Diffraction

(Optics, 1st year, SDG)

When a beam of light passes through a narrow slit, it spreads out to a certain extent into the region of the geometrical shadow. This effect is one of the simplest examples of *diffraction*, i.e., of the failure of light to travel in straight lines. It can be satisfactorily explained only by assuming a wave character for light, and here we shall investigate quantitatively the *diffraction pattern*, or distribution of intensity of the light behind the aperture, using the Huygens – Fresnel Principle

The Huygens – Fresnel Principle

It states that every unobstructed point of a wavefront, at a given instant, serves as a source of spherical secondary wavelets (with same frequency as that of the primary wave). The amplitude of the optical field (i.e, electric field) at any point beyond is the superposition of all these wavelets (considering their amplitudes and relative phases).

Distinction between Interference and Diffraction

Sl No.	Interference	Diffraction
1)	In interference, two separate wavefronts originating from two coherent sources superpose at a point on the screen.	In diffraction, secondary wavelets originating from different parts of the same wavefront superpose.
2)	The width of the interference fringes may or may not be the same.	The diffraction bands are never of equal width.
3)	All bright fringes have the same intensity.	The intensity of the bright fringe usually decreases with the increase in order number.

Classification of Diffraction

Diffraction phenomena are conveniently divided into **two general classes**, **(1)** those in which the **source of light and the screen** on which the pattern is observed are effectively at **infinite distances from the aperture** causing the diffraction --- ***Fraunhofer diffraction*** and **(2)** those in which **either the source or the screen, or both, are at finite distances from the aperture -----** ***Fresnel diffraction***.

Sl No.	Fresnel Diffraction	Fraunhofer Diffraction
1)	The distance of the source, or the screen or both from the diffracting elements are at finite.	The distance of the source and the screen from the diffracting elements are effectively infinite.
2)	Either Spherical or Cylindrical wavefront.	Plane wavefront.
3)	No mirror or lenses are used for study.	Converging lens is used to focus parallel rays.
4)	In the plane of the aperture, the phase of the secondary wavelets is not same at all points.	The secondary wavelets are in same phase at every point in the plane of aperture.

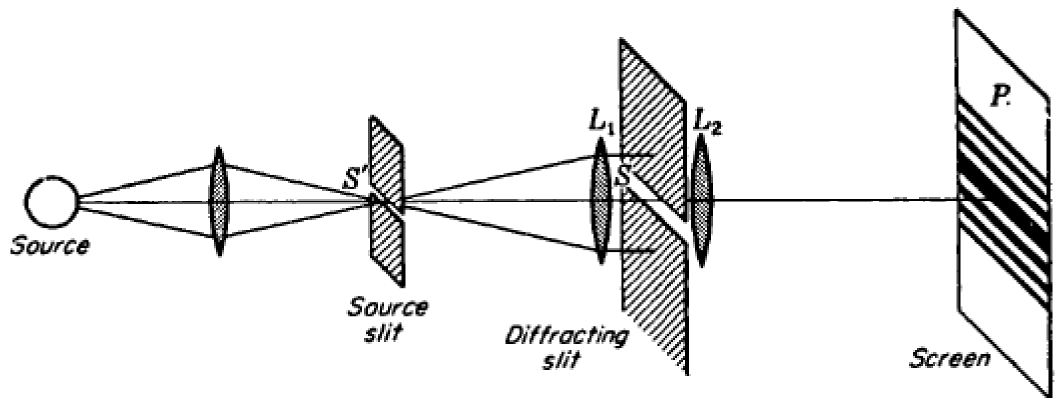

Fig 1: Experimental arrangement for obtaining the Fraunhofer diffraction pattern due to a single slit.

It is easily observed in practice by rendering the light from a source parallel with a lens and focusing it on a screen with another lens placed behind the aperture, an arrangement which effectively removes the source and screen to infinity. A slit is a rectangular aperture of length large compared to its breadth. Consider a slit S to be set up as in Fig.1, with its long dimension perpendicular to the plane of the page, and to be illuminated by parallel monochromatic light from the narrow slit *S'*, at the principal focus of the lens L_1. The light focused by another lens L_2 on a screen or photographic plate *P* at its principal focus will form a diffraction pattern, as indicated schematically.

<u>**Fraunhofer Diffraction by a Single Slit**</u>

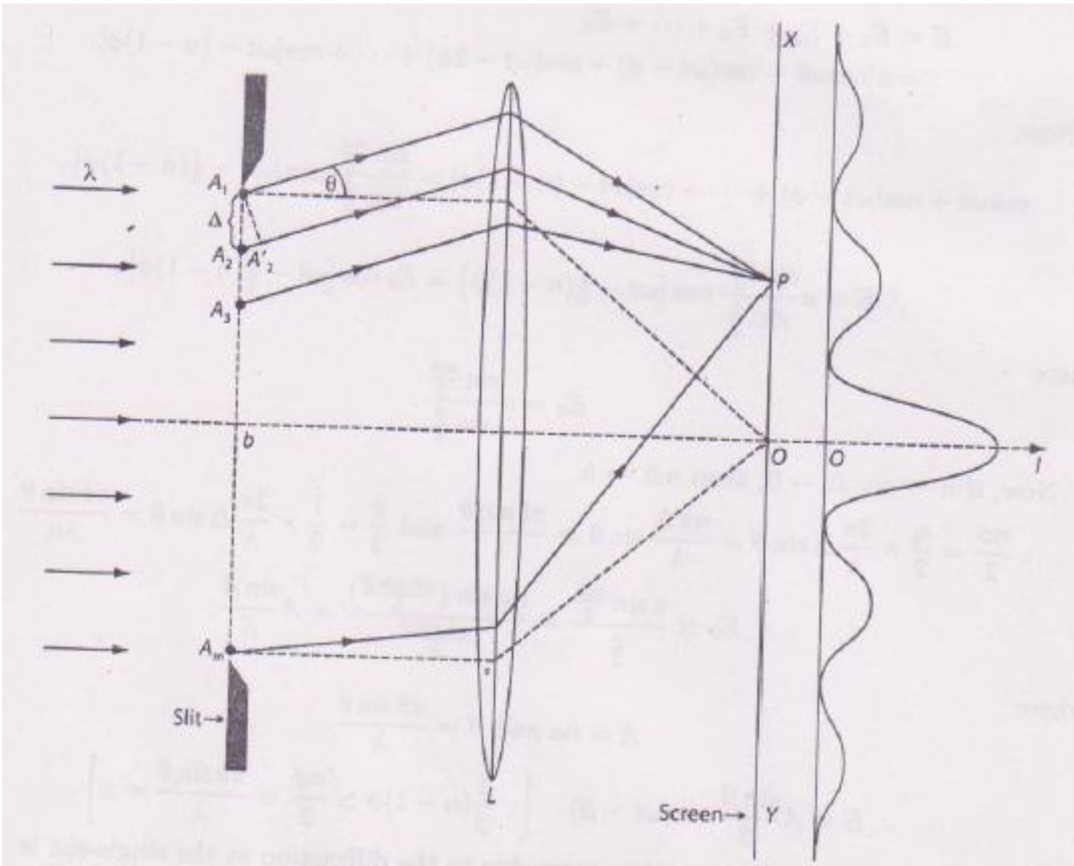

Fig 2: Fraunhofer diffraction of a single slit.

