5.2 PLANE, STRAIGHT LINE AND SPHERE

5.2.1. Introduction.

In the previous chapter we discussed three dimensional co-ordinate system where the position of a point as well as the position of a vector was determined in space by means of three dimensional co-ordinates. These led us to study the equation of straight line and the equation of several surfaces in the space. In this chapter we give the equation of Plane, Straight line and Sphere in three dimensional space with the help of vector. Two types of equation of these geometrical items are established. One is Cartesian equation and another is Vector equation.

5.2.2. Direction Cosines of a Straight line.

If a straight line makes angles α, β and γ with the positive direction of X axis, Y axis and Z axis in three dimensional co-ordinate system then $\cos\alpha, \cos\beta, \cos\gamma$ are called the *Direction Cosines (d.c.) of the line.*

An important property of the direction cosines is

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Note. The direction cosines of X axis is $\cos 0^0, \cos 90^0$ and $\cos 90^0$ *i.e.,* 1, 0, 0.

Similarly the d.c of Y axis is 0, 1, 0 and those of Z axis is 0, 0, 1.

Direction Ratio of a Straight line

Let g be a straight line in space. $\vec{\alpha}$ be a vector parallel to g having co-ordinate (a, b, c). Then a, b, c (or, (a,b,c)) are called *Direction Ratio (d.r.) of the line g.*

Obviously direction ratio of a line is not unique.

Note.(1). Since the vector $\vec{i} = (1, 0, 0)$ is along X axis so the direction ratio of X axis is 1, 0, 0. Similarly the direction ratio of Y axis and Z axis are 0, 1, 0 and 0, 0, 1.

Relation between two sets of direction ratio of a line

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are direction ratios of a straight line then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

(2) Direction Ratio of a line joininig two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ = co-ordinate of $\vec{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Proof. Since (a_1, b_1, c_1) is a direction ratio of a line g (say) so there exists a vector $\vec{\alpha}$ parallel to g having co-ordinate (a_1, b_1, c_1). Similarly $\vec{\beta}$ is another vector parallel to g having co-ordinate (a_2, b_2, c_2). Since both of $\vec{\alpha}$ and $\vec{\beta}$ are parallel to g so $\vec{\alpha}$ and $\vec{\beta}$ are parallel.

Hence $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ (Using Remark (3), page 5-6)

Relation between direction cosines and direction ratio of a line.

The direction cosines and the direction ratio of a straight line are proportional.

Proof. Let $\cos\alpha, \cos\beta, \cos\gamma$ be the direction cosines and (a, b, c) be the direction ratio of a line g. Then there exists a vector $\vec{\rho}$ collinear with g whose co-ordinate is (a, b, c).

Then using relation (1) of Art 5.1.4 we get $a = |\vec{\rho}|\cos\alpha$, $b = |\vec{\rho}|\cos\beta$ and $c = |\vec{\rho}|\cos\gamma$ *i.e.,* $\dfrac{a}{\cos\alpha} = |\vec{\rho}|, \dfrac{b}{\cos\beta} = |\vec{\rho}|$ and $\dfrac{c}{\cos\gamma} = |\vec{\rho}|$

These give $\dfrac{a}{\cos\alpha} = \dfrac{b}{\cos\beta} = \dfrac{c}{\cos\gamma}$. Hence proved.

Corollary. The above property tells that every direction cosines can be regarded as direction ratio though the converse is not true.

Condition of Parallelism and Perpendicularity.

Let g_1 and g_2 be two straight lines having direction ratios l_1, m_1, n_1 and l_2, m_2, n_2.

(i) g_1 and g_2 are parallel if and only if $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$.

(ii) g_1 and g_2 are perpendicular to each other if and only if $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

Proof. The vector $\vec{\alpha} = (l_1, m_1, n_1)$ is parallel to g_1 and $\vec{\beta} = (l_2, m_2, n_2)$ is parallel to g_2.

(i) Since g_1 and g_2 are parallel so $\vec{\alpha}$ and $\vec{\beta}$ are parallel.

$$\therefore \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

(ii) Since g_1 and g_2 are perpendicular so $\vec{\alpha}$ and $\vec{\beta}$ are perpendicular to each other. Hence $\vec{\alpha}.\vec{\beta} = 0$

or, $(l_1, m_1, n_1).(l_2, m_2, n_2) = 0$

or, $l_1l_2 + m_1m_2 + n_1n_2 = 0$

5.2.3. Illustrative Examples.

Ex. 1. *Show that the triangle formed by the points* $(-1,6,6),(-4,9,6)$ *and* $(0,7,10)$ *is isosceles right-angled.*

Let $A = (-1,6,6)$, $B = (-4,9,6)$, $C = (0,7,10)$.

$$\therefore AB = \sqrt{(-1+4)^2 + (6-9)^2 + (6-6)^2} = 3\sqrt{2}$$

$$BC = \sqrt{(0+4)^2 + (7-9)^2 + (10-6)^2} = 6$$

$$CA = \sqrt{(0+1)^2 + (7-6)^2 + (10-6)^2} = 3\sqrt{2}$$

$\therefore AB = CA$ and $AB^2 + CA^2 = BC^2$.

So the triangle formed by the given points is isosceles right angled.

Ex. 2. *A straight line makes* 60°, 60° *with y and z-axes respectively. What angle does it make with x-axis?*

Let the straight line makes an angle α with x-axis. So the d.c's of the straight line is $\cos\alpha$, $\cos 60^\circ$, $\cos 60^\circ$

i.e., $\cos\alpha, \frac{1}{2}, \frac{1}{2}$

$$\therefore \cos^2\alpha + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$$

or, $\cos^2\alpha = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$

or, $\cos\alpha = \frac{1}{\sqrt{2}}$ (assuming α acute) $\therefore \alpha = 45^\circ$

So the straight line makes an angle 45° with x-axes.

Ex. 3. *Find the d.c.'s of a line which is perpendicular to the lines whose direction ratios are* $-1, 1, 2$ *and* $1, 2, 3$.

Let l, m, n be the d.c.'s of a line which is perpendicular to the given lines whose direction ratios are $-1, 1, 2$ and $1, 2, 3$.

$\therefore -l + m + 2n = 0$ and $l + 2m + 3n = 0$.

Doing cross-multiplication between the two we get,

$$\frac{l}{3-4} = \frac{m}{2+3} = \frac{n}{-2-1} \quad i.e., \ \frac{l}{-1} = \frac{m}{5} = \frac{n}{-3}$$

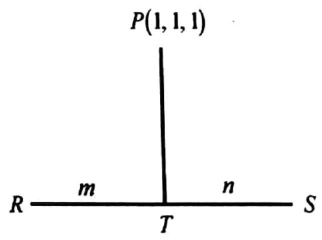
i.e., $$\frac{l}{1} = \frac{m}{-5} = \frac{n}{3} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+(-5)^2+3^2}} = \frac{1}{\sqrt{35}}$$

$$\therefore l = \frac{1}{\sqrt{35}}, \ m = \frac{-5}{\sqrt{35}}, \ n = \frac{3}{\sqrt{35}}.$$

So the d.c.'s of the line are $\frac{1}{\sqrt{35}}, \frac{-5}{\sqrt{35}}, \frac{3}{\sqrt{35}}$.

Ex. 4. *Find the co-ordinates of the foot of the perpendicular from* $P(1,1,1)$ *on the line joining* $R(1,4,6)$ *and* $S(5,4,4)$.

Let the foot of the perpendicular T divides RS in the ratio $m : n$.



Then the co-ordinates of T are

$$\left(\frac{5m+n}{m+n}, \frac{4m+4n}{m+n}, \frac{4m+6n}{m+n}\right).$$

So the direction ratios of PT = co-ordinate of $\vec{PT}$

$$= \left(\frac{5m+n}{m+n} - 1, \ \frac{4m+4n}{m+n} - 1, \ \frac{4m+6n}{m+n} - 1\right)$$

i.e., d.r of PT are $\frac{4m}{m+n}, \frac{3m+3n}{m+n}, \frac{3m+5n}{m+n}$

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Also the direction ratio of RS are $5-1,\ 4-4,\ 4-6$ *i.e.,* $4,\ 0,\ -2$

As PT is perpendicular to RS, so

$$\frac{4m}{m+n}\times 4+\frac{3m+3n}{m+n}\times 0+\frac{3m+5n}{m+n}\times(-2)=0$$

or, $16m-6m-10n=0$

or, $10m=10n\ \therefore m=n$.

So the foot of perpendicular T is

$$\left(\frac{5n+n}{n+n},\frac{4n+4n}{n+n},\frac{4n+6n}{n+n}\right)\quad i.e.,\ (3,4,5).$$

Ex. 5. *A, B, C and D are points of $(\alpha, 3, -1)$, $(3, 5, -3)$, $(1, 2, 3)$ and $(3,5,7)$ respectively. If AB is perpendicular to CD, then find the value of α.* [W.B.U.Tech.2006]

The direction ratios of the line AB are $3-\alpha,\ 5-3,\ -3+1$ *i.e.,* $3-\alpha,\ 2,\ -2$ and that of CD are $3-1,\ 5-2,\ 7-3$ *i.e.,* $2,3,4$. Since AB is perpendicular to CD, we have

$$(3-\alpha)\times 2+2\times 3+(-2)\times 4=0$$

or, $6-2\alpha+6-8=0$

or, $2\alpha=4\ \therefore\ \alpha=2$

Ex. 6. *If α, β, γ are the angles which a line makes with the co-ordinate axes, prove that* $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$. [W.B.U.Tech.2005]

Since α, β, γ are the angles which a line makes with the co-ordinate axes, so we must have

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

i.e., $$1-\sin^2\alpha+1-\sin^2\beta+1-\sin^2\gamma=1.$$

$$\therefore \sin^2\alpha+\sin^2\beta+\sin^2\gamma=2.$$

5.2.4. Cartesian Equation of a Plane in Space.

Let π be a plane in space and $P(x, y, z)$ be a variable point on π. As P moves on π the values of x, y and z get changed but there would have an algebraic relation among them. This relation is called Cartesian Equation of the plane π.

Illustration. Suppose π be a plane which passes through the three fixed points $(3, 3, 1)$, $(-3, 2, -1)$ and $(8, 6, 3)$. If $P(x, y, z)$ be the running point on π then it can be shown that the relation among x, y and z would be $4x+2y-13z=5$.

