1.1 EXACT EQUATIONS AND INTEGRATING FACTOR

1.1.1. Introduction. An ordinary differential equation of first order and first degree is of the form $\dfrac{dy}{dx} = f(x,y)$ . . . (1)

which can also be written as $Mdx + Ndy = 0$ . . . (2)

where M and N are functions of x and y or constants. All differential equations of first order and first degree cannot be solved. However, in this chapter, we shall consider the following special types of equations of first order and first degree which can be solved by some standard methods:

(i) Exact equations.

(ii) Linear equations.

1.1.2. Exact Equations.

A first order differential equation of the form

$$M\,dx + N\,dy = 0 \qquad \ldots (3)$$

where both M and N are functions of x, y, is said to be exact, if there exist a function $u\,(x,y)$ such that

$$M\,dx + N\,dy = du\,. \qquad \ldots (4)$$

Then equation (3) becomes $du = 0$, which on integration gives

$u\,(x,y) = c$, c being a constant.

Therefore $u\,(x,y) = c$ is a solution of (3).

For example $\log x\,dy + \dfrac{y}{x}dx = 0$ is an exact differential, since

$$\log x\,dy + \frac{y}{x}\,dx = d\,(y\log x).$$

Hence $y\log x = c$, c being a constant, is a general solution of the equation.

Theorem. The necessary and sufficient condition for the differential equation $M\,dx + N\,dy = 0$ to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Proof. Let the equation $M\,dx + N\,dy = 0$ be exact.

Then there exist a function $u(x,y)$ such that $Mdx + Ndy = du$.

But $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$

So $Mdx + Ndy = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy.$

Therefore $\frac{\partial u}{\partial x} = M, \frac{\partial u}{\partial y} = N.$

$\therefore \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y}, \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}.$

But, if both $\frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial x \partial y}$ are continuous, then $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

Thus the condition is necessary.

Conversely, let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

We are to show that $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = F(x,y).$ $\quad \ldots \quad (5)$

$\therefore \quad \frac{\partial F}{\partial x} = M.$

So $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

$\therefore \quad \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right).$

$\therefore \quad N = \frac{\partial F}{\partial y} + f(y).$ $\quad \ldots \quad (6)$

$$\therefore \quad Mdx + Ndy = \frac{\partial F}{\partial x}dx + \left\{\frac{\partial F}{\partial y} + f(y)\right\}dy$$
$$= \left(\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy\right) + f(y)dy$$
$$= dF + f(y)dy = d\left[F + \int f(y)dy\right]$$
$$= du, \quad \ldots \quad (7)$$

where $u = F + \int f(y)dy.$ $\quad \ldots \quad (8)$

which shows that $Mdx + Ndy = 0$ is exact.

Note. By (8), the solution of $Mdx + Ndy = 0$ is $u(x,y) =$ constant.

i.e., $F + \int f(y)dy =$ constant

i.e., $\int Mdx + \int f(y)dy =$ constant, by (5)

i.e., $\int Mdx + \int$(terms of N not containing x) $dy =$ constant. $\ldots$ (9)

Illustrative Examples:

Ex. 1. *Show that* $(3x + 4y + 5)dx + (4x - 3y + 3)dy = 0$ *is an exact equation and hence solve it.*

Here $M = 3x + 4y + 5,\ N = 4x - 3y + 3.$

$\therefore \quad \frac{\partial M}{\partial y} = 4, \ \frac{\partial N}{\partial x} = 4.$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

So the given equation is an exact equation.

Hence the solution of the equation is

$\int Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c/2$, c being a constant.

or, $\int (3x + 4y + 5)dx + \int (-3y + 3)dy = c/2$

or, $\dfrac{3x^2}{2} + 4xy + 5x - \dfrac{3y^2}{2} + 3y = c/2$

$\therefore$ $3x^2 + 8xy - 3y^2 + 10x + 6y = c$, which is the required solution.

Ex. 2. *Solve :* $e^x \sin y dx + (e^x + 1)\cos y dy = 0$.

