Quantum Physics : PART ONE

§ 1. Introduction

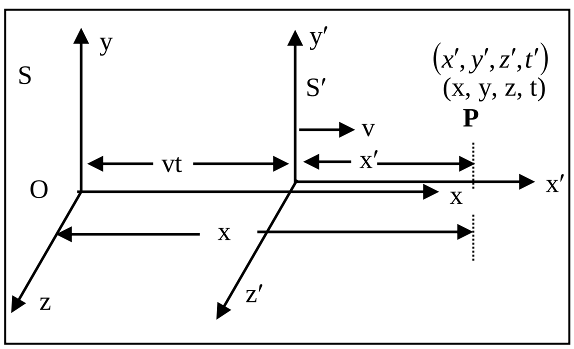
Classical physics is based on Newton's law of motion and mainly deals with the macroscopic bodies and the phenomenon related to them. The postulates of classical mechanics could not explain the phenomenon like black body radiation, Photoelectric effect, Compton effect, stability of atom etc. as the theoretical concepts of classical physics were found to be contrary to the experimentally observed facts. This led to the formulation of new concept to provide satisfactory explanation of the above. These concepts are now developed to an extent of forming an entirely different branch of physics called Quantum Physics.To understand Quantum Physics we need to have a basic knowledge of special theory of relativity which are discussed in the subsequent section.

Basic Knowledge of Special theory of relativity

§ 2. Galelian transformation

When a physical phenomenon is observed in two inertial frames moving with uniform velocity relative to each other and the time interval registered in both the frames is same, then datas of results in one frame of reference can be transformed to these in the second frame. This process is known as Galilean transformation

Galilean Co-ordinate Transformation



Let us assume S frame is stationary while S' frame is moving with uniform velocity v along (+ve) x axis. Initially both the co-ordinate axes of both the frames coincide at O at time t = 0.

After a time t = t S' frame has traveled at a distance vt during time interval t. An event occurred at point P is measured by the observers from both the frames.

In S' frame the co-ordinate and time of P measured as (x', y', z', t')

In S frame the co-ordinate and time of P measured as (x, y, z, t).

Since the co-ordinate frame S' is moving along (+ve) x direction the relationship between the measurement (x, y, z, t) of P and the measurements (x', y', z', t') of P for a particular event is obtained by

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\}$$

These above four equations are called the Galilean co-ordinates transformations.

Speciality of Galilean transformation

Space and time coordinates do not mix, ***importance of these equation*** is that they ensure the physical laws that are invariant with respect to these equations are valid everywhere and at all times (if we use our common sense ideas of space and time)

§ 3. Galilean velocity transformations

According to Galilean transformation, co-ordinate transformations are,

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

Let the velocity components (u_x, u_y, u_z) and $\left(u'_x, u'_y, u'_z\right)$ are measured from frame S and S' respectively. The relationship between (u_x, u_y, u_z) and $\left(u'_x, u'_y, u'_z\right)$ is obtained from time differentiation of the Galilean co-ordinate transformation $u'_x = \frac{dx'}{dt'} = \frac{d}{dt}(x - vt)\frac{dt}{dt'}$

$$u'_x = \left(\frac{dx}{dt} - v\right)1 \qquad [\text{as } t = t'\,]$$

$$\boxed{u'_x = u_x - v} \quad u'_y = u_y \quad \& \quad u'_z = u_z \quad [\text{as } y' = y \quad \& \quad z' = z\ ]$$

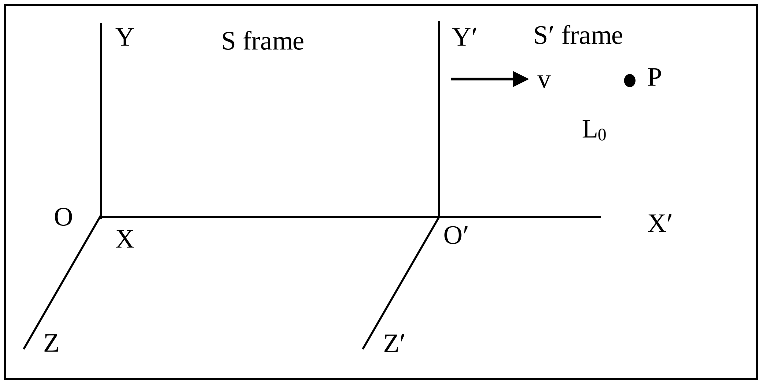
which are required velocity transformations .

Limitation: When two photons are moving from opposite direction with the speed *c* then the relative velocity of one with respect to other is 2c which is against the postulate of the special theory of relativity. To have a correct interpretation we will learn another transformation i.e., Lorentz transformation.

Postulates of Special theory of relativity:

1) All the laws of physics are same in all inertial frames of reference having an uniform translational velocity w.r.t one onther
2) The velocity of light is the same in all inertial frames and is independent of the motion of the source and observer.

§ 4. Lorentz Transformation



Let us consider two inertial frames of reference S and S', which, for simplicity assume have parallel axes in Cartesian systems, and which have their X axis, coincident with origins O and O' which we assume also coincident at the inertial instant of motion, t = t' =0

A point P be represented by (x, y, z, t) as measured by a observer in S frame and that of (x', y', z', t') in S' frame. The frame S' have a constant relative speed of v with respect to S frame along common X axis. Let we imagine at the time t = t' = 0 when the origin O' coincides with the origin O a spherical pulse of light leaves the common origin of S and S'. Since the velocity of light is invariant each observer sees a spherical wave expanding outwards with the speed c in his own system of measuring instruments.

