5.2 LAPLACE TRANSFORMS

5.2.1. Introduction.

Laplace transform method provide easy and effective ways for the solution of many problems arising in various fields of science and technology. Because of this, the theory of Laplace Transform has recently become an important part of the mathematical background required by engineers, physicists and other scientists. This chapter has three articles and begins with definitions and contains several theorems, formulae on Laplace Transform.

5.2.2. Definition of Laplace Transform.

Let $F(t)$ be a function for $t > 0$. Then $L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t) dt$

is called Laplace Transform of $F(t)$, where the parameter s is positive real.

If the above improper integral does not exist, we say the Laplace Transform of $F(t)$ does not exist.

Illustrative Examples.

(i) *Prove that* $L(1) = \frac{1}{s}$

By definition,

$$L(1) = \int_0^\infty e^{-st} dt = \lim_{X\to\infty} \int_0^X e^{-st} dt = \lim_{X\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^X$$

$$= -\frac{1}{s} \lim_{X\to\infty} \left(e^{-sX} - 1\right) = \frac{1}{s}$$

(ii) *Prove that* $L(t^n) = \frac{n!}{s^{n+1}}$.

By definition, $L(t^n) = \int_0^\infty e^{-st} t^n dt = I_n$ (say) . . . (1)

Now, $I_n = \lim_{X\to\infty} \int_0^X e^{-st} t^n dt \quad = \lim_{x\to\infty} \left\{ \left[t^n . \frac{e^{-st}}{-s} \right]_0^X - \int_0^X n t^{n-1} \frac{e^{-st}}{-s} dt \right\}$

$$= \lim_{X\to\infty} \left\{ -\frac{1}{s} . \frac{X^n}{e^{sX}} + \frac{n}{s} \int_0^X t^{n-1} e^{-st} dt \right\}$$

$$= \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt \quad \left[\because \lim_{X\to\infty} \frac{X^n}{e^{sX}} = 0 \right] \quad = \frac{n}{s} I_{n-1}$$

Thus, $I_n = \frac{n}{s} I_{n-1}$ . . . (2)

Therefore, $I_{n-1} = \frac{n-1}{s} I_{n-2}$, replacing n by $n-1$ in (2)

Similarly, $I_{n-2} = \frac{n-2}{s} I_{n-3}$

....

....

$I_2 = \frac{2}{s} I_1$

$I_1 = \frac{1}{s} I_0$

By successive substitution, we have

$$I_n = \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} . \ldots . \frac{2}{s} . \frac{1}{s} I_0$$

Now, $I_0 = \int_0^\infty e^{-st} dt$, from (1)

$= \frac{1}{s}$ [by Illustration (i)]

$$\therefore I_n = \frac{n.(n-1).(n-2)\ldots.2.1}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

From (1), $L(t^n) = \frac{n!}{s^{n+1}}$

(iii) *Show that* $L(t) = \frac{1}{s^2}$

Follows from Illustration (ii)

(iv) *Show that* $L\left(e^{at}\right)=\frac{1}{s-a}$

$$L\left(e^{at}\right)=\int_0^{\infty} e^{-st}e^{at}dt=\underset{X\to\infty}{Lim}\int_0^{X} e^{(a-s)t}dt=\underset{X\to\infty}{Lim}\left[\frac{e^{(a-s)t}}{a-s}\right]_0^X$$

$$=\frac{1}{a-s}\underset{X\to\infty}{Lim}\left\{\frac{1}{e^{(s-a)X}}-1\right\}=\frac{1}{a-s}(0-1)=\frac{1}{s-a}.$$

(v) *Prove that* (*a*) $L(\sin at)=\frac{a}{s^2+a^2}$, (*b*) $L(\cos at)=\frac{s}{s^2+a^2}$.

We have $L\left(e^{iat}\right)=\frac{1}{s-ai}$ (by Illustration (*iv*) above)

i.e., $\int_0^{\infty} e^{-st}(\cos at+i\sin at)dt=\frac{s+ai}{s^2+a^2}$

Equating real and imaginary parts, we get,

$\int_0^{\infty} e^{-st}\cos at\,dt=\frac{s}{s^2+a^2}$ i.e., $L(\cos at)=\frac{s}{s^2+a^2}$

and $\int_0^{\infty} e^{-st}\sin at\,dt=\frac{a}{s^2+a^2}$ i.e., $L(\sin at)=\frac{a}{s^2+a^2}$

(vi) *Prove that* $L(\sinh at)=\frac{a}{s^2-a^2}$ [supposing the two improper integrals are convergent]

$$L(\sinh at)=\int_0^{\infty} e^{-st}\sinh at\,dt$$

$$=\int_0^{\infty} e^{-st}\frac{e^{at}-e^{-at}}{2}dt \qquad \left[\because \sinh z=\frac{e^z-e^{-z}}{2}\right]$$

$$=\frac{1}{2}\left\{\int_0^{\infty} e^{-st}e^{at}dt-\int_0^{\infty} e^{-st}e^{-at}dt\right\}$$

$$=\frac{1}{2}\left\{L\left(e^{at}\right)-L\left(e^{-at}\right)\right\}=\frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)=\frac{a}{s^2-a^2}$$

(vii) *Prove that* $L(\cosh at)=\frac{s}{s^2-a^2}$

Left to the reader $\left(\text{Hint. } \cosh z=\frac{e^z+e^{-z}}{2}\right)$

Note. (i) Laplace transform of a function $F(t)$ is an another function $f(s)$ of s.

(ii) s can be considered as complex number also.

In light of above illustrations we get the following formulae :

I. $L(1)=\frac{1}{s}$ **II.** $L\left(t^n\right)=\frac{n!}{s^{n+1}}$

III. $L(t)=\frac{1}{s^2}$ **IV.** $L\left(e^{at}\right)=\frac{1}{s-a}$

V. $L(\sin at)=\frac{a}{s^2+a^2}$ **VI.** $L(\cos at)=\frac{s}{s^2+a^2}$

VII. $L(\sinh at)=\frac{a}{s^2-a^2}$ **VIII.** $L(\cosh at)=\frac{s}{s^2-a^2}$.

5.2.3. Existence of Laplace transform.

Theorem. If a function $F(t)$ satisfies the following two conditions, then its Laplace transform exists:

(i) Every interval $[0, N]$ can be sub-divided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

(ii) A real constant $M>0$ and ν exists such that for all $t>N$,

$$\left|e^{-\nu t}F(t)\right|<M \text{ or, } |F(t)|<Me^{\nu t}$$

Proof: For any positive integer N,

$$\int_0^{\infty} e^{-st}F(t)dt=\int_0^{N} e^{-st}F(t)dt+\int_N^{\infty} e^{-st}F(t)dt$$

By condition (i), the first integral on the right exists. Again since by condition (ii) $\left|e^{-st}F(t)\right|<M$, so the second integral on

the right is absolutely convergent i.e. the second integral on the right exists. So

$$\int_0^{\infty} e^{-st}F(t)\,dt \text{ exists} \qquad \text{i.e.,} \quad L\{F(t)\} \text{ exists.}$$

Illustrative Examples .

(i) *Show that $L(t^2)$ exists.*

Here $F(t)=t^2$ is continuous everywhere. So it is continuous in [0, N] for each positive integer N.

$$\text{Now, } e^{3t} = 1+3t+\frac{3^2t^2}{2!}+\frac{3^3t^3}{3!}+\ldots.$$

$$\text{or, } e^{3t}-t^2 = 1+3t+\frac{7}{2}t^2+\frac{3^3t^3}{3!}+\ldots..$$

Since $t>0$, so R. H. S. >0

Therefore, $e^{3t}-t^2>0$ i.e., $e^{3t}>t^2$

i.e., $e^{-3t}.t^2<1 \therefore \left|e^{-3t}.t^2\right|<1$

Thus $F(t)=t^2$ satisfies both the two sufficient conditions for existence of Laplace Transform. So $L(t^2)$ exists.

(ii) *Show that the function $F(t)=e^{t^3}$ does not satisfy the second sufficient condition for existence of Laplace Transform.*

$\left|e^{-vt}e^{t^3}\right| = e^{t^3-vt} \to \infty$ as $t\to\infty$ for all v. Hence proved.

5.2.4. Linear property of Laplace Transformations.

Theorem. If $F_1(t)$ and $F_2(t)$ be two functions whose Laplace transform exists, then $L\{c_1F_1(t)+c_2F_2(t)\}=c_1L\{F_1(t)\}+c_2L\{F_2(t)\}$ where c_1, c_2 are constants.

Proof. $L\{c_1F_1(t)+c_2F_2(t)\} = \int_0^{\infty} e^{-st}\{c_1F_1(t)+c_2F_2(t)\}dt$

$$= c_1\int_0^{\infty} e^{-st}F_1(t)dt + c_2\int_0^{\infty} e^{-st}F_2(t)dt \;= c_1L\{F_1(t)\}+c_2L\{F_2(t)\}$$

Illustrative Example.

Find the Laplace transform of the function $5t^3+3\cos 2t+7e^{-t}$.

$L\left(5t^3+3\cos 2t+7e^{-t}\right)$

$=5L\left(t^3\right)+3L(\cos 2t)+7L\left(e^{-t}\right)$, [By the above linear properties]

$$=5\frac{3!}{s^{3+1}}+3.\frac{s}{s^2+2^2}+7\frac{1}{s+1}=\frac{30}{s^4}+\frac{3s}{s^2+4}+\frac{7}{s+1}$$

5.2.5. Shifting property of Laplace transformation.

Theorem 1 (First shifting property) :

If $L\{F(t)\}=f(s)$ then $L\{e^{at}F(t)\}=f(s-a)$

Proof. We know $L\{F(t)\}=\int_0^{\infty} e^{-st}F(t)\,dt=f(s)$

Now $L\{e^{at}F(t)\}=\int_0^{\infty} e^{-st}e^{at}F(t)\,dt=\int_0^{\infty} e^{-(s-a)t}F(t)\,dt=f(s-a)$

Illustrative Examples.

Find $L\{e^{-5t}\sin 2t\}$

We know $L\{\sin 2t\}=\dfrac{2}{s^2+2^2}=\dfrac{2}{s^2+4}=f(s)$ (say)

Therefore, by first shifting property,

$L\{e^{-5t}\sin 2t\}=f(s-(-5))=f(s+5)$

$$=\frac{2}{(s+5)^2+4}=\frac{2}{s^2+10s+29}$$

Theorem 2 (Second shifting property) :

If $L\{F(t)\}=f(s)$ and

$$G(t)=F(t-a), \quad t>a$$
$$=0 \quad , \quad t<a$$

Then $L\{G(t)\} = e^{-as} f(s)$

Proof. $L\{G(t)\} = \int_0^\infty e^{-st} G(t) dt = \int_0^a e^{-st}.0.dt + \int_a^\infty e^{-st} F(t-a) dt$

$$= \int_a^\infty e^{-st} F(t-a) dt$$

$$= \int_0^\infty e^{-s(z+a)} F(z) dz \text{, by putting } t = z + a$$

$$= \int_0^\infty e^{-sz}.e^{-sa} F(z) dz = e^{-sa} \int_0^\infty e^{-sz} F(z) dz$$

$$= e^{-as} \int_0^\infty e^{-st} F(t) dt = e^{-as} L\{F(t)\} = e^{-as} f(s)$$

Illustrative Example.

