned}$$

independent of z. This tends to 0, and this gives uniform convergence since we found a bound tending to 0 independent of z. So the limit is holomorphic.

One can check that the Riemann ζ function converges for $\Re s > 1$. But this convergence is not uniform. It is uniform in the region $\Re s > s_0$ for some fixed $s_0 > 1$. This is because the error is given by

$$\frac{1}{|m|^s} + \frac{1}{|m+1|^s} + \cdots \le \frac{1}{m^{s_0}} + \frac{1}{(m+1)^{s_0}} + \cdots,$$

which tends to 0. We see that ζ is holomorphic for $\Re s > 1$ since the union of the regions on which it converges is $\Re s > 1$.

The series

$$\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

certainly converges for real $s > 0$, but what about in general?

Proposition 10.6. *Suppose*

$$\frac{a_1}{1^s} + \frac{a_2}{2^s} + \cdots$$

converges for $s = s_0$. Then it converges uniformly in a region that can be made arbitrarily close to $\Re s > 0$ such that it is holomorphic in the region $\Re s > \Re s_0$.

Remark 10.7 (Abel's theorem)**.** We can write

$$\begin{aligned} &a_m b_m + a_{m+1} b_{m+1} + \cdots + a_n b_n \\ &= (b_m - b_{m+1}) a_m + (b_{m+1} - b_{m+2})(a_m + a_{m+1}) + \cdots + (b_n - 0)(a_m + a_{m+1} + \cdots + a_n). \end{aligned}$$

Remark 10.8.

$$\left|\frac{1}{(m+1)^s} - \frac{1}{m^s}\right| \le \left(\left|\frac{1}{m^s}\right| - \left|\frac{1}{(m+1)^s}\right|\right)\frac{|s|}{\Re s}$$

Proof. Let's take $s_0 = 0$. We know that $a_1 + a_2 + a_3 + \cdots$ converges, and we want to show that

$$\frac{a_1}{1^s} + \frac{a_2}{2^s} + \cdots$$

converges uniformly in the region that can be made arbitrarily close to the region $\Re s > 0$. If m is sufficiently large then the partial sums $a_m + \cdots + a_n$ will be less than some $\varepsilon > 0$. We get

$$\begin{aligned}
&\left|\frac{a_m}{m^s} + \frac{a_{m+1}}{(m+1)^s} + \cdots + \frac{a_n}{n^s}\right| \\
&= a_m\left(\frac{1}{m^s} - \frac{1}{(m+1)^s}\right) + (a_m + a_{m+1})\left(\frac{1}{(m+1)^s} - \frac{1}{(m+2)^s}\right) + \cdots + (a_m + \cdots + a_n)\frac{1}{n^s} \\
&\le \varepsilon\frac{|s|}{\Re s}\left|\frac{1}{m^s}\right|,
\end{aligned}$$

and this is bounded for some s in the region approximating $\Re s > 0$, so that Dirichlet series converges uniformly on that region. Hence these Dirichlet series are holomorphic for $\Re s > 0$. □

11 Residue calculus

Theorem 11.1. *Suppose f is holomorphic except at finitely many points a_k. Then*

$$\int_C f(z)\,dz = 2\pi i\sum_k \operatorname{Res}(f, a_k)$$

where the residue at a_k

$$\operatorname{Res}(f, a_k) := \frac{1}{2\pi i}\int_{C_k} f(z)\,dz$$

and C_k is a small region around a_k.

This makes sense. But what is the residue of $f(z)\,dz$ at a point p? We expand f as a Laurent series:

$$f(z) = a_{-n}(z-p)^{-n} + \cdots + a_{-1}(z-p)^{-1} + a_0 + a_1(z-p) + \cdots .$$

Then

$$\operatorname{Res}(f,p) = a_{-1}.$$

Take $p = 0$. Recall that

$$\int_C z^n \, dz = \begin{cases} 2\pi i & n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

1. We can transform a contour integral into a sum of residues;
2. We can transform a sum into a contour integral.

Example 11.2.

$$\int_{-\infty}^{\infty} \frac{1}{1+z^2} \, dz.$$

Introductory methods show that this integral is π. A good contour to use is a semicircle from $-R$ to R as $R \to \infty$. In the limit we can ignore the "curved section" of the semicircle. This is not holomorphic at $z = \pm i$, and $z = -i$ is not in the contour. Hence

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{1+z^2} \, dz &= 2\pi i \operatorname{Res}(f,i) \\ &= 2\pi i(1/2i) \\ &= \pi. \end{aligned}$$

Example 11.3.

$$\int_{-\infty}^{\infty} \frac{\cos z}{1+z^2} \, dz.$$

The function $(\cos z)/(1+z^2)$ does not have an elementary antiderivative. We *cannot* use the semicircle since $(\cos z)/(1+z^2)$ is large in the "curved section" (so we cannot ignore it). We can see that

$$\int_{-\infty}^{\infty} \frac{\cos z}{1+z^2} \, dz = \Re \int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2} \, dz.$$

Since $\left|e^{iz}\right| \le 1$, the integral over the "curved section" approaches 0 in the limit, so we ignore it. Hence this integral is $2\pi i \operatorname{Res}(f,i)$. We get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos z}{1+z^2} \, dz &= 2\pi i \operatorname{Res}(f,i) \\ &= 2\pi i(e^{-1}/2i) \\ &= \pi/e. \end{aligned}$$

Example 11.4.