[*W.B.U.T. 2005, 2006*]

Here $M = e^x \sin y,\ N = (e^x + 1)\cos y$

$\therefore$ $\dfrac{\partial M}{\partial y} = e^x \cos y,\ \dfrac{\partial N}{\partial x} = e^x \cos y$

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

So the given equation is an exact equation. Hence the solution of the equation is

$\int Mdx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$.

or, $\int e^x \sin y dx + \int \cos y dy = c$, being a constant,

$\therefore$ $e^x \sin y + \sin y = c$, c which is the required solution.

Ex. 3. *Solve :* $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$.

Here $M = y^2 e^{xy^2} + 4x^3, N = 2xy\, e^{xy^2} - 3y^2$

$\therefore$ $\dfrac{\partial M}{\partial y} = 2ye^{xy^2} + 2y^3 x\, e^{xy^2}, \dfrac{\partial N}{\partial x} = 2ye^{xy^2} + 2xy^3\, e^{xy^2}$

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

So the equation is an exact equation.

Hence the solution of the equation is given by

$\int Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c$, c being a constant.

or, $\int (y^2 e^{xy^2} + 4x^3)dx + \int (-3y^2)dy = c$

$\therefore$ $e^{xy^2} + x^4 - y^3 = c$, which is the required solution.

Ex. 4. *Show that the equation* $3y(x^2 - 1)dx + (x^3 + 8y - 3x)dy = 0$ *is exact and its particular solution when* $x = 0,\ y = 1$ *is* $xy(x^2 - 3) = 4(1 - y^2)$.

Here $M = 3y(x^2 - 1),\ N = x^3 + 8y - 3x$

$\therefore$ $\dfrac{\partial M}{\partial y} = 3(x^2 - 1),\ \dfrac{\partial N}{\partial x} = 3x^2 - 3 = 3(x^2 - 1)$

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Thus the given equation is exact.

Hence the solution of the equation is

$\int 3y(x^2 - 1)dx + \int 8y dy = c$.

$3y\left(\dfrac{x^3}{3} - x\right) + 8 \cdot \dfrac{y^2}{2} = c$

$xy(x^3 - 3) + 4y^2 = c$

When $x=0,\ y=1$

$\therefore \quad c=0+4.1$

$\therefore \quad c=4$

Therefore the particular solution is

$$xy(x^3-3)+4y^2=4$$

$$\therefore \quad xy(x^3-3)=4(1-y^2)$$

1.1.3. Integrating factor (I. F).

Differential equation which are not exact can sometimes be made exact after multiplying by a suitable factor (a function of x and / or y) called *Integrating factor*.

For example, the equation $xdy-ydx=0$ is not exact. But after multiplying the equation by $\frac{1}{x^2}$, the equation becomes

$$\frac{xdy-ydx}{x^2}=0.$$

or, $d\left(\frac{y}{x}\right)=0,$

which is an exact and solution is $\frac{y}{x}=c$, a constant.

Note : It can be proved that if a differential equation has one integrating factor, the equation has an infinite number of integrating factors.

Rules for finding integrating factors.

Here we consider a differential equation of the form

$$Mdx+Ndy=0 \qquad \ldots \quad (10)$$

which is not exact i.e., $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

Rule 1. If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$, a function of x only, then $e^{\int f(x)dx}$ is an integrating factor of the differential equation (10).

Rule 2. If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=g(y)$, a function of y only, then $e^{-\int g(y)dy}$ is an integrating factor of the differential equation (10).

Rule 3. If M and N are both homogeneous functions in x, y of same degree and $Mx+Ny\neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor of (10).

Rule 4. If the equation (10) is of the form $yf(xy)dx+xg(xy)dy=0$ and $Mx-Ny\neq 0$, then $\frac{1}{Mx-Ny}$ is an integrating factor of the equation.

Rule 5. If the equation (10) is of the form

$$x^a y^b(mydx+nxdy)+x^{a'}y^{b'}(m'ydx+n'xdy)=0$$

where a,b,a',b',m,n,m',n' are all constants, then $x^h y^k$ is an integrating factor of the equation where

$$\frac{a+h+1}{m}=\frac{b+k+1}{n},\frac{a'+h+1}{m'}=\frac{b'+k+1}{n'}.$$

Illustrative Examples.

