Course Name : Mathematics II					
Course Code: MATH1201					
Contact hrs per week:	L	T	P	Total	Credit points
	3	1	0	4	4

Module I [10 L]

Ordinary differential equations (ODE)-

First order and first degree: Exact equations, Necessary and sufficient condition of exactness of a first order and first degree ODE (statement only), Rules for finding Integrating factors, Linear and non-linear differential equation, Bernoulli's equation. General solution of ODE of first order and higher degree (different forms with special reference to Clairaut's equation).

Second order and first degree:

General linear ODE of order two with constant coefficients, C.F. & P.I., D-operator methods for finding P.I., Method of variation of parameters, Cauchy-Euler equations.

Module II:[10L]

Basics of Graph Theory

Graphs, Digraphs, Weighted graph, Connected and disconnected graphs,Complement of a graph, Regular graph, Complete graph, Subgraph,; Walks, Paths, Circuits, Euler Graph,Cut sets and cut vertices, Matrix representation of a graph, Adjacency and incidence matrices of a graph,Graph isomorphism, Bipartite graph.

Tree:

Definition and properties, Binary tree, Spanning tree of a graph, Minimal spanning tree, properties of trees, Algorithms: Dijkstra's Algorithm for shortest path problem, Determination of minimal spanning tree using DFS, BFS, Kruskal's and Prim's algorithms.

Introduction to Graph Theory

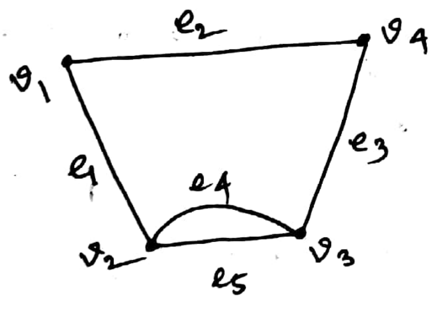
Graph theory is an abstract mathematical structure. The interest to study this branch of mathematics has increased rapidly due to its vast application in different fields of science like social science, social networking, artificial intelligence, transportation system, operations research etc.

Defn of Graph:

A graph $G(V, E, g)$ or simply $G(V, E)$ is a mathematical structure consisting of two sets V and E.

where V is a non empty set ie, $|V| \neq \phi$ $V = \{v_1, v_2, \ldots v_n\}$ are called the set of vertices and $E = \{e_1, e_2, \ldots e_n\}$ is called the set of edges.

where g is called the edge-end point function Generally denoted as $g(e_k) = \{v_i, v_j\}$ which maps every edge of the edge set E to two vertices of the vertex set V. Such that e_k is the edge incident to v_i & v_j.



$G(V, E, g)$

$V = \{v_1, v_2, v_3, v_4\}$

$E = \{e_1, e_2, e_3, e_4, e_5\}$

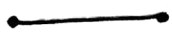
$g(e_1) = \{v_1, v_2\}$

$g(e_2) = \{v_1, v_4\}$

$g(e_3) = \{v_3, v_4\}$

$g(e_4) = \{v_2, v_3\}$

$g(e_5) = \{v_2, v_3\}$

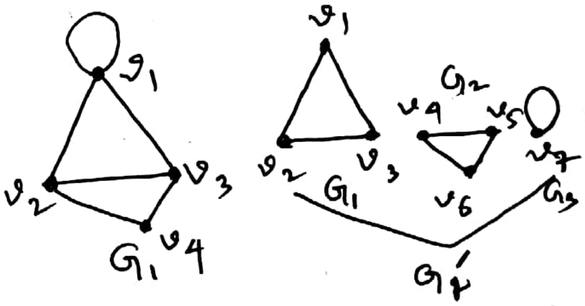
- Null Graph / Trivial Graph:
  A graph with single vertex and with no edges.
- Proper Edge:  An unique edge betn two distinct vertices.
- Parallel Edge:  More than one edge betn same set of pair of vertices.
- Self loop:  An edge from one vertex to itself
- Degree of a Vertex: Degree of a vertex is counted by the no of edges incident to it.
- Isolated Vertex: A vertex whose degree is zero ie, no edge is incident to the vertex.
- Pendant Vertex: A vertex whose degree is one.

Different Classification of Graph:

Finite / Infinite Graph:

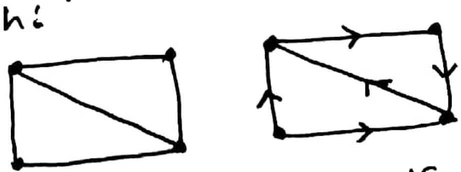
A graph with vertex and edge set finite is finite graph otherwise infinite.

Connected / Disconnected Graph:



When all the vertices of a graph are connected then the graph is connected, otherwise disconnected. Disconnected graphs are said to have components.

