Computing the Probability of Striking a Battleship

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What is Battleship?

- 2-player pen and paper game.
- Sometimes played using special plastic boards too.

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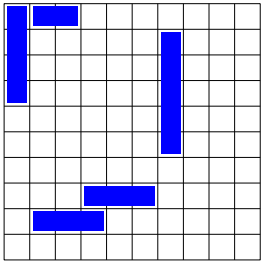

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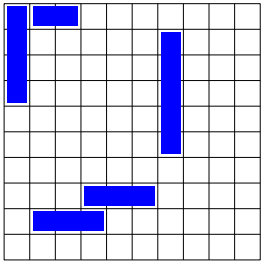
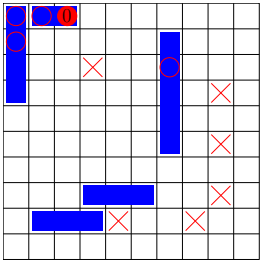


In general, we denote the number of ships as n.

Battleship Game Rules (2)



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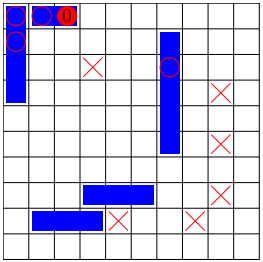

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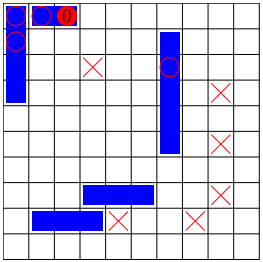

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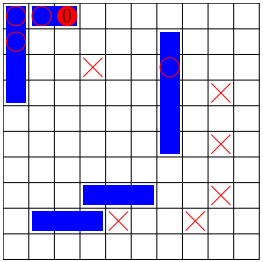

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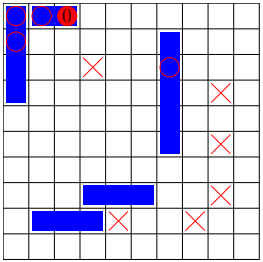

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- Attack a square on the opponent's grid.
- Three possible results:
  - Miss.
  - Hit.
  - Sink! Not only was something sunk, but every other square of that ship was already hit. The attacking player also learns exactly which ship was sunk ("You sunk my battleship.").

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Another idea: Minimize the maximum number of attacks necessary. Fiat and Shamir made some progress on a special case of this.

Probabilities

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Ultimate Goal: Compute these probabilities in about the same amount of time a human would take to make a move (up to about 10 seconds).

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- Assume your opponent in a game of battleship chose a ship configuration uniformly at random.
- Given: The attacks you have made so far, and their results (miss, hit, sink).
- For each square s on the opponent's board, output the probability that s has a ship on top of it.

Related Work

This is not the first attempt at a variation of this problem. C. Liam Brown attempted this problem as well, but his full implementation of this problem takes about a day to run. He instead used MCMC methods to estimate the probabilities.

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Our solution does not employ a significantly different approach, but rather uses a long series of optimizations and changes to drastically improve the performance overall (success by a thousand cuts).
We eventually achieve an implementation that can run in just a few seconds.

Placement Graph

Create a graph (the 'placement graph' G_1):

- Vertex for every ship + placement pair.

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The *n-cliques* in this graph correspond to legal configurations at the start of the game.

Placement Graph (Example)

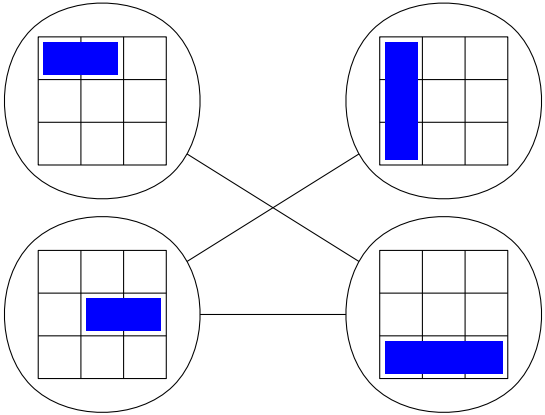

Figure: For a 3x3 board with ship sizes $\{2, 3\}$, the above diagram represents a subgraph of the board-induced graph.

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Given the attacks and results so far, we take an induced subgraph of the placement graph (the *valid placement graph* $G_2 \subset G_1$) using vertices corresponding to placements that still could be possible.

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Now, the remaining legal configurations correspond to cliques in G_2 that cover all squares that have received *hit* results from attacks.