$$\int_{-\infty}^{\infty} \frac{\sin z}{z}\, dz.$$

The semicircle doesn't work again, since $\sin z$ is "big" on the curved section. Then we try

$$\Im \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\, dz = \int_{-\infty}^{\infty} \frac{\sin z}{z}\, dz.$$

But at $z = 0$, $(\cos z)/z$ is infinite. Also, e^{iz}/z is holomorphic except at $z = 0$. Now consider the semicircle contour, but we avoid the point $z = 0$ with a smaller semicircle. The imaginary part of the integral on the "straight section" of this contour approaches

$$\int_{-\infty}^{\infty} \frac{\sin z}{z}\, dz.$$

We want to show that the "outside curved section" approaches 0. In the "middle part" of this section $\left|e^{iz}\right|$ is small and on the "ends" $|1/x|$ is small. Pick $M \in \mathbf{R}$ such that $\left|e^{iz}\right| < \varepsilon$ when $\Im z > M$. Then

$$\int_{\text{above } M} f(z)\, dz \le \varepsilon\pi \to 0$$

and

$$\int_{\text{below } M} f(z)\, dz \le 2M(1/R) \to 0.$$

So we can ignore this section. Now consider the "inside curved section," which is a semicircle of radius r. If C is a circle around the origin, then

$$\int_C \frac{e^{iz}}{z}\, dz = 2\pi i \operatorname{Res}(f, 0) = 2\pi i.$$

If D is the contour formed by the contour in question, then

$$\int_D \frac{e^{iz}}{z}\, dz \to -\frac{1}{2}(2\pi i \operatorname{Res}(f, 0)) = -\pi i.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\sin z}{z}\, dz = \pi.$$

12 Summing series

EXAMPLE 12.1 (Basel problem). Show that

$$\sum_k 1/k^2 = \pi^2/6.$$

We want to find a function whose residues correspond to $1/k^2$. The function $f(z) = 1/(\tan z)$ has singularities at $\pi\mathbf{Z}$. The residues at each of these points is 1 because for $z \approx 0$ then $\tan z \approx z$, so $1/(\tan z) \approx 1/z$, which has residue 1. Now for $g(z) = 1/z^2(\tan z)$, the residues at πk is $1/(\pi k)^2$ for $k \in \mathbf{Z} \setminus \{0\}$. Now at 0, the pole is of order 3. We see that

$$\sum_k \operatorname{Res}(g, a_k) = \operatorname{Res}(g, 0) + \frac{2}{\pi^2}\left(\sum_k 1/k^2\right).$$

Let's calculate the sum of the residues first (we want to show that this sum is 0). Consider the contour C given by a square with sidelength $2R$ centred at the origin where $R = \pi/2 + \pi k$. In particular,

$$\begin{aligned}\left|\int_C \frac{1}{z^2 \tan z}\, dz\right| &\le |C| \max_C f \\ &= 8R\left(\max_C 1/z^2\right)\left(\max_C 1/\tan z\right) \\ &= 8R(1/R^2)M \to 0\end{aligned}$$

where M is independent of R. Hence

$$\sum_k \operatorname{Res}(g, a_k) \to 0.$$

What about that mysterious residue? We see that

$$\begin{aligned}\frac{1}{\tan z} &= \frac{\cos z}{\sin z} \\ &= \frac{1 - z^2/2! + \cdots}{z - z^3/3! + \cdots} \\ &\approx z^{-1} - z/3.\end{aligned}$$

So

$$\frac{1}{z^2 \tan z} \approx z^{-3} - z^{-1}/3.$$

But recall that the residue is given by the coefficient of z^{-1} in the Laurent series expansion. So $\operatorname{Res}(g,0) = -1/3$. Hence

$$0 = -1/3 + \frac{2}{\pi^2}\left(\sum_k 1/k^2\right),$$

and, therefore,

$$\sum_k 1/k^2 = \pi^2/6.$$

Exercise 12.2. Show that

$$\sum_k 1/k^4 = \pi^4/90.$$

(Hint: Use $g(z) = 1/(z^4 \tan z)$.)

What about $\sum_k 1/k^3$? To compute $\sum_n r(n)$ where r is a rational function this way we need $\deg r \le -2$.

Exercise 12.3. Compute

$$\sum_k (-1)^k/(2k+1) = (1/2)\left(\cdots + 1/(-3)^3 - 1/(-1)^3 + 1/1^3 - 1/3^3 + \cdots\right).$$

(Hint: Use $f(z) = 1/(\cos z)$ whose residues are sometimes -1 and then $g(z) = 1/(z^3 \cos z)$.)

13 Zeta function functional equation

Here, we will use the residue theorem to relate $\zeta(s)$ and $\zeta(1-s)$. We will use the Bromwich contour. Often, the integral around the "little circle" of radius r tends to 0 as r tends to 0 and the integral around the "big circle" of radius R tends to 0 as R tends to infinity. The integrals along the real axis do not necessarily cancel out, since functions of the form $z^s f(z)$ are multivalued and change by a factor of $\exp(2\pi i s)$ with every revolution around the origin. Watch out for signs.

