II. Superposition of two SHM's.

In view of the kinematical consideration presented in the last section we can claim that a uniform circular motion of angular frequency ω_0 can be resolved ~~is~~ into two simple harmonic motions along the two mutually perpendicular co-ordinate axes. The SHM's thus obtained have

(i) Identical frequencies.

(ii) Identical amplitudes.

(iii) $\frac{\pi}{2}$ phase separation.

In the discussion that follows we'll consider just the reverse — the composition of two mutually perpendicular SHM's by retaining the condition (i) and relaxing (ii) and (iii) giving rise to what is known as Lissajous figures.

a. Perpendicular Superposition

Definition-7: The diagram representing the trajectory of the resultant motion due to superposition of two mutually perpendicular simple harmonic motions is called a Lissajous figure.

We'll consider various cases of Lissajous figures when the superposing motions are of identical frequencies ~~and~~ but different amplitudes and phase separations, in view of the following theorem.

Theorem-2: Let's consider two SHM's of identical frequency about the origin in X-Y plane, ~~interacting~~ given by

$$\left.\begin{aligned} x(t) &= a_1 \cos(\omega_0 t + \delta_1) \quad \cdots \\ y(t) &= a_2 \cos(\omega_0 t + \delta_2) \quad \cdots \end{aligned}\right\} [14\ a, b]$$

The trajectory representing the resultant motion due to their superposion (perpendicular) is an ellipse.

Proof : From [14a]

$$\left.\begin{aligned} \frac{x}{a_1} &= \cos\omega_0 t \cos\delta_1 - \sin\omega_0 t \sin\delta_1 \\ \frac{y}{a_2} &= \cos\omega_0 t \cos\delta_2 - \sin\omega_0 t \sin\delta_2 \end{aligned}\right\} [15a, b]$$

[15-a] × $\sin\delta_2$ − [15-b] × $\sin\delta_1$ yields.

$$\frac{x}{a_1}\sin\delta_2 - \frac{y}{a_2}\sin\delta_1 = \cos\omega t \sin(\delta_2-\delta_1) \quad \text{---[16a]}$$

Again, [15-a] × $\cos\delta_2$ − [15-b] × ~~sin~~ $\cos\delta_1$ yields.

$$\frac{x}{a_1}\cos\delta_2 - \frac{y}{a_2}\cos\delta_1 = \cancel{\cos\omega t}\ \sin\omega t \sin(\delta_2-\delta_1) \quad \text{-[16b]}$$

$[16a]^2 + [16b]^2$ yields.

$$\boxed{\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1a_2}\cos(\delta_2-\delta_1) = \sin^2(\delta_2-\delta_1)} \quad \text{..[17]}$$

Equation-[17] represents a conic.
Now, the determinant

$$\begin{vmatrix} \frac{1}{a_1^2} & -\frac{1}{a_1a_2}\cos(\delta_2-\delta_1) \\ -\frac{1}{a_1a_2}\cos(\delta_2-\delta_1) & \frac{1}{a_2^2} \end{vmatrix} = \frac{\sin^2(\delta_2-\delta_1)}{a_1^2a_2^2} \geq 0$$

This means eqn. [17] represents an ellipse.

Remark: 1. The resultant trajectory is therefore independent of the ~~pass~~ phases of the individual oscillations but depends upon the phase difference $|\delta_2-\delta_1| (=\delta)$. (absolute value)

2. For $\delta = 0$ eqn. [17] becomes

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1a_2}\cos 0^\circ = \sin^2 0^\circ$$

$$\Rightarrow \quad y = \frac{a_2}{a_1}x \quad \text{(straight line) (Diagram – 2a)}$$

For $\delta = \pi$, $y = -\frac{a_2}{a_1}x$. ~~(Diagram~~

3. $\delta = \frac{\pi}{2}, \frac{3\pi}{2}$ the eqn. [17] becomes

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1. \quad \text{(Diagram-(2b)}$$

4. The direction of the superposed motion can be determined by locating the co-ordinates over the time period, as illustrated by the following example. (example-~~2~~ 2a)

5. As $\max|x(t)| = a_1$ and $\max|y(t)| = a_2$ the ellipse will always be confined in the region $-a_1 \leq x \leq a_1$ and $-a_2 \leq y \leq a_2$. It touches the the bounds at points, $(\pm a_1, \pm a_2\cos\delta)$ and $(\pm a_1\cos\delta, \pm a_2)$. Example-2 will clarify the matter.

• *Example - 2 a.* $x(t) = a_1\cos\omega t$; $y(t) = a_2\cos\omega t$. $a_1, a_2 > 0$, $a_1 > a_2$

$$y = \frac{a_2}{a_1}x \text{ with } \delta = 0$$

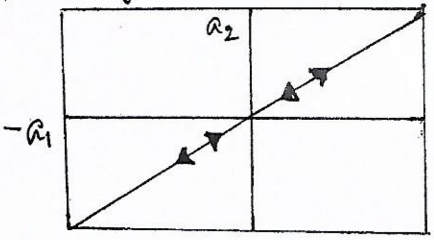

Diagram - 2a

t	0	$\frac{\pi}{4\omega}$	$\frac{\pi}{2\omega}$	$\frac{3\pi}{4\omega}$	$\frac{\pi}{\omega}$	$\frac{5\pi}{4\omega}$	$\frac{3\pi}{2\omega}$	$\frac{7\pi}{4\omega}$	$\frac{2\pi}{\omega}$	$0+$
X	a_1	$\frac{a_1}{\sqrt{2}}$	0	$-\frac{a_1}{\sqrt{2}}$	$-a_1$	$-\frac{a_1}{\sqrt{2}}$	0	$\frac{a_1}{\sqrt{2}}$	$a_1 \rightarrow$	
Y	a_2	$\frac{a_2}{\sqrt{2}}$	0	$-\frac{a_2}{\sqrt{2}}$	$-a_2$	$-\frac{a_2}{\sqrt{2}}$	0	$\frac{a_2}{\sqrt{2}}$	$a_2 \rightarrow$	

Table - 2a

The trajectory is a straight line inclined at an angle $\tan^{-1}\frac{a_2}{a_1}$ w.r.t. the x-axis.

• *Remark*: For $\delta = \pi$, i.e.; $x(t) = a_1\cos\omega t$ $y(t) = a_2\cos(\pi+\omega t)$ the st. line would have been $y = -\frac{a_2}{a_1}x$.

• *Example - 2.b*: $x(t) = a_1\cos\omega t$; $y(t) = a_2\sin\omega t$, $a_1 > a_2 > 0$

Comparing with the standard form

$$x(t) = a_1\cos(\omega t + 0)$$
$$y(t) = a_2\cos\left(\omega t - \frac{\pi}{2}\right)$$

giving $\delta = \frac{\pi}{2}$ and

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1.$$

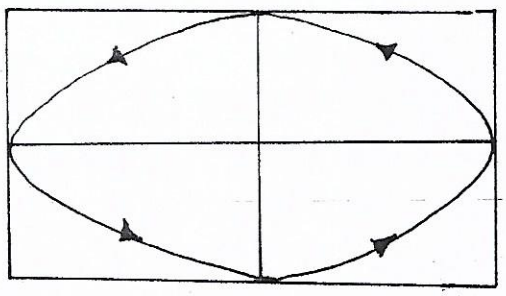

Diagram - 2b.

The resultant ellipse touches the bounds at $(\pm a_1, 0)$ and $(0, \pm a_2)$. The direction of motion can be understood by the following table.

t	0	$\frac{\pi}{4\omega}$	$\frac{\pi}{2\omega}$	$\frac{3\pi}{4\omega}$	$\frac{\pi}{\omega}$	$\frac{5\pi}{4\omega}$	$\frac{3\pi}{2\omega}$	$\frac{7\pi}{4\omega}$	$\frac{2\pi}{\omega}$
X	a_1	$\frac{a_1}{\sqrt{2}}$	0	$-\frac{a_1}{\sqrt{2}}$	$-a_1$	$-\frac{a_1}{\sqrt{2}}$	0	$\frac{a_1}{\sqrt{2}}$	a_1
Y	0	$\frac{a_2}{\sqrt{2}}$	a_2	$\frac{a_2}{\sqrt{2}}$	0	$-\frac{a_2}{\sqrt{2}}$	$-a_2$	$-\frac{a_2}{\sqrt{2}}$	0

• *Remark* 1. For $a_1 = a_2 = a >$ the ellipse will be circle

2. For $x(t) = a_1\sin$ and $y(t) = a_2\cos\omega t$, the motion will get reversed. (clockwise)

→ Table 2-b

- **Example-2c**: Let $x(t) = 3 \sin \omega t$; $y(t) = 2 \cos\left(\omega t + \frac{\pi}{4}\right)$

Comparing with the standard form

$$x(t) = 3 \cos\left(\omega t - \frac{\pi}{2}\right)$$
$$y(t) = 2 \cos\left(\omega t + \frac{\pi}{4}\right)$$
$$a_1 = 3, \quad a_2 = 2, \quad \delta = 3\pi/4$$

Hence, the general equation of ellipse takes the form

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} - \frac{2xy}{3.2} \cos\left(\frac{3\pi}{4}\right) = \sin^2 \frac{3\pi}{4}$$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{4} + \frac{\sqrt{2}\,xy}{6} = \frac{1}{2} \quad \cdots\cdots [18]$$

The ellipse described by equation [18] is confined within the region $-3 \leq x \leq +3$ and $-2 \leq y \leq +2$

The points at which it touches $x = \pm 3$ and $y = \pm 2$ can be found by solving [18] and the corresponding equation.