Find $L\{F(t)\}$ where $F(t)$ is defined as

$$F(t) = \cos\left(t - \frac{2\pi}{3}\right), \quad t > \frac{2\pi}{3}$$

$$= 0 \quad , \quad t < \frac{2\pi}{3}$$

We have $L(\cos t) = \dfrac{s}{s^2+1} = f(s)$, say

Therefore, by Second shifting property

$$L\{F(t)\} = e^{-\frac{2\pi}{3}s} f(s) = e^{-\frac{2\pi}{3}s}.\frac{s}{s^2+1} = \frac{se^{-\frac{2\pi}{3}s}}{s^2+1}$$

5.2.6. Change of scale property.

Theorem. If $L\{F(t)\} = f(s)$, then $L\{F(at)\} = \dfrac{1}{a} f\left(\dfrac{s}{a}\right)$

Proof. $L\{F(t)\} = \int_0^\infty e^{-st} F(t) dt = f(s)$

$$\therefore L\{F(at)\} = \int_0^\infty e^{-st} F(at) dt = \frac{1}{a}\int_0^\infty e^{-s\frac{z}{a}} F(z) dz \text{, by putting } z = a\,t$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}t} F(t) dt = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Illustrative Example.

If $L\{F(t)\} = \dfrac{s+1}{s^2+3s-1}$, *find* $L\{F(3t)\}$

Using the change of scale property, we get

$$L\{F(3t)\} = \frac{1}{3}\left\{\frac{\frac{s}{3}+1}{\left(\frac{s}{3}\right)^2 + 3.\left(\frac{s}{3}\right) - 1}\right\} = \frac{s+3}{s^2+9s-9}$$

5.2.7. Laplace Transformation on derivative.

Theorem 1. (On first order derivative)

If $L\{F(t)\} = f(s)$ and satisfies the following conditions:

(i) $F(t)$ is continuous on $[0, N]$,

(ii) there exists some real number M and v such that $|F(t)| < Me^{vt}$,

(iii) $F'(t)$ exists and sectionally continuous on $[0, N]$, then

$$L\{F'(t)\} = sf(s) - F(0)$$

Proof. $L\{F'(t)\} = \int_0^\infty e^{-st} F'(t) dt = \underset{X\to\infty}{Lim} \int_0^X e^{-st} F'(t) dt$

$$= \underset{X\to\infty}{Lim}\left\{\left[e^{-st} F(t)\right]_0^X + s\int_0^X e^{-st} F(t) dt\right\}$$

$$= \underset{X\to\infty}{Lim}\left\{e^{-sX} F(X) - F(0) + s\int_0^X e^{-st} F(t) dt\right\}$$

$$= \underset{X\to\infty}{Lim}\, e^{-sX} F(X) - F(0) + s \underset{X\to\infty}{Lim} \int_0^X e^{-st} F(t)\, dt \quad \cdots (1)$$

Now, $\left|e^{-sX} F(X)\right| = e^{-sX} \left|F(X)\right| < e^{-sX} M e^{vX} = M e^{(v-s)X}$

$$= \frac{M}{e^{(s-v)X}} \to 0 \text{ as } X \to \infty, \text{ for } s > v$$

$$\therefore \underset{X\to\infty}{Lim}\, e^{-sX} F(X) = 0$$

So, from (1), we have $L\{F'(t)\} = 0 - F(0) + s\int_0^\infty e^{-st} F(t) dt$

$$= sL\{F(t)\} - F(0)$$

$$\therefore L\{F'(t)\} = sf(s) - F(0)$$

Illustrative Example.

Using Laplace Transformation of $\sin 6t$ *find the Laplace Transformation of* $\cos 6t$.

We know $L(\sin 6t) = \dfrac{6}{s^2+6^2} = \dfrac{6}{s^2+36} = f(s)$, (say)

Since $\dfrac{d}{dt}(\sin 6t) = 6\cos 6t$,

$L\left\{\dfrac{d}{dt}(\sin 6t)\right\} = sf(s) - \sin(6\times 0)$ or, $L\{6\cos 6t\} = s.\dfrac{6}{s^2+36} - 0$

or, $6L\{\cos 6t\} = \dfrac{6s}{s^2+36}$ $\quad \therefore L(\cos 6t) = \dfrac{s}{s^2+36}$.

Note.(i) If in the above theorem $F(t)$ is not continuous at $t=0$, then we would have $L\{F'(t)\} = sf(s) - \underset{t\to 0+}{Lim} F(t)$,

(ii) If in the above theorem $F(t)$ is discontinuous at some point, say $x = a$, then

$$L\{F'(t)\} = sf(s) - F(0) - e^{-as}\left\{\underset{t\to a+}{Lim} F(t) - \underset{t\to a-}{Lim} F(t)\right\}.$$

Theorem 2. (On second order derivative)

If $L\{F(t)\} = f(s)$ and $F(t)$ satisfies the following conditions:

(i) $F(t)$ and $F'(t)$ are continuous on $[0, N]$,

(ii) there exists some real number M and v such that $|F(t)| < Me^{vt}$ and $|F'(t)| < Me^{vt}$,

(iii) $F''(t)$ exists and sectionally continuous on $[0, N]$,

then $L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$.

Proof. $L\{F''(t)\} = L\left\{\dfrac{d}{dt}F'(t)\right\} = sL\{F'(t)\} - F'(0)$, by Theorem (1)

$$= s\{sf(s) - F(0)\} - F'(0) = s^2 f(s) - sF(0) - F'(0)$$

Illustrative Example.

Using Laplace Transform on second order derivative, show that $L(\sin at) = \dfrac{a}{s^2+a^2}$.

Let $F(t) = \sin at$, $L\{F(t)\} = f(s)$, $F'(t) = a\cos at$ and $F''(t) = -a^2 \sin at$.

We know, $L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

or, $L(-a^2 \sin at) = s^2 L\{F(t)\} - s.0 - a.1$

or, $-a^2 L(\sin at) = s^2 L(\sin at) - a$ or, $(s^2 + a^2) L(\sin at) = a$

$$\therefore L(\sin at) = \frac{a}{s^2+a^2}$$

Note. If $F(t)$ and $F'(t)$ are not continuous, appropriate modification can be made in the result of Theorem 2 (as discussed in Note of Theorem 1).

Theorem 3. (On n^{th} order derivative)

If $L\{F(t)\} = f(s)$ and $F(t)$ satisfies the following conditions:

(i) $F(t), F'(t), \ldots\ldots F^{n-1}(t)$ are continuous on $[0, N]$.

(ii) there exists some real number M and v such that

$|F(t)|, |F'(t)|, \ldots |F^{(n-1)}(t)| < Me^{vt}$ for $t > N$,

(iii) $F^{(n)}(t)$ exists and sectionally continuous on $[0, N]$, then $L\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \dots$

$$\dots - sF^{(n-2)}(0) - F^{(n-1)}(0)$$

Proof. Detail proof is omitted. But it can be proved by method of induction.

Illustrative Example. *If* $F(t) = t^2, \quad 0 < t \le 1$
$= 0 \; , \quad t > 1$

find $L\{F''(t)\}$.

$$F'(t) = 2t, \quad 0 < t < 1$$
$$= 0 \; , \quad t > 1$$

But the left derivative $LF'(1) = \left[\frac{d}{dt}(t^2)\right]_{t=1} = 2$ and the right derivative $RF'(1) = \underset{h \to 0+}{Lim} \frac{F(1+h) - F(1)}{h} = \underset{h \to 0+}{Lim} \frac{0-1}{h} = -\infty$

$\therefore F'(1)$ does not exist

$\therefore F'(t) = 2t, \quad 0 < t < 1$
$= 0 \; , \quad t > 1$

Similarly, $F''(t) = 2, \quad 0 < t < 1$
$= 0, \quad t > 1$

$$\therefore L\{F''(t)\} = \int_0^1 e^{-st} \times 2\,dt + \int_1^\infty e^{-st} \times 0 \times dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^1 = \frac{2}{s}\left(1 - e^{-s}\right)$$

Note. Here we note that the Theorem 2 cannot be applied because $F(t)$ is not continuous at $t = 1$, even $F'(t)$ does not exist at $t = 1$.

5.2.8. Laplace Transform on Integrals.

Theorem. If $L\{F(t)\} = f(s)$, then $L\left\{\int_0^t F(x)dx\right\} = \frac{1}{s}f(s)$.

Proof. Let $G(t) = \int_0^t F(x)dx \quad \therefore G'(t) = F(t)$ and $G(0) = 0$

Now $L\{G'(t)\} = sL\{G(t)\} - G(0) = sL\{G(t)\}$

$$\therefore L\left\{\int_0^t F(x)dx\right\} = \frac{1}{s}L\{F(t)\} = \frac{1}{s}f(s)$$

Illustrative Example. *Find* $L\left\{\int_0^t e^t \cos 2t\,dt\right\}$

Let $L(e^t \cos 2t) = f(s)$

Then $L\left\{\int_0^t e^t \cos 2t\,dt\right\} = \frac{1}{s}f(s) = \frac{1}{s}L(e^t \cos 2t) = \frac{1}{s}\phi(s-1)$, by shifting property,

where $\phi(s) = L\{\cos 2t\} = \frac{s}{s^2 + 2^2} = \frac{s}{s^2 + 4}$

$$\therefore L\left\{\int_0^t e^t \cos 2t\,dt\right\} = \frac{1}{s} \cdot \frac{s-1}{(s-1)^2 + 4} = \frac{s-1}{s(s^2 - 2s + 5)}$$

Note: Using the above theorem improper integrals of some functons can be evaluated. See Ex.18,19,20,21(pages 5-72 to 5-74)

5.2.9. Multiplication by t^n.

Theorem. If $L\{F(t)\} = f(s)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$,

where n is positive integer. [*W.B.U.Tech.2015, 2005*]

Proof. We have $f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$

Differentiating both sides w. r. t. s, we get

$$\frac{df}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} F(t)dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}[e^{-st} F(t)dt] \quad \left[\text{by Leibnitz's rule for differentiation under integral sign}\right]$$

$$= \int_0^\infty -te^{-st} F(t)dt = -\int_0^\infty e^{-st}\{t\,F(t)\}\,dt = -L\{t\,F(t)\}$$

$$\therefore L\{t\,F(t)\} = -\frac{df}{ds} \quad \text{or,} \quad L\{t\,F(t)\} = -\frac{d}{ds}[L\{F(t)\}] \quad \cdots(1)$$

Hence, $L\{t.t\,F(t)\} = -\dfrac{d}{ds}L\{t\,F(t)\}$

i.e., $L\{t^2 F(t)\} = (-1)^2 \dfrac{d}{ds}\left(\dfrac{df}{ds}\right)$, by (1)

$$\therefore L\{t^2 F(t)\} = (-1)^2 \frac{d^2 f}{ds^2}$$

So, the theorem is true for $n = 1, 2$,
Let us assume that the theorem is true for $n = m$,

Then, $L\{t^m F(t)\} = (-1)^m \dfrac{d^m f}{ds^m}$

or, $\displaystyle\int_0^\infty e^{-st} t^m F(t)dt = (-1)^m \frac{d^m f}{ds^m}$

Differentiating both sides w. r. t. s, we get,

$$\frac{d}{ds}\int_0^\infty e^{-st}\, t^m\, F(t)dt = (-1)^m \frac{d^{m+1} f}{ds^{m+1}}$$

or, $\displaystyle\int_0^\infty \frac{\partial}{\partial s}\left[e^{-st}\, t^m\, F(t)\right]dt = (-1)^m \frac{d^{m+1} f}{ds^{m+1}}$

or, $\displaystyle\int_0^\infty -te^{-st}\, t^m\, F(t)dt = (-1)^m \frac{d^{m+1} f}{ds^{m+1}}$

or, $\displaystyle\int_0^\infty -e^{-st}\, t^{m+1} F(t)dt = (-1)^m \frac{d^{m+1} f}{ds^{m+1}}$

$$\therefore L\{t^{m+1} F(t)\} = (-1)^{m+1} \frac{d^{m+1} f}{ds^{m+1}}$$

which shows that the theorem is ture for $n = m + 1$.
Hence by mathematical induction, the theorem is ture for all positive integer n.

