• IV Forced Oscillation

A free or damped oscillation may undergo the effect of external force which, in so many practical situations, is a function of time. In the following we shall consider the effect of such force on damped oscillation.

• *The inhomogeneous differential equation*: Let's consider the equation of motion for free damped vibration:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \quad \cdots\cdots [30]$$

where, γ = damping constant

ω_0 = natural frequency of vibration.

In presence of a time varying external force $F(t)$ eq^n [30] will be modified by an acceleration term $f(t) = F(t)/_m$ leading to,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) \quad \cdots\cdots [30]'$$

• *Remark*: 1. Eq [30]′ is an inhomogeneous differential equation in the sense it contains a term $f(t)$ on the right hand side which is independent of x and its derivative.

2. Eq^n·[30]′ is linear for obvious reason and its solution can be determined in view of the following theorem.

• *Theorem – 6*: Let X_p be a particular solution of [30]′. The general solution X_g of eq^n·[30]′ is given by

$$X_g = X_h + X_p \quad \cdots\cdots [31]$$

where X_h is the solution of the homogeneous equation eq^n·[30].

Proof: Let $x_1(t)$ and $x_2(t)$ be two solutions of [30]′

Then $\ddot{x}_1 + 2\gamma\dot{x}_1 + \omega_0^2 x_1 = f(t)$

and $\ddot{x}_2 + 2\gamma\dot{x}_2 + \omega_0^2 x_2 = f(t)$

$$\Rightarrow (\ddot{x}_1 - \ddot{x}_2) + 2\gamma(\dot{x}_1 - \dot{x}_2) + \omega_0^2(x_1 - x_2) = 0$$

Hence, the difference $x_1 - x_2$ is a solution of the homogeneous eq^n·[30]. Therefore

$$x_1 - x_2 = X_h$$

The above equation is valid for any pair of solution of [30]$'$. Choosing $x_1 = X_g$ and $x_2 = X_p$

$$X_g - X_p = X_h \Rightarrow X_g = X_h + X_p.$$

• *Remark* : 1. Relation [31] is valid for any linear inhomo--geneous equation.

2. The form of $X_h(t)$ may change depending upon various types of damping.

3. We shall consider eq^n. [30]$'$ when $f(t)$ itself is harmonic with frequency ω i.e.

$$f(t) = f_0 \cos \omega t$$

because al-most all functions of t can be represested in terms of such type of harmonic functions. Eq^n. [30]$'$ therefore, takes the form,

$$\boxed{\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f_0 \cos \omega t \quad \text{-----} \quad [32]}$$

Equation [32] is called the equation of forced damped vibration.

• Solution of eq^n. [32] : $X_h(t)$ for eq^n. [32] is always of the form :-

$$X_h(t) = \exp(-\gamma t) g(t)$$

a term always dominated by the exponential part and hence at $t \to \infty$ $X_h(t) \to 0$. So for large t values the general solution $X_g(t)$ retains only the $X_p(t)$ part leading to the following definitions.

• Definition - 11 : Let the solution of eq^n. [32] be given by $X_g(t) = X_h(t) + X_p(t)$. As $\lim_{t \to \infty} X_h(t) = 0$

$$\lim_{t \to \infty} X_g(t) = X_p(t)$$

$X_h(t)$ is called transient solution and the state represented by it has at all any relevance in the initial moments of oscillation. Soon it starts to be fed by the driving force and for large t the system ~~st~~ likes to oscillate with driving frequency irrespective of the value of the damping constant. Such a state is called steady state.

From now on we shall call $x_p(t)$ as steady state solution and consider the following theorem regarding its behaviour.

• Theorem-7 : The steady-state solution $x_p(t)$ of eq^n [32] is

(i) Oscillatory with a phase difference $\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)$ with respect to the driving force.