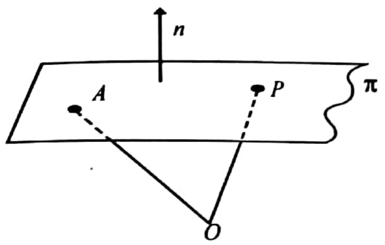
So this equation is the cartesian equation of the plane π.

Determination of the Cartesian Equation of a Plane.

Form 1: The equation of plane passing through the point (x_0, y_0, z_0) and having a normal with direction ratio a, b, c is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Proof. Let $A=(x_0, y_0, z_0)$ be fixed point on the plane π. $\vec{n}$ is the vector parallel to the normal of π having co-ordinate $\vec{n}=(a,b,c)$. Let $P(x, y, z)$ be the variable point on π.



Then $\vec{OP}=(x-0, y-0, z-0)=(x, y, z)$

and $\vec{OA}=(x_0-0, y_0-0, z_0-0)=(x_0, y_0, z_0)$

Now, $\vec{OA}+\vec{AP}=\vec{OP}$

or, $\vec{AP}=\vec{OP}-\vec{OA}=(x, y, z)-(x_0, y_0, z_0)$

$=(x-x_0, y-y_0, z-z_0)$

Since the vector $\vec{n}$ is perpendicular on π so it is perpendicular on the vector $\vec{AP}$ lying on π.

So, $\vec{n}.\vec{AP}=0$ or, $(a,b,c).(x-x_0, y-y_0, z-z_0)=0$

or, $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$.

Form 2: (General form of Plane)

The general equation of a plane is $ax+by+cz+d=0$ where (a, b, c) is the direction ratio of a normal to the plane.

Proof. The equation obtained in the above form is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

or, $ax+by+cz+(-ax_0-by_0-cz_0)=0$

or, $ax+by+cz+d=0$, putting $d=-ax_0-by_0-cz_0$.

Form 3: The equation to the plane passing through the three given points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0$$

Proof. Omitted

Form 4: (Intercept form)

If a plane cuts off intercepts a, b, c from X, Y and Z axes respectively then the equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Proof. The plane cuts X axis at $(a, 0, 0)$, Y axis at $(0, b, 0)$ and Z axis at $(0, 0, c)$. Therefore the equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix}$$

Expanding the determinant we shall get the equation as $bcx+acy+abz=abc$

or, $$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Form 5: (Normal form)

If $\cos\alpha, \cos\beta, \cos\gamma$ are direction cosines of the normal of a plane whose perpendicular distance from origin O is p then the equation of the plane is

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p$$

Proof. **Omitted.**

Corollary.

(1) The equation to the plane passing through the origin is $ax+by+cz=0$.

Proof. Let $ax+by+cz+d=0$ be the equation of the plane. Since it passes through $(0, 0, 0)$ so

$$a\times 0+b\times 0+c\times 0+d=0$$

or, $d=0$

$\therefore$ The equation becomes $ax+by+cz=0$.

(2) The equation to the xy **plane,** yz **plane and** zx **plane are** $z=0, x=0$ **and** $y=0$.

Proof. xy plane means XOY co-ordinate plane which passes through origin and whose normal is z-axis. The direction ratio of z-axis is $(0, 0, 1)$. Therefore the equation of XOY plane is $ox+oy+1z=0$ or, $z=0$.

Similarly the other two equations can be derived.

(3) The equation to the plane parallel to xy **plane is** $z=k$; **the equation to the plane parallel to** yz **plane is** $x=k$ **and the equation to the plane parallel to** zx **plane is** $y=k$

Proof. z-axis is a normal to the plane parallel to xy plane. So the direction ratio of the normal to such plane is $(0, 0, 1)$. Therefore the equation of this plane is

$$ox+oy+1z+d=0$$

or, $z=-d$ or, $z=k$ (putting $-d=k$)

(4) Equation of the plane parallel to x**-axis is** $by+cz+d=0$. **Equation to the plane parallel to** y**-axis is** $ax+cz+d=0$ **and equation to the plane parallel to** z**-axis is** $ax+by+d=0$

Proof. **Omitted.**

(5) Equation to the plane parallel to the plane $ax+by+cz+d=0$ **is** $ax+by+cz+k=0$.

Proof. Since two parallel planes have same normal the result is obvious.

(6) If the two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are perpendicular to each other then $a_1a_2+b_1b_2+c_1c_2=0$.

Proof. The vector $\vec{n_1}=(a_1,b_1,c_1)$ is perpendicular to the plane $a_1x+b_1y+c_1z+d_1=0$ and the vector $\vec{n_2}=(a_2,b_2,c_2)$ is perpendicular to the plane $a_2x+b_2y+c_2z+d_2=0$. Since the two planes are perpendicular to each other so $\vec{n_1}$ and $\vec{n_2}$ are also perpendicular to each other.

So $\vec{n_1}.\vec{n_2}=0$ or, $a_1a_2+b_1b_2+c_1c_2=0$

(7) The equation of the plane passing through the line of intersection of the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is

$(a_1x+b_1y+c_1z+d_1)+k(a_2x+b_2y+c_2z+d_2)=0$ where $k(\neq 0)$ is a real number.

Proof. Obvious.

Illustration.

(i) The equation of a plane passing through the point (–1, 6, 5) is $a(x+1)+b(y-6)+c(z-5)=0$.

(ii) The plane $x-5z=3$ is $x-5z-3=0$ which is parallel to y-axis.

(iii) The plane $4x+3y-2z-13=0$ can be written as

$$\frac{x}{\frac{13}{4}}+\frac{y}{\frac{13}{3}}+\frac{z}{-\frac{13}{2}}=1.$$

So this plane cuts X axis at $\left(\frac{13}{4},0,0\right)$, Y axis at $\left(0,\frac{13}{3},0\right)$ and Z axis $\left(0,0,-\frac{13}{2}\right)$.

(iv) $3\vec{i}-\vec{j}+2\vec{k}$ is a vector perpendicular to the plane $3x-y+2z=2$.

Distance of a point from a plane.

The perpendicular distance of the point (x_1,y_1,z_1) from the plane $ax+by+cz+d=0$ is

$$\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

Proof. Beyond the scope of the book.

Illustration. The perpendicular distance of the point (–3, 0, 2) from the plane $5x+y+9z-6=0$ is $\left|\frac{5(-3)+0+9\times 2-6}{\sqrt{5^2+1^2+9^2}}\right|=\frac{3}{\sqrt{107}}$

Equation of the planes bisecting the angle between two planes.

The equation of the planes bisecting the angles between the planes

$$a_1x+b_1y+c_1z+d_1=0$$

and $a_2x+b_2y+c_2z+d_2=0$

are $$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}.$$

of these two bisecting planes, one bisects the acute angle and the other bisects the obtuse angle between the given two planes.

If d_1,d_2 are of same sign, the bisector obtained by taking (+) sign on the RHS bisects the angle containing the origin and so the other bisector bisects the angle between the given planes which does not contain the origin.

Illustration.

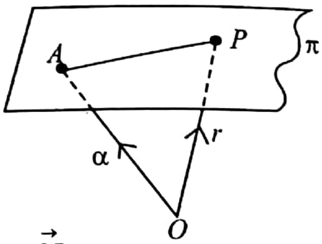
The bisector of the angle between the planes $3x+2y-6z=2$ and $x+2y+2z=5$ are $\frac{3x+2y-6z-2}{\sqrt{9+4+36}}=\pm\frac{x+2y+2z-5}{\sqrt{1+4+4}}$

Taking + sign we get $2x-8y-32z+29=0$ which bisects the angle containing origin (as the two constant terms –2 and –5 are of same sign); taking – sign we get $16x+20y-4z-41=0$ which bisects the angle which does not contain origin.

5.2.5. Vector Equation of a plane in space.

Let π be a plane in space and P be a variable point on π having position vector $\vec{r}$. As P moves on π the vector $\vec{r}$ get changed but it would satisfy some vector equation. This equation is called Vector Equation of the plane π.

Illustration. Let A be a point on the plane π whose position vector is $\vec{\alpha}=3\vec{i}+2\vec{j}-\vec{k}$. The vector $\vec{\gamma}=\vec{i}+\vec{j}+\vec{k}$ be perpendicular on π. Let P be any point on the plane having position vector $\vec{r}$.



Now, $\vec{OA}+\vec{AP}=\vec{OP}$

or, $\vec{AP}=\vec{r}-\vec{\alpha}$.

Now $\vec{\gamma}$ is perpendicular on π so it is perpendicular to the vector $\vec{AP}$.

Hence $\vec{\gamma}.\vec{AP}=0$

or, $\left(\vec{i}+\vec{j}+\vec{k}\right).\left(\vec{r}-\vec{\alpha}\right)=0$

or, $\left(\vec{i}+\vec{j}+\vec{k}\right).\vec{r}-\left(\vec{i}+\vec{j}+\vec{k}\right).\left(3\vec{i}+2\vec{j}-\vec{k}\right)=0$

or, $\left(\vec{i}+\vec{j}+\vec{k}\right).\vec{r}-(3+2-1)=0$

or, $\left(\vec{i}+\vec{j}+\vec{k}\right).\vec{r}=4$

Since this relation is satisfied by any point P, so this is the vector equation of the plane π.

Vector Equation of a plane passing through a given point and parallel to two given non-parallel vectors.

The vector equation of a plane which passes through a point whose position vector is $\vec{\alpha}$ and which is parallel to two non-parallel vectors $\vec{\beta}$ and $\vec{\gamma}$ is $\vec{r}=\vec{\alpha}+s\vec{\beta}+t\vec{\gamma}$

Proof. Let π be the plane, A be the point having position vector $\vec{\alpha}$. Without loss of generality we may assume $\vec{\beta}$ and $\vec{\gamma}$ lie on π. Let P be arbitrary point on π having position vector $\vec{r}$. O be origin.

Now, $\vec{OA}+\vec{AP}=\vec{OP}$

or, $\vec{\alpha}+\vec{AP}=\vec{r}$ ... (1)

Now since $\vec{AP}$ is coplanar with $\vec{\beta}$ and $\vec{\gamma}$ so we can express $\vec{AP}$ as $\vec{AP}=s\vec{\beta}+t\vec{\gamma}$ where s, t are independent scalars.

Then from (1) we get

$$\vec{\alpha}+s\vec{\beta}+t\vec{\gamma}=\vec{r}$$

or, $\vec{r}=\vec{\alpha}+s\vec{\beta}+t\vec{\gamma}$

Vector Equation of a plane passing through three given non-collinear points.

