Graphs and Linear Algebra

Introduction

JOHNS HOPKINS
WHITING SCHOOL of ENGINEERING

Topics

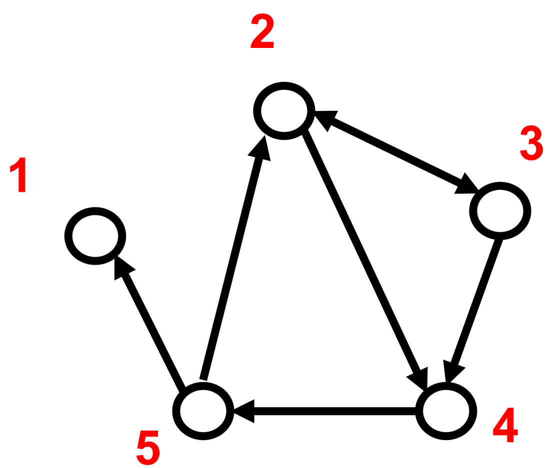
Topics:

- **Matrix representation of networks**
- **Matrix properties: Trace, transpose, determinant**
- **Eigenvalues and eigenvectors**
- **Properties of Adjacency & Laplacian Matrices**

Directed Network

Links can be represented as matrix elements:

$$A_{ij}\ :$$



Directed graph

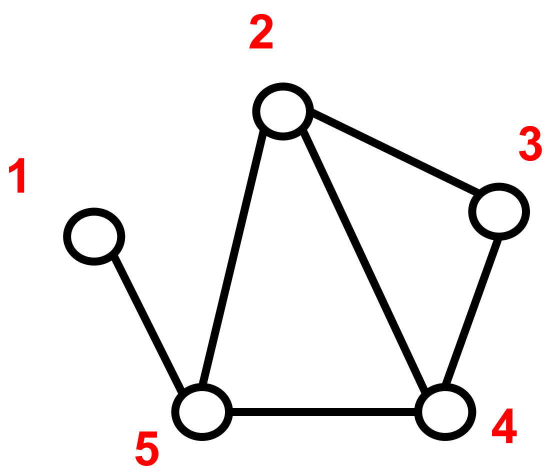
$$A = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Matrix representation

Undirected Network

When graph is undirected links can be traversed in either direction.

This results in a symmetric matrix:



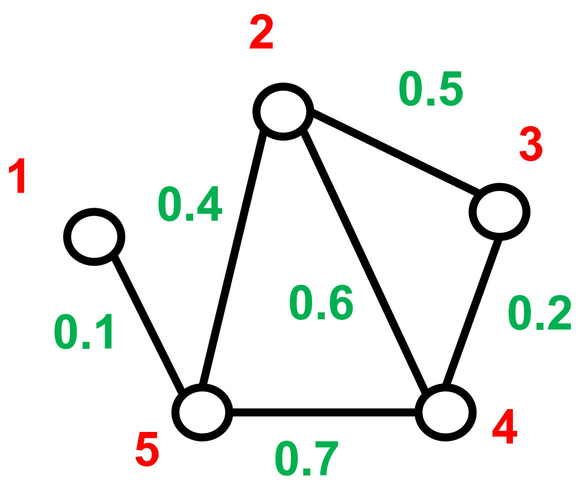
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Undirected graph $\longrightarrow$ Matrix representation

Weighted Network

Often links have weights (distance, strength, probability etc).

Adjacency matrix can represent this information:



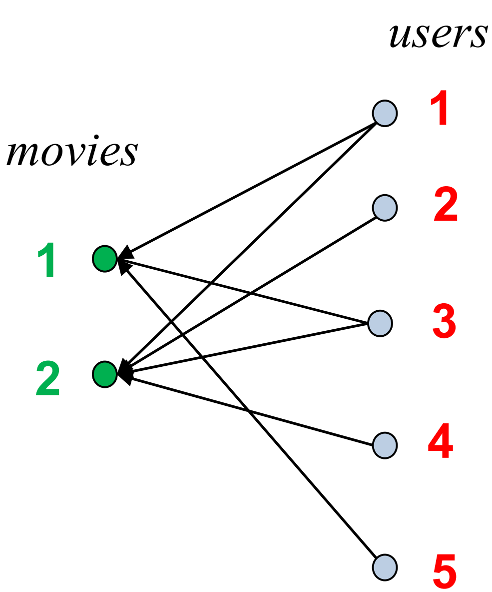
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0.5 & 0.6 & 0.4 \\ 0 & 0.5 & 0 & 0.2 & 0 \\ 0 & 0.6 & 0.2 & 0 & 0.7 \\ 0.1 & 0.4 & 0 & 0.7 & 0 \end{bmatrix} \end{array}$$

Undirected graph $\longrightarrow$ Matrix representation

Bipartite Network

Some networks are bipartite, which means there are 2 types of nodes (like users and movies).