For example solving eq^n 18 and $x = 3$ we get $y = -\sqrt{2}$

Similarly, we get four touching points $(3, -\sqrt{2})$ $(-3, \sqrt{2})$, $(-3/\sqrt{2}, 2)$ and $\left(\frac{3}{\sqrt{2}}, -2\right)$. The direction of motion over the time period can again be determined from the following table. (Table-2-c and Diagram 2-c)

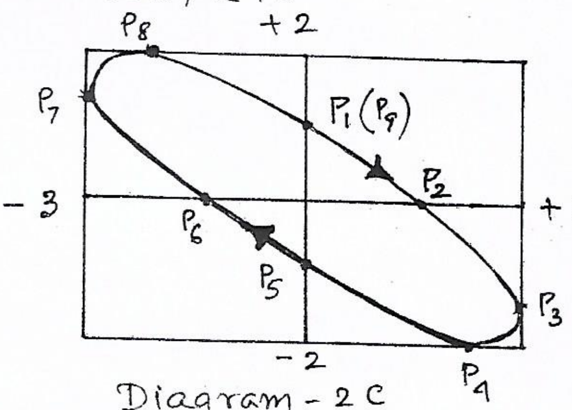

Diagram - 2C

t	0	$\pi/4\omega$	$\pi/2\omega$	$3\pi/4\omega$	π/ω	$5\pi/4\omega$	$3\pi/2\omega$	$7\pi/4\omega$	$2\pi/\omega$
X	0	$\frac{3}{\sqrt{2}}$	3	$\frac{3}{\sqrt{2}}$	0	$-\frac{3}{\sqrt{2}}$	-3	$-\frac{3}{\sqrt{2}}$	0
Y	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2	$\sqrt{2}$
	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9

Table 2-C

- *Remark* 1. Lissajous' figures have direct correspondence to the phenomenon of polarization where the behavior of light vector is usually understood as the superposition of two components of the said vector (having identical frequencies) in the ~~respective~~ plane of polarization.

2. A broad class of Lissajous' figures can be obtained when the superposing waves are of different frequencies.

• III. Damped Oscillation

In most of the real situations the free oscillation never sustains but dies down due to the presence of a several types of decaying force leading to a steady decrease of the total energy of the system. The presence of such damping force, as we will see, can affect the system in various ways (even destroying the very oscillatory character of the system under certain specific situation) causing the motion to cease eventually. In the following we will consider various cases of damped motion characterized by the relations between the natural frequency of vibration (ω_0) and the so-called damping constant.

• *Constitutive equation of damped motion*

Recalling the energy theorem (Theorem-1) we can write the rate of change of total energy in presence of damping as

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2\right) < 0$$

The left hand side can be equated with the power dissipated due the presence of damping force. In various physical situations the damping force $F_{damping} = +\Gamma v$, $\Gamma (>0)$ being a const. Hence, the power dissipated is given by

$$P = F_{damping}\, v = \Gamma v^2 = \Gamma \dot{x}^2$$

and,

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2\right) = -\Gamma\dot{x}^2$$

$$\Rightarrow \quad \dot{x}\ddot{x} + \omega_0^2 x\dot{x} = -\frac{\Gamma}{m}\dot{x}^2$$

$$\Rightarrow \quad \boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0} \quad \cdots\cdots [20]$$

• *Remark*: 1. Equation-[20] can also be constructed by considering the damping force $(\Gamma\dot{x})$ acting along the same direction of the restoring force thus constituting the Newton's law for the system.

2. Considering $\gamma\dot{x}$ has the dimension of acceleration i.e.

$$[\gamma][\dot{x}] = \cancel{[M]}[L][T]^{-2}$$

$$\Rightarrow [\gamma][L][T]^{-1} = [L][T]^{-2} \Rightarrow [\gamma] = [T]^{-1}$$

So γ has the dimension of frequency (unit = sec^{-1})

3. The effect of the damping constant γ on the free oscillati[on] of frequency ω_0 can be understood in the following theorem

- <u>Theorem - 4</u>: The solution of eqn.-[20] under the initial condition

$$x(t=0) = 0$$
$$v(t=0) = v_0 \quad \text{is}$$

(i) Non-oscillatory for $\gamma \geq \omega_0$ *

(ii) Oscillatory for $\gamma < \omega_0$

Proof : (i) Given. $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$ ······ [20]

We take a trial solution $x(t) = \exp(-\gamma t) f(t)$

$$\dot{x} = \frac{dx}{dt} = \exp(-\gamma t)\dot{f}(t) - \gamma \exp(-\gamma t) f(t)$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \exp(-\gamma t)\ddot{f}(t) - \gamma \exp(-\gamma t)\dot{f}(t)$$
$$- \gamma \exp(-\gamma t)\dot{f}(t) + \gamma^2 \exp(-\gamma t) f(t)$$

Substituting all these results in eqn. [20] we get,

$$\exp(-\gamma t)\left[\ddot{f}(t) - 2\gamma\dot{f}(t) + \gamma^2 f(t)\right]$$
$$+ 2\gamma \exp(-\gamma t)\left[\dot{f}(t) - \gamma f(t)\right] + \omega_0^2 e^{-\gamma t} f(t)$$
$$= 0$$

$$\Rightarrow \exp(-\gamma t)\left[\ddot{f}(t) + (\omega_0^2 - \gamma^2) f(t)\right] = 0 \quad \cdots [21]$$

As eqn. [21] is valid for all $t \in [0, \infty]$

$$\boxed{\ddot{f}(t) + (\omega_0^2 - \gamma^2) f(t) = 0} \quad \cdots\cdots [22]$$

a. For $\gamma > \omega_0$, $f(t) = A\exp(\alpha t) + B\exp(-\alpha t)$

where, $\alpha = \sqrt{\gamma^2 - \omega_0^2} > 0$

Hence, $x(t) = A\exp(\alpha - \gamma)t + B\exp(-\alpha - \gamma)t$

Now, $x(0) = A + B = 0$ (Given)

$\dot{x}(0) = A(\alpha - \gamma) - B(\alpha + \gamma) = v_0$ (Given)

---

* A solution is said to be nonoscillatory if it crosses $x=0$ not more once.

Solving $A = -B = \frac{v_0}{2\alpha}$

Hence $x(t) = \exp(-\gamma t)\left[\frac{v_0}{2\alpha}\left\{\exp(\alpha t) - \exp(-\alpha t)\right\}\right]$

$$\Rightarrow \boxed{x(t) = \exp(-\gamma t)\frac{v_0}{\alpha}\sinh(\alpha t)} \quad \text{--- [23]}$$

b. For $\gamma = \omega_0$ equation-[22] yields, $\ddot{f} = 0$ i.e; $f(t) = At + B$.

This gives, $x(t) = \exp(-\gamma t) f(t) = \exp(-\gamma t)(At + B)$

Now, $\left.\begin{array}{l} x(0) = 0 = B \\ \dot{x}(0) = v_0 = A \end{array}\right\}$ (Given)

Hence, $\boxed{x(t) = v_0 t \exp(-\gamma t)}$ ---- [24]

(ii) For $\gamma < \omega_0$ equation-[22] yields

$$\ddot{f} = -\tilde{\omega}^2 f \quad \text{i.e; } f(t) = A\exp(+i\tilde{\omega} t) + B\exp(-i\tilde{\omega} t)$$

where $\tilde{\omega} = (\omega_0^2 - \gamma^2)^{1/2} > 0$

and $i = \sqrt{-1}$

This gives $x(t) = A\exp\left[-(\gamma - i\tilde{\omega})t\right] + B\exp\left[-(\gamma + i\tilde{\omega})t\right]$

Now $x(0) = 0 = A + B$.