Ex. 1. *Solve :* $(xy^2-e^{\frac{1}{x^3}})dx-x^2ydy=0.$ [W.B.U.T. 2012, 2014]

Here $M=xy^2-e^{\frac{1}{x^3}},N=-x^2y$

$$\therefore \quad \frac{\partial M}{\partial y}=2xy,\frac{\partial N}{\partial x}=-2xy$$

$\therefore \quad \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}.$

So the equation is not exact but

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{4xy}{-x^2y}=-\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

Hence I. F. $= e^{\int -\frac{4}{x}dx} = e^{-4\log x} = e^{\log x^{-4}} = \dfrac{1}{x^4}.$

Multiplying the equation by $\dfrac{1}{x^4}$, we get

$$\left(\frac{y^2}{x^3}-\frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx-\frac{y}{x^2}dy=0, \text{ which is an exact equation.}$$

Therefore the solution is

$$\int\left(\frac{y^2}{x^3}-\frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx+0=\frac{c}{6}, \text{ since there is no term which}$$

does not contain x in N.

i.e., $\quad y^2\displaystyle\int\frac{dx}{x^3}+\frac{1}{3}\int e^{\frac{1}{x^3}}d\left(\frac{1}{x^3}\right)=\frac{c}{6}$

i.e., $\quad -\dfrac{y^2}{2x^2}+\dfrac{1}{3}e^{\frac{1}{x^3}}=\dfrac{c}{6}$

$\therefore \quad 2x^2e^{\frac{1}{x^3}}-3y^2=cx^2$ which is the required solution.

Ex. 2. *Solve :* $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0.$

[W.B.U.T. 2014]

Here $M=3x^2y^4+2xy,\ N=2x^3y^3-x^2$

$\therefore \quad \dfrac{\partial M}{\partial y}=12x^2y^3+2x, \dfrac{\partial N}{\partial x}=6x^2y^3-2x$

$\therefore \quad \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}.$

So the equation is not exact but

$$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{6x^2y^3+4x}{3x^2y^4+2xy}=\frac{2}{y} \text{ which is a function of } y \text{ only.}$$

Hence I. F. $= e^{-\int\frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = \dfrac{1}{y^2}.$

Multiplying the equation by $\dfrac{1}{y^2}$, we get

$$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0 \text{ which is an exact}$$

equation.

Therefore the solution is

$$\int\left(3x^2y^2+\frac{2x}{y}\right)dx+0=c, \text{ since there is no term which does}$$

not contain x in N.

$\therefore \quad x^3y^2+\dfrac{x^2}{y}=c,$ which is the required solution.

Ex. 3. *Solve :* $(x^4+y^4)\,dx-xy^3dy=0.$

Here $M=x^4+y^4,\ N=-xy^3.$

So M, N are both homogeneous functions in x, y of degree 4.

Now, $\dfrac{\partial M}{\partial y}=4y^3,\ \dfrac{\partial N}{\partial x}=-y^3.$

$\therefore \quad \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}.$

So the equation is not exact, but $Mx+Ny=x^5\neq 0.$

Hence $\dfrac{1}{x^5}$ is an integrating factor.

Multiplying the equation by $\frac{1}{x^5}$, we get $\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx-\frac{y^3}{x^4}dy=0$

which is an exact equation.

Therefore the solution is

$$\int\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx=\frac{c}{4},$$

since there no term which does not contain x in N.

or, $\log x-\frac{y^4}{4x^4}=\frac{c}{4}$.

$\therefore$ $4x^4\log x-y^4=cx^4$ which is the required solution.

Ex. 4. *Solve :* $y(xy+2x^2y^2)dx+(xy-x^2y^2)xdy=0$.

Here $M=y(xy+2x^2y^2)$, $N=(xy-x^2y^2)x$.

Also the equation is of the form $yf(xy)dx+xg(xy)dy=0$.

Moreover $Mx-Ny=3x^3y^3$.