Illustrative Examples.

(i) Find $\{t^2 \cos 5t\}$

We know $L\{\cos 5t\} = \dfrac{s}{s^2 + 5^2}$

$$\therefore L\{t^2 \cos 5t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+25}\right) = \frac{2s^3 - 150s}{(s^2+25)^3}$$

(ii) Find $\{(t^2 - 3t + 2)e^{5t}\}$

Now, $L\{(t^2 - 3t + 2)e^{5t}\}$

$= L\{t^2 e^{5t}\} - 3L\{t\,e^{5t}\} + 2L\{e^{5t}\}$ [$\because F(u)$ is periodic with period T]

$$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-5}\right) - 3(-1)^1 \frac{d}{ds}\left(\frac{1}{s-5}\right) + 2.\frac{1}{s-5}$$

$$\left[\because L(e^{5t}) = \frac{1}{s-5}\right]$$

$$= \frac{2}{(s-5)^3} - \frac{3}{(s-5)^2} + \frac{2}{(s-5)}$$

5.2.10. Division by t.

Theorem. If $L\{F(t)\} = f(s)$, then $L\left\{\dfrac{F(t)}{t}\right\} = \displaystyle\int_s^\infty f(u)du$, provided $\displaystyle\lim_{t\to 0}\frac{F(t)}{t}$ exist finitely.

Proof. Since $L\{F(t)\} = f(s)$, so $\displaystyle\int_0^\infty e^{-st} F(t)dt = f(s)$

or, $\int_s^\infty \left\{\int_0^\infty e^{-st} F(t)dt\right\} ds = \int_s^\infty f(s)ds$

or, $\int_0^\infty \left\{\int_s^\infty e^{-st} F(t)ds\right\} dt = \int_s^\infty f(u)du$ [We interchange the order of integration since the required coditions hold here]

or, $\int_0^\infty F(t)\left\{\int_s^\infty e^{-st} ds\right\} dt = \int_s^\infty f(u)du$ $\cdots(1)$

Now, $\int_s^\infty e^{-st}\, ds = \lim_{X\to\infty} \int_s^X e^{-st}\, ds = \lim_{X\to\infty}\left[\frac{e^{-st}}{-t}\right]_s^X$

$= \lim_{X\to\infty}\left(\frac{e^{-Xt}}{-t} + \frac{e^{-st}}{t}\right) = \frac{1}{t}e^{-st}$

$\therefore$ From (1) we get, $\int_0^\infty F(t)\frac{1}{t}e^{-st}dt = \int_s^\infty f(u)du$

or, $\int_0^\infty e^{-st}\frac{F(t)}{t}dt = \int_s^\infty f(u)du$

$\therefore L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\, du$

Illustrative Examples. (i) *Find* $L\left\{\frac{\sin 5t}{t}\right\}$

We know, $L\{\sin 5t\} = \frac{5}{s^2+25} = f(s)$, (say)

$\therefore L\left\{\frac{\sin 5t}{t}\right\} = \int_s^\infty f(u)du = \int_s^\infty \frac{5}{u^2+25}du = \lim_{X\to\infty} 5\int_s^X \frac{du}{u^2+25}$

$= 5\lim_{X\to\infty}\frac{1}{5}\left[\tan^{-1}\frac{u}{5}\right]_s^X$

$= \lim_{X\to\infty}\left(\tan^{-1}\frac{X}{5} - \tan^{-1}\frac{s}{5}\right)$

$= \frac{\pi}{2} - \tan^{-1}\frac{s}{5} = \cot^{-1}\frac{s}{5} = \tan^{-1}\frac{5}{s}$

(ii) *Prove that* $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$. [*W.B.U.T 2007, 2016*]

Let $F(t) = \sin t$ $\quad\therefore L\{F(t)\} = \frac{1}{s^2+1} = f(s)$, (say)

Therefore,

$L\left(\frac{\sin t}{t}\right) = \int_s^\infty f(u)du = \int_s^\infty \frac{1}{u^2+1}du = \lim_{X\to\infty}\int_s^X \frac{du}{u^2+1}$

$= \lim_{X\to\infty}\left[\tan^{-1}u\right]_s^X = \lim_{X\to\infty}\left(\tan^{-1}X - \tan^{-1}s\right) = \frac{\pi}{2} - \tan^{-1}s$

$\therefore \int_0^\infty \frac{\sin t}{t}e^{-st}dt = \frac{\pi}{2} - \tan^{-1}s$

Putting $s=0$ we get $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$.

5.2.11. Laplace Transformation of Periodic Function.

Theorem. *Let F(t) be a periodic function of period T (> 0), then*

$$L\{F(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)\,dt.$$

Proof. $L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt = \int_0^T e^{-st}F(t)dt + \int_T^{2T} e^{-st}F(t)dt +$

$$\int_{2T}^{3T} e^{-st}F(t)\,dt+\ldots\infty \;=\sum_{r=1}^{\infty}\int_{(r-1)T}^{rT} e^{-st}F(t)dt \qquad \ldots(1)$$

Put $t=u+(r-1)T$

$\therefore dt=du$, since r, T are constants in each integral.

Now, $t=(r-1)T \Rightarrow u=0$ and $t=rT \Rightarrow u=T$

$\therefore$ From (1), we get

$$L\{F(t)\}=\sum_{r=1}^{\infty}\int_0^T e^{-s\{u+(r-1)T\}}F\{u+(r-1)T\}du$$

$$=\sum_{r=1}^{\infty}\int_0^T e^{-su}.e^{-s(r-1)T}F(u)du$$

$$=\sum_{r=1}^{\infty}e^{-s(r-1)T}\int_0^T e^{-su}F(u)du$$

$$=\int_0^T e^{-su}F(u)du.\sum_{r=1}^{\infty}e^{-s(r-1)T}$$

$$=\int_0^T e^{-su}F(u)\,du\left\{1+e^{-sT}+e^{-2sT}+\ldots\right\}$$

$$=\int_0^T e^{-su}F(u)du\left(1-e^{-sT}\right)^{-1}$$

$$=\frac{1}{1-e^{-sT}}\int_0^T e^{-su}F(u)\,du=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)\,dt$$

Illustrative Example. *Given* $F(t)=\sin t$, $0<t<\pi$
$=0$, $\pi<t<2\pi$

and extended periodically with period 2π. *Find* $L\{F(t)\}$.
[W.B.U.Tech 2008]

Here $F(t)$ is a periodic function with period 2π. Therefore, by above theorem,

$$L\{F(t)\}=\frac{1}{1-e^{-2\pi s}}\int_0^{2\pi}e^{-st}F(t)dt$$

$$=\frac{1}{1-e^{-2\pi s}}\left\{\int_0^{\pi}e^{-st}\sin t\,dt+\int_{\pi}^{2\pi}e^{-st}.0\,dt\right\}$$

$$=\frac{1}{1-e^{-2\pi s}}\int_0^{\pi}e^{-st}\sin t\,dt \qquad \ldots(1)$$

Let $I=\int_0^{\pi}e^{-st}\sin t\,dt=\left[\frac{e^{-st}}{s^2+1}(-s\sin t-\cos t)\right]_0^{\pi}=\frac{e^{-s\pi}+1}{1+s^2}$

$\left[\because \int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)\right]$

$\therefore$ From (1), we get $L\{F(t)\}=\frac{1}{1-e^{-2\pi s}}.\frac{e^{-\pi s}+1}{1+s^2}$

$$=\frac{1}{\left(1-e^{-\pi s}\right)\left(s^2+1\right)}.$$

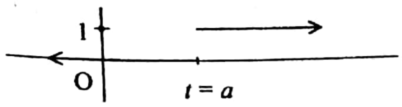
5.2.12. Laplace Transformation on unit step function.

Definition of unit step function.

The function u defined by

$$u(t-a)=1\ ,\quad t\ge a$$
$$=0\ ,\ t<a$$

is called unit step function or Heaviside's unit step function. Its graph is shown below.



Note. The unit step function may also be defined as

$$u(t)=1\,,\qquad t\ge a$$
$$=0\,,\qquad t<a.$$

Theorem 1. The function

$$F(t)=F_1(t), \quad t<a$$
$$=F_2(t), \quad t>a$$

can be expressed by a unit step function like

$$F(t)=F_1(t)+\{F_2(t)-F_1(t)\}u(t-a)$$

Proof. Consider the function $F(t)=F_1(t)+\{F_2(t)-F_1(t)\}u(t-a)$ where $u(t-a)$ is a unit step function. From definition of unit step function we can write

$$F(t)=F_1(t)+\{F_2(t)-F_1(t)\}\times 1, \ t\geq a$$
$$=F_1(t)+\{F_2(t)-F_1(t)\}\times 0, \ t<a$$

i.e., $F(t)=F_2(t), \ t\geq a$

$$=F_1(t), \ t<a$$

$\therefore \ F(t)=F_1(t), \quad t<a$

$$=F_2(t), \quad t\geq a$$

Theorem 2. If $F(t)=F_1(t), \ t<a_1$

$$=F_2(t), \ a_1<t<a_2$$
$$=F_3(t), \ a_2<t,$$

then $F(t)$ can be expressed as

$$F(t)=F_1(t)+\{F_2(t)-F_1(t)\}u(t-a_1)+\{F_3(t)-F_2(t)\}u(t-a_2)$$

Proof. Beyond the scope of the book.

Note : The above theorem can be extended to step function having several steps.

Illustrations.

(*i*) *Given* $\phi(t)=t^2, \quad t<2$

$$=\sin t, \ t>2$$

$\phi(t)$ *can be expressed with a single branch.*

With the help of Theorem 1, we can write $\phi(t)=t^2+(\sin t-t^2)u(t-2)$

where $u(t-2)=1, \quad t>2$

$$=0, \quad t<2$$

(*ii*) *We can express the function*

$$F(t)=e^t, \quad t<3$$
$$=t^3, \quad 3<t<5$$
$$=\cos 2t, \ 5<t$$

in terms of unit step function.

$$F(t)=e^t+(t^3-e^t)u(t-3)+(\cos 2t-t^3)u(t-5)$$

where $u(t-3), u(t-5)$ are two unit step function.

Theorem 3. If $u(t-a)$ is a unit step function, then $L\{u(t-a)\}=\dfrac{e^{-as}}{s}$.

Proof. $L\{u(t-a)\}=\int_0^\infty e^{-st}u(t-a)dt$

$$=\int_0^a e^{-st}\times 0\,dt+\int_a^\infty e^{-st}\times 1\,dt=\lim_{X\to\infty}\int_a^X e^{-st}dt$$

$$=\lim_{X\to\infty}\left[\frac{e^{-st}}{-s}\right]_a^X=-\frac{1}{s}\lim_{X\to\infty}\left(e^{-sX}-e^{-as}\right)$$

$$=\frac{e^{-as}}{s}$$

Illustrative Example. *Find* $L\{F(t)\}$ *where* $F(t)=e^{-t}, \ 0<t<3$

$$=0, \ t>3$$

[*W.B.U.Tech.2004*]

We can express $F(t)$ in terms unit step function like

$$F(t)=e^{-t}+(0-e^{-t})u(t-3)$$

where $u(t-3)$ is a unit step function

Then $L\{F(t)\}=L\{e^{-t}\}-L\{e^{-t}u(t-3)\}$ ... (1)

Now $L\{e^{-t}\}=\dfrac{1}{s-(-1)}=\dfrac{1}{s+1}$

and $L\{e^{-t}u(t-3)\}=f\{s-(-1)\}=f(s+1)$, by first shifting property

where $L\{u(t-3)\}=f(s)$ or, $\dfrac{e^{-3s}}{s}=f(s)$, by Theorem 3

$$\therefore L\{e^{-t}u(t-3)\}=\frac{e^{-3(s+1)}}{s+1}$$

$\therefore$ From (1) we get $L\{F(t)\}=\dfrac{1}{s+1}-\dfrac{e^{-3(s+1)}}{s+1}$

Theorem 4. Let $L\{F(t)\}=f(s)$ and $u(t-a)$ be a unit step function. Then $L\{F(t-a)u(t-a)\}=e^{-as}f(s)=e^{-as}L\{F(t)\}$.