Directed (Di) Graph / Undirected Graph:



A graph where every edges has a particular direction is digraph otherwise undirected graph.

Undirected Graph

Directed Graph

Theorems:

Th 1: Handshaking Theorem

The sum of degrees of all the vertices of an undirected graph is twice the sum of the edges of the graph

$$\sum_{i=1}^{n} d(v_i) = 2e$$

Proof: An edge e be it proper edge or self loop contributes degree 2 to the sum of all the degrees.

Hence all the edges togeather contributes degree 2e to the sum of the degrees.

ie, $\sum_{i=1}^{n} d(v_i) = 2e$

Th. 2: The number of odd degree vertices in a graph is always even

Proof: By handshaking theorem

$$\sum_{i=1}^{n} d(v_i) = 2e$$

Let v_o denotes the vertices of odd degree

v_e " " " even "

$\therefore \quad \sum d(v_i) = \sum d(v_o) + \sum d(v_e) = 2e$

as 2e is an even number

$\sum d(v_e)$ is also an even number

Therefore $\sum d(v_o)$ must be an even number

it is only possible when the no of odd vertics is even.

Hence the proof.

Th. 3 Degree of any vertex in a simple graph is max. $(n-1)$

Proof: In a simple graph all the edges are proper. Therefore one vertex can be adjacent to at max rest $(n-1)$ vertices. Therefore the result holds.

Th. 4. In a simple graph no. of edges is max $^nC_2$

Proof: By handshaking Theorem

$$\sum_{i=1}^{n} d(v_i) = 2e$$

or, $d(v_1) + d(v_2) + \dots + d(v_n) = 2e$ ——— (1)

as it is simple graph so, by Th. 3 $d(v_i) \leq (n-1)$

$\therefore$ $d(v_1) \leq (n-1)$, $d(v_2) \leq (n-2)$ ... $d(v_n) \leq (n-1)$

$\therefore (1) \Rightarrow (n-1) + (n-1) + \dots + (n-1) \geq 2e$

$\therefore$ $n(n-1) \geq 2e$

$\therefore$ $e \leq {}^nC_2$ Hence the result.

The. 5 A complete graph with n vertices has exactly $^nC_2$ edges.

Proof: In a complete graph, there exists exactly one edge between every distinct pair of vertices.

So, out of n vertices there must be exactly $^nC_2$ edges.

Hence the proof

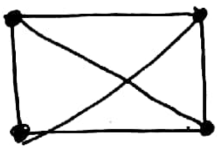
Def ③ :- A complete graph is a simple graph with highest possible edges.

- In a complete graph with n verteces there exists ${}^nC_2$ edges exactly. [explanation: Bet^n every 2 vertices there is one proper edge. So, out of n vertices there will be exactly ${}^nC_2$ edges].

- The degree of all the vertices of a complete graph with n vertices is $(n-1)$. Therefore a complete graph is a regular graph.

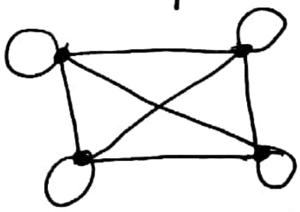
- Every complete graph is regular But every regular graph may not be complete

Eg.



Complete & regular

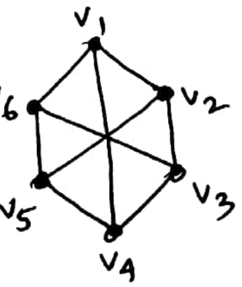
Regular but not complete



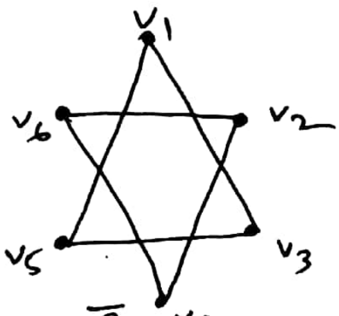
Regular but not complete

Complement of a Graph:

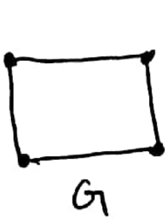
The complement of a simple graph G is another simple graph $\overline{G}$ with same set of vertices such that if any two vertices say u & v are adjacent in G they must be non-adjacent in $\overline{G}$ and vice-versa.



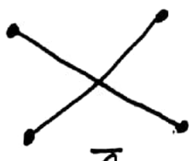
G



$\overline{G}$



G



$\overline{G}$

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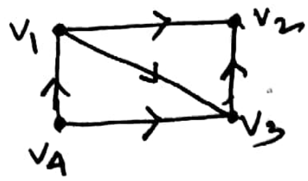
- **Degree of vertex in Directed Graph:**



In Digraph degree of vertex has two types. In degree $d_i(v)$ & Out degree $d_o(v)$

In deg is → how many edges are coming into it

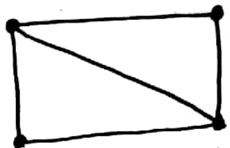
Out deg is → how many edges are coming out of it.

