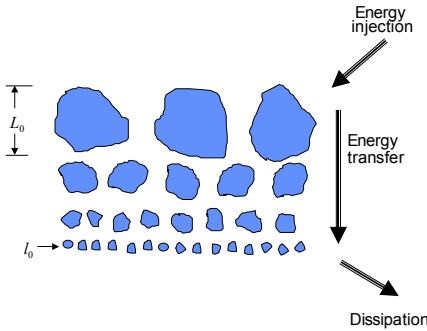

Kolmogorov Theory of Turbulence

Classical studies of **turbulence** were concerned with fluctuations in the velocity field of a viscous fluid. In particular, it was observed that the longitudinal wind velocity associated with the turbulent atmosphere fluctuates randomly about its mean value. That is, the wind velocity field assumes the nature of a random or stochastic field, which means that at each point in space and time within the flow the velocity may be represented by a random variable.

A statistical approach has been fruitful over the years in describing both atmospheric turbulence and its various effects on optical/IR systems. For the purpose of mathematical simplification, it is often necessary in such statistical approaches to assume that point separations within certain scale sizes exhibit the important characteristics of statistical homogeneity and isotropy. In general, **statistical homogeneity** of the random velocity field implies that the mean value of the field is constant and that correlations between random fluctuations in the field from point to point are independent of the chosen observation points, depending only on their vector separation. Moreover, if the random fluctuations are also statistically **isotropic**, then point-to-point correlations depend only on the magnitude of the vector separation between observation points.

Classical Turbulence

Classical turbulence is associated with the random **velocity fluctuations** of a viscous fluid such as the atmosphere. The atmosphere has two distinct states of motion: laminar and turbulent. Mixing does not occur in **laminar flow**, but **turbulent flow** is characterized by dynamic mixing and acquires random subflows called **turbulent eddies**.



Reynolds number: $Re = Vl/\nu$, where V [m/s] and l [m] are the velocity (speed) and “dimension” of the flow, and ν [m²/s] is the kinematic viscosity. Transition from laminar to turbulent flow takes place at a **critical Reynolds number**. Close to the ground $Re \sim 10^5$, considered highly turbulent.

Kolmogorov turbulence theory is the set of hypotheses that a small-scale structure is statistically homogeneous, isotropic, and independent of the large-scale structure. The source of energy at large scales is either wind shear or convection. When the wind velocity is sufficiently high that the critical Reynolds number is exceeded, large unstable air masses are created.

Energy cascade theory – unstable air masses under the influence of inertial forces break up into smaller eddies to form a continuum of eddy size for the transfer of energy from a macroscale L_0 (**outer scale of turbulence**) to a microscale l_0 (**inner scale of turbulence**).

Inertial range – family of eddies bounded by L_0 above and l_0 below.

Dissipation range – scale sizes smaller than l_0 . The remaining energy in the fluid motion is **dissipated as heat**.

Velocity Fluctuations

Velocity structure function of wind velocity satisfies the power laws

$$D_{RR}(R) = \langle (V_1 - V_2)^2 \rangle = \begin{cases} C_V^2 R^{2/3}, & l_0 \ll R \ll L_0, \\ C_V^2 l_0^{-4/3} R^2, & R \ll l_0, \end{cases}$$

where V_1, V_2 denote the velocity at two points separated by distance R . Here, C_V^2 is the velocity **structure constant** (or **structure parameter** [units of $\text{m}^{4/3}/\text{s}^2$]), related to the average energy dissipation rate ϵ [units of m^2/s^3] by

$$C_V^2 = 2\epsilon^{2/3}.$$

In the surface layer up to ~ 100 m, L_0 is on the order of height above ground of the observation point. Only eddies of scale sizes smaller than L_0 are assumed **statistically homogeneous** and **isotropic**. As the turbulent eddies become smaller, the relative amount of energy dissipated by viscous forces increases until the energy dissipated matches that supplied by the kinetic energy of the parent flow. The associated eddy size then defines the inner scale of turbulence, typically on the order of 1 to 10 mm near the ground.

Inner scale, l_0 : Related to energy dissipation rate ϵ and viscosity ν by $l_0 = (\nu^3/\epsilon)^{1/4}$. Strong turbulence has smaller inner scale and weak turbulence has larger inner scale.

Outer scale, L_0 : Proportional to $\epsilon^{1/2}$; increases and decreases directly with the strength of turbulence. It is the distance over which the mean flow velocity changes appreciably.

Power spectrum: Equivalent to the $2/3$ power law of the structure function in the inertial range in three dimensions

$$\Phi_{RR}(\kappa) = 0.033 C_V^2 \kappa^{-11/3}, \quad 1/L_0 \ll \kappa \ll 1/l_0.$$

The power spectrum exhibits a $-11/3$ power law, which corresponds to a 1D spectrum with a $-5/3$ power law.

