Improper Integral :

* The Definite Integral as the Limit of a Sum :

Let $f(x)$ be a bounded real-valued function defined on a closed interval $[a,b]$ — some finite interval. We divide the interval $[a,b]$ into n finite parts — not necessarily all equal — by means of a partition P with arbitrary set of points $a = x_0, x_1, \ldots\ldots x_{r-1}, x_r, x_{r+1}, \ldots, x_{n-1}, x_n = b$ where $x_0 < x_1 < \ldots\ldots < x_n$.

Let δ_r or $\Delta x_r \equiv [x_{r-1}, x_r]$ or $\equiv x_r - x_{r-1}$. Thus $\delta_1, \delta_2, \ldots, \delta_n$ are the respective sub-intervals into which the interval $[a,b]$ is divided. Take $\xi_1, \xi_2, \ldots, \xi_r, \ldots, \xi_n$, to be arbitrary set of points in the respective intervals $\delta_1, \delta_2, \ldots, \delta_r, \ldots, \delta_n$.

Now, form the sum $\sum_{r=1}^{n} f(\xi_r)\delta_r = f(\xi_1)\delta_1 + \cdots + f(\xi_n)\delta_n$

which depends on the choice of the interval δ_r and of the point ξ_r of δ_r.

Now, let $\delta = \max\{\delta_1, \delta_2, \ldots, \delta_n\}$. Next, let n increases indefinitely in such a manner that δ tends to zero. If in this case $\sum_{r=1}^{n} f(\xi_r)\delta_r$ tends to a definite limit, being independent of the choice of the interval δ_r, and of the point ξ_r of δ_r then this limit is said to be the definite integral of $f(x)$ over $[a,b]$ denoted by $\int_a^b f(x)\,dx$; i.e.

$$\lim_{\delta \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \int_a^b f(x)\,dx .$$

* Improper Integrals :

The definition of a definite integral $\int_a^b f(x)\,dx$ presupposes (i) that the limits a, b are finite, (ii) that the integrand is bounded and integrable in $a \le x \le b$. Hence when either (or both) of these assumptions are not satisfied, that is, when a limit is infin[ite] or the integrand becomes infinite in $a \le x \le b$, we need to modify the previous definition, if such integrals, called improper integra[ls] are to have a meaning.

Examples:
1. Does the int[...]
Sol: N[...]

* **Types of Improper Integrals:**

Improper integrals are of two main types:

(1) The interval is infinite

(2) The integrand has a finite no. of infinite discontinuities.

(A) Type - I :

Under type I, we have three kinds of unbounded ranges over which integrals may be taken are symbolised and defined as follows:

(i) $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{B\to\infty}\int_a^{B} f(x)\,dx$, $f(x)$ bdd and integrable in $a \le x \le B$.

The improper integral on the LHS is said to converge or to exist if the limit on the RHS exists finitely.

(ii) $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{A\to-\infty}\int_A^{b} f(x)\,dx$, $f(x)$ bdd. and integrable in $A \le x \le b$.

The improper integral on the LHS is said to converge or to exist if the limit on the RHS exists finitely.

(iii) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{a} f(x)\,dx}_{(ii)} + \underbrace{\int_a^{\infty} f(x)\,dx}_{(i)}$

Examples:

1. Does the improper integral $\int_0^{\infty} \frac{1}{1+x^2} dx$ exist.

Sol^n: To determine whether this integral is convergent or not, we see that $\frac{1}{1+x^2}$ is bounded and integrable in $0 \le x \le B$ for every $B>0$ and

$$\lim_{B\to\infty} \int_0^B \frac{1}{1+x^2} dx = \lim_{B\to\infty} \left[\tan^{-1}x\right]_0^B$$
$$= \lim_{B\to\infty} \left(\tan^{-1}B - \tan^{-1}0\right)$$
$$= \lim_{B\to\infty} \tan^{-1}B = \pi/2.$$

2. Evaluate $\int_1^{\infty} \frac{1}{x^2} dx$, if it converges.

Sol^n: Here, first of all, we note that no attention need to be paid to the singularity at $x=0$, since it does not lie within the range of integration. The singularity is only at the upper limit.
Now, $\lim_{B\to\infty} \int_1^B \frac{1}{x^2} dx = \lim_{B\to\infty} \left[-\frac{1}{x}\right]_1^B = \lim_{B\to\infty} \left[-\frac{1}{B}+1\right] = 1$

3. Evaluate $\int_a^{\infty} \sin x\, dx$, if it exists.

Sol^n: Here $\lim_{B\to\infty} \int_a^B \sin x\, dx = \lim_{B\to\infty} \left[-\cos x\right]_a^B = \lim_{B\to\infty} (\cos a - \cos B)$ oscillates finitely. Therefore, $\int_a^{\infty} \sin x\, dx$ is oscillatory.