$d_i(v_1) = 1$ $\quad d_o(v_1) = 2$

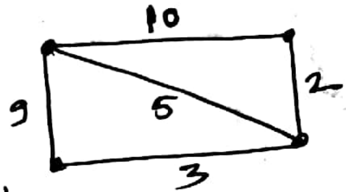
$d_i(v_4) = 0$ $\quad d_o(v_4) = 2$

■ **Weighted & Unweighted Graph**

Some graphs are such that all the edges in the graph are associated with a real no that is referred as the weight of the edge. They are called weighted graph, otherwis unweighted.

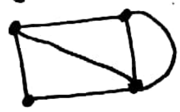


Unweighted



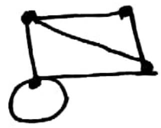
Weighted

■ **Simple Graph:** A graph which has no self loops or parallel edges is a simple graph. ie, a graph where all the edges are proper edges.



Not Simple (Multigraph)

Simple

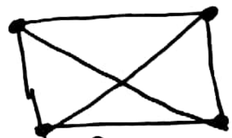


Not simple (Psedugraph)

■ **Regular Graph:**

A graph whose all vertices are of same degree.

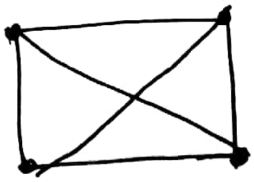
Regular



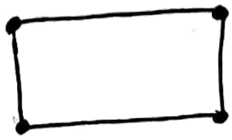
Regular

■ **Complete Graph**

A simple graph in which there exist exactly one edge between each pair of distinct vertices.



Complete graph



Not complete graph

Eg Problems: ④

1) How many vertices are necessary to construct a graph with exactly 12 edges where each vertex are of degree 3.

Sol^n $\sum_{i=1}^{n} d(v_i) = 2e \Rightarrow 3 \times n = 2 \times 12 \Rightarrow n = 8$

2) Is it possible to have a group of 9 people in a group such that each of them is friend with exactly 5 others in the group?

Sol^n Lets the 9 people are the 9 vertices of a graph and friendship between two people is an edge. Then each person has friendship with exactly 5 others means every vertex is of degree 5. There are 9 vertices in the graph. So this is a contradiction as odd degree vertices in a graph must be even in no.

So, it is not possible.

3) If G is a non-directed graph with 12 edges. If G has 6 vertices each of degree 3 and rest have degree less than 3. Find min possible vertices G may have?

Sol^n Say G has n vertices $v_1, v_2, \ldots, v_n$

$\sum_{i=1}^{n} d(v_i) = 2e \Rightarrow d(v_1) + \ldots + d(v_6) + d(v_7) + \ldots + d(v_n) = 2 \times 12$

$3 \times 6 + (n-3) \times ? = 2 \times 12$

Let $d(v_7) = \ldots = d(v_n) = 3$

So, $3 \times 6 + (n-3) \times 3 > 24$

$\Rightarrow n > 8$

So min no of vertices is $8 + 1 = 9$

④ Prove that a simple graph with ($n \geq 2$) vertices has atleast two vertices of same degree.

Sol: Say the vertices are $v_1, v_2, \ldots, v_n$
as the graph is simple, so the max ~~deg~~ deg of any vertex be $(n-1)$
Now say no two vertex in the graph has same degree

So, $d(v_1) = 0$
$d(v_2) = 1$
$\vdots$
$d(v_n) = n-1$

Now this is a contradiction, as if there is one vertex v_n having degree $(n-1)$, so this vertex must be adjacent to rest all other vertices, so, there may not exist any vertex of degree 0.

⑤ If a simple-regular graph with n vertices and 24 edges exist, find all possible values of n