(ii) A bounded function for $\gamma \neq 0$

Proof :(i) Let, $x_p(t) = A\cos\omega t + B\sin\omega t$, A, B consts.

$$\dot{x}_p(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\ddot{x}_p(t) = -\omega^2(A\cos\omega t + B\sin\omega t)$$

Substituting all these results in equation [32] we ge

$$-\omega^2(A\cos\omega t + B\sin\omega t) + 2\gamma(-A\omega\sin\omega t + B\omega\cos\omega t)$$
$$+\omega_0^2(A\cos\omega t + B\sin\omega t) = f_0\cos\omega t$$

Equating the co-efficients of $\cos\omega t$ and $\sin\omega t$ * we get,

$$\left.\begin{aligned} \cos\omega t :&= A(\omega_0^2-\omega^2) + 2\gamma\omega B = f_0 \\ \sin\omega t :&= B(\omega_0^2-\omega^2) - 2\gamma\omega A = 0 \end{aligned}\right\} \quad [33\ a, b]$$

Solving equation [33] for A and B

$$A = \frac{f_0(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

$$B = \frac{f_0\ 2\gamma\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

Giving, $$x_p(t) = \frac{f_0}{(\omega_0^2-\omega^2)+4\gamma^2\omega^2}\left[(\omega_0^2-\omega^2)\cos\omega t + 2\gamma\omega\sin\omega t\right]$$

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* If $c_1\cos\omega t + c_2\sin\omega t = 0$; differentiating we get, $-c_1\sin\omega t + c_2\cos\omega t = 0$. Multiplying the 1st eq^n by $\sin\omega t$ and 2nd by $\cos\omega t$ and adding we get $c_2 = 0$ & so $c_1 = c$

$$\Rightarrow x_p(t) = \frac{f_0}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}}\left[\frac{(\omega_0^2-\omega^2)\cos\omega t}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}} + \frac{2\gamma\omega\sin\omega t}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]}\right]$$

Identifying,
$$\cos\phi = \frac{\omega_0^2-\omega^2}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}} \quad \text{---[32a]}$$

and
$$\sin\phi = \frac{2\gamma\omega}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}} \quad \text{---[32b]}$$

we get
$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2-\omega^2}$$

or
$$\boxed{\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)} \quad \text{---[33]}$$

where as

$$x_p(t) = \tilde{A}(\omega,\omega_0,\gamma)\left[\cos\phi\cos\omega t + \sin\phi\sin\omega t\right]$$

or
$$x_p(t) = \tilde{A}(\omega,\omega_0,\gamma)\cos(\omega t-\phi) \quad \text{----[34]}$$

where
$$\boxed{\tilde{A}(\omega,\omega_0,\gamma) = \frac{f_0}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}}} \quad \text{--[35]}$$

(ii) The solution $x_p(t)$ is bounded provided $\tilde{A}(\omega,\omega_0,\gamma)$ is bounded for all ω, given ω_0 and γ fixed. We shall show that for $\gamma \neq 0$ $\tilde{A}$ indeed has an extremum i.e.;

$D = (\omega_0^2-\omega^2)^2+4\gamma^2\omega^2$ has an extremum (minimum for our purpose) say at $\omega=\omega_r$

$$\left.\frac{dD}{d\omega}\right|_{\omega=\omega_r} = \left.\left[8\gamma^2\omega - 4\omega(\omega_0^2-\omega^2)\right]\right|_{\omega_r} = 0$$

$$\Rightarrow \omega_r\left[2\gamma^2-\omega_0^2+\omega_r^2\right] = 0$$

Giving $\omega_r = 0$ or $\omega_r^2 = \omega_0^2 - 2\gamma^2$, where the 1st case is meaningless. Hence

$$\boxed{\omega_r^2 = \omega_0^2 - 2\gamma^2} \quad \text{----} [36]$$

Hence $$D(\omega)\Big|_{\omega=\omega_r} = \left(\omega_0^2 - \omega_0^2 + 2\gamma^2\right)^2 + 4\gamma^2\left(\omega_0^2 - 2\gamma^2\right)$$
$$= 4\gamma^2\omega_0^2 - 4\gamma^4 = 4\gamma^2(\omega_0^2 - \gamma^2)$$