$\dot{x}(0) = v_0 = -A(\gamma - i\tilde{\omega}) - B(\gamma + i\tilde{\omega})$

Solving $A = -B = \frac{v_0}{2i\tilde{\omega}}$

Hence $x(t) = \frac{v_0}{2i\tilde{\omega}}\left[\exp(-\gamma t)\left\{e^{+i\tilde{\omega}t} - e^{-i\tilde{\omega}t}\right\}\right]$

$$\Rightarrow \boxed{x(t) = \frac{v_0}{\tilde{\omega}}\exp(-\gamma t)\sin(\tilde{\omega} t)} \quad \text{----- [25]}$$

Equation [23] and [24] are nonoscillatory solutions while Equation [25] is giving an oscillatory solution.

• *Remark*: 1. Equation-[23] represents motion inpresence of large damping ($\gamma > \omega_0$) when the oscillator starts from the origin ($x=0$) with an initial velocity (v_0). The displacement always takes up positive values. For small values of t the exponential (γt) is close to unity and $x(t)$, therefore, increases as $\sinh(\alpha t)$ is increasing. Finally, the $\exp(-\gamma t)$ dominates and the ~~oscillation~~ motion dies down to zero. The turning point happens when,

$$\dot{x}(t) = \frac{v_0}{\alpha}\left[e^{-\gamma t}\alpha \cosh \alpha t - \gamma e^{-\gamma t} \sinh \alpha t\right] = 0$$

$$\Rightarrow \quad \tanh \alpha t_0 = \frac{\alpha}{\gamma}$$

$$\Rightarrow \quad \boxed{t_0 = \frac{1}{\alpha}\tanh^{-1}\left(\frac{\alpha}{\gamma}\right)}$$

Such type of nonoscillatory motion is called <u>dead beat</u>. (Represented in the diagram - 4a)

2. Equation-[24] represents what is known as <u>critical damping</u>. ($\gamma = \omega_0$). For small values of t, $x(t)$ increases almost linearly as the exponential term is close to 1. At some point of time t_0, $x(t)$ falls exponentially to zero. The turning point happens when

$$\dot{x}(t) = v_0\left[e^{-\gamma t} - \gamma t e^{-\gamma t}\right] = 0$$

$$\Rightarrow \quad \boxed{t_0 = \frac{1}{\gamma} = \frac{1}{\omega_0}}$$

critical damping is often preferred over the dead beat when we desire a high rate of decay without oscillation. (see diagram — 4-b)

3. Equation-[25] represents an oscillatory behavior known as <u>weak damping</u>. The oscillatory behavior is ensured by $\sin(\tilde{\omega} t)$ which crosses the time axis when-ever $t = \frac{n\pi}{\tilde{\omega}}$. <u>The so-called amptitude part now depends upon time exponentially and vanishes for large t, thus ceasing the oscillation.</u> Though the motion is no longer periodic one can define a time interval between two successive zeros of $x(t)$, which is clearly $\frac{\pi}{\tilde{\omega}}$. This is also the time interval between a maximum and the next minimum value of the displacement, but maxima & minima are not exactly half-way between the zeros. For maxima or minima, $\dot{x} = 0$

giving $\cos\tilde{\omega}t - \frac{\gamma}{\tilde{\omega}}\sin\tilde{\omega}t = 0$ or

$$\boxed{\tan\tilde{\omega}t = \frac{\tilde{\omega}}{\gamma}}$$

For $\gamma/\tilde{\omega} \to 0$ $\quad \tilde{\omega}t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\tilde{\omega}}$

See diagram- 4-c

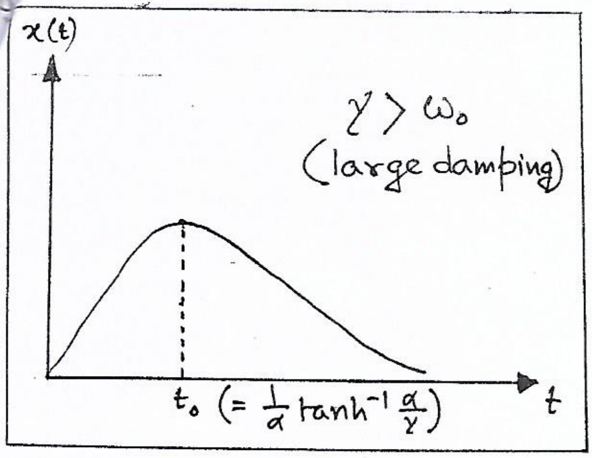

Diagram 4-a

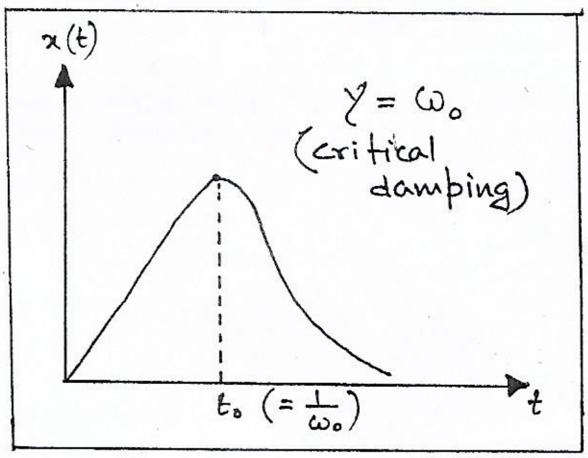

Diagram 4-b.

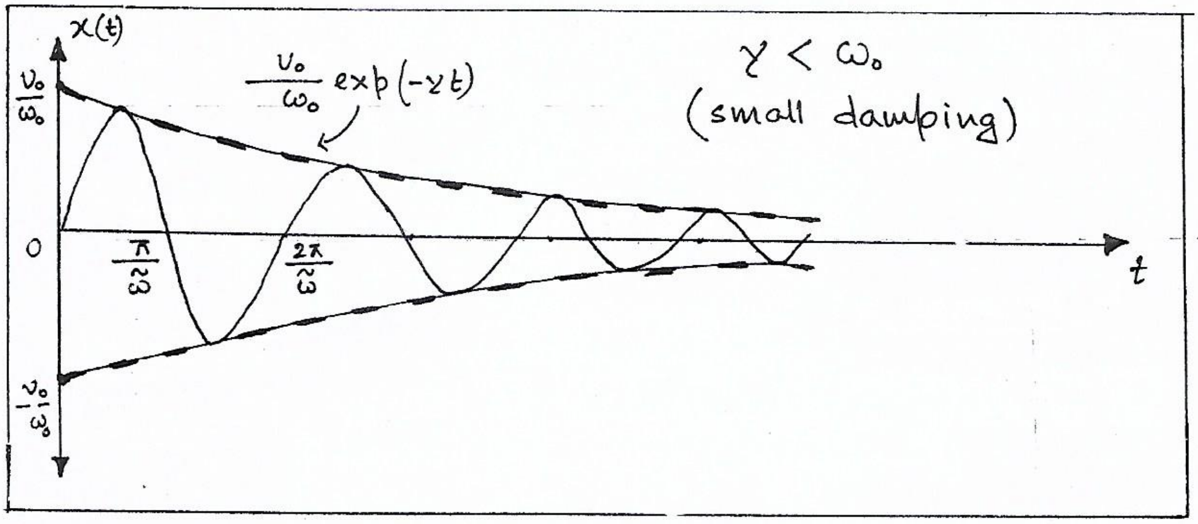

Diagram -4-c

5. The present theorem (Th-4) and the diagrams we obtain are all based on initial condition : at $t=0$ $x(t)=0$ and $\dot{x}(t) = v_0$. One can always start with various other initial conditions (for example at $t=0$ $x(t)=x_0$, $\dot{x}=0$) The decaying part of the above diagrams are almost identical in nature but there can be differences in the initial regimes. In particular, when ~~$x \neq 0$~~ at $t=0$ $x = x_0$ and $\dot{x} = v_0$ it is not in general possible to conclude (in case of large damping) that whether the particle will shoot past the origin or whether it will merely fall down to the origin without crossing it (like diagram-4). It all depends upon how γ is related to the initial condition (x_0 & v_0). For some γ The solution remains to be non-oscillatory crossing $x=0$ once and for all.

- ## *Study of weakly damped oscillation*

In our preceeding study of weakly damped oscillation we have already observed that the dynamical character of the system is chiefly controlled by two parameters.

(i) The natural frequency of oscillation (ω_0)

(ii) The damping const (γ).