Proof. $L\{F(t-a)u(t-a)\}=\int_0^\infty e^{-st}F(t-a)u(t-a)dt$

$$=\int_0^a e^{-st}F(t-a)\times 0\,dt+\int_a^\infty e^{-st}F(t-a)\times 1\,dt$$

$$=\int_a^\infty e^{-st}F(t-a)dt=\int_0^\infty e^{-s(z+a)}F(z)dz,$$

by putting $z=t-a$ $\therefore dz=dt$

$$=e^{-sa}\int_0^\infty e^{-st}F(t)dt=e^{-sa}L\{F(t)\}=e^{-as}f(s).$$

Corollary. $L\{F(t)u(t-a)\}=e^{-as}L\{F(t+a)\}$

Note. This is an important formula.

Illustrative Example. *Find* $L\{F(t)\}$ *where* $F(t)=\cos t,\ 0<t<\pi$

$=\cos 2t,\quad \pi<t<2\pi$

$=\cos 3t,\quad 2\pi<t$

In terms of unit step function, we can express

$$F(t)=\cos t+(\cos 2t-\cos t)u(t-\pi)+(\cos 3t-\cos 2t)u(t-2\pi)$$

$$\therefore L\{F(t)\}=L(\cos t)+L\{\cos 2t\,u(t-\pi)\}$$
$$-L\{\cos t\,u(t-\pi)\}+L\{\cos 3t\,u(t-2\pi)\}$$
$$-L\{\cos 2t\,u(t-2\pi)\} \qquad \ldots(1)$$

Now, $L(\cos t)=\dfrac{s}{s^2+1}$

$L\{\cos 2t\,u(t-\pi)\}=e^{-\pi s}L\{\cos 2(t+\pi)\}$ (using the above corollary)

$$=e^{-\pi s}L\{\cos(2t+2\pi)\}$$

$$=e^{-\pi s}L(\cos 2t)=e^{-\pi s}\frac{s}{s^2+4}$$

$L\{\cos t\ u(t-\pi)\}=e^{-\pi s}L\{\cos(t+\pi)\}$

$$=e^{-\pi s}L(-\cos t)=-e^{-\pi s}L(\cos t)$$

$$=-e^{-\pi s}\frac{s}{s^2+1}$$

Similarly $L\{\cos 3t\,u(t-2\pi)\}=e^{-2\pi s}\dfrac{s}{s^2+9}$

$L\{\cos 2t\,u(t-2\pi)\}=e^{-2\pi s}\dfrac{s}{s^2+4}$

From (1), we get

$$L\{F(t)\}=\frac{s}{s^2+1}+\frac{se^{-\pi s}}{s^2+4}+\frac{se^{-\pi s}}{s^2+1}+\frac{se^{-2\pi s}}{s^2+9}-\frac{se^{-2\pi s}}{s^2+4}$$

5.2.13. Miscellaneous Examples

Ex 1. *Find, from definition, the Laplace Transform of the function F(t) defined by*

$F(t)=0,\quad 0<1\le 1$

$=t,\quad 1<t\le 2$ [W.B.U.Tech.2005]

$=0,\quad t>2$

According to the definition of Laplace Transform,

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$$

$$= \int_0^1 0.e^{-st}dt + \int_1^2 t.e^{-st}dt + \int_2^\infty 0.e^{-st}dt = \int_1^2 t.e^{-st}dt$$

$$= \left[-\frac{t}{s}e^{-st}\right]_1^2 + \frac{1}{s}\int_1^2 e^{-st}dt$$

$$= -\frac{1}{s}\left(2e^{-2s} - e^{-s}\right) - \frac{1}{s^2}\left[e^{-st}\right]_1^2$$

$$= -\frac{1}{s}\left(2e^{-2s} - e^{-s}\right) - \frac{1}{s^2}\left(e^{-2s} - e^{-s}\right)$$

$$= \left(\frac{1}{s} + \frac{1}{s^2}\right)e^{-s} - \left(\frac{1}{s^2} + \frac{2}{s}\right)e^{-2s}.$$

Ex. 2 . *Find Laplace transform of*

$f(t) = \sin t, 0 < t < \pi$

$= 0, \qquad t > \pi$ [WBUT 2008]

We know

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\pi e^{-st}\sin t\, dt + \int_\pi^\infty e^{-st}\,.\,0\, dt$$

$$= \int_0^\pi e^{-st}\sin t\, dt$$

Let $I = \int_0^\pi e^{-st}\sin t\, dt$

$$= \left[-e^{-st}\cos t\right]_0^\pi - \int_0^\pi e^{-st}(-s)(-\cos t)\, dt$$

$$= -e^{-s\pi}\cos\pi + 1 - s\int_0^\pi e^{-st}\cos t\, dt$$

$$= -e^{-s\pi}(-1) + 1 - s\left\{\left[e^{-st}\sin t\right]_0^\pi - \int_0^\pi e^{-st}(-s)\sin t\, dt\right\}$$

$$= -e^{-s\pi} + 1 - s\left\{0 + s\int_0^\pi e^{-st}\sin t\, dt\right\}$$

$\therefore \quad I = e^{-s\pi} + 1 - s^2 I$

or, $\quad I\left(1+s^2\right) = 1 + e^{-s\pi}$

$\therefore \quad I = \dfrac{1+e^{-s\pi}}{1+s^2}$

$\therefore \quad L\{f(t)\} = \dfrac{1+e^{-s\pi}}{1+s^2}$

Ex. 3. *Find* $L\left\{\left(5e^{2t}-3\right)^2\right\}$.

$$L\left\{\left(5e^{2t}-3\right)^2\right\} = L\left\{25e^{4t} - 30e^{2t} + 9\right\}$$

$$= 25L\left(e^{4t}\right) - 30L\left(e^{2t}\right) + 9L(1)$$

$$= \frac{25}{s-4} - \frac{30}{s-2} + \frac{9}{s}$$

Ex. 4. *Evaluate* $L\{\sin(at+b)\}$.

$$L\{\sin(at+b)\} = L\{\sin at.\cos b + \cos at.\sin b\}$$

$$= \cos b L(\sin at) + \sin b L(\cos at)$$

$$= \cos b.\frac{a}{s^2+a^2} + \sin b.\frac{s}{s^2+a^2}$$

$$= \frac{a\cos b + s\sin b}{s^2+a^2}$$

Ex 5. *Find the Laplace transfrom of* $4\cos^2 2t$.

$$L(4\cos^2 2t) = L(2+2\cos 4t)$$
$$= 2L(1) + 2L(\cos 4t)$$
$$= 2.\frac{1}{s} + 2.\frac{s}{s^2+16}$$
$$= \frac{2}{s} + \frac{2s}{s^2+16}$$

Ex 6. *Find* $L\{\sin^3 t\}$.

$$L(\sin^3 t) = L\left\{\frac{1}{4}(3\sin t - \sin 3t)\right\} = \frac{3}{4}L(\sin t) - \frac{1}{4}L(\sin 3t)$$
$$= \frac{3}{4}.\frac{1}{s^2+1} - \frac{1}{4}.\frac{3}{s^2+9} = \frac{6}{(s^2+1)(s^2+9)}$$

Ex 7. *Evaluate* $L\{e^{-t}\sin^2 t\}$.

$$L(\sin^2 t) = L\left\{\frac{1}{2}(1-\cos 2t)\right\} = \frac{1}{2}\{L(1) - L(\cos 2t)\}$$
$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) = \frac{2}{s(s^2+4)}$$

$\therefore$ **By first shifting property,**

$$L(e^{-t}\sin^2 t) = \frac{2}{(s+1)\{(s+1)^2+4\}} = \frac{2}{(s+1)(s^2+2s+5)}$$

Ex 8. *If* $L\{F(t)\} = f(s)$, *find* $L\{F(t)\sin 4t\}$.

Since $\sin 4t = \frac{1}{2i}(e^{4it} - e^{-4it})$,

$$L\{F(t)\sin 4t\} = L\left\{\frac{1}{2i}(e^{4it} - e^{-4it})F(t)\right\}$$
$$= \frac{1}{2i}\{L(e^{4it}F(t)) - L(e^{-4it}F(t))\}$$

$$= \frac{1}{2i}\{f(s-4i) - f(s+4i)\}$$,by 1st shifting property

Ex. 9. *Find* $L\{F(t)\}$ *where*

$$F(t) = \sin\left(t - \frac{\pi}{3}\right),\ t > \frac{\pi}{3}$$
$$= 0, \qquad t < \frac{\pi}{3}$$ [WBUT 2004]

We know $L(\sin t) = \frac{1}{s^2+1} = f(s)$, say

$\therefore$ By second shifting property,

$$L\{F(t)\} = e^{-\frac{\pi}{3}s} f(s)$$
$$= e^{-\frac{\pi}{3}s}\cdot\frac{1}{s^2+1}$$

Ex 10. *Find* $L\{\phi(t)\}$, *where* $\phi(t)$ *is defined as*

$$\phi(t) = e^{t-a},\quad t > a$$
$$= 0 \quad , \quad t < a.$$

We have, $L(e^t) = \frac{1}{s-1} = f(s)$, say

Since $\phi(t) = e^{t-a}$, $t > a$

$$= 0 \quad , t < a$$

So, by second shifting property, we have

$$L\{\phi(t)\} = e^{-as}f(s) = e^{-as}.\frac{1}{s-1} = \frac{e^{-as}}{s-1}.$$

Ex 11. *Evaluate* $L(\cos 5t)$ *by changing scale.*

We have $L(\cos t) = \frac{s}{s^2+1} = f(s)$, say

$\therefore$ By change of scale property,

$$L(\cos 5t) = \frac{1}{5} f\left(\frac{s}{5}\right) = \frac{1}{5} \frac{\frac{s}{5}}{\left(\frac{s}{5}\right)^2 + 1} = \frac{s}{s^2 + 25}$$

Ex 12. *If* $L\{F(t)\} = \frac{e^{-\frac{1}{s}}}{s}$, *prove that* $L\{e^{-t} F(4t)\} = \frac{e^{-\frac{4}{s+1}}}{s+1}$.

Since $L(F(t)) = \frac{e^{-\frac{1}{s}}}{s} = f(s)$, say

By change of scale property,

$$L\{F(4t)\} = \frac{1}{4} f\left(\frac{s}{4}\right) = \frac{1}{4} \frac{e^{-\frac{4}{s}}}{\frac{s}{4}} = \frac{e^{-\frac{4}{s}}}{s} = \phi(s)$$, say

Then applying first shifting property, we get,

$$L\{e^{-t} F(4t)\} = \phi(s+1) = \frac{1}{s+1} e^{-\frac{4}{s+1}}$$

Ex 13. *If* $L\{F''(t)\} = \tan^{-1}\left(\frac{1}{s}\right)$, $F(0) = 2$ *and* $F'(0) = -1$, *find* $L\{F(t)\}$.