Therefore, $$\boxed{\tilde{A}(\omega, \omega_0, \gamma)\Big|_{\omega=\omega_r} = \frac{f_0}{2\gamma(\omega_0^2 - \gamma^2)^{1/2}} \neq \infty \text{ for all } \gamma > 0,\ \gamma \neq 0} \quad \text{---} [37]$$

Remark: 1. Whether the condition $-[36]$ at all represents a minima of $D(\omega)$ and hence a maxima for $\tilde{A}$ can be verified by the following,

$$\frac{d^2 D(\omega)}{d\omega^2}\Big|_{\omega=\omega_r} = \left\{4\left[2\gamma^2 - \omega_0^2 + \omega^2\right] + 4\omega\left[2\omega\right]\right\}\Big|_{\omega=\omega_r}$$
$$= 8\left[\omega_0^2 - 2\gamma^2\right] > 0 \text{ for } \omega_0 > \sqrt{2}\,\gamma$$

which is the case in almost all practical situation.

2. The quantity ω_r is called <u>resonance frequency</u> and for small damping constant γ, ω_r is close to natural frequency of oscillation ω_0. The system is said to be in <u>resonance</u> or properly in <u>amplitude resonance</u> when it oscillates with the frequency ω_r.

3. It is obvious that at resonance the amplitude has finite maximum value so long $\gamma \neq 0$. For $\gamma = 0$, $\tilde{A}_{max}$ become infinite at resonance (see eqn. 37). Therefore at resonance it the damping factor that provides bound and prevent the system from ~~ultimate~~ an eventual breakdown.

4. Therefore, <u>close to resonance</u> the system is said to be <u>resistance controlled</u>.

4. When $\omega \ll \omega_0$ i.e.; driving frequency is much less than the natural frequency of vibration we can write from eqn. [35]

$$\tilde{A}(\omega,\omega_0,\gamma)\Big|_{\omega\ll\omega_0} = \frac{f_0}{\omega_0^2\left[\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+4\gamma^2\frac{\omega^2}{\omega_0^4}\right]^{1/2}}$$

$$\simeq \frac{F_0/m}{\omega_0^2} = \frac{F_0}{K}$$

In this regime the oscillation is controlled by the stiffness constant K. The resultant motion is called stiffness controlled.

5. When $\omega \gg \omega_0$ eqn. [35] can be represented as

$$\tilde{A}(\omega,\omega_0,\gamma)\Big|_{\omega\gg\omega_0} = \frac{f_0}{\omega^2\left[\left(1-\frac{\omega_0^2}{\omega^2}\right)^2+4\frac{\gamma^2}{\omega^4}\right]^{1/2}}$$

$$\simeq \frac{F_0/m\omega^2}{\left[1+4\frac{\gamma^2}{\omega^4}\right]^{1/2}} \simeq F/m\omega^2$$

In this regime the oscillation is chiefly controlled by the mass m. The resultant motion is called mass controlled.

6. The following diagram is very crucial in demonstrating the effect of forcing frequency (ω) on the amplitude $\tilde{A}$ in defferent regime.

wrong.