For these types of networks indicence matrix can capture the relationship in a more concise structure:



$$B = \begin{array}{c} \\ \\ \end{array}\overset{\textit{users}}{\begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}} \textit{movies}$$

General Properties: Transpose

Transpose: $C = A^T \Rightarrow C_{ij} = A_{ji}$

$$(AB)^T = B^T A^T$$

Proof:

$$[(AB)^T]_{ij} = [\,(AB)]_{ji} = \sum_k A_{jk} B_{ki}$$

$$[(AB)^T]_{ij} = \sum_k B^T{}_{ik} A^T{}_{kj} = B^T A^T$$

General Properties: Trace

Trace:

$$Tr(A) \equiv \sum_i A_{ii}$$

$$Tr(AB) = Tr(BA)$$

Proof:

$$Tr(AB) = \sum_i \sum_k A_{ik} B_{ki} = \sum_k \sum_i A_{ik} B_{ki} = \sum_k \sum_i B_{ki} A_{ik} = \sum_k (BA)_{kk}$$

$$\Longrightarrow \quad Tr(AB) = Tr(BA)$$

General Properties: Determinant

Determinant:

$$\det(A) = \sum_{p} \sigma(p) a_{1p_1} a_{2p_2} \dots a_{np_n}$$

$$\sigma(p_1, p_2 \dots, p_n) = \begin{cases} +1 \text{ If p is an even permutation} \\ -1 \text{ If p is an odd permutation} \end{cases}$$

Properties:

$$\det(A) = \det(A^T)$$

$$\det(AB) = \det(A)\det(B)$$

Example Uses of Graph Matrices

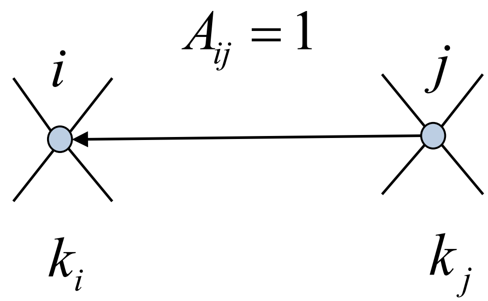
Degree using Adjacency Matrix

A measure of local connections:

Out-degree: $\delta_j^+ = I^T A = \sum_i A_{ij}$

In-degree: $\delta_i^- = AI = \sum_j A_{ij}$

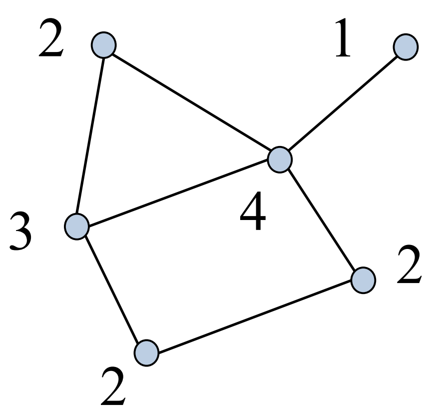
$$\mathrm{I} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



$$M \equiv \frac{1}{2}\sum_{i=1}^{N} k_i$$

Directionality:

- **Out: Maybe a sign of access and connection**
- **In: Maybe a sign of popularity or authority**
- **Undirected: Measure of well connectedness**



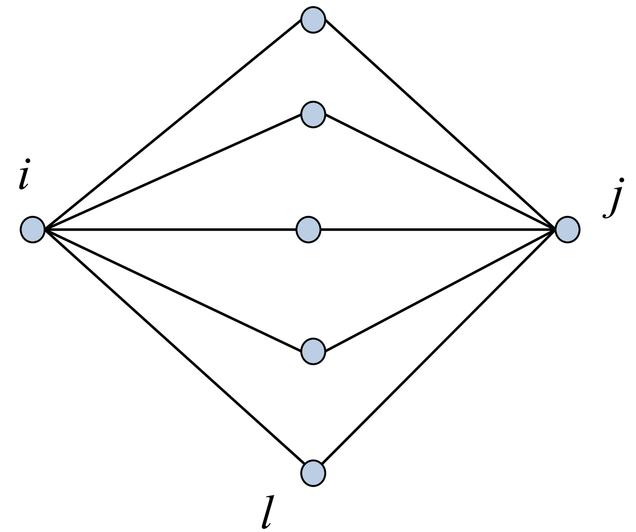
Comments:

- **Significance determined by degree distribution**
- **If links are probabilistic definition generalizes without any modification**

Walks

Let us consider number of 2 hop walks on an undirected network

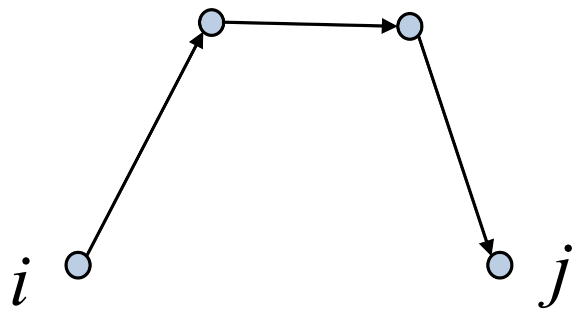
$$A_{ij}^2 = \sum\nolimits_l A_{il} \, A_{lj}$$

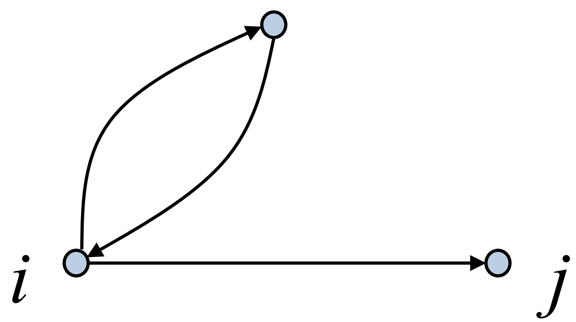


Idea can be extended to k order walks: A_{ij}^k

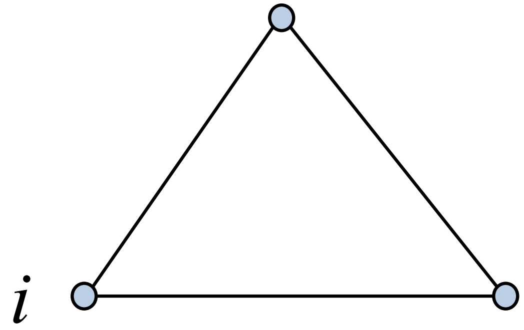
Walks

Example: A_{ij}^{3} **includes simple paths as well as potentially undesirable loops**