Let $L\{F(t)\} = f(s)$ Now, $L\{F''(t)\} = \tan^{-1}\left(\frac{1}{s}\right)$

$\therefore$ using the result $L\{F''(t) = s^2 f(s) - sF(0) - F'(0)\}$

$$f(s) = \frac{2s + \tan^{-1}\left(\frac{1}{s}\right) - 1}{s^2}.$$

Ex 14. *Evaluate* $L\{t(3\sin 2t - 2\cos 2t)\}$.

We have $L(3\sin 2t - 2\cos 2t) = 3L(\sin 2t) - 2L(\cos 2t)$

$$= 3.\frac{2}{s^2+4} - 2.\frac{s}{s^2+4} = \frac{2(3-s)}{s^2+4}$$

$$\therefore L\{t(3\sin 2t - 2\cos 2t)\} = -\frac{d}{ds}\left\{\frac{2(3-s)}{s^2+4}\right\} = -\frac{2(s^2+6s+4)}{(s^2+4)^2}$$

Ex 15. *Prove that* $L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$.

We have $L\{e^{at}\} = \frac{1}{s-a}$

$$\therefore L\{t^n e^{at}\} = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s-a}\right) = (-1)^n \frac{(-1)^n . n!}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}}$$

Ex 16. *Evaluate* $L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$.

$$L\{e^{-at} - e^{-bt}\} = L(e^{-at}) - L(e^{-bt}) = \frac{1}{s+a} - \frac{1}{s+b}$$

$= f(s)$, say

$$\therefore L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_s^\infty f(u)\,du = \int_s^\infty \left(\frac{1}{u+a} - \frac{1}{u+b}\right) du$$

$$= \lim_{X\to\infty} \int_s^X \left(\frac{1}{u+a} - \frac{1}{u+b}\right) du = \lim_{X\to\infty} [\log(u+a) - \log(u+b)]_s^X$$

$$= \lim_{X\to\infty}\left(\log\frac{X+a}{X+b} - \log\frac{s+a}{s+b}\right) = \lim_{X\to\infty} \log\left(\frac{1+\frac{a}{X}}{1+\frac{b}{X}}\right) + \log\frac{s+b}{s+a} = \log\frac{s+b}{s+a}$$

Ex 17. *Prove that* $L\left(\frac{\sin^2 t}{t}\right) = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$.

$$L(\sin^2 t) = \frac{1}{2} L(1 - \cos 2t) = \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\} = f(s), \text{ say}$$

$$\therefore L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty f(u)\,du = \frac{1}{2}\int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2+4}\right) du$$

$$=\frac{1}{2}\lim_{X\to\infty}\int_s^X\left(\frac{1}{u}-\frac{u}{u^2+4}\right)du=\frac{1}{2}\lim_{X\to\infty}\left[\log u-\frac{1}{2}\log\left(u^2+4\right)\right]_s^X$$

$$=\frac{1}{2}\lim_{X\to\infty}\left\{\log\frac{X}{\sqrt{X^2+4}}-\log\frac{s}{\sqrt{s^2+4}}\right\}$$

$$=\frac{1}{2}\lim_{X\to\infty}\left(\log\frac{1}{\sqrt{1+\frac{4}{X^2}}}-\log\frac{s}{\sqrt{s^2+4}}\right)$$

$$=-\frac{1}{2}\log\frac{s}{\sqrt{s^2+4}}=\frac{1}{2}\log\sqrt{\frac{s^2+4}{s^2}}$$

$$=\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right).$$

Ex. 18. *Evaluate* $\int_0^\infty e^{-3t}\sin t\cos t\,dt$ using Laplace transform

[WBUT 2011]

Now, $L\left(e^{-3t}\sin t\cos t\right)$

$$=\tfrac{1}{2}L\left(e^{-3t}\sin 2t\right)$$

We know $L\{\sin 2t\}=\frac{2}{s^2+2^2}=\frac{2}{s^2+4}=f(s)$, say

$\therefore$ By first shifting property

$$L\left(e^{-3t}\sin 2t\right)=f(s-(-3))=f(s+3)$$

$$=\frac{2}{(s+3)^2+4}$$

$$=\frac{2}{s^2+6s+13}$$

$\therefore\ L\left(e^{-3t}\sin t\cos t\right)$

$$=\tfrac{1}{2}\cdot\frac{2}{s^2+6s+13}$$

$$=\frac{1}{s^2+6s+13}$$

$$\therefore\ \int_0^\infty e^{-st}e^{-3t}\sin t\cos t\,dt=\frac{1}{s^2+6s+13}$$

Putting $s=0$ in the above result we get

$$\int_0^\infty e^{-3t}\sin t\cos t\,dt=\frac{1}{13}$$

Ex 19. *Evaluate* $\int_0^\infty t^3e^{-t}\sin t\,dt$.

We suppose the integral is convergent.

Now, $L(\sin t)=\frac{1}{s^2+1}$

$$\therefore\ L\left(t^3\sin t\right)=(-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$$

$$=24s\left(s^2+1\right)^{-3}\left\{-1+2s^2\left(s^2+1\right)^{-1}\right\}$$

$\therefore\ L\left(e^{-t}t^3\sin t\right)$

$$=24(s+1)\left\{(s+1)^2+1\right\}^{-3}\times\left[-1+2(s+1)^2\left\{(s+1)^2+1\right\}^{-1}\right]$$

$=f(s)$, say

Therefore $\int_0^\infty t^3e^{-1}\sin t\,e^{-st}dt=f(s)$

Putting $s=0$, $\int_0^\infty t^3 e^{-t}\sin t\,dt = f(0) = 0$.

Ex 20. *Evaluate* $\int_0^\infty \frac{e^{-t}\sin t}{t}dt$.

We have, $L(\sin t) = \frac{1}{s^2+1}$

$$\therefore L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$$

$$\therefore L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty \frac{1}{u^2+2u+2}du$$

$$\therefore \int_s^\infty \left(\frac{e^{-t}\sin t}{t}\right)e^{-st}dt = \int_s^\infty \frac{1}{u^2+2u+2}du$$

Putting $s=0$, $\int_0^\infty \frac{e^{-t}\sin t}{t}dt = \int_0^\infty \frac{du}{u^2+2u+2}$

$= \frac{\pi}{4}$ (detail calculation is not shown)

Ex 21. *Evaluate the improper integral* $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}dt$, *assuming its convergency.*

We have $L(e^{-t}\sin^2 t) = \frac{2}{(s+1)(s^2+2s+5)}$ [See Ex - 7.]

$$\therefore L\left(\frac{e^{-t}\sin^2 t}{t}\right) = \int_s^\infty \frac{2du}{(u+1)(u^2+2u+5)}$$

$$\therefore \int_0^\infty \frac{e^{-t}\sin^2 t}{t}e^{-st} = \int_s^\infty \frac{2}{(u+1)(u^2+2u+5)}du$$

Putting $s=0$ we get $\int_0^\infty \frac{e^{-t}\sin^2 t}{t}dt = \int_0^\infty \frac{2du}{(u+1)(u^2+2u+5)}$

$$= 2\lim_{X\to\infty}\int_0^X \frac{du}{(u+1)\{(u+1)^2+4\}}$$

$$= 2\lim_{X\to\infty}\int_1^{X+1} \frac{dz}{z(z^2+4)}, \qquad \ldots (1)$$

[putting $u+1=z$]

Let $\frac{1}{z(z^2+4)} = \frac{A}{z} + \frac{Bz+C}{z^2+4}$ $\quad \therefore 1 = A(z^2+4) + Bz^2 + Cz$

Equating the coefficients of z^2, z and the constant terms, we have, $A+B=0, C=0, 4A=1$

$$\therefore A = \frac{1}{4}, B = -\frac{1}{4} \qquad \therefore \frac{1}{z(z^2+4)} = \frac{1}{4z} - \frac{z}{4(z^2+4)}$$

$$\therefore \int_1^{X+1}\frac{dz}{z(z^2+4)} = \frac{1}{4}[\log z]_1^{X+1} - \frac{1}{8}\left[\log(z^2+4)\right]_1^{X+1}$$

$$= \frac{1}{4}\log(X+1) - \frac{1}{8}\log(X^2+2X+5) + \frac{1}{8}\log 5$$

$$= \frac{1}{4}\left\{\log\frac{X+1}{\sqrt{X^2+2X+5}} + \frac{1}{2}\log 5\right\}$$

$\therefore$ From (1), $\displaystyle\int_0^{\infty}\frac{e^{-t}\sin^2 t}{t}dt = \frac{1}{2}\lim_{X\to\infty}\left\{\log\frac{1+\frac{1}{X}}{\sqrt{1+\frac{2}{X}+\frac{5}{X^2}}} - \frac{1}{2}\log 5\right\}$

$$= \frac{1}{2}\left(\log 1 - \frac{1}{2}\log 5\right) = \frac{1}{4}\log 5$$

Ex. 22. *Express the following function in terms of unit step function :*

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$$

and find its Laplace transform. [WBUT 2002]

The given function can be written as

$$f(t) = (t-1) - \{(3-t)-(t-1)\}u(t-2)$$

i.e., $f(t) = (t-1) - 2(2-t)u(t-2)$

where $u(t-2)\begin{cases} 1, & t \geq 2 \\ 0, & t < 2 \end{cases}$

$\therefore \quad L\{f(t)\} = L(t-1) + 2L\{(2-t)u(t-2)\}$

$$= L(t) - L(1) + 2e^{-2s}L\{2-(t+2)\}$$

$$= \frac{1}{s^2} - \frac{1}{s} - 2e^{-2s}L(t)$$

$$= \frac{1}{s^2} - \frac{1}{s} - 2e^{-2s}\cdot\frac{1}{s^2}$$

$$= \frac{1}{s^2}\left(1-2e^{-2s}\right) - \frac{1}{s}$$

Ex 23. *Find* $L\{F(t)\}$ *where*

$F(t) = t, \quad 0 < t < 1$

$\quad = 0, \quad 1 < t < 2$

and $F(t+2) = F(t)$. [WBUT 2015]

Here $F(t)$ is a periodic function of period 2.
So, by theorem of Art 5.2.11, we have,

$$L\{F(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}F(t)dt$$

$$= \frac{1}{1-e^{-2s}}\left\{\int_0^1 e^{-st}t\,dt - \int_1^2 e^{-st}\times 0\,dt\right\}$$

$$= \frac{1}{1-e^{-2s}}\int_0^1 te^{-st}dt = \frac{1}{1-e^{-2s}}\left\{\left[\frac{te^{-st}}{-s}\right]_0^1 + \frac{1}{s}\int_0^1 e^{-st}dt\right\}$$

$$= \frac{1}{1-e^{-2s}}\left\{-\frac{e^{-s}}{s} - \frac{1}{s^2}\left[e^{-st}\right]_0^1\right\}$$

$$= \frac{1}{1-e^{-2s}}\left\{-\frac{1}{se^s} - \frac{1}{s^2}\left(\frac{1}{e^s}-1\right)\right\} = \frac{e^s\left(e^s-s-1\right)}{s^2\left(e^{2s}-1\right)}.$$

Ex. 24. *Find Laplace transform of a periodic function* $f(t)$ *given by*

$f(t) = t$ for $0 < t < c$

$\quad = 2c - t$ for $c < t < 2c$ [WBUT 2003]

Here $F(t)$ is a periodic function of period $2c$

$\therefore \quad L\{F(t)\}$

$$= \frac{1}{1-e^{-2sc}}\int_0^{\infty} e^{-st}F(t)dt$$

$$= \frac{1}{1-e^{-2sc}}\left[\int_0^c te^{-st}dt + \int_0^{2c}(2c-t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2sc}}\left\{\left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{-s^2}\right]_0^c + \left[\frac{(2c-t)e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right]_c^{2c}\right\}$$

$$= \frac{1}{1-e^{-2sc}}\left\{\frac{ce^{-sc}}{-s} - \frac{e^{-sc}}{s^2} + \frac{1}{s^2} + \frac{e^{-2sc}}{s^2} - \frac{ce^{-sc}}{s} - \frac{e^{-sc}}{s^2}\right\}$$

$$= \frac{1}{1-e^{-2sc}}\left(\frac{1-e^{-sc}}{s}\right)^2 = \frac{1-e^{-sc}}{s^2\left(1-e^{-sc}\right)}$$

Ex 25. *If* $F(t) = \sin t$, $t > \pi$

$= \cos t$, $0 < t < \pi$

Find $L\{F(t)\}$.