Example 13.1. Recall

$$\Gamma(s) = \int_0^\infty e^{-z} z^{s-1}\, dz, \;\; \Re s > 0.$$

The integral around the "little circle" of the Bromwich contour C tends to 0 if $\Re s > 0$. Then

$$\int_C e^{-z} z^{s-1}\, dz = (1 - e^{2\pi i s})\Gamma(s).$$

We see that this is holomorphic for all s (we're avoiding $s = 0$ with the choice of contour). The quantity $1 - e^{2\pi is}$ is nonzero for $s \notin \mathbf{Z}$, and this gives the analytic continuation of Γ to the plane. Equivalently (more symmetrically),

$$\int_C = e^{-z}(-z)^{s-1}\,dz = (e^{-\pi is} - e^{\pi is})\Gamma(s).$$

Then

$$\begin{aligned} -2i\sin(\pi s)\Gamma(s)/n^s &= \int_C e^{-z}(-z/n)^{s-1}\,dz/n \\ &= \int_C e^{-nz}(-z)^{s-1}\,dz. \end{aligned}$$

Now, we sum over $n > 0$. We get

$$-2i\sin(\pi s)\Gamma(s)\zeta(s) = \int_C \frac{1}{e^z - 1}(-z)^{s-1}\,dz.$$

This gives the analytic continuation of ζ.

Consider this integral over the Bromwich contour C. The integral around the "big circle" tends to 0 as R tends to infinity if $\Re s < 0$. Further, $e^z - 1 = 0$ if $z \in 2\pi i\mathbf{Z}$. So the integral around the "big circle" is not 0:

$$\int_{\text{around } 2n\pi i} f(z)\,dz = (-2\pi ni)^{s-1}(-2\pi i).$$

Therefore, we get

$$\begin{aligned} 2\sin(\pi s)\Gamma(s)\zeta(s) &= (2\pi)^s \sum_{n>0} n^{s-1}((-i)^{s-1} + i^{s-1}) \\ 2\sin(\pi s)\Gamma(s)\zeta(s) &= (2\pi)^s \zeta(1-s)((-i)^{s-1} + i^{s-1}). \end{aligned}$$

If we write $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, the functional equation above simplifies to $\zeta^*(s) = \zeta^*(1-s)$.

We see that $\zeta^*(s)$ is real if $\Re s = 1/2$ since

$$\overline{\zeta^*(s)} = \zeta^*(\overline{s}) = \zeta^*(1 - \overline{s}) = \zeta^*(s).$$

Exercise 13.2. Compute

$$\int_0^\infty \frac{x^{s-1}}{x+1}\,dx$$

provided $0 < \Re s < 1$. (Hint: Take the the Bromwich contour, show that the "integrals around the circles" are 0, relate the integral along the real axis to the integral above, compute the residue at $x = -1$, and put everything together.)

14 Singularities

A *singularity* of a function f is a point where it f is not holomorphic.

- Isolated singularities (f is holomorphic in a neighbourhood of the singularity):
  - Removable singularities;
  - Poles (nice);
  - Essential singularities (nasty).
- Non-isolated singularities:
  - Branch points;
  - Limits of singular points;
  - Natural boundaries.

Example 14.1 (Removable singularities)**.**

- One can rewrite $(\sin x)/x$ as a power series that converges at 0 even though it is singular there.
- The function

  $$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

  is singular at $x = 0$, but just change the 1 to a 0.

Suppose f is holomorphic for $0 < |z| < \varepsilon$. If f is bounded here, then a singularity at 0 is removable. To show this, put

$$g(w) := \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w} \, dz.$$

The function g is holomorphic for $|w| < \varepsilon$, and if f is bounded, then $f = g$ in the region $0 < |z| < \varepsilon$.

A *pole* is something that, locally, looks like $f(z)/z^n$ for f holomorphic (this is for $z = 0$). This looks like

$$g(z) := \frac{f(z)}{z^n} = a_{-n}z^{-n} + a_{1-n}z^{1-n} + \cdots + a_0 + a_1 z + \cdots$$

a Laurent series. Notice that $|g(z)| \to \infty$ as $z \to 0$.

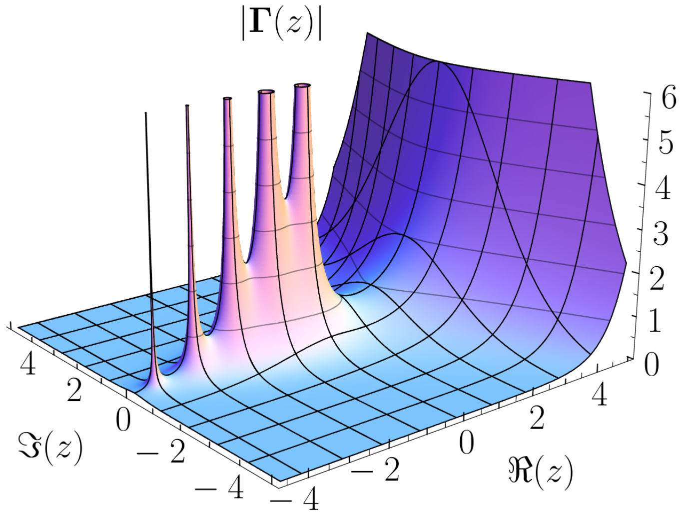

Figure 2: One can see the poles of $\Gamma(z)$ at $\mathbf{Z}_{<0}$. It is meromorphic.

Definition 14.2. A function f is *meromorphic* if all of its singularities are poles.

If f has a pole at $z = 0$ then $1/f$ is holomorphic at $z = 0$.

Jahnke and Emde's book[2] illustrates functions such as the gamma function, Jacobi/Weierstrass elliptic functions, and the Riemann zeta function. (Order-two poles are "fatter" than order-one poles.)