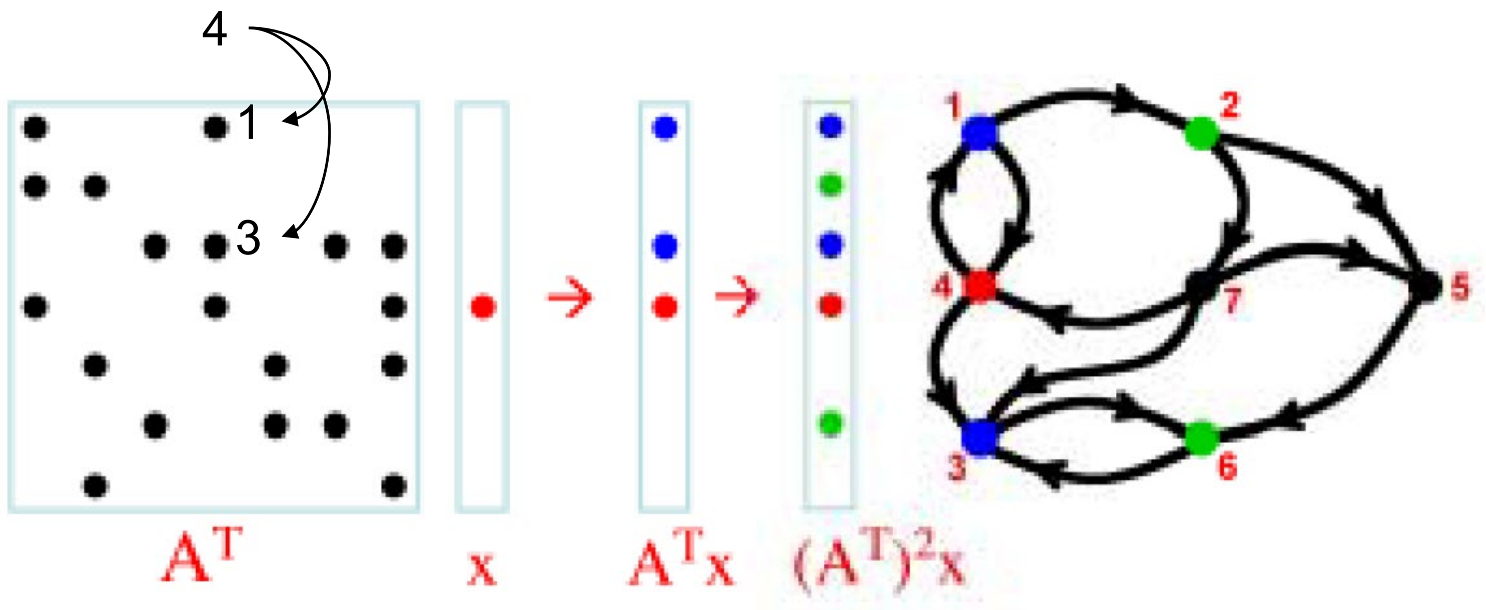


Example: A_{ii}^{3} **provides number of triangles with one corner at i**

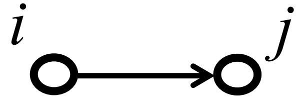


Breadth First Search

- ***Breadth first search with Linear Algebra.***



Convention: A_{ij}



"*Graph algorithms in the Language of Linear Algebra*" J. Kepner, J. Gilbert, 2011

Block Matrices

Block Matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} & & A_{1r} \\ A_{21} & A_{22} & & \\ & & & \\ A_{s1} & & & A_{sr} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} & & B_{1t} \\ B_{21} & B_{22} & & \\ & & & \\ B_{r1} & & & B_{rt} \end{pmatrix}$$

Multiplication of Block Matrices:

$$C_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj} \quad \longleftarrow$$

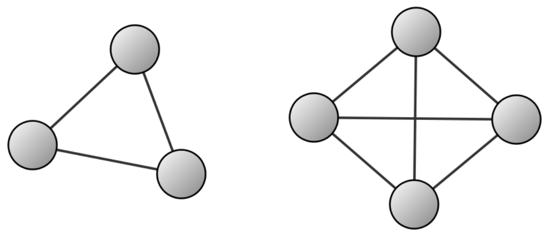
Each term is a matrix

Block Matrices are particularly useful when there are patterns:

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \Rightarrow \qquad AB = \begin{pmatrix} A_{11}B_{11} & 0 \\ 0 & A_{22}B_{22} \end{pmatrix}$$

Walks and identification of graph components

Graph with 2 components



$$A=\begin{matrix} * & 1 & 1 & * & * & * & * \\ 1 & * & 1 & * & * & * & * \\ 1 & 1 & * & * & * & * & * \\ * & * & * & * & 1 & 1 & 1 \\ * & * & * & 1 & * & 1 & 1 \\ * & * & * & 1 & 1 & * & 1 \\ * & * & * & 1 & 1 & 1 & * \end{matrix}$$