We write the function as

$$F(t) = \cos t, \quad 0 < t < \pi$$
$$= \sin t, \quad t > \pi$$

By Theorem 1 of Art 5.2.12, we can write

$$F(t) = \cos t + (\sin t - \cos t)u(t-\pi)$$

where $u(t-\pi) = 1, \quad t > \pi$

$= 0, \quad t < \pi$

is unit step function.

$$\therefore L\{F(t)\} = L(\cos t) + L\{\sin t\, u(t-\pi)\} - L\{\cos t\, u(t-\pi)\} \quad \ldots(1)$$

Now, $L(\cos t) = \dfrac{s}{s^2+1}$

Also by Corollary of Theorem 4 of Art 5.2.12

$$L\{\sin t\, u(t-\pi)\} = e^{-\pi s}L\{\sin(t+\pi)\} = e^{-\pi s}L\{-\sin t\}$$

$$= -e^{-\pi s}L\{\sin t\} = -e^{-\pi s}.\frac{1}{s^2+1} = -\frac{e^{-\pi s}}{s^2+1}$$

Similarly $L\{\cos t\, u(t-\pi)\} = -\dfrac{se^{-\pi s}}{s^2+1}$

$\therefore$ From (1), we get $L\{F(t)\} = \dfrac{s}{s^2+1} - \dfrac{e^{-\pi s}}{s^2+1} + \dfrac{se^{-\pi s}}{s^2+1}$

$$= \frac{s+(s-1)e^{-\pi s}}{s^2+1}.$$

Ex 26. *Verify directly that* $L\left\{\int_0^t \left(u^2 - u + e^{-u}\right)du\right\} = \dfrac{1}{s}L\left(t^2 - t + e^{-t}\right)$.

$$\int_0^t \left(u^2 - u + e^{-u}\right)du = \left[\frac{u^3}{3} - \frac{u^2}{2} - e^{-u}\right]_0^t = \frac{t^3}{3} - \frac{t^2}{2} - e^{-t} + 1$$

$$\therefore L\left\{\int_0^t \left(u^2 - u + e^{-u}\right)du\right\}$$

$$= L\left(\frac{t^3}{3} - \frac{t^2}{2} - e^{-t} + 1\right) = \frac{1}{3}L\left(t^3\right) - \frac{1}{2}L\left(t^2\right) - L\left(e^{-t}\right) + L(1)$$

$$= \frac{1}{3}.\frac{3!}{s^4} - \frac{1}{2}.\frac{2!}{s^3} - \frac{1}{s+1} + \frac{1}{s}$$

$$= \frac{2}{s^4} - \frac{1}{s^3} + \frac{1}{s(s+1)} \quad \ldots(1)$$

Again $\dfrac{1}{s}L\left(t^2 - t + e^{-t}\right) = \dfrac{1}{s}\left\{L\left(t^2\right) - L(t) + L\left(e^{-t}\right)\right\}$

$$= \frac{1}{s}\left\{\frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s+1}\right\} = \frac{2}{s^4} - \frac{1}{s^3} + \frac{1}{s(s+1)} \quad \ldots(2)$$

$\therefore$ From (1) and (2), we have

$$L\left\{\int_0^t \left(u^2 - u + e^{-u}\right)\right\}du = \frac{1}{s}L\left(t^2 - t + e^{-t}\right)$$

Ex 27. *Prove that* $L\left\{\int_0^t \dfrac{1-e^{-x}}{x}dx\right\} = \dfrac{1}{s}\log\dfrac{s+1}{s}$.

$$L\left(1-e^{-t}\right) = L(1) - L\left(e^{-t}\right) = \frac{1}{s} - \frac{1}{s+1} = f(s), \quad \text{say}$$

$$\therefore L\left(\frac{1-e^{-t}}{t}\right) = \int_s^\infty f(u)du$$

$$= \int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du$$

$$= \lim_{X\to\infty} \int_s^X \left(\frac{1}{u} - \frac{1}{u+1}\right) du$$

$$= \lim_{X\to\infty} \left[\log \frac{u}{u+1}\right]_s^X$$

$$= \lim_{X\to\infty} \left(\log \frac{X}{X+1} - \log \frac{s}{s+1}\right)$$

$$= \lim_{X\to\infty} \left(\log \frac{1}{1+\frac{1}{X}} - \log \frac{s}{s+1}\right)$$

$$= 0 - \log \frac{s}{s+1} = \log \frac{s+1}{s} = \phi(s), \text{ say}$$

Therefore $L\left\{\int_0^t \frac{1-e^{-x}}{x} dx\right\} = \frac{1}{s}\phi(s) = \frac{1}{s} \log \frac{s+1}{s}$.

Ex 28. *Prove that* $\int_{t=0}^\infty \int_{u=0}^t \frac{e^{-t} \sin u}{u} du\, dt = \frac{\pi}{4}$.

$$\int_{t=0}^\infty \int_{u=0}^t \frac{e^{-t} \sin u}{u} du\, dt = \int_{t=0}^\infty e^{-t} \left\{\int_{u=0}^t \frac{\sin u}{u} du\right\} dt \quad \ldots (1)$$

Now $L(\sin t) = \frac{1}{s^2+1} = f(s)$, say

$$\therefore L\left(\frac{\sin t}{t}\right) = \int_s^\infty f(u) du \qquad \left[\because \lim_{t\to 0} \frac{\sin t}{t} \text{ exists}\right]$$

$$= \int_s^\infty \frac{1}{u^2+1} du = \lim_{X\to\infty} \int_s^X \frac{1}{u^2+1} du$$

$$= \lim_{X\to\infty} \left[\tan^{-1} u\right]_s^X = \lim_{X\to\infty} \left(\tan^{-1} X - \tan^{-1} s\right)$$

$$= \frac{\pi}{2} - \tan^{-1} s = \phi(s), \text{ say}$$

$$\therefore L\left\{\int_0^t \frac{\sin u}{u} du\right\} = \frac{1}{s}\phi(s) = \frac{\pi}{2s} - \frac{1}{s}\tan^{-1} s$$

$$\therefore L\left\{e^{-t} \int_0^t \frac{\sin u}{u} du\right\} = \frac{\pi}{2(s+1)} - \frac{\tan^{-1}(s+1)}{s+1}, \left[\text{by shifting property}\right]$$

$$= \psi(s)$$

$$\therefore \int_0^\infty e^{-t} \left\{\int_0^t \frac{\sin u}{u} du\right\} dt = \psi(0)$$

$$\therefore \int_{t=0}^\infty \int_0^t \frac{e^{-t} \sin u}{u} du\, dt = \frac{\pi}{2} - \tan^{-1}(1), \quad \text{by (1)}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Ex 29. *Find* $L\left\{\cosh t \int_0^t e^x \cosh x\, dx\right\}$.

$$\int_0^t e^x \cosh x\, dx = \int_0^t e^x \left(\frac{e^x + e^{-x}}{2}\right) dx$$

$$= \frac{1}{2}\int_0^t \left(e^{2x} + 1\right) dx = \frac{1}{2}\left(\frac{e^{2t}}{2} + t - \frac{1}{2}\right)$$

$$\therefore L\left\{\cosh t \int_0^t e^x \cosh x\, dx\right\} = L\left\{\cosh t \frac{1}{2}\left(\frac{e^{2t}}{2} + t - \frac{1}{2}\right)\right\}$$

$$= \frac{1}{4} L\left(e^{2t} \cosh t\right) + \frac{1}{2} L\left(t \cosh t\right) - \frac{1}{4} L\left(\cosh t\right)$$

$$=\frac{1}{4}\frac{s-2}{(s-2)^2-1}+\frac{1}{2}(-1)\frac{d}{ds}\left(\frac{s}{s^2-1}\right)-\frac{1}{4}\frac{s}{s^2-1}$$

$$=\frac{1}{4}\cdot\frac{s-2}{s^2-4s+3}+\frac{s^2+1}{2(s^2-1)^2}-\frac{s}{4(s^2-1)}.$$

EXERCISE

[I] SHORT ANSWER QUESTIONS

1. Find, from definition, the Laplace transform of the function $f(t)=2$.
2. Obtain a reduction formula for $L(t^n)$.
3. Find, from definition, $L(e^{5t})$
4. State why the Laplace transform of the function $f(t)=t$ does exist?
5. Find the Laplace transform of t^2+5t-7
6. Find $L(\cos^2 t)$
7. Find $L(t^2e^{-2t})$.
8. State the second shifting property of Laplace transform.
9. If $L\{F(t)\}=f(s)$ then prove that $L\left\{F\left(\frac{t}{5}\right)\right\}=5f(5s)$
10. If $L\{f(t)\}=\phi(s)$ then prove that $L\{e^{at}f(t)\}=\phi(s-a)$
11. Prove that $L\{Af(t)+B\phi(t)\}=AL\{f(t)\}+BL\{\phi(t)\}$ where A, B are two constants.
12. Applying second shifting property of Laplace transform evaluate $L\{F(t)\}$ where $F(t)=(t-1)^3, t>1$
$\qquad = 0, 0<t<1$.
13. Find the Laplace transform of the function $f(t)=\cos\left(t-\frac{2\pi}{3}\right)$, $t>\frac{2\pi}{3}$ $=0,\ t<\frac{2\pi}{3}$

14. Prove that $L\{tF(t)\}=-\frac{d}{ds}L\{F(t)\}$
15. Find the Laplace transform of $t\sinh 2t$
16. Given $L(f(t))=\frac{1}{s^2}+e^{-4s}\left(\frac{1}{s}-\frac{1}{s^2}\right)$ where $f(t)=t,\ 0\le t\le 4$
$\qquad =5,\ t>4$
Find $L\left\{\frac{d}{dt}f(t)\right\}$.
17. Evaluate the Laplace transform of $\frac{\sin^2 t}{t}$
18. Using Laplace transform evaluate $\int_0^\infty t\cos t\,dt$.
19. Using Laplace transform evaluate $\int_0^\infty t\cosh t\,dt$.
20. Determine the Laplace transform of $f(t)$ that is periodic and defined on one period as
$f(t)=1,\ 0\le t\le 2$
$\quad =0,\ 2\le t<3$
21. Determine $L\{f(t)\}$ where $f(t)$ is periodic and defined on one period as $f(t)=t,\ 0\le t\le 1$
$\quad =0,\ 1\le t<2$
22. $f(t)=2t,\ 0<t<\pi$
$\quad =1 \quad t>\pi$
(a) Express this function in terms of unit step function
(b) Find $L\{F(t)\}$.
23. Evaluate $L\{e^{t-1}u(t-1)\}$ where $u(t-1)$ is unit step function.