$\therefore$ For observer in S frame:

$$x^2 + y^2 + z^2 = c^2t^2 \quad \text{--------------(1)}$$

Similarly for the observer in S' frame,

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \quad \text{----------(2)}$$

Since the motion of $S^/$ is along the $XX^/$ axes, it follows that the y and z coordinates measured in the two frames are always equal. That is, they are unaffected by the motion along x and the therefore $y=y^/$ and $z=z^/$

Hence subtracting (2) from (1) we get $x^2 - x'^2 = c^2t^2 - c^2t^{/2}$

$$x^2 - c^2t^2 = x'^2 - c^2t^{/2} \text{------------(3)}$$

Now, the inertial frames S(x,y,z,t) and $S'(x^/, y^/, z^/, t^/)$ will be equivalent if the transformation equations are symmetrical w.r.t both. If the quantities (*x,y,z,t*) be finite , then $(x^/, y^/, z^/, t^/)$ must also be finite. These conditions require that the transformations between S(x,y,z,t) and $S'(x^/, y^/, z^/, t^/)$ must be linear.

Hence , we take the transformation $\quad x^/ = \gamma\left(x - vt\right) \quad$ ----------(4)

And $\quad x' = \gamma^/\left(x^/ + vt^/\right) \quad$ --------(5)

Where γ and $\gamma^/$ are two undetermined constants.

Now from (5) we have $t' = \frac{1}{v}\left(\frac{x}{\gamma'} - x'\right) = \frac{1}{v}\left(\frac{x}{\gamma'} - \gamma\left(x - vt\right)\right)$ (replacing x' in (4))

Or, $$t' = \frac{1}{\gamma}\left(t - \frac{x}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)\right) \text{ --------(6)}$$

Substituting the value of x' and t' from equation (4) and (6) in equation (3) we have

$$x^2 - c^2t^2 = \gamma^2\left(x - vt\right)^2 - c^2\gamma^2\left(t - \frac{x}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)\right)^2 \text{ -------(7)}$$

Equating the coefficients of t^2 and x^2 from both sides of equation (7) we get

$$-c^2 = \gamma^2 v^2 - c^2\gamma^2 \text{ --------------------(8)}$$

And $$1 = \gamma^2 - c^2\gamma^2\frac{1}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2 \text{ ---------------(9)}$$

From (8) we have $\boldsymbol{\gamma} = \frac{c}{\sqrt{c^2 - v^2}}$ or $\boldsymbol{\gamma} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ ……………(10)

From equation (9) $\frac{1}{\gamma^2} = 1 - \frac{c^2}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2$ or, $1 - \frac{v^2}{c^2} = 1 - \frac{c^2}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2$ (replacing γ from 10)

$\frac{v^2}{c^2} = \frac{c^2}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2$. or, $\frac{v^4}{c^4} = \left(1 - \frac{1}{\gamma\gamma'}\right)^2$ or, $\frac{v^2}{c^2} = \left(1 - \frac{1}{\gamma\gamma'}\right)$ or, $\frac{1}{\gamma\gamma'} = 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$

Or, $$\boldsymbol{\gamma} = \boldsymbol{\gamma}'\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Hence the required traformations are $x' = \gamma\left(x - vt\right) = \frac{\left(x - vt\right)}{\sqrt{1 - \frac{v^2}{c^2}}}$, $y' = y$, $z' = z$

And $t' = \frac{1}{\gamma}\left(t - \frac{x}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)\right) = \gamma\{t - \frac{x}{v}.\frac{v^2}{c^2}\}$ or, $t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$

Therefore ***Lorentz transformation*** equations are

$\boxed{x' = \frac{x - v\,t}{\sqrt{1 - v^2/c^2}}}$; $\boxed{t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}}$; $\boxed{y' = y}$ & $\boxed{z' = z}$

The ***Inverse Lorentz transformation equations*** are obtained by interchanging the coordinates and replacing v by –v the above equations becomes

$$\boxed{x = \frac{x' + v' t'}{\sqrt{1 - v^2/c^2}}} \quad \boxed{t = \frac{t' + v/c^2\, x'}{\sqrt{1 - v^2/c^2}}} \quad \boxed{y' = y\ ;} \quad \boxed{z' = z}$$

Special case: In the low velocity region i.e., (v<<c) , $\frac{v^2}{c^2} << 1$, Lorentz transformation reduce to

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Which is Galilean transformation

§ 5. Relativistic Velocity Addition

So far we have reached at the idea that Galilean transformation are no more valid in Relativity, so simple vector addition for velocities ($\vec{v}_1 + \vec{v}_2$) law followed in Newtonian mechanics will not be hold good. Let we imagine a particle moving with a constant velocity $\vec{v}$ in frame S, with components u_x, u_y, u_z along the axes. Then we have,

$$u_x = \frac{dx}{dt}; \qquad u_y = \frac{dy}{dt}; \qquad u_z = \frac{dz}{dt} \quad \text{in S frame.}$$

$$u'_x = \frac{dx'}{dt'}; \qquad u'_y = \frac{dy'}{dt'}; \qquad u'_z = \frac{dz'}{dt'} \quad \text{in } S' \text{ frame}$$

From inverse Lorentz transformation,

$$x = \frac{x' + v' t'}{\sqrt{1 - v^2/c^2}} \ ; \qquad y = y' \ ; \qquad z = z' \ ; \qquad t = \frac{t' + v/c^2\, x'}{\sqrt{1 - v^2/c^2}}$$

$$\therefore \qquad dx = \frac{dx' + v'\, dt'}{\sqrt{1 - v^2/c^2}} \ ; \qquad dy = dy' \ ; \qquad dz = dz' \ ; \qquad dt = \frac{dt' + v/c^2\, dx'}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \quad u_x = \frac{dx}{dt} = \frac{\dfrac{dx' + v'\, dt'}{\sqrt{1 - v^2/c^2}}}{\dfrac{dt' + v/c^2\, dx'}{\sqrt{1 - v^2/c^2}}} = \frac{\dfrac{dx'}{dt'} + v}{1 + \dfrac{v}{c^2}\dfrac{dx'}{dt'}} \quad \text{or,} \quad u_x = \frac{u'_x + v}{1 + v u'_x/c^2}$$

