ASSIGNMENT- GRAPH THEORY.

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1. True or False : If every vertex of a Graph G has degree 2, then it must be a cycle. Explain.

True.

Let us assume, for contradiction, that G has no cycle. Considering the longest path, P (in G), let v be the final vertex. v has 2 edges (out of which one, say e_1 is the last edge of the path, connecting v to the previous vertex). Now, the second edge (e_2) from v cannot be incident on any of the intermediate vertices (since they have degree 2 ; and that is already satisfied, for our chosen path). The only possibility is that e_2 is incident to v and the starting vertex (which has degree 1, before this). But this creates a cycle.

∴ our assumption is wrong.

Hence proved.

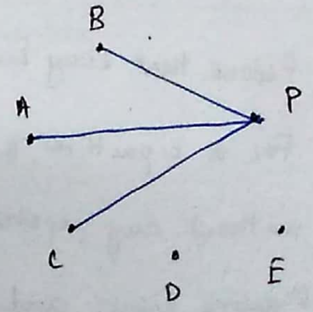
2. Prove that among any six people, there are three mutual acquaintances or three mutual strangers.

Let us consider a Graph of 6 vertices (say, A, B, C, D, E, P).

Let it be a complete graph (K_6), with each edge coloured either red or blue (say red is when two people (vertices) are mutual strangers, and blue indicates that two people are mutual acquaintances.)

Let us choose P. 5 edges leave P (each is either red or blue) Now, by the pigeonhole principle, <u>at least</u> 3 of them must be of same colour (either blue or red, lets say, blue for instance).

Let these edges be PA, PB and PC.



Now, if any one of the edges AB or BC or CA is blue (or even if all are blue), then we have a triangle of only blue edges [3 mutual friends].

Otherwise, if all of AB, BC, CA are red, we again have a triangle of only red edges [3 mutual strangers].

Hence proved.

3 The complete bipartite graph whose partite sets have m and n elements is Eulerian if and only if m and n are both even or one of them is 0.

An Euler graph is one in which there is a closed trail, consisting of all the edges present in G (exactly once, each).

Now, for a closed trail, no. of edges has to be even (since it is a bipartite graph, an even no. of edges is required to come back to the starting point). This happens when both m and n are even or both m and n are odd.

We omit the case of both m and n being odd, since, the degree of any vertex would then be odd, and we would not have a complete graph.

Also, if either m or n is 0, there would be no edges. Hence, satisfied.

Hence proved.

4 Prove that every Eulerian bipartite graph has an even number of edges.

For a bipartite graph to be Eulerian, we must have at least one cycle, without any repetition of edges. For a cycle, we must come back to the starting point, and in a bipartite graph, to do so, we require an even number of edges (because every edge connects one vertex each, from the two vertex sets of the bipartite graph).

Hence proved.

5. True or False: Every Eulerian simple graph, having an even number of vertices has an even number of edges. Explain.

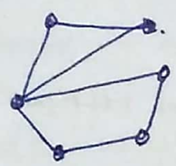
False.

Let us consider two separate cycles one of odd length, another of even length.
(say 3) (say 4).

Let us connect them at one vertex only. ∴ The resulting graph is Eulerian.

It has 6 vertices but 7 edges.

eg:



6 A graph having exactly two vertices of odd degree must contain a path from one to the other.

True. We'll prove it by contradiction.

Let G be a graph with exactly 2 vertices of odd degree (say u and v), and there is no u-v path in G.

∴ G is disconnected and u and v lie in separate connected components of G.

Let u be part of a subgraph H (which is a component of G). All other vertices in H are of even degree.

∴ Sum of degrees of all vertices in H = an odd number which is not possible for a graph.

∴ our assumption is wrong.

∴ A graph having exactly two vertices of odd degree must contain a path from one to another.

Proved.

7. If every vertex of G has even degree, then G has no cut edge.

True. We will prove it by contradiction.

Let there be a graph G, whose each vertex has an even degree, and G has a cut edge. Let, on removal of the cut-edge, the two subparts obtained (disconnected, but internally connected) be H and K.