In the following we will consider how the decay in energy and the decay in amplitude can be understood in relation to these two parameters. We will assume the solution —

$$x(t) = \frac{v_0}{\tilde{\omega}} \exp(-\gamma t) \sin(\tilde{\omega} t) \quad \cdots [25]$$

where $\tilde{\omega} = (\omega_0^2 - \gamma^2)^{1/2}$ and replacing the amplitude by $A(t)$ we can also write

$$x(t) = A(t) \sin(\tilde{\omega} t) \quad \cdots [25]'$$

- *Theorem-5*: For a weakly dampled oscillation $(\gamma << \omega_0)$

(i) The time average of total energy $\langle E(t) \rangle$ decays exponentialy.

(ii) $\langle E(t) \rangle = (2\gamma)^{-1} \langle P(t) \rangle$, where $\langle P(t) \rangle$ denotes the time average of power dissipated.

Proof: (i) The total energy $E(t) = T + V$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$$

$$= \frac{1}{2} m \left[\frac{v_0^2}{\tilde{\omega}_0^2} e^{-2\gamma t} \left\{ \tilde{\omega}^2 \cos^2 \tilde{\omega} t + \gamma^2 \sin^2 \tilde{\omega} t - 2 \tilde{\omega} \gamma \cos \tilde{\omega} t \sin \tilde{\omega} t \right\} \right]$$

$$+ \frac{1}{2} m \omega_0^2 \left[\frac{v_0^2}{\tilde{\omega}^2} \cdot \exp(-2\gamma t) \sin^2 \tilde{\omega} t \right]$$

Now to find $\langle E(t) \rangle$ we'll take the time average of the above quantity. For small γ we can take $e^{-2\gamma t}$ outside the integral and use the following resu

$$\left.\begin{aligned} &1.\ \langle \cos^2 \tilde{\omega} t \rangle = \tfrac{1}{2} \\ &2.\ \langle \sin^2 \tilde{\omega} t \rangle = \tfrac{1}{2} \\ &3.\ \langle \sin 2\tilde{\omega} t \rangle = 0 \end{aligned}\right\}.$$

Using all these result we get

$$\langle E(t)\rangle = \frac{1}{\tilde{T}}\int_0^{\tilde{T}} E(t)\,dt \ , \quad \text{where } \tilde{T} = \frac{2\pi}{\tilde{\omega}}$$

$$= \frac{m v_0^2}{4\tilde{\omega}^2}\, e^{-2\gamma t}\left[\tilde{\omega}^2 + \omega_0^2 + \gamma^2\right]$$

As $\tilde{\omega}^2 = \omega_0^2 - \gamma^2$

$$\langle E(t)\rangle = \frac{1}{2} m A_0^2 \omega_0^2\, e^{-2\gamma t} \ ; \quad A_0 = \frac{v_0^2}{\tilde{\omega}^2}$$

(ii) By definition power consumed by the damping force $m2\gamma\dot{x}$ is

$$P(t) = m2\gamma\dot{x}\cdot\dot{x} = m2\gamma\dot{x}^2$$

$$\langle P(t)\rangle = m2\gamma\langle\dot{x}^2\rangle = 2\gamma A_0^2 e^{-2\gamma t}\frac{1}{2}m\left[\frac{\tilde{\omega}^2+\gamma^2}{2}\right]$$

$$= \frac{1}{2} m A_0^2\omega_0^2 (2\gamma)\, e^{-2\gamma t}$$

$$= (2\gamma)\langle E(t)\rangle$$

$$\Rightarrow \boxed{E(t) = (2\gamma)^{-1}\langle P(t)\rangle}$$

- ***Remark*** : 1. The time dependence of average energy is given by the following diagram. (Diagram-5)

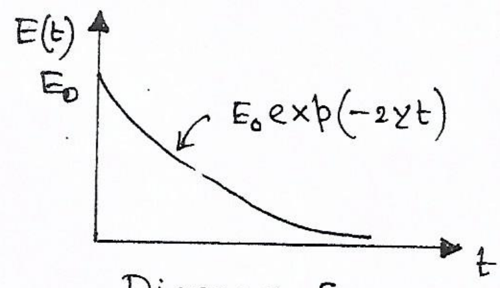

Diagram-5.

- ***Logarithmic Decrement*** : Let's consider the following diagram representing a weakly damped motion

$$x(t) = A_0 \exp(-\gamma t)\sin\tilde{\omega}t$$

$$\Rightarrow x(t) = A_0\exp(-\gamma t)\sin\frac{2\pi}{\tilde{T}}t \quad \cdots [26]$$

where $\tilde{T} = \frac{2\pi}{\tilde{\omega}}$

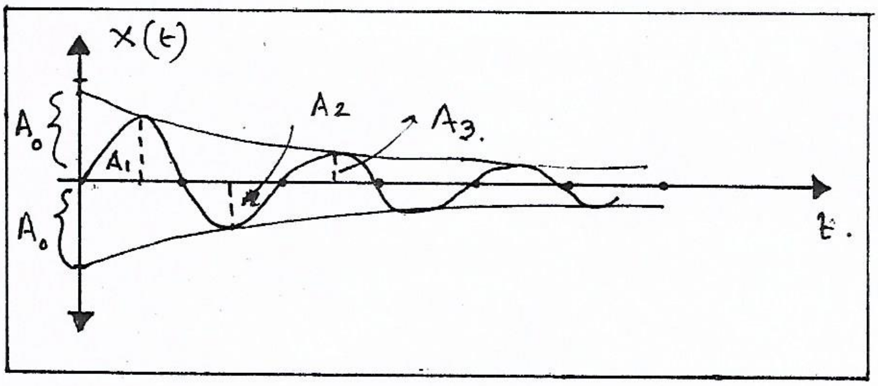

Diagram - 6

Let us call maxi displacement from the origin between two successive zeros of the solution as the 'amplitude' from diagram - 6 we can construct a decreasing sequence of amplitudes values (irrespective of sign) as $\{A_n \mid n=1,2,3\ldots\}$ occuring at $\{(2n-1)\frac{\tilde{T}}{4} \mid n=1,2,3\cdots\}$ respectively. Then from equation - 26 we can write,

1. $$\left|x\left(t=\frac{\tilde{T}}{4}\right)\right| = \left|A_0 \exp\left(-\frac{\gamma\tilde{T}}{4}\right)\sin\left(\frac{2\pi}{\tilde{T}}\,\frac{\tilde{T}}{4}\right)\right|$$
$$= \left|A_0 \exp\left(-\frac{\gamma\tilde{T}}{4}\right)\right| = A_1.$$

2. $$\left|x\left(t=\frac{3\tilde{T}}{4}\right)\right| = \left|A_0 \exp\left(-\frac{3\gamma\tilde{T}}{4}\right)\sin\left(\frac{2\pi}{\tilde{T}}\,\frac{3\tilde{T}}{4}\right)\right|$$
$$= \left|-A_0 \exp\left(-\frac{3\gamma\tilde{T}}{4}\right)\right| = A_2$$

3. $$\left|x\left(t=\frac{5\tilde{T}}{4}\right)\right| = \left|A_0 \exp\left(-\frac{5\gamma\tilde{T}}{4}\right)\right| = A_3.$$

and so on.

Thus we see that

$$\frac{A_1}{A_2} = \frac{A_2}{A_3} = \frac{A_3}{A_4} = \cdots = \frac{A_{n-1}}{A_n} = \exp\left(\frac{\gamma\tilde{T}}{2}\right) = \Delta$$
(const

$$\Rightarrow \ln\Delta = \gamma\tilde{T}/2.$$

One can also express

$$\frac{A_1}{A_n} = \frac{A_1}{A_2}\cdot\frac{A_2}{A_3}\cdot\frac{A_3}{A_4}\cdots\frac{A_{n-2}}{A_{n-1}}\,\frac{A_{n-1}}{A_n}$$

$$\Rightarrow \frac{A_1}{A_n} = \cancel{(n-1)}\,\Delta^{n-1} = \cancel{(n-1)\exp\left(\frac{\gamma\tilde{T}}{2}\right)}$$

$$\Rightarrow \ln\frac{A_1}{A_n} = (n-1)\ln\Delta$$

$$\Rightarrow \boxed{\ln\Delta = \frac{1}{n-1}\ln\frac{A_1}{A_n}} \cdots [27]$$

This leads to the following definition—

- *Definition - 8:* The quantity $\ln \Delta$ is the natural logarithm of the ratio of two successive amplitudes that are seperated by a half period $\tilde{T}/2$, the larger amplitude being the numerator. We denote $\lambda = \ln \Delta = \gamma \tilde{T}/2$, and call it logarithmic decrement.