Hence $\frac{1}{3x^3y^3}$ is an integrating factor.

Multiplying the equation by $\frac{1}{3x^3y^3}$, we get

$\frac{1}{3}\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\frac{1}{3}\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0$ which is an exact equation. Therefore the solution is given by

$$\frac{1}{3}\int\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\frac{1}{3}\int\left(-\frac{1}{y}\right)dy=\frac{c}{3},\ c \text{ being a constant.}$$

or, $-\frac{1}{xy}+2\log x-\log y=c$.

$\therefore$ $\log\frac{x^2}{y}=\frac{1}{xy}-c$, which is the required solution.

Ex. 5. *Solve :* $3ydx-2xdy+x^2y^{-1}(10ydx-6xdy)=0$.

Comparing the given equation with the equation

$$x^ay^b(mydx+nxdy)+x^{a'}y^{b'}(m'ydx+n'xdy)=0$$

we have $a=b=0,\ m=3,\ n=-2,\ a'=2,\ b'=-1,\ m'=10,\ n'=-6$.

Let x^hy^k be an integrating factor.

Then $\frac{a+h+1}{m}=\frac{b+k+1}{n}$

i.e., $\frac{0+h+1}{3}=\frac{0+k+1}{-2}$

i.e., $2h+3k=-5$. ... (i)

and $\frac{a'+h+1}{m'}=\frac{b'+k+1}{n'}$

$\therefore$ $\frac{2+h+1}{10}=\frac{-1+k+1}{-6}$

$\therefore$ $3h+5k=-9$. ... (ii)

Solving (i) and (ii), we get $h=2,\ k=-3$.

Therefore x^2y^{-3} is an integrating factor.

Multiplying the equation by x^2y^{-3}, we get

$(3x^2y^{-2}+10x^4y^{-3})dx-(2x^3y^{-3}+3x^5y^{-4})dy=0$ which is an exact equation.

Therefore the solution is given by

$\int(3x^2y^{-2}+10x^4y^{-3})dx=c,$ since there is no term which does not contain x in N.

$\therefore$ $x^3y^{-2}+2x^5y^{-3}=c$, which is the required solution.

Ex. 6. *Solve* $\left(x^2y-2xy^2\right)dx+\left(3x^2y-x^3\right)dy=0$ [W.B.U.T. 2008]

Here $M=x^2y-2xy^2$, $N=3x^2y-x^3$ are both homogeneous functions in x, y of same degre 3.

Now $Mx+Ny=\left(x^2y-2xy^2\right)x+\left(3x^2y-x^3\right)y$

$$=x^2y^2$$

$$\therefore \quad I.F.=\frac{1}{x^2y^2}$$

Multiplying the given differential equation by $\frac{1}{x^2y^2}$, we get,

$$\frac{1}{x^2y^2}\left(x^2y-2xy^2\right)dx+\frac{1}{x^2y^2}\left(3x^2y-x^3\right)dy=0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(\frac{3}{y}-\frac{x}{y^2}\right)dy=0$$

which is an exact equation.

Hence the solution of the equation is

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

or, $\frac{x}{y}-2\log x+3\log y=c$

$\therefore \quad \frac{x}{y}+\log\frac{y^3}{x^2}=c$, c being a constant.

Ex. 7. *Solve,* $\left(2xy+e^x\right)y\,dx-e^x dy=0$

Hence $M=\left(2xy+e^x\right)y$, $N=-e^x$

$$\therefore \quad \frac{\partial M}{\partial y}=4xy+e^x,\ \frac{\partial N}{\partial x}=-e^x$$

$$\therefore \quad \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$\therefore \quad \frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=2\left(2xy+e^x\right)$$

$$\therefore \quad \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\frac{2}{y}$$

$$\therefore \quad I.F.=e^{-\int\frac{2}{y}dy}=e^{-2\log y}=e^{\log y^{-2}}=\frac{1}{y^2}$$

Multiplying the given equation by $\frac{1}{y^2}$, we get,

$$\frac{1}{y}\left(2xy+e^x\right)dx-\frac{e^x}{y^2}dy=0$$

i.e., $\left(2x+\frac{e^x}{y}\right)dx-\frac{e^x}{y^2}dy=0$

which is an exact equation.