Now, in H, exactly one vertex (from where cut edge was removed) has odd degree, rest are even. $\therefore$ Sum of degrees in H = odd. Which is not possible by the first theorem of Graph Theory. $\therefore$ There is a contradiction.

$\therefore$ If every vertex of G has even degree, then G has no cut edge.

Proved.

8. True or False: Every graph with fewer edges than vertices has a component that is a tree.

True. Let G be a graph with $|n \text{ edges}| < |n \text{ vertices}|$. Let G have k components $(1 \to k)$. $\{G_1, G_2, \ldots G_k\}$. [disconnected components].

Let no component have a tree.

$\therefore$ For $G_1, G_2, \ldots, G_k$, we have (individually):

$|n \text{ edges}| \geq |n \text{ vertices}|$

$\therefore$ taking summation for all components (from $1 \to k$),

we say, that for G, $|n \text{ edges}| \geq |n \text{ vertices}|$,

which is a contradiction to our definition of G.

$\therefore$ our assumption is wrong.

$\therefore$ Every graph with fewer edges than vertices has a component that is a tree.

Proved.

9. Prove that a graph is a tree if and only if it is connected and every edge is a cut edge.

A tree is a simple, connected graph, (such that there is a unique path between two vertices), with no cycles.

Obviously, if the graph is disconnected, there is no path between certain vertices $\therefore$ it has to be connected.

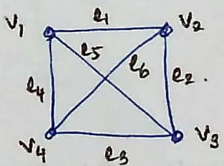
Now, let there be a non-cut edge between vertices v_i and v_j. So, if we remove this edge, there is still a path from v_i to v_j. $\therefore$ addition of this edge (back into the graph) creates a cycle. Hence not possible.

$\therefore$ every edge is a cut-edge.

Hence proved.

10. Show that the complete graph having four vertices has (i) a walk that is not a trail and (ii) a trail that is not closed and is not a path.

K_4 =



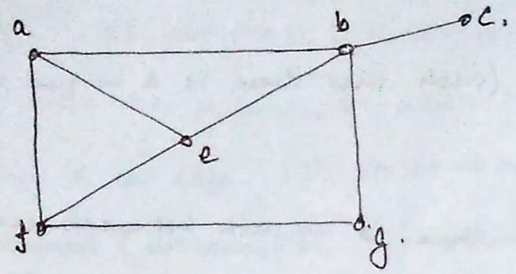
i) The walk $w = v_1, e_1, v_2, e_6, v_4, e_3, v_3, e_2, v_2, e_1, v_1$

does not have a trail since e_1 is repeated.

ii) The walk $w = v_1, e_5, v_3, e_2, v_2, e_6, v_4, e_3, v_3$

is a trail (all edges distinct), which is not closed (since initial and terminal vertex are different), and is not a path (since v_3 is repeated).

11. Determine the adjacency matrix of the following graph:



The Adjacency matrix of the above graph G is:

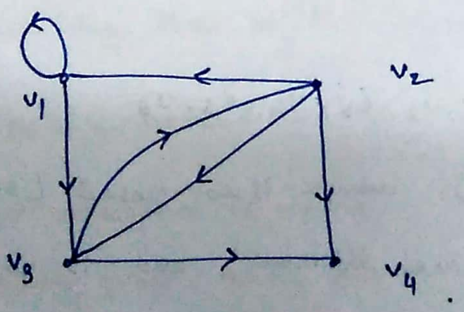
$$A = \begin{array}{c} \\ a \\ b \\ c \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & e & f & g \end{array} \\ \left[\begin{array}{cccccc} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

12. Construct the Graph G whose adjacency matrix is given below:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Since it isn't symmetric, the matrix represents a Di-Graph where $a_{ij} = 1$ implies there is an edge $(i \to j)$.