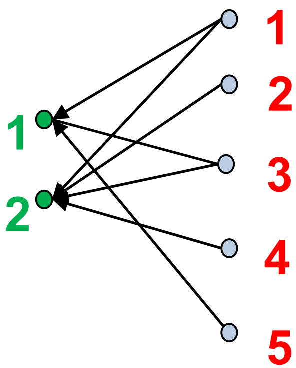
$$[A \cup B]_{ij} = \begin{cases} 0 & if\ A_{ij} = B_{ij} = 0 \\ 1 & otherwise \end{cases}$$

$$\mathrm{B} \stackrel{\text{def}}{=} A \cup A^2$$

$$B_2=\begin{matrix} 1 & 1 & 1 & * & * & * & * \\ 1 & 1 & 1 & * & * & * & * \\ 1 & 1 & 1 & * & * & * & * \\ * & * & * & 1 & 1 & 1 & 1 \\ * & * & * & 1 & 1 & 1 & 1 \\ * & * & * & 1 & 1 & 1 & 1 \\ * & * & * & 1 & 1 & 1 & 1 \end{matrix}$$

Incidence Matrix Based Inference

Incidence matrix can be used to derive user to user or group to group relationships



users

$$B = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \quad \textit{movies}$$

$$B^T B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

User similarity matrix

$$B^T B = \begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Group similarity matrix

$$BB^T = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$$

Eigenvalue Equation

Eigenvalues

Eigenvalue equation for A:

$$Ax = \lambda x$$

Eigenvalue equation for the transpose:

$$A^T y = \beta y \quad \Rightarrow \quad \det\left(A^T - \beta I\right) = 0$$

Are eigenvalues related?

$$\det\left(A - \lambda I\right) = 0 \quad \Rightarrow \quad \det\left(\left(A - \lambda I\right)^T\right) = 0$$

$$\Rightarrow \quad \det\left(A^T - \lambda I\right) = 0$$

Equation that defines eigenvalues of the transpose

$$\therefore \left\{\lambda\right\} = \left\{\beta\right\}$$

Matrix and its transpose have the same set of eigenvalues

Eigenvalues of Real Symmetric Matrices

Real Symmetric Matrix:

Real $A = A^*$ Symmetric $A = A^T$

$$Ax = \lambda x \quad \Rightarrow \quad x^{*T} A x = x^{*T} \lambda x = \lambda x^2$$

$$x^{*T} A x = \left(A^{*T} x\right)^{*T} x = \left(Ax\right)^{*T} x = \left(\lambda x\right)^{*T} x = \lambda^* x^2$$

$$\therefore \lambda = \lambda^* \quad \Rightarrow$$ **Eigenvalues are real**

Eigenvectors of Real Symmetric Matrices

Real Symmetric Matrix:

Real $A = A^*$ Symmetric $A = A^T$

$$Ax = \lambda x \qquad Ay = \beta y \qquad \lambda \neq \beta$$

$$x^T A y = x^T \beta y = \beta x^T y$$

$$x^T A y = \left(A^T x\right)^T y = (Ax)^T y = (\lambda x)^T x = \lambda x^T y$$

$$\lambda x^T y = \beta x^T y \qquad \therefore x^T y = 0 \Longrightarrow$$ **Eigenvectors are orthogonal**

Eigenvector Decomposition

Eigenvector decomposition:

$$Ax = \lambda x \qquad A = \sum_{i=1}^{N} \lambda_i x_i x_i^T \qquad x_i^T x_j = \delta_{ij}$$

$$Ax_j = \sum_{i=1}^{N} \lambda_i x_i x_i^T x_j = \sum_{i=1}^{N} \lambda_i x_i \delta_{ij} = \lambda_j x_j \quad \checkmark$$

$$U \equiv \begin{bmatrix} x_1 & x_2 & x_3 & \dots \\ \downarrow & \downarrow & \downarrow & \end{bmatrix}$$

$$A^{-1} = \sum_{i=1}^{N} \frac{1}{\lambda_i} x_i x_i^T$$

$$D \equiv \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & \\ 0 & 0 & 0 & \dots \end{bmatrix}$$

$$AA^{-1}A = \sum_{j=1}^{N} \frac{1}{\lambda_j} Ax_j x_j^T A \quad \Rightarrow \quad A = \sum_{i=1}^{N} \lambda_i x_i x_i^T \quad \checkmark$$

Positive Semidefinite Matrices

Positive Semidefinite Matrix:

$$x^{*T} A x \geq 0$$

$$x^{*T} A x = x^{*T} \lambda x = \lambda x^2 \geq 0$$

$$\therefore \lambda \geq 0 \quad \Rightarrow$$

Eigenvalues are non-negative

Eigenvalues Relation with Trace

Relationship of Trace with eigenvalues:

$$\det(AB) = \det(A)\det(B)$$

$$\det(A - \lambda I) = \det(UU^T AUU^T - \lambda I) = \det(UDU^T - \lambda I)$$

$$\det(U^T U) = \det(UU^T)$$

$$\det(UDU^T - \lambda I) = \det(D - \lambda I) = \prod_{i=1}^{N} (\lambda_k - \lambda)$$

$$A - \lambda I \equiv \begin{bmatrix} a_{11} - \lambda & & & \dots \\ & a_{22} - \lambda & & \\ & & \dots & \\ \dots & & & \dots \end{bmatrix}$$

$$\det(A - \lambda I) = \prod_{i=1}^{N} (\lambda_k - \lambda)$$

$$D \equiv \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & \\ 0 & 0 & 0 & \dots \end{bmatrix}$$

$$(A_{11} - \lambda)(A_{22} - \lambda)\dots(A_{NN} - \lambda) \qquad (\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_N - \lambda)$$

$$O(\lambda^{N-1}): \qquad \lambda^{N-1}\sum_{i=1}^{N} A_{ii} \qquad \lambda^{N-1}\sum_{i=1}^{N} \lambda_i \Rightarrow \boxed{tr(A) = \sum_{i=1}^{N} \lambda_i}$$

Gerschgorin Circles

Can we bound eigenvalues of A without solving for them ?