A function with an essential singularity is $f(z) = \sum_{k \in \mathbf{Z}} a_k z^k$ in which the number of terms with a negative exponent might not be finite. Expanding $e^{1/z} + e^z$ as a Laurent series, you see that something like this occurs. Any function with an essential singularity has this property[3].

The function $\exp(1/z)$ has an essential singularity, and so does $\exp(-1/z^2)$.

If f has an essential singularity at 0, then $\{f(z) : 0 < |z| < \varepsilon\}$ is dense in $\mathbf{C}$. Suppose α is not a limit point of values of $f(z)$. Then $|f(z) - \alpha| > \delta$, so $|1/(f(z) - \alpha)|$ is bounded, and $1/(f(z) - \alpha)$ is holomorphic. (See Great Picard's theorem.)

Now we're on to non-isolated singularities. Limits of singularities are one: Take, for example,

$$f(z) = \frac{1}{\sin(1/z)}.$$

[2] See Tables of Functions with Formulae and Curves fig. 5, fig. 49, fig. 55, fig. 150, etc.

[3] See `https://youtu.be/MbPOKnwgL-8?t=702`.

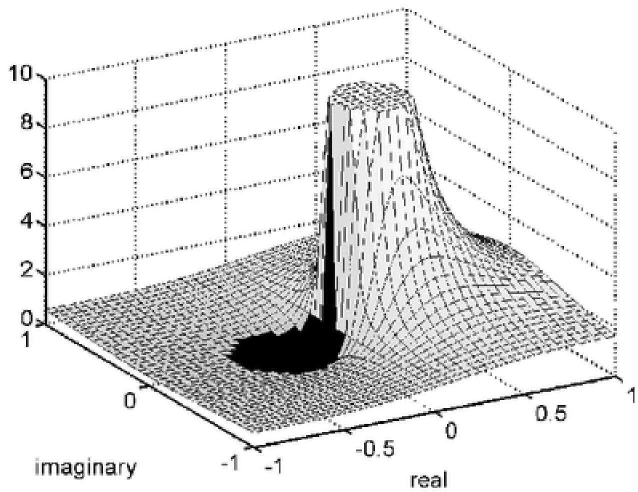

Figure 3: An illustration of the essential singularity of $\exp(1/z)$.

Branch points are another: Take $\log z$ or z^s (where $s \notin \mathbf{Z}$). Hankel functions are another example. Values change depending on from where you approach the point.

Then, we have natural boundaries. The function $f(z) = \sum_k z^k$ converges for $|z| < 1$. It has a pole at $z = 1$, but it is nonsingular elsewhere because we can continue it to a holomorphic function by $z/(1-z)$. Now consider $f(z) = \sum_k z^{2^k}$. There's obviously a pole at $z = 1$. At $z = -1$, there's also a pole. At $z = i$? Pole. For all 2^kth roots of unity, there is a pole. But $|f(z)|$ does not tend to ∞ as $|z|$ tends to 1. If you approach a non–2^kth root of unity, we do not need to tend to ∞ at all. This is a natural boundary—a sort of "wall of poles."

15 Gamma function

Euler defined the gamma function as follows:

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\, dt.$$

It has the fundamental properties that

$$s\Gamma(s) = \Gamma(s+1)$$

and

$$\Gamma(1) = 1.$$

This defines a holomorphic function for $\Re s > 0$. Using its defining properties, one can extend it to a holomorphic function with poles at nonpositive integers.

Analyzing Figure 2, one notices that the poles of $\Gamma(s)$ get thinner as s gets more negative. Further, $|\Gamma(s)|$ seems to get smaller as $|\Im s|$ increases. Wherefore?

The pole at $s = 0$ has residue 1, since $\Gamma(s) = s^{-1}\Gamma(s+1)$ and $\Gamma(1) = 1$. At $s = -1$, since $\Gamma(s-1) = (s-1)^{-1}\Gamma(s)$ and the previously-mentioned pole has residue 1, the pole has residue $1/(-1) = -1$. At $s = -2$, since $\Gamma(s-2) = (s-2)^{-1}\Gamma(s-1)$, the pole has residue $(-1)/(-2) = 1/2$. At $s = -3$, the residue is $-1/6$. One sees that at $s = -n$, the pole has residue $(-1)^n/n!$.

Further,

$$\begin{aligned} |\Gamma(s)| &= \left| \int_0^\infty e^{-t} t^{s-1}\, dt \right| \\ &\leq \int_0^\infty \left| e^{-t} t^{s-1} \right| \, dt \\ &= \int_0^\infty e^{-t} t^{\Re(s-1)}\, dt \\ &= \Gamma(\Re s). \end{aligned}$$

So $\Gamma(s)$ is bounded for $\alpha \leq \Re s \leq \beta$ where $0 < \alpha < \beta$, since $\Gamma(s)$ is bounded for $\alpha \leq s \leq \beta$. Now $\Gamma(s) = s^{-1}\Gamma(s+1)$, and $\Gamma(s+1)$ is bounded for $\alpha \leq \Re s \leq \beta$, so $s^{-1}\Gamma(s+1)$ tends to 0 as $|\Im s|$ tends to infinity. And $|(\Im s)^n \Gamma(s)|$ is bounded for $n > 0$. So $\Gamma(s)$ is rapidly decreasing in vertical strips $\alpha \leq \Re s \leq \beta$.