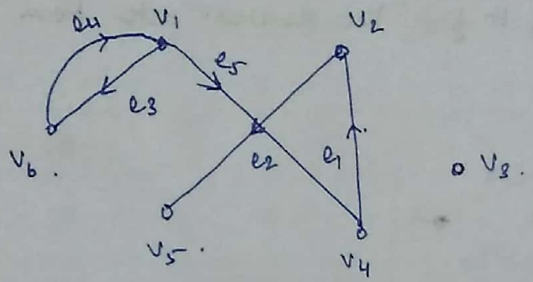
∴ The Graph G is:



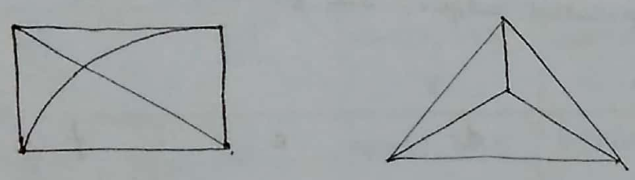
13. Draw the graph whose incidence matrix is:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

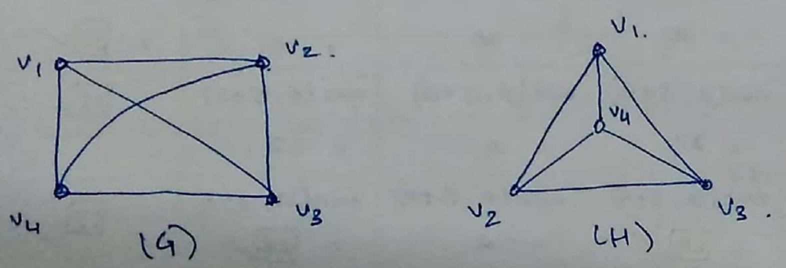
The Graph:



14. Are the given graphs isomorphic?

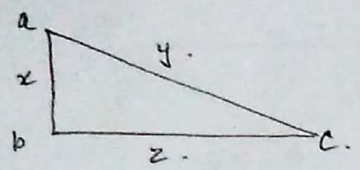


Yes, they are isomorphic.



Plotting the vertices for both graphs, we see that for every edge between two vertices in G, there is an edge between the corresponding vertices in H. Basically, they're both K4. ∴ isomorphic.

15. Represent the given graph using incidence matrix:

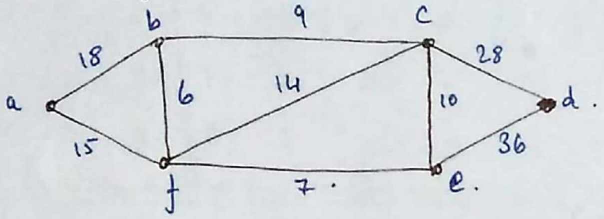


The matrix,

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

(Ans).

16. Apply Djikstra's Algorithm to find the shortest path from a to d in the given graph:



Since, there are no loops / parallel edges, we go to the algorithm directly.

a	b	c	d	e	f
$\boxed{0}$ ✓	∞	∞	∞	∞	∞
$\boxed{0}$	$\min(\infty, 0+18) = 18$	$\min(\infty, 0+\infty) = \infty$	$\min(\infty, 0+\infty) = \infty$	$\min(\infty, 0+\infty) = \infty$	$\min(\infty, 0+15) = \boxed{15}$ ✓
$\boxed{0}$.	$\min(18, 15+6) = \boxed{18}$ ✓	$\min(\infty, 15+14) = 29$	$\min(\infty, 15+\infty) = \infty$	$\min(\infty, 15+7) = 22$.	$\boxed{15}$
$\boxed{0}$	$\boxed{18}$	$\min(29, 18+9) = 27$	$\min(\infty, 18+\infty) = \infty$	$\min(22, 18+\infty) = \boxed{22}$ ✓	$\boxed{15}$
$\boxed{0}$	$\boxed{18}$	$\min(27, 22+10) = \boxed{27}$ ✓	$\min(\infty, 22+36) = 58$	$\boxed{22}$	$\boxed{15}$
$\boxed{0}$	$\boxed{18}$	$\boxed{27}$	$\min(58, 27+28) = \boxed{55}$ ✓	$\boxed{22}$	$\boxed{15}$

By backtracking process, the path we get is $d \to c \to b \to a$.