ANSWERS

1. $\frac{2}{s}$ 2. $L(t^n)=\frac{n}{s}L(t^{n-1})$ 3. $\frac{1}{s-5}$ 5. $\frac{2}{s^3}+\frac{5}{s^2}-\frac{7}{s}$

6. $\frac{1}{2s}+\frac{s}{2s^2+8}$ 7. $\frac{2}{(s+2)^3}$ 12. $\frac{6e^{-s}}{s^4}$

13. $e^{-\frac{2\pi s}{3}}\cdot\frac{s}{s^2+1}$ 15. $\frac{4s}{(s^2-4)^2}$ 16. $\frac{1-e^{-4s}}{s}$

17. $\frac{1}{4}\log\frac{s^2+4}{s^2}$ 18. -1 19. 1 20. $\frac{e^{-2s}-1}{s(e^{-3s}-1)}$

21. $\frac{1}{1-e^{-2s}}\left\{\frac{1}{s^2}-\frac{e^{-s}}{s^2}-\frac{e^{-s}}{s}\right\}$

22. (a) $(1-2t)u(t-\pi)+2tu(t-0)$ (b) $\frac{2}{s^2}+\left(\frac{1-2\pi}{s}-\frac{2}{s^2}\right)e^{-\pi s}$

23. $\frac{e^{-s}}{s-1}$

[II] Long Answer Questions

1. Find, from definition, the Laplace Transform of the function
   a) $F(t)=\sin t,\quad 0<t<\pi$
   $\quad\quad =0\quad,\quad t>\pi$
   b) $F(t)=e^t,\quad 0<t<5$
   $\quad\quad =3,\quad t>5$
   c) $F(t)=1,\quad t>a$
   $\quad\quad =0,\quad t<a$ [W.B.U.Tech.2006]
2. Prove that the function e^{t^2} does not satisfy the second sufficient condition for existence of Laplace Transform.
3. From the definition of Laplace Transform evaluate $L\{f(t)\}$ where

$$f(t)=(t-1)^2,\ t>1$$
$$=0\quad,\ 0<t<1$$

4. Evaluate $L\{4t^2-3\cos 2t+5e^{-t}\}$.

5. Evaluate $L\{2\cos 2t-3\sin 4t+6t^3+4e^{5t}\}$.
6. Evaluate $L\{(t^2+1)^2\}$.
7. Evaluate $L\{7e^{2t}+9e^{-2t}+5\cos t+5\sin 3t+7t^3+2\}$.
8. Evaluate $L\{\sin t\cos t\}$.
9. Evaluate $L\{\sin^2 at\}$
10. Evaluate $L\{-3\cosh 5t+4\sinh 5t\}$
11. Evaluate $L(\sin t-\cos t)^2$
12. Evaluate $L(t^3e^{-3t})$
13. (a) Evaluate $L(2e^{3t}\sin 4t)$
    (b) $L\{e^t\cos t\sin t\}$ [W.B.U.Tech.2006]
14. Evaluate $L\left[(1+te^{-t})^3\right]$
15. Evaluate $L\{(t+1)^2e^t\}$.
16. Evaluate $L(G(t))$ where $G(t)=(t-2)^3,\ t>2$
    $=0,\quad t<2$
17. Find $L\{F(t)\}$ if $F(t)=(t-1)^2,\ t>1$
    $=0\quad,\ 0<t<1$
18. If $L\{F(t)\}=\frac{s^2-s+1}{(s-1)(2s+1)^2}$ then prove that $L\{F(2t)\}=\frac{s^2-2s+4}{4(s+1)^2(s-2)}$ by applying change of scale. [W.B.U.Tech 2002]
19. Find $L\{F(t)\}$ where

$$F(t)=\sin\left(t-\tfrac{2\pi}{3}\right),\quad t>\tfrac{2\pi}{3}$$
$$=0,\quad t<\tfrac{2\pi}{3}$$

20. Find $L\{F(t)\}$ for $F(t)=(t-1)^3, \quad t>1$

$= 0 \quad , \quad 0<t<1$

21. Find $L\{F(t)\}$ for $F(t)=e^{2(t-3)}\sin 3(t-3), \quad t>3$

$= 0 \quad , \quad t<3$

22. If $F(t)=2t, \quad 0\le t\le 1$

$= t, \quad t>1$; find

(i) $L\{F(t)\}$,

(ii) $L\{F'(t)\}$

(iii) Does the result $L\{F'(t)\}=sL\{F(t)\}-F(0)$ hold for this case?

23. Given $f(t)=t+1, \quad 0<t<2$

$= 3 \quad , \quad t>2$;

find $L\{f(t)\}$ and $L\{f'(t)\}$.

24. Find $L\{t\sin mt\}$.

25. Find $L\{t^2e^{-3t}\cos at\}$.

26. Evaluate $L(t^2e^t\sin t)$.

27. Evaluate $L(t\sinh 2t)$.

28. Evaluate $L(t^3\cos t)$

29. Evaluate $L\{(t^2-3t+2)\sin 3t\}$.

30. Evaluate $L\{\sin at - at\cos at\}$.

31. Find (a) $L\left(\frac{\cos \alpha t-\cos \beta t}{t}\right)$. (b) $L\left(\frac{1-\cos t}{t^2}\right)$

32. Prove that $L\left(\frac{\sin t}{t}\right)=\tan^{-1}\frac{1}{s}$. Hence find $L\left(\frac{\sin at}{t}\right)$.

[W.B.U.Tech.2003]

33. Evaluate the improper integral $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$, assuming its convergency.

34. Evaluate $\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}dt$.

35. Evaluate $\int_0^\infty \frac{1-\cos t}{t^2}dt$.

36. Evaluate $\int_0^\infty t\sin t e^{-3t}dt$

37. (a) Evaluate $\int_0^\infty \frac{\sin^3 t}{t}dt$.

(b) Find the Laplace Tranform of $\frac{\sin at}{t}$.

Hence show that $\int_0^\infty \frac{\sin t}{t}dt=\frac{\pi}{2}$ [W.B.U.Tech.2005]

38. Find $\int_0^\infty te^{-2t}\cos t\,dt$.

39. Find $\int_0^\infty te^t\cos t\,dt$.

40. Find $L\{F(t)\}$ where

$F(t)=3t, \quad 0<t<2$

$= 6, \quad 2<t<4$ and $F(t)$ has period 4.

41. (a) Let $F(t)=t, \quad 0<t<\pi$

$= \pi - t, \quad \pi<t<2\pi$,

where $F(t)=F(t+2\pi)$; find $L\{F(t)\}$.

Draw its graph.

42. Find $L\{F(t)\}$ if $F(t)=e^t, 0<t<2\pi$ and $F(t)$ is periodic of period 2π.

43. Find $L\{F(t)\}$ if $F(t) = -1,\ a \le t \le 2a$

$= 1\ ,\ 0 \le t < a$

where $F(t+2a) = F(t)$ for all t.

44. Find $L\{F(t)\}$, if $F(t) = \frac{t}{k}$ and $F(t+k) = F(t)$ for all values of t

45. If $F(t) = 5\sin 3\left(t - \frac{\pi}{4}\right),\quad t > \frac{\pi}{4}$

$= 0 \qquad ,\ t < \frac{\pi}{4}$

find $L\{F(t)\}$.

46. If $F(t) = t^2,\quad 0 < t < 2$

$= 4t,\quad t > 2,$ find $L\{F(t)\}$

47. If $F(t) = \sin t,\quad 0 < t < \pi$

$= \sin 2t,\quad \pi < t < 2\pi$

$= \sin 3t,\quad t > 2\pi,$ find $L\{F(t)\}$.

48. Evaluate $L\{t^2 u(t-2)\}$.

49. Find $L\{g(t)\}$ where $g(t) = 2t, 0 \le t \le 5\ = 1, t > 5$

50. Find $L\{F(t)\}$ where

$F(t) = t, 0 < t < 4$

$= 5, t > 4$

51. Find $L\left\{\int_0^t \frac{\sin u}{u} du\right\}$.

52. Find $L\left\{t \int_0^t \frac{\sin u}{u} du\right\}$

53. Evaluate $L\left\{\int_0^t e^{-3t} \sin 2t\,dt\right\}$.

54. Evaluate $L\left\{\int_0^t t e^{-3t} \cos 4t\,dt\right\}$.

55. Evaluate $L\left\{\int_0^t \frac{e^{2t} \sin t}{t} dt\right\}$.

ANSWERS

1. (a) $\frac{e^{-s\pi}+1}{s^2+1}$ (b) $\frac{3}{5}e^{-5s} - \frac{1-e^{-5(s-1)}}{s-1}$ (c) $\frac{e^{-as}}{s}$ **3.** $2e^{-s}/s^3$

4. $\frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$ **5.** $\frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} - \frac{2s}{s^2-4}$

6. $s^4 + 4s^2 + 24/s^5,\ s > 0$ **7.** $\frac{16s-4}{s^2-4} + \frac{5s}{s^2+1} + \frac{15}{s^2+9} + \frac{42+2s^3}{s^4}$

8. $\frac{1}{s^2+4},\ s > 0$ **9.** $\frac{2a^2}{s(s^2+4a^2)}$ **10.** $\frac{20-3s}{s^2-25}, s > 5$

11. $\frac{s^2-2s+4}{s(s^2+4)}, s > 0$ **12.** $6/(s+3)^4$

13. (a) $8/(s^2-6s+25)$ (b) $\frac{1}{s^2-2s+5}$

14. $\frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$ **15.** $s^2 + 1/(s-1)^3$

16. $6e^{-2s}/s^4$ **17.** $\frac{2e^{-s}}{s^3}$

19. $\frac{e^{-2\pi s/3}}{s^2+1}$ **20.** $\frac{6e^{-s}}{s^4}$ **21.** $\frac{3e^{-3s}}{s^2-4s+13}$

22. (i) $\frac{2}{s^2}-\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}$ (ii) $\frac{2}{s}-\frac{e^{-s}}{s}$

23. $\frac{1}{s}+\frac{1}{s^2}\left(1-e^{-2s}\right)$, $\frac{1}{s}\left(1-e^{-2s}\right)$

24. $\frac{2ms}{\left(s^2+m^2\right)^2}$

25. $-\frac{s^3-9s^2+\left(3a^2+9\right)s+5a^2+33}{\left(s^2+2s+a^2+9\right)^3}$

26. $\frac{6s^2-12s+4}{s^2-2s+2}$

27. $\frac{4s}{\left(s^2-4\right)^2}$

28. $\frac{6s^4-36s^2+6}{\left(s^2+1\right)^4}$

29. $\frac{6s^4-18s^3+126s^2-162s+432}{\left(s^2+9\right)^3}$

30. $\frac{2a^3}{\left(s^2+a^2\right)^2}$

31. (a) $\frac{1}{2}\log\frac{s^2+\beta^2}{s^2+\alpha^2}$ (b) $\cot^{-1}s-\frac{s}{2}\log\frac{s^2+1}{s^2}$

32. $\tan^{-1}\frac{a}{s}$

33. $\log 3$

34. $\log 2$

35. $\frac{\pi}{2}$

36. $\frac{3}{50}$

37. (a) $\frac{\pi}{4}$, (b) $\cot^{-1}\frac{s}{a}$

38. $\frac{3}{25}$

39. 0

40. $\frac{3-3e^{-2s}-6se^{-4s}}{\left(1-e^{-4s}\right)s^2}$

41. (a) $\frac{1-e^{-\pi s}\left(\pi s+1\right)}{\left(1+e^{-\pi s}\right)s^2}$ (b) $\frac{1}{s^2}\frac{1-e^{-sc}}{1+e^{-sc}}$

42. $\frac{1-e^{2(1-s)\pi}}{(s-1)\left(1-e^{-2\pi s}\right)}$

43. $\frac{1-e^{-as}}{s\left(1+e^{-as}\right)}$

44. $\frac{1}{k\left(1-e^{-ks}\right)}\left[-\frac{k}{s}e^{-ks}-\frac{1}{s^2}e^{-ks}+\frac{1}{s^2}\right]$