Hence the required solution is

$$\int\left(2x+\frac{e^x}{y}\right)dx+0=c$$

or, $2\cdot\frac{x^2}{2}+\frac{e^x}{y}=c$

$$\therefore \quad x^2+\frac{e^x}{y}=c$$

Ex. 8. *Show that the I.F. of the equation*

$$\left(x^2+y^2+2x\right)dx+2y\,dy=0 \text{ is } e^x$$

and its particular solution $x^2+y^2=2e^{1-x}$ *when* $x=y=1$.

Multiplying the given equation by e^x we get,

$$e^x\left(x^2+y^2+2x\right)dx+2ye^x dy=0 \qquad \ldots \text{(i)}$$

which is of the form $Mdx+Ndy=0$

$$\therefore \quad M=e^x\left(x^2+y^2+2x\right),\ N=2ye^x$$

$$\therefore \quad \frac{\partial M}{\partial y}=e^x.2y,\ \frac{\partial N}{\partial x}=2ye^x$$

$$\therefore \quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

So the equation (i) is exact.

Hence e^x is an integrating factor.

Therefore the solution of the given equation is

$$\int e^x\left(x^2+y^2+2x\right)dx+0=c$$

or, $\int e^x x^2 dx+y^2\int e^x dx+2\int xe^x dx=c$

or, $x^2e^x-\int 2xe^x+y^2e^x+2\int xe^x dx=c$

$\therefore \quad x^2+y^2=ce^{-x}$

When $x=y=1$, we have $1+1=ce^{-1}$

$\therefore \quad c=2e$

Thus the particular solution is

$$x^2+y^2=2ee^{-x}$$

i.e., $x^2+y^2=2e^{1-x}$

Ex. 9. *Prove that* $(x+y+1)^{-4}$ *is an integrating factor of the equation* $(2xy-y^2-y)dx+(2xy-x^2-x)dy=0$ *and hence solve it.*

Multiplying the given equation by $(x+y+1)^{-4}$ we get

$$\frac{2xy-y^2-y}{(x+y+1)^4}dx+\frac{2xy-x^2-x}{(x+y+1)^4}dy=0. \qquad \ldots \text{(i)}$$

which is of the form $Mdx+Ndy=0$

$$\therefore \quad M=\frac{2xy-y^2-y}{(x+y+1)^4},\ N=\frac{2xy-x^2-x}{(x+y+1)^4}$$

$$\therefore \quad \frac{\partial M}{\partial y}=\frac{(2x-2y-1)(x+y+1)^4-4(x+y+1)^3(2xy-y^2-y)}{(x+y+1)^8}$$

$$=(2x^2+2y^2-8xy+x+y+1)/(x+y+1)^5$$

and $$\frac{\partial N}{\partial x}=\frac{(2y-2x-1)(x+y+1)^4-4(2xy-x^2-x)(x+y+1)^3}{(x+y+1)^8}$$

$$=(2x^2+2y^2-8xy+x+y+1)/(x+y+1)^5$$

$$\therefore \quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

So the equation (i) is exact. Hence $(x+y+1)^{-4}$ is an integrating factor.

Therefore the solution of the equation is given by

$\int\frac{2xy-y^2-y}{(x+y+1)^4}dx+0=c$, since there is no term which does not contain x.

or, $$2y\int\frac{(x+y+1)-(y+1)}{(x+y+1)^4}dx-\int\frac{y(y+1)}{(x+y+1)^4}dx=c$$

or, $2y\int\frac{dx}{(x+y+1)^3}-3y(y+1)\int\frac{dx}{(x+y+1)^4}=c$

or, $-\frac{y}{(x+y+1)^2}+\frac{y(y+1)}{(x+y+1)^3}=c$

or, $xy=c\,(x+y+1)^3$ which is the required solution.