Let a plane parallel monochromatic light of wavelength λ be incident normally on a narrow slit AB of width b placed perpendicular to plane of the paper **(fig 2)**. We have to calculate the intensity distribution due to diffraction on the screen XY placed at the focal plane of lens L. We assume that (i) each unobstructed point on the slit is a source of secondary wavelets. Interference takes place between the wavelets originating from two such secondary waves. (ii) The slit AB consists of a large number of secondary point sources A_1 , A_2 , …..A_n, which are equi-spaced.
If Δ be the separation between any two consecutive point sources, then,
b = (n-1)Δ ….(1); where n is the no of point sources.

$$[A_1 A_2 = A_2 A_3 = \ldots\ldots = A_{n-1} A_n = \Delta \ldots(2)]$$

As the point P on the screen is at a large distance from the slit, the amplitudes of the secondary waves from A_1 and A_2 will be nearly the same. If the diffracted rays from A_1 and A_2 make an angle **θ with the normal to the plane of the slit (diffraction angle)**, then the path difference between them is

$$A_2A_2' = \Delta \sin\theta, \text{ thus the corresponding phase difference } \varphi = \frac{2\pi}{\lambda}\Delta \sin\theta \ldots.(3)$$

Similarly, the path difference between A_1 and A_3 is
$2\boldsymbol{\Delta}\,\mathbf{sin}\,\boldsymbol{\theta}$, thus the corresponding phase difference : $\frac{2\pi}{\lambda}2\Delta sin\theta = 2\varphi \ldots.(4)$
Thus, the phase difference between A_1 and A_n = (n-1)φ …..(5)

So, if the electric field at P due to disturbance coming from A_1 be represented by,
$E_1 = E_0 \cos \omega t \ldots (6)$, where E_0 is the amplitude and ω is angular frequency. Then the field at P due to disturbance from A_2 is $E_2 = E_0 \cos(\omega t - \varphi) \ldots (7)$,
Similarly, the fields at P due to the disturbances from A_3, A_4 ,…. , A_n are,

$E_3 = E_0 \cos(\omega t - 2\varphi) \ldots (8)\,, thus, E_n = E_0 \cos(\omega t - (n-1)\varphi) \ldots (9)$,

Thus, the resultant field at P is,
$\boldsymbol{E = E_1 + E_2 + \ldots. + E_n}$ **[using equation (6), (7), (8) and (9), we can write]**
$or, E = E_0[\cos \omega t + \cos(\omega t - \varphi) + \cos(\omega t - 2\varphi) + \cdots + \cos(\omega t - (n-1)\varphi)] \ldots (10)$,

$$\text{Or, } E = Real\left[E_0\left[e^{i\omega t} + e^{i(\omega t-\varphi)} + e^{i(\omega t - 2\varphi)} + \cdots + e^{i(\omega t-(n-1)\varphi)}\right]\right]$$

$$= Real\left[E_0 e^{i\omega t}\left[1 + e^{-i\varphi} + e^{-2i\varphi} + \cdots + e^{-i(n-1)\varphi}\right]\right]$$

$$= Real\left[E_0 e^{i\omega t}\left\{\frac{(1-e^{-in\varphi})}{(1-e^{-i\varphi})}\right\}\right] = Real\left[E_0 e^{i\omega t}\left\{\frac{(e^{in\varphi/2} - e^{-in\varphi/2})}{(e^{i\varphi/2} - e^{-i\varphi/2})}\right\}\frac{e^{-in\varphi/2}}{e^{-i\varphi/2}}\right]$$

$$= Real\left[E_0 e^{i\omega t}\left\{\frac{2i\ \sin n\varphi/2}{2i\ \sin\varphi/2}\right\} e^{-i(n-1)\varphi/2}\right] = Real\left[E_0\left\{\frac{\sin n\varphi/2}{\sin\varphi/2}\right\} e^{-i\{\omega t - \frac{(n-1)\varphi}{2}\}}\right]$$

$$= E_0\left\{\frac{\sin n\varphi/2}{\sin\varphi/2}\right\}\cos\{\omega t - \frac{(n-1)\varphi}{2}\}\ldots(11a)$$

Here, the modified amplitude of the resultant electric field is given by $E_0\left\{\frac{\sin n\varphi/2}{\sin\varphi/2}\right\}$ and modified phase is $\{\omega t - \frac{(n-1)\varphi}{2}\}$.
Therefore, the resultant intensity (I) at P will be:

$$I = E_0^2\, {sin^2 n\varphi/2}\Big/{sin^2\varphi/2} \ \ldots..(11\ b)$$

In the limit of $n \to \infty$: $\Delta \to 0, b \cong n\Delta$ (from equation (1)),
which gives, $\varphi = \frac{2\pi}{\lambda}\Delta sin\theta \to 0$ means $sin\varphi \cong \varphi$ **….. (12)**

Therefore, combining equ (3) and (12) we can write,

$$\frac{n\varphi}{2} = \frac{n\ 2\pi}{\lambda}\frac{b}{n} sin\theta = {\pi b\ sin\theta}/{\lambda} = \beta\ (say) \ldots (13)$$

Thus, the resultant intensity (I) at P can be expressed as**,**

$$I = E_0^2\,{sin^2 n\varphi/2}\Big/{sin^2\varphi/2} = n^2E_0^2\,{sin^2\beta}\Big/{\beta^2} = I_0\frac{sin^2\beta}{\beta^2}, where\ I_0 = n^2E_0^2 \ \ldots..(14)$$

$$\boxed{I = I_0\frac{sin^2\beta}{\beta^2}\ ,\quad where, \beta = {\pi b\ sin\theta}/{\lambda}}$$

Condition of Principal Maximum (Important)

As $\boldsymbol{\beta \to 0}, \lim_{\beta\to 0} {}^{\sin\beta}/_{\beta} = \mathbf{1}$, thus the intensity of principal maximum $\boldsymbol{I = I_0} \ldots . (\mathbf{15})$

Condition of Minima (Important): $\sin\boldsymbol{\beta} = \mathbf{0}, \boldsymbol{or}, \boldsymbol{\beta} = \boldsymbol{m\pi}\ (\boldsymbol{m} = \pm\mathbf{1}, \pm\mathbf{2}, \ldots)$
[m=0 is excluded as this corresponds to the condition of principal maximum]
$\boldsymbol{or}, \boldsymbol{b\ sin\theta} = \boldsymbol{m\lambda}\ (\boldsymbol{m} = \pm\mathbf{1}, \pm\mathbf{2}, \ldots)\ \{\boldsymbol{using\ equ}\ (\mathbf{13})\} \ldots\ldots (\mathbf{16})$.