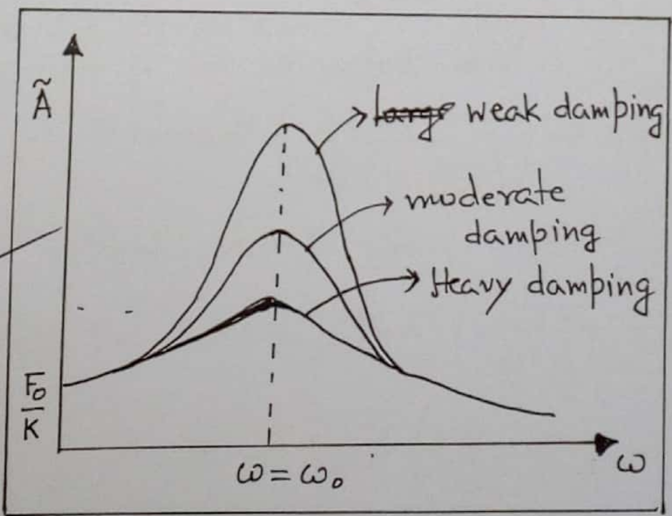

Diagram-7.

It is to be noted that

(i) For heavy damping the peak of the amplitude curve is slightly displaced from $\omega = \omega_0$ line, otherwise the resonance always occurs at $\omega = \omega_0$

(ii) Apart from resonance the steady state motion is not much sensitive to the damping factor.

7. The dependence of phase angle ϕ on the driving frequency ω can be represented by the following diagram.

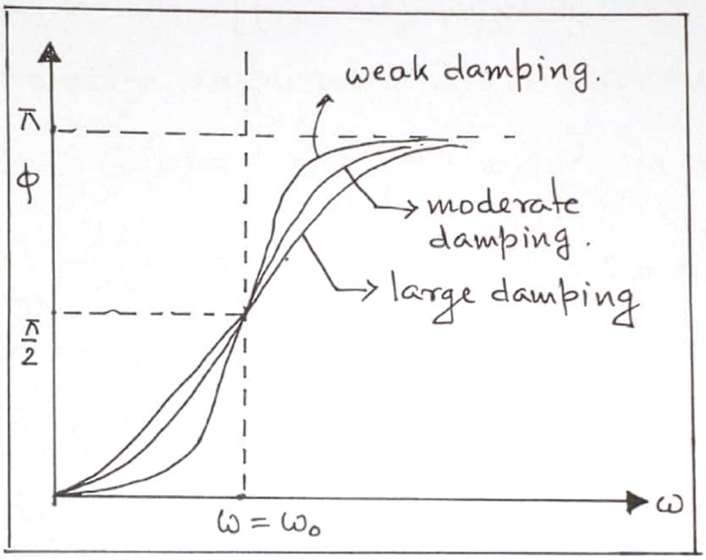

Diagram – 8

The angle $\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right)$ denotes the phase by which the displacement lags behind the driving force. It is to be noted that

(i) $\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2\left(1 - \frac{\omega}{\omega_0^2}\right)}\right) \to \tan^{-1}\frac{2\gamma\omega}{\omega_0^2}$ when $\omega_0 >> \omega$. $\phi = 0$ when $\omega_0 >> \omega$

(ii) For $\omega >> \omega_0$, $\tan^{-1}\left(\frac{2\gamma\omega}{\omega^2\left(\frac{\omega_0^2}{\omega^2} - 1\right)}\right) \to \tan^{-1}\frac{2\gamma}{\omega}$

$$= -\tan^{-1}\left(\frac{2\gamma}{\omega_0}\frac{\omega_0}{\omega}\right) \doteq \pi$$ for weak damping $(\gamma < \omega_0)$

(iii) At resonance $\phi = \tan^{-1}\infty = \frac{\pi}{2}$

8. The steady-state velocity is given by

$$v(t) = \dot{x}_p(t) = \frac{-f_0 \omega}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}} \sin(\omega t - \phi)$$

$$= \frac{f_0 \omega}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}} \cos\left(\omega t - \phi + \frac{\pi}{2}\right)$$

$$= V_0 \cos\left(\omega t - \phi + \frac{\pi}{2}\right)$$

where,

$$V_0 = \frac{f_0 \omega}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}}$$ is called

the velocity amplitude. It is maximum when

$$G(\omega) = \frac{(\omega_0^2-\omega^2)^2}{\omega^2} + 4\gamma^2$$ is minimum.