Ex. 10. *Solve* $2\sin y^2dx+xy\cos y^2dy=0,\ y(2)=\sqrt{\frac{\pi}{2}}$

Here $M=2\sin y^2,\ N=xy\cos y^2$

$\therefore\ \frac{\partial M}{\partial y}=4y\cos y^2,\ \frac{\partial N}{\partial x}=y\cos y^2$

$\therefore\ \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{3y\cos y^2}{xy\cos y^2}$

$=\frac{3}{x}$ which is a function of x only.

$\therefore I.F=e^{\int\frac{3}{x}dx}=e^{3\log x}=x^3$.

Multiplying the equation by x^3 we get

$$2x^3\sin y^2dx+x^4y\cos y^2dy=0$$

which is an exact equation.

Therefore the solution is

$\int 2x^3\sin y^2dx+0=\frac{c}{2}$, since there is no term which does not contain x in N.

i.e., $x^4\sin y^2=c$

As we are given $y(2)=\sqrt{\frac{\pi}{2}}$, so $2^4\cdot\sin\frac{\pi}{2}=c$

$\therefore\ c=16$.

So the required solution is $x^4\sin y^2=16$.

EXERCISE

[I] SHORT ANSWER QUESTIONS

1. Find the IF of the differential equation $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{\sqrt{1+x^2}}$.

2. Solve the differential equation $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$.

3. Solve the differential equation $xdy+ydx=xydx$ when $y(1)=1$

4. Find the geometrical locus represented by the differential equation $xdy-ydx=0$.

5. Find whether the differential equation $ydx-xdy=x^2ydx$ is exact.

6. Find whether the differential equation

$$(x+y-1)dx+(2x+2y-3)dy=0 \text{ is exact.}$$

7. Find the geometric locus represented by the differential equation $y\frac{dy}{dx}+x=k$.

8. Find the IF of the differential equation

$$x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1.$$

9. Is the differential equation $(3xy+2y^3)dx+(4x^2+6xy^2)dy=0$ exact ?

10. Find the IF of the differential equation

$$(2x^2+y^2+x)dx+xydy=0.$$

[Hint : $IF=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$]

11. Find the IF of the differential equation $(x^3+y^3)dx-xy^2dy=0$

12. Examine whether the differential equation

$$(9-2xy-y^2)dx-(x+y)^2dy=0 \text{ is exact}$$

13. Examine whether the differential equation $\left(x^2y-2xy^2\right)dx+\left(3x^2y-x^3\right)dy=0$ is exact. If not find the factor which makes it exact after multiplication.

14. Find the IF of the equation

$$\left(3x^2y^4+2xy\right)dx+\left(2x^3y^3-x^2\right)dy=0.$$

15. Find the IF of the equation

$$(xy\sin xy+\cos xy)ydx+(xy\sin xy-\cos xy)xdy=0$$

ANSWERS

1. x 2. $e^y=e^x+\frac{x^3}{3}+c$ 3. $y=\frac{e^x}{ex}$

4. Family of st. line passing through origin

5. not exact

6. not

7. family of circles centred at origin

8. $x\sec x$

9. no

10. x

11. $\frac{1}{x^4}$ 12. exact 13. $x^{-2}y^{-2}$

14. $\frac{1}{y^2}$ 15. $\frac{1}{2xy\cos xy}$

[II] LONG ANSWER QUESTIONS

Solve the following equations (1-26) :

1. $(x^4-2xy^2+y^4)dx=(2x^2y-4xy^3+\sin y)dy$.

2. $y\sin 2xdx-(1+y^2+\cos^2 x)dy=0$.

3. $(x^3y^2+x)dy+(x^2y^3-y)dx=0$.

4. $(y^2+2x^2y)dx+(2x^3-xy)dy=0$.

5. $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$.

6. $ye^{xy}dx+(xe^{xy}+2y)dy=0$.

7. $(\sin x\cos y+e^{2x})dx+(\cos x\sin y+\tan y)dy=0$.

[W.B.U.T. 2016]

8. $(1+xy)ydx+(1-xy)xdy=0$.

9. $(xy^2+2x^2y^3)dx+(x^2y-x^3y^2)dy=0$.