Also, $u_y = \frac{dy}{dt} = \frac{dy'}{\frac{dt' + \frac{v}{c^2} dx'}{\sqrt{1-\frac{v^2}{c^2}}}} = \frac{\frac{dy'}{dt'}}{\frac{\left(1+\frac{v}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}}$ or, $u_y = \frac{u'_y}{\frac{1+u'_x \frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}}$

Similarly $u_z = \frac{u'_z \sqrt{1-\frac{v^2}{c^2}}}{1+u'_x \frac{v}{c^2}}$

$$\Rightarrow u_y = \frac{u'_y}{\frac{1+u'_x \frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}} \; ; \quad u_y = \frac{u'_y \sqrt{1-\frac{v^2}{c^2}}}{1+u'_x \frac{v}{c^2}} \quad \text{and} \quad u_z = \frac{u'_z \sqrt{1-\frac{v^2}{c^2}}}{1+u'_x \frac{v}{c^2}}$$

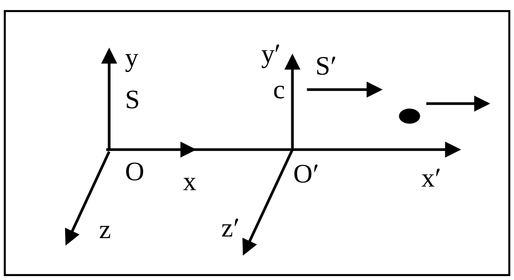
This is the relativistic law of addition of velocities

***Example Problem 1 Show that* velocity of light is a constant in all inertial frame.**

Solution : Let a photon is moving with a velocity c in the frame S' and S' is moving with velocity c relative to S along positive X – axis direction

Using velocity addition formula,

$$u_x = \frac{c+c}{1+c\frac{c}{c^2}} = c \quad \left[\text{As, } u'_x = c \,;\, v = c\right]$$



Example Problem 2: Show that addition of any velocity to c leaves the velocity c only.

Solution:

In the Example 1 velocity of the frame S' be v but velocity of photon is c, in frame S calculate velocity of photon in frame S.

Again, $$u_x = \frac{c+v}{1+c\,{}^{v}\!/\!{}_{c^2}} = \left(\frac{c+v}{c+v}\right)\cdot \mathrm{c} = \mathrm{c}$$

§ 6. Mass- Energy Realation in Relativity

According to the special theory of relativity, the mass of an object in a frame of reference at rest is called its rest mass m_o. if this mass is measured by an observation moving with a constant speed v relative to the object, then it will not remain constant if the speed v is comparable to c. The mass m in the moving frame will vary according to the mass variation given by: $m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$

Above equation is the relativistic formula for the variation of mass with velocity. When $v \to c$, $m \to \infty$ i.e., an object traveling at the speed of light would have infinite mass. Thus, no material particle can have a velocity equal to or greater than the velocity of light.

§ 7. Einstein's mass-energy relation $E = mc^2$

Let a body of mass m (variable) moving with velocity v. Let a constant force F acts on the body at rest which produces a displacement ds of the body. Work done on the body is $dW = \vec{F}\cdot\vec{\mathrm{ds}}$

Force and the displacement are along the same direction. So $dW = Fds$

Due to this work done, the corresponding kinetic energy

$$K = \int_0^S F\mathrm{ds} = \int_0^S \frac{d}{dt}(mv)\mathrm{ds} = \int_0^S \left(m\frac{dv}{dt} + v\frac{dm}{dt}\right)ds \qquad \ldots\ldots\ldots\ldots\ldots(i)$$

Now from , $m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$ we have $mc^2\left(1-\dfrac{v^2}{c^2}\right) = m_0c^2$ or, $m^2c^2 - m^2v^2 = m_0c^2$

Now, $\dfrac{d}{dt}\left(m^2c^2 - m^2v^2\right) = 0$ so, $2mc^2\dfrac{dm}{dt} - 2mv^2\dfrac{dm}{dt} - 2m^2v\dfrac{dv}{dt} = 0$

Or, $$\frac{c^2}{v}\frac{dm}{dt} = v\frac{dm}{dt} + m\frac{dv}{dt} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$$

Substituting (ii) in (i) we have

So, $$K = \int_0^S \frac{c^2}{v}\frac{dm}{dt}ds = \int_0^S \frac{c^2}{v}dm\left(\frac{ds}{dt}\right) = \int_0^v \frac{c^2}{v}dm.dv = \int_{m_0}^{m} c^2dm = mc^2 - m_0c^2$$

So, expression for relativistic kinetic energy $\boxed{K.E = mc^2 - m_0c^2}$

Now, putting $m = \dfrac{m_0}{\sqrt{1-v^2\ /c^2}}$ we have, $K.E = \dfrac{m_0c^2}{\sqrt{1-v^2/c^2}} - mc^2$

We define $\dfrac{m_0c^2}{\sqrt{1-\mathrm{v}^2\ /c^2}} = mc^2 = E$ is the total energy and m_0c^2 = rest energy. So we can write,

$\boxed{K = E - m_0c^2}$ which is another expression of kinetic energy

Kinetic energy = Total energy - Rest energy
Total energy = Rest energy + Kinetic energy

i.e, $E = m_0c^2 + K$ and $E = mc^2$

So, $\boxed{mc^2 = m_0c^2 + K.E}$

§ 7.1. Rest energy of electron, Proton and neutron

The rest energy of the subatomic particles can be calculated from $E_0 = m_0c^2$. Now we calculate the rest energy of the following subatomic particles as follows.