The vector equation of a plane passing through three non-collinear points having position vectors $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ is

$$\vec{r}=(1-s-t)\vec{\alpha}+s\vec{\beta}+t\vec{\gamma}$$

Proof. Let A, B, C be the three points on the plane having position vectors $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ respectively. Let O be origin and P be arbitrary point on the plane having position vector $\vec{r}$

Now, $\vec{OA}+\vec{AB}=\vec{OB}$

or, $\vec{\alpha}+\vec{AB}=\vec{\beta}$

or, $\vec{AB}=\vec{\beta}-\vec{\alpha}$

Simlarly $\vec{AC}=\vec{\gamma}-\vec{\alpha}$.

Using the previous result we get the equation of the plane as

$\vec{r}=\vec{\alpha}+s\vec{AB}+t\vec{AC}$

or, $\vec{r}=\vec{\alpha}+s\left(\vec{\beta}-\vec{\alpha}\right)+t\left(\vec{\gamma}-\vec{\alpha}\right)$

or, $\vec{r}=(1-s-t)\vec{\alpha}+s\vec{\beta}+t\vec{\gamma}$

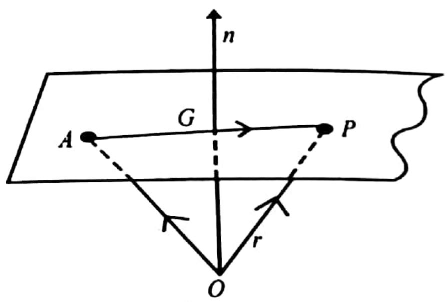
Vector Equation of a plane if a unit normal to the plane and the perpendicular distance of origin from the plane is given.

Let $\vec{n}$ be the unit vector perpendicular to the plane and p be the perpendicular distance of the plane from origin. Then the vector equation of the plane is $\vec{r}.\vec{n} = p$

Proof. A be a point on the plane, O be origin. Take P be arbitrary point on the plane having position vector $\vec{r}$.

Now, $\vec{OA} + \vec{AP} = \vec{OP}$

or, $\vec{AP} = \vec{r} - \vec{OA}$



Since the vector $\vec{n}$ is perpendicular to the plane so it is perpendicular to the vector $\vec{AP}$.

$\therefore \vec{AP}.\vec{n} = 0$

or, $\left(\vec{r} - \vec{OA}\right).\vec{n} = 0$

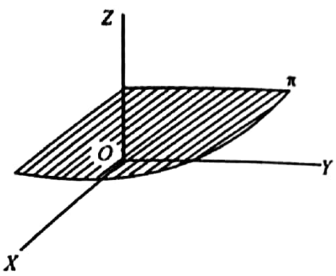
or, $\vec{r}.\vec{n} = \vec{OA}.\vec{n}$

= Projection of $\vec{OA}$ on the line of $\vec{n}$

$= OG = p$

$\therefore \vec{r}.\vec{n} = p$

Illustration. Consider the plane π which is parallel to the co-ordinate plane XOY. Let its distance from origin is 5 unit. The unit vector $\vec{k}$ is perpendicular to π. Then the vector equation of is $\vec{r}.\vec{k} = 5$



5.2.6. Illustrative Examples.

Ex. 1. *Find the equation of a plane through the points* $(3, 3, 1)$, $(-3, 2, -1)$ *and* $(8, 6, 3)$.

The cartesian equation of the plane passing through the given points is given by

$$\begin{vmatrix} x & y & z & 1 \\ 3 & 3 & 1 & 1 \\ -3 & 2 & -1 & 1 \\ 8 & 6 & 3 & 1 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} x-3 & y-3 & z-1 & 0 \\ 6 & 1 & 2 & 0 \\ -11 & -4 & -4 & 0 \\ 8 & 6 & 3 & 1 \end{vmatrix} = 0,$$ using row operations

or, $$\begin{vmatrix} x-3 & y-3 & z-1 \\ 6 & 1 & 2 \\ -11 & -4 & -4 \end{vmatrix} = 0$$

or, $(x-3)(-4+8)-(y-3)(-24+22)+(z-1)(-24+11)=0$

or, $4(x-3)+2(y-3)-13(z-1)=0$

$\therefore 4x+2y-13z=5$

Ex. 2. *Find the equation of a plane through* $(5,-2,7)$ *and parallel to the plane* $3x+4y-7z=6$.

Let the equation of the plane parallel to the given plane be $3x+4y-7z=k$ which passes through the point $(5,-2,7)$.

$\therefore 3\cdot5+4(-2)-7\cdot7=k \quad \therefore k=-42$.

So the required equation of the plane is $3x+4y-7z+42=0$.

Ex. 3. *Find the equation of a plane through the point $(5,1,4)$ and intersects the positive x and y-axes at distances 3, 2 units respectively.*

Let the equation of a plane which intersects the positive x and y-axes at distances 3, 2 units respectively be $\frac{x}{3}+\frac{y}{2}+\frac{z}{c}=1$.

This plane passes through the point $(5,1,4)$.

$\therefore \frac{5}{3}+\frac{1}{2}+\frac{4}{c}=1$

or, $\frac{4}{c}=-\frac{7}{6}$

$\therefore c=\frac{-24}{7}$.

So the required equation of the plane is $\frac{x}{3}+\frac{y}{2}-\frac{7z}{24}=1$.

$\therefore 8x+12y-7z=24$.

Ex. 4. *Find the equation of the plane which passes through the point $(-4,2,3)$ and normal to the line joining the points $(6,2,-4)$ and $(3,-4,1)$.*

The co-ordinate of the vector joining the points $(6,2,-4)$ and $(3,-4,1)$ is $(6-3,2+4,-4-1)$ *i.e.,* $(3,6,-5)$. Thus the direction ratios of the normal to the plane is $(3,6,-5)$.

So let the equation of the plane be $3x+6y-5z+d=0$, which passes through the point $(-4,2,3)$.

$\therefore 3(-4)+6\cdot 2-5(3)+d=0$

$\therefore d=15$.

Thus the required equation of the plane is $3x+6y-5z+15=0$.

Ex. 5. *Find the direction cosines of the normal to the plane $2x+4y+5z-20=0$. Also find the length of the perpendicular from the origin to the plane.*

Let the normal form of the plane $2x+4y+5z-20=0$ be $x\cos\alpha+y\cos\beta+z\cos\gamma=p$

Comparing the two we get

$$\frac{\cos\alpha}{2}=\frac{\cos\beta}{4}=\frac{\cos\gamma}{5}=\frac{p}{20}$$

$\therefore \cos\alpha=\frac{p}{10};\cos\beta=\frac{p}{5}$ and $\cos\gamma=\frac{p}{4}$

We know $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

or, $\frac{p^2}{100}+\frac{p^2}{25}+\frac{p^2}{16}=1$ or, $p^2=\frac{80}{9}$ or, $p=\frac{4\sqrt{5}}{3}$

$\therefore \cos\alpha=\frac{1}{10}\times\frac{4\sqrt{5}}{3}=\frac{2\sqrt{5}}{15}$,

$\cos\beta=\frac{1}{5}\times\frac{4\sqrt{5}}{3}=\frac{4\sqrt{5}}{15}$; $\cos\gamma=\frac{1}{4}\times\frac{4\sqrt{5}}{3}=\frac{\sqrt{5}}{3}$

$\therefore$ Direction cosines of the normal to the plane are $\frac{2\sqrt{5}}{15},\frac{4\sqrt{5}}{15},\frac{\sqrt{5}}{3}$.

Length of perpendicular from origin to the plane is $\frac{4\sqrt{5}}{3}$.

Ex. 6. *A variable plane moves so that the sum of reciprocal of its intercepts on the three co-ordinates axes is constant. Prove that the plane always passes through a fixed point.*

Let the equation of the variable plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ... (1)

which intercepts on the axes are a, b, c.

Now by the given condition we have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=$ constant $=\frac{1}{k}$ (say).

$\therefore \frac{k}{a}+\frac{k}{b}+\frac{k}{c}=1$ ... (2)

In virtue of (2), we can say that the plane (1) always passes through a point (k,k,k) which is a fixed point.

Ex. 7. *Show that the equation of the plane which passes through the point $(-1,1,-4)$ and perpendicular to each of the planes $-2x+y+z+2=0$, $x+y-3z+1=0$ is $4x+5y+3z+11=0$.*

Let the equation of the plane through the point $(-1,1,-4)$ be $a(x+1)+b(y-1)+c(z+4)=0$ which is perpendicular to the given planes.

$\therefore -2a+b+c=0$ and $a+b-3c=0$

By cross multiplication, we get

$$\frac{a}{-3-1}=\frac{b}{1-6}=\frac{c}{-2-1}=k \text{ (say)}$$

$\therefore a=-4k$, $b=-5k$, $c=-3k$

So the required equation of the plane is

$-4k(x+1)-5k(y-1)-3k(z+4)=0$

$\therefore 4x+5y+3z+11=0$.

Ex. 8. *Show that the equation of the planes parallel to the plane $2x-2y-z=3$ and at a distance 7 units from it are $2x-2y-z-24=0$ and $2x-2y-z+18=0$.*

The distance of the plane $2x-2y-z=3$

from the origin is $\dfrac{3}{\sqrt{4+4+1}}=1$. ... (1)

Let the equation of the plane parallel to (1) be

$2x-2y-z=k$ ... (2)

whose distance from the origin is

$$\frac{k}{\sqrt{4+4+1}}=\frac{k}{3}.$$

By the given condition

$\dfrac{k}{3}-1=\pm 7$

i.e., $\dfrac{k}{3}=8$ or, -6

$\therefore k=24$ or, -18

Thus the required equation of the planes are

$2x-2y-z=24$ and $2x-2y-z=-18$

i.e., $2x-2y-z+18=0$.

Ex. 9. *A variable plane is at a constant distance p from the origin and meets co-ordinate axes in A, B and C. The planes are drawn through A, B and C and parallel to co-ordinate axes. Show that the locus of their point of intersection shall be*

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$$ [W.B.U.Tech.2006]

Let the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{... (1)}$$

which meets co-ordinate axes in A, B, C.

$\therefore A\equiv(a,0,0),\ B\equiv(0,b,0),\ C\equiv(0,0,c)$.

Since (1) is at a constant distance p from the origin, so we must have

$$p=\frac{1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$

$$\therefore \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2} \quad \text{... (2)}$$

Now the planes drawn through A, B and C and parallel to co-ordinate axes are

$x=a,\ y=b,\ z=c$ respectively.