45. $\frac{15e^{-\frac{\pi s}{4}}}{s^2+9}$

46. $\frac{2}{s^3}+\frac{2e^{-2s}}{s^3}\left(2s^2-1\right)$

47. $\frac{1}{s^2+1}-e^{-\pi s}\frac{2}{s^2+4}+\frac{e^{-\pi s}}{s^2+1}+\frac{3e^{-2\pi s}}{s^2+9}-\frac{2e^{-2\pi s}}{s^2+4}$

48. $\frac{2}{s^3}-\frac{2e^{-2s}}{s^3}\left(1+2s+2s^2\right)$

49. $\frac{2}{s^2}\left(1-e^{-5s}\right)-\frac{9}{s}e^{-5s}$, $s>0$

50. $\frac{1+(s-1)e^{-4s}}{s^2}$

51. $\frac{1}{s}\tan^{-1}\frac{1}{s}$

52. $\frac{1}{s\left(1+s^2\right)}+\frac{1}{s^2}\cot^{-1}s$

53. $\frac{2}{s\left(13+6s+s^2\right)}$

54. $\frac{s^2+6s-7}{s\left(s^2+6s+25\right)^2}$

55. $\frac{1}{s}\cot^{-1}(s-2)$

[III] MULTIPLE CHOICE QUESTIONS

1. Laplace tranform of the function sin at is

(a) $\frac{s}{s^2+a^2}$ (b) $\frac{s}{s^2-a^2}$

(b) $\frac{a}{s^2+a^2}$ (c) $\frac{a}{s^2-a^2}$ [WBUT 2011]

2. $L\{\sinh at\}=$

(a) $\frac{a}{s^2-a^2}$ (b) $\frac{a}{s^2+a^2}$

(c) $\frac{s}{s^2-a^2}$ (d) $\frac{s}{s^2+a^2}$

3. $L\{\cosh 3t\} =$

(a) $\frac{3}{s^2+9}$ (b) $\frac{3}{s^2-9}$

(c) $\frac{s}{s^2+9}$ (d) $\frac{s}{s^2-9}$

4. Laplace tranform of the function $\cos(at)$ is

(a) $\frac{s}{s^2-a^2}$ (b) $\frac{a}{s^2+a^2}$

(c) $\frac{s}{s^2+a^2}$ (d) $\frac{1}{s^2-a^2}$ [WBUT 2007]

5. $L\{\sin^2 t\} =$

(a) $\frac{2s}{(s^2+4)}$ (b) $\frac{2}{s(s^2+4)}$

(c) $\frac{2}{s^2+4}$ (d) none of these

6. $L\{\cos^2 at\} =$

(a) $\frac{s^2+2a^2}{s(s^2+4a^2)}$ (b) $\frac{s}{s^2+4a^2}$

(c) $\frac{2s^2}{s(s+4a^2)}$ (d) none of these

7. If $L\{f(t)\} = \frac{1}{s+1}$, then $L\{f(3t)\} =$

(a) $\frac{3}{s+3}$ (b) $\frac{1}{s+3}$

(c) $\frac{1}{3s+3}$ (d) $\frac{3}{3s+1}$

8. The laplace transform of $e^{-3t}\sin 4t$ is

(a) $\frac{4}{s^2+6s-7}$ (b) $\frac{s}{s^2+6s-7}$

(c) $\frac{1}{s^2+6s-7}$ (d) $\frac{4}{s^2-6s-25}$

[WBUT 2012]

9. The Laplace transform of a unit step function $u(t-a)$ is

(a) $\frac{e^{-s}}{s}$ (b) $\frac{e^{-sa}}{s-a}$

(c) $\frac{e^{-sa}}{s}$ (d) none of these

10. $L(\sin 3t \sin 2t) =$

(a) $\frac{s}{(s^2+1)(s^2+25)}$ (b) $\frac{12s}{(s^2+25)(s^2+1)}$

(c) $\frac{12s}{(s^2+1)^2}$ (d) none of these.

11. If $L(f(t)) = \frac{s^2}{4s+1}$ then the Laplace transform of the function $f(5t)$ is

(a) $\frac{s^2}{4s+5}$ (b) $\frac{s^2}{100s+5}$

(c) $\frac{s^2}{25(4s+5)}$ (d) none

12. $L(e^{-2t}\cos t) =$

(a) $\frac{s+2}{s^2+4s+5}$ (b) $\frac{s}{s^2+4s+5}$

(c) $\frac{s+1}{s^2+4s+1}$ (d) none [WBUT 2010, 2015]

13. $L\left\{e^{-3t}(2\cos 5t - 3\sin 5t\right\} =$

(a) $\dfrac{s-9}{s^2+6s+34}$ (b) $\dfrac{2s-9}{s^2+6s+34}$

(c) $\dfrac{2s+9}{s^2+6s+34}$ (d) none

14. $L(t\cos t) =$

(a) $\dfrac{s}{s^2+1}$ (b) $\dfrac{s+1}{s^2+1}$

(c) $\dfrac{2s}{s^2+1}$ (d) $\dfrac{s^2-1}{(s^2+1)^2}$

15. $L\left(te^{2t}\right)$ is equal to

(a) $\dfrac{1}{s-2}$ (b) $2(s-2)^2$

(c) $\dfrac{1}{(s-2)^2}$ (d) $\dfrac{2!}{s^2}$ [WBUT 2006]

16. $L(t^3e^{-3t}) =$

(a) $\dfrac{1}{(s+3)^4}$ (b) $\dfrac{6}{(s+3)^3}$

(c) $\dfrac{6}{(s+3)^4}$ (d) $\dfrac{2}{(s+3)^4}$

17. $L\left(\dfrac{t\sin t}{e^t}\right) =$

(a) $\dfrac{s+1}{(s^2+2s+2)}$ (b) $\dfrac{s+1}{(s^2+2s+2)^2}$

(c) $\dfrac{2(s+1)}{(s^2+2s+2)^2}$ (d) $\dfrac{s+2}{(s^2+2s+2)^2}$

18. $L\left\{\dfrac{\sin t}{t}\right\} = \tan^{-1}\left(\dfrac{1}{s}\right)$. Then $L\left\{\dfrac{\sin at}{t}\right\}$ is

(a) $\tan^{-1}\dfrac{1}{s^2}$ (b) $\tan^{-1}\dfrac{a}{s}$

(c) $\tan^{-1}\left(\dfrac{1}{as}\right)$ (d) $\tan^{-1}\left(\dfrac{1}{s^2+a^2}\right)$

[WBUT 2006]

19. $L\left(\dfrac{\sin 2t}{t}\right) =$

(a) $\sin^{-1}\dfrac{s}{2}$ (b) $\tan^{-1}\dfrac{s}{2}$

(c) $\cot^{-1}\dfrac{s}{2}$ (d) $\cos^{-1}\dfrac{s}{2}$

20. $\int_0^\infty t\cos 2t\, dt =$

(a) $\dfrac{1}{4}$ (b) $-\dfrac{1}{4}$

(c) 0 (d) $\dfrac{3}{4}$

21. $L\left\{\int_0^t t\sin 3t\,dt\right\} =$

(a) $\dfrac{6}{s^2+9}$ (b) $\dfrac{6}{(s^2+9)^2}$

(c) $\dfrac{2s}{(s^2+9)^2}$ (d) $\dfrac{3}{s^2+9}$

22. The value of $\int_0^\infty \dfrac{\sin t}{t}\,dt$ is equal to

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{6}$

(c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{2}$ [WBUT 2009]

23. If $L\{f(t)\} = \tan^{-1}\left(\frac{1}{p}\right)$, then $L\{tf(t)\}$ is

(a) $\tan^{-1}\left(\frac{1}{p^2}\right)$ (b) $\frac{1}{1+p^2}$

(c) $\frac{1}{1+p}$ (d) $\tan^{-1}\left(\frac{2}{\pi p}\right)$

[WBUT 2008]

24. $\int_0^\infty t^3 e^{5t}\, dt =$

(a) $\frac{1}{625}$ (b) $\frac{6}{25}$

(c) $\frac{6}{625}$ (d) none

25. $L\left\{\int_0^t e^t \cosh t\, dt\right\} =$

(a) $\frac{s-1}{s^2(s-2)}$ (b) $\frac{s}{s-2}$

(c) $\frac{s-1}{s^3(s-2)}$ (d) none

26. Laplace transform of the function

$$f(t) = \begin{cases} 2, & 9 < t < a \\ 0, & t \geq a \end{cases} \text{ is}$$

(a) $\frac{2}{s}\left(1-e^{-sa}\right)$ (b) $\frac{2}{s}\left(1+e^{-sa}\right)$

(c) e^{-sa} (d) $\frac{1}{s}\left(1-e^{-sa}\right)$

27. If $f'(t) = 1,\ t \geq 5$

$= 0,\ t < 5$ then $L\{f(t)\} =$

(a) $\frac{e^{5s}}{5}$ (b) e^{-5s}

(c) $\frac{e^{-5s}}{s}$ (d) $\frac{e^{-s}}{5}$

28. $L\{\sin(2t+10)u(t-5)\} =$

(a) $\frac{e^{-5s}}{s^2+4}$ (b) $\frac{2}{s^2+4}$

(c) $\frac{2e^{-s}}{s^2+4}$ (d) $\frac{2e^{-5s}}{s^2+4}$

[$u(t-5)$ is unit step function]

29. If $f(t) = t$ when $0 \leq t \leq 2$ and $= 0$ when $2 < t < \infty$ then $L\{f(t)\} =$

(a) $\frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}$ (b) $\frac{1}{s^2} - \frac{2e^{-2s}}{s}$

(c) $\frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{3e^{-2s}}{s}$ (d) none

30. If $f(t) = 3\sin 5t,\ 0 \leq t \leq \frac{\pi}{5}$

$= 0,\ t > \frac{\pi}{5}$; then $L\{f(t)\} =$

(a) $3(s^2+5^2)^{-1}\left(1+e^{-\frac{3\pi s}{5}}\right)$ (b) $2(s^2+4)^{-1}(1+e^{-\frac{\pi s}{5}})$

(c) $15(s^2+25)^{-1}(1+e^{-\frac{\pi s}{5}})$ (d) none

31. If $f(t)$ is a period function with period T, then $L\{f(t)\}$ is

(a) $\dfrac{1}{1+e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

(b) $\dfrac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

(c) $\dfrac{1}{1+e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

(d) none of these

32. Value of $L\{f(t)\}$ where $f(t)$ is defined as $f(t)=\sin t$, $0 \le t \le \pi$ and is periodic with period π, is

(a) $\dfrac{1+e^{-\pi s}}{1-e^{-\pi s}}\cdot\dfrac{1}{s^2+1}$

(b) $\dfrac{1-e^{-\pi s}}{1+e^{-\pi s}}\cdot\dfrac{1}{s^2+1}$

(c) $\dfrac{1-e^{-\pi s}}{s^2+1}$

(d) none of these

33. If $f(t)$ is periodic with period 2π and

$f(t)=1,\ 0<t<\pi$

$\quad = -1,\ \pi<t<2\pi$ then $L\{f(t)\}=$

(a) $\dfrac{1}{(e^{-2\pi s}-1)}\left\{2e^{-\pi s}-e^{-2\pi s}-1\right\}$

(b) $\dfrac{1}{s(e^{-2\pi s}-1)}\left\{2e^{-\pi s}-e^{-2\pi s}-1\right\}$

(c) $e^{-\pi s}$

(d) none

ANSWERS

1.c	2.a	3.d	4.c	5.b	6.a	7.b	8.d
9.?	10.b	11.c	12. a	13.b	14.d	15.c	16.c
17.c	18.b	19.c	20.b	21.b	22.d	23.b	24.c
25.a	26.a	27.c	28. d	29.a	30.c	31.?	32.a
33.b							