Electron: $$E_0 = \frac{9.109 \text{ x } 10^{-31} \times \left(3\times10^8\right)^2}{1.6\times10^{-19}} eV = 0.511 \text{ MeV}$$

Proton : $$E_0 = \frac{1.6726 \text{ x } 10^{-27} \times \left(3\times10^8\right)^2}{1.6\times10^{-19}} eV = 938.28 \text{ MeV}$$

Neutron: $$E_0 = \frac{1.6750 \text{ x } 10^{-27} \times \left(3\times10^8\right)^2}{1.6\times10^{-19}} eV = 939.57 \text{ MeV}$$

Example Problem: : **Kinetic energy of a relativistic particle reduces to classical energy value for low speeds.**

Solution:

$\therefore$ Kinetic energy = $mc^2 - m_0c^2$

$$\therefore \text{K.E.} = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2$$

$$\therefore \text{K.E.} = m_0c^2 \left[1 - v^2/c^2\right]^{-1/2} - m_0c^2$$

$$= m_0c^2 \left[1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + ...\right] - m_0c^2 \qquad \left[\text{Expanding } \left(1 - v^2/c^2\right)^{-\frac{1}{2}}\right]$$

If v << c then v/c <<1 so, $\left(\frac{v}{c}\right)^4 \to 0$

$$\text{K.E.} = m_0c^2 \left[1 + \frac{1}{2}\frac{v^2}{c^2}\right] - m_0c^2 \qquad \text{[neglecting higher order terms]}$$

$$= m_0c^2 + \frac{1}{2}m_0v^2 - m_0c^2$$

$\text{K.E.} = \frac{1}{2}m_0v^2$ Which is the classical value of kinetic energy of a particle.

§ 8. Energy & momentum of relativistic particle

If a body moving with a velocity v of mass m (varying) where m = $\frac{m_0}{\sqrt{1-v^2/c^2}}$ Momentum of the particle, $p = mv\ \frac{m_0 v}{\sqrt{1-v^2/c^2}}$

So, $p^2c^2 = \frac{m_0^2 v^2 c^2}{1-v^2/c^2}$

Now $p^2c^2 + m_0^2c^4 = \frac{m_0^2v^2c^2}{1-v^2\ /c^2} + m_0^2c^4 = \frac{m_0^2v^2c^2 + m_0^2c^4 - m_0^2v^2c^2}{1-v^2\ /c^2}$

Or, $\mathrm{p}^2\mathrm{c}^2 + \mathrm{m}_0^2c^4 = \left(mc^2\right)^2 \left[\text{As,}\quad m\ \frac{m_0}{\sqrt{1-v^2/c^2}}\right] \therefore$ $\boxed{E^2 = p^2c^2 + m_0^2c^4}$

Example :

How much does a proton gain in mass when it is accelerated to a kinetic energy of *500 MeV*?

Answer:

We know kinetic energy of particle in relativistic case, $K.E = mc^2 - m_0c^2 = \left(m - m_0\right)c^2$

$K.E = \Delta mc^2$ (Δm is difference of relativistic mass and rest mass)

putting $K.E.$ = 500 MeV we have, $K.E = 500\times10^6 \times 1.6\times10^{-19}\ Newton\ mt$

$\therefore\ \frac{500\times10^6\times1.6\times10^{19}}{c^2} = \Delta m$

$\Rightarrow\ \Delta m = \frac{500\times10^6\times1.6\times10^{-19}}{(3\times10^8)^2} = \frac{8\times10^{-11}}{9\times10^{16}}Kg.$

$\Rightarrow \Delta m = 0.888\times10^{-27}$ Kg. Hence $\Delta m = 8.9\times10^{-28}$ Kg.

Which is the required gain in mass of the proton.

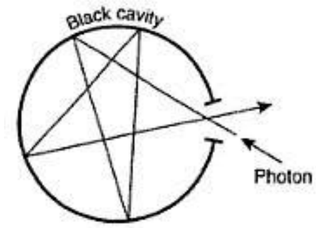
Assignment Problems

1. An electrons speed is doubled from 0.2C to 0.4C. By what ratio does its momentum increase? What happens to the momentum ratio when the electron's speed is doubled again from 0.4C to 0.8C? [**Ans**: $P_2 = 2.14P_1$; $P_2 = 3.06P_1$]
2. Verify that $\frac{1}{1-\frac{v^2}{c^2}} = 1 + \frac{p^2}{{m_0}^2c^2}$ [**Hint**: $E^2 = p^2c^2 + m_0^2c^4 = m^2c^4$]
3. A certain quantity of ice at 0°C melts into water at 0°C and in so doing gains 1.00 kg of mass. What was its initial mass?

4. At what speed does the kinetic energy of a particle equal its rest energy? [**Ans**: $v=\frac{\sqrt{3}}{2}c$]
5. An electron has a kinetic energy of 0.100 MeV. Find its speed according to classical and relativistic mechanics. [**Ans**: $1.64\times10^{8}m/s$]
6. A particle has a kinetic energy 20 times its rest energy. Find speed of the particle in terms of C. [**Ans**: v=0.998c]
7. What is the energy of a photon whose momentum is the same as that of a proton whose kinetic energy is 10.0 MeV? [**Ans**: 137.4 MeV]
8. Find the momentum (in MeV/c) of an electron whose speed is 0.600C. [**Ans**: 0.416 MeV/c]
9. Find the total energy and kinetic energy (in GeV) and the momentum (in GeV/c) of a proton whose speed is 0.900C. The mass of the proton is 0.938 GeV/c^2. [**Ans**: Total energy =2.51 GeV]
10. Find the momentum of an electron whose kinetic energy equals its rest energy of 511keV. [**Ans**: 0.885 MeV/c]
11. Find the speed and momentum (in GeV/c) of a proton whose total energy is 3.500GeV. [**Ans**: v=0.963 c; P=3.372 GeV/c]
12. Find the total energy of a neutron (m=0.940GeV/c^2) whose momentum is 1.200GeV/c. [**Ans**: E=1.524 GeV]
13. A body of rest mass m_0 breaks up into two parts spontaneously with rest masses m_{01} and m_{02} and speeds v_1 and v_2 respectively. Prove that $m_0 > (m_{01} + m_{02})$.
[**Hint**: $m_0c^2=m_1c^2+m_2c^2=\frac{m_{01}c^2}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{m_{02}c^2}{\sqrt{1-\frac{v^2}{c^2}}}$]
14. If 1 gm of a substance is fully converted into energy, how mach calories of heat will be produced. [**Ans**: $2.14\times10^{13}cal$]
15. Show that for low speed the relativistic kinetic energy of a body reduces to classical energy value. [done in class]
16. In an electric field the speed of an electron changes from 0.95 C to 0.98 C. Calculate the change of mass and the work done on the electron to change the velocity. [**Ans**: $(m_2-m_1)c^2$]
17. Show that the rest mass m_0 of a particle of momentum P and kinetic energy T are related by $m_0=\frac{P^2c^2-T^2}{2Tc^2}$ [**Hint**: $E^2=p^2c^2+m_0^2c^4 \rightarrow \left(T+m_0c^2\right)^2=p^2c^2+m_0^2c^4$]
18. Calculate the velocity of electrons accelerated by a potential of 1 MeV. [**Ans**: 0.94c]
19. An electron and a photon both have momentum 2 MeV/c. Find the total energy of each. [**Ans**: 939 MeV]
20. How fast must an electron move in order that its mass equal the rest mass of the proton? [**Ans**: $v\approx c$]