Maximum order of minima are restricted by equ (16),

since $|\sin\boldsymbol{\theta}| \leq \mathbf{1}, \boldsymbol{thus}\ |\boldsymbol{m}| \leq {}^{\boldsymbol{b}}/_{\boldsymbol{\lambda}} \ldots (\mathbf{17})$

Condition of Secondary Maxima (Important

$$\frac{\boldsymbol{d}}{\boldsymbol{d\beta}}\left\{\frac{\boldsymbol{sin^2\beta}}{\boldsymbol{\beta^2}}\right\} = \mathbf{0} \text{ gives,} \qquad \boldsymbol{tan\beta} = \boldsymbol{\beta} \ \ldots\ldots (\mathbf{18})$$

The values of β satisfying this relation (17) are easily found graphically as the intersections of the curve $y = \tan\beta$ and the straight line $y = \beta$ curve [fig 3] and first few values of β are given by,
$\boldsymbol{\beta} = \pm\mathbf{1.43}\boldsymbol{\pi}, \pm\mathbf{2.46}\boldsymbol{\pi} \ldots$ *and so on* …(19)

Thus we have seen that from the *principal maximum* the intensity falls to zero at $\boldsymbol{\beta = \pm\pi}$, then passes through several *secondary maxima,* with equally spaced points of zero intensity at $\boldsymbol{\beta} = \pm\boldsymbol{\pi}, \pm 2\boldsymbol{\pi} \ldots$,, or in general $\boldsymbol{\beta} = \boldsymbol{m\pi}\ (\boldsymbol{m} = \pm\mathbf{1}, \pm\mathbf{2}, \ldots)$. The secondary maxima do not fall halfway between these points, but are displaced toward the center of the pattern by an amount which decreases with increasing *m*. the exact values of *β* for these maxima can be found by differentiating Eq(14) with respect to β and equating to zero.

Important
Ratio of the intensities of 1st Secondary Maxima and principal maximum is 4.5%

Intensity of **Principal Maximum is I_0** and of the **1st Secondary Maxima I_{SM1}** for

$(\boldsymbol{\beta} = \pm\mathbf{1.43}\boldsymbol{\pi} \cong \frac{3\pi}{2})$ gives, $\boldsymbol{I_{SM1}} = \boldsymbol{I_0}\ {}^{\boldsymbol{sin^2}\frac{3\pi}{2}}\Big/_{(\frac{3\pi}{2})^2} = \frac{\boldsymbol{I_0}}{\mathbf{22}}$,

Important

If θ_1 is the diffraction angle corresponding to the first minima, then $\boldsymbol{sin\,\theta_1} = {}^{\boldsymbol{\lambda}}\!/_{\boldsymbol{b}}$ ... (**20**). In practice $\boldsymbol{\theta_1}$ is usually a very small angle, so we may put the **sine equal to the angle.** Therefore, $\boldsymbol{\theta_1} = {}^{\boldsymbol{\lambda}}\!/_{\boldsymbol{b}}$ ... (21).

The above relation (21) shows at once how the dimensions of the pattern vary with λ. and *b*. The ***linear* width** of the pattern on a screen will be proportional to the **slit-screen distance**, which is the **focal length (f)** of a **lens placed close to the slit**. **The linear distance *x* between successive minima corresponding to the angular separation** $\boldsymbol{\theta_1} = {}^{\boldsymbol{\lambda}}\!/_{\boldsymbol{b}}$ is thus $\boldsymbol{x} = {}^{\boldsymbol{f\lambda}}\!/_{\boldsymbol{b}}$ .... (**22**)

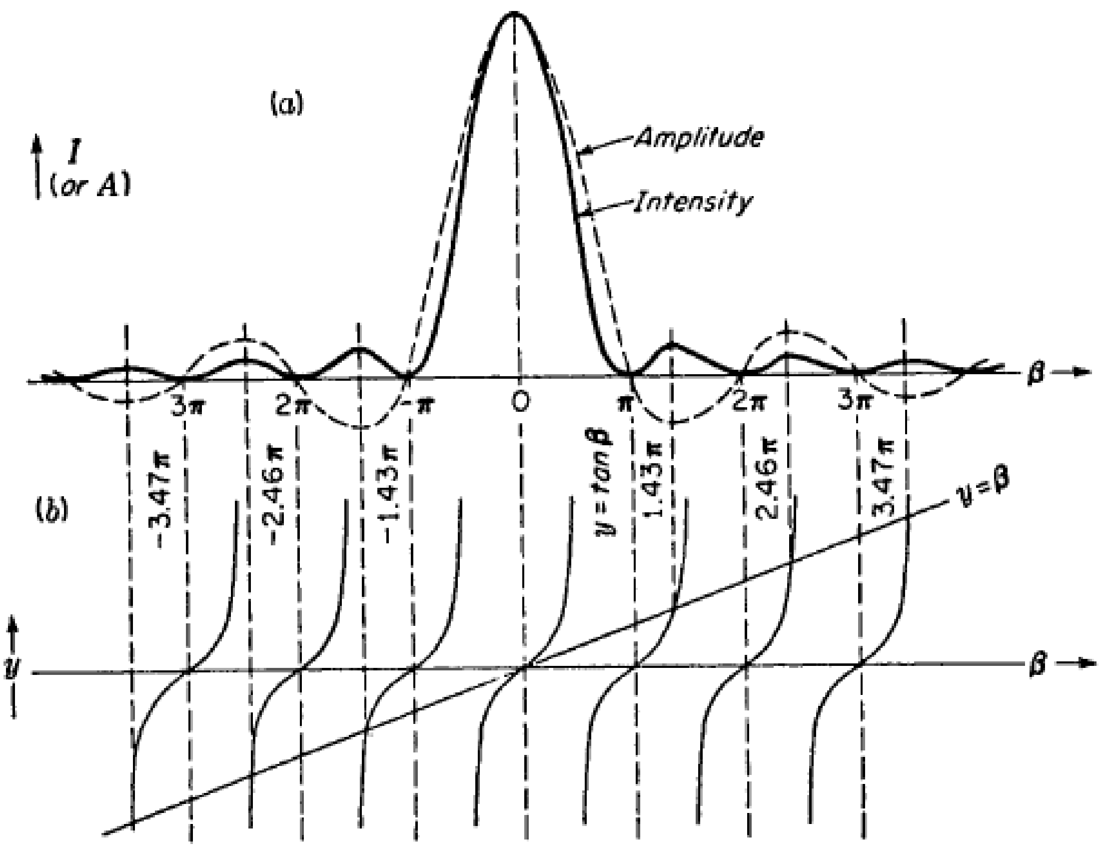

Fig 3: Intensity pattern of a single slit diffraction

Numerical Important

1) A convex lens of focal length of 20 cm is placed after a slit of width (b) 0.6mm. If a plane wave of wavelength 6000Å falls on the slit normally, calculate the separation between the second minima on either side of the principal maximum.

Ans. The condition of nth minimum on the screen is b $\sin\theta_n = n\lambda$, where θ is the angle of diffraction.

Here, λ=6000Å, n=2, f=20cm and width of the slit b=0.06 cm.
Therefore, $\sin\theta 2 = 2\lambda/b = 2*6000*10^{-8}/0.06 = 0.002$

Since the screen is placed at the focal plane of the lens, thus for small value of θ,

$\sin\theta_2 = \tan\theta_2 = x/f = 2\lambda/b$ where x is the distance between the principal maximum

and 2nd minima.

Therefore, $x = (2\lambda/b)*f = 0.002*20$ cm = **0.04 cm**

2) A monochromatic light of wavelength 5500Å is incident on a single slit of width 0.3 mm and gets diffracted. Find the diffraction angles for the first minima and the next maximum.