$\therefore$ the shortest path from a to d is via $a \to b \to c \to d$.

It's edge weight is $18 + 9 + 28 = 55$.

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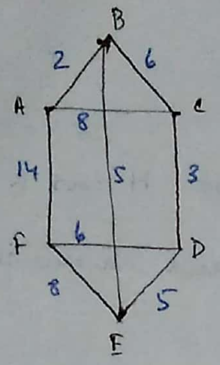
17. By Dijkstra's Algorithm, find the length of the shortest path from the vertex A to D in the following graph. Show the shortest path.

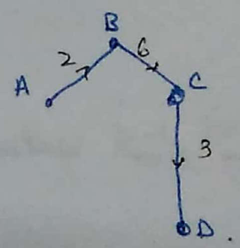


Since there are no loops / parallel edges, we go to the algorithm directly.

A	B	C	D	E	F
$\boxed{0}$ ✓	∞	∞	∞	∞	∞
$\boxed{0}$	$\min(\infty, 0+2) = \boxed{2}$ ✓	$\min(\infty, 0+8) = 8$	$\min(\infty, 0+\infty) = \infty$	$\min(\infty, 0+\infty) = \infty$	$\min(\infty, 0+14) = 14$
$\boxed{0}$	$\boxed{2}$	$\min(8, 2+6) = 8$	$\min(\infty, 2+\infty) = \infty$	$\min(\infty, 2+5) = \boxed{7}$ ✓	$\min(14, 2+\infty) = 14.$
$\boxed{0}$	$\boxed{2}$	$\min(8, 7+\infty) = \boxed{8}$ ✓	$\min(\infty, 7+5) = 12$	$\boxed{7}$	$\min(14, 7+8) = 14.$
$\boxed{0}$	$\boxed{2}$	$\boxed{8}$	$\min(12, 8+3) = \boxed{11}$ ✓	$\boxed{7}$	$\min(14, 8+\infty) = 14.$

By backtracking, the path is: D → C → B → A.

∴ Shortest path from A to D: A → B → C → D



length of shortest path is: 11.

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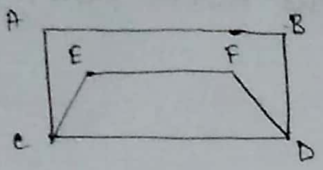
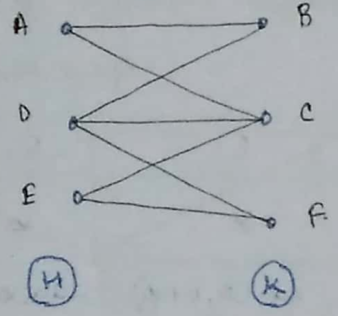
18. Is the given graph bipartite? Justify your logic.



Yes, the graph is bipartite. Taking the partite sets to be H and K

Let H E { A, D, E } and K E { B, C, F }. ∴ Rearranging and redrawing we have:



The same graph has been represented as shown above. Hence it is a bipartite graph with the partite sets H and K.

19. Justify whether the matrix $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is an adjacency matrix of a graph or not.

$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Clearly, adjacency matrix of a graph has to be Symmetric, because if v_n shares an edge with v_m; then v_m also shares an edge with v_n.

In the given matrix, we see that $a_{34} = 0$ and $a_{43} = 1$.

$\therefore a_{34} \neq a_{43}$.

Hence not symmetric.

∴ it cannot be the adjacency matrix of a graph.

However, it might be the adjacency matrix of a Di-graph.

20. The adjacency matrix of a graph is a symmetric matrix. True or False?

True.

Adjacency matrix $X = (x_{ij})_{n \times n}$ is defined as:

$$x_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are connected by an edge.} \\ 0, & \text{if there is no edge between } v_i \text{ and } v_j. \end{cases}$$

By this logic, x_{ij} and x_{ji} are both 1 if v_i and v_j are connected by an edge, or both 0 otherwise.