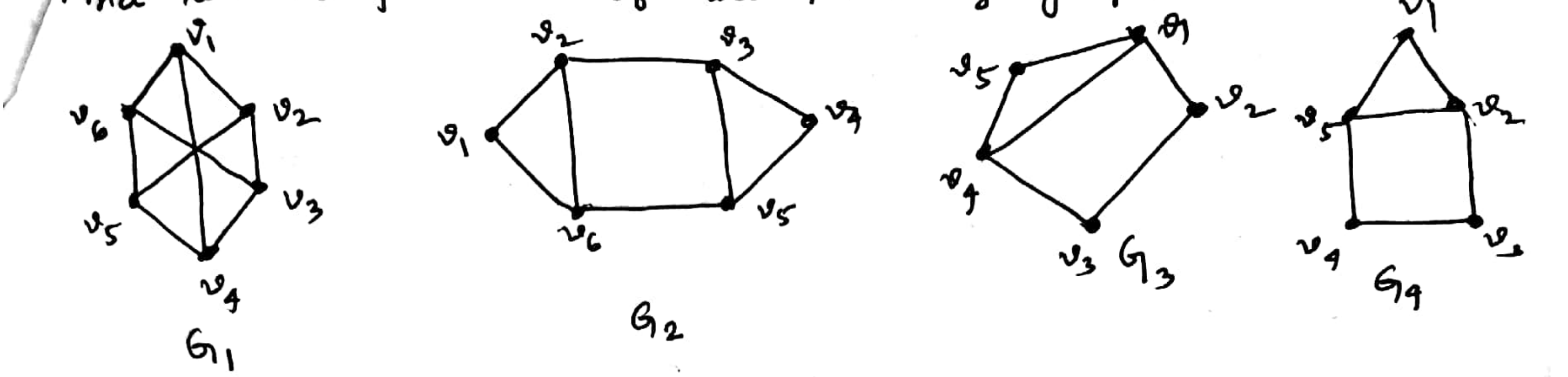
Sol: Say the graph is k-regular
Then $n \times k = 2e \Rightarrow n \times k = 2 \times 24$ —(1)
as this is simple graph so, ${}^nC_2 \geq e = 24$
$\therefore n(n-1) \geq 2 \times 24$ —(2)

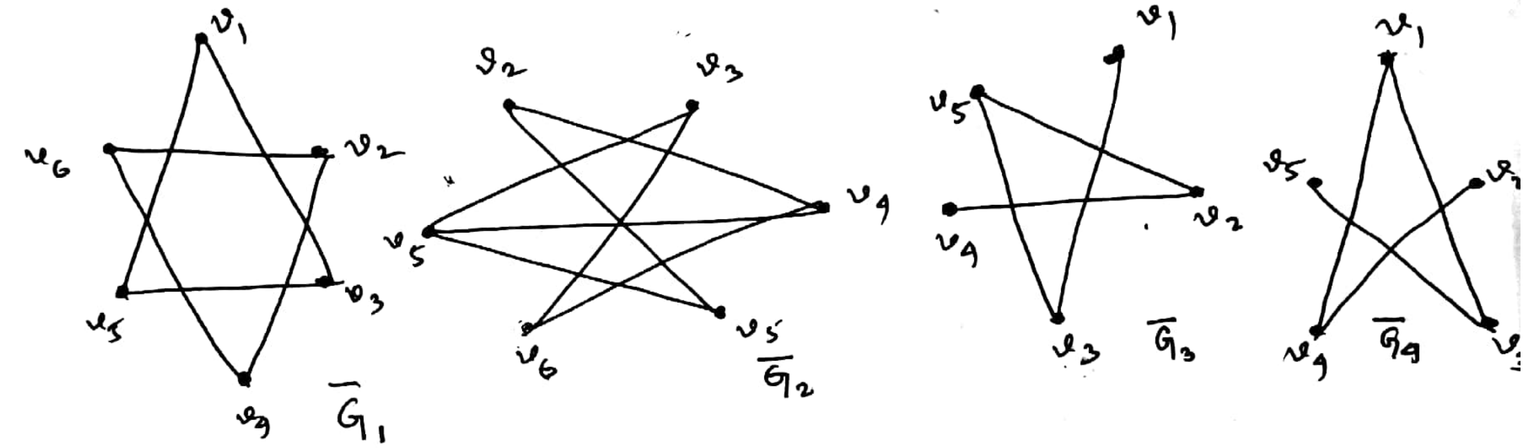
Let $k=1 \Rightarrow n = 48$ $\therefore 48(48-1) \geq 48$ ✓
$k=2 \Rightarrow n = 24$ $\therefore 24(24-1) \geq 48$ ✓
$k=3 \Rightarrow n = 16$ $\therefore 16(16-1) \geq 48$ ✓
$k=4 \Rightarrow n = 12$ $\therefore 12(12-1) \geq 48$ ✓
$k=5 \Rightarrow$ Not possible
$k=6 \Rightarrow n = 8$ $\therefore 8(8-1) \geq 48$ ✓
$k=7 \Rightarrow$ Not possible
$k=8 \Rightarrow n=6$ $\quad 6 \times (6-1) \ngeq 48$

Possible no of vertices

Problems on Complement of Graph

Find the complements of the following graphs



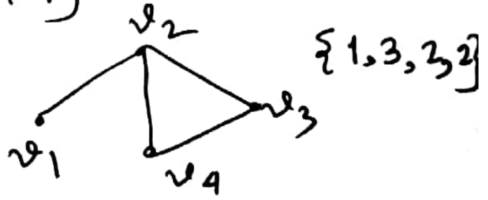


Th.-6 If G be a simple graph with n vertices and $\overline{G}$ be its complement
Prove that any arbitray vertex v in G
$d(v)$ in G + $d(v)$ in $\overline{G}$ = $n-1$

Proof: Let $d(v)$ in $G = k$ This implies v is adjacent to k no. of vertices in G

Therefore $d(v)$ in $\overline{G} = n-1-k$ ie, v must be adjacent to $n-1-k$ no of vertices in $\overline{G}$

$\therefore$ $d(v)$ in G + $d(v)$ in $\overline{G} = k - n - k - 1 = (n-1)$



Degree Sequence: If the degrees of the vertices of a graph are written in a seqn. $\{d(v_i)\}$ then this called degree seqn.