Ans. Diffraction angle for 1st minima: $b\sin\theta_1 = 1*\lambda$
Or, $\sin\theta_1 = \lambda/b = (5500*10^{-8} / 0.03)$
$\theta_1 = \mathbf{0.105^0}$

For the first secondary maxima, $b\sin\theta_1' = 3\lambda/2$
Or, $\theta_1' = \sin^{-1}(3*\lambda/2*b)$
$= \sin^{-1}(3*5500*10^{-8}/2*0.03)$
$= \mathbf{0.158^0}$

Fraunhofer Double Slit Diffraction

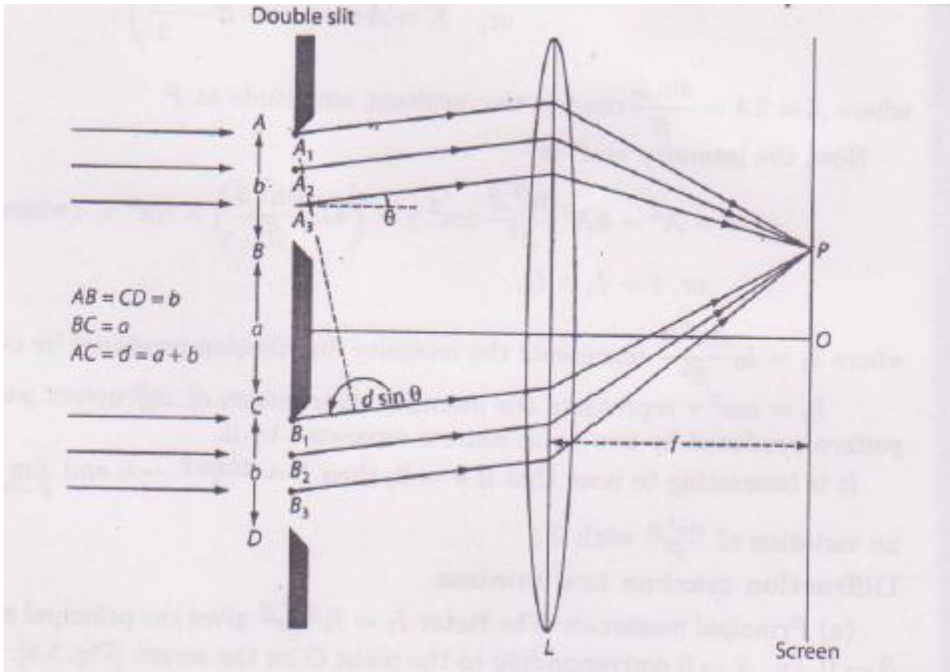

Fig 4: Fraunhofer double slit diffraction

Let AB and CD be two slits of small width ***b*** and the width of the opaque space between the two slits is ***a*** **[consider (a+b) =d]** and plane parallel monochromatic light of wavelength λ be incident normally on the slits **(fig 4).** We assume that (i) each unobstructed point on the slit is a source of secondary wavelets. (ii) The slits AB and CD consist of a large number of secondary point sources which are equi-spaced.

Let A_1 , A_2 , …..A_n, and B_1 , B_2 , …..B_n be the point sources in silt AB and CD respectively as shown in **fig-4**. If the diffracted rays make an angle **θ (diffraction angle)** with the normal to the slits, then the path difference between the waves reaching the point P on the screen from two consecutive points in a slit is $\boldsymbol{\Delta\ sin\theta}$**, where** Δ is the separation between any two consecutive point sources on either slit.

The field produced by the slit AB at P is given by (equ (11a))

$$\boldsymbol{E_1 = E_0\left\{\frac{\sin n\varphi/2}{\sin \varphi/2}\right\} \cos\{\omega t - \frac{(n-1)\varphi}{2}\}}$$

In the limit of $\boldsymbol{n} \rightarrow \infty$: $\boldsymbol{\Delta} \rightarrow \boldsymbol{0}, \boldsymbol{b} \cong \boldsymbol{n\Delta}$ (from equation (1)), which gives,

$\boldsymbol{\varphi = \frac{2\pi}{\lambda}\Delta sin\theta} \rightarrow \boldsymbol{0}$ means $\boldsymbol{sin\varphi \cong \varphi}$ and $\boldsymbol{\frac{n\varphi}{2} = \frac{n\,2\pi}{\lambda}\frac{b}{n}sin\theta = {\pi b\, sin\theta}/{\lambda} = \beta}\ (\boldsymbol{say})$

and $\boldsymbol{\frac{(n-1)\varphi}{2} \cong \frac{n\varphi}{2}}$**. ….(23)**

Therefore, the field produced by the slit AB at P is given by

$$\boldsymbol{E_1 = nE_0\frac{sin\,\beta}{\beta}\cos\{\omega t - \beta\}} \textbf{ ….(24)}$$

The path difference between two **corresponding points, (two points which are separated by a distance (a+b) =d =A_1B_1 = A_2B_2 =…=A_nB_n) is given by *d sinθ*,**
thus the phase difference becomes**,**

$$\varphi_1 = \frac{2\pi}{\lambda} d \sin\theta \ \dots\dots (25)$$

Therefore, the field produced by the slit CD at P is given by,

$$E_2 = nE_0 \frac{\sin\beta}{\beta} \cos\{\omega t - \beta - \varphi_1\} \ \dots. (26)$$

Therefore, the **resultant electric field** at P due to both slit will be **$E=E_1+E_2$**

$$E = nE_0 \frac{\sin\beta}{\beta}[\cos(\omega t - \beta) + \cos(\omega t - \beta - \varphi_1)]$$

$$= 2nE_0 \frac{\sin\beta}{\beta}[\cos(\omega t - \beta + \omega t - \beta - \varphi_1)/2 * \cos\{\omega t - \beta - (\omega t - \beta - \varphi_1)\}/2]$$

$$= 2nE_0 \frac{\sin\beta}{\beta}\left[\cos(\omega t - \beta - \varphi_1/2)\cos\left\{{}^{\varphi_1}/_2\right\}\right] \dots (27)$$

Therefore, the resultant intensity at P becomes,

$$I = 4n^2E_0^2 \frac{\sin^2\beta}{\beta^2} \cos^2 {}^{\varphi_1}/_2 = 4I_0 \frac{\sin^2\beta}{\beta^2} \cos^2\gamma \ \dots. (28)$$

Where, $\gamma = \frac{\varphi_1}{2} = {}^{\pi d \sin\theta}/_{\lambda} \ and \ n^2E_0^2 = I_0 \ \dots (29)$

$$\boxed{\begin{gathered} I = 4I_0 \frac{\sin^2\beta}{\beta^2} \cos^2\gamma \\ \gamma = {}^{\pi d \sin\theta}/_{\lambda} \ and \ \beta = {}^{\pi b \sin\theta}/_{\lambda} \end{gathered}}$$

The factor ${}^{\sin^2\beta}/_{\beta^2}$ in equation (28) is just that derived for the single slit of width ***b***.

