Powerful Property of Gaussian Quadrature: Fast Convergence

Reeshad Mohammed

March 7, 2025

Properties of Gaussian-Legendre Polynomials

In order to prove the error estimate for Gaussian-Legendre Polynomials, we need to use the properties below, which are in the Brief Overview and Developing Gaussian Quadrature sections. Note that Gaussian-Legendre Polynomials have the conditions that $[a, b] = [-1, 1]$, so the below is the Gaussian-Legendre approximation.

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} w_i f(x_i) \tag{1}$$

We will denote the right hand side approximation as $G(f)$. We will also use the fact that the weights, w_i, are the integral of the Lagrangian from a to b, which is -1 to 1, which was shown in the Developing Gaussian Quadrature Section.

$$w_i = \int_{-1}^{1} \prod_{j=1 j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\, dx \tag{2}$$

Gaussian-Legendre Quadrature Error Estimate

Let $P_n(x)$ denote the Legendre polynomial of degree n. Let x_i for $i = 1, \ldots, n$ be the roots of $P_n(x)$.
We will prove the following. Let $f \in C^{2n}[-1, 1]$. Then there exists $c \in (-1, 1)$ such that

$$\int_{-1}^{1} f(x)dx = G(f) + \frac{f^{(2n)}(c)}{(2n)!} \int_{-1}^{1} P_n^2(x)dx, \tag{3}$$

where G(f) is

$$G(f) = \sum_{i=1}^{n} w_i f(x_i) \tag{4}$$

Notice that G(f) is the same as in the section above.

We will use Hermite polynomials to justify the error estimate for Gaussian quadrature. Let $H_{2n-1}(x)$ denote a polynomial of degree $2n - 1$ such that

$$H_{2n-1}(x_i) = f(x_i)$$
$$H'_{2n-1}(x_i) = f'(x_i)$$

From Hermite interpolation, we know such polynomial exists and is unique. The difference between $f(x)$ and a Hermite polynomial $H_n(x)$ is

$$f(x) = H_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n)!} \prod_{i=1}^{m} (x - x_i)^{k_i}. \tag{5}$$

for some $\xi(x) \in (a, b)$, where n is the degree of the Hermite polynomial, m is the number of x_i and k_i is the known number derivatives of x_i for $i = 1, \ldots, n$
For $H_{2n-1}(x)$, we know two derivatives at each x_i for $i = 1, \ldots, n$, so $k = 2$, and the following holds for some $\xi(x) \in (a, b)$

$$f(x) = H_{2n-1}(x) + \frac{f^{(2n)}(\xi)}{(2n)!} \prod_{i=1}^{n} (x - x_i)^2 \tag{6}$$

We can see that $(x - x_1) \ldots (x - x_n)$ is $P_n(x)$ with roots x_i for $i \in 1, \ldots, n$. So taking the integral from -1 to 1 of the equation above yields

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} H_{2n-1}(x)dx + \int_{-1}^{1} \frac{f^{(2n)}(\xi)}{(2n)!} P_n^2(x)dx \tag{7}$$

From the 2n-1 degree precision of the Gaussian Quadrature, we get the following

$$\int_{-1}^{1} H_{2n-1}(x)dx = G(H_{2n-1}) \tag{8}$$

Then using the definition of G, we get

$$G(H_{2n-1}) = \sum_{i=1}^{n} H_{2n-1}(x_i) \tag{9}$$

Lastly, since $H_{2n-1}(x)$ interpolates $f(x)$, the following summation holds

$$\sum_{i=1}^{n} H_{2n-1}(x_i) = \sum_{i=1}^{n} f(x_i) = G(f). \tag{10}$$

Combining the previous three equations yields

$$\int_{-1}^{1} H_{2n-1}(x)dx = G(f) \tag{11}$$

Since $P_n^2(x)$ is nonnegative, we can use Mean Value Theorem to conclude

$$\int_{-1}^{1} \frac{f^{(2n)}(\xi(x))}{(2n)!} P_n^2(x)dx = \frac{f^{(2n)}(c)}{(2n)!} \int_{-1}^{1} P_n^2(x)dx \tag{12}$$

for some $c \in (-1, 1)$.
Substituting the right hand sides of the two equations above, we can conclude

$$\int_{-1}^{1} f(x)dx = G(f) + \frac{f^{(2n)}(c)}{(2n)!} \int_{-1}^{1} P_n^2(x)dx \tag{13}$$

completing the proof.

1 Implications of Error Estimate on When To Use Gaussian Quadrature

As we can see from the error estimate, Gaussian-Legendre Quadrature has a strong order of convergence as sene with the inclusion of the factorial term that grows quickly as well as the integral of the squared Legendre- polynomial having a strong order of convergence.

The main conditions are $f \in C^{2n}[-1, 1]$ and the integral being from -1 to 1.

The proof above generalizes to Gaussian Quadrature on [a,b], so the second condition is not much of an issue with the following result where w(x) is no longer equal to 1.

Note that the $P_n(x)$ used below is more restrictive than the Legendre polynomial. For more robust analysis on this general form, see page 22 of this source on page 22: (https://www.math.umd.edu/ mariakc/AMSC466/LectureNotes/quadrature.pdf)

$$\int_a^b f(x)w(x)dx = Q(f) + \frac{f^{(2n)}(c)}{(2n)!} \int_a^b p_n^2(x)w(x)dx \tag{14}$$

Then it follows in general that the extremely strong order of convergence for Gaussian Quadratures, including Gaussian-Legendre Quadratures, can be taken advantage of as long as we have highly smooth functions.

In the experimental section, we will try a variety of functions to see how Gaussian Quadrature performs relative to other Quadratures such as Trapezoidal Rule.

Based on the theoretical results above, for smooth functions, we should see Gaussian Quadrature dominates most quadratures.