The point of intersection of these planes is (α,β,γ) where

$\alpha=a,\ \beta=b,\ \gamma=c$

Hence, from (2), $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\gamma^2}=\dfrac{1}{p^2}$

Thus the required locus of the point of intersection is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$$

Ex. 10. *Find the equation of the plane which contains the line of intersection of the planes $2x-3y+z=1$ and $x+y-4z=5$ and is perpendicular to the plane $4x+9y+z=-2$*

Let the equation of the plane passing through the planes $2x-3y+z=1$ and $x+y-4z=5$ be

$(2x-3y+z-1)+k(x+y-4z-5)=0$

i.e., $(2+k)x+(k-3)y+(1-4k)z-(5k+1)=0$

which is perpendicular to the plane $4x+9y+z=-2$

$\therefore 4(2+k)+9(k-3)+1\cdot(1-4k)=0$

or, $9k=18 \therefore k=2$

So the required equation of the plane is

$(2+2)x+(2-3)y+(1-4\times 2)z-(5\times 2+1)=0$

$\therefore 4x-y-7z=11$

Ex. 11. *A variable plane is at constant distance p from the origin and meets the axes at A, B, C. Show that the locus of the centroid of the tetrahedron OABC is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$*

Let the equation of the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

which meets the axes in A, B, C. (1)

$\therefore\ A\equiv(a,0,0),\ B\equiv(0,b,0),\ C\equiv(0,0,c)$.

Also $O\equiv(0,0,0)$.

$$\text{Now, } p=\frac{1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}} \quad (2)$$

Let (α,β,γ) be the centroid of the tetrahedron $OABC$.

Then $\alpha=\dfrac{a+0+0+0}{4}$

$\therefore\ a=4\alpha$

Similarly $b=4\beta$, $c=4\gamma$.

$$\therefore \text{ From (2), } p=\frac{1}{\sqrt{\left(\frac{1}{4\alpha}\right)^2+\left(\frac{1}{4\beta}\right)^2+\left(\frac{1}{4\gamma}\right)^2}}$$

or, $$\frac{1}{(4\alpha)^2}+\frac{1}{(4\beta)^2}+\frac{1}{(4\gamma)^2}=\frac{1}{p^2}$$

$$\therefore\ \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{16}{p^2}$$

Thus the required locus is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$$

i.e., $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$

Ex. 12. *The plane $x+y-z=5$ is rotated through a right angle about its line of intersection with the plane $3x+y+7z=12$. Find the equation of the plane in its new position.*

Since the required plane passes through the line of intersection of the given planes ...

$$x+y-z=5 \quad \ldots$$

and, $3x+y+7z=12$

so let its equation be

$$(x+y-z-5)+k(3x+y+7z-12)=0.$$

or, $(1+3k)x+(1+k)y+(-1+7k)z-(5+12k)=0$. ... (3)

Now (1) is perpendicular to the plane (3),

$\therefore\ (1+3k)\cdot 1+(1+k)\cdot 1+(-1+7k)(-1)=0$ or, $k=1$

$\therefore$ From (3), we get the required equation of the plane in its new position as

$$(1+3\cdot 1)x+(1+1)y+(-1+7\cdot 1)z-(5+12\cdot 1)=0.$$

$\therefore\ 4x+2y+6z=17$

Ex. 13. *Find the bisector of the angle containing the origin between the planes $2x+2y-z-11=0$ and $3x+2y-6z+3=0$*

The given equation of the planes are

$$2x+2y-z-11=0$$

and, $$-3x-2y+6z-3=0$$

So the equation of the planes bisecting the angles between the given planes are

$$\frac{2x+2y-z-11}{\sqrt{4+4+1}}=\pm\frac{-3x-2y+6z-3}{\sqrt{9+4+36}}$$

or, $$\frac{2x+2y-z-11}{3}=\pm\frac{-3x-2y+6z-3}{7}$$

or, $14x+14y-7z-77=\pm(-9x-6y+18z-9)$

Taking +ve sign we get

$$23x+20y-25z-68=0$$

Taking –ve sign we get

$$5x+8y+11z-86=0.$$

Thus the equation of the plane bisecting the angle containing the origin is $23x+20y-25z-68=0$.

Ex. 14. *Find the points on y-axis which are equidistant from the two planes $3x-y+5z=5$ and $2x+4y-z+2=0$.*

Let $(0,\alpha,0)$ be any point on y-axis which is equidistant from the two given planes.

$$\therefore \frac{3\cdot 0-\alpha+5\cdot 0-5}{\sqrt{9+1+25}}=\pm\frac{2\cdot 0+4\cdot\alpha-0+2}{\sqrt{4+16+1}}.$$

or, $\dfrac{-\alpha-5}{\sqrt{35}}=\pm\dfrac{4\alpha+2}{\sqrt{21}}$

or, $\dfrac{\alpha+5}{\sqrt{5}}=\mp\dfrac{4\alpha+2}{\sqrt{3}}$

or, $\alpha\sqrt{3}+5\sqrt{3}=\mp\left(4\sqrt{5}\,\alpha+2\sqrt{5}\right)$

$$\therefore \alpha=\frac{-\left(2\sqrt{5}+5\sqrt{3}\right)}{\sqrt{3}+4\sqrt{5}}=\frac{-\left(2\sqrt{15}+15-40-20\sqrt{15}\right)}{3-80}$$

$$=\frac{-\left(-18\sqrt{15}-25\right)}{-77}=-\frac{1}{77}\left(18\sqrt{15}+25\right)$$

and $\alpha=\dfrac{2\sqrt{5}-5\sqrt{3}}{\sqrt{3}-4\sqrt{5}}=\dfrac{2\sqrt{15}+40-15-20\sqrt{15}}{3-80}$

$$=\frac{-18\sqrt{15}+25}{-77}=\frac{1}{77}\left(18\sqrt{15}-25\right).$$

Thus the required points are

$$\left(0,-\frac{18\sqrt{15}+25}{77},0\right),\left(0,\frac{18\sqrt{15}-25}{77},0\right).$$

Ex. 15. *Obtain the vector equation of the plane passing thourgh the three non-collinear points (2,2,4), (1, –2, 1) and (–1, 1, 2).*

The position vectors of the three points are

$\vec{\alpha}=(2,2,4)$, $\vec{\beta}=(1,-2,1)$ and $\vec{\gamma}=(-1,1,2)$

∴ the vector equation of the required plane is

$$\vec{r}=(1-s-t)\vec{\alpha}+s\vec{\beta}+t\vec{\gamma}$$

or, $\vec{r}=(1-s-t)(2,2,4)+s(1,-2,1)+t(-1,1,2)$

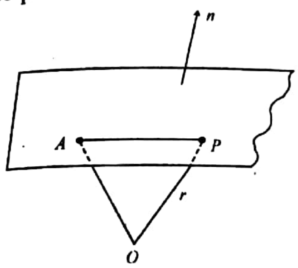
$$=(2-2s-2t+s-t,2-2s-2t-2s+t,4-4s-4t+s+2t)$$

or, $\vec{r}=(2-s-3t,2-4s-t,4-3s-2t)$

where s, t are scalars.

Ex. 16. *Find the vector equation of the plane through the point (4,3,–1) and perpendicular to the vector* $3\vec{i}-4\vec{j}+\vec{k}$.

Let $A(4,3,-1)$ lies on the plane. O be origin. P be arbitrary point on the plane whose position vector is $\vec{r}$.



Now, $\vec{OA}+\vec{AP}=\vec{OP}$

or, $\vec{AP}=\vec{r}-\vec{OA}=\vec{r}-(4,3,-1)$

Obvious $\vec{AP}$ and the vector $\vec{n}=3\vec{i}-4\vec{j}+\vec{k}$ are perpendicular to each other.

$\therefore\ \vec{AP}\cdot\vec{n}=0$

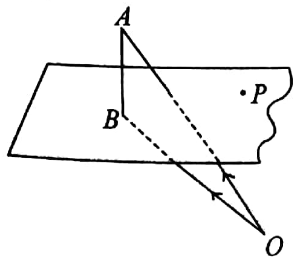
or, $\left\{\vec{r}-(4,3,-1)\right\}\cdot(3,-4,1)=0$

or, $(3,-4,1)\cdot\vec{r}-(4,3,-1)\cdot(3,-4,1)=0$

or, $(3,-4,1)\cdot\vec{r}-(12-12-1)=0$

or, $(3,-4,1)\cdot\vec{r}=-1$ which is the required eqution of the plane.

Ex. 17. *The position vectors of two points A and B are* $3\vec{i}+\vec{j}+2\vec{k}$ *and* $\vec{i}-2\vec{j}-4\vec{k}$ *respectively. Find the equation of the plane through B and perpendicular to AB.*



Let P be arbitrary point on the plane whose position vector is $\vec{r}$.

Now, $\vec{OP}+\vec{PB}=\vec{OB}$ (O is origin)

or, $\vec{PB}=\vec{OB}-\vec{OP}=\vec{i}-2\vec{j}-4\vec{k}-\vec{r}$

Now, $\vec{BA}=(3i+j+2k)-(i-2j-4k)$

$=2i+3j+6k$

Obviously $\vec{PB}$ is perpendicular with $\vec{BA}$

$\therefore \vec{PB}\cdot\vec{BA}=0$

or, $\left(i-2j-4k-\vec{r}\right)\cdot(2i+3j+6k)=0$

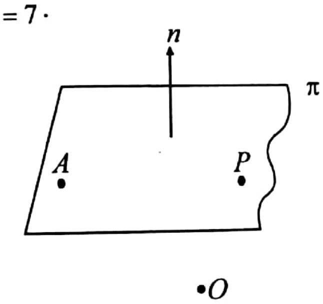
or, $(i-2j-4k)\cdot(2i+3j+6k)=\vec{r}\cdot(2i+3j+6k)$

or, $2-6-24=\vec{r}\cdot(2i+3j+6k)$

or, $\vec{r}\cdot(2i+3j+6k)=-28$ which is the required equation.

Ex. 18. *Find the vector equation of the plane through the point $2\vec{i}-\vec{j}-4\vec{k}$ and parallel to the plane $\vec{r}\cdot\left(4\vec{i}-12\vec{j}-3\vec{k}\right)=7$.*

The vector $\vec{n}=4\vec{i}-12\vec{j}-3\vec{k}$ is perpendicular to the plane $\vec{r}\cdot(4\vec{i}-12\vec{j}-3\vec{k})=7$.



So $\vec{n}$ is perpendicular to the required plane π (say).

Let A be the point on π whose position vector $\vec{OA}=2i-j-4k$.