10. $(3xy-2ay^2)dx+(x^2-2axy)dy=0$.

11. $(2x^2y-3y^4)dx+(3x^3+2xy^3)dy=0$.

12. $(e^x\sin y+e^{-y})dx+(e^x\cos y-xe^{-y})dy=0$.

13. $(2x^2y^2+y)dx-(x^3y-3x)dy=0$.

14. $(\cos x-x\cos y)dy-(\sin y+y\sin x)dx=0$

15. $(\cos x\cos y-\cot x)dx-\sin x\sin y\,dy=0$

16. $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$

17. $\left(\sin x\sin y-xe^y\right)dy=\left(e^y+\cos x\cos y\right)dx$

18. $\left(3x^2y^4+2xy\right)dx+\left(2x^3y^3-x^2\right)dy=0$

19. $y\left(y^2-2x^2\right)dx+x\left(2y^2-x^2\right)dy=0$

20. $\left(x^2y^2+xy+1\right)ydx+\left(x^2y^2-xy+1\right)xdy=0$

21. $x^3y^3(2ydx+xdy)-(5ydx+7xdy)=0$

22. $3\left(x^2+y^2\right)dx+x\left(x^2+3y^2+6y\right)dy=0$

23. $x(4ydx+2xdy)+y^3(3ydx+5xdy)=0$

24. Solve $(5x^2+xy-1)dx+(\frac{1}{2}x^2-y+2y^2)dy=0$; given $y=1$ when $x=0$.

25. $\dfrac{dy}{dx}=\dfrac{y-2x}{2y-x},\ y(1)=2$

26. Find an integrating factor of the equation $(y^2+2x^2y)dx+(2x^3-xy)dy=0$ and hence solve it.

27. Prove that e^{x^2} is an I. F. of the equation

$$(x^2+xy^4)dx+2y^3dy=0$$ and hence solve it.

28. Show that $\dfrac{1}{3x^3y^3}$ is an I.F. of

$$y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy=0$$

29. Show that $\dfrac{1}{x(x^2-y^2)}$ is an integrating factor of the equation $(x^2+y^2)dx-2xydy=0$ and hence solve the equation.

ANSWERS

1. $x^5-5x^2y^2+5xy^4+5\cos y=c$.
2. $3y\cos 2x+6y+2y^3=c$.
3. $\log(y/x)+\frac{1}{2}x^2y^2=c$.
4. $4(xy)^{1/2}-\frac{2}{3}\left(\frac{y}{x}\right)^{3/2}=c$.
5. $xy+\dfrac{2x}{y^2}+y^2=c$.
6. $e^{xy}+y^2=c$.
7. $-\cos x\cos y+\frac{1}{2}e^{2x}+\log\sec y=c$.
8. $-\dfrac{1}{xy}+\log\dfrac{x}{y}=c$.
9. $-\dfrac{1}{xy}+\log\left(x^2/y\right)=c$.
10. $x^2(ay^2-xy)=c$.
11. $5x^{-\frac{36}{13}}y^{\frac{24}{13}}-12x^{-\frac{10}{13}}y^{-\frac{15}{13}}=c$.
12. $e^x\sin y+xe^{-y}=c$.
13. $4x^2y=5+cx^{4/7}y^{12/7}$.
14. $y\cos x-x\sin y=c$
15. $\sin x\cos y=\log(c\sin x)$

16. $x^4-y^3+e^{xy^2}=c$
17. $xe^y+\sin x\cos y=c$
18. $x^3y^2+\dfrac{x^2}{y}=c$
19. $x^2y^2\left(y^2-x^2\right)=c$
20. $xy-\dfrac{1}{xy}+\log\left(\dfrac{x}{y}\right)=c$
21. $x^3y^3+2=cx^{\frac{5}{3}}y^{\frac{7}{3}}$
22. $x\left(x^2+3y^2\right)=ce^{-y}$
23. $x^4y^2+x^3y^5=c$
24. $10x^3+3x^2y-6x-3y^2+4y^3-1=0$.
25. $x^2-xy+y^2=3$
26. I. F. $=x^{-5/2}y^{-1/2};\ 6\sqrt{xy}-x^{3/2}y^{3/2}=c$.
27. $\int x^2e^{x^2}dx+\frac{1}{2}y^4e^{x^2}=c$.
29. $x^2-y^2=cx$.