Using Clique Iteration

The most obvious solution is now to iterate through all cliques in G_2, count up how many times each vertex appears in the cliques, and then compute how many times each square appears.

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In order to attain our goal of a few seconds per input, we're going to need to perform **lots** of performance optimizations.

Clique Iteration Background

Bron and Kerbosch have a well-known method for clique iteration for general graphs, but a more naive approach works much better in our case:

```
For each placements p_1 of ship s_1:
  For each placements p_2 of ship s_2 compatible with the previous ones:
    For
      ...
        Record the clique if it corresponds to a valid configuration.
```

Note: `placements p_i of ship s_i` corresponds exactly to the vertices in part i of the graph.

Benchmarks

Benchmark methodology:

- Query-less board has the most cliques, so used for benchmark.
- All other inputs have a subset of the cliques of the empty board.

Benchmarks ran on an AMD Ryzen 5 1400 CPU and an NVIDIA RTX 3080 GPU.

We test on the most common variant of a 10×10 board with ships of sizes $\{5, 4, 3, 3, 2\}$.

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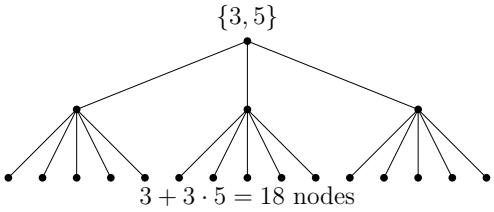

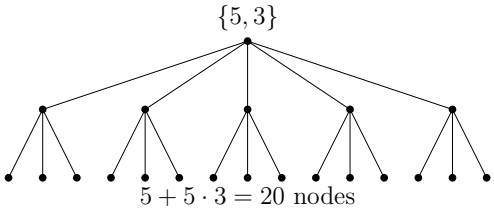

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We can now maintain a *currently valid* set for each part of the graph:When choosing a vertex corresponding to (s_1, p_1) in each of the for loops, intersect the generated sets above with the currently valid sets for every choice of s_2.

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In addition, this actually allows us to remove all heap allocations from the critical path of our code **(zero-alloc code!)**.

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Variadic C++ Templates make this easy!

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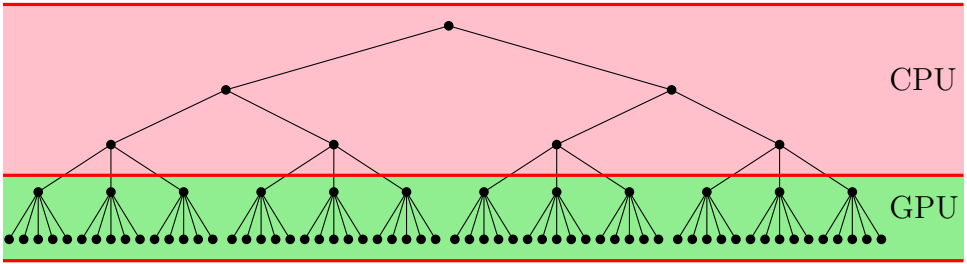

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Source code:
https://github.com/jacketsj/gpgpu-fun/tree/master/battleship/rewrite

Probabilities

If you were curious what we were computing for the benchmark, the most common variant has the following probabilities for the blank board:

0.080	0.115	0.143	0.159	0.167	0.167	0.159	0.143	0.115	0.080
0.115	0.143	0.166	0.178	0.184	0.184	0.178	0.166	0.143	0.115
0.143	0.166	0.184	0.194	0.199	0.199	0.194	0.184	0.166	0.143
0.159	0.178	0.194	0.203	0.208	0.208	0.203	0.194	0.178	0.159
0.167	0.184	0.199	0.208	0.214	0.214	0.208	0.199	0.184	0.167
0.167	0.184	0.199	0.208	0.214	0.214	0.208	0.199	0.184	0.167
0.159	0.178	0.194	0.203	0.208	0.208	0.203	0.194	0.178	0.159
0.143	0.166	0.184	0.194	0.199	0.199	0.194	0.184	0.166	0.143
0.115	0.143	0.166	0.178	0.184	0.184	0.178	0.166	0.143	0.115
0.080	0.115	0.143	0.159	0.167	0.167	0.159	0.143	0.115	0.080

Applicability to Extensions of Battleship

This approach also extends perfectly to other variations of Battleship, e.g.:

- The variation where no two ships can touch.
- Battleship Outer Space
- Hexagonal Battleship

The Last Slide

Fin.