P be arbitrary point on the plane. Now $\vec{OA}+\vec{AP}=\vec{OP}$

or, $\vec{AP}=\vec{r}-(2i-j-4k)$

Obviously $\vec{AP}$ and $\vec{n}$ are perpendicular to each other.

$\therefore \vec{AP}\cdot\vec{n}=0$

or, $\left\{\vec{r}-(2i-j-4k)\right\}\cdot(4i-12j-3k)=0$

or, $\vec{r}(4i-12j-3k)=(2i-j-4k)\cdot(4i-12j-3k)$

or, $\vec{r}\cdot(4i-12j-3k)=8+12+12$

or, $\vec{r}\cdot(4i-12j-3k)=32$ which is the required equation of the plane.

Ex. 19. *Find the intercepts on the co-ordinate axes made by the plane $\vec{r}\cdot(3i-2j+k)=7$.*

If $P(x,y,z)$ is on the plane whose position vector is $\vec{r}$,

$\vec{r}=(x-0,y-0,z-0)=(x,y,z)$

Then the equation of the plane becomes

$(x,y,z)\cdot(3,-2,1)=7$

or, $3x-2y+z=7$

or, $\dfrac{x}{7/3}+\dfrac{y}{7/-2}+\dfrac{z}{7}=1$

$\therefore$ the intercepts on X axis, Y axis and Z axis are $\dfrac{7}{3},-\dfrac{7}{2}$ and 7 repectively

Ex. 20. *Find the angle between the planes*

$\vec{r}\cdot(2i+3j+k)=7$ *and* $\vec{r}\cdot(3i-2j+5k)=5$

The vector $\vec{\alpha}=2i+3j+k$ is normal to the first plane and the vector $\vec{\beta}=3i-2j+5k$ is normal to the second.

So the angle between the two planes is nothing but the angle between $\vec{\alpha}$ and $\vec{\beta}$.

If θ be the angle then,

$\vec{\alpha}\cdot\vec{\beta}=\cos\theta\left|\vec{\alpha}\right|\left|\vec{\beta}\right|$

or, $2\times3+3\times(-2)+1\times5=\cos\theta\sqrt{2^2+3^2+1^2}\cdot\sqrt{3^2+(-2)^2+5^2}$

or, $6-6+5=\cos\theta\sqrt{14}\sqrt{38}$

or, $\cos\theta = \dfrac{5}{\sqrt{14}\sqrt{38}}$

or, $\theta = \cos^{-1}\dfrac{5}{\sqrt{14}\sqrt{38}}$ is the required angle.

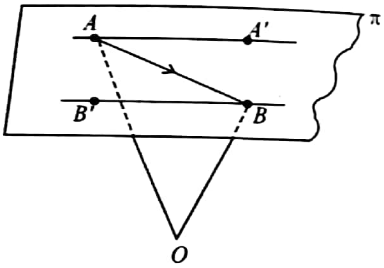
Ex. 21. *Find the equation of the plane through the two parallel straight lies whose vector equations are $\vec{r} = \vec{r}_1 + s\vec{\alpha}$ and $\vec{r} = \vec{r}_2 + t\vec{\alpha}$ where s, t are scalars.*

Let the two parallel straight lines be AA' and BB'.

Equation of AA' is $\vec{r} = \vec{r}_1 + s\vec{\alpha}$ and equation of BB' is $\vec{r} = \vec{r}_2 + t\vec{\alpha}$.

Let the position vector of A be $\vec{r}_1$ and position vector of B be $\vec{r}_2$. O is origin.

Now, $\vec{OA} + \vec{AB} = \vec{OB}$ or, $\vec{AB} = \vec{r}_2 - \vec{r}_1$



Let π be the required plane. Then π passes through A whose position vector is $\vec{r}_1$. π is parallel to $\vec{\alpha}$ [$\because AA'$ is parallel to $\vec{\alpha}$] and $\vec{AB}$.

$\therefore$ The vector equation of the plane is

$$\vec{r} = \vec{r}_1 + s\vec{\alpha} + t\,\vec{AB}$$

or, $\vec{r} = \vec{r}_1 + s\vec{\alpha} + t\left(\vec{r}_2 - \vec{r}_1\right)$.

Ex. 22. *Find the equation of the plane which contains the line $\vec{r} = t\vec{\alpha}$ and is perpendicular to the plane containing the lines $\vec{r} = t_1\vec{\beta}$ and $\vec{r} = t_2\vec{\gamma}$ where t_1, t_2 are scalars; $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are given vectors.*

$\vec{r} = t\vec{\alpha}$ can be arranged as $\vec{r} = 0 + t\vec{\alpha}$

$\therefore$ the origin lies on the line and the line is parallel to the vector $\vec{\alpha}$.

On the other hand the normal to the given plane is $\vec{\beta}\times\vec{\gamma}$, i.e. the required plane is parallel to $\vec{\beta}\times\vec{\gamma}$. Thus the required plane passes through the point whose position vector is O and parallel to the two vectors $\vec{\alpha}$ and $\vec{\beta}\times\vec{\gamma}$.

$\therefore$ the equation of the plane is

$$\vec{r} = O + s\vec{\alpha} + t\left(\vec{\beta}\times\vec{\gamma}\right)$$

or, $\vec{r} = s\vec{\alpha} + t\left(\vec{\beta}\times\vec{\gamma}\right)$

5.2.7. Cartesian Equation of a straight line in space.

Let g be a straight line in space and $P(x, y, z)$ be a variable point on g. As P moves on g the values of x, y, and z get changed but there would have an algebraic relation among them. This relation is called cartesian equation of the straight g.

Illustration : Suppose g be a straight line passing through the two fixed points $A(8, 2, -7)$ and $B(11, 0, 2)$. If $P(x, y, z)$ be the running point on g then it can be shown that whenever P goes on g the relation among x, y and z would be

$$\frac{x-8}{3} = \frac{y-2}{-2} = \frac{z+7}{9}$$

So this equation is the cartesian equation of the straight line g.

Determination of Cartesian Equation of a straight line.

Form 1. The equation of the straight line passing through the point (x_1, y_1, z_1) and having direction ratio a, b, c is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}.$$

Proof. Let $A(x_1, y_1, z_1)$ be the fixed point on the line g (say). $P(x, y, z)$ be running point on g. Then $\vec{AP} = (x-x_1, y-y_1, z-z_1)$ is a vector along g.

So $\vec{PA}$ and $\vec{\alpha}$ are parallel.

Hence $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$.

Form 2. The equation of the straight line passing through the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$$

Proof. If $A=(x_1, y_1, z_1)$ and $B=(x_2, y_2, z_2)$

then $\vec{BA}=(x_1-x_2, y_1-y_2, z_1-z_2)$ is along the line.

So $(x_1-x_2, y_1-y_2, z_1-z_2)$ is direction ratio of the line. Hence the equation of the line is $\dfrac{x-x_1}{x_1-x_2}=\dfrac{y-y_1}{y_1-y_2}=\dfrac{z-z_1}{z_1-z_2}$ (using Form1)

Note. The equation of X axis, Y axis, Z axis are respectively $\dfrac{x-0}{1}=\dfrac{y-0}{0}=\dfrac{z-0}{0}$ i.e., $\dfrac{x}{1}=\dfrac{y}{0}=\dfrac{z}{0}$, $\dfrac{x}{0}=\dfrac{y}{1}=\dfrac{z}{0}$ and $\dfrac{x}{0}=\dfrac{y}{0}=\dfrac{z}{1}$ respectively.

Illustration. The equation of a straight line through the point $(5, 3, -2)$ and having direction cosines $\dfrac{3}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, -\dfrac{1}{\sqrt{14}}$ is

$$\frac{x-5}{\frac{3}{\sqrt{14}}}=\frac{y-3}{\frac{2}{\sqrt{14}}}=\frac{z+2}{-\frac{1}{\sqrt{14}}}$$

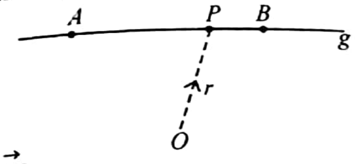
or, $\dfrac{x-5}{3}=\dfrac{y-3}{2}=\dfrac{z+2}{-1}$

5.2.8. Vector Equation of a Straight line in space.

Let g be a straight line in space and P be a variable point on g having position vector $\vec{r}$. As P moves on g the vector $\vec{r}$ gets changed but it would satisfy some vector equation. This equation is called vector equation of the straight line g.

Illustration. Let g be a straight line on which A and B are two fixed points having co-ordinate (1, 3, 0) and (–2,1, 0) respectively.

Let P be running point on g whose position vector is $\vec{r}$.



Now $\vec{OA}+\vec{AP}=\vec{OP}$

or, $\vec{AP}=\vec{r}-\vec{OA}$,

where $\vec{OA}=(1-0, 3-0, 0-0)$

$=(1, 3, 0)$

Now, $\vec{AB}=\lambda\,\vec{AP}$ where λ is a scalar depending on P.

or, $(-2-1, 1-3, 0-0)=\lambda\left(\vec{r}-\vec{OA}\right)$

or, $(-3, -2, 0)=\lambda\left(\vec{r}-(1,3,0)\right)$ which is the vector equation of the line g.

Vector Equation of a line passing through a given point and parallel to a given vector.

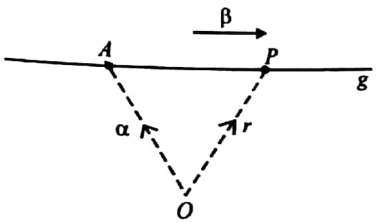
If a line passes through a fixed point having position vector $\vec{\alpha}$ and parallel to the vector $\vec{\beta}$ then the vector equation of the line is $\vec{r}=\vec{\alpha}+t\,\vec{\beta}$

Proof. Let g be the line and A be the fixed point whose position vector is $\vec{\alpha}$.

P be the running point on g whose position vector is $\vec{r}$.

Now, $\vec{OA}+\vec{AP}=\vec{OP}$ ... (1)

Since the two vectors $\vec{AP}$ and $\vec{\beta}$ are parallel so $\vec{AP}=t\,\vec{\beta}$ where t is a scalar whose value depends on P.



Then from (1) we get $\vec{\alpha}+t\vec{\beta}=\vec{r}$

or, $\vec{r}=\vec{\alpha}+t\vec{\beta}$

which is the vector equation of the line g.