§ 9. Black Body Radiation

The thermal radiation emitted by a hot body, in general, depend on the composition and the temperature of the body. However, there is a class of bodies, called black bodies, which emit thermal radiation whose quality and quantity depend only on their temperature. For this reason the radiation emitted by such bodies is called the thermal radiation.



A heated body not emitted radiation but also absob radiation incident on it. According to Kirchoff's law " The ratio of the emissive to the absorptive power of a body is constant at a given temperature.

***Emissive power* (*E*):** The amount of energy radiated through a body per unit surface area per unit time .

***Absorptive power* (A) :** $\dfrac{\text{The energy absorbed by the body}}{\text{The total energy incident on the body}}$ at a given temperature

When a body absorb all the enrgy incident on it, then its absorptive power is 1. Such a body can be treated as a black body.

Lamp black and platinum black are nearest to ideal black bodies. When a black body is maintained at constant high temperature, the emitted radiation is called the black body radiation. For experimental purposes a cavity having a small hole can be regarded as a perfect black body.

If radiation is allowed to enter such cavity, it is reflected back and forth at the inner walls of the cavity and each reflection some portion of energy is absorbed. After suffering a large number of reflections at the walls it is completely absorbed in the cavity. Therefore at lower temperature the hole appears to be black. When the cavity is maintained at higher temperature the radiation that comes out of the hole is similar to that emitted by black body at the same temperature. Thus a cavity with a small hole acts like a black body.

§ 9.1 Experimental Observation

Experimental results show that the blackbody radiation has a continuous spectrum as shown graphically.

The intensity of the emitted radiation E_λ is plotted as a function of the wavelength for different temperatures. The wavelength of the emitted radiation ranges continuously from zero to infinity (high value). The intensity of the emitted radiation E_λ increases with increasing temperature for all wavelengths .

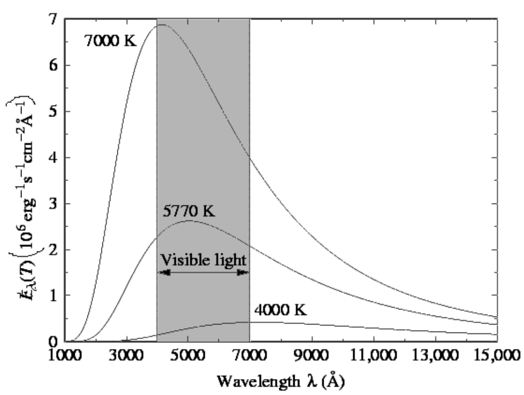


Figure : Spectrum of blackbody radiation

Wien's displacement law: E_λ has very low values for both very short and very long wavelengths and has a maximum in between at some wavelength λ_m. The value of λ_m depends on the temperature of the blackbody and decreases with increasing temperature. As the temperature increases, the wavelength λ_m corresponding to maximum energy radiation shifts towards the shorter wavelength region. It was experimentally found by Wien $\lambda_m T = \text{constant}$ Where λ_m stands for the wavelength at which intensity of radiation is maximum. The above empirical relationship is known as ***Wien's displacement law***.

Stefan's law:

The value of the constant is $2.898 \times 10^{-3}\ mK$ As the temperature is increased the total energy radiated per unit area, i.e., energy density for all wavelength increases. It was found that the total energy density, which is equal to the area under the curve is proportional to the forth power of the temperature i.e.,

$$\boxed{E = \int_0^\infty E_\lambda d\lambda = \sigma T^4}$$

which is known as ***Stefan-Boltzmann law***. Where $\sigma = 5.57 \times 10^{-8}$ watt per m^2 per K^4.

Wien's law:

Wien in 1893, from thermodynamic reasoning alone showed that energy density in black body radiation is given by

$$\boxed{E_\lambda d\lambda = \frac{c_1}{\lambda^5}(e^{-c_2/\lambda T})\, d\lambda}$$

where c_1 and c_2 are empirical constants. By proper choice of these constants Wien's law can be made fit the experimental curve in the shorter wavelength region alone but fails in the longer wavelength region.

§ 9.2 Rayleigh –Jeans Law

The British physicists Lord Rayleigh (1842-1919) and James Jeans (1877-1946) made an attempt to derive a better radiation law on the basis of the following assumptions.