Prob: Is it possible to draw a graph with 4 vertices & 4 edges with degree seqn $\{1, 2, 3, 4\}$

Sol: $\Sigma d(v_i) = d(v_1) + d(v_2) + d(v_3) + d(v_4) = 1+2+3+4 = 10$

$\Sigma d(v_i) = 2e \Rightarrow 10 = 2e \Rightarrow e = 5$

But there are 4 edges, So not possible

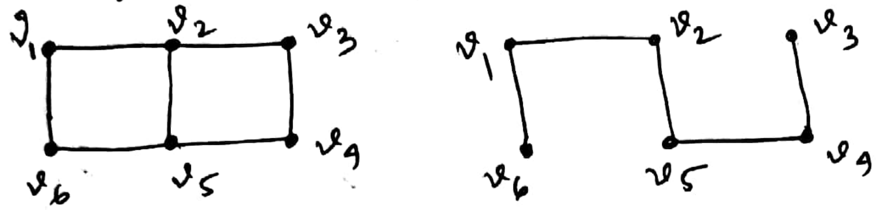
§ Subgraph:

Say $G(V, E)$ be a graph

Then $H(V', E')$ be a subgraph of G

if $V' \subseteq V$ and $E' \subseteq E$.

i.e, if V' is subset of V and E' is subset of E.



§ Spanning Subgraph: If $V' = V$ Then it is a spanning subgraph. i.e, all the vertices of G are present in H.

§ Walk, Path, Trail, Circuit & cycle:

$\boxed{\text{Walk}}$: Finite alternating sequence of vertices and edges, beginning and ending with vertices

$\boxed{\text{Path}}$: An open walk (intial & terminal vertices are diff) with no repeated vertex.

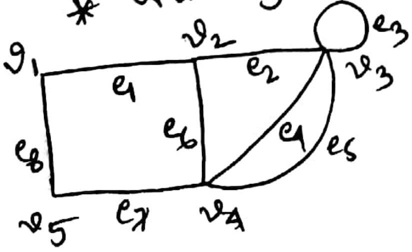
$\boxed{\text{Trail}}$: An open walk with no repeated edge.

$\boxed{\text{Cycle}}$: A path with no repeated vertices except initial and terminal vertex.

$\boxed{\text{Circuit}}$: A trail with no repeated edges where initial and terminal vertices are same.

* All paths are trail but all trails are not path

* All cycle are circuit but all circuit are not cycle.



$W = \{v_1, e_1, v_2, e_6, v_4, e_4, v_3, e_3, v_3, e_5, v_4, e_7, v_5\}$

open walk

$P = \{v_1, e_1, v_2, e_6, v_4\}$

$T = \{v_1, e_1, v_2, e_6, v_4, e_4, v_3, e_5, v_4\}$

P is T but T is not P.

cycle $\rightarrow \{v_1, e_1, v_2, e_6, v_4, e_7, v_5, e_8, v_1\} \rightarrow$ Circuit

The minimum no. of edges in a connected graph with n vertices is $n-1$.

Proof We will prove this result by the method of mathematical induction on the number of edges.

Let the no of edge is m. Need to prove $\boxed{m \geq n-1}$

Case I when $m=0$, the graph consist only one isolated vertex $\therefore n=1$ $\therefore$ edge $= 1-1=0$ $\therefore m \geq n-1$

II when $m=1$, the graph consist only one edge.

so, $n=2$ $\therefore m \geq 2-1$ ie $m \geq n-1$

Now say the result holds good for $m=k$ no. of edges $\therefore$ a graph with n vertices and k edges

$k \geq n-1$

Now if we can show the result for $m=k+1$, we are done.

Let G be any graph with $k+1$ edges and n vertices. We need to show $k+1 \geq n-1$

Now let e be any edge in G.

Now $G \setminus \{e\}$ is a sub graph of G.

Now $G \setminus \{e\}$ has k edges & n vertices

Case-I so, $G \setminus \{e\}$ → If $G \setminus \{e\}$ remains connected

so, by our hypothesis $k \geq n-1$

Now $k+1 \geq n-1+1 = n > n-1$

Case-II If $G \setminus \{e\}$ becomes disconnected.

Let $G \setminus \{e\}$ has two components with k_1 & k_2 no. of edges & n_1 & n_2 no of vertices $n_1+n_2=n$

s.t $k_1+k_2=k$

Now by our hypothesis $k_1 \geq n_1-1$ $k_2 \geq n_2-1$

$\therefore k_1+k_2 \geq (n_1-1)+(n_2-1) = n_1+n_2-2 = n-2$

$\therefore k_1+k_2 = k \geq n-2$

$\therefore k+1 \geq n-1$.

Prob Find the minimum and maximum number of edges in a simple connected graph with 15 vertices.

Sol: We know max. edge of a simple connected graph with n vertex is $^nC_2 = {^{15}C_2} =$

and min edge $n-1 = 15-1$

Result: If G be a simple disconnected graph with n vertices and k components then minimum no of edges G may have is $(n-k)$

Proof: Suppose G be a disconnected graph with k components $g_1, g_2, \ldots, g_k$ with $n_1, n_2, \ldots, n_k$ vertices

Since each of these components are simple and connected so, each components have min $(n_i - 1)$ edges

$\therefore$ Total edges should be $\sum_{i=1}^{n} (n_i - 1) = n-k$ min in the graph G.

Result: If G be a simple disconnected graph with n vertices and k components then maximum no of edges G may have is $\frac{(n-k)(n-k+1)}{2}$ [Proof Not needed]

Prob: Find the number of minimum and maximum no of edges in a disconnected graph with 20 vertices and 5 components

Sol: min edges $n-k = 20-5 = 15$

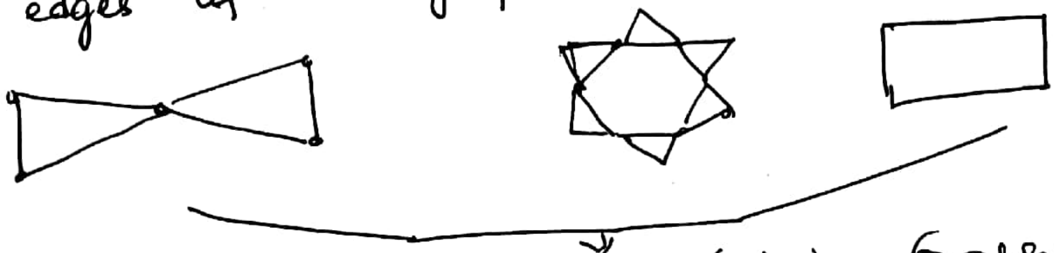
max. edges $\frac{(n-k)(n-k+1)}{2} = \frac{(20-5)(20-5+1)}{2}$

$= \frac{15 \times 16}{2} = 120$

Eulerian Graph:

A graph is called Eulerian graph if it contains an Eulerian circuit.

An Eulerian circuit is circuit that contains all the edges of the graph.



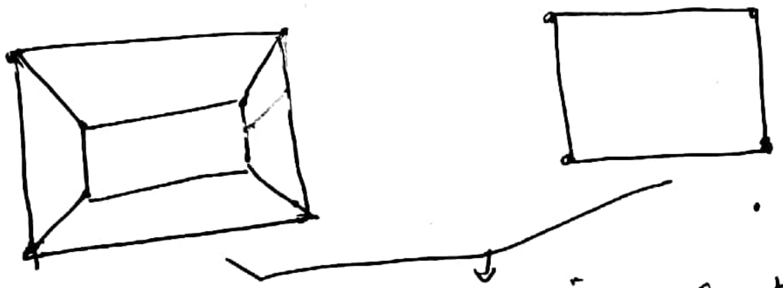
all are Eulerian Graphs

* A graph is Eulerian if the degree of all its vertices are even.

Hamiltonian Graph:

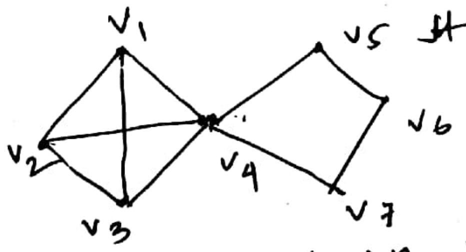
A graph is called Hamiltonian if it contains a Hamiltonian cycle.

An cycle is Hamiltonian if that contains all the vertices of the graph.



Hamiltonian Graphs

Prob



Find distance betn v_2 & v_7.
Find the diameter of the graph

$d(v_2, v_7) = 2$

$\text{Dim}(G) = 3$

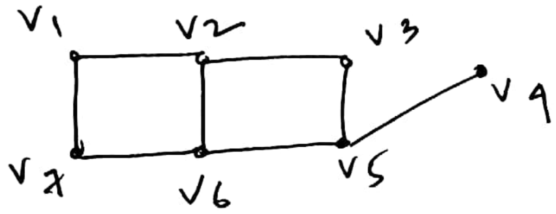
§ Bipartite graph:

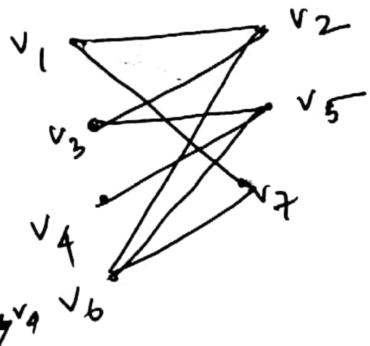
A graph $G(V,E)$ is called bipartite if the vertex set V can be partitioned into two disjoint sets V_1 & V_2 such that any edge of E connects a vertex from V_1 & V_2, but there exists no edge betn the pair of vertices belonging V_1 or V_2.

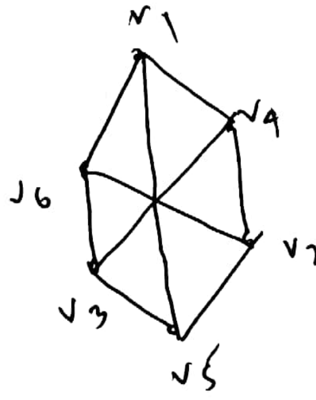
§ Complete Bipartite Graph:

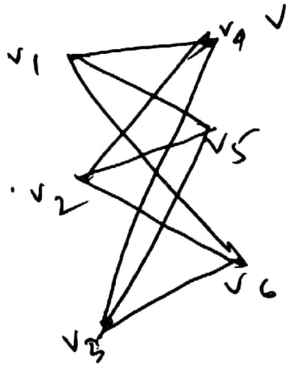
A bipartite graph is called complete if there exists an edge between every pair of the vertices of V_1 & V_2.

§ Check whether the following graphs are bipartite.









Th A Bipartite graph can't contain a cycle of odd length

Let $G(V,E)$ be a bipartite graph with $V = V_1 \cup V_2$

Say G has a cycle $C = \{v_1, e_1, v_2, e_2, v_3, e_3, \ldots, v_n, e_n, v_1\}$ of length n. We need to show n is even.

If $v_1 \in V_1$ & $v_2 \in V_2$ as $v_2 \in V_2$ then $v_3 \in V_1$. So, Similarly if $v_1 \in V_1$ then $v_n \in V_2$

So, all the $v_i \in V_2$ if i is even

So, n is even.

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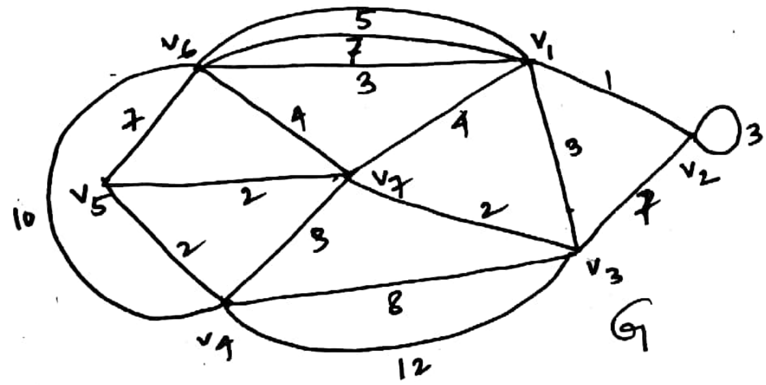
Dijkstra's Algorithm:

This algorithm determines the shortest path between two given vertices of a weighted graph. The shortest path means the path bearing the smallest weight.

Suppose v_i and v_j are the vertices which are adjacent to each other and they are connected by an edge e_{ij}, and say the weight of the edge e_{ij} is w_{ij}. The algorithm follows certain steps that we mention as follows:

Eg: Find the shortest path in the graph G between the vertices v_2 and v_5.



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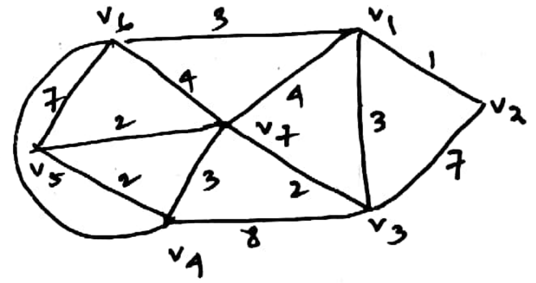

Step I If the given graph is not simple first turn it into simple by discarding all self loops if exist, and also discard the parallel edge except that one which bear the smallest weight among the rest of others.

Step - II Labeling of all the vertices and writing the weights

$$w_{ij} = \begin{cases} \text{weight of } e_{ij}, \text{ if } v_i \,\&\, v_j \text{ are adjacent} \\ \infty \quad \text{if } v_i \,\&\, v_j \text{ are non-adjacent} \end{cases}$$

also label the starting vertex (v_2) permanently as 0.

Step - III In this step we follow an iterative scheme of re-labelling the vertices in each step.

Initially the starting vertex is labelled as zero (0) and remaining all vertices are labelled as ∞ (infinity)

Next step the label will be changed, as per the rule

label of v_i = min[label of v_i at the preceeding step, permanent label of vertex $v_j + w_{ij}$]

where v_j is the vertex that has been permanently labelled in the preceeding step.

In every successive steps one vertex is permanently labelled. The vertex bearing the smallest label value gets permanently labelled. Once a vertex is permanently labelled in the successive step that vertex will not change its label ever.

In this way labelling the vertices will be done following the iterative scheme. We will not stop this process till all vertices are permanently labelled. We will create a tabel to do this labelling.

<u>Step-II</u> Once all the vertices are permanently labled, we look in the table and find for the vertex of the path ie the terminal vertex, here for example it is v_5. We mark the step in the table where this v_5 has got permanently labelled. The we look straight upward in the column, search where the value got changed, identify the step and find the vertex that got permanently labelled in this step, include that vertex in the path, for example here it will be v_7. Next we will keep repeating the same process untill we reach to the intial vertex ie, v_2.

Step	v_1	v_2	v_3	v_4	v_5	v_6	v_7
I	∞	$\boxed{0}$	∞	∞	∞	∞	∞
II	min $[\infty, 0+1]$ $= \boxed{1}$	×	min $[\infty, 0+7]$ $= 7$	min $[\infty, 0+\infty]$ $= \infty$	min $[\infty, 0+\infty]$ $= \infty$	min $[\infty, 0+\infty]$ $= \infty$	min $[\infty, 0+\infty]$ $= \infty$
III	×	×	min $[7, 1+3]$ $= \boxed{4}$	min $[\infty, 1+\infty]$ $= \infty$	min $[\infty, 1+\infty]$ $= \infty$	min $[\infty, 1+3]$ $= 4$	min $[\infty, 1+4]$ $= 5$
IV	×	×	×	min $[\infty, 4+8]$ $= 12$	min $[\infty, 4+\infty]$ $= \infty$	min $[4, 4+\infty]$ $= \boxed{4}$	min $[5, 4+2]$ $= 5$
V	×	×	×	min $[12, 4+10]$ $= 12$	min $[\infty, 4+7]$ $= 11$	×	min $[5, 4+4]$ $= \boxed{5}$
VI	×	×	×	min $[12, 5+3]$ $= 8$	min $[11, 5+2]$ $= \boxed{7}$	×	×
VII	×	×	×	min $[8, 7+2]$ $= \boxed{8}$	×	×	×
VIII	×	×	×	×	×	×	×

Hence the path will be $v_5 \rightarrow v_7 \rightarrow v_1 \rightarrow v_2$ weight is $2+4+1=7$
writing in reverse way $v_2 — v_1 — v_7 — v_5$

Theorems of Graph

1. The sum of degrees of all the vertices is twice the no of edges. (Handshaking Theorem / First theorem of graph theory).

2. The maximum degree of any vertex in a simple graph with n vertices is $(n-1)$.

3. The maximum no of edges in a simple connected graph with n vertices is $^nC_2 = \frac{n(n-1)}{2}$

4. The no of odd degree vertices in a graph is always even.

5. The complete graph with n vertices has exactly $^nC_2 = \frac{n(n-1)}{2}$ edges.

6. The minimum no of edges in a simple connected graph is $(n-1)$.

7. The minimum no of edges in a simple (but not necessarily connected) graph with n vertices is $(n-k)$ where k is the no of connected components of the graph.

8. In a digraph (directed graph) the sum of out degrees of all vertices = the sum of the indegrees of all vertices = no of edges in the graph.

9. If a graph has exactly two vertices of odd degree then there must exists a path joining these two vertices.

10. Def^n of Disconnected Graph:

A graph G with vertex set V is said to be disconnected iff there exists two non-empty disjoint sets V_1 & V_2 s.t $V = V_1 \cup V_2$ and there exists no edge in G whose one end vertex belongs to the set V_1 & other to V_2.

11. A simple graph with n vertices and k components can have maximum $\frac{(n-k)(n-k+1)}{2}$ edges.
(Proof Not required)

12. A bipartite graph can't contain a cycle of odd length.

13. If a simple graph does not contain any cycle of odd length then the graph is bipartite.

14. A connected graph is an Euler graph iff all the vertices are of even degree.

15. Every $u-v$ Trail contains a $u-v$ path

16. All Paths are Trail but all trails are not paths

17. All cycles are circuits but all circuits are not cycles.

18. Every vertex except the initial (origin) and the terminal vertices of a trail are of even degree.

* Proof Not required of Th. 8, 11, 14, 15, 16, 17, 18