Chapter 1
Analysis of the Orszag-Mclaughlin System and Generalization to $\mathbb{R}^N$

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Abstract In the following paper, we present the results of numerical solutions to the Orzsag-McLaughlin nonlinear system via 4th order Runge-Kutta integration. We verify the system is indeed chaotic in that it is highly sensitive to perturbations in initial conditions. We also verify that it is a conservative system, despite being non-time-reversible. We present findings of our analysis regarding dependence on initial conditions, number of dimensions, and the nuances between the cases of even and odd dimension.

1.1 Introduction

In this paper we will investigate the dynamics of the Orszag-McLaughlin system, a nonlinear dynamical system analogous to the Navier-Stokes (Euler) equations. This analysis was inspired by a paper by Dr. Ookie Ma and Dr. J. B. Marston [2] and seeing the interesting advances made in analogy to quantum mechanics. While the specific advances they made may be out of the realm of an undergraduate curriculum, in this paper we investigate the general dynamics of the Orszag-Mclaughlin system, using published lecture notes from Dr. Orszag [3] as a guide. In the following sections, we will discuss some background and motivation for the Orszag-McLaughlin system, starting from the theory of turbulent flow and analogy to the Navier-Stokes equations. After some discussion of the fundamental physics of both systems, we will move to numerical integration of the system for the canonical 5-dimensional case. We will then present some of our findings regarding the system's dynamical behavior in generalized N-dimensions, and some of the differences between the cases of even and odd N.

1.2 Turbulent Flow

Turbulent flow is the motion of a highly viscous (inviscid) fluid, which possesses complex and seemingly random structure. In physics, turbulent flow is more precisely defined as fluid motion characterized by chaotic changes in its pressure and flow velocity. One of the key properties of turbulent flow is its enhanced transport properties: momentum, energy, and particle transport rates may exceed the transport rates of the molecules carrying these properties.

Examples of turbulent flow include the plume of smoke rising from a cigarette or incense candle, the mixing of pollution into the atmosphere to create smog, the flow of air around an airplane, and the mixing of martinis.

1.2.1 Homogeneous Turbulence

Turbulence is called *homogeneous* if all points in space are statistically equivalent. In other words, the only boundary conditions or external forces are at the edge of the system.

1.2.2 Isotropic Turbulence

Additionally, turbulence is said to be *isotropic* if all directions are statistically equivalent, meaning the statistical properties are invariant for a full rotation group, which includes rotations and reflections of the coordinate axes. Moreover, homogeneous isotropic turbulence is an idealized version of the realistic turbulence, but is amenable to analytical studies and is the type of turbulence analyzed in this paper.

1.3 Background of the Navier-Stokes Equations

The Navier-Stokes equations, named after French engineer/physicist Claude-Louis Navier and Anglo-Irish physicist/mathematician George Gabriel Stokes, are a set of second order nonlinear partial differential equations that describe the motion of incompressible Newtonian fluids.

The equations are a set of coupled differential equations that are intended to be solved in order to describe the fluid flow of a system. Solving the equations provides the fluid velocity and pressure of a dynamical fluid system. Even so, complete solutions to the Navier-Stokes equations have only been found for a maximum of two dimensional fluid flow. Above two dimensions, chaotic behavior of solutions is unmanageable. As such, it is only possible to approximate solutions to the Navier-Stokes equations above three dimensions via numerical analysis. Given the rigorous nature of the topic, general solutions to the Navier-Stokes equations have been chosen as a "Millennium Problem" by the Clay Mathematics Institute, and solving it rewards one million dollars. The Navier-Stokes equations can be used by environmental scientists to model the weather, biomedical engineers to track blood flow in the human body, or even civil engineers to design windmills.

The Navier-Stokes equations contain a time-dependent continuity equation to account for the conservation of mass, three time-dependent conservation of momentum equations, and a time dependent conservation of energy equation. Naturally, in three dimensions, such a system of equations would depend on spatial coordinates (x,y,z) and time t.

Continuity Equation

$$\overrightarrow{\nabla} \cdot \overrightarrow{V} = 0 \tag{1.1}$$

Momentum Equations

$$\rho \frac{D\overrightarrow{V}}{Dt} = -\overrightarrow{\nabla} p + \rho \overrightarrow{g} + \mu \nabla^2 \overrightarrow{V} \tag{1.2}$$

Where $-\overrightarrow{\nabla} p$ is the pressure gradient. This term accounts for the fact that fluid flows in the direction of the largest change of pressure. $\rho \overrightarrow{g}$ is the body force term, which accounts for external forces on the field. $\mu \overrightarrow{\nabla}^2$ is the term which accounts for the diffusion of momentum, as viscosity is constant in a Newtonian fluid. The expression $\frac{D\overrightarrow{V}}{Dt}$ is known as the total derivative, and can be expressed as follows:

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V \tag{1.3}$$

Where $\frac{\partial V}{\partial t}$ represents the change of velocity with respect to time and $(V \cdot \nabla)V$ is a corrective term.

Euler Equations

More specifically, the Navier-Stokes equations represented with zero viscosity and zero thermal conductivity are known as the Euler equations. The Euler equations are a set of quasilinear hyperbolic partial differential equations expressed as follows.

$$\frac{Du}{Dt} = -\overrightarrow{\nabla} \omega + g \tag{1.4}$$

$$\overrightarrow{\nabla} \cdot u = 0 \tag{1.5}$$

Where ω is the thermodynamic work, or the internal source term.

1.4 Relevance of the Navier-Stokes Equations to the Orszag-McLaughlin System

Fluctuation velocities in the atmosphere, an incompressible turbulent system, generally have error of only a few meters per second. As such, the incompressible Navier-Stokes equation is applicable to turbulence. However, trouble arises when sound is introduced, as error in sound velocity occurs in a window of hundreds of meters per second. The Finite-Mode Model, as will be discussed in Section 1.6, does not account for sound, as there is no accurate way to approximate it.