- *Relaxation time:* For weakly damped motion the time dependence of amplitude is given by

$$A(t) = A_0 \exp(-\gamma t)$$

Putting $t = \frac{1}{\gamma}$ we find $A\left(t = \frac{1}{\gamma}\right) = A_0 e^{-1}$. We denote the time $1/\gamma$ by τ defined as below.

- *Definition - 9:* Let γ be the damping factor of a weakly damped oscillation. The time $\tau = \frac{1}{\gamma}$ is required for the amplitude to reduce to $1/e$ th of its initial value. τ is called the relaxation time.

- *Remark:* 1. We know that $\lambda = \frac{\gamma \tilde{T}}{2}$, and $\tau = \frac{1}{\gamma}$. This gives $\boxed{\lambda \tau = \tilde{T}/2}$

2. Hence λ is the ratio of half the period of oscillation and the relaxation time.

- *Quality factor or Q-value:* We have a relation between the average energy stored $\langle E(t) \rangle$ and that dissipated given by the formula,

$$\langle E(t) \rangle = (2\gamma)^{-1} \langle P(t) \rangle$$

Considering one period interval this equation becomes

$$\tilde{T} \langle E(t) \rangle = \tilde{T} (2\gamma)^{-1} \langle P(t) \rangle$$

$$\Rightarrow \frac{2\pi}{\tilde{\omega}} \langle E(t) \rangle = \frac{2\pi}{\tilde{\omega}} (2\gamma)^{-1} \langle P(t) \rangle$$

$$\Rightarrow 2\pi \left(\frac{2\gamma}{\tilde{\omega}}\right) \langle E(t) \rangle = \frac{2\pi}{\tilde{\omega}} \langle P(t) \rangle$$

$$\Rightarrow \frac{2\pi}{Q} \langle E(t) \rangle = \tilde{T} \langle P(t) \rangle$$

$$\Rightarrow Q = 2\pi \frac{\langle E(t) \rangle}{\tilde{T} \langle P(t) \rangle} \quad \text{--- [28]}$$

In view of equation -[28] we get the following definition

• *Definition-10*: The quality factor Q of a damped oscillation with frequency $\tilde{\omega}$ and damping const γ is defined as

$$Q = \frac{\tilde{\omega}}{2\gamma} = 2\pi \times \frac{\text{average energy stored in one period}}{\text{average energy lost in one period.}} \quad \text{----[29]}$$

• *Remark*: 1. For $\omega_0 >> \gamma$, $\tilde{\omega} \simeq \omega_0$ and $Q = \frac{\omega_0}{2\gamma}$

i.e.; lower the damping higher the Q value.

2. We have seen that the loss of energy is given by $\langle E(t)\rangle = E_0 \exp(-2\gamma t)$. Using quality factor $Q = \frac{\omega_0}{2\gamma}$

$$\langle E(t)\rangle = E_0 \exp\left(-\frac{\omega_0 t}{Q}\right)$$

Let, Γ be the time by which the energy falls $\frac{1}{e}$th of its initial value,

$$\frac{1}{e} E_0 = E_0 e\left(-\frac{\omega_0 \Gamma}{Q}\right)$$

$$\Rightarrow \frac{\omega_0 \Gamma}{Q} = 1$$

$$\Rightarrow \boxed{\Gamma = \frac{Q}{\omega_0} = \frac{1}{2\gamma} = \frac{\tau}{2}} \quad \text{----[29a]}$$

T is called the mean decay time.

II. Damped Oscillation.

1. A massless spring suspended from a rigid rod carrie a mass of 200 g at its lower end. It is observed the the system oscillates with a time period of 0.2 s and the amplitude of oscillations reduces to half of its initial value in 30 seconds. Calculate,
   (i) Relaxation time
   (ii) Quality factor
   (iii) spring constant. and lograrilhmic decreme

[Hint: (i) Considering the equation of motion

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0$$

where $\omega_0^2 = \frac{K}{m}$, K = spring constant.

Given $T = \frac{2\pi}{\omega_0} = 0.2$ s., $m = 200$ g

Relaxation time $\tau = \frac{1}{\gamma}$

Now $A(t) = A_0 \exp(-\gamma t)$

$$\Rightarrow -\gamma t = \ln \frac{A(t)}{A_0}, \quad t = 30 \quad A(t) = \frac{1}{2} A_0$$

$$\Rightarrow \gamma = \frac{\ln 2}{30} = 0.023 \text{ s}^{-1}$$

$$\Rightarrow \tau = \frac{1}{\gamma} = \frac{30}{\ln 2} = \boxed{43.28 \text{ s}}$$

(ii) Quality factor $Q = \frac{\tilde{\omega}}{2\gamma} = \frac{\sqrt{\omega_0^2 - \gamma^2}}{2\gamma}$

$$= \frac{1}{2}\left(\left(\frac{\omega_0}{\gamma}\right)^2 - 1\right)^{1/2}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi \text{ s}^{-1} = \cancel{98.6}\ 31.4 \text{ s}^{-1}$$

$$\gamma = \frac{\ln 2}{30} = 0.023 \text{ s}^{-1}$$

Hence, $$Q = \frac{1}{2}\left[\left(\frac{\cancel{98.6}\ 31.4}{0.023}\right)^2 - 1\right]^{1/2} = \boxed{683}$$

(iii) Spring constant $K = m\omega_0^2 = 200 \times (31.4)^2$

$$= \boxed{197192 \text{ dyne/cm}}$$

Now logarithmic decrement

$$\lambda = \frac{\gamma \tilde{T}}{2} = \frac{\gamma\, 2\pi}{2\tilde{\omega}} = \frac{\gamma \pi}{\tilde{\omega}}$$

$$= \frac{0.023 \times 3.14}{\sqrt{(31.4)^2 - (0.023)^2}} = \boxed{0.0023}$$

2. Calculate the frequency, relaxation time and th quality factor of an LCR circuit with

$L = 1$ mH
$C = 5\,\mu F$
$R = 0.5\ \Omega$.

[Hint: $L = 1$ mH $= 10^{-3}$ H
$C = 5\,\mu F = 5 \times 10^{-6}$ F
$R = 0.5\,\Omega$.

Considering the equation: $\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q =$
and comparing it with $\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = 0$, th
frequency $\tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2} = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{1/2}$

$$= \boxed{1.414 \times 10^4 \text{ rad/}}$$

frequency $\tilde{\nu} = \frac{\tilde{\omega}}{2\pi} = \boxed{0.225 \times 10^4 \text{ Hz.}}$

Relaxation time $\tau = \frac{1}{\gamma} = \frac{2L}{R} = \boxed{4 \times 10^{-3} \text{ s}}$

Quality factor $Q = \frac{\tilde{\omega}}{2\gamma} = 28.3.$]

3. Show that the amplitude of a weakly dampe oscillator reduces to half of its initial value i time $t = \tau \ln 2$, τ being the relaxation time.

[Hint: $A(t) = A_0 \exp(-\gamma t)$ and $\tau = \frac{1}{\gamma}$

Hence $A(t) = A_0 \exp(-t/\tau)$

$\Rightarrow A(t = \tau \ln 2) = A_0 \exp(-\ln 2)$

$\Rightarrow A = A_0/2$]

• IV Forced Oscillation

A free or damped oscillation may undergo the effect of external force which, in so many practical situations, is a function of time. In the following we shall consider the effect of such force on damped oscillation.

• *The inhomogeneous differential equation*: Let's consider the equation of motion for free damped vibration:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \quad \cdots\cdots [30]$$

where, γ = damping constant
ω_0 = natural frequency of vibration.

In presence of a time varying external force $F(t)$ eq^n [30] will be modified by an acceleration term $f(t) = F(t)/m$ leading to,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) \quad \cdots\cdots [30]'$$

• *Remark*: 1. Eq [30]$'$ is an inhomogeneous differential equation in the sense it contains a term $f(t)$ on the right hand side which is independent of x and its derivative.

2. Eq^n.[30]$'$ is linear for obvious reason and its solution can be determined in view of the following theorem.

• *Theorem – 6*: Let X_p be a particular solution of [30]$'$ The general solution X_g of eq^n. [30]$'$ is given by

$$X_g = X_h + X_p \quad \cdots\cdots [31]$$

where X_h is the solution of the homogeneous equation - eq^n. [30].