The second factor $\cos^2\gamma$ is characteristic of the interference pattern produced by two beams of equal intensity and phase difference φ_1**as shown in fig 4.**
The resultant intensity will be zero when either of the two factors is zero. For the first factor this will occur when $\beta = \pm\pi, \pm 2\pi \dots,,$ and for the second factor when $\gamma = \pm\pi/2, \pm 3\pi/2 \dots,,$ That the two variables β and γ are not independent.

Condition of Principal Maximum (Important)

As $\boldsymbol{\beta \to 0, \lim_{\beta\to 0} {}^{\sin\beta}/_{\beta} = 1}$, $(\mathbf{30})$

Condition of Minima (Important)

Diffraction Minima: $\boldsymbol{\sin\beta = 0, or, \beta = m\pi\ (m = \pm1, \pm2, ..)}$
[m=0 is excluded as this corresponds to the condition of principal maximum]
$\boldsymbol{or, b\ sin\theta = m\lambda\ (m = \pm1, \pm2, ..) \dots (31)}$.

Interference Minima: $\boldsymbol{cos^2\gamma = 0, or, \gamma = (2n+1)\pi/2\ (n = 0, \pm1, \pm2, ..)}$

Or, $\boldsymbol{d\ sin\theta = (2n+1)\lambda/2\ (n = 0, \pm1, \pm2, ..) \dots (32)}$

Condition of Secondary Maxima (Important)

$$\boldsymbol{\frac{d}{d\beta}\left\{\frac{sin^2\beta}{\beta^2}\right\} = 0} \text{ gives,} \qquad \boldsymbol{tan\beta = \beta \ \dots\dots (32)}$$

Condition of Interfernce Maxima (Important)

$$\boldsymbol{cos^2\gamma = 1, or, \gamma = n\pi\ (n = 0, \pm1, \pm2, ..)}$$
$$\textbf{Or, } \boldsymbol{d\ sin\theta = n\lambda\ (n = 0, \pm1, \pm2, ..) \dots (33)}$$

Thus in a double slit, interference of the beams from the two slits produces the narrow maxima and minima given by the $\boldsymbol{cos^2\gamma}$ factor **(fig 5a),** and diffraction, represented by $\boldsymbol{{}^{sin^2\beta}/_{\beta^2}}$,

(fig 5b), modulates the intensities of these interference fringes **(fig 5c),**.

(Important) Show that double slit diffraction pattern reduces two Youngs' double slit interference as the width of the slit goes to zero.

$$\textbf{As b} \to \mathbf{0}, \boldsymbol{{}^{\pi b\ sin\theta}/_{\lambda} = \beta \to 0, thus\ \lim_{\beta\to 0} {}^{\sin\beta}/_{\beta} = 1}$$

Therefore, equation (28) can be written as $\boldsymbol{I = 4I_0 cos^2\gamma}$

Thus, double sit diffraction pattern reduces to interference pattern as the width of the slit goes to zero.

Missing Order (Important)

If the slit-width ***b*** **is kept constant** and **the separation between the slits *a, thus d=a+b*** is varied, the scale of the interference pattern varies, but that of the diffraction pattern remains the same. And **when the condition for a maximum of the interference, Eq. (33), and for a minimum of the diffraction, Eq. (31), are both fulfilled for the same value of θ, that is for the same angle of diffraction, that interference maxima will be absent in the spectra and termed as the missing order.**

$$\boldsymbol{d\ sin\theta = n\lambda\ (n = 0, \pm 1, \pm 2, ..)\ and\ b\ sin\theta = m\lambda\ (m = \pm 1, \pm 2, ..)}$$

Since ***m*** **and n are both integers**, ***d/b*** must be in the ratio of two integers if we are to have missing orders. This ratio determines the orders which are missing, in such a way that when ***d/b* = 2,** orders **2, 4, 6**, ... are missing; when ***d/b* = 3**, orders **3, 6, 9**, ...are missing; etc.

Condition of missing order: $\boldsymbol{\dfrac{d}{b} = \dfrac{n}{m}}$ **....(34)**

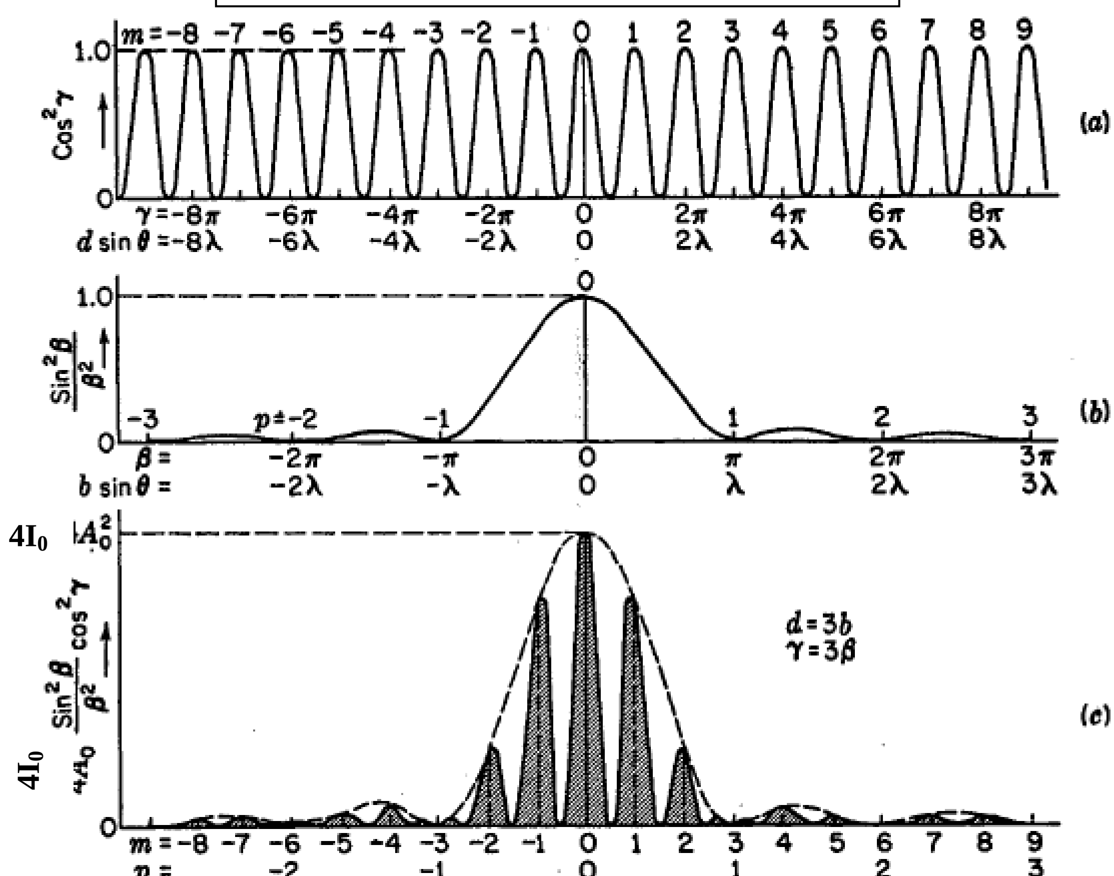

Fig 5: Fraunhofer double-slit diffraction pattern, where *a=2b*, thus *d/b = (a+b)/b* = 3.

