6. Two beam interference pattern, Thin film

Imp.

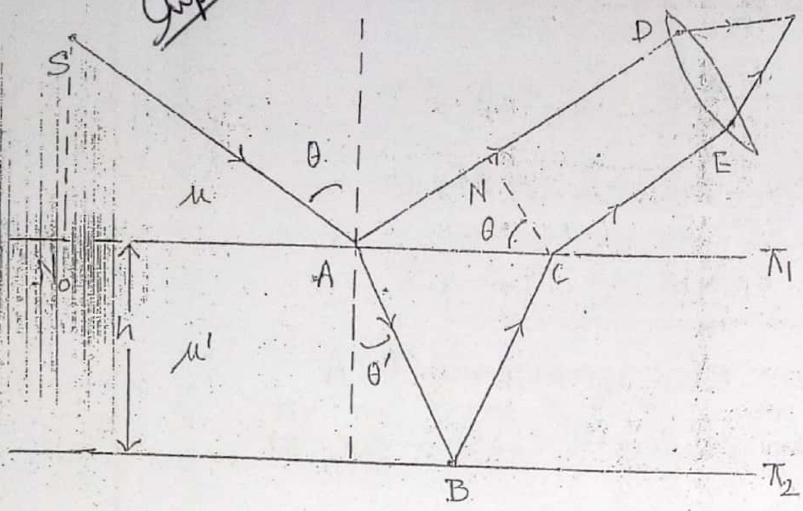


Diagram-7

π_1 and π_2 are are two surfaces (upper and lower respectively) of a plate made of transparent material. π_1 and π_2 are parallel. The plate is illuminated by a point source S of quasi-monochromatic light. The point P is reached by two rays — one reflected by π_1 and other by π_2. We get the following observations.

(i) A nonlocalized interference pattern on the same side of the plate as S.

(ii) Due to symmetry around SN_0, the fringes in the plane parallel to the plate are circular around SN_0, so that at any position of the point P, the run perpendicular to the plane SN_0P.

Explanation

The optical path difference between the rays SADP and SABCE is

$$\Delta = \mu'(AB + BC) - \mu . AN \quad \text{- - -} \quad (i)$$

From geometrical consideration $AB = BC = \frac{h}{\cos\theta'}$ - - (ii)

and $AN = AC \sin\theta = 2h \tan\theta' \sin\theta$ - - - (iii)

From Snell's law of refraction

$$\mu' \sin\theta' = \mu \sin\theta \quad \text{- - -} \quad (iv)$$

Hence from equation-(i)

$$\Delta = \mu' \frac{2h}{\cos\theta'} - \mu 2h \tan\theta' \sin\theta$$

$$\Rightarrow \Delta = 2h\left[\frac{\mu'}{\cos\theta'} - \tan\theta' \mu' \sin\theta'\right]$$

$$\Rightarrow \Delta = 2\mu' h\left[\frac{1}{\cos\theta'} - \frac{\sin^2\theta'}{\cos\theta'}\right]$$

$$\Rightarrow \boxed{\Delta = 2\mu' h \cos\theta'} \quad (\text{cosine law})$$

A phase change can occur due to reflection either from π_1 or from π_2. Hence an extra phase difference $\pm\pi$ is to be added. Hence the net phase difference

$$\delta = \frac{2\pi}{\lambda}\Delta \pm \pi$$

$$= \frac{4\pi}{\lambda}\mu' h \cos\theta' \pm \pi$$

For bright fringe: $\frac{4\pi}{\lambda}\mu' h \cos\theta' \pm \pi = 2m\pi$

$$\Rightarrow \boxed{2\mu' h \cos\theta' = (2m \pm 1)\frac{\lambda}{2}}, \; m = 0, 1, 2$$

For dark fringe: $\frac{4\pi}{\lambda}\mu' h \cos\theta' \pm \pi = (2n+1)\pi$

$$\Rightarrow 2\mu' h \cos\theta' = (2n \pm 2)\frac{\lambda}{2}$$

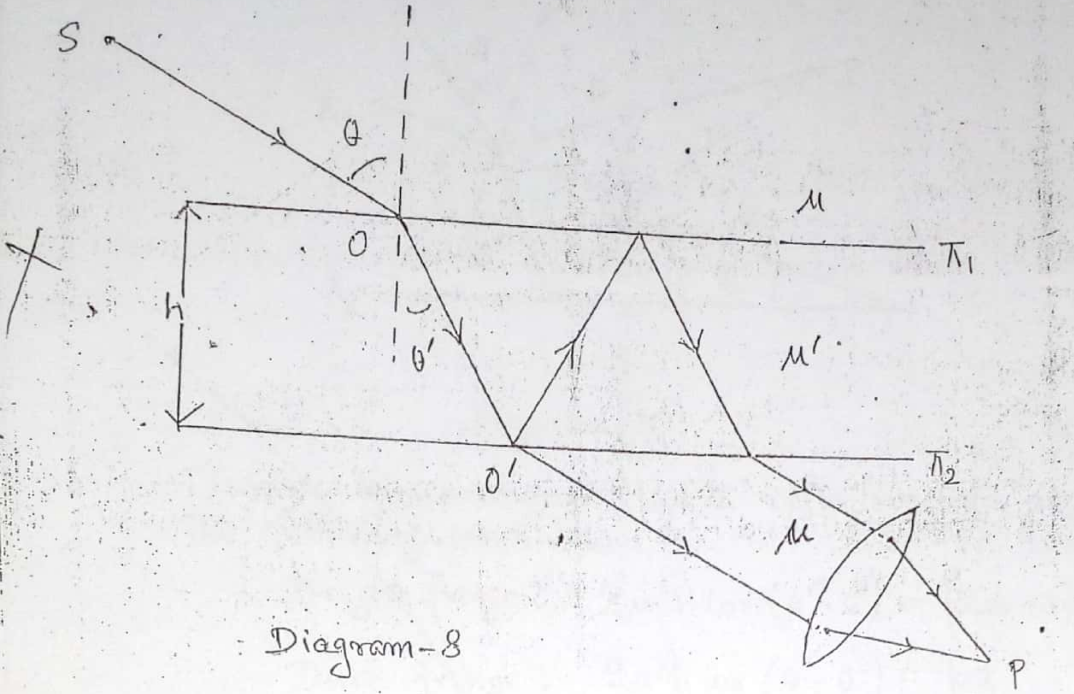
$$\Rightarrow \boxed{2\mu' h \cos\theta' = p\lambda}, \; p = 0, 1, 2 \cdots$$

Remark: 1. As θ is determined only by the position of P in the focal plane of the telescope, δ is independent of the position of the source S. It follows that fringes are as distinct with an extended source as with the point source.

2. As the fringes are characterized by the value of θ and θ', the are called fringes of equal inclination.

3. When the axis of telescope objective is normal to the plate, the fringes are concentric circles about the focal point for normally reflected light $(\theta = \theta' = 0)$.

4. If we consider the interference of transmitted waves, the following diagram results :-



: Diagram-8

The path difference can be calculated as usual giving

$$\Delta = 2\mu' h \cos\theta'$$

There is no additional phase difference from the phase change on reflection considering O' as the starting point. Hence, the phase difference

$$\delta = \frac{2\pi}{\lambda}\Delta = \frac{4\pi}{\lambda}\mu' h \cos\theta'$$

Bright fringe : $\frac{4\pi}{\lambda}\mu' h \cos\theta' = 2m\pi$

$$\Rightarrow 2\mu' h \cos\theta' = m\lambda \;:\; m = 0, 1, 2 \cdots$$

Dark fringe : $\frac{4\pi}{\lambda}\mu' h \cos\theta' = (2n+1)\pi$

$$\Rightarrow 2\mu' h \cos\theta' = (2n+1)\lambda/2 \;:\; n = 0, 1, 2 \cdots$$

The conditions are just reversed as that with the reflected rays.

5. <u>Wedge-Shaped Film</u>: If there is a nonzero angle θ_0 between the planes π_1 and π_2. The ~~phase~~ path difference between the relevant reflected rays can be shown to be equal to

$$\boxed{\Delta = 2\mu' h \cos(\theta' - \theta_0)}$$

Imp.

by considering the following diagram.

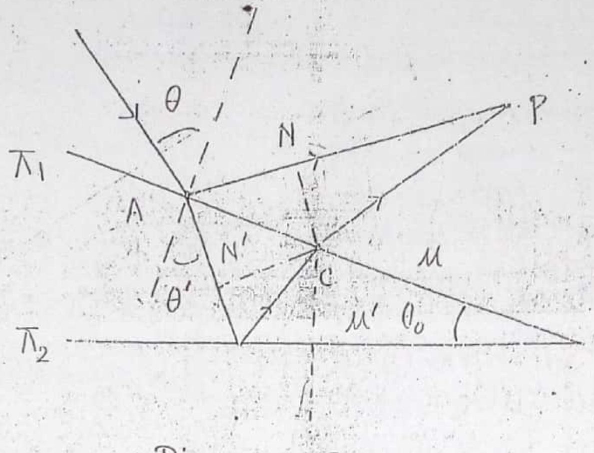


Diagram-7

Taking phase change due to reflection into account we get the following conditions:

Bright fringe : $2\mu' h \cos(\theta' - \theta^{\circ}) = (2m+1)\frac{\lambda}{2}$

Dark fringe : $2\mu' h \cos(\theta' - \theta^{\circ}) = p\lambda$

a. For small wedge-angle $\cos(\theta' - \theta_0)$ can be averaged over the points of the source which contribute light to P. $\langle \cos(\theta' - \theta_0) \rangle$ remains fixed in that case and the fringes are of equal thick--ness.

b. For $h \to 0$, the path difference $= \lambda/2$, the film surface will be perfectly dark even with the white light.

c. When the white light is incident on the film λ, θ' and μ' will be different for different colours of light. So at a particular point all the wave lengths may not satisfy the condition of maxima and minima. Hence some of the colours may be absent in the reflected beam.

Imp. C. Newton's Ring

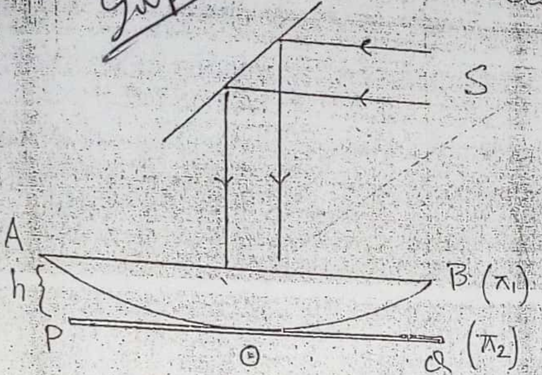


Diagram – 10

A plano-convex lens AOB is placed on a glass plate POQ. We are considering the interference due to thin film (air) captured between the surface π_1 (curved surface of the lens AOB) and the surface π_2 (glass plate POQ) is a wedge-shaped region and interference is achieved by division of amplitude. The light reflected from the surface AOB and that reflected from the surface POQ will therefore interfere as path difference develops due to

(i) Extra path traversed by the ray reflected by the surface POQ.

(ii) A phase difference that develops due to the reflection from optically denser medium.

Neglecting the wedge angle, the path difference

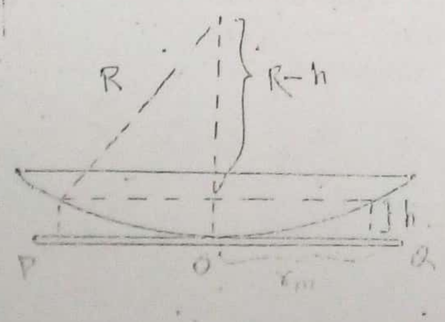
Bright fringe $\Rightarrow 2\mu' h = (2m+1)\frac{\lambda}{2} \quad | \; m = 0, 1, \ldots$

Dark fringe $\Rightarrow 2\mu' h = m\lambda \quad | \; m = 0, 1, 2$

Remark : 1. As the thickness at the point of contact is zero, the central fringe will be ~~zero~~ dark.

2. Since, the convex side of the lens is a spherical surface, the thickness of the air film will be constant over a circle. As a result concentric dark and bright fringes will be available.

3. The radius of the m-th fringe can be calculated by the following consideration.



Let, r_m be the radius of the m-th fringe and h_0 be the thickness of the film at a distance r_m from the point of contact O. From geometrical consideration,

$$r_m^2 = R^2 - (R-h_0)^2 = h_0(2R-h_0)$$
$$\simeq 2h_0 R$$

Now for m-th bright fringe $2\mu' h_c = (2m+1)\lambda/2$

$$\Rightarrow 2\mu' \frac{r_m^2}{2R} = (2m+1)\frac{\lambda}{2}$$

$$\Rightarrow \boxed{r_m^2 = \frac{2m+1}{2\mu'}\lambda R}$$

Similarly for m-th dark fringe $\boxed{r_m^2 = \frac{m\lambda R}{\mu'}}$

a. The above calculation shows that the radii of the ring vary as square root of natural numbers. This fact reflects into the observation that the rings appear close to each other with radius increasing.

b. The difference in square of the radii of m-th and (m+p)-th fringes is

$$r_{m+p}^2 - r_m^2 = p\lambda R, \ (\mu' = 1)$$

$$\Rightarrow \lambda = \frac{r_{m+p}^2 - r_m^2}{pR}$$

$$\Rightarrow \boxed{\lambda = \frac{D_{m+p}^2 - D_m^2}{4pR}} \ (D = \text{diameter})$$

c. Gradual lifting of the lens along the upward direction will result in the collapse of the fringes. An upward shift by an amount $\lambda/4$ will result causing the next fringe to occupy the position of the previous fringe of smaller radius.

d. There can be available a pattern due to the transmitted light but their visibility status is very low.

e. If white light is used in Newton's ring experiment the central fringe will be dark as usual, but away from centre, the pattern from different monochromatic components of the source become increasingly out of step — coloured ring will be observed in a characteristic sequence known as Newton's colour.

II. Superposition of two SHM's.

In view of the kinematical consideration presented i the last section we can claim that a uniform circular motion of angular frequency ω_0 can be resolved ~~ix~~ into two simple harmonic motions along the two mutualy perpendicular co-ordinate axes. The SHM's thus obtained have

(i) Identical frequencies.

(ii) Identical amplitudes.

(iii) $\frac{\pi}{2}$ phase separation.

In the discussion that follows we'll consider just the reverse — the composition of two mutually perpendicular SHM's by retaining the condition (i) and relaxing (ii) and (iii) giving rise to what is known as Lissajous figures.

• a. Perpendicular Superposition

• *Definition-7*: The diagram representing the trajectory of the resultant motion due to superposition of two mutually perpendicular simple harmonic motions is called a Lissajous figure.

We'll consider various cases of Lissajous figures when the superposing motions are of identical frequencies ~~and~~ but different amplitudes and phase separations, in view of the following theorem.

Theorem-2: Let's consider two SHM's of identical frequency about the origin in X-Y plane, ~~interacting~~ given by

$$\left.\begin{aligned} x(t) &= a_1 \cos(\omega_0 t + \delta_1) \quad \cdots \\ y(t) &= a_2 \cos(\omega_0 t + \delta_2) \quad \cdots \end{aligned}\right\} [14\ a, b]$$

The trajectory representing the resultant motion due to their superposion (perpendicular) is an ellipse.

Proof: From [14a]

$$\left.\begin{aligned} \frac{x}{a_1} &= \cos\omega_0 t \cos\delta_1 - \sin\omega_0 t \sin\delta_1 \\ \frac{y}{a_2} &= \cos\omega_0 t \cos\delta_2 - \sin\omega_0 t \sin\delta_2 \end{aligned}\right\} [15a, b]$$

$[15-a] \times \sin\delta_2 - [15-b] \times \sin\delta_1$ yields.

$$\frac{x}{a_1}\sin\delta_2 - \frac{y}{a_2}\sin\delta_1 = \cos\omega t \sin(\delta_2-\delta_1) \quad \cdots [16a]$$

Again, $[15-a] \times \cos\delta_2 - [15-b] \times \cancel{\sin\delta}\cos\delta_1$ yields.

$$\frac{x}{a_1}\cos\delta_2 - \frac{y}{a_2}\cos\delta_1 = \cancel{\cos\omega t}\ \sin\omega t \sin(\delta_2-\delta_1) - [16b]$$

$[16a]^2 + [16b]^2$ yields.

$$\boxed{\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos(\delta_2-\delta_1) = \sin^2(\delta_2-\delta_1)} \quad \cdots [17]$$

Equation-[17] represents a conic.

Now, the determinant

$$\begin{vmatrix} \frac{1}{a_1^2} & -\frac{1}{a_1 a_2}\cos(\delta_2-\delta_1) \\ -\frac{1}{a_1 a_2}\cos(\delta_2-\delta_1) & \frac{1}{a_2^2} \end{vmatrix} = \frac{\sin^2(\delta_2-\delta_1)}{a_1^2 a_2^2} \geq 0$$

This means eqn. [17] represents an ellipse.

Remark : 1. The resultant trajectory is therefore independent of the ~~pass~~ phases of the individual oscillations but depends upon the phase difference $|\delta_2-\delta_1| (=\delta)$. (absolute value)

2. For $\delta = 0$ eqn. [17] becomes

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos 0^\circ = \sin^2 0^\circ$$

$$\Rightarrow \quad y = \frac{a_2}{a_1}x \text{ (straight line) (Diagram — 2a)}$$

For $\delta = \pi$, $y = -\frac{a_2}{a_1}x$ ~~(Diagram~~

3. $\delta = \frac{\pi}{2}, \frac{3\pi}{2}$ the eqn. [17] becomes

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1. \text{ (Diagram-(2b)}$$

4. The direction of the superposed motion can be determined by locating the co-ordinate over the time period, as illustrated by the following example. (example-~~2~~ 2a)

5. As $\max|x(t)| = a_1$ and $\max|y(t)| = a_2$ the ellipse will always be confined in the region $-a_1 \le x \le a_1$ and $-a_2 \le y \le a_2$. It touches the the bounds at points, $(\pm a_1, \pm a_2 \cos\delta)$ and $(\pm a_1 \cos\delta, \pm a_2)$. Example-2 will clarify the matter.

• *Example - 2 a.* $x(t) = a_1 \cos\omega t$; $y(t) = a_2 \cos\omega t$. $a_1, a_2 > 0$

$a_1 > a_2$

$y = \frac{a_2}{a_1} x$ with $\delta = 0$

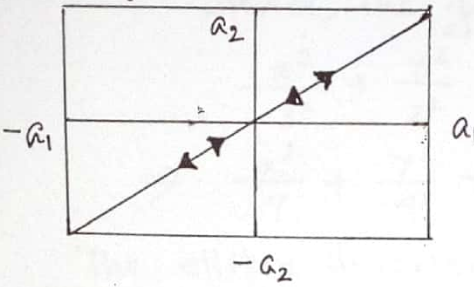


Diagram - 2a

t	0	$\frac{\pi}{4\omega}$	$\frac{\pi}{2\omega}$	$\frac{3\pi}{4\omega}$	$\frac{\pi}{\omega}$	$\frac{5\pi}{4\omega}$	$\frac{3\pi}{2\omega}$	$\frac{7\pi}{4\omega}$	$\frac{2\pi}{\omega}$	$0+$
X	a_1	$\frac{a_1}{\sqrt{2}}$	0	$-\frac{a_1}{\sqrt{2}}$	$-a_1$	$-\frac{a_1}{\sqrt{2}}$	0	$\frac{a_1}{\sqrt{2}}$	$a_1 \to$	
Y	a_2	$\frac{a_2}{\sqrt{2}}$	0	$-\frac{a_2}{\sqrt{2}}$	$-a_2$	$-\frac{a_2}{\sqrt{2}}$	0	$\frac{a_2}{\sqrt{2}}$	$a_2 \to$	

Table - 2a

The trajectory is a straight line inclined at an angle $\tan^{-1}\frac{a}{a}$ w.r.t. the x-axis.

• *Remark*: For $\delta = \pi$, i.e.; $x(t) = a_1 \cos\omega t$ $y(t) = a_2 \cos(\pi + \omega t)$ the st. line would have been $y = -\frac{a_2}{a_1} x$.

• *Example - 2.b*: $x(t) = a_1 \cos\omega t$; $y(t) = a_2 \sin\omega t$, $a_1 > a_2 > 0$

Comparing with the standard form

$$x(t) = a_1 \cos(\omega t + 0)$$
$$y(t) = a_2 \cos\left(\omega t - \frac{\pi}{2}\right)$$

giving $\delta = \frac{\pi}{2}$ and

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1.$$

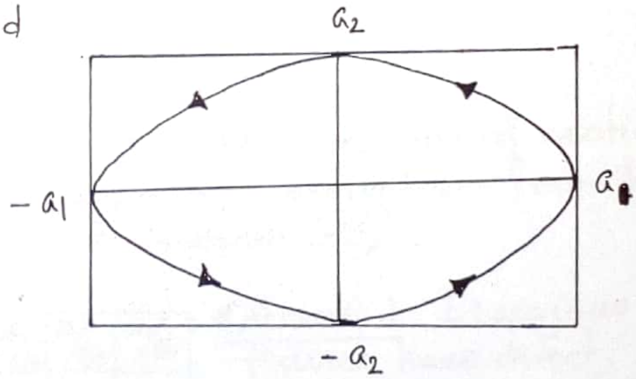


Diagram - 2b.

The resultant ellipse touches the bounds at $(\pm a_1, 0)$ and $(0, \pm a_2)$. The direction of motion can be understood by the following table.

t	0	$\frac{\pi}{4\omega}$	$\frac{\pi}{2\omega}$	$\frac{3\pi}{4\omega}$	$\frac{\pi}{\omega}$	$\frac{5\pi}{4\omega}$	$\frac{3\pi}{2\omega}$	$\frac{7\pi}{4\omega}$	$\frac{2\pi}{\omega}$
X	a_1	$\frac{a_1}{\sqrt{2}}$	0	$-\frac{a_1}{\sqrt{2}}$	$-a_1$	$-\frac{a_1}{\sqrt{2}}$	0	$\frac{a_1}{\sqrt{2}}$	a_1
Y	0	$\frac{a_2}{\sqrt{2}}$	a_2	$\frac{a_2}{\sqrt{2}}$	0	$-\frac{a_2}{\sqrt{2}}$	$-a_2$	$-\frac{a_2}{\sqrt{2}}$	0

• *Remark* 1. For $a_1 = a_2 = a > 0$ the ellipse will be circle

2. For $x(t) = a_1 \sin\omega t$ and $y(t) = a_2 \cos\omega t$, the motion will get reversed. (clockwise)

→ Table 2-b

- *Example-2c*: Let $x(t) = 3\sin\omega t$; $y(t) = 2\cos\left(\omega t + \frac{\pi}{4}\right)$

Comparing with the standard form

$$x(t) = 3\cos\left(\omega t - \frac{\pi}{2}\right)$$
$$y(t) = 2\cos\left(\omega t + \frac{\pi}{4}\right)$$
$$a_1 = 3,\ a_2 = 2,\ \delta = 3\pi/4$$

Hence, the general equation of ellipse takes the form

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} - \frac{2xy}{3.2}\cos\left(\frac{3\pi}{4}\right) = \sin^2\frac{3\pi}{4}$$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{4} + \frac{\sqrt{2}\,xy}{6} = \frac{1}{2} \quad \cdots\cdots [18]$$

The ellipse described by equation [18] is confined within the region $-3 \leq x \leq +3$ and $-2 \leq y \leq +2$

The points at which it touches $x = \pm 3$ and $y = \pm 2$ can be found by solving [18] and the corresponding equation.

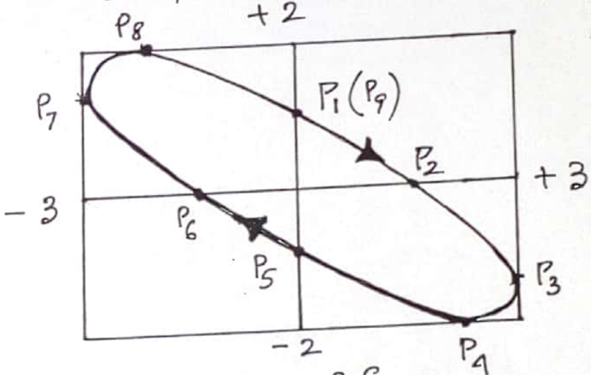


Diagram - 2 C

For example solving eq^n 18 and $x = 3$ we get $y = -\sqrt{2}$. Similarly, we get four touching points $(3, -\sqrt{2})$, $(-3, \sqrt{2})$, $(-3/\sqrt{2}, 2)$ and $\left(\frac{3}{\sqrt{2}}, -2\right)$. The direction of motion over the time period can again be determined from the following table. (Table-2-c and Diagram 2-c)

t	0	$\pi/4\omega$	$\pi/2\omega$	$3\pi/4\omega$	π/ω	$5\pi/4\omega$	$3\pi/2\omega$	$7\pi/4\omega$	$2\pi/\omega$
X	0	$\frac{3}{\sqrt{2}}$	3	$\frac{3}{\sqrt{2}}$	0	$-\frac{3}{\sqrt{2}}$	-3	$-\frac{3}{\sqrt{2}}$	0
Y	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2	$\sqrt{2}$
	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9

Table 2-C

- *Remark* 1. Lissajous' figures have direct correspondence to the phenomenon of polarization where the behavior of light vector is usually under-stood as the superposition of two components of the said vector (having identical frequencies) in the ~~respective~~ plane of polarization.

2. A broad class of Lissajous' figures can be obtained when the superposing waves are of different frequencies.