$$\Rightarrow \frac{dG}{d\omega} = 0 \text{ i.e.; } -\frac{2\omega_0^4}{\omega^3} + 2\omega = 0 \Rightarrow \omega = \omega_0$$

and $G_{min} = 4\gamma^2$ and so

$$(V_0)_{max} = \frac{f_0}{2\gamma} = \frac{F_0}{2m\gamma} \quad \text{.... [36]}$$

It is to be noted that,

(i) Though the maximum of velocity at resonance is always controlled by the damping factor γ the condition of resonance $(\omega = \omega_0)$ is independent of γ unlike amplitude resonance. The resonance thus obtained is called velocity resonance. The following diagram will clarify the matter.

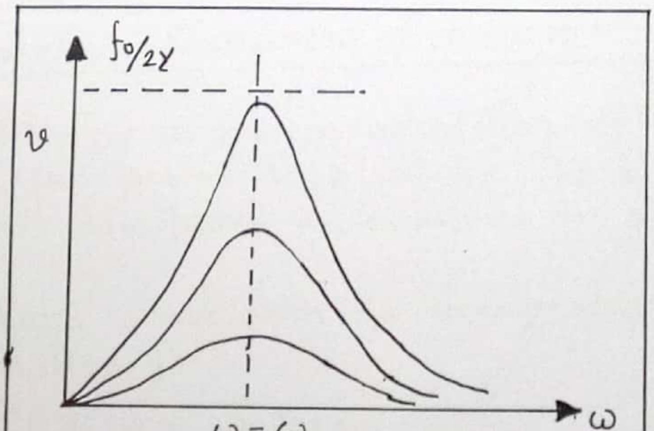

Diagram-9

9. The phase of the velocity relative to the driving force is given by

$$\delta = \phi - \frac{\pi}{2}$$

Clearly for $\omega << \omega_0$ i.e.; $\phi = 0$ $\delta = -\pi/2$, the velocity lags behind the driving force, while for $\omega >> \omega_0$, $\phi = \pi$ $\delta = \pi - \pi/2 = \pi/2$, the velocity leads the force. Finally for resonance $\omega = \omega_0$ the velocity becomes in phase with the driving force. The dependence of ~~v on ω~~ δ on ω is given in the following diagram.

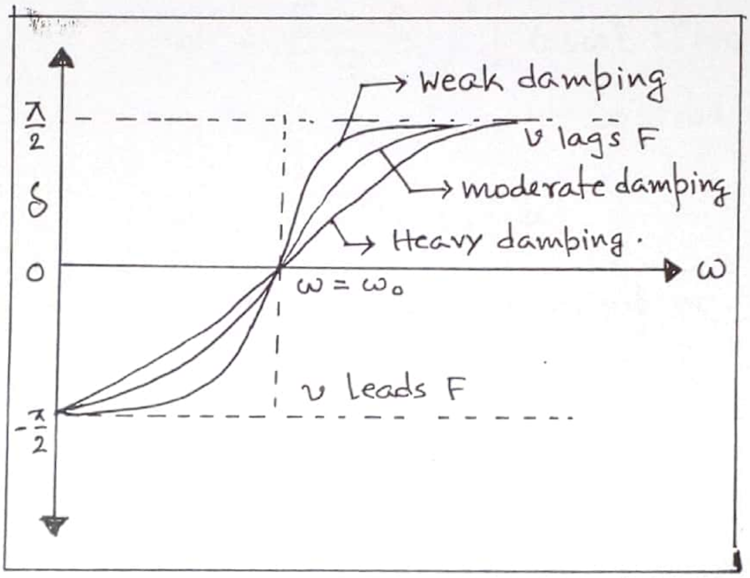

Diagram-10

10. When, $\omega = \omega_0$ i.e.; $\delta = 0$ as the velocity and the driving force are in ~~pa~~ phase, it is the most favourable situation to transfer maximum energy from the driving system. The following section will clarify the matter.