Proposition 15.1 (Euler's reflection formula)**.**

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

Proof.

1. Write $\varphi(s) := \Gamma(s)\Gamma(1-s)$. Notice that $\varphi(s) = -\varphi(s+1)$. Also, φ has a pole of residue 1 at $s = 0$. Further, $\varphi(s)$ is rapidly decreasing in vertical strips $\alpha \leq \Re s \leq \beta$ where $\Im s > \varepsilon$. The same is true for $\pi/\sin(\pi s)$. Therefore, $\psi(s) := \varphi(s) - \pi/\sin(\pi s)$ satisfies $\psi(s+1) = -\psi(s)$, is rapidly decreasing, and has no poles: it is holomorphic everywhere. Since ψ is periodic and bounded in vertical strips, it is bounded. Therefore, by Theorem 8.1 (Liouville), ψ is constant. Since both functions are rapidly decreasing, $\psi = 0$.

2.

$$
\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \int_0^\infty e^{-t}t^{s-1}\,dt \int_0^\infty e^{-u}u^{1-(s-1)}\,du \\
&= \int_0^\infty \int_0^\infty e^{-(t+u)}(t/u)^s t^{-1}\,dt\,du \\
&= \int_0^\infty \int_0^\infty e^{-u(t+1)}t^s t^{-1}\,dt\,du && t \longmapsto tu \\
&= \int_0^\infty \frac{1}{t+1}t^{s-1}\,dt \\
&= \frac{\pi}{\sin(\pi s)} && \text{Exercise 13.2.}
\end{aligned}
$$

□

Remark 15.2. A theme of Riemann: To study a complex function, you should analyze its singularities and growth rate.

Remark 15.3.

$$\Gamma(1/2) = \sqrt{\pi}$$

Remark 15.4.

$$\Gamma(s) \neq 0$$

Proposition 15.5. *The function $\Gamma(s)$ is the only meromorphic function such that*

1. $s\Gamma(s) = \Gamma(s+1)$;
2. $\Gamma(s)$ *is bounded in strips* $\alpha \leq \Re s \leq \beta$ *where* $\Im s \geq 1$;
3. $\Gamma(1) = 1$.

Proof. Suppose Γ_0 has these properties. Then

$$\Gamma_0(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

since the proof of the reflection formula only required the three properties in question. Hence $\Gamma_0 = \Gamma$. □

The function $\varphi(s) := \Gamma(s/2)\Gamma((s+1)/2)$ has the "same poles" as $\Gamma(s)$. Now

$$
\begin{aligned}
\varphi(s+1) &= \Gamma((s+1)/2)\Gamma(s/2+1) \\
&= (s/2)\Gamma((s+1)/2)\Gamma(s/2) \\
&= (s/2)\varphi(s).
\end{aligned}
$$

Now, set $\psi(s) := 2^s\Gamma(s/2)\Gamma((s+1)/2)$. Then $\psi(s+1) = s\psi(s)$. The function ψ is bounded in vertical strips (except near poles, as with Γ). So $\psi(s) = \alpha\Gamma(s)$ for some constant α. Now, with $s = 1$, we find

$$\begin{aligned} 2^1\Gamma(1/2)\Gamma(1) &= \alpha\Gamma(1) \\ \alpha &= 2\sqrt{\pi}. \end{aligned}$$

Therefore,

$$2^s\Gamma(s/2)\Gamma((s+1)/2) = 2\sqrt{\pi}\Gamma(s).$$

(This is called the duplication formula.)

Exercise 15.6. Prove that

$$m^s \prod_{k=1}^{m} \Gamma\left(\frac{s+k-1}{m}\right) = (2\pi)^{(m-1)/2}\sqrt{m}\Gamma(s).$$

16 Maximum modulus principle

Suppose f is holomorphic and U is an open set. If $|f(z)|$ is maximal at $z \in U$ then, f is constant.

Proof. Suppose $f(z)$ is maximal at $z = 0$. Let C be a curve around 0. Then

$$\begin{aligned} f(0) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z}\,dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}} ire^{i\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\,d\theta \\ &= \text{average of } f \text{ on } C. \end{aligned}$$

Suppose $|f(0)| = M$, so $|f(z)| \le M$. But $f(0)$ is the average of f on C, so $f(z) = f(0)$ for all z on C. So f is constant. $\square$

There is a physical interpretation. Recall that if f is holomorphic, then $\Re f$ is harmonic, so it satisfies the steady-state heat equation. Think of a plate U. The heat at a point of maximum temperature on U will flow out to cooler areas. So the heat must be the same throughout the plate.

Theorem 16.1 (Fundamental theorem of algebra). *Any non-constant polynomial f has a root.*

Proof. Suppose f has no root and $f(0) = M$. Then $1/f(z) \to 0$ as $|z| \to \infty$ and $1/f(z)$ is holomorphic. Then pick the maximum value of $f(z)$ for $|z| \le M/2$, which you can do since we're dealing with a continuous function on a compact region. Further, f cannot be constant since it tends to 0, so we obtain a contradiction: $1/f(z)$ is not holomorphic, so f vanishes somewhere.

One might also use Liouville's theorem. □

Consider $U = \{z \in \mathbf{C} : |z| < 1\}$. One symmetry of U is given by $z \longmapsto \exp(i\theta)z$, which rotates U by θ. Any symmetry is a map $f : U \longrightarrow U$ that is holomorphic with an inverse $f^{-1} : U \longrightarrow U$. One can check that

$$f(z) = \frac{z - \alpha}{1 - \overline{\alpha}z}$$

is another symmetry for $|\alpha| < 1$. We get a three-dimensional group of symmetries given by

$$z \longmapsto \exp(i\theta)\frac{z - \alpha}{1 - \overline{\alpha}z}.$$

(These are called Möbius transformations.) Are there others?