Illustration. Vector equation of a straight line passing through the origin and parallel to the vector $2\vec{i}+\vec{j}+\vec{k}$ will be

$\vec{r}=\vec{O}+t\vec{\beta}$ where $\vec{O}$ is null vector and $\vec{\beta}=2\vec{i}+\vec{j}+\vec{k}$

That is $\vec{r}=t\left(2\vec{i}+\vec{j}+\vec{k}\right)$

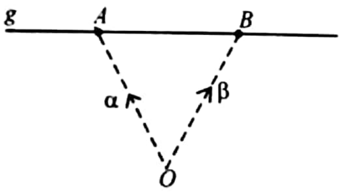
or, $\vec{r}=2t\,\vec{i}+t\,\vec{j}+t\,\vec{k}$

Vector Equation of a straight line passing through two points whose position vectors are given.

If $\vec{\alpha}$ and $\vec{\beta}$ be the position vectors of two fixed points on a line then the vector equation of the line is

$\vec{r}=(1-t)\vec{\alpha}+t\vec{\beta}$ where t is a scalar.

Proof. Let g be the line; A and B be the points on g whose position vectors are $\vec{\alpha}$ and $\vec{\beta}$.



Now, $\vec{OA}+\vec{AB}=\vec{OB}$

or, $\vec{AB}=\vec{\beta}-\vec{\alpha}$.

$\therefore$ the vector $\vec{\beta}-\vec{\alpha}$ is collinear with g and we may suppose g parallel to $\vec{\beta}-\vec{\alpha}$. So using the previous result we can write the equation of g as

$\vec{r}=\vec{\alpha}+t\left(\vec{\beta}-\vec{\alpha}\right)$

or, $\vec{r}=(1-t)\vec{\alpha}+t\vec{\beta}$

Illutration. The vector equation of the line on which two points having position vector $\vec{i}+\vec{k}$ and $2\vec{i}-\vec{j}+3\vec{k}$ lie is

$$\vec{r}=(1-t)\left(\vec{i}+\vec{k}\right)+t\left(2\vec{i}-\vec{j}+3\vec{k}\right)$$

or, $\vec{r}=(1-t+2t)\vec{i}-t\vec{j}+(1-t+3t)\vec{k}$

or, $\vec{r}=(1+t)\vec{i}-t\vec{j}+(1+2t)\vec{k}$

where t is a scalar depending on $\vec{r}$.

5.2.9 Illustrative Examples.

Ex. 1. *Find the equation of a st. line through the point $(-2,3,5)$ and making $60^\circ, 45^\circ$ with the positive direction of x-, y-axis.*

Let the st. line makes an angle γ with the positive direction of z-axes. Then the d.c's of the line is

$\cos 60^\circ, \cos 45^\circ, \cos\gamma$

i.e. $\frac{1}{2}, \frac{1}{\sqrt{2}}, \cos\gamma$

$$\therefore \left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\cos^2\gamma=1$$

or, $\cos^2\gamma=\frac{1}{4}$

$\therefore \cos\gamma=\frac{1}{2}$ (assuming γ as acute angle)

So the d.c's of the line are $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$ (which are also regarded as d.r) and passes through the point $(-2,3,5)$. Thus the equation of the line is

$$\frac{x+2}{\frac{1}{2}}=\frac{y-3}{\frac{1}{\sqrt{2}}}=\frac{z-5}{\frac{1}{2}}$$

i.e
$$\frac{x+2}{1}=\frac{y-3}{\sqrt{2}}=\frac{z-5}{1}.$$

Ex. 2. *Show that the equation of a line passing through $(-2,5,6)$ and perpendicular to the plane $x-5y+2z-9=0$ is $\frac{x+2}{1}=\frac{y-5}{-5}=\frac{z-6}{2}$.*

As the st. line is perpendicular to the plane $x-5y+2z-9=0$, so the direction ratios of the line are $1,-5,2$. Also the line passes through $(-2,5,6)$. So the equation of the required line is $\frac{x+2}{1}=\frac{y-5}{-5}=\frac{z-6}{2}$.

Ex. 3. *Find the point where the line joining the points $(-1,3,5)$ and $(5,0,7)$ cuts the plane $2x-y+z+15=0$.*

The equation of the line joining the points $(-1,3,5)$ and $(5,0,7)$ is

$$\frac{x+1}{5+1}=\frac{y-3}{0-3}=\frac{z-5}{7-5}$$

i.e $\frac{x+1}{6}=\frac{y-3}{-3}=\frac{z-5}{2}=r$ (say) ... (1)

$\therefore x=6r-1,\ y=-3r+3,\ z=2r+5$

So the co-ordinate of any point on (1) is $(6r-1,-3r+3,2r+5)$.

If this point lies on the plane $2x-y+z+15=0$,

then $2(6r-1)-(-3r+3)+(2r+5)+15=0$

or, $17r=-15$

$\therefore r=-15/17$

So the required point is $\left(-107/17, 96/17, 55/17\right)$.

Ex. 4. *Find the distance of the point $(1,-2,3)$ from the plane $x-y+z=5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$.*

The equation of a st. line passing through the point $(1,-2,3)$ and parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$ is

$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}=r$ (say) ... (1)

Then the co-ordinates of any point on (1) are

$$(2r+1,3r-2,-6r+3). \quad ... (2)$$

If this point lies on the plane $x-y+z=5$,
then $2r+1-(3r-2)+(-6r+3)=5$

or, $-7r=-1$

$\therefore r=\frac{1}{7}$.

So the point of intersection of (1) and (2) is

$\left(2\cdot\frac{1}{7}+1, 3\cdot\frac{1}{7}-2, -6\cdot\frac{1}{7}+3\right)$ i.e $\left(\frac{9}{7},\frac{-11}{7},\frac{15}{7}\right)$.

Hence the required distance is the distance between the points

$(1,-2,3)$ and $\left(\frac{9}{7},\frac{-11}{7},\frac{15}{7}\right)$ and is given by

$$\sqrt{\left(\frac{9}{7}-1\right)^2+\left(\frac{-11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2}$$

$$=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1.$$

Ex. 5. *Show that the straight lines $\frac{x+1}{3}=\frac{y+3}{2}=\frac{z}{5}$ and $2x+y-3z=5$, $3x+2y+5z=8$ are at right angles.*

Let l, m, n be the direction ratios of the st. line given by $2x+y-3z=5,\ 3x+2y+5z=8$.

$\therefore 2l+m-3n=0$

$3l+2m+5n=0$

By cross multiplication, we get $\frac{l}{5+6}=\frac{m}{-9-10}=\frac{n}{4-3}$

i.e $\frac{l}{11}=\frac{m}{-19}=\frac{n}{1}$

So the direction ratios of the 2nd st. line are $11,-19,1$ and that of the 1st st. line are $3,2,5$

Now $11\times3+(-19)\times2+1\times5=0$.

Hence the given st. lines are at right angles.

Ex. 6. *Find the equation of the plane through the line $\frac{x-1}{3}=\frac{y-4}{2}=\frac{z-4}{-2}$ and parallel to the line $\frac{x+1}{2}=\frac{y-1}{-4}=\frac{z+2}{1}$.*

Obviously the plane passes through $(1,4,4)$

So the equation of the plane be

$a(x-1)+b(y-4)+c(z-4)=0$ ... (1)

where $3a+2b-2c=0$ ... (2)

as the vector (a,b,c) and (3,2,–2) are per perpendicular to each other

As the plane (1) is parallel to the line

$\frac{x+1}{2}=\frac{y-1}{-4}=\frac{z+2}{1}$, ... (3)

so the normal of the plane (1) is perpendicular to the line (3).

$\therefore 2a-4b+c=0$ ... (4)

From (2) and (4), we have $\frac{a}{2-8}=\frac{b}{-4-3}=\frac{c}{-12-4}$

i.e $\frac{a}{6}=\frac{b}{7}=\frac{c}{16}=k$ (say)

$\therefore a=6k$, $b=7k$, $c=16k$.

$\therefore$ From (1), we get $6k(x-1)+7k(y-4)+16k(z-4)=0$

$\therefore 6x+7y+16z=98$

which is the required equation of the plane.

Ex. 7. *Show that the lines*

$$\frac{x+1}{-3}=\frac{y-3}{2}=z+2 \text{ and } x=\frac{y-7}{-3}=\frac{z+7}{2} \text{ intersect.}$$

Find the co-ordinates of the point of intersection and the equation to the plane containing them.

Let the co-ordinates of any point on the st. line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ be $(-3r-1,2r+3,r-2)$, r being the parameter.

If this pt. lies on the second line $x=\frac{y-7}{-3}=\frac{z+7}{2}$,

then $\frac{-3r-1}{1}=\frac{2r+3-7}{-3}=\frac{r-2+7}{2}$

i.e $-3r-1=\frac{2r-4}{-3}=\frac{r+5}{2}$ ... (1)

From first two members, we get $9r+3=2r-4$ i.e $r=-1$ and from last two members we get $4r-8=-3r-15$ i.e $r=-1$.

Thus relation (1) holds for $r=-1$. Hence the two given lines intersect and point of intersection is

$(-3(-1)-1,2(-1)+3,(-1)-2)$

i.e $(2,1,-3)$.

Again the equation of the plane containing the given st. lines is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}=0$$

i.e $(x+1)(4+3)-(y-3)(-6-1)+(z+2)(9-2)=0$

$\therefore x+y+z=0$.

Ex. 8. *Obtain the equation of the plane through straight line $3x-4y+5z-10=0, 2x+2y-3z-4=0$ and parallel to the line $x=2y=3z$* [W.B.U.Tech 2006]

Let the equation of the plane through the given straight line be

$(3x-4y+5z-10)+\lambda(2x+2y-3z-4)=0$

or, $(3+2\lambda)x+(2\lambda-4)y+(5-3\lambda)z-(4\lambda+10)=0$ (1)

which is parallel to the line $x=2y=3z$

i.e. $\frac{x}{6}=\frac{y}{3}=\frac{z}{2}$

Obviously the two vector $(3+2\lambda,2\lambda-4,5-3\lambda)$ and (6,3,2) are perpendicular to each other

$\therefore 6(3+2\lambda)+3(2\lambda-4)+2(5-3\lambda)=0$

or, $18+12\lambda+6\lambda-12+10-6\lambda=0$

or, $12\lambda=-16$

$\therefore \lambda=-4/3$

$\therefore$ From (1), the required equation of the plane is

$\left(3-\frac{8}{3}\right)x+\left(-\frac{8}{3}-4\right)y+(5+4)z-\left(-\frac{16}{3}+10\right)=0$

$\therefore x-20y+27z-14=0$

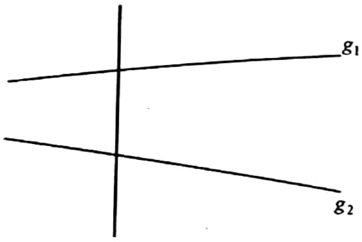
Ex. 9. *A straight line with direction ratios 2,7 and −5 is drawn to intersect the lines* $\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}$ *and* $\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}$. *Find the co-ordinates of the point of intersection and length intercepted on it.* [W.B.U.Tech.2005]

Let

$$g_1:\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}$$

$$g_2:\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}$$

be the two lines intersected by the line g having d.r. 2,7,−5. Let (x_1,y_1,z_1) be the point of intersection of g and g_1.