Temperature Fluctuations

The basic ideas of velocity fluctuations have also been applied to passive scalars such as **potential temperature** θ , related to absolute temperature T by $\theta = T + \alpha_a h$, where α_a is the adiabatic rate of decrease of the temperature and h is height above the Earth's surface. An associated l_0 and L_0 of the small-scale temperature fluctuations form the lower and upper boundaries of the **inertial-convective range**.

Temperature structure function:

$$D_T(R) = \langle (T_1 - T_2)^2 \rangle = \begin{cases} C_T^2 R^{2/3}, & l_0 \ll R \ll L_0 \\ C_T^2 l_0^{-4/3} R^2, & R \ll l_0, \end{cases}$$

where T_1, T_2 are the temperature at two points separated by distance R ; C_T^2 is the temperature **structure constant** [in $\text{deg}^2/\text{m}^{2/3}$]. (The structure “constant” is also referred to as the structure “parameter.”)

Inner scale: $l_0 = 5.8(D^3/\epsilon)^{1/4}$,

where D is the diffusivity of heat in air (in m^2/s).

The three-dimensional **power spectrum** of temperature fluctuations takes the $-11/3$ power-law form

$$\begin{aligned} \Phi_T(\kappa) &= \frac{1}{4\pi} \beta \chi \epsilon^{-1/3} \kappa^{-11/3} \\ &= 0.033 C_T^2 \kappa^{-11/3}, \quad 1/L_0 \ll \kappa \ll 1/l_0, \end{aligned}$$

where β is the Obukhov-Corrsin constant and χ is the rate of dissipation of mean-squared temperature fluctuations.

The temperature spectrum at high wave numbers actually contains a small “bump” near $1/l_0$ that can have important consequences on a number of applications involving optical wave propagation.

Optical Turbulence

At a point \mathbf{R} in space the **index of refraction** is written as

$$n(\mathbf{R}) = 1 + 79 \times 10^{-6} [P(\mathbf{R})/T(\mathbf{R})] = 1 + n_1(\mathbf{R}),$$

where $n(\mathbf{R})$ has been normalized by its mean value n_0 , P is pressure, T is temperature, and the small dependence on optical wavelength λ is neglected. Temperature-induced fluctuations in the atmospheric refractive index is called **optical turbulence**, which has properties of **statistical homogeneity** and **isotropy** within the inertial subrange.

Refractive-index structure function:

$$D_n(R) = \langle [n(\mathbf{R}_1) - n(\mathbf{R}_2)]^2 \rangle = \begin{cases} C_n^2 R^{2/3}, & l_0 \ll R \ll L_0 \\ C_n^2 l_0^{-4/3} R^2, & R \ll l_0, \end{cases}$$

where $\mathbf{R}_1, \mathbf{R}_2$ denote the refractive index at two points separated by distance R , and $\langle \rangle$ is an ensemble average.

Refractive-index inner scale: $l_0 = 7.4(v^3/\epsilon)^{1/4}$.

The **structure constant** C_n^2 [units of $\text{m}^{-2/3}$] can be deduced from the temperature structure constant C_T^2 by

$$C_n^2 = (79 \times 10^{-6} P/T^2)^2 C_T^2.$$

The **power spectral density** for refractive-index fluctuations over the inertial subrange is defined by

$$\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3}, \quad 1/L_0 \ll \kappa \ll 1/l_0.$$

The **Kolmogorov spectrum** is widely used in theoretical calculations. However, it is limited to the **inertial subrange** ($1/L_0 \ll \kappa \ll 1/l_0$) so other models of the spectrum for refractive-index fluctuations are required in some calculations.

Structure Parameter and Inner Scale

The **structure parameter** C_n^2 is a measure of turbulence strength.

Inner scale l_0 has a strong impact on scintillation.

Path-averaged values of C_n^2 and l_0 can be obtained simultaneously by optical measurements over a short path length (typically ~ 150 m) using a **scintillometer**.

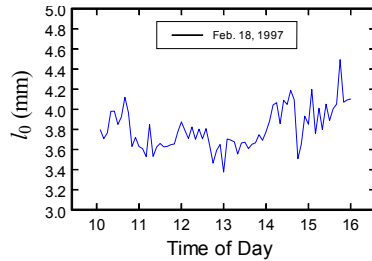
Near ground level, C_n^2 data over a 24-hr period would show a diurnal cycle with peaks during midday hours, near constant values at night, and minima near sunrise and sunset.

Weak turbulence: $C_n^2 \sim 10^{-17} \text{ m}^{-2/3}$ or less

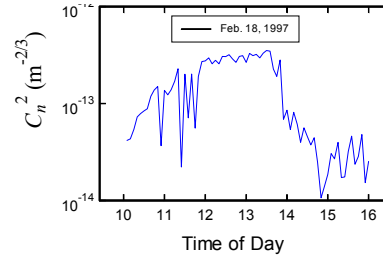
Strong turbulence: $C_n^2 \sim 10^{-13} \text{ m}^{-2/3}$ or more

For **vertical** or **slant paths** the structure parameter C_n^2 varies as a function of height above ground and must be described by an altitude profile model. Formulated from a series of measurements made over many years, several such models have now been developed, most based on specific geographic locations.