Hence it is a symmetric matrix.

21 Does there exist a tree with five vertices and two vertices of degree 3?

If possible, let there be a tree with five vertices and two vertices of degree 3. Let us choose the degree of remaining 3 vertices to be (1, 1, 2) {this is for the minimum sum of degrees of vertices, keeping in mind that the sum has to be even}.

$\therefore \Sigma$ (degrees of vertices) $= 3+3+1+1+2 = 10$.

$\therefore$ no. of edges $= 10/2 = 5$.

But in a tree with 5 vertices, there should be $(5-1) = 4$ edges.

$\therefore$ the no. of edges for the minimum case only, doesn't tally with the criterion of a tree.

Hence, it is not possible to have a tree with 5 vertices and two vertices of degree 3.

22 Are the following trees possible to draw?

Contd.

(i) A tree having 9 vertices and 9 edges.

A tree of n vertices has (n-1) edges. Hence, not possible to have a tree with 9 vertices and 9 edges.

(ii) A tree with 6 vertices where sum of all the degrees of the vertices is 14.

Σ (degrees of all vertices) = 14.

$\therefore$ No. of edges = $14/2 = 7$.

Clearly, with 6 vertices in a tree, 7 edges are not allowed.

$\therefore$ not possible.

(iii) A tree with all vertices of degree 2.

Such a tree, having n vertices (say), will have

Σ (degrees of all edges) = 2n

$\therefore$ no. of edges = $2n/2 = n$ = no. of edges.

which is not allowed for a tree.

$\therefore$ not possible.

(iv) A tree with 6 vertices with degree sequence {1, 1, 1, 1, 3, 3}.

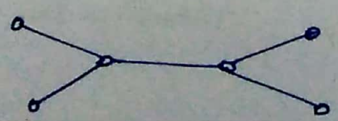
Sum of degrees = 10.

$\therefore$ no. of edges = $10/2 = 5$

for 6 vertices, we have $(6-1) = 5$ edges.

$\therefore$ It is possible to have such a tree.

It will look like:



23. Find the following answers:

(i) How many pendant vertices a binary tree should have that has 15 vertices?

No. of pendant vertices, in a binary tree of n vertices $= \frac{n+1}{2}$.

$\therefore$ No. of pendant vertices $= \frac{15+1}{2} = 8$.

(ii) How many total vertices a binary tree has, if it has 18 internal vertices.

18 internal vertices.

$\therefore$ no. of pendant vertices $= 18+1 = 19$.

$\therefore$ total no. of vertices $= 18+19 = 37$.

(iii) How many internal vertices a binary tree should have, with 20 pendant vertices.

No. of internal vertices $=$ no. of pendant vertices -1

$= 20-1 = 19$.

24. If a simple graph G and it's complement G' both are tree, then find the number of vertices in it.

No. of vertices cannot be 3 since complement of a 3 vertex tree is disconnected (only the pendant vertices are joined; and hence not a tree).

However taking an example for $n=4$: [n → vertices]

(G) (G')

we see that both G and G' are trees. This is true for all $n \geq 4$ [integer values].

Contd.

in G',

This happens because in a tree with $n \geq 4$, the pendant vertices (apart from joining each other), may also be connected to an internal vertex [to which it wasn't connected in G].

Hence, no. of vertices ≥ 4.

25. A tree has 4 vertices of degree 2, 3 vertices of degree 3, 3 vertices of degree 4. How many pendant vertices the tree should have?

Let the number of pendant vertices $= x$.

Now $2 \times$ no. of edges $= \Sigma$ (degree of all vertices)

$$= x \cdot 1 + 4(2) + 3(3) + 3(4)$$

$$= x + 29 \quad \ldots\ldots (i)$$

Also for a tree,

no. of edges $=$ no. of vertices $- 1$

$$= (x + 4 + 3 + 3) - 1$$

$$= x + 9 \quad \ldots\ldots (ii)$$

From (i) and (ii):

$$\frac{x + 29}{2} = x + 9.$$

$\Rightarrow x + 29 = 2x + 18$

$\Rightarrow 11 = x.$

$\therefore$ No. of pendant vertices $= 11$.