(i) The radiation in a cavity is electromagnetic in nature. In metallic cavity whose walls are perfectly reflecting, the superposition of incident and reflected waves of each frequency results in the formation of standing waves with nodes at the walls. The number of standing waves (or modes) per unit volume in the frequency range υ and $\upsilon + d\upsilon$ is given by (derivation not included in the present syllabus)

$$N(\nu)d\nu = \frac{8\pi}{c^3}\nu^2 d\nu \qquad \ldots(2)$$

We can realize that the number of modes is proportional to the square of the frequency of the radiation. The theorem of equipartition of energy is also valid for electromagnetic waves, according to which the average contribution of each degree of freedom to the total energy of a system is $\frac{1}{2}kT$ where k is Boltzmann constant and T is the absolute temperature of the system. A standing wave is a system of two degrees of freedom, one corresponding to the electric field and the other to the magnetic field. Hence the average energy of each standing wave (or mode) is kT

The energy density of radiation in the frequency range υ and $\upsilon + d\upsilon$ in a cavity maintained at temperature T is $u_\nu d\nu = \frac{8\pi\nu^2}{c^3} kTd\nu$ ...(3)

Now using $c = \nu\lambda$ we have $\lambda = \frac{c}{\nu} \rightarrow d\lambda = -\frac{c}{\nu^2} d\nu$ and substituting this in (3) we have

$$u_\lambda d\lambda = \frac{8\pi kT}{c^3}\left(\frac{c}{\lambda}\right)^2\left(-\frac{c}{\lambda^2}\right)d\lambda$$

As wavelength increases frequency decreases so -ve sign is now adjusted . So in terms of wavelength this relation is expressed as

$$\boxed{u_\lambda d\lambda = \frac{8\pi}{\lambda^4} kT\, d\lambda} \qquad ...(4)$$

Intensity of radiation:

In any black body radiation, the intensity of the emitted radiation (E_λ) is measured from the standard relation $E_\lambda = \frac{c}{4} u_\lambda$ ...(5)

Putting the value of u_λ from (4) into (5) we have So, intensity of radiation will be $\boxed{E_\lambda = \frac{2\pi c}{\lambda^4} kT\, d\lambda}$

This is famous Rayleigh-Jeans formula for black body radiation.

§ 9.3 Failure of classical theory to explain black body radiation : Ultraviolet Catastrophe

A glance at the Rayleigh-Jeans formula, which is a rigorous consequence of classical physics, reveals that it fails to explain the experimental results in the higher frequency (lower wave length) region. Instead of finite energy density, it predicts infinite energy density at extremely short wavelengths viz. ultraviolet, X-rays and Gama-rays. This discrepancy between the theory and the experiment was dramatically called ***ultraviolet catastrophe***

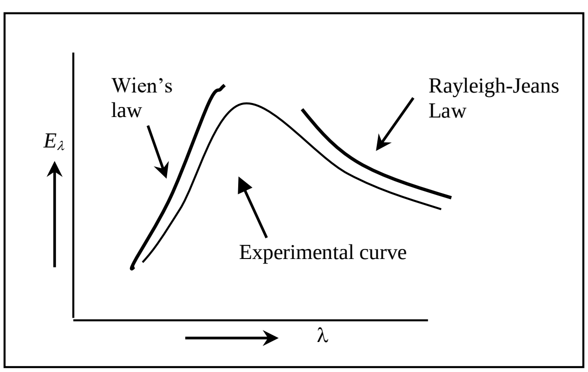


Figure : comparison of theoretical and experimental curve

The failure of Rayleigh-Jeans law means the failure of the basic assumption derived from well-established laws of classical physics. Max Planck in Germany was the first man who looked the matter beyond the framework of classical physics. He proposed a revolutionary hypothesis, according to which the emission and absorption of electromagnetic energy takes place in the form of packets (bundles) called (quanta), This concept of quantisation of energy is the door step of quantum theory.

§ 9.4 Planck's Black Body Radiation

In 1900, in the process of giving a theoretical explanation to observations on the black body radiation, Max Planck put forward a totally new hypothesis known as Planck's quantum hypothesis.

Assumptions

1. Planck's assumed that the walls of the cavity consist of microscopic oscillators.
2. In thermal equilibrium the absorption and emission of radiation by these oscillators take place at equal rate.
3. According to Planck's hypothesis the emission and absorption of radiation by an oscillator take place in the form of discrete packets of energy called photons, whose energy is proportional to the frequency of radiation. The energy ε of a photon of frequency ν is $\varepsilon = h\nu$ Where h is planck's constant. Since an oscillator can absorb whole number of photons, the allowed values of energy of oscillators are

$$0, \varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon \cdots\cdots \quad , n\varepsilon$$

i.e., $0, h\nu, 2h\nu, 3h\nu ....nh\nu$ (where n is integer)

Let $N_0, N_1, N_3 ....N_n$ be the number of oscillators with energy $0, \varepsilon, 2\varepsilon ....n\varepsilon$ So it is quite obvious that the total number of oscillators N and the total energy E of the system are given by

$$N = N_0 + N_1 + N_2 ....N_n$$

and $$E = 0N_0 + \varepsilon N_1 + 2\varepsilon N_2 + 3\varepsilon N_3 + ...n\varepsilon N_n = \sum_{n=0}^{\infty} n\varepsilon N_n$$

4. The number of modes of standing waves in the cavity is equal to the number of oscillators in the walls and the average energy per mode of standing wave is equal to the average energy of an oscillator.

Average energy calculation:

The number of oscillators in an energy state $\varepsilon_n = nh\nu$ is determined by the well-known Maxwell-Boltzmann distribution function:

$$N_n = N_0 e^{-\frac{\varepsilon_n}{kT}} = N_0 e^{-\frac{nh\nu}{kT}}$$

where N_0 is the number of oscillator in the ground state & N_n decreases exponentially with increasing energy ε_n

Average energy of the oscillator having frequency υ can be calculated as follows

$$\bar{\varepsilon} = \frac{\text{total energy}}{\text{total no of oscillators}} = \frac{\sum_0^{\infty} n\varepsilon N_n}{\sum_0^{\infty} N_n} = \frac{\sum_0^{\infty} n\varepsilon N_0 e^{-\frac{n\varepsilon}{kT}}}{\sum_0^{\infty} N_0 e^{-\frac{n\varepsilon}{kT}}}$$

$$\bar{\varepsilon} = \frac{\sum_0^\infty nh\nu\, e^{-\frac{nh\nu}{kT}}}{\sum_0^\infty e^{-\frac{nh\nu}{kT}}} = \frac{h\nu\, e^{-\frac{h\nu}{kT}} + 2h\nu\, e^{-\frac{2h\nu}{kT}} + 3h\nu\, e^{-\frac{3h\nu}{kT}} + ...}{e^{-\frac{h\nu}{kT}} + e^{-\frac{2h\nu}{kT}} + e^{-\frac{3h\nu}{kT}} + ...}$$

let $x = e^{-\frac{h\nu}{kT}}$ so the above expression becomes

$$\bar{\varepsilon} = \frac{\left[h\nu x + 2h\nu x + 3h\nu x^2 + 4h\nu x^3 + ...\right]}{\left[x + x^2 + x^3 + ...\right]}$$

$$\bar{\varepsilon} = \frac{h\nu x\left[1 + 2x + 3x^2 + 4x^3 + ...\right]}{\left[1 + x + x^2 + x^3 + ...\right]}$$

$$\bar{\varepsilon} = h\nu x \frac{(1-x)^{-2}}{(1-x)^{-1}} = \frac{h\nu\, x}{1-x} = \frac{h\nu}{\frac{1}{x} - 1} = \frac{h\upsilon}{e^{\frac{h\upsilon}{kT}} - 1} \qquad \ldots(6)$$

As per assumption (4) if the expression (6) is multiplied by (2) we will get the total energy in terms of radiation density (energy density) i.e., $u_\nu\, d\nu = \frac{8\pi}{c^3}\nu^2 d\nu . \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}$

$$u_\nu\, d\nu = \frac{8\pi h\upsilon^3}{c^3} . \frac{1}{e^{\frac{h\upsilon}{kT}} - 1} d\upsilon \qquad \ldots(7)$$

[which is radiation density in terms of frequency]

Now using $c = \nu\lambda$ we have $\lambda = \frac{c}{\nu} \rightarrow d\lambda = -\frac{c}{\nu^2} d\nu$ and substituting this in above relation we have

$$\boxed{u_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5} . \frac{1}{e^{hc\lambda\, /\, kT} - 1} d\lambda} \qquad \ldots(8)$$

[which is radiation density in terms of wavelength]

Expression (7) or (8) is Planck's radiation density formula, agrees well with the experimental results, both for the long wavelength and the short wavelength ends of the energy spectrum. It can be easily seen that it reduces to R-J law and when $\lambda \rightarrow \infty$ and to Wien's law when $\lambda \rightarrow 0$.

Intensity of radiation:

Same as before using $E_\lambda = \frac{c}{4} u_\lambda$ we get the formula for intensity of radiation will be

$$\boxed{E_\lambda d\lambda = \frac{2\pi c^2}{\lambda^5} \frac{1}{e^{hc\lambda/kT} - 1} d\lambda} \qquad \ldots(9)$$

§ 9.4.1 Deduction of Stefan's law from Planck's radiation

The total energy density of radiation emitted by a black body from Planck's radiation is given by

$$u = \int_0^\infty u_\nu \, d\nu = \frac{8\pi h}{c^3} . \int_0^\infty \frac{\nu^3}{e^{\frac{h\upsilon}{kT}} - 1} d\nu \qquad \ldots(10)$$

Substituting $x = \frac{h\nu}{kT}$ $dx = \frac{hd\nu}{kT}$ and $\nu = \frac{kT}{h} x \to d\nu = \frac{kT}{h} dx$ in expression (10)

$$u = \frac{8\pi h}{c^3} . \left(\frac{kT}{h}\right)^3 \left(\frac{kT}{h}\right) \int_0^\infty \frac{x^3}{e^x - 1} dx$$

$$Now \int_0^\infty \frac{x^3}{e^x - 1} dx = \int_0^\infty x^3 (e^{-x} + e^{-2x} + e^{-3x} + ...) \, dx = \sum_p \int_0^\infty x^3 e^{-px} \, dx$$

As per the mathematical formula,

$$\sum_p \int_0^\alpha x^n e^{-px} dx = \frac{n!}{p^{n+1}} \quad \& \sum_p 1/p^4 = \frac{\pi^4}{90}$$

We have $\qquad \sum_p \int_0^\infty x^3 e^{-px} \, dx = \sum_p \frac{3!}{p^4}$

So using the above mentioned mathematical formula we have

$$u = \frac{8\pi k^4 T^4}{c^3 h^3} 3! \left(\frac{\pi^4}{90}\right) = \frac{8\pi k^4 T^4}{c^3 h^3} \left(\frac{\pi^4}{15}\right) = \frac{8\pi^5 k^4}{15 c^3 h^3} T^4$$

If we define $\frac{8\pi^5 k^4}{15 c^3 h^3} = a(say) = \text{a constant}$. Then $u = aT^4$

Intensity E of the black body radiation is related to energy density with the relation $E = \frac{c}{4} u$. Hence

$$E = \frac{ca}{4} T^4 = \frac{2\pi^5 k^4}{15 h^3 c^2} T^4 = \sigma T^4 \qquad ...(11)$$

where $\sigma = \frac{2\pi^5 k^4}{15 h^3 c^2} = 5.67 \times 10^{-8} \, Wm^{-2} K^{-4}$ is known as Stefan constant. Expression (11) is known as **Stefan's law** of black body radiation

§ 9.4.2 Deduction of Wien's displacement law from Planck's radiation law

Planck's radiation law, in terms of λ is $u_\lambda d_\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$ …(12)

According to Plack's law u_λ is maximum when $\lambda = \lambda_m$ (say)

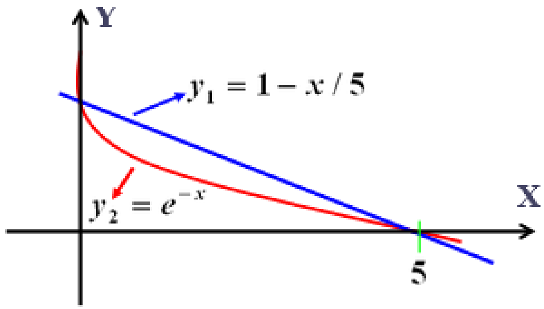
$$So, \; u_\lambda d_\lambda = \frac{8\pi hc}{y} d\lambda \qquad \text{where } y = \lambda^5 (e^{\frac{hc}{\lambda kT}} - 1)$$

But since y is a function of λ so using method of calculus we have $\frac{dy}{d\lambda} = 0$ $\lambda = \lambda_m$

$$\therefore \quad \frac{dy}{d\lambda} = 5\lambda^4 (e^{\frac{hc}{\lambda kT}} - 1) - \lambda^5 \frac{hc}{\lambda^2 kT} e^{\frac{hc}{\lambda kT}} = 0$$

Then $\quad 1-e^{-\left(\frac{hc}{\lambda_m kT}\right)} = \frac{hc}{5\lambda_m kT} \quad$ where $\quad x = hc/\lambda_m kT$

This is a transcendental equation which cannot be solved analytically. It can be solved graphically.



intersection points:
$x = 0, x = 4.965$

On solving we get

$$\lambda_m T = \frac{hc}{kx} = \frac{hc}{4.9651k} = 0.29cm\,K = constant$$

i.e., $\boxed{\lambda_m T = \text{constant}}$ Which is Wien's displacement law

§ 9.4.3 Deduction of Rayleigh-Jeans law from Planck's radiation

As we know that $e^{\frac{hc}{\lambda kT}} - 1 \approx 1 + \frac{hc}{\lambda kT} + \left(\frac{hc}{\lambda kT}\right)^2 + \ldots\ldots - 1 \;= \frac{hc}{\lambda kT}$ when $\lambda \to \infty$. So putting this in expression (12) for $\lambda \to \infty$ we have

$$\boxed{E_\lambda d\lambda = \frac{8\pi}{\lambda^4} kTd\lambda}$$ Which is Rayleigh Jeans law

§ 9.4.4 Deduction of Wien's law from Planck's radiation

Wein's displacement law.In (12) when λ tends to 0

$e^{\frac{hc}{\lambda kT}} - 1 \approx e^{\frac{hc}{\lambda kT}}$ and expression (12) becomes $\quad u_\lambda d\lambda = \frac{8\pi}{\lambda^5}.hc^3 e^{-\frac{hc}{\lambda kT}} d\lambda \qquad \ldots(13)$

Further Wein in 1893, from thermodynamic reasoning alone showed that energy density in black body radiation is given by

$$\boxed{E_\lambda d\lambda = \frac{C_1}{\lambda^5}\left(e^{-\frac{c_2}{\lambda T}}\right) d\lambda} \qquad \ldots(14)$$

where c_1 and c_2 are empirical constants. By analogy equation (13) and (14) are equivalent. By proper choice of these constants Wein's law can be made fit the experimental curve in the shorter wavelength region i.e., $\lambda \to 0$ alone but fails in the longer wavelength region.

Assignment Problems

1. At what wavelength does a cavity at 6000°K radiate most per unit wavelength?
[**Ans**: 1.356×10^6 J/m^4]
2. A cavity radiator at 6000°K has a hole 10.0mm in diameter drilled in its wall. Find the power radiated through the hole in the range 5500-5510Å. [**Ans**:7.51Watt]
3. a) Assuming the surface temperature of the sun to be 5700°K, use Stefan's law, to determine the rest mass lost per second to radiation by the sun. Take the sun's diameter to be 1.4×10^9m.
b) What fraction of the sun's rest mass is lost each year from electromagnetic radiation? Take the sun's rest mass to be 2.0×10^{30}kg. [**Ans**: a) 4.094×10^9 kg/sec, b) 6.5×10^{-14}]
4. At a given temperature, λ_{max}=6500Å for a blackbody cavity. What will λ_{max} be if the temperature of the cavity walls is increased so that the rate of emission of spectral radiation is doubled?
[**Ans**: 5466Å]
5. Show that, at the wavelength λ_{max}, where $U(\lambda)$ has its maximum $U(\lambda_{max})=170\pi(kT)^5/(hc)^4$
6. A tungsten sphere 2.30 cm in diameter is heated to 2000°C. At this temperature tungsten radiates only about 30% of the energy radiated by a blackbody of the same size and temperature. (a) Calculate the temperature of a perfectly black spherical body of the same size that radiates at the same rate as the tungsten sphere. (b) Calculate the diameter of a perfectly black spherical body at the same temperature as the tungsten sphere that radiates at the same rate.
[**Ans**: a) 1409°C, b) 1.26 cm]
7. Find the temperature of a cavity having a radiant energy density at 2000 A° that is 3.82 times the energy density at 4000 A°. [**Ans**: 17950K]
8. For heating metals, a very small hole in an electric furnace acting as a 'black body is used. If the area of the hole is 100 mm^2 and it is desired to maintain the metal at 1100°C, how much energy travels per sec through this hole? [**Ans**: 20.13 Watt]
9. An electric heater emits 1500W of thermal radiation. Assuming that the coils of the heater radiates like blackbody, determine its temperature. [Given: surface area of the coil=0.030m^2 and $\sigma=5.67\times10^{-8}$W/m^2/K^4.] [**Ans**: 969.1K]
10. Taking sun's temperature to be 5902 K, calculate the wavelength corresponding to the maximum emission of sun. [**Ans**: 4913Å]
11. Estimating the average human body to have a total surface area of 1.5 m^2 and skin temperature of 30°C, find the energy that one would lose in space in 30 sec (Assume the emissivity of the skin surface to be 0.9) [**Ans**: 1.936×10^4 Joule]