The equations of g is $\frac{x-x_1}{2}=\frac{y-y_1}{7}=\frac{z-z_1}{-5}$ (1)

Now the three vectors (x_1+3,y_1-3,z_1-6), $(2,7,-5)$ and $(-3,2,4)$ are collinear.

$$\text{so}\begin{vmatrix} x_1+3 & y_1-3 & z_1-6 \\ 2 & 7 & -5 \\ -3 & 2 & 4 \end{vmatrix}=0$$

or, $38x_1+7y_1+25z_1=57$ ... (2)

Since (x_1,y_1,z_1) lies on g_1 so $\frac{x_1-5}{3}=\frac{y_1-7}{-1}=\frac{z_1+2}{1}=k$ (say)

i.e. $x_1=5+3k,\ y_1=7-k$ and $z_1=-2+k$

Putting these in (2) we get

$$38(5+3k)+7(7-k)+25(-2+k)=57$$

or, $k=-1$.

$\therefore x_1=2,\ y_1=8,\ z_1=-3$.

Thus g and g_2 becomes

$$g:\frac{x-2}{2}=\frac{y-8}{7}=\frac{z+3}{-5} \quad \dots (3)$$

$$\text{and } g_2:\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4} \quad \dots (4)$$

From the first two of (3) and (4) we get respectively

$$7x-2y=-2$$

$$2x+3y=3$$

Solving we get $x=0,\ y=1$

Putting this in (3) or (4) we get $z=2$

Hence the two points of intersections are $(2,8,-3)$ and $(0,1,2)$.

The length of intercept = distance between the two points

$$\sqrt{(2-0)^2+(8-1)^2+(-3-2)^2}=\sqrt{78}$$

Ex. 10. *Find the vector equation of a straight line passing through the origin and parallel to the vector* $2i+j+k$.

Position vector of origin is null vector 0. Since the line is parallel to the vector $\vec{\beta}=2i+j+k$ so the vector equation of the line is

$\vec{r}=0+t\vec{\beta}$

or, $\vec{r}=t(2i+j+k)$

Ex. 11. *Find the vector equation of a straight line passing through the point whose position vector is* $3i-j+k$ *and parallel to the vector* $2i+j-4k$

Let $\vec{\alpha}=3i-j+k$ and $\vec{\beta}=2i+j-4k$

Then the vector equation of the required straight line is

$\vec{r}=\vec{\alpha}+t\vec{\beta}$

or, $\vec{r}=(3i-j+k)+t(2i+j-4k)$

or, $\vec{r}=(3+2t)i+(-1+t)j+(1-4t)k$

Ex. 12. *Find the vector equation of a straight line passing through the points* $i+j+k$ *and* $3i+2j-k$.

Let $\vec{\alpha} = i + j + k$ and $\vec{\beta} = 3i + 2j - k$

$\therefore$ the vector equation of the required line is

$$\vec{r} = (1-t)\vec{\alpha} + t\vec{\beta}$$
$$= (1-t)(i+j+k) + t(3i+2j-k)$$
$$= (1-t+3t)i + (1-t+2t)j + (1-t-t)k$$

or, $\vec{r} = (1+2t)\vec{i} + (1+t)\vec{j} + (1-2t)\vec{k}$

5.2.10. Cartesian Equation of a Sphere

Let $P(x, y, z)$ be a variable point on a sphere. As P moves on the sphere x, y and z get changed but there would have an algebraic relation among them. This relation is called cartesian equation of the sphere.

Determination of the Cartesian Equation of Sphere.

The equation of a sphere having centre at (x_0, y_0, z_0) and radius r is

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$$

Proof. Let $C(x_0, y_0, z_0)$ be the centre and $P(x, y, z)$ be variable point on the sphere. Then $\vec{CP} = (x-x_0, y-y_0, z-z_0)$. Since the radius is r so $\left|\vec{CP}\right| = r$

or, $\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = r$

or, $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$

General Form of Sphere

Simplifying the above equation of the sphere it is seen that the equation of sphere takes the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (1)$$

where u, v, w and d are constants.

Now (1) can be arranged as

$$(x+u)^2 + (y+v)^2 + (z+w)^2 = \left(\sqrt{u^2+v^2+w^2-d}\right)$$

Illustration. The equation $x^2 + y^2 + z^2 - x - 7y + 5z + 14 = 0$ may represent a sphere.

Here $2u = -1, 2v = -7, 2w = 5$ and $d = 14$

i.e. $u = -\frac{1}{2}, v = -\frac{7}{2}, w = \frac{5}{2}$

$\therefore$ its centre $= \left(\frac{1}{2}, \frac{7}{2}, -\frac{5}{2}\right)$

and radius $= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{7}{2}\right)^2 + \left(\frac{5}{2}\right)^2 - 14}$

$$= \sqrt{\frac{1}{4} + \frac{49}{4} + \frac{25}{4} - 14} = \frac{\sqrt{19}}{2}.$$

Equation of a sphere passing through a given circle.

The section of a sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ by any plane

$ax + by + cz + d' = 0$ is a circle.

The equation of a sphere passing through this circle is

$$(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d) + \lambda(ax + by + cz + d') = 0$$

where λ is a constant.

Illustration. The equation of any sphere through the circle $x^2 + y^2 + z^2 - x + 2y + 9z - 1 = 0$ and $2x + y - 3z = 5$ is

$$(x^2 + y^2 + z^2 - x + 2y + 9z - 1) + \lambda(2x + y - 3z - 5) = 0$$

If this sphere passes through the point $(2, 0, 1)$ then

$$(2^2 + 0^2 + 1^2 - 2 + 2\times 0 + 9\times 1 - 1) + \lambda(2\times 2 + 0 - 3\times 1 - 5) = 0$$

or, $\lambda = \frac{11}{4}$

Then the equation of the sphere passing through the given circle becomes

$$x^2 + y^2 + z^2 - x + 2y + 9z - 1 + \frac{11}{4}(2x + y - 3z - 5) = 0$$

or, $4x^2 + 4y^2 + 4z^2 - 4x + 8y + 36z - 4 + 22x + 11y - 33z - 55 = 0.$

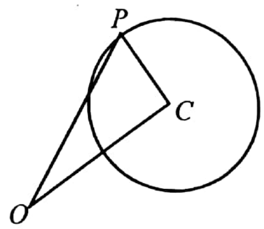
5.2.11 Vector Equation of a Sphere

Let P be a variable point on the sphere having position vector $\vec{r}$. As P moves on the sphere the vector $\vec{r}$ gets changed but it would satisfy some vector equation. This equation is called vector equation of the sphere.

Vector Equation of a Sphere whose Centre and Radius are given

If $\vec{c}$ be the position vector of the centre of a sphere and a be radius then the vector equation of the sphere is $\left(\vec{r}-\vec{c}\right)^2 = a^2$

Proof. Let O be origin, C be centre, P be arbitrary point on the sphere whose position vector is $\vec{r}$. Then $\vec{OC}=\vec{c}$ and $\vec{OP}=\vec{r}$



Now $\vec{OC}+\vec{CP}=\vec{OP}$

or $\vec{CP}=\vec{r}-\vec{c}$

or, $\vec{r}-\vec{c}=\vec{CP}$

or $\left|\vec{r}-\vec{c}\right|=\left|\vec{CP}\right|$

or, $\left|\vec{r}-\vec{c}\right|=a$ (since radius is a)

or $\left|\vec{r}-\vec{c}\right|^2=a^2$

$\left(\vec{r}-\vec{c}\right)^2=a^2$

Corollary. If the centre is origin then $\vec{c}$ is null vector O. Then the vector equation of the sphere becomes $\vec{r}^{\,2}=a^2$

5.2.12 Illustrative Examples.

Ex. 1. *Find the centre and radius of the sphere* $3x^2+3y^2+3z^2+x+6y-7z+6=0$.

The given equation can be written as

$x^2+y^2+z^2+\frac{1}{3}x+2y-\frac{7}{3}z+2=0$.

So the centre of the sphere is at $\left(-\frac{1}{6},-1,\frac{7}{6}\right)$ and radius

$=\sqrt{\left(-\frac{1}{6}\right)^2+(-1)^2+\left(\frac{7}{6}\right)^2-2}$

$=\sqrt{\frac{1}{36}+1+\frac{49}{36}-2}=\frac{\sqrt{14}}{6}$

Ex. 2. *The plane* $2x+3y+4z=12$ *meets the co-ordinate axes at P, Q, R. Find the equation of the sphere which passes through the point O, P, Q, R. Also determine the centre and the radius of the sphere.*

The given equation of the plane can be written as $\frac{x}{6}+\frac{y}{4}+\frac{z}{3}=1$. So the co-ordinates of P, Q, R are $(6,0,0)$, $(0,4,0)$ and $(0,0,3)$ respectively.

Let the equation of the sphere through the origin O be

$x^2+y^2+z^2+2ux+2vy+2wz=0$

which passes through the point P, Q, R.

$\therefore 36+2u\cdot 6=0 \qquad \therefore u=-3$

$16+8v=0 \qquad \therefore v=-2$

$9+6w=0 \qquad \therefore w=-3/2$

So the required equation of the sphere is

$x^2+y^2+z^2-6x-4y-3z=0$.

Ex. 3. *Show that the point* $(-1,0,5)$ *lies inside the sphere* $x^2+y^2+z^2+4x+5y-6z+2=0$.

The centre of the sphere is at $P\left(-2,-5/2,3\right)$ and radius,

$r=\sqrt{4+\frac{25}{4}+9-2}=\frac{\sqrt{69}}{2}$.

Now the distance of $A(-1,0,5)$ from the centre P is

$$AP=\sqrt{(-2+1)^2+\left(\frac{-5}{2}-0\right)^2+(3-5)^2}$$

$$=\sqrt{1+\frac{25}{4}+4}=\frac{\sqrt{45}}{2}<\frac{\sqrt{69}}{2}=r.$$

So the point $(-1,0,5)$ lies inside the sphere.