Proof: Let $x_1(t)$ and $x_2(t)$ be two solutions of [30]$'$

Then $\ddot{x}_1 + 2\gamma\dot{x}_1 + \omega_0^2 x_1 = f(t)$

and $\ddot{x}_2 + 2\gamma\dot{x}_2 + \omega_0^2 x_2 = f(t)$

$$\Rightarrow (\ddot{x}_1 - \ddot{x}_2) + 2\gamma(\dot{x}_1 - \dot{x}_2) + \omega_0^2(x_1 - x_2) = 0$$

Hence, the difference $x_1 - x_2$ is a solution of the homogeneous eq^n. [30]. Therefore

$$x_1 - x_2 = X_h$$

The above equation is valid for any pair of solution [30]′. Choosing $x_1 = X_g$ and $x_2 = X_p$

$$X_g - X_p = X_h \Rightarrow X_g = X_h + X_p.$$

- *Remark* : 1. Relation [31] is valid for any linear inhomo--geneous equation.

2. The form of $X_h(t)$ may change depending upon various types of damping.

3. We shall consider eqn. [30]′ when $f(t)$ itself is harmonic with frequency ω i.e.

$$f(t) = f_0 \cos \omega t$$

because al-most all functions of t can be represested in terms of such type of harmonic functions. Eqn. [30]′ therefore, takes the form,

$$\boxed{\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f_0 \cos \omega t \quad \cdots\cdots [32]}$$

Equation [32] is called the equation of forced damped vibration.

- <u>Solution of eqn. [32]</u> : $X_h(t)$ for eqn. [32] is always of the form :-

$$X_h(t) = \exp(-\gamma t)\, g(t)$$

a term always dominated by the exponential part and hence at $t \to \infty$ $X_h(t) \to 0$. So for large t values the general solution $X_g(t)$ retains only the $X_p(t)$ part leading to the following definitions.

- <u>Definition-11</u> : Let the solution of eqn-[32] be given by $X_g(t) = X_h(t) + X_p(t)$. As $\lim_{t\to\infty} X_h(t) = 0$

$$\lim_{t\to\infty} X_g(t) = X_p(t)$$

$X_h(t)$ is called <u>transient</u> solution and the state represented by it has at all any relevance in the initial moments of oscillation. Soon it starts to be fed by the driving force and for large t the system ~~st~~ likes to oscillate with driving frequency irrespective of the value of the damping constant. Such a state is called <u>steady state</u>.

From now on we shall call $x_p(t)$ as steady state solution and consider the following theorem regarding its behaviour.

• *Theorem-7* : The steady-state solution $x_p(t)$ of eqn [32] is

(i) Oscillatory with a phase difference $\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)$ with respect to the driving force.

(ii) A bounded function for $\gamma \neq 0$

Proof :(i) Let, $x_p(t) = A\cos\omega t + B\sin\omega t$, A, B consts.

$$\dot{x}_p(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\ddot{x}_p(t) = -\omega^2(A\cos\omega t + B\sin\omega t)$$

Substituting all these results in equation [32] we get

$$-\omega^2(A\cos\omega t + B\sin\omega t) + 2\gamma(-A\omega\sin\omega t + B\omega\cos\omega t) + \omega_0^2(A\cos\omega t + B\sin\omega t) = f_0\cos\omega t$$

Equating the co-efficients of $\cos\omega t$ and $\sin\omega t$ * we get,

$$\left.\begin{aligned} \cos\omega t :&= A(\omega_0^2-\omega^2) + 2\gamma\omega B = f_0 \\ \sin\omega t :&= B(\omega_0^2-\omega^2) - 2\gamma\omega A = 0 \end{aligned}\right\} \quad [33\ a, b]$$

Solving equation [33] for A and B

$$A = \frac{f_0(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

$$B = \frac{f_0\, 2\gamma\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

Giving, $$x_p(t) = \frac{f_0}{(\omega_0^2-\omega^2)+4\gamma^2\omega^2}\left[(\omega_0^2-\omega^2)\cos\omega t + 2\gamma\omega\sin\omega t\right]$$

---

* If $c_1\cos\omega t + c_2\sin\omega t = 0$; differentiating we get, $-c_1\sin\omega t + c_2\cos\omega t = 0$, Multiplying the 1st eqn. by $\sin\omega t$ and 2nd by $\cos\omega t$ and adding we get $c_2 = 0$ & so $c_1 = 0$

$$\Rightarrow x_p(t) = \frac{f_0}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}\left[\frac{(\omega_0^2-\omega^2)\cos\omega t}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}} + \frac{2\gamma\omega\sin\omega t}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]}\right]$$

Identifying, $$\cos\phi = \frac{\omega_0^2-\omega^2}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}} \quad \text{--- [32 a]}$$

and $$\sin\phi = \frac{2\gamma\omega}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}} \quad \text{--- [32 b]}$$

we get $$\tan\phi = \frac{2\gamma\omega}{\omega_0^2-\omega^2}$$

or $$\boxed{\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)} \quad \text{--- [33]}$$

where as

$$x_p(t) = \tilde{A}(\omega,\omega_0,\gamma)\left[\cos\phi\cos\omega t + \sin\phi\sin\omega t\right]$$

or $$x_p(t) = \tilde{A}(\omega,\omega_0,\gamma)\cos(\omega t - \phi) \quad \text{---- [34]}$$

where $$\boxed{\tilde{A}(\omega,\omega_0,\gamma) = \frac{f_0}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}} \quad \text{-- [35]}$$

(ii) The solution $x_p(t)$ is bounded provided $\tilde{A}(\omega,\omega_0,\gamma)$ is bounded for all ω, given ω_0 and γ fixed. We shall show that for $\gamma \neq 0$ $\tilde{A}$ indeed has an extremum i.e.;

$D = (\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2$ has an extremum (minimum for our purpose) say at $\omega = \omega_r$

$$\left.\frac{dD}{d\omega}\right|_{\omega=\omega_r} = \left.\left[8\gamma^2\omega - 4\omega(\omega_0^2-\omega^2)\right]\right|_{\omega_r} = 0$$

$$\Rightarrow \omega_r\left[2\gamma^2 - \omega_0^2 + \omega_r^2\right] = 0$$

Giving $\omega_r = 0$ or $\omega_r^2 = \omega_0^2 - 2\gamma^2$, where the 1st case is meaningless. Hence

$$\boxed{\omega_r^2 = \omega_0^2 - 2\gamma^2} \quad \cdots\cdots [36]$$

Hence $$D(\omega)\Big|_{\omega=\omega_r} = \left(\omega_0^2 - \omega_0^2 + 2\gamma^2\right)^2 + 4\gamma^2\left(\omega_0^2 - 2\gamma^2\right)$$
$$= 4\gamma^2\omega_0^2 - 4\gamma^4 = 4\gamma^2\left(\omega_0^2 - \gamma^2\right)$$

Therefore, $$\boxed{\tilde{A}(\omega, \omega_0, \gamma)\Big|_{\omega=\omega_r} = \frac{f_0}{2\gamma(\omega_0^2 - \gamma^2)^{1/2}} \neq \infty \text{ for all } \gamma > 0 \text{ } \gamma \neq 0} \quad \cdots [37]$$

Remark : 1. Whether the condition – [36] at all represents a minima of $D(\omega)$ and hence a maxima for $\tilde{A}$ can be verified by the following,

$$\frac{d^2 D(\omega)}{d\omega^2}\Bigg|_{\omega=\omega_r} = \left\{4\left[2\gamma^2 - \omega_0^2 + \omega^2\right] + 4\omega\left[2\omega\right]\right\}\Bigg|_{\omega=\omega_r}$$
$$= 8\left[\omega_0^2 - 2\gamma^2\right] > 0 \text{ for } \omega_0 > \sqrt{2}\,\gamma$$

which is the case in almost all practical situations.

2. The quantity ω_r is called <u>resonance frequency</u> and for small damping constant γ, ω_r is close to natural frequency of oscillation ω_0. The system is said to be in <u>resonance</u> or more properly in <u>amplitude resonance</u> when it oscillates with the frequency ω_r.

3. It is obvious that at resonance the amplitude has finite maximum value so long $\gamma \neq 0$. For $\gamma = 0$, $\tilde{A}_{max}$ become infinite at resonance (see eqn. 37). Therefore at resonance it the damping factor that provides bound and prevent the system from ~~ultimate~~ an eventual breakdown.

4. Therefore, <u>close to resonance</u> the system is said to be <u>resistance controlled</u>.

4. When $\omega << \omega_0$ i.e.; driving frequency is much less than the natural frequency of vibration we can write from eqn. [35]

$$\tilde{A}(\omega,\omega_0,\gamma)\Big|_{\omega<<\omega_0} = \frac{f_0}{\omega_0^2\left[\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+4\gamma^2\frac{\omega^2}{\omega_0^4}\right]^{1/2}}$$

$$\simeq \frac{F_0/m}{\omega_0^2} = \frac{F_0}{K}$$

In this regime the oscillation is controlled by the stiffness constant K. The resultant motion is called stiffness controlled.