[III] MULTIPLE CHOICE QUESTIONS

1. A first order first degree equation of the form $Mdx+Ndy=0$ is exact if

(a) $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ (b) $\dfrac{\partial M}{\partial x}=\dfrac{\partial N}{\partial y}$

(c) $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$ (d) $\dfrac{\partial M}{\partial x}\neq\dfrac{\partial N}{\partial y}$

2. The differential equation

$$(a_1x-b_1y)dx+(c_2x-b_2y)dy=0$$

is exact if

(a) $a_1=b_2$ (b) $b_1=b_2$

(c) $a_1=-b_2$ (d) $a_2=-b_1$

3. For the differential equation $f(x,y)\frac{dy}{dx}+g(x,y)=0$ to be exact if

(a) $\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x}$ (b) $\frac{\partial f}{\partial x}=\frac{\partial g}{\partial y}$

(c) $\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 g}{\partial y^2}$ (d) none of these [WBUT 2011]

4. The equation $(3x^2+py)dx+(-6y^2+qx)dy=0$ is exact if

(a) $p+q=0$ (b) $p-q=0$

(c) $3p+q=0$ (d) $p\neq q$

5. The equation $xdy-ydx=0$ is exact. The statement is

(a) True (b) False

6. The equation $(1+xy)ydx+(1-xy)xdy=0$ is not exact. The statement is

(a) True (b) False

7. The differential equation

$xdx+ydy+\frac{xdy-ydx}{x^2+y^2}=0$ is exact. The statement is

(a) True (b) False

8. The integrating factor of

$$ydx-xdy+4x^3y^2e^{x^4}dx=0 \quad \text{is}$$

(a) $\frac{1}{y}$ (b) y^2

(c) xy^2 (b) $\frac{1}{y^2}$

9. If IF of $x(1-x^2)dy+(2x^2y-y-ax^3)dx=0$ is $e^{\int Pdx}$ then P is

(a) $2x^3-1$ (b) $\frac{2x^2-1}{x(1-x^2)}$

(c) $\frac{2x^2-1}{ax^3}$ (d) $\frac{2x^2-ax^3}{x(1-x^2)}$

10. If $\frac{dy}{dx}=e^{-2y}$ and $y(5)=0$ then $y(a)=3$. The value of a is

(a) e^5 (b) e^6+1

(c) $\frac{1}{2}(e^6+9)$ (d) $\log_e 6$

11. If the differential equation $\left(y+\frac{1}{x}+\frac{1}{x^2y}\right)dx+\left(x-\frac{1}{y}+\frac{A}{xy^2}\right)dy=0$ is exact then the value of A is

(a) 2 (b) 1

(c) 0 (d) -1

[WBUT 2014, 2012]

12. To make the equation $(2xy-3y^3)dx+(4x^2+6xy^2)dy=0$ exact it will have to be multiplied by

(a) x^2y (b) x^2y^2

(c) xy^2 (d) xy^3 [WBUT 2012]

13. The differential equation $(xe^{axy}+2y)\frac{dy}{dx}+ye^{xy}=0$ is exact for $a=$

(a) 3 (b) 1

(c) 2 (d) 0

14. If $x^m y^n$ be an IF of the equation

$(2ydx+3xdy)+2xy(3ydx+4xdy)=0$ then the value of m, n are respectively

(a) 1, 3
(b) 2, 1
(c) 2, 2
(d) 1, 2

15. If $x^m y^n$ be an IF of the equation

$\left(3x^{-1}+2y^{4}\right)dx-\left(xy^{3}-3y^{-1}\right)dy=0$ then the values of m,n are respectively

(a) –3, –3
(b) –3, 3
(c) 3, –3
(d) none

Answers

1.a	2.d	3. b	4.b	5.b	6. a	7.a	8.d
9.b	10.c	11.b	12.a	13.b	14. d	15. a	