4. Evaluate $\int_0^{\infty} e^x dx$, if it converges.

Sol^n: Now $\lim_{B\to\infty} \int_0^B e^x dx = \lim_{B\to\infty} (e^B - 1)$. Since (e^B-1) increases beyond all bounds as $B\to\infty$, this integral diverges.

5. Evaluate : $\int_{-\infty}^{\infty} x \cdot e^{-x^2} dx$, if it converges.

Solⁿ. For convenience, we split this infinite range into two parts as,

$$I = \int_{-\infty}^{0} x e^{-x^2} dx + \int_{0}^{\infty} x e^{-x^2} dx.$$

Now, $\lim_{A \to -\infty} \int_{A}^{0} x e^{-x^2} dx + \lim_{B \to \infty} \int_{0}^{B} x \cdot e^{-x^2} dx$

$$= \lim_{A \to -\infty} \left[-\frac{1}{2} e^{-x^2} \right]_{A}^{0} + \lim_{B \to \infty} \left[-\frac{1}{2} e^{-x^2} \right]_{0}^{B}$$

$$= \lim_{A \to -\infty} \left(+\frac{1}{2} e^{-A^2} - \frac{1}{2} \right) + \lim_{B \to \infty} \left(\frac{1}{2} - \frac{1}{2} e^{-B^2} \right)$$

$$= 0.$$

Thus, $\int_{-\infty}^{\infty} x e^{-x^2} dx = 0.$

(B) Type - II

Under type-II we have the following kinds of integrals:

(1) If $f(x)$ has an infinite discontinuity only at the left hand end-point a, then by $\int_a^b f(x)\, dx$ we shall mean $\lim_{\epsilon \to 0+} \int_{a+\epsilon}^{b} f(x)\, dx$, $0 < \epsilon < b-a$.

(2) If $f(x)$ has an infinite discontinuity only at b, by $\int_a^b f(x)\, dx$ we shall mean $\lim_{\epsilon \to 0+} \int_{a}^{b-\epsilon} f(x)\, dx$, $0 < \epsilon < b-a$.

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(a)(3) If $f(x)$ has an infinite discontinuity at the point $x=c$, where $a<c<b$, then by $\int_a^b f(x)\,dx$, we shall mean

$$\lim_{\epsilon\to 0+}\int_a^{c-\epsilon} f(x)\,dx + \lim_{\delta\to 0+}\int_{c+\delta}^{b} f(x)\,dx.$$

If either of these limits fail to exist, we say that the integral does not exist.

Examples:

6. Evaluate $\int_0^1 \frac{1}{x}\,dx$, if it converges.

Solⁿ: Here $\frac{1}{x}$ has an infinite discontinuity at $x=0$.

So, we evaluate $\int_\epsilon^1 \frac{1}{x}\,dx = \log_e 1 - \log_e \epsilon = -\log_e \epsilon$

As $\epsilon \to 0+$, $\ln \epsilon \to -\infty$.

Hence $\lim_{\epsilon\to 0+}\int_\epsilon^1 \frac{1}{x}\,dx$ does not exist and the integral diverges.

7. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, if it converges.

Solⁿ: Since the integrand becomes infinite as $x\to 1$, we evaluate

$$\int_0^{1-\epsilon}\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(1-\epsilon).$$

As $\epsilon\to 0+$, $\sin^{-1}(1-\epsilon) \to \sin^{-1}1 = \pi/2$. Hence $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi/2$.

~~8. Show that~~ $\cancel{\int_{-1}^2 \frac{dx}{x}}$ ~~does not exist but P.~~

two parts

(**) Note: In case 3 - Type-II, if we make $\epsilon=\delta$ and say that

$$\int_a^b f(x)\,dx \text{ means } \lim_{\epsilon\to 0+}\left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx\right]$$

We have what is called the <u>Cauchy Principal Value</u> of $\int_a^b f(x)\,dx$ and write it as

$$P_c\int_a^b f(x)\,dx = \lim_{\epsilon\to 0+}\left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx\right]$$

It may sometimes happen that, the Cauchy principal value of the integral exists when according to the general definition the integral does not exist.