Power Transfer – Sharpness of resonance – Quality factor.

- *Theorem-8*: For a system admitting eqn.[32] as equation of motion, the power supplied by the driving force is equal to that dissipated by damping in steady state.

Proof: Let's start with the steady-state solution of [32] given by equation [34]

$$x_p(t) = \tilde{A} \cos(\omega t - \phi)$$

$$\Rightarrow \quad \dot{x}_p(t) = -\tilde{A}\omega \sin(\omega t - \phi)$$

Power delivered by the driving force,

$$P_{input} = F(t)\,\dot{x}_p(t)$$
$$= mf_0 \cos\omega t\left(-\tilde{A}\omega \sin(\omega t-\phi)\right)$$
$$= -\tilde{A}m\omega f_0 \cos\omega t \sin(\omega t-\phi)$$

So, $$\langle P_{input}\rangle = -\frac{\tilde{A}m\omega f_0}{T}\int_0^T \cos\omega t \sin(\omega t-\phi)\,dt$$
$$= -\frac{\tilde{A}m\omega f_0}{T}\int_0^T [\cos\omega t \sin\omega t\cos\phi - \cos^2\omega t\sin\phi]\,dt.$$
$$= \frac{\tilde{A}m\omega f_0}{2}\sin\phi$$

Power dissipated due to the damping force,

$$P_{dissipated} = 2m\gamma\left(\dot{x}_p\right)^2$$
$$= 2m\gamma\tilde{A}^2\omega^2\sin^2(\omega t-\phi)$$

So, $$\langle P_{dissipated}\rangle = 2m\gamma\tilde{A}^2\omega^2\frac{1}{T}\int_0^T \sin^2(\omega t-\phi)\,dt.$$
$$= \frac{2m\gamma\tilde{A}^2\omega^2}{2} = \frac{\tilde{A}m\omega f_0}{2}\sin\phi$$

[Using [32b]]

Hence, $\langle P_{input}\rangle = \langle P_{dissipated}\rangle = \langle P\rangle$ (let's call)

• Remark: 1. We can write $\langle P\rangle = \frac{\tilde{A}m\omega f_0}{2}\sin\phi$

$$\Rightarrow \langle P\rangle = \frac{\tilde{A}m\omega f_0}{2}\,\frac{2\gamma\omega}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{1/2}}$$

$$\Rightarrow \langle P\rangle = \frac{m\omega f_0}{2}\,\frac{2f_0\gamma\omega}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]}$$

$$\Rightarrow \langle P\rangle = \gamma m f_0^2\left[\frac{\omega^2}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\right]$$

Therefore, $\langle P \rangle$ is maximum when $\omega = \omega_0$ and so,

$$\langle P \rangle_{max} = \frac{m f_0^2}{4\gamma}$$

Hence, $$\langle P \rangle = \langle P \rangle_{max} \frac{4\omega^2\gamma^2}{\left[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]} \quad \text{-- [37]}$$

The following diagram will illustrate the dependence of $\langle P \rangle$ on ω.