- **Eigenvalues of A are contained in the union of Gerschgorin circles:**

$$r_i \equiv \sum_{i \neq j} |a_{ij}| \qquad \Rightarrow \qquad |z - a_{ii}| \leq r_i$$

- **If a union *U* of *k* circles do not touch the rest of the *n-k* circles than there are exactly k eigenvalues in *U*:**

- **Since the spectrum of A is same as its Transpose column sums have the same constraint:**

$$c_i \equiv \sum_{i \neq j} |a_{ij}| \qquad \Rightarrow \qquad |z - a_{ii}| \leq c_i$$

Gerschgorin Circles

- **Eigenvalues of A are contained in the intersection of column and row Gerschgorin circles:**

$$Ax = \lambda x \qquad |x| = 1 \qquad |x_i| \geq |x_j| \qquad x_i > 0$$

Proof:

$$\lambda x_i = \sum_j a_{ij} x_j \quad \Rightarrow \quad (\lambda - a_{ii}) x_i = \sum_{j,i \neq i} a_{ij} x_j$$

$$|(\lambda - a_{ii}) x_i| = \left| \sum_{j,i \neq i} a_{ij} x_j \right| \quad \Rightarrow \quad |(\lambda - a_{ii})| = \left| \sum_{j,i \neq i} a_{ij} \frac{x_j}{x_i} \right|$$

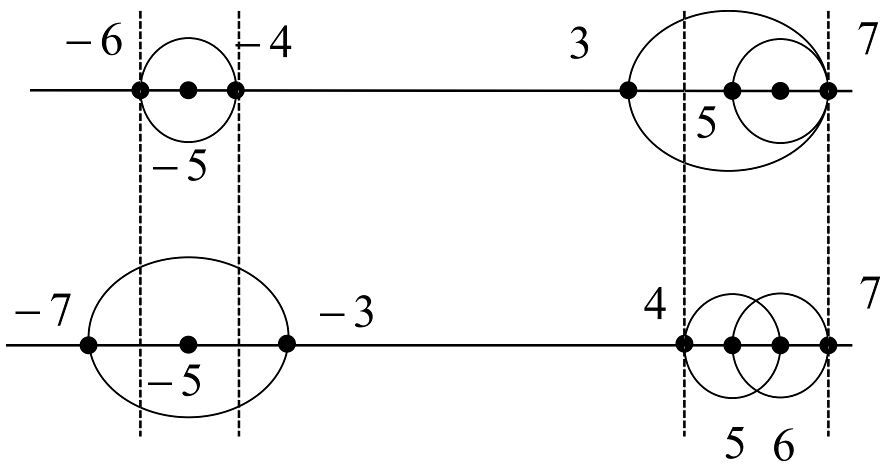
$$|(\lambda - a_{ii})| \leq \sum_{j,i \neq i} |a_{ij}| \left| \frac{x_j}{x_i} \right| \leq \sum_{j,i \neq i} |a_{ij}| = r_i \quad \Rightarrow \quad -r_i + a_{ii} \leq \lambda \leq r_i + a_{ii}$$

Gerschgorin Circles

Example:

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 0 & 6 & 1 \\ 1 & 0 & -5 \end{pmatrix} \qquad r_i \equiv \sum_{i \neq j} |a_{ij}|$$

$$a_{11} = 5$$
$$a_{22} = 6$$
$$a_{33} = -5$$



$$r_1 = 2$$
$$r_2 = 1$$
$$r_3 = 1$$

$$c_1 = 1$$
$$c_2 = 1$$
$$c_3 = 2$$

$$-6 \leq \lambda_3 \leq -4 \qquad 4 \leq \lambda_{1,2} \leq 7$$

Perron-Frobenous Summary

Positive: ⇨ $A_{ij} > 0$

Non-negative: ⇨ $A_{ij} \geq 0$

Primitive: ⇨ A^k is positive

Reducible: ⇨ A is block diagonal

Positive

- **Y** → $|\lambda| < \lambda_{Max}$

  $$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

  $$\lambda = \{0,2\}$$

- **Nonnegative** → Irreducible
  - **Y** → Primitive
    - **Y** → $|\lambda| < \lambda_{Max}$