5. When $\omega >> \omega_0$ eqn. [35] can be represented as

$$\tilde{A}(\omega,\omega_0,\gamma)\Big|_{\omega>>\omega_0} = \frac{f_0}{\omega^2\left[\left(1-\frac{\omega_0^2}{\omega^2}\right)^2+4\frac{\gamma^2}{\omega^4}\right]^{1/2}}$$

$$\simeq \frac{F_0/m\omega^2}{\left[1+4\frac{\gamma^2}{\omega^4}\right]^{1/2}} \simeq F_0/m\omega^2$$

In this regime the oscillation is chiefly controlled by the mass m. The resultant motion is called mass controlled.

6. The following diagram is very crucial in demonostrating the effect of forcing frequency (ω) on the amplitude $\tilde{A}$ in defferent regime.

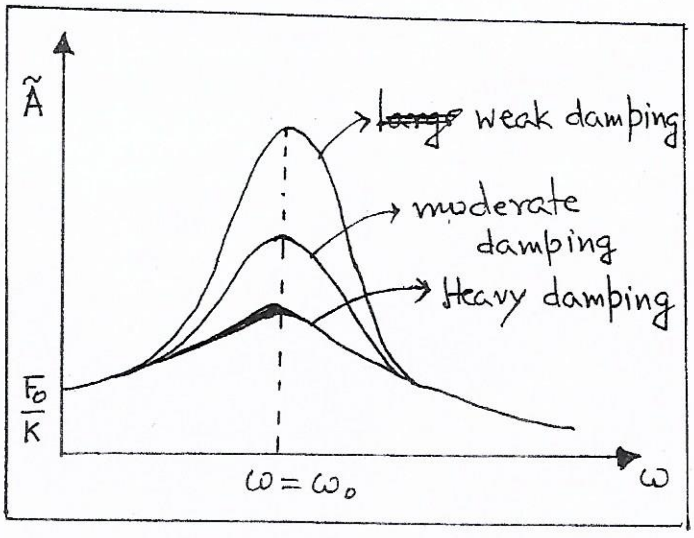

Diagram-7.

It is to be noted that

(i) For heavy damping the peak of the amplitude curve is slightly displaces from $\omega = \omega_0$ line, otherwise the resonance always occurs at $\omega = \omega_0$

(ii) Apart from resonance the steady state motion is not much sensitive to the damping factor.

7. The dependence of phase angle ϕ on the driving frequency ω can be represented by the following diagram.

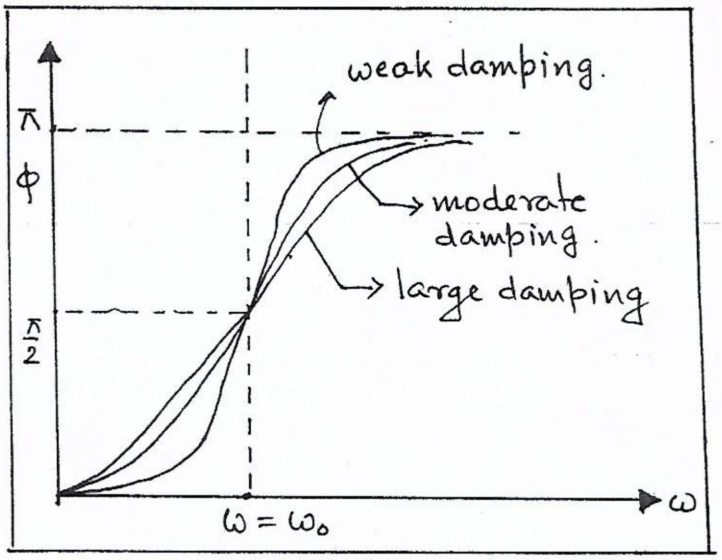

Diagram – 8

The angle $\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right)$ denotes the phase by which the displacement lags behind the driving force. It is to be noted that

(i) $\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2\left(1 - \frac{\omega}{\omega_0^2}\right)}\right) \to \tan^{-1}\frac{2\gamma\omega}{\omega_0^2}$ when $\omega_0 >> \omega$. $\phi = 0$ when $\omega_0 >> \omega$

(ii) For $\omega >> \omega_0$, $\tan^{-1}\left(\frac{2\gamma\omega}{\omega^2\left(\frac{\omega_0^2}{\omega^2} - 1\right)}\right) \to \tan^{-1}\frac{2\gamma}{\omega}$

$$= -\tan^{-1}\left(\frac{2\gamma}{\omega_0}\frac{\omega_0}{\omega}\right) \doteq \pi \text{ for weak}$$

damping $(\gamma < \omega_0)$

(iii) At resonance $\phi = \tan^{-1}\infty = \frac{\pi}{2}$

8. The steady-state velocity is given by

$$v(t) = \dot{x}_p(t) = \frac{-f_0\,\omega}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}} \sin(\omega t - \phi)$$

$$= \frac{f_0\,\omega}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}} \cos\left(\omega t - \phi + \frac{\pi}{2}\right)$$

$$= V_0 \cos\left(\omega t - \phi + \frac{\pi}{2}\right)$$

where,

$$V_0 = \frac{f_0\,\omega}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}}$$ is called

~~the~~ velocity amplitude. It is maximum when

$$G(\omega) = \frac{(\omega_0^2-\omega^2)^2}{\omega^2} + 4\gamma^2$$ is minimum.

$$\Rightarrow \frac{dG}{d\omega} = 0 \text{ i.e.; } -\frac{2\omega_0^4}{\omega^3} + 2\omega = 0 \Rightarrow \omega = \omega_0$$

and $G_{min} = 4\gamma^2$ and so

$$(V_0)_{max} = \frac{f_0}{2\gamma} = \frac{F_0}{2m\gamma} \quad \text{- - - - [36]}$$

It is to be noted that,

(i) Though the maximum of velocity at resonance is always controlled by the damping factor γ the condition of resonance $(\omega = \omega_0)$ is independent of γ unlike amplitude resonance. The resonance thus obtained is called velocity resonance. The following diagram will clarify the matter.

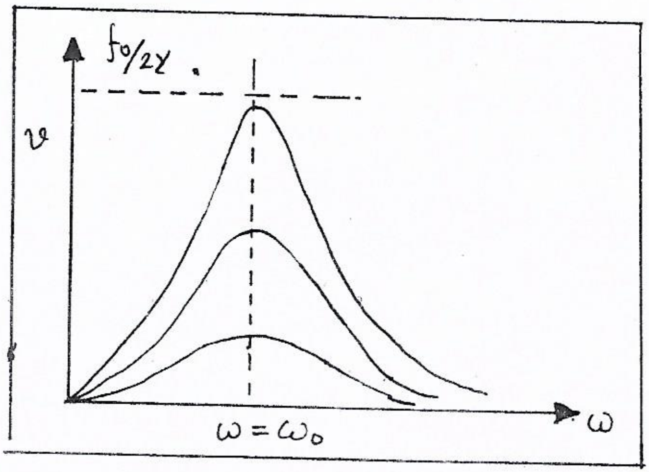

Diagram – 9

9. The phase of the velocity relative to the driving force is given by

$$\delta = \phi - \frac{\pi}{2}$$

clearly for $\omega << \omega_0$ i.e.; $\phi = 0$ $\delta = -\pi/2$, the velocity lags behind the driving force, while for $\omega >> \omega_0$, $\phi = \pi$ $\delta = \pi - \pi/2 = \pi/2$, the velocity leads the force. Finally for resonance $\omega = \omega_0$ the velocity becomes in phase with the driving force. The dependence of ~~v on ω~~ δ on ω is given in the following diagram.