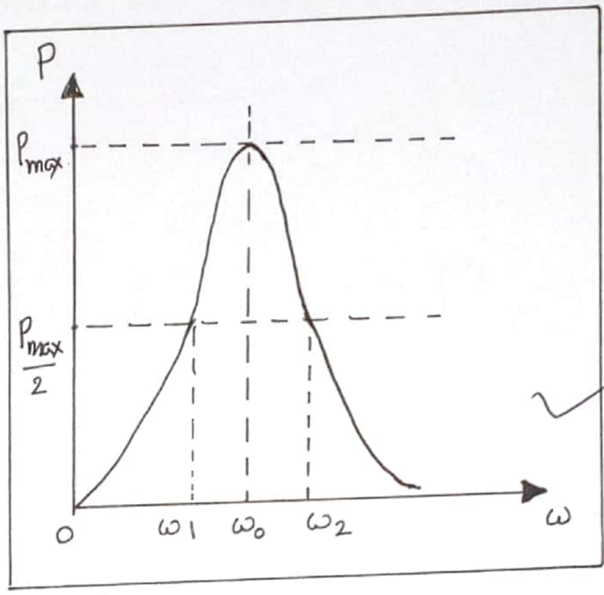

Diagram-11

Now $\langle P \rangle = \frac{1}{2}\langle P \rangle_{max}$ happens when

$$\frac{1}{2} = \frac{4\omega^2\gamma^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$$

$$\Rightarrow (\omega_0^2 - \omega^2)^2 = 4\omega^2\gamma^2$$

$$\Rightarrow \omega = \pm\gamma \pm \sqrt{\gamma^2 + \omega_0^2}$$

ω being positive we can have two possibilities

$$\left.\begin{array}{l} \text{(i)} \ \omega_1 = -\gamma + \sqrt{\gamma^2 + \omega_0^2} \\ \text{(ii)} \ \omega_2 = \gamma + \sqrt{\gamma^2 + \omega_0^2} \end{array}\right\}$$

Hence we can define the so called band width

$$\boxed{\Delta\omega = \omega_2 - \omega_1 = 2\gamma = \frac{1}{T}} \quad (T = \text{mean decay time}) \quad \text{--- [38]}$$

2. The so called sharpness of resonance can be under-stood with the help of what is known as quality factor defined as follows,

$$Q = \frac{\text{Resonant frequency}}{\text{Band width}} = \frac{\omega_0}{\Delta\omega}$$

or $$\boxed{Q = \frac{\omega_0}{2\gamma} = \omega_0 \Gamma} \quad \text{- - - [39]}$$

Expressing the steady-state amplitude $\tilde{A}$ in terms Q,

$$\tilde{A} = \frac{f_0}{\left[(\omega_0^2 - \omega^2)^2 + 4\omega^2\gamma^2\right]^{1/2}}$$

$$\Rightarrow \tilde{A} = \frac{f_0/2\omega\gamma}{\left[1 + \frac{(\omega_0^2-\omega^2)^2}{4\omega^2\gamma^2}\right]^{1/2}}.$$

$$\Rightarrow \tilde{A} = \frac{f_0 Q/\omega\omega_0}{\left[1 + Q^2\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2\right]^{1/2}} \quad \text{- - - - [40]}$$

The following diagram represents the dependence of $\tilde{A}$ on ω.

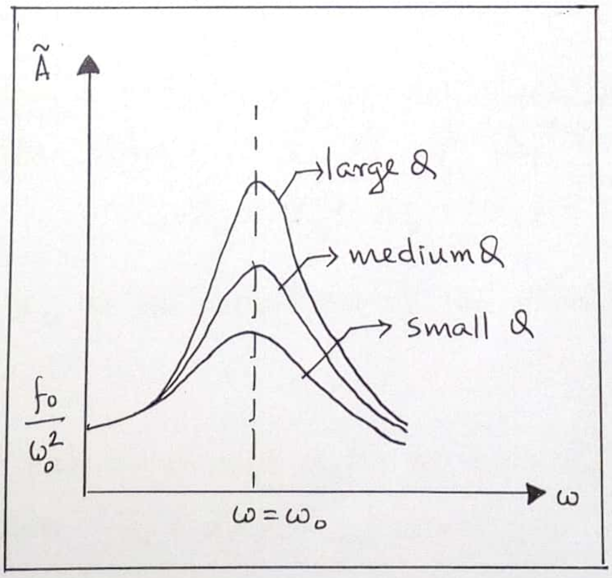

Diagram-12

It is to be noted that for larger Q values not only the peak of the resonance is higher but also the curve falls off faster and faster on both sides of the resonace frequency. So higher Q values is an indicator of the sharpness of resonance.