Suppose $f : U \longrightarrow U$. We can compose f with a Möbius transformation to make $f(0) = 0$. Now put $g(z) := f(z)/z$. The function g is holomorphic since it has a removable singularity at 0. We want to show that if $|z| < 1$, then $|g(z)| \le 1$ (where $=$ occurs only if g is constant). Now $|g(0)| \le 1$, and if $|g(0)| = 1$, then g is constant by the maximum modulus principle. Suppose that f has an inverse $f^{-1} : U \longrightarrow U$ with $f(0) = 0$. Then

$$f'(0) = \frac{1}{(f^{-1})'(0)},$$

and $f'(0)$ and $(f^{-1})'(0)$ have absolute value at most 1, so $|f'(0)| = 1$. And $f'(0) = g(0)$, so g is constant and f is linear. Therefore, $f(z) = \exp(i\theta)z$. That is, all symmetries of U are Möbius transformations.

What about the structure of this group? First, notice that we can identify it with the upper-half plane by

$$\begin{aligned} w &\longmapsto \frac{z - i}{z + i} \\ i\frac{1 + w}{1 - w} &\longleftarrow z. \end{aligned}$$

So we get a group of three-dimensional symmetries of the upper-half plane. These are given by $\mathrm{PSL}_2(\mathbf{R})$, which acts on the upper-half plane by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \frac{a\tau + b}{c\tau + d}.$$

The more profound idea here is to find a group $\Gamma \subseteq \mathrm{PSL}_2(\mathbf{Z})$ and look for functions on $\mathbf{H}$ invariant under Γ (these are called modular functions).

17 Elliptic functions

Elliptic functions are doubly-periodic functions. Recall that a doubly-periodic function φ satisfies $\varphi(z) = \varphi(z+\omega_1) = \varphi(z+\omega_2)$ for two $\mathbf{R}$–linearly independent numbers $\omega_1, \omega_2 \in \mathbf{C}$.

Are there holomorphic elliptic functions? Well, no: If φ is holomorphic and elliptic then it is constant, since φ is bounded in the fundamental domain in the lattice generated by two linearly independent periods (which is compact if you include the boundary points) and φ is continuous. That is, φ is bounded on $\mathbf{C}$ by periodicity. So, by Theorem 8.1, φ is constant.

Let's find a meromorphic function φ such that $\varphi(z) = \varphi(m\omega_1 + n\omega_2 + z)$ for all $m, n \in \mathbf{Z}$ (that is, φ is doubly-periodic). Here, we get a group of translations isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$. A straightforward way to get a function φ invariant under a group is to take the average of some function ψ:

$$\varphi(z) = \sum_{m,n \in \mathbf{Z}} \psi(z + m\omega_1 + n\omega_2).$$

So, we have an elliptic function—right? Wrong! This function φ is elliptic if the series is absolutely convergent. Let's figure out when that is the case.

Suppose that $\psi(z) \leq \text{constant}/z^\alpha$ for $|z|$ large. We will estimate the sum in an annulus. Fix a natural number r. Then the number of points in the annulus $\{z : |z| < r+1\} \setminus \{z : |z| < r\}$ of the form $m\omega_1 + n\omega_2$ is at most a constant times r. So the sum is bounded by $\sum_{r \in \mathbf{N}} \text{constant}\, r/r^\alpha$, which converges for $\alpha > 2$.

Now if ψ is a rational function of degree at most -3, then it is elliptic. For instance,

$$\psi(z) := \frac{1}{(z-\alpha)(z-\beta)(z-\gamma)}$$

has poles at $\{\alpha, \beta, \gamma\} + m\omega_1 + n\omega_2$. That is, we need at least 3 poles in the fundamental domain.

But are there elliptic functions with 1 or 2 poles in the fundamental domain? Notice that an elliptic function φ is determined by its roots and poles up to constant multiplication.

We require the argument principle. Suppose f is meromorphic in a region U. Write $f(z) = a_n z^n + \cdots$, so $f'(z) = n a_n z^{n-1} + \cdots$. Then $f'(z)/f(z) = nz^{-1} + \cdots$, and the first term has residue n, which is the order of the root of f at 0. Then

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial U} \frac{f'(z)}{f(z)}\, dz &= \#\text{roots} - \#\text{poles} \\ &= \frac{1}{2\pi} \Delta(\arg f(z)). \end{aligned}$$

Why? Well,

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}(\log f(z)),$$

and even though log is multivalued, it changes by a constant, but differentiating kills that constant. So that integral above is $(1/2\pi i) \times \Delta(\log f(z))$. But log only changes in its imaginary part around this contour, so we get the result above.

Now, suppose f is elliptic. How many roots and poles does it have in its fundamental domain? Well,

$$(\#\text{roots} - \#\text{poles})\Big|_{\text{fundamental domain}} = \frac{1}{2\pi i} \int_{\partial(\text{fundamental domain})} \frac{f'(z)}{f(z)}\, dz.$$

But $f(z) = f(z+\omega_1) = f(z+\omega_2)$, and we are integrating along the fundamental domain, so the integrals on its parallel sections cancel. Hence

$$\#\text{roots} = \#\text{poles}$$

is in the fundamental domain. Now, if a root or pole is on the boundary of the fundamental domain, we slightly manipulate the contour of integration to avoid the root/pole. Therefore, we only count a root/pole once if it is on the boundary. Do not over-count!