<u>Examples:</u>

8. Prove that $\int_{-1}^{1}\frac{1}{x^3}\,dx$ exists in Cauchy principal value sense but not in general sense.

<u>Solⁿ</u>: The integrand is unbounded as $x\to 0$. Therefore, we evaluate

$$\lim_{\epsilon\to 0+}\int_{-1}^{-\epsilon}\frac{1}{x^3}\,dx + \lim_{\delta\to 0+}\int_{\delta}^{1}\frac{1}{x^3}\,dx = \lim_{\epsilon\to 0+}\left[-\frac{1}{2x^2}\right]_{-1}^{-\epsilon} + \lim_{\delta\to 0+}\left[-\frac{1}{2x^2}\right]_{\delta}^{1}$$

$$= \lim_{\epsilon\to 0+}\left\{\frac{1}{2}-\frac{1}{2\epsilon^2}\right\} + \lim_{\delta\to 0+}\left\{-\frac{1}{2}+\frac{1}{2\delta^2}\right\}$$

Now since $\lim_{\epsilon\to 0+}\frac{1}{2\epsilon^2}$ and $\lim_{\delta\to 0+}\frac{1}{2\delta^2}$ do not exist, the general integral does not exist. If however, we consider Cauchy principal value, we can find

$$\lim_{\epsilon\to 0+}\left[\int_{-1}^{-\epsilon}\frac{1}{x^3}\,dx + \int_{\epsilon}^{1}\frac{1}{x^3}\,dx\right] = \lim_{\epsilon\to 0}\left\{\left(\frac{1}{2}-\frac{1}{2\epsilon^2}\right)+\left(-\frac{1}{2}+\frac{1}{2\epsilon^2}\right)\right\}$$

$$=0.$$

An useful Theorem in Evaluating Improper Integral.

If (i) $f(x)$ be bounded and integrable in $0 < x \le a$ and tends to ∞ only when $x \to 0+$ or $f(x)$ is bounded and integrable in $0 \le x < a$ and tends to ∞ only when $x \to a-$ and (ii) $\int_0^a f(x)\,dx$ converges, then

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

Example

9. Assuming the integrals to be convergent show that

$$\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = \pi/2 \log 1/2.$$

Sol^n^: The only singularity is at $x=0$. The integral has been assumed to be convergent, hence by

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx,$$

we have

$$I = \int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\sin(\pi/2 - x)\,dx$$

$$= \int_0^{\pi/2} \log\cos x\,dx.$$

$$\therefore\ 2I = \int_0^{\pi/2} (\log\sin x + \log\cos x)\,dx = \int_0^{\pi/2} \log\left(\tfrac{1}{2}\sin 2x\right)dx$$

(addition is valid, since both the integrals are convergent)

$$= \int_0^{\pi/2} \log 1/2\,dx + \int_0^{\pi/2} \log\sin 2x\,dx$$

$$= \pi/2 \log 1/2 + \int_0^{\pi/2} \log\sin 2x\,dx.$$

Now since $\log \frac{1}{2}$ is convergent, let us find the convergent integral

$$\int_0^{\pi/2} \log \sin 2x \, dx.$$

Actually we are to calculate $\int_{\epsilon}^{\pi/2-\delta} \log \sin 2x \, dx$, when $\epsilon, \delta \to 0^+$

Put $2x = z$, then it becomes $\frac{1}{2}\int_{2\epsilon}^{\pi-2\delta} \log \sin z \, dz$ when $\epsilon, \delta \to 0^+$

Thus,

$$\int_0^{\pi/2} \log \sin 2x \, dx = \frac{1}{2}\int_0^{\pi} \log \sin x \, dx.$$

$$\therefore \; 2I = \pi/2 \log \tfrac{1}{2} + \frac{1}{2}\int_0^{\pi} \log \sin x \, dx$$

$$= \pi/2 \log \tfrac{1}{2} + \frac{1}{2}\cdot 2\int_0^{\pi/2} \log \sin x \, dx \quad (*)$$

$$= \pi/2 \log \tfrac{1}{2} + I$$

$$\Rightarrow I = \pi/2 \log \tfrac{1}{2}.$$

Some More Results :

(*) 1. $\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$, when $f(x) \to \infty$ as $x \to 0+$ or $f(x) \to \infty$ as $x \to a-$, $f(x)$ being bounded and integrable in $0 < x \le a$ in the first case and in $0 \le x < a$ in the second case provided $\int_0^a f(x)\,dx$ converges.

2. $\int_a^b \{f(x) \pm \phi(x)\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b \phi(x)\,dx$, provided any two of the integrals be convergent.

However, equation (2), is not always true if only one of the three integrals be convergent. Thus $\int_2^{\infty}\left(-\frac{1}{x}\right)dx$ and $\int_2^{\infty}\frac{dx}{x-1}$ are divergent, but $\int_2^{\infty}\left(\frac{1}{x-1}-\frac{1}{x}\right)dx$ is convergent.

Example:

10. Evaluate: $\displaystyle\int_0^\infty \log\left(x+\frac{1}{x}\right)\cdot\frac{dx}{1+x^2}$.

Solⁿ: The singularities exist at both ends. Hence we write

$$I = \int_0^\infty \log\left(x+\frac{1}{x}\right)\cdot\frac{dx}{1+x^2}$$

$$= \int_0^1 \log\left(x+\frac{1}{x}\right)\cdot\frac{dx}{1+x^2} + \int_1^\infty \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}.$$

provided both integrals at the right be convergent.