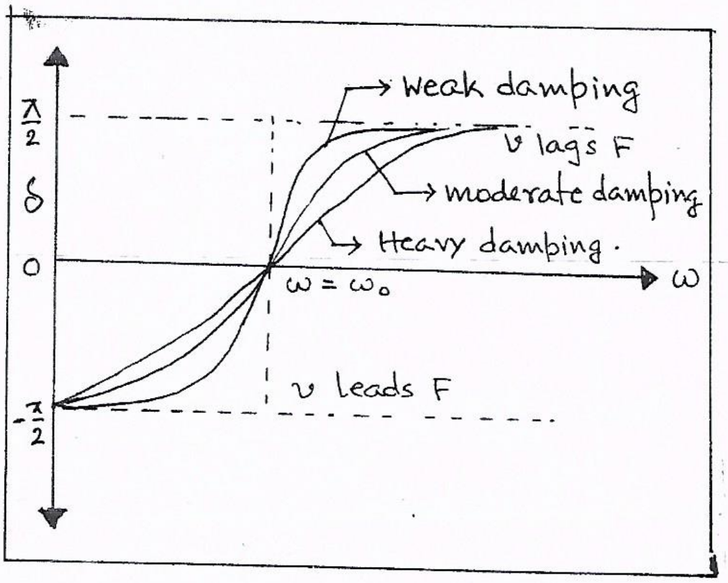

Diagram – 10

10. When, $\omega = \omega_0$ i.e.; $\delta = 0$ as the velocity and the driving force are in ~~pa~~ phase, it is the most favourable situation to transfer maximum energy from the driving system. The following section will clarify the matter.

- *Power Transfer – Sharpness of resonance – Quality factor.*

Theorem – 8 : For a system admitting eq^n·[32] as equation of motion, the power supplied by the driving force is equal to that dissipated by damping in steady state.

Proof : Let's start with the steady-state solution of [32] given by equation [34]

$$x_p(t) = \tilde{A} \cos(\omega t - \phi)$$

$$\Rightarrow \quad \dot{x}_p(t) = -\tilde{A} \omega \sin(\omega t - \phi)$$

Power delivered by the driving force,

$$P_{input} = F(t)\,\dot{x}_p(t)$$
$$= mf_0 \cos\omega t\left(-\tilde{A}\omega \sin(\omega t - \phi)\right)$$
$$= -\tilde{A} m\omega f_0 \cos\omega t \sin(\omega t - \phi)$$

So, $$\langle P_{input}\rangle = -\frac{\tilde{A} m\omega f_0}{T}\int_0^T \cos\omega t \sin(\omega t-\phi)\,dt$$
$$= -\frac{\tilde{A} m\omega f_0}{T}\int_0^T [\cos\omega t \sin\omega t \cos\phi - \cos^2\omega t \sin\phi]\,dt.$$
$$= \frac{\tilde{A} m\omega f_0}{2}\sin\phi$$

Power dissipated due to the damping force,

$$P_{dissipated} = 2m\gamma\left(\dot{x}_p\right)^2$$
$$= 2m\gamma\,\tilde{A}^2\omega^2\sin^2(\omega t - \phi)$$

So, $$\langle P_{dissipated}\rangle = 2m\gamma\tilde{A}^2\omega^2\frac{1}{T}\int_0^T \sin^2(\omega t-\phi)\,dt.$$
$$= \frac{2m\gamma\tilde{A}^2\omega^2}{2} = \frac{\tilde{A} m\omega f_0}{2}\sin\phi$$

[Using [32b]]

Hence, $\langle P_{input}\rangle = \langle P_{dissipated}\rangle = \langle P\rangle$ (let's call)

- Remark: 1. We can write $\langle P\rangle = \frac{\tilde{A} m\omega f_0}{2}\sin\phi$

$$\Rightarrow \langle P\rangle = \frac{\tilde{A} m\omega f_0}{2}\,\frac{2\gamma\omega}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}$$

$$\Rightarrow \langle P\rangle = \frac{m\omega f_0}{2}\,\frac{2f_0\gamma\omega}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]}$$

$$\Rightarrow \langle P\rangle = \gamma m f_0^2\left[\frac{\omega^2}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\right]$$

Therefore, $\langle P \rangle$ is maximum when $\omega = \omega_0$ and so,

$$\langle P \rangle_{max} = \frac{m f_0^2}{4\gamma}$$

Hence, $$\langle P \rangle = \langle P \rangle_{max} \frac{4\omega^2\gamma^2}{\left[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]} \quad \text{-- [37]}$$

The following diagram will illustrate the dependence of $\langle P \rangle$ on ω.

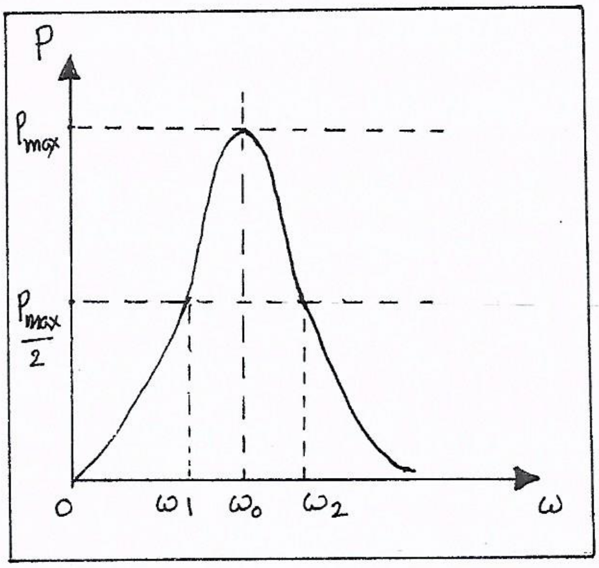

Diagram-11

Now $\langle P \rangle = \frac{1}{2} \langle P \rangle_{max}$ happens when

$$\frac{1}{2} = \frac{4\omega^2\gamma^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$$

$$\Rightarrow (\omega_0^2 - \omega^2)^2 = 4\omega^2\gamma^2$$

$$\Rightarrow \omega = \pm\gamma \pm \sqrt{\gamma^2 + \omega_0^2}$$

ω being positive we can have two possibilities

$$\left.\begin{array}{l} \text{(i)} \ \omega_1 = -\gamma + \sqrt{\gamma^2 + \omega_0^2} \\ \text{(ii)} \ \omega_2 = \gamma + \sqrt{\gamma^2 + \omega_0^2} \end{array}\right\}$$

Hence we can define the so called <u>band width</u>

$$\boxed{\Delta\omega = \omega_2 - \omega_1 = 2\gamma = \frac{1}{T}} \quad (T = \text{mean decay time}) \quad \text{--- [38]}$$

2. The so called sharpness of resonance can be under-stood with the help of what is known as quality factor defined as follows,

$$Q = \frac{\text{Resonant frequency}}{\text{Band width}} = \frac{\omega_0}{\Delta\omega}$$

or $$\boxed{Q = \frac{\omega_0}{2\gamma} = \omega_0 \Gamma} \quad \text{---} [39]$$

Expressing the steady-state amplitude $\tilde{A}$ in terms Q,

$$\tilde{A} = \frac{f_0}{\left[(\omega_0^2 - \omega^2)^2 + 4\omega^2\gamma^2\right]^{1/2}}$$

$$\Rightarrow \tilde{A} = \frac{f_0/2\omega\gamma}{\left[1 + \frac{(\omega_0^2-\omega^2)^2}{4\omega^2\gamma^2}\right]^{1/2}}$$

$$\Rightarrow \tilde{A} = \frac{f_0 Q/\omega\omega_0}{\left[1 + Q^2\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2\right]^{1/2}} \quad \text{----} [40]$$

The following diagram represents the dependence of $\tilde{A}$ on ω.

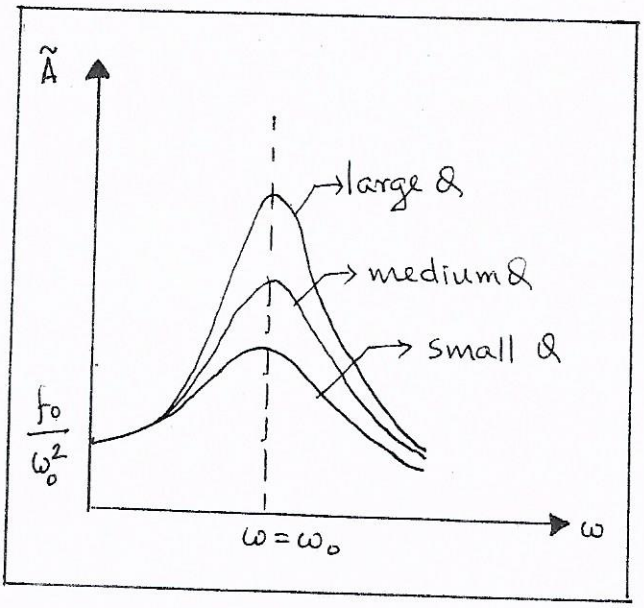

Diagram-12

It is to be noted that for larger Q values not only the peak of the resonance is higher but also the curve falls off faster and faster on both sides of the resonace frequency. So higher Q values is an indicator of the sharpness of resonance.