Let's take this further. With the residue calculus, we see that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} g(z)\, dz = \sum_{\text{roots/poles } p} \operatorname{Res}(f'/f,\, p) g(p)$$

if g is holomorphic. Now, suppose f is elliptic. Then

$$\begin{aligned}
\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} z\, dz &= \frac{1}{2\pi i} \left(\int_{\omega_1}^{\omega_1+\omega_2} \frac{f'(z)}{f(z)} z\, dz - \int_0^{\omega_2} \frac{f'(z)}{f(z)} z\, dz + \text{similar} \right) \\
&= \frac{1}{2\pi i} \left(\int_0^{\omega_1} \frac{f'(z)}{f(z)} ((z+\omega_1) - z)\, dz + \text{similar} \right) \\
&= \frac{1}{2\pi i} \left(\omega_1 \int_0^{\omega_2} \log f(z)\, dz + \omega_2 \int_0^{\omega_1} \log f(z)\, dz \right) \\
&= m\omega_1 + n\omega_2 \qquad\qquad m, n \in \mathbf{Z}.
\end{aligned}$$

Therefore,

$$\sum_{\text{roots/poles } p} p \operatorname{Res}(f'/f, p) = m\omega_1 + n\omega_2$$

for some integers m and n.

Let's show that f cannot have exactly 1 pole. Suppose f has a pole at α and a root at β. Then $\alpha - \beta = m\omega_1 + n\omega_2$. This is saying that (supposing f has a pole at α) that f has both a pole and a root at α, which is a contradiction.

What about 2 poles in the fundamental domain? Write $n_p := \operatorname{Res}(f'/f, p)$. Then we have the following conditions for an elliptic function:

- $\sum n_p = 0$;
- $\sum p n_p = m\omega_1 + n\omega_2$.

These are sufficient conditions. In fact, we can have 2 poles for an elliptic function.

18 Weierstrass elliptic functions

If we want an elliptic function with a pole of order 3 at 0, we might take

$$\sum_{m,n\in\mathbf{Z}} \frac{1}{(z+m\omega_1+n\omega_2)^3}.$$

This is okay, but, for a pole of order two, we cannot take

$$\sum_{m,n\in\mathbf{Z}} \frac{1}{(z+m\omega_1+n\omega_2)^2}$$

because this does not converge absolutely. Forevermore,

$$\lambda := m\omega_1 + n\omega_2 \in \Lambda$$

for a lattice Λ. Now, near $z = 0$,

$$\frac{1}{(z+\lambda^2)} \approx \frac{1}{\lambda^2},$$

so we can try to subtract this "divergent part":

$$\sum_{\lambda\in\Lambda} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right).$$

What about at $\lambda = 0$, though? Second patch:

$$\frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right).$$

The summand is, approximately, of degree -3 in λ, so it converges absolutely. Let Λ be a lattice. Then, we define

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right).$$

But, now, this is not invariant under $z \longmapsto z + \lambda$—right? Well, it actually is:

$$\wp'(z) = -2 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(z+\lambda)^2}.$$

Term-by-term differentiation is valid: $\wp$ is locally uniformly convergent, local uniform convergence preserves integrals, and the derivative variation on the Cauchy integral formula shows that a derivative can be written as an integral. Therefore,

$$\wp(z) - \wp(z+\omega_1) = \alpha.$$

The function $\wp$ is even, so, letting $z = -\omega_1/2$, $\alpha = 0$. Therefore,

$$\wp(z) = \wp(z+\omega_1) = \wp(z+\omega_2).$$

That is, $\wp$ is elliptic, and it has a pole of order 2 at $z = 0$.

Abiding by this procedure for a pole of order 1, we get

$$\zeta(z) := \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right),$$

which is Weierstrass's ζ function—not Riemann's. It is absolutely convergent and has a pole of order 1. Now, let's continue to check if it's elliptic:

$$\zeta'(z) = -\wp(z),$$

so

$$\zeta(z+\omega_1) - \zeta(z) = \beta.$$

Now, ζ is odd, and $\beta \neq 0$. So ζ is "elliptic up to constants."

Nevertheless,

$$\zeta(z-\alpha) - \zeta(z-\beta)$$

is elliptic.

The function ζ defined this way is an example of a Mittag–Leffler series.

Any elliptic function with no poles is constant, and this fact generates legions of identities[4]. Some light hand-waving[5] shows that

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3.$$

[4] See A Course of Modern Analysis.

[5] See https://youtu.be/p1dts9PrDtI?t=1243.

19 Classification of elliptic functions

We can classify elliptic functions in three ways:

1. They are all rational functions of $\wp$ and $\wp'$;

2. They are determined by their roots and poles (up to a constant);

3. They are determined by their singularities (up to a constant).

Suppose f is elliptic. Notice that f is the sum of an odd function and an even function. But an odd function divided by $\wp'$ is even, so this all reduces to the case when f is even.

Then, we get rid of all poles not on Λ by multiplying by $\wp(z) - \wp(\alpha)$ (if there is a pole at $z = \alpha$).