      $$A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$$

      $$\lambda = \{-1,2\}$$

    - **N** → $|\lambda| \leq \lambda_{Max}$

      $$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

      $$\lambda = \{-1,1\}$$

  - **N** → $|\lambda| \leq \lambda_{Max}$

    $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

    $$\lambda = \{1,1\}$$

Properties of Adjacency and Laplacian Matrices

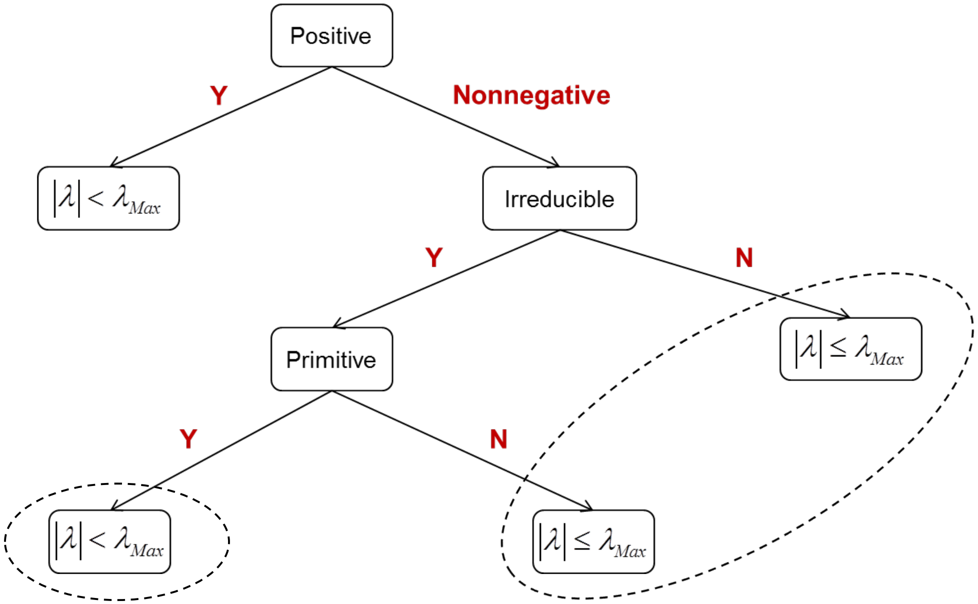
Properties of Adjacency Matrix

Adjacency Matrix is Real and non-negative

For undirected graphs it is also symmetric

In general largest eigenvalue of A does not dominate the rest:

$$-\lambda_{\max} \leq \lambda \leq \lambda_{\max}$$



If A is primitive largest eigenvalue of A dominates the rest:

$$|\lambda| < \lambda_{\max}$$

Properties of Adjacency Matrix

Maximum eigenvalue: $\lambda_{\max} \le d_{\max} = N-1$

Proof: $Ax = \lambda x$ *Normalize x such that:* $x_k = 1 \ge x_i$ $0 \le |x_i| \le 1$

$$[Ax]_i = \sum_{k=1}^{N} A_{ik} x_k \le \sum_{k=1}^{N} A_{ik} = d_i \le d_{\max} \qquad d_i \equiv \sum_{k=1}^{N} A_{ik}$$

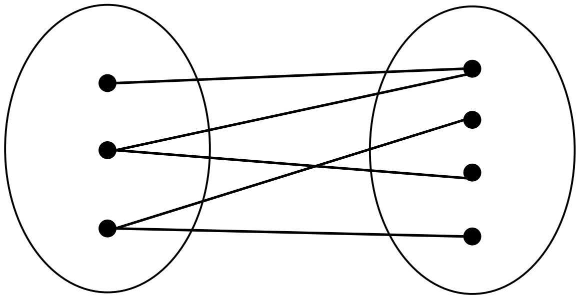
For all i: $\lambda x_i \le d_{\max}$ $x_k = 1 \ge x_i$ $\Rightarrow$ $\lambda_{\max} \le d_{\max} = N-1$

Alternative Proof: Use Gerschgorin Circles:

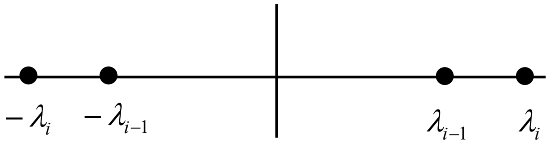
$$r_i \equiv \sum_{i \ne j} |A_{ij}| \le d_{\max} \qquad A_{ii} = 0 \qquad -d_{\max} \le \lambda_{\max} \le d_{\max}$$

Adjacency Matrices of Special Graphs

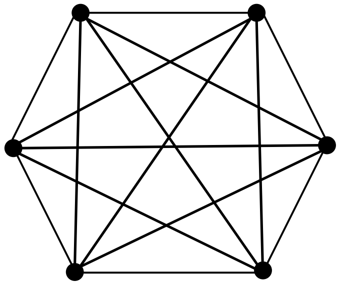
A bipartite graph:

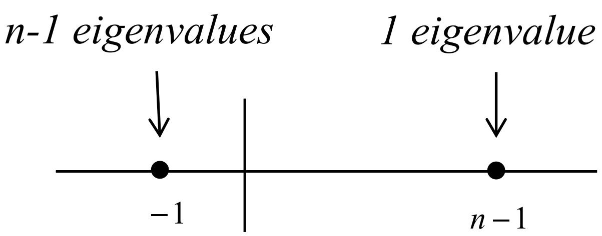


Symmetric eigenvalues:



A complete graph:





Connection with Isomorphism:

If eigenvalues of 2 graphs do not match they are not isomorphic (opposite is not true)

Properties of Laplacian Matrix

Laplacian Matrix: $$L = D - A$$

Eigenvalues: $$0 \le \lambda \le 2d_{\max}$$

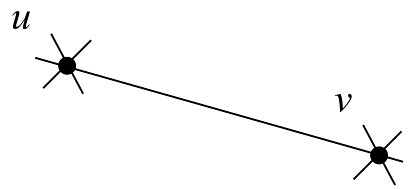
$$D \equiv \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & \dots \end{bmatrix}$$