Numerical Important

1) The width of each slit of a double-slit is 0.15mm and the width of the opaque space between the two slits is 0.45mm. If double-slit produces Fraunhofer diffraction pattern, find the missing orders.

   Ans. Here width of each slit, b=0.15 mm and width of the opaque space between the two slits, a = 0.45 mm.
   When n^{th} order interference maximum coincides with m^{th} order diffraction minimum, then one will get the missing orders.

   Therefore, n/m= (a+b)/b = (0.45+0.15)/0.15=4
   or, **n=4m**.

   For m=1, 2, 3, 4 etc. n = 4, 8, 12, 16 etc. This indicates that the **4^{th}, 8^{th}, 12^{th}, 16^{th}** order interference maxima will be missed from the diffraction spectra.

2) Fraunhofer double-slit diffraction pattern is observed in the focal plane of a lens (f=0.5m). The wavelength (λ) of incident light is 5000Å. The distance between two maxima adjacent two zero order maximum is 5mm and the fourth order maximum is missing. Find the width of each slit and the distance between their centres.

Ans. Since the 4^{th} order maximum is missing, we have

$$(a+b)/b=4 \text{ or, } a=3b$$

For the first order maximum, the condition is $(a+b)\sin\theta_1 = \lambda$

$$\text{Or, } \theta_1 \approx \lambda/(a+b) = \lambda/4b$$

Now, the distance between two maxima adjacent to zero order maximum is,

$$x = f*2\theta_1 = f*(2*\lambda/ 4*b)$$

$$\text{or, } b=f*\lambda/2*x = (0.5*5000*10^{-8})/(2*5*10^{-3}) = \mathbf{0.025\ mm}$$

Therefore, a=3*b = **0.075mm**.

Fraunhofer N- Slit Diffraction (Diffraction Grating)

A diffraction grating is an arrangement of large number of equidistant parallel rectangular slits of equal width ***b*** and width of the opaque space between the two consecutive slits is ***a***.
A plane transmission grating may be formed by ruling equidistant parallel lines with a fine diamond point on a plane glass-plate. The ruled lines are opaque to light while the space between two lines is transparent.
Let ABCDEFGH represents the section of a plane transmission grating placed perpendicular to the plane of the paper **(Fig-6). Let the points A and C, C and E, E and G etc. be separated by a distance d=a+b, where a= width of the opaque space and b=that of the transparent space. The points separated by d are called pair of corresponding points.**
Suppose a plane parallel monochromatic light of wavelength λ be incident normally on the surface of the grating (on the slits) **(fig 6).** We assume that (i) each unobstructed point on the slits is a source of secondary wavelets. (ii) The slits consist of a large number (n) of secondary point sources which are equi-spaced.
The lens (L) focuses the diffracted rays at an angle of diffraction θ at P on the screen.

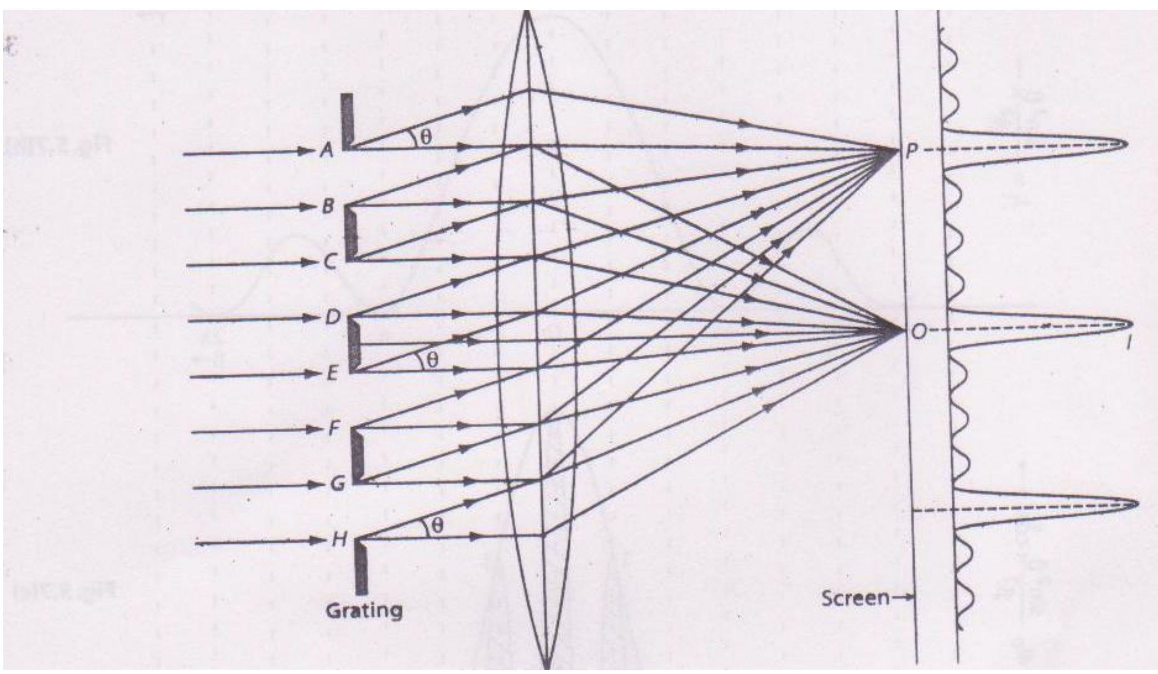

Fig 6: Fraunhofer N- slit diffraction

The field produced by the slit AB at P is given by equation (24),

$$\boldsymbol{E_1 = nE_0 \frac{\sin\beta}{\beta} \cos\{\omega t - \beta\}} \text{ ....(35)}$$

where, the path and phase difference between the waves reaching the point P on the screen from two consecutive points in a slit are $\boldsymbol{\Delta\ sin\theta}$ **and** $\boldsymbol{\varphi = \frac{2\pi}{\lambda}\Delta sin\theta}$ **respectively.**
Here, Δ is the separation between any two consecutive secondary point sources on each slit and in the limit of $\boldsymbol{n \to \infty}$: $\boldsymbol{\Delta \to 0, b \cong n\Delta}$ (from equation (1)), which gives,

$$\frac{n\varphi}{2} = \pi b\ sin\theta / \lambda = \beta \ \ldots\mathbf{(36)}$$

The path difference between two **corresponding points, (two points which are separated by a distance (a+b) =d =$A_1B_1 = A_2B_2 = \ldots = A_nB_n$) is given by** $\boldsymbol{d\ sin\theta}$**,**
Thus the phase difference becomes**,**

$$\varphi_1 = \frac{2\pi}{\lambda} d\ sin\theta \text{ and we consider, } \frac{\varphi_1}{2} = \gamma = \pi d\ sin\theta / \lambda \ \ldots .(37)$$

Therefore, the field produced by the slit CD at P is given by equation (26),

$$E_2 = nE_0 \frac{\sin\beta}{\beta} \cos\{\omega t - \beta - \varphi_1\} \ \ldots . \ (38)$$

Similarly, the field produced by the slit EF at P is given by,

$$E_3 = nE_0 \frac{\sin\beta}{\beta} \cos\{\omega t - \beta - 2\varphi_1\} \ldots .(39)$$