Now, for the pole at 0, we note that $f(z) = a_n z^{-2n} + \cdots$, and then we can write

$$f(z) = a_n \wp(z)^n + \text{something with a smaller pole.}$$

We can continue to reduce the size of the pole by subtracting polynomials in $\wp$ until f is zero. So every elliptic function is a rational function of $\wp$ and $\wp'$.

Recall that

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3.$$

If $y = \wp'(z)$ and $x = \wp(z)$, we get a map

$$\begin{aligned} \mathbf{C}/\Lambda &\longrightarrow C : y^2 = 4x^3 - g_2 x - g_3 \\ z &\longmapsto (\wp(z), \wp'(z)). \end{aligned}$$

This is almost an isomorphism, except 0 is taken to the point at infinity.

The second classification: An elliptic function f is determined by its roots and poles. Suppose f has roots at $z = p_i$ of order n_i (if $n_i < 0$ then f has a pole of order $-n_i$ at $z = p_i$). Some time ago, we put some conditions on these roots and poles:

1. $$\#\text{roots} = \#\text{poles} \iff \sum_i n_i = 0;$$

2. $$\sum_i p_i n_i \in \Lambda.$$

We showed that these are necessary for f to exist, but, now, it's time to show that they are sufficient.

Recall that Weierstrass's zeta function has poles of order 1 at $\lambda \in \Lambda$ and $\zeta'(z) = -\wp(z)$. Weierstrass's sigma function is defined by

$$\sigma(z) := \exp\left(\int_\alpha^z \zeta(z)\, dz\right)$$

or[6]

$$\frac{d}{dz} \log \sigma(z) := \zeta(z)$$

and

$$\lim_{z \to 0} \frac{\sigma(z)}{z} := 1.$$

The integral has logarithmic singularities at Λ, but these "ambiguities" are nullified by taking exp. This function has roots of order 1 at Λ and no other roots or poles. It is not elliptic, but it satisfies

$$\sigma(z + 2\omega_1) = -\sigma(z) \exp(2\eta_1(z + \omega_1));$$
$$\sigma(z + 2\omega_2) = -\sigma(z) \exp(2\eta_2(z + \omega_2));$$

where

$$\eta_1 := -\frac{\pi^2 \vartheta_1'''}{12\omega_1 \vartheta_1'};$$
$$\eta_2 := -\frac{\pi^2 \omega_2 \vartheta_1'''}{12\omega_1^2 \vartheta_1'} - \frac{\pi i}{2\omega_1}.$$

Dr. Borcherds writes this as

$$\sigma(z + \omega_1) = \sigma(z) \exp(A_1 z + B_1).$$

Now,

$$\prod_i \sigma(z - p_i)^{n_i}$$

has roots of order n_i at p_i. Applying $z \longmapsto z + \omega_1$ gives

$$\prod_i \sigma(z - p_i)^{n_i} \exp(A_1(z - p_i) + B_1)^{n_i}.$$

[6] See https://mathworld.wolfram.com/WeierstrassSigmaFunction.html.

Since $\sum n_i p_i \in \Lambda$, write $\sum n_i p_i = 0$. This, along with $\sum n_i = 0$, means that

$$\prod_i \sigma(z - p_i)^{n_i}$$

is invariant under $z \longmapsto z + \omega_1$, so is elliptic (provided $\sum n_i p_i = 0$). So we can find elliptic functions with given roots and poles if and only if they satisfy the two conditions in question.

Exercise 19.1. Show that

$$\wp(z) - \wp(\alpha) = -\frac{\sigma(z+\alpha)\sigma(z-\alpha)}{\sigma^2(z)\sigma^2(\alpha)}.$$

(Hint: Do not use the definitions. Consider the roots and poles of both sides, show they are the same, and show that the constant by which they differ is 0.)

Can we find an elliptic function with singularities at specified points in the fundamental domain? Not always: With an appropriate choice of contour C, we can make

$$\frac{1}{2\pi i} \int_C f(z)\, dz = 0,$$

so

$$\sum_{\text{points } p} \operatorname{Res}(f, p) = 0.$$

This condition is sufficient. (Notice that this also proves that you cannot have an elliptic function with only 1 pole of order 1 in its fundamental domain.)

Notice that $\wp^{(n)}(z - \beta)$ has a pole of order $n + 2$, so we can use these to kill poles of order greater than 1. Now, recall that

$$\zeta(z - \alpha) - \zeta(z - \beta)$$

has poles of order 1 at α and β. We use a function of this kind to "move all poles to one point." Therefore, we can assume that the function has one pole of order 1. Now, recall that the sum of the residues must be 0, so f must be holomorphic and constant.

These Weierstrass functions have trigonometric analogues in 1-dimensional lattices[7]. Recall:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \neq 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

[7] See https://en.wikipedia.org/wiki/Weierstrass_functions.

Similarly,

$$\sum_{n\in\mathbf{Z}} \frac{1}{(z-n\pi)^2} = \frac{1}{\sin^2 z},$$

and this can be seen by analyzing poles, periodicity, and behaviour as $|\Im z|$ grows.

Integrating $\wp$, we get $-\zeta$, and integrating $-\csc^2 z$, we get

$$\frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right) = \frac{1}{\tan z}.$$

Finally, integrating and exponentiating ζ, we get σ. Doing the same to $\cot z$, we get

$$z\prod_{n\neq 0}\left(1 - \frac{z^2}{n^2\pi^2}\right) = \sin z.$$

(Recall that Euler used this informally to resolve the Basel problem.)

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