Ex. 4. *Show that the equation of the sphere passing through the point $(1,2,-3)$ and through the circle $x^2+y^2+z^2+2x-5y+9z+3=0$, $x+2y-7z-8=0$ (a sphere and a plane together give a circle) is $x^2+y^2+z^2+3x-3y+2z-5=0$.*

Let the equation of the sphere through the given circle be

$$x^2+y^2+z^2+2x-5y+9z+3+\lambda(x+2y-7z-8)=0$$

i.e $x^2+y^2+z^2+x(\lambda+2)+y(2\lambda-5)+z(9-7\lambda)+3-8\lambda=0$

which passes through the point $(1,2,-3)$.

$\therefore 1+4+9+\lambda+2+2(2\lambda-5)-3(9-7\lambda)+3-8\lambda=0$

i.e $18\lambda-18=0$

$\therefore \lambda=1$

So the required equation of the sphere is

$$x^2+y^2+z^2+3x-3y+2z-5=0.$$

Ex. 5. *Obtain the equation of the sphere which passes through the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and has its centre on the plane $x+y+z=6$.*

Let the equation of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ... (1)

which passes through the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

$\therefore 1+2u+d=0$ ... (2)

$1+2v+d=0$ ... (3)

$1+2w+d=0$ ... (4)

Also the centre of the sphere $(-u,-v,-w)$ lies on $x+y+z=6$

$\therefore -u-v-w=6$ ... (5)

Solving (2) to (5) we get $u=v=w=-2$, $d=3$.

So the required equation of the sphere is

$$x^2+y^2+z^2-4x-4y-4z+3=0.$$

Ex. 6. *Find the equation to the sphere containing the circle $x^2+y^2+z^2-4x-2y-28=0$, $y+z-5=0$ and passing through the centre of the sphere $x^2+y^2+z^2-2x+4y-10z-1=0$.*

Let the equation of the sphere containing the given circle be

$$(x^2+y^2+z^2-4x-2y-28)+\lambda(y+z-5)=0.$$

i.e. $x^2+y^2+z^2-4x+y(\lambda-2)+\lambda z-28-5\lambda=0$ (1)

Now the centre of the sphere $x^2+y^2+z^2-2x+4y-10z-1=0$ is $(1,-2,5)$.

So the sphere (1) passes through the point $(1,-2,5)$

$\therefore 1+4+25-4-2(\lambda-2)+5\lambda-28-5\lambda=0.$

$\therefore \lambda=1.$

Thus the required equation of the sphere is, using (1),

$$x^2+y^2+z^2-4x-y+z-33=0.$$

Ex. 7. *Find the equation of the sphere having the circle $x^2+y^2+z^2-10y+4z-8=0$, $x+y+z=6$ as a great circle.*

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Let the equation to any sphere passing through the given circle be

$$(x^2+y^2+z^2-10y+4z-8)+\lambda(x+y+z-6)=0$$

i.e. $x^2+y^2+z^2+\lambda x+y(\lambda-10)+z(\lambda+4)-8-6\lambda=0$ (1)

whose centre is at $\left(-\frac{\lambda}{2},-\frac{\lambda-10}{2},-\frac{\lambda+4}{2}\right)$.

If the given circle is a great circle of the sphere (1), then the centre of sphere (1) lies on the plane $x+y+z-6=0$

$$\therefore -\frac{\lambda}{2}-\frac{\lambda-10}{2}-\frac{\lambda+4}{2}-6=0$$

$\therefore \lambda=-2$

So the required equation of the sphere is

$$x^2+y^2+z^2-2x-12y+2z+4=0$$

Ex. 8. *A plane passing through a fixed point (α,β,γ) cuts the axes at A,B,C. Show that the locus of the centre of the sphere $OABC$ is*

$$\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z}=2$$

Let the equation of the plane ABC be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \quad \cdots (1)$$

$\therefore A\equiv(a,0,0),\ B\equiv(0,b,0),\ C\equiv(0,0,c)$.

Also (1) passes through the point (α,β,γ)

$$\therefore \frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1 \quad \cdots (2)$$

Now let the equation of the sphere $OABC$ be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

which passes through the point $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$

$\therefore d=0,\ u=-a/2,\ v=-b/2,\ w=-c/2$.

So the equation of the sphere is

$$x^2+y^2+z^2-ax-by-cz=0.$$

Let (x_1,y_1,z_1) be the centre of the sphere

$\therefore x_1=\frac{a}{2}$

$\therefore a=2x_1$.

Similarly $b=2y_1$ and $c=2z_1$.

$\therefore$ From (2),

$$\frac{\alpha}{2x_1}+\frac{\beta}{2y_1}+\frac{\gamma}{2z_1}=1.$$

i.e. $\frac{\alpha}{x_1}+\frac{\beta}{y_1}+\frac{\gamma}{z_1}=2$.

Thus the required locus is $\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z}=2$.

Ex. 9. *Find whether the plane $x-y+z+1=0$ cuts or touches or neither the sphere $(x-1)^2+y^2+(z-2)^2=16$.*

The centre of the sphere is $(1,0,2)$. The perpendicular distance from $(1,0,2)$ to the plane $x-y+z+1=0$ is $\frac{1-0+2+1}{\sqrt{1^2+(-1)^2+1^2}}=\frac{4}{\sqrt{3}}$ which is less than the radius 4.

So the plane cuts the sphere (at a circle).

Ex. 10. *Find the vector equation of a sphere whose centre is $2i+3j-k$ and radius is 5.*

The position vector of the centre, $\vec{c}=2i+3j-k$ and radius $a=5$

$\therefore$ the vector equation of the sphere becomes $\left(\vec{r}-\vec{c}\right)^2=5^2$

or, $\left(\vec{r}-2i-3j+k\right)^2=25$

Ex.11. *Find the vector equation of a sphere having the points $i-2j+4k$ and $3i+2j-k$ as the extremities of a diameter.*

Let A and B be the two extremities; O be origin

$\therefore \vec{OA}=i-2j+4k$ and $\vec{OB}=3i+2j-k$

If C be centre then C being the mid point of AB,

$$\vec{OC}=\frac{1}{2}\{(i-2j+4k)+(3i+2j-k)\}$$

$$=\frac{1}{2}(4i+3k)$$

Now $\vec{AB}=\vec{OB}-\vec{OA}=(3i+2j-k)-(i-2j+4k)$

$$=2i+4j-5k$$

$\therefore \left|\vec{AB}\right|=\sqrt{2^2+4^2+(-5)^2}=\sqrt{45}=3\sqrt{5}$ which is diameter of the sphere.

$\therefore$ the radius $=\frac{3}{2}\sqrt{5}$

$\therefore$ the vector equation of the sphere is

$$\left\{\vec{r}-\frac{1}{2}(4i+3k)\right\}^2=\left(\frac{3}{2}\sqrt{5}\right)^2=\frac{45}{4}$$

$$\text{or, }\left\{2\vec{r}-(4i+3k)\right\}^2=45$$

EXERCISES

[I] SHORT ANSWER QUESTIONS

1. Show that the straight line joining the points $(3,4,-1)$ and $(4,2,2)$ is parallel to the straight line joining the points $(2,1,-5)$ and $(4,-3,1)$.

2. For what value of k, the lines PQ and RS are parallel where $P(1,2,4)$, $Q(5,4,-6)$, $R(3,5,k)$, $S(7,7,-2)$ are the points.

3. Find the angle between the lines whose direction cosines are proportional to 1, 2, 1 and 2, –3, 6.

4. Find the ratio in which the line joining the points $(2,4,5)$, $(3,5,-4)$ is divided by yz-plane.

5. Find the ratio in which the line joining $(2,4,16)$ and $(3,5,-4)$ is divided by the plane $2x-3y+z+6=0$.

6. Find the equation of the plane that passes through $(2,-3,1)$ and is perpendicular to the line joining the points $(3,4,-1)$ and $(2,-1,5)$.

7. Find the angle between the planes $2x-y+z=6$ and $x+y+2z=3$.

8. Find the distance between the parallel planes $2x-2y+z+3=0$ and $4x-4y+2z+5=0$.

9. Find the equations of the planes parallel to the plane $x-2y+2z-3=0$ whose perpendicular distance from $(1,2,3)$ is 1.

10. Find the equation to the plane passing through the line of intersection of the planes $3x-y-z-4=0$ and $x+y+z-2=0$ and parallel to the x-axis

11. Find the planes which are parallal to the plane $3x-2y+6z+8=0$ and at a distance of 2 units from it

12. Show that the foot of the perpendicular from the point $(-1,3,2)$ to the plane $x+2y+2z-3=0$ is $\left(-\frac{5}{3},\frac{5}{3},\frac{2}{3}\right)$

13. Find the equation of the planes through the straight line $2x-y+3z+2=0=3x+2y-z+3$ parallel to the co-ordinate axes.

14. Find the symmetrical form of the straight line $x+2y-z-3=0$, $2x-2y+3z-2=0$.

15. Show that the straight line $x=t-2$, $y=3-4t$, $z=5t+6$ is parallel to the plane $x-y-z=1$.

16. Find the co-ordinates of the point of intersection of the line $\frac{x+1}{1}=\frac{y+3}{3}=\frac{z-2}{-2}$ with the plane $3x+4y+5z=5$.

17. Show that the straight line $\frac{x-1}{-1}=\frac{y+4}{3}=\frac{z+5}{2}$ meets the plane

$$2x-3y+4z+5=0 \text{ at } \left(\frac{4}{3},-5,-\frac{17}{3}\right).$$

18. Show that the line $\frac{x+1}{-2}=\frac{y+2}{3}=\frac{z+5}{4}$ lies on the plane $x+2y-z=0$.

19. A plane cuts the axes at A,B,C and the centroid of the triangle ABC is (a,b,c). Show that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$.

20. Find the centre and the radius of the sphere

$$3x^2+3y^2+3z^2+2x-4y-2z-1=0.$$

21. Find the equation of the sphere with centre $(2,-2,3)$ and passing through $(7,-3,5)$.

22. Find the equation of the sphere having the points $(1,-2,4)$ and $(3,2,-1)$ as the extremities of a diameter.

23. Determine whether the point $(2,3,4)$ lies outside or inside the sphere

(i) $x^2+y^2+z^2-4x+6y-2z-2=0$

(ii) $x^2+y^2+z^2-2x-4y-6z-25=0$

24. Find the equation to the sphere at $(2,4,7)$ and which touches the x-axis

25. Find whether the plane $z=17$ cuts, touches or does not cut the sphere

$$x^2+y^2+z^2+4x-6y-14z+58=0.$$