And by the Nth slit will be,

$$E_N = nE_0 \frac{\sin\beta}{\beta} \cos\{\omega t - \beta - (N-1)\varphi_1\} \ldots (40)$$

Thus, the resultant field at P due to N-slit diffraction is given by,

$\boldsymbol{E = E_1 + E_2 + \ldots . + E_N}$ **[using equation (35), (38), (39) and (40), we can write]**

$$or, E = nE_0 \frac{sin\ \beta}{\beta}[\cos(\omega t - \beta) + \cos(\omega t - \beta - \varphi_1) + \cos(\omega t - \beta - 2\varphi_1) + \cdots + \cos(\omega t - \beta - (N-1)\varphi_1)] \ldots (41),$$

$$\text{Or, } E = Real\left[nE_0 \frac{sin\ \beta}{\beta}\left[e^{i(\omega t-\beta)} + e^{i(\omega t-\beta-\varphi_1)} + e^{i(\omega t-\beta-2\varphi_1)} + \cdots + e^{i(\omega t-\beta-(N-1)\varphi_1)}\right]\right]$$

$$= Real\left[nE_0 \frac{sin\ \beta}{\beta} e^{i(\omega t-\beta)}\left[1 + e^{-i\varphi_1} + e^{-2i\varphi_1} + \cdots + e^{-i(N-1)\varphi_1}\right]\right]$$

$$= Real\left[nE_0 \frac{sin\ \beta}{\beta} e^{i(\omega t-\beta)}\left\{\frac{(1 - e^{-iN\varphi_1})}{(1 - e^{-i\varphi_1})}\right\}\right]$$

$$= Real\left[nE_0 \frac{sin\ \beta}{\beta} e^{i(\omega t-\beta)}\left\{\frac{(e^{in\varphi_1/2} - e^{-in\varphi_1/2})}{(e^{i\varphi_1/2} - e^{-i\varphi_1/2})}\right\} \frac{e^{-in\varphi_1/2}}{e^{-i\varphi_1/2}}\right]$$

$$= Real\left[nE_0 \frac{sin\ \beta}{\beta} e^{i(\omega t-\beta)}\left\{\frac{2i\ \sin\frac{N\varphi_1}{2}}{2i\ \sin\frac{\varphi_1}{2}}\right\} e^{-\frac{i(N-1)\varphi_1}{2}}\right]$$

$$= Real\left[nE_0 \frac{sin\ \beta}{\beta}\left\{\frac{\sin N\varphi_1/2}{\sin\varphi_1/2}\right\} e^{-i\{\omega t - \beta - \frac{(N-1)\varphi_1}{2}\}}\right]$$

$$= nE_0 \frac{\sin\beta}{\beta}\left\{\frac{\sin N\varphi_1/2}{\sin\varphi_1/2}\right\} \cos\{\omega t - \beta - \frac{(N-1)\varphi_1}{2}\}......(42)$$

Here, the modified **amplitude** of the resultant electric field is given by,
$nE_0 \frac{\sin\beta}{\beta}\left\{\frac{\sin N\varphi_1/2}{\sin\varphi_1/2}\right\}$ and modified **phase is** $\{\omega t - \beta - \frac{(N-1)\varphi_1}{2}\}$**.**

Therefore, the resultant intensity (I) at P will be:

$$I = n^2E_0^2 \frac{sin^2\beta}{\beta^2}\frac{sin^2N\gamma}{sin^2\gamma} = I_0 \frac{sin^2\beta}{\beta^2}\frac{sin^2N\gamma}{sin^2\gamma} \ldots..(43)$$

Where, $\gamma = \frac{\varphi_1}{2} = {}^{\pi d\ sin\theta}/_{\lambda}\ and\ n^2E_0^2 = I_0$

$$I = I_0 \frac{sin^2\beta}{\beta^2}\frac{sin^2N\gamma}{sin^2\gamma},$$

$$\gamma = {}^{\pi d\ sin\theta}/_{\lambda}\ and\ \beta = {}^{\pi b\ sin\theta}/_{\lambda}$$

and (a+b)=d is defined as the grating element.

By putting N=1 and N=2 , in equation (43) we can obtain the intesity expression for single and double-slit respectively.

Condition of Principal Maximum

The new factor $\frac{sin^2N\gamma}{sin^2\gamma}$ may be said to represent the *interference* term for *N* slits. It possesses maximum values equal to N^2 for $\gamma = \mathbf{0}, \boldsymbol{\pi}, \mathbf{2}\boldsymbol{\pi}, \ldots$, Although the quotient becomes indeterminate at these values, this result can be obtained by noting that,

$$\lim_{\gamma\to n\pi} \frac{sin^2N\gamma}{sin^2\gamma} = \pm N$$

Thus, $d\ sin\theta = n\lambda\ (n = 0, \pm 1, \pm 2, ..)$ ... (44)

They are more intense, however, in the ratio of the square of the number of slits. The relative intensities of the different orders *n* are in all cases governed by the single slit diffraction envelope $\frac{sin^2\beta}{\beta^2}$, hence the relation between β and γ in terms of slit width and slit separation remains unchanged, as does the condition for missing orders (equation (34)).

Condition of Minima and Secondary Maxima

To find the minima of the function $\frac{sin^2N\gamma}{sin^2\gamma}$ we note that the numerator becomes zero more often than the denominator, and this occurs at the values ***Nγ* = 0, *π*, *2π***, ... or, in general, ***sπ.*** In the special cases when **s = 0, *N*, *2N*, ... , γ will be *0, n, 2n, ...*** *;* so for these values the denominator will also vanish, and we have the principal maxima described above. The other values of s give zero intensity, since for these the denominator does not vanish at the same time. Hence the condition for a minimum is,

$$\sin N\boldsymbol{\gamma} = \mathbf{0}, \boldsymbol{or}, \boldsymbol{N\gamma} = \boldsymbol{s\pi}\ (\boldsymbol{s} \neq \mathbf{0}, \pm\boldsymbol{N}, \pm\mathbf{2}\boldsymbol{N}, ..)$$

$$\text{Or, } \boldsymbol{d\ sin\theta} = \boldsymbol{s\lambda}\ (\boldsymbol{s} \neq \mathbf{0}, \pm\boldsymbol{N}, \pm\mathbf{2}\boldsymbol{N}, ..) \ldots\ldots (\mathbf{45})$$

Diffraction Minima: $\sin\boldsymbol{\beta} = \mathbf{0}, \boldsymbol{or}, \boldsymbol{\beta} = \boldsymbol{m\pi}\ (\boldsymbol{m} = \pm\mathbf{1}, \pm\mathbf{2}, ..)$
[m=0 is excluded as this corresponds to the condition of principal maximum]

$$\boldsymbol{or}, \boldsymbol{b\ sin\theta} = \boldsymbol{m\lambda}\ (\boldsymbol{m} = \pm\mathbf{1}, \pm\mathbf{2}, ..) \ldots (\mathbf{46}) .$$

Between two adjacent principal maxima there will hence be *N* - I points of zero intensity. The two minima on either side of a principal maximum are separated by twice the distance of the others.