Furthermore, the theory behind the Navier-Stokes equations above two dimensions is generally incomplete, as there is no established governance of existence and uniqueness of solutions that show nice behavior of solutions. It is possible to modify the Navier-Stokes equations for above three dimensions. For example, Ladyzhenskaya proposes introducing a biharmonic damping term to the momentum equations as discussed in [3]. This would guarantee existence and uniqueness for positive values of the biharmonic damping term. Nevertheless, despite the seemingly unsolvable enigma of the Navier-Stokes equations above two dimensions, it is still possible to analyze a turbulent system without using them by looking at models such as those discussed in Section 1.6.

1.5 Ergodicity

Ergodicity is the expression of the idea that a point of a moving system will eventually visit all parts of the space the system moves in given enough time. This movement implies that the average behavior of a system can be deduced from the trajectory of a point. So, a sufficiently large collection of random samples from a process can represent the average statistical properties of the entire process. In essence, ergodicity is a mathematical construct that provides a technical argument for the everyday notion of randomness, such as going back to the examples from Section 1.2 for smoke coming up from a cigarette and, if given enough time, the smoke would fill the room. Further reading on Ergodicity can be found in [1].

1.6 Finite-Mode Model (Orszag-McLaughlin System)

Due to the inherent "random" solutions to the differential equations found in the Navier-Stokes equations, an N-dimensional dynamical system was devised by Orszag and McLaughlin [3] to better understand the random nature of turbulence. This system is:

$$\frac{dx_i}{dt} = x_{i+1}x_{i+2} + x_{i-1}x_{i-2} - 2x_{i+1}x_{i-1}, (i = 1, \dots, N) \tag{1.6}$$

Where the periodic boundary condition $x_i = x_{i+N}$ is assumed for this system with N modes, dimensions, or degrees of freedom (DOF). The system (1.6) is analogous to the inviscid Navier-Stokes (Euler) equations in a few respects:

1. The system involves only quadratic interaction among degrees of freedom
2. The Orzag-McLaughlin system conserves energy in the sense:

$$\frac{d}{dt}\frac{1}{2}\sum_{i=1}^{N} x_i^2 = 0 \tag{1.7}$$

3. The system obeys Liouville's theorem in the following manner, despite being non-time-reversible:

$$\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{dx_i}{dt}\right) = 0 \tag{1.8}$$

Liouville's theorem is a key theorem in Hamiltonian mechanics which asserts that the phase-space distribution function is constant along the trajectories of the system. In other words, Liouville's theorem states that the density of a system points in the vicinity of a given point traveling through phase-space constant in time. Since the system (1.6) above obeys (1.8), it is an example of an incompressible dynamical (or conservative) system. An incompressible dynamical system occurs when the total volume occupied in phase space cannot be compressed, as in it keeps the same volume. So although the shape of a given volume-element may become distorted over time, its total volume always remains unchanged. It is also important to note that all of the system's trajectories lie on a hypersphere of radius $R = \sqrt{2E}$ [2]. The system also appears to be ergodic for all initial conditions except a set of measure zero, also discussed in [2].

The purpose of this investigation is to understand the Orszag-McLaughlin system, so we first investigate its time-evolution in different dimensions. While many dimensions exhibit intriguing dynamical behavior, it is important to note that for 3 DOF, the system becomes static. The following sections will discuss more interesting systems with higher DOFs.

1.7 Numerical Techniques

In order to analyze (1.6), a numerical integration technique is necessary. In our analysis, we use 4-th order Runge-Kutta (RK) numerical integration in Python. With the assistance of helper functions that enforce the periodic boundary for the subscripts of plus/minus one or two in (1.6), 4-th order RK is easily implemented as follows:

$$\boldsymbol{k}_1 = h\boldsymbol{f}(\boldsymbol{x}, t)$$

$$\boldsymbol{k}_2 = h\boldsymbol{f}(\boldsymbol{x} + \frac{1}{2}\boldsymbol{k}_1, t + \frac{1}{2}h)$$

$$\boldsymbol{k}_3 = h\boldsymbol{f}(\boldsymbol{x} + \frac{1}{2}\boldsymbol{k}_2, t + \frac{1}{2}h)$$

$$\boldsymbol{k}_4 = h\boldsymbol{f}(\boldsymbol{x} + \frac{1}{2}\boldsymbol{k}_3, t + h)$$

$$\boldsymbol{x}(t+h) = \boldsymbol{x}(t) + \frac{1}{6}(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4) \tag{1.9}$$

Where h is the time step and now we are using boldface to refer to vectors for convenience.

$$h = T_f/dt \tag{1.10}$$

Even without any prior understanding of numerical integration techniques, one can see how (1.9), starting at $t = 0$ and iterating through however many timesteps one chooses, can determine the state of the system at any given time.

1.8 Lyapunov Exponents and the Lyapunov Spectrum

While it is important to be able to use numerical techniques, more tools are needed to better understand the system and if it is chaotic or not. One way to determine this is by looking at the rate of separation of infinitesimally close trajectories. This quantity is known as a Lyapunov exponent. To be quantitative, if there is some initial condition x_0 with a nearby point considered as $x_0 + \delta_0$, where δ_0 is an extremely small initial separation, then δ_n would be the seperation after n iterations. If $|\delta_n| = |\delta_0|e^{n\lambda}$, λ would be the Lyapunov exponent. This is discussed in detail in [4]. From here, we will often use $\delta x(t)$ to refer to

$$\delta x(t) = \delta_n = |\boldsymbol{x} - \boldsymbol{y}|, (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N$$

. We will relate t to n by a time step referenced in (1.10)

$$h = t/n$$

Since the rate of this separation can be different for different orientations of initial separation vectors, this implies there is a spectrum of Lyapunov exponents that are equal in number to the dimensionality of the phase space. The largest exponent, known as the Maximal Lyapunov Exponent (MLE), determines a notion of predictability for whether a system becomes chaotic or not. If the MLE is positive, it is an indication that the system is chaotic. n = t/dt

The MLE can be defined quantitatively as the largest value gained from:

$$\begin{aligned}\lambda &\approx |\frac{1}{n}\ln\frac{\delta_n}{\delta_0}| \\ &= \frac{1}{n}|\ln\frac{f^n(x_0+\delta_0)-f^n(x_0)}{\delta_0}| \\ &= \frac{1}{n}\ln|((f^n)'(x_0)|\end{aligned} \tag{1.11}$$

Where the limit $\delta_0 \to 0$ has been taken in the previous step.

In this paper, the MLE is determined for multiple degrees of freedom to see if the system would be chaotic or not and to see how the Lyapunov spectrum might evolve for various degrees of freedom.

1.9 Orszag-McLaughlin: N=5

In the case of N=5 DOF, the Orszag-McLaughlin system becomes dynamical. We first investigate the chaotic behavior of the system by examining two different trajectories with *near* identical initial conditions. We choose to investigate the initial conditions suggested by Orzsag in a canonical example [3]

$$\boldsymbol{x}(0) = (0.54, -1.5, -0.68, -1.18, -0.676)$$

We'll perturb the system by changing the value of $x_1(0)$ very slightly.

$$\boldsymbol{y}(0) = (0.5401, -1.5, -0.68, -1.18, -0.676)$$

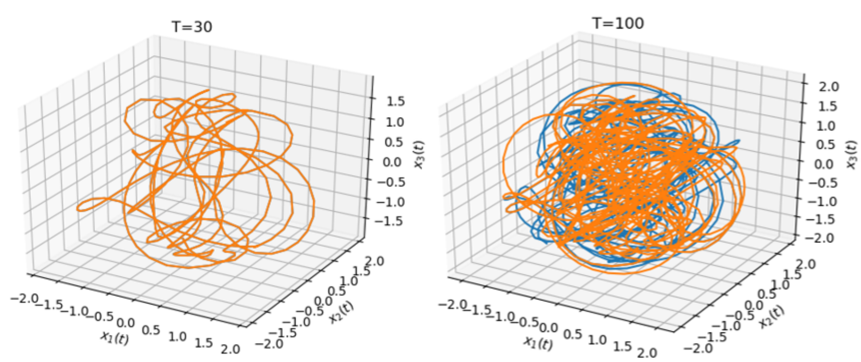

Fig. 1.1 3D phase portrait (x_1, x_2, x_3) of $\boldsymbol{x}$(t), $\boldsymbol{y}$(t) for T=30 and T=100 with a 0.1s timestep. The blue line is $\boldsymbol{x}$(t) and the orange $\boldsymbol{y}$(t).

In Fig. 1.1, the divergent behavior of the two trajectories does not become apparent for T=30, but is clear by T=100. Note that the initial difference between these two trajectories is only 0.0001 for one of their 5 DOF. This is indicative of our hypothesis that the Orszag-McLaughlin system is chaotic, but we should examine the Lyapunov spectra to get a better quantitative picture of the chaos of the system.

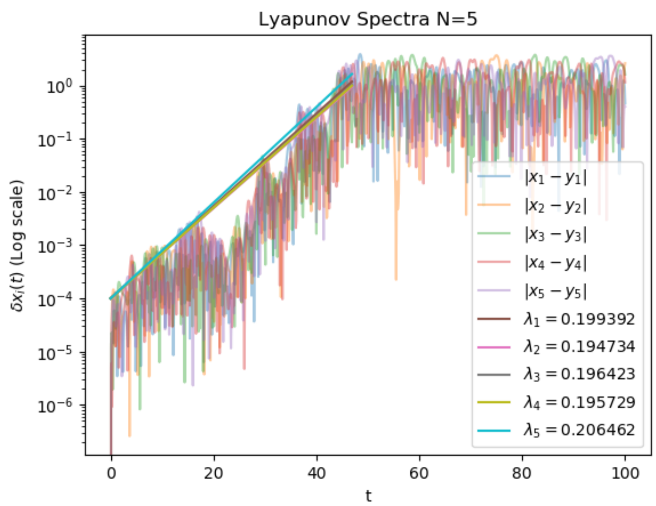

Fig. 1.2 Lyapunov spectra, λ_i for each of the 5 dimensions of $\boldsymbol{x}$(t), $\boldsymbol{y}$(t).

In Fig. 1.2, we fit each of the displacements $\delta_i(t) = x_i(t) - y_i(t)$ up to 470 indices (t=47), where the logarithmic difference becomes negligible. The fact that the largest λ_i (MLE) is positive means that the system is chaotic by definition [4]. In Fig. 1.3 we see the *global* Lyapunov exponent, where the displacement $\delta\boldsymbol{x}(t)$ is calculated as the total difference between $\boldsymbol{x}$(t), $\boldsymbol{y}$(t) in their 5-dimensional phase space.

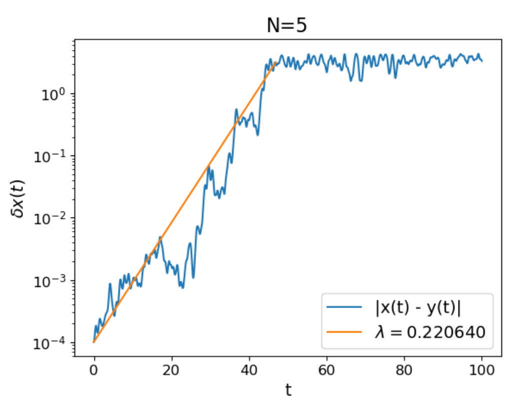

Fig. 1.3 Global Lyapunov exponent for $\delta\boldsymbol{x}(0) = 0.0001$. Measured $\lambda = 0.220640 \pm 0.000725$

1.9.1 Energy Conservation and Error

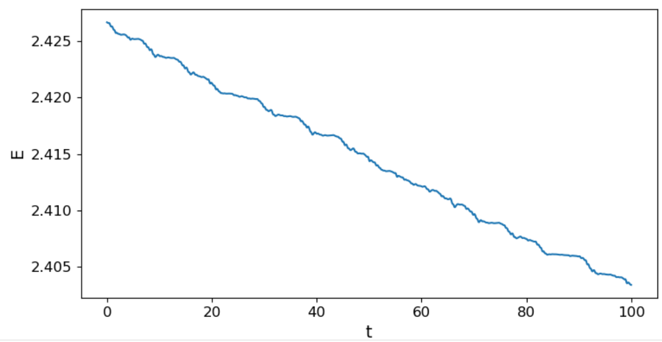

Fig. 1.4 The total energy $E = \frac{1}{2}\sum_{i=1}^{N} x_i^2$ over time for $\boldsymbol{x}(0) = (0.54, 1.5, 0.68, 1.18, 0.676)$. By T=100, the total energy has decreased by $\sigma E \simeq 0.02$

Recall equation (1.7) defines an energy-like quantity that is conserved in the system.

$$\frac{1}{2}\sum_{i=1}^{N} x_i^2 = E \tag{1.12}$$

The energy computed according to (1.12) is shown in Fig. 1.4. Over the integration up to T=100, the phase-space energy E decreases by 0.02. This loss of energy is due to the imperfection of numerical integration, and should be accounted for by the error in our RK4 method, which has a total truncated error on the order of $O(h^4)$.

With a timestep of $h = 0.1$, we expect an error of on the order of 0.0001. However, because we are squaring the each x_i, we can expect an error on the order of $\sqrt{(h^4)} = 0.01$, consistent with the $\sigma E \simeq 0.02$.

Fundamental to the Orszag-McLaughlin system is that fact that all trajectories lie on the hypersphere of radius

$$R = \sqrt{2E} \tag{1.13}$$

Fig. 1.4 indicates $E \simeq 2.4$. In Fig. 1.1, we see the trajectories in 3D phase space lie within the box of

$$(x, y, z) \in \{\boldsymbol{r}, |x_i| < 2, \forall x_i \in \boldsymbol{r}\}$$

This gives good intuitive (visual) confirmation of (1.13), as $R \simeq \sqrt{2 * 2.4} \simeq 2.19$.

1.9.2 Dependence on Initial Conditions

For the case of $\boldsymbol{x}(0) = (0.54, 1.5, 0.68, 1.18, 0.676)$ and $\delta\boldsymbol{x}(0) = 0.0001$ we measured a global $\lambda = 0.220640 \pm 0.000725$. The question arises:

- *Does λ change depending on $\boldsymbol{x}(0)$?*

Consider a new set of vectors, arbitrarily chosen.

$$\boldsymbol{x}(0) = (1.0, 2.0, 0.5, -0.3, 1.2)$$

$$\boldsymbol{y}(0) = (1.0001, 2.0, 0.5, -0.3, 1.2)$$

Again, with $\delta\boldsymbol{x}(0) = 0.0001$. The trajectory in 2D-component phase space (x_1, x_2) for these new $\boldsymbol{x}$, $\boldsymbol{y}$ compared to those in Sec. 1.8 are shown in Fig. 1.5. Clearly, the trajectories are both ergodic, but have unique behavior in both cases.

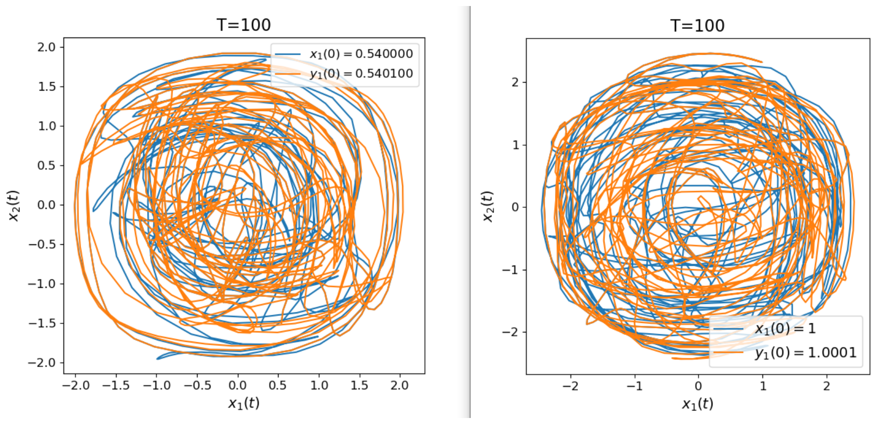

Fig. 1.5 2D phase space (x_1, x_2) for (left) $\boldsymbol{x}(0) = (0.54, 1.5, 0.68, 1.18, 0.676)$, (right) $\boldsymbol{x}(0) = (1.0, 2.0, 0.5, -0.3, 1.2)$, both with $\delta x_1(0) = 0.0001$.

Fitting the Lyapunov exponent in this case also brings new questions to light. In Fig. 1.6 we see the Lyapunov exponent for two differnet timescales. The green line is fit to t=31, where the logarithmic difference visually seems to stop. This fit yields $\lambda = 0.317919 \pm 0.003008$ with a reduced $\chi^2 = 0.00037$. The orange line is fit to t=47, as before, and yields $\lambda = 0.220624 \pm 0.003244$ with a $\chi^2 = 0.00264$.

The second result, with a smaller χ^2, compares favorably to our first set of initial conditions fit to the same timescale, which yielded $\lambda = 0.220640 \pm 0.000725$. This agreement between the two values obtained from comparison of trajectories with wildly different initial conditions, only agreeing in $\delta x(0)$, implies that the Lyapunov exponent is a function of the system and $\delta x(0)$.

1.9.2.1 Varying $\delta x(0)$

For completeness, we now compare with the Lyapunov exponent for a set of $\boldsymbol{x}$, $\boldsymbol{y}$ with $\delta x(0) = 0.002$ as opposed to $\delta x(0) = 0.0001$ before. We'll arbitrarily define some new vectors in $\mathbb{R}^5$:

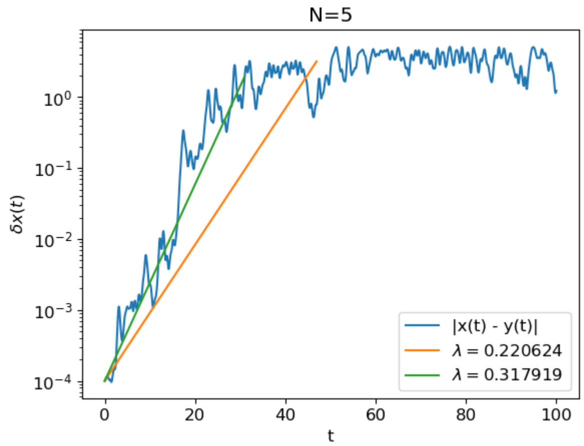

Fig. 1.6 $\delta x(t)$ for $\boldsymbol{x}(0) = (1.0, 2.0, 0.5, -0.3, 1.2)$, $\boldsymbol{y}(0) = (1.0001, 2.0, 0.5, -0.3, 1.2)$ fit to t=31 (green) and t=47 (orange).

$$\boldsymbol{x} = (0.7, 0.75, -1.9, -0.34, -0.45)$$

$$\boldsymbol{y} = (0.7, 0.752, -1.9, -0.34, -0.45)$$

Our new system behaves as shown in Fig. 1.7. The Lyapunov behavior is cut off far more quickly, at t=10, yielding a greater Lyapunov exponent of $\lambda = 0.776199 \pm 0.007194$. Thus, it is clear λ depends on the initial separation.

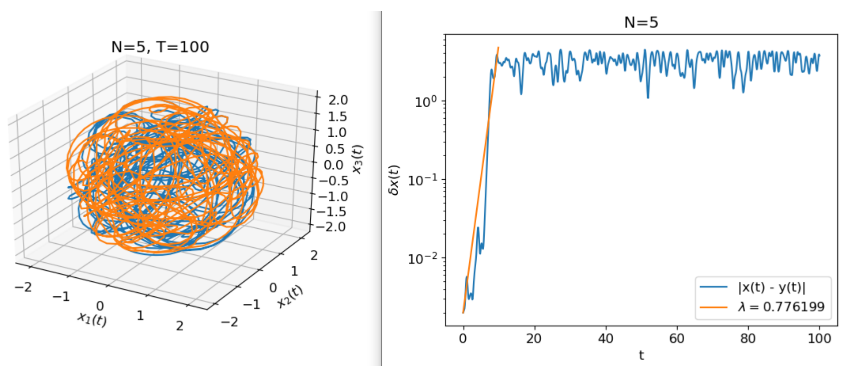

Fig. 1.7 $\delta x(t)$ for $\boldsymbol{x}(0) = (1.0, 2.0, 0.5, -0.3, 1.2)$, $\boldsymbol{y}(0) = (1.0001, 2.0, 0.5, -0.3, 1.2)$ fit to t=31 (green) and t=47 (orange).

1.10 Lyapunov Spectra - Odd N-Dependence

As we will see in later sections, even dimensional iteration of (1.6) do not exhibit chaotic behavior. Thus, in considering the behavior of the Lyapunov spectrum's dependence on DOF, N, we will only consider odd N.

First, we replicate the canonical initial conditions

$$\boldsymbol{x}(0) = (0.54, -1.5, -0.68, -1.18, -0.676)$$

$$\boldsymbol{y}(0) = (0.5401, -1.5, -0.68, -1.18, -0.676)$$

Then, we increase the number of dimensions, adding simply $x_i(0) = y_i(0) = 1$ for $i = 6, 7, ...N$. The behavior separations for this system are shown in Fig. 1.8(a). The slopes of the spectra before they plateau give the values of λ_N. λ_5 is the smallest, followed by N=9, and then N=7. At higher N, λ stabilizes around $\lambda_N \simeq 1$. It is interesting to note that λ is not proportional to N.

Fig. 1.8(b) shows a similar case, except now for $i > 5$,

$$x_i(0) = y_i(0) = \begin{cases} 1 & \text{for odd } i \\ 0.1 & \text{for even } i \end{cases}$$

Here, λ_5 is again the smallest, but the second smallest is $\lambda_1 5$. It is clear that N is not directly proportional to λ_N for the system.

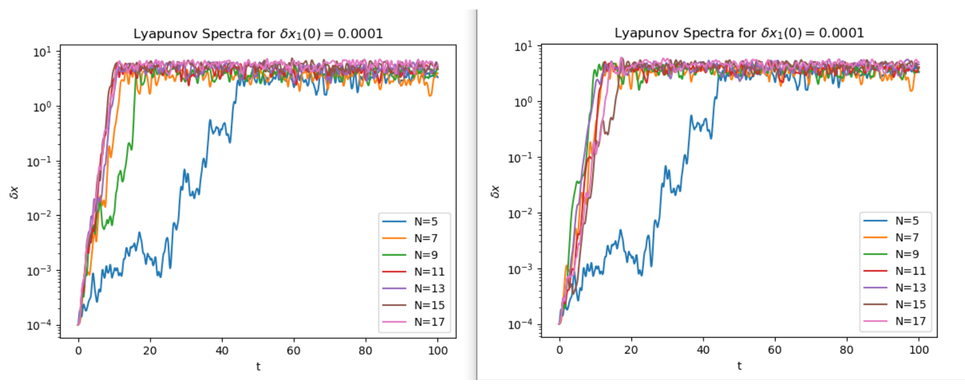

Fig. 1.8 (a) Lyapunov spectra for higher $x_i(0) = y_i(0) = 1$; (b) Lyapunov spectra for even $x_i(0) = y_i(0) = 1$ and for odd, 0.1

1.11 Orszag-McLaughlin: N=6

We also think it's intriguing to see what would would happen to the Orszag-McLaughlin system in even dimensions. Specifically if we look at N=6. While the literature exclusively focused on odd N, for completeness we should examine the behavior of (1.6) for even N. Here, we consider the case N=6.

The initial conditions for N=6 are similar to those specified in 1.9 with the additional parameter of $x_6 = 5.0$. Meaning the initial conditions for N=6 are:

$$\boldsymbol{x}(0) = (0.54, -1.5, -0.68, -1.18, -0.676, 5.0)$$

$$\boldsymbol{y}(0) = (0.5401, -1.5, -0.68, -1.18, -0.676, 5.0)$$

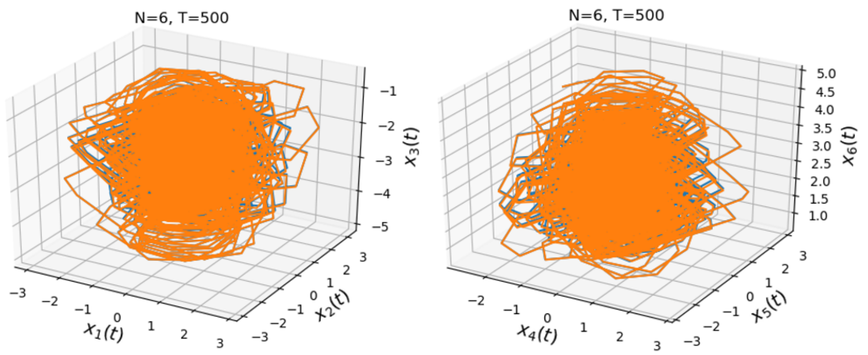

Fig. 1.9 3D phase portraits of (x_1, x_2, x_3) & (x_4, x_5, x_6) of $\boldsymbol{x}$(t), $\boldsymbol{y}$(t) for T=500 with a 0.1s timestep is on the left and right respectively. The trajectories appear to oscillate between each other so that they cannot be clearly distinguished.

Fig. 1.9 shows the behavior of the system up to T=500. Note there is no noticeable divergent behavior of the two trajectories in any dimension. The lack of divergence between trajectories forces us to question if the energy of the system is conserved, as is required for the Orszag-McLaughlin system discussed in [3]. Fig. 1.10, the energy appears to be approaching $\simeq 5.4$ which violates (1.7) discussed in Section 1.6 and is therefore not a conserved system.

Given the lack of conservation of energy, we should now check to see if the system is chaotic. Taking a look at the Lyapunov exponents for each dimension, Fig. 1.11, we can see that the system is not chaotic because not a single dimension has a Lyapunov exponent at all! Instead, the trajectories oscillate around each other. A visualization of this can be seen in Fig. 1.12, where the separations between the trajectories oscillates between $\frac{1}{100}$ and $\frac{1}{10000}$.

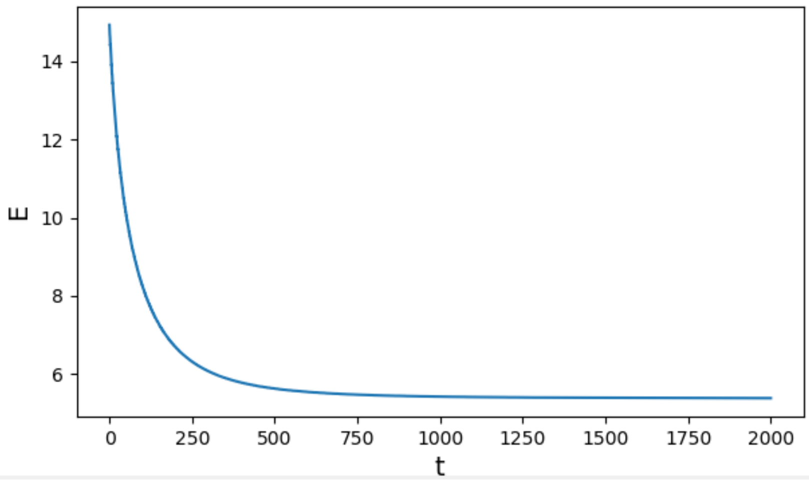

Fig. 1.10 The total energy $E = \frac{1}{2}\sum_{i=1}^{N} x_i^2$ over time for $\boldsymbol{x}(0) = (0.54, 1.5, 0.68, 1.18, 0.676, 5.0)$. By T=2000, the total energy appears to be approaching $\simeq 5.4$.

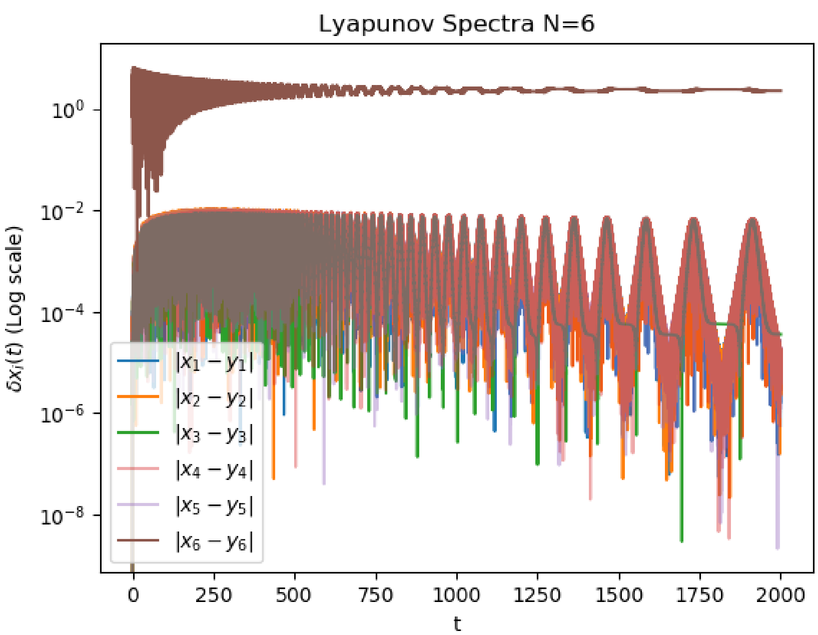

Fig. 1.11 Lyapunov spectra for N=6 Orszag-McLaughlin System.

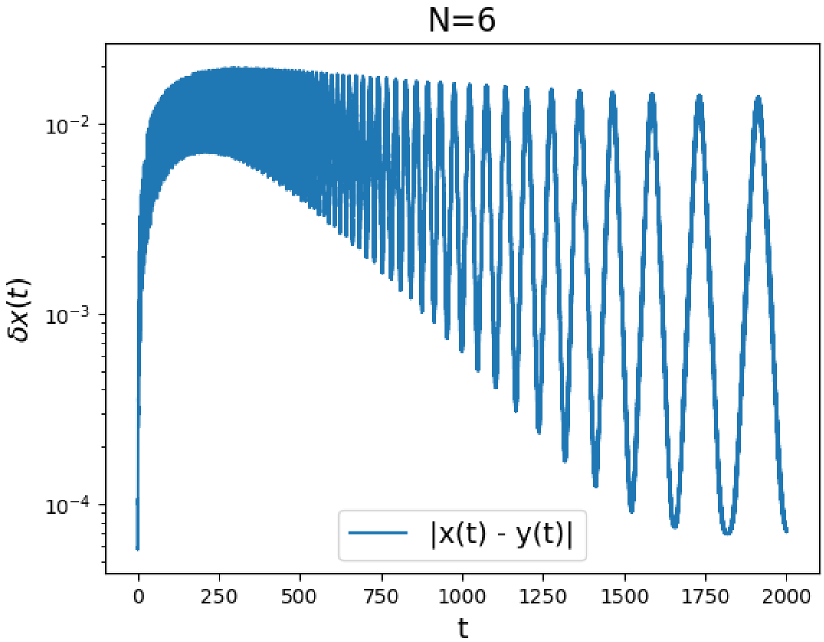

Fig. 1.12 Difference between the trajectories for N=6, which oscillate between $\frac{1}{50}$ and $\frac{1}{10000}$

1.11.1 N=6 Dependence on h

The oscillatory behavior observed in Fig. 1.12 should draw one's attention back to Fig. 1.9 to look for the alleged oscillations between the trajectories. While they can be seen faintly, the critical reader might note that the trajectories have jagged edges, and may inquire what timestep h was used in our RK4 method. In the previous section, a timestep of $h = 0.1$ was used.

Here, we will examine the same quantities, but for a more prpecise time step $h = 0.01$, which should lead to an error in our RK4 method of $O(10^{-8})$ rather than $O(10^{-4})$. Observing Fig. 1.14, we can see that while the oscillations between $\boldsymbol{x}$(t) and $\boldsymbol{y}$(t) still exist, they are occur at a much greater separation that $\delta x(0)$. While difficult to fit to a Lyapunov exponent, the constant overall increase in the total phase space separation indicates the system actually is chaotic for N=6.

There is a myriad of other interesting behavior in the case of N=6 with a more precise RK4 timestep. For example, consider Fig. 1.15, which shows the energy of the system over time. Now, as opposed to the $h = 0.1$ case, it appears energy is conserved, barring some error due to our integration method. Perhaps most intriguing of all the behavior for N=6, is the case of the (x_1, x_2) phase portrait (Fig. 1.16). Note that the behavior of $x_1(t), x_2(t)$ is non-ergodic, in that it totally avoids the origin.

The discrepancy between analyses for $h = 0.1$ and $h = 0.01$ makes sense, as in the case of $h = 0.1$, the error due to RK4 is $O(10^{-4})$, which is comparable to our initial separation $\delta x(0) = 0.0001 = 10^{-4}$. With $h = 0.01$, the error of $O(10^{-8})$ is far greater than our initial separation, thus yielding a more accurate picture of the behavior.

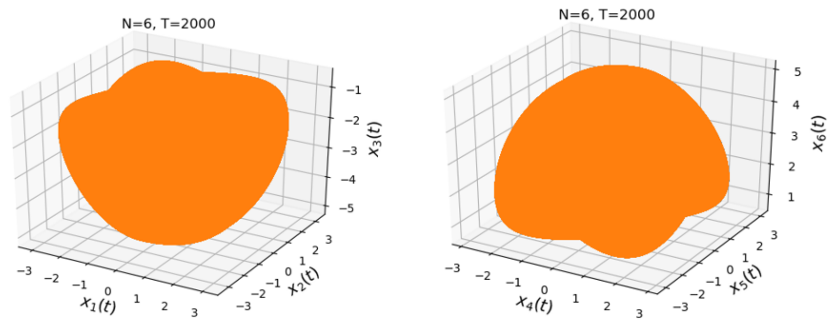

Fig. 1.13 Phase portraits: Left - (x_1, x_2, x_3) for N=6, Right - (x_4, x_5, x_6). Both for RK4 timestep $h = 0.01$

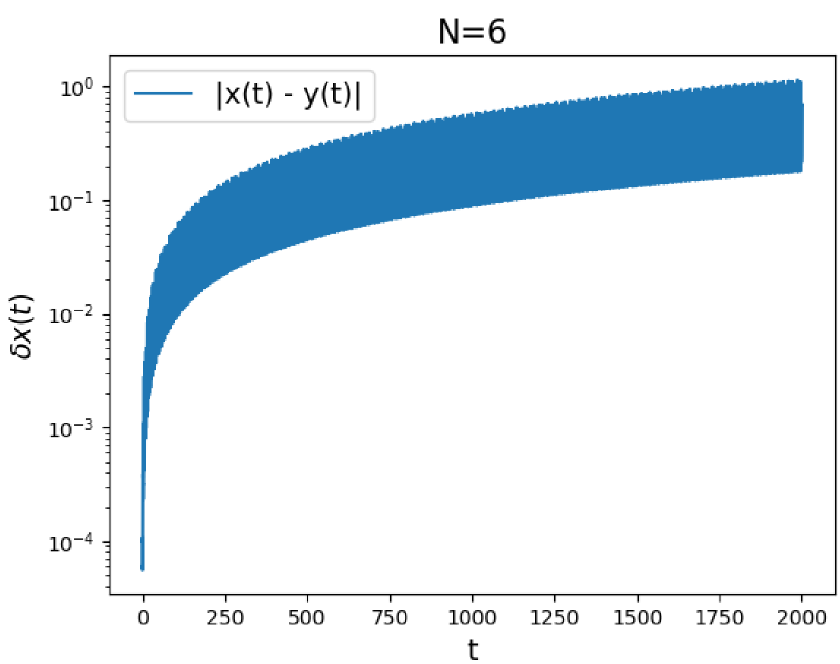

Fig. 1.14 Phase-space difference between the trajectories for N=6, RK4 $h = 0.01$.

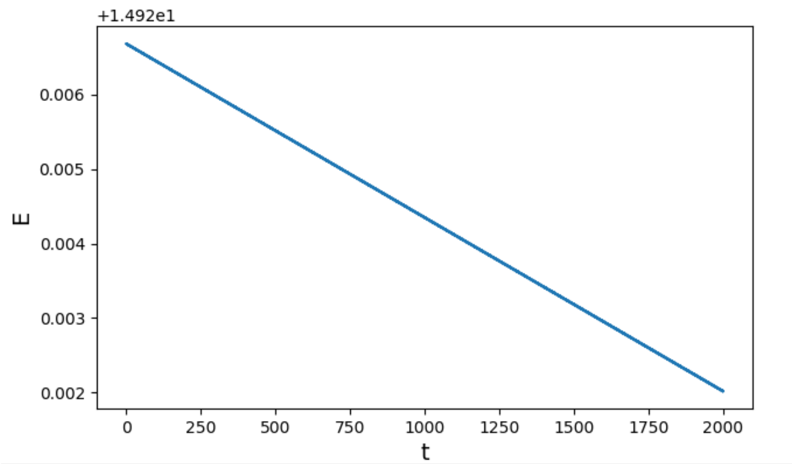

Fig. 1.15 Phase-space difference between the trajectories for N=6, RK4 h = 0.01.

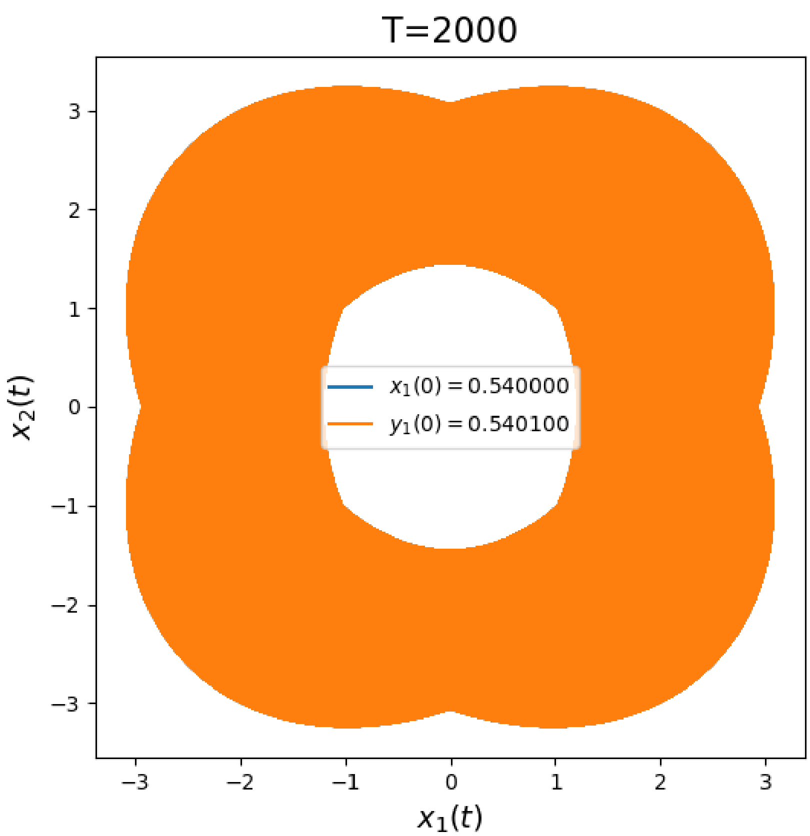

Fig. 1.16 Phase-space difference between the trajectories for N=6, RK4 h = 0.01.

1.12 Conclusions

There is indeed a plethora of remarkable behavior in the dynamics of the Orszag-McLaughlin (1.6) system. In the previous sections, we have numerically shown that the system indeed obeys conservation of energy and Liouville's theorem. We have also shown that for odd $N > 3$, the system is chaotic and extremely sensitive to small perturbations.

In the case of even $N > 2$, the system still fits the definition of chaos, and also conserves energy. However, even N are much less sensitive to differences in initial conditions compared to odd N. Furthermore, the error in even N's energy is much greater compared to odd N. For $h = 0.1$, and N even, we observed $O(10)$ error (Fig. 1.10). For N odd and same h, we observe $O(10^{-2})$ (Fig. 1.4) . Also, it seems some dimensions of even N are non-ergodic (Fig. 1.16), but without a rigorous proof, we can only speculate.

Our analysis also leads one to note the special case of $N = 5$, where a wide range of Lyapunov exponents can be found. On the other hand, for higher order N, the range of Lyapunov exponents is much smaller, and tends to stabilize around $\lambda \simeq 1$. We also find that for a given N, λ is independent of initial conditions, and depends only on the *difference* in initial conditions $\delta x(0)$.

We hope that this paper serves as a comfortable introduction to the chaos theory of turbulent flows, and specifically to the Orszag-McLaughlin dynamical system. Our hope is that having read this paper, the reader will have developed an appreciation for the nuances of the Orszag-McLaughlin system, and chaotic systems in general. We also hope that for the reader interested in numerical techniques and the study of nonlinear dynamics, this paper can act as a illustration (if not a guide) of some of the more fundamental aspects of numerical analysis.

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