Thus we are only to calculate

$$\lim_{\epsilon\to 0^+}\int_\epsilon^1 \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2} + \lim_{B\to\infty}\int_1^B \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}$$

Putting $x = \tan\theta$,

$$= -\lim_{\epsilon\to 0^+}\int_{\tan^{-1}\epsilon}^{\pi/4}\left(\log\sin\theta+\log\cos\theta\right)d\theta$$

$$-\lim_{B\to\infty}\int_{\pi/4}^{\tan^{-1}B}\left(\log\sin\theta+\log\cos\theta\right)d\theta$$

$$= -\int_0^{\pi/4}\left(\log\sin\theta+\log\cos\theta\right)d\theta - \int_{\pi/4}^{\pi/2}\left(\log(\sin\theta)+\log\cos\theta\right)d\theta$$

(both the integrals are convergent)

$$= -\int_0^{\pi/2}\left(\log\sin\theta+\log\cos\theta\right)d\theta$$

$$= -2\cdot\frac{\pi}{2}\cdot\log\frac{1}{2} = \pi\log 2. \quad \text{(by Example 9.)}$$

11. Show that $\int_0^{\pi/2} \sin x \log \sin x \, dx$ converges and find its val[ue]

Sol. The only singularity is at $x=0$. Now,

$$\int_\epsilon^{\pi/2} (\log \sin x) \sin x \, dx$$
$$= \left[-\cos x \log \sin x\right]_\epsilon^{\pi/2} - \int_\epsilon^{\pi/2} (\sin x - \operatorname{cosec} x) \, dx$$
$$= \cos\epsilon \log \sin\epsilon - \int_\epsilon^{\pi/2} (\sin x - \operatorname{cosec} x) \, dx$$
$$= \cos\epsilon \log \sin\epsilon + \left[\cos x + \log \tan \frac{x}{2}\right]_\epsilon^{\pi/2}$$
$$= \cos\epsilon \log \sin\epsilon - \cos\epsilon - \log \tan \epsilon/2$$
$$\to \log 2 - 1 \text{ as } \epsilon \to 0^+$$

Since $\lim_{\epsilon \to 0+} \left(\cos\epsilon \log \sin\epsilon - \cos\epsilon - \log \tan \frac{\epsilon}{2}\right)$

[Writing $\sin\epsilon = 2\sin\epsilon/2 \cos\epsilon/2$, $\tan\epsilon/2 = \sin\epsilon/2 / \cos\epsilon/2$]

$$= \lim_{\epsilon \to 0+} \left\{ (\cos\epsilon - 1)\log \sin \epsilon/2 + \cos\epsilon \log 2\cos\epsilon/2 + \log \cos\epsilon/2 - \cos\epsilon \right\}$$

and $\lim_{\epsilon \to 0+} (\cos\epsilon - 1)\log \sin\epsilon/2$

$$= \lim_{\epsilon \to 0+} \frac{\log \sin \epsilon/2}{-1/2 \operatorname{cosec}^2 \epsilon/2} \quad \left(\frac{\infty}{\infty}, \text{ by L'Hospital's Rule}\right)$$

$$= \lim_{\epsilon \to 0^+} \frac{1}{2} \sin^2 \epsilon/2 = 0$$

And also, $\lim_{\epsilon \to 0+} \left\{ \cos\epsilon \log 2\cos\epsilon/2 + \log \cos\epsilon/2 - \cos\epsilon \right\}$

$$= \log 2 - 1$$

$\therefore$ The integral converges and its value $= \log 2 - 1$.

Problems :

1. Evaluate : $\int_a^\infty \frac{dx}{x^n}$, $(a>0)$ (Ans. $\frac{1}{(n-1)a^{n-1}}$; $n>1$
∞ ; $n<1$
∞ ; $n=1$)

2. Evaluate : $\int_a^b \frac{dx}{(x-a)^n}$, $(n>0)$

3. Evaluate : $\int_a^b \frac{dx}{(b-x)^n}$, $(n>0)$

4. Examine the convergence of the following integrals and if possible evaluate them,

(i) $\int_0^2 \frac{dx}{x(2-x)}$ (Ans. dvg)

(ii) $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ (Ans. π)

(iii) $\int_{-\infty}^\infty x e^{-x^2} dx$ (Ans. 0)

(iv) $\int_1^\infty \frac{dx}{x^2(x+1)}$ (Ans. $1-\log 2$)

(v) $\int_0^{1/e} \frac{dx}{x(\log x)^2}$ (Ans. 1)

(vi) $\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}\, dx$ (Ans. π)