Inner scale values range in size from 1–2 mm up to 1 cm or more near the ground. Also, the inner-scale size increases and decreases inversely with the structure parameter C_n^2 .



Measured values of the inner scale.



Measured values of C_n^2 .

C_n^2 Profile Models

For applications involving propagation along a horizontal path, assume C_n^2 is constant. Propagation along a **vertical** or **slant path**, however, requires a C_n^2 profile model as a function of altitude h .

Hufnagle-Valley Model: one most often used by researchers

$$C_n^2(h) = 0.00594(w/27)^2(10^{-5}h)^{10} \exp(-h/1000) \\ + 2.7 \times 10^{-16} \exp(-h/1500) + A \exp(-h/100),$$

where $A = C_n^2(0)$ is a ground-level value of C_n^2 , and w is **rms wind speed** often modeled by

$$w = \left[\frac{1}{15 \times 10^3} \int_{5 \times 10^3}^{20 \times 10^3} \left\{ \omega_s h + w_g + 30 \exp \left[- \left(\frac{h - 9400}{4800} \right)^2 \right] \right\}^2 dh \right]^{1/2}.$$

Here, w_g is the ground wind speed and ω_s is the beam **slew rate**.

Two additional proposed models are given below.

SLC Day Model –

$$C_n^2(h) = 1.7 \times 10^{-14}, \quad 0 < h < 18.5 \text{ m}, \\ = 3.13 \times 10^{-13}/h^{1.05}, \quad 18.5 < h < 240 \text{ m}, \\ = 1.3 \times 10^{-15}, \quad 240 < h < 880 \text{ m}, \\ = 8.87 \times 10^{-7}/h^3, \quad 880 < h < 7200 \text{ m}, \\ = 2.0 \times 10^{-16}/h^{1/2}, \quad 7200 < h < 20000 \text{ m}.$$

SLC Night Model –

$$C_n^2(h) = 8.4 \times 10^{-15}, \quad 0 < h < 18.5 \text{ m}, \\ = 2.87 \times 10^{-12}/h^2, \quad 18.5 < h < 110 \text{ m}, \\ = 2.5 \times 10^{-16}, \quad 110 < h < 1500 \text{ m}, \\ = 8.87 \times 10^{-7}/h^3, \quad 1500 < h < 7200 \text{ m}, \\ = 2.0 \times 10^{-16}/h^{1/2}, \quad 7200 < h < 20000 \text{ m}.$$

Power Spectrum Models

Power spectrum: The Fourier transform of the refractive-index covariance function.

Kolmogorov spectrum: The conventional model for refractive-index fluctuations; theoretically valid only over the inertial subrange $1/L_0 \ll \kappa \ll 1/l_0$.

Tatarskii spectrum: For wavenumbers $\kappa > 1/l_0$, based on

$$\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3} \exp(-\kappa^2/\kappa_m^2), \quad \kappa \gg 1/L_0,$$

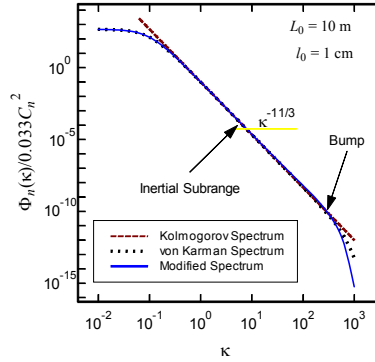
where $\kappa_m = 5.92/l_0$. For $\kappa < 1/L_0$, the spectrum is **anisotropic** and its form is not known.

Von Kármán spectrum: For $\kappa < 1/L_0$, based on

$$\Phi_n(\kappa) = 0.033 C_n^2 \frac{\exp(-\kappa^2/\kappa_m^2)}{(\kappa^2 + \kappa_0^2)^{11/6}}, \quad 0 \leq \kappa < \infty,$$

where $\kappa_0 = 2\pi/L_0$.

None of these models contain the rise (“bump”) at high wave numbers near $1/l_0$ that appears in temperature spectral data.



Several power spectrum models for inner-scale $l_0 = 1$ cm and outer-scale $L_0 = 10$ m.

Modified atmospheric spectrum: An analytic approximation to the bump spectrum is given by

$$\Phi_n(\kappa) = 0.033 C_n^2 \left[1 + 1.802(\kappa/\kappa_l) - 0.254(\kappa/\kappa_l)^{7/6} \right] \frac{\exp(-\kappa^2/\kappa_l^2)}{(\kappa^2 + \kappa_0^2)^{11/6}}, \quad 0 \leq \kappa < \infty,$$

where $\kappa_l = 3.3/l_0$, $\kappa_0 = 2\pi/L_0$.