26. If a tree has exactly 2 pendant vertices, then prove that degree of every other vertex is exactly 2.

Let the no. of vertices in the tree be n.

$\therefore$ no. of edges $= n - 1$.

Contd.

Now

$2 \times$ no. of edges $= \Sigma$ (degree of all n vertices).

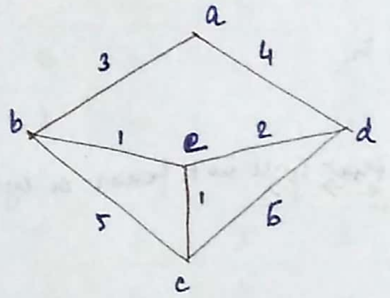
$\Rightarrow 2 \times (n-1) = 2 \times 1 + \Sigma$ (degree of remaining $(n-2)$ vertices)

$\Rightarrow \Sigma$ (degree of remaining $(n-2)$ vertices) $= 2n - 4$

$= 2(n-2)$.

$\therefore$ degree of each of the $(n-2)$ vertices is 2.

27. Find the minimal spanning tree for the following graph by Kruskal's Algorithm.



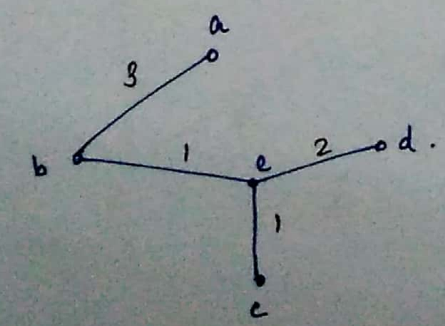
G contains 5 vertices. $\therefore$ the shortest spanning tree has 4 edges.

Arranging the edges in non-decreasing order of weight:

Edges:	(be)	(ec)	(ed)	(ab)	(ad)	(bc)	(cd)
Weight:	1	1	2	3	4	5	6

Selecting the edge with smallest weight, and continuing to select weights in the above order, till we have 4 edges (omitting those edges which would make a cycle), we have:

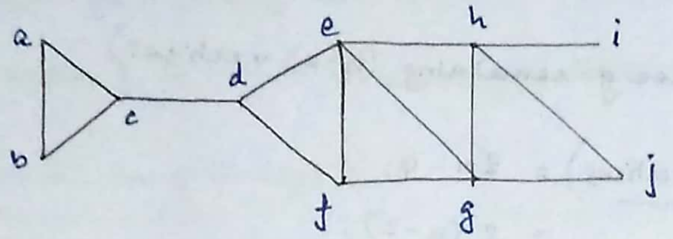
The minimal spanning tree:



the edges selected (in order):
be, ec, ed, ab

weight of this spanning tree
$= 3+1+1+2 = 7$.

28. Construct a spanning tree by DFS algorithm from the following graph:



Arbitrarily, let's choose the vertex a [since there are no loops or multiple edges, we start from the given graph itself].

We make a path from a by adding edges and vertices.

$a \to c \to d \to e \to h \to i$

Backtracking to h, we find a path from h

$h \to j$ and $j \to g$. and $g \to f$.

Backtracking to h, (we only take paths that will not form a cycle),

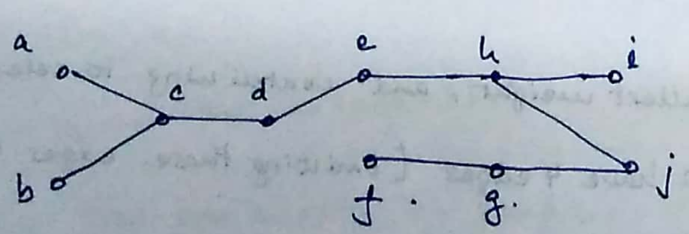
and then,

Backtracking to c, we take $(c \to b)$.