Proof: Use Gerschgorin Circles:

$$r_i \equiv \sum_{i \ne j} \left| L_{ij} \right| = d_i \qquad L_{ii} = d_i$$

$$r_i - L_{ii} \le \lambda \le r_i + L_{ii} \quad \Rightarrow \quad 0 \le \lambda \le 2d_{\max} \quad \checkmark$$

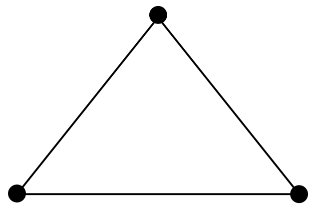
A tighter bound: Anderson & Morley 1985:



$$0 \le \lambda \le \max[d(u) + d(v)] \le 2d_{\max}$$

An Example of Laplacian Matrix

Triangle Matrix:

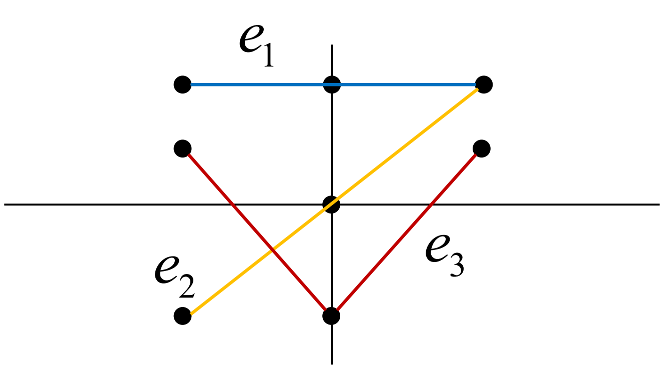


$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\lambda = (0,3,3)$$

$$\lambda_1 = 0 \Rightarrow e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \lambda_2 = 3 \Rightarrow e_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \lambda_3 = 3 \Rightarrow e_3 = \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}$$

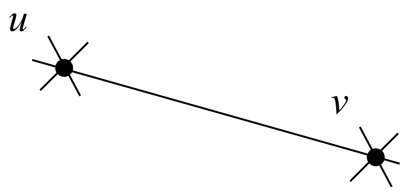
Eigenvectors plotted:



Normalized Laplacian Matrix

Eigenvalues of Laplacian turns out to be useful for graph resiliency analysis. In order to be able to compare different graphs it is necessary to "normalize" the eigenvalues such that they can be compared.

$$L_N = D^{-1/2} L D^{-1/2}$$



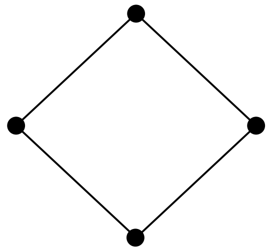
$$L_N(u,v) = \begin{cases} 1 & \text{if } u = v \;\; d(v) \neq 0 \\ -\dfrac{1}{\sqrt{d(u)d(v)}} & \text{if } (u,v) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Bounds on eigenvalues: $0 \le \lambda \le 2$

Eigenvalues of Adjacency & Laplacian Matrices

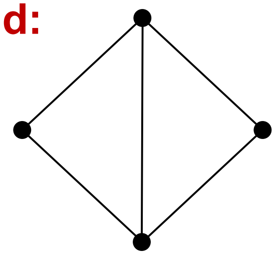
Examples:

Cycle:



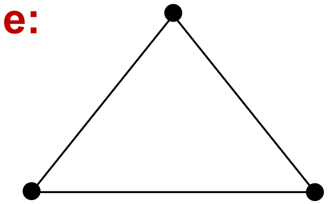
$$d_{\max} = 2$$
$$A = (-2,0,0,2)$$
$$L = (0,2,2,4)$$

Diamond:



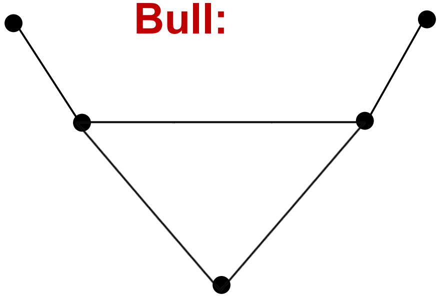
$$d_{\max} = 3$$
$$A = (-1.5,-1,0,2.56)$$
$$L = (0,2,4,4)$$

Triangle:



$$d_{\max} = 2$$
$$A = (-1,-1,2)$$
$$L = (0,3,3)$$

Bull:

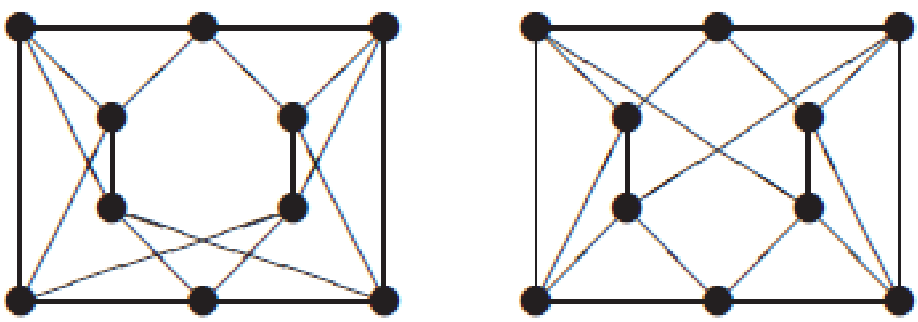


$$d_{\max} = 3$$
$$A = (-1.6,-1.3,0,0.6,2.3)$$
$$L = (0,0.7,1.38,3.6,4.3)$$

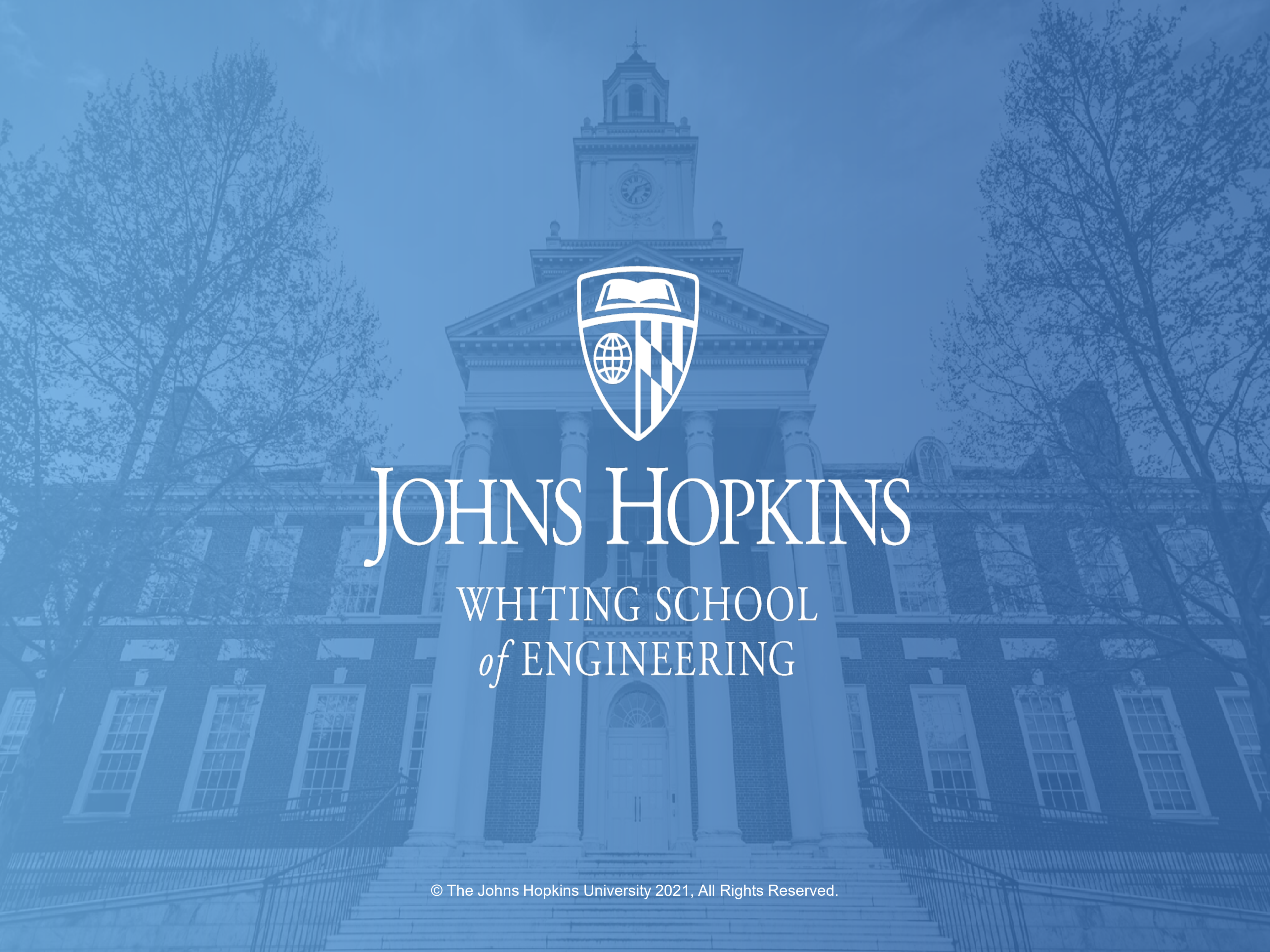
Cospectral Graphs

For small graphs eigenvalue spectrum may exactly match. This does not imply that graphs are structurally equivalent.

Example [1]: Below graphs have the same spectrum but are not isomorphic



[1] A. Brouwer, W. Haemers, “Spectra of Graphs”, Springer



JOHNS HOPKINS
WHITING SCHOOL of ENGINEERING

Example: Eigenvalues & Eigenvectors

Eigenvalue equation: $Ax = \lambda x \qquad D \equiv A - \lambda I \qquad Dx = 0$

In order to have a solution: $\det\left(A - \lambda I\right) = 0$

Ex: $A = \begin{bmatrix} 7 & -4 \\ 5 & -2 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 7-\lambda & -4 \\ 5 & -2-\lambda \end{bmatrix}$

$$\det\left(A - \lambda I\right) = \lambda^2 - 5\lambda + 6 = 0 \qquad \Rightarrow \qquad \lambda = \{2,3\}$$

$$\lambda = 2: \qquad D \equiv A - \lambda I = \begin{bmatrix} 5 & -4 \\ 5 & -4 \end{bmatrix} \qquad\qquad \lambda = 3: \qquad D \equiv A - \lambda I = \begin{bmatrix} 4 & -4 \\ 5 & -5 \end{bmatrix}$$

$$D\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{41}}\begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad\qquad D\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$