Between the other minima the intensity rises again, but the secondary maxima thus produced are of much smaller intensity than the principal maxima. **(Fig 7)** shows a plot for six slits of the quantities $\boldsymbol{sin^2N\gamma}$ and$\boldsymbol{sin^2\gamma}$, and also of their quotient, which gives the intensity distribution in the interference pattern. The intensity of the principal maxima is $\boldsymbol{N^2}$ **or 36**, so that the lower figure is drawn to a smaller scale. The intensities of the secondary maxima are also shown. These secondary maxima are not of equal intensity but fall off as we go out on either side of each principal maximum. Nor are they in general equally spaced, the lack of equality being due to the fact that the maxima are not quite symmetrical. This lack of symmetry is greatest for the secondary maxima immediately adjacent to the principal maxima, and is such that the secondary maxima are slightly shifted toward the adjacent principal maximum.

Secondary Maxima

$$\frac{\boldsymbol{d}}{\boldsymbol{d\gamma}}\left\{\frac{\boldsymbol{sin^2N\gamma}}{\boldsymbol{sin^2\gamma}}\right\} = \mathbf{0} \text{ gives,} \qquad \boldsymbol{Ntan\gamma} = \boldsymbol{tan\ N\gamma}\ \ldots\ldots (\mathbf{47})$$

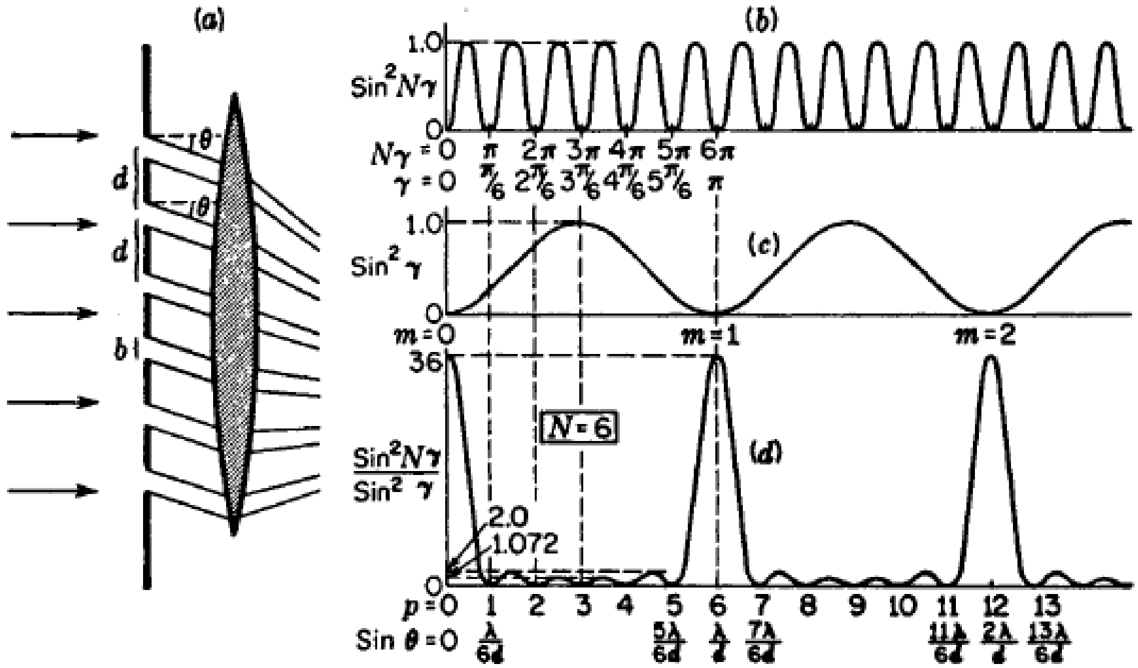

Fig 7: Intensity pattern of a N- slit diffraction (for N=6)

<u>Angular Dispersive Power</u>

The separation of any two colors, such as λ_1 and λ_2, increases with the order number. To express this separation the quantity frequently used is called the ***angular dispersion***, which is defined as the **rate of change of angle with change of wavelength**. An expression for this quantity is obtained by differentiating **Eq. (44)** with respect to λ we get**,**

$$d\, sin\theta = n\lambda$$

Or,

$$cos\theta\ d\theta = \frac{n\ d\lambda}{d} = pn\ d\lambda, where\ p = \frac{1}{d} is\ the\ no\ of\ lines\ per\ cm\ of\ a\ grating$$

$$or, \frac{d\theta}{d\lambda} = \frac{np}{cos\ \theta} \dots\dots (48)$$ **(Important)**

Resolving Power of a Grating

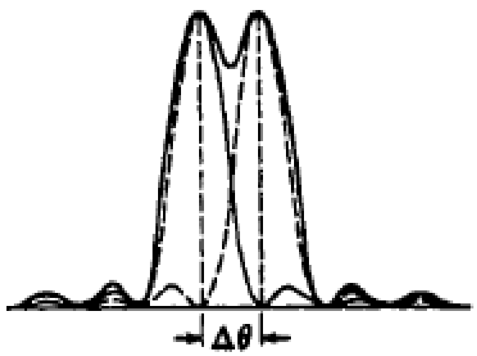

Fig 8: Two spectra are just resolved

When N is many thousands, as in any useful diffraction grating, the maxima are extremely narrow. The resolving power **Δλ/λ** is correspondingly high. To evaluate it, we apply the Rayleigh criterion. **The images formed in two wavelengths that are barely resolved must be separated by the angle $\Delta\theta$. Consequently the light of wavelength $\lambda+\Delta\lambda$ must form its principal maximum of order m at the same angle as that for the first minimum of wavelength λ in that order** (**fig 8**). Hence we can equate the extreme path differences in the two cases and obtain

$$(mN+1)\lambda = mN(\lambda+\Delta\lambda)$$

$$\textit{Or, } \lambda/_{\Delta\lambda} = mN, \dots \dots ..(49)$$ **(Important)**

$$\textit{where } m \textit{ is the order of the spectra}$$
$$\textit{and } N \textit{ is the no of slits (or total no lines in the grating)}$$

Numerical Important

1) A parallel beam of light is incident normally on a plane grating having 4250 lines per cm and the second order spectrum is formed at an angle 30^0. Calculate the wavelength of the monochromatic light.

Ans. No. Of lines/cm, p =1/(a+b), where a+b is the grating element.

Therefore, a+b = 1/p =1/4250 = $\mathbf{2.35*10^{-4}}$ **cm**

Now, the condition for mth order principal maximum in a grating is (a+b) sinθ = mλ
Here, m=2, θ=30^0 and (a+b) = 2.35 *10^{-4} cm

So, λ = (a+b)sinθ/m = (2.35 *10^{-4} *0.5)/2 = **5875 Å**

2) What is the maximum number of lines of a grating, which will resolve 3^{rd} order spectrum of two lines having wavelengths 5890 Å and 5896 Å.

Ans. Resolving power of a grating RP=λ/dλ = mN.

Here m=3 and λ= (5890+5896)/2 Å = **5893 Å** and dλ = (5896-5890) Å = **6 Å**

So, the required maximum number of lines of the grating = 5893/6 =3N

or, **N = 327**