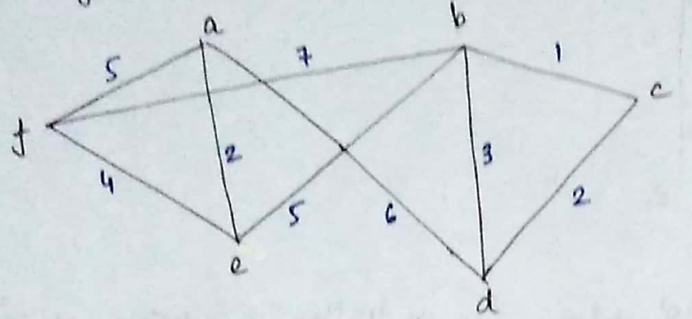
($b \to a$ is discarded as it would form a cycle).

This completes the DFS.

The spanning tree obtained is:



29. Find the minimal spanning tree for the following graph, by Prim's algorithm



Since there are no loops or parallel edges, we go to the algorithm directly.

	a	b	c	d	e	f
a	0	∞	∞	6	2	5
b	∞	0	1	3	5	7
c	∞	~~1~~	0	2	∞	∞
d	6	3	2	0	∞	∞
e	2	5	∞	∞	0	4
f	5	7	∞	∞	~~4~~	0

starting from row a, minimum is 2 (ae).

In row a and e, minimum is 4 (ef).

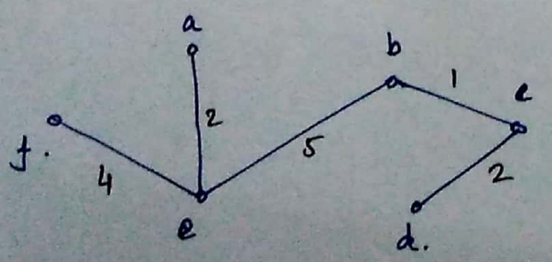
In row a, e, f, minimum is 5 (eb) [af, fa are discarded, as they would make a cycle].

In rows a, e, f, b, minimum is 1 (cb)

In rows a, e, f, b, c, minimum is 2 (cd)

There were 6 vertices. We got (6-1) = 5 edges. ∴ we stop.

∴ the spanning tree:

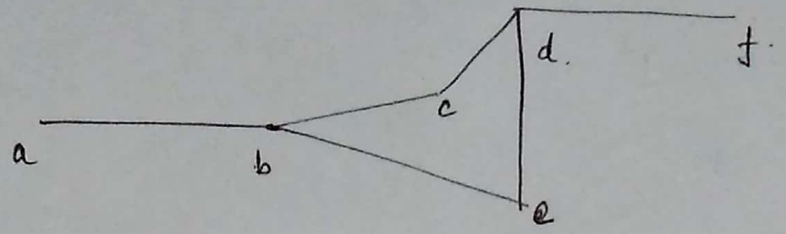


The weight of the minimal spanning tree is:

4+2+5+1+2

= 14.

30. Constuct a spanning tree by BFS Algorithm from the following graph:



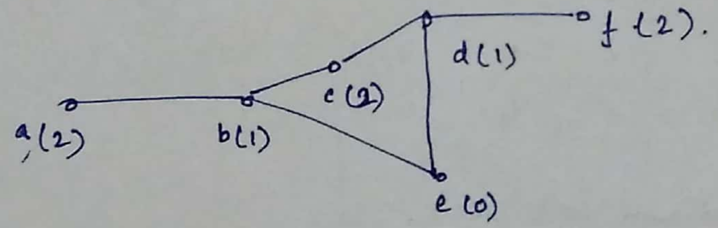
Since, there are no loops / parallel edges, we go to the algorithm directly.

We choose e as the starting vertex (0).

∴ b, d are (1) ; c, f, a are (2).

We join the vertices stage to stage [i.e from 0 → 1 ; from 1 → 2],

taking care not to include edges that would make it a cycle.



[Labelling only].

∴ The spanning tree: