2.1 D–OPERATOR METHOD OF SOLUTION

2.1.1. Introduction. A linear differential equation of second order with constant coefficients is of the form

$$\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y=X \qquad \ldots \qquad (1)$$

where P_1, P_2 are constants and X is either constant or a function of x only.

Using the symbol D for the differential operator $\frac{d}{dx}$, the above equation can be written in symbolic form as

$$(D^2+P_1D+P_2)\,y=X \qquad \ldots \qquad (2)$$

or, $$f(D)y=X \qquad \ldots \qquad (3)$$

where $f(D)=D^2+P_1D+P_2$ is a polynomial in D.

When $X=0$ then $f(D)y=0$ ... (4)

is called the homogeneous equation.

2.1.2. Theorems.

Theorem 1. If y_1, y_2 are two linearly independent solutions of the differential equation $(D^2+P_1D+P_2)y=0$, ... (i)

then $u=c_1y_1+c_2y_2$ is also its solution, where c_1, c_2 are arbitrary constants.

Proof. Left as an exercise.

Note : Since this solution contains two arbitrary constants, it is the *general or complete solution* of the equation (i).

Theorem 2. If $y=u$ is the general solution of the equation $f(D)y=0$ and $y=v$ is a particular solution (containing no arbitrary constants) of the equation $f(D)y=X$, then the general solution of the equation $f(D)y=X$ is $y=u+v$.

Proof. Since u is the general solution of the equation $f(D)y = 0$

so

$f(D)u = 0.$ ... (i)

Also, since $y = v$ is a particular solution of the equation $f(D)y = X$, so

$f(D)v = X.$ ... (ii)

Adding (i) and (ii), we have $f(D)(u+v) = X$

which shows that $y = u + v$ satisfies the equation $f(D)y = X$. Hence $y = u + v$ is the general solution of the equation $f(D)y = X$.

Here $y = u$ is called the **complementary function (C.F.)** and $y = v$ is called the **particular integral (P.I.)** of the equation $f(D)y = X$.

Thus the complete or general solution of the equation (3) i.e. of $\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = X$ is $y = C.F. + P.I.$

Note. Sometimes the symbols y_c and y_p are used to denote the C.F. and P.I. of the differential equation.

2.1.3. Rules for finding C.F.

Consider the differential equation

$$(D^2 + P_1 D + P_2)y = 0. \quad \ldots (5)$$

The general solution of this is nothing but the C. F. of (2) or (3).

Let $y = e^{mx}$ be a trial solution of (5).

Then $Dy = me^{mx}, D^2y = m^2 e^{mx}$.

Hence from (5), we get

$$\left(m^2 + P_1 m + P_2\right)e^{mx} = 0$$

i.e., $m^2 + P_1 m + P_2 = 0$, since $e^{mx} \neq 0$ ... (6)

which is called the *auxiliary equation* (A.E.) for the differential equation (5).

Let m_1, m_2 be the roots of the auxiliary equation (6). The solution of the equation (5) depends upon the nature of roots of the A. E. (6). The following cases arise :

Case (i) Let m_1, m_2 be real and distinct. Then the general solution of (5) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$, where c_1, c_2 are arbitrary constants.

Case (ii) Let the roots of the A. E. are real and equal (m_1). Then the general solution of the equation (5) is

$$y = (c_1 + c_2 x)e^{m_1 x}$$

where c_1, c_2 are arbitrary constants.

Case (iii) Let the roots of A. E. be imaginary which are $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$.

Then the general solution of the equation (5) is

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

where c_1, c_2 are arbitrary constants.

Illustrative Examples.

Ex. 1. *Solve :* $\frac{d^2y}{dx^2} - 24\frac{dy}{dx} + 144y = 0.$

Let $y = e^{mx}$ be a trial solution.

Then the auxiliary equation is $m^2 - 24m + 144 = 0$

or, $(m-12)^2 = 0$

$\therefore \quad m = 12, 12.$

Hence the general solution is $y = (c_1 + c_2 x)e^{12x}$ where c_1, c_2 are arbitrary constants.

Ex. 2. *Solve :* $(D^2 - 3D + 2)y = 0$.

Let $y = e^{mx}$ be a trial solution.

Then the auxiliary equation is $m^2 - 3m + 2 = 0$

or, $(m-1)(m-2) = 0$

$\therefore \quad m = 1, 2$

Hence the general solution is

$y = c_1 e^{x} + c_2 e^{2x}$

where c_1, c_2 are arbitrary constants.

Ex. 3. *Solve :* $\dfrac{d^2y}{dx^2} + 4y = 0$. [W.B.U.T.2014]

Let $y = e^{mx}$ be a trial solution.

Then the auxiliary equation is $m^2 + 4 = 0$

$\therefore \quad m = 0 \pm 2i$.

Hence the general solution is

$y = e^{0.x}[c_1 \cos 2x + c_2 \sin 2x]$

$= c_1 \cos 2x + c_2 \sin 2x$, where c_1, c_2 are arbitrary constants.

Ex. 4. *Solve the equation* $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 0$ *and find the particular solution if* $x = 3$ *when* $t = 0$ and $x = 8$ when $t = \log 2$.

Let $x = e^{mt}$ be a trial solution.

Then the auxiliary equation is $m^2 - 5m + 6 = 0$.

$\therefore \quad m = 2, 3$

So the general solution is

$x = c_1 e^{2t} + c_2 e^{3t}$ where c_1, c_2 are arbitrary constants.

Now $x = 3$ when $t = 0$.

$\therefore \quad 3 = c_1 + c_2$. ... (i)

Also it is given that $x = 8$ when $t = \log 2$.

$\therefore \quad 8 = c_1 e^{2\log 2} + c_2 e^{3\log 2}$

$= c_1 e^{\log 4} + c_2 e^{\log 8} = 4c_1 + 8c_2$

$\therefore \quad c_1 + 2c_2 = 2$. ... (ii)

Solving (i) and (ii), we get $c_1 = 4,\ c_2 = -1$.

Hence the particular solution is $x = 4e^{2t} - e^{3t}$.

Ex. 5. *Solve* $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4y = 0$, $y(0) = 1$, $\left(\dfrac{dy}{dt}\right) = 0$

Let $y = e^{mt}$ be a trial solution. Then the auxiliary equation is

$m^2 + 4m + 4 = 0$

or, $(m+2)^2 = 0$

$\therefore \quad m = -2, -2$

So the general solution is

$y = (c_1 + c_2 t)e^{-2t}$

$\therefore \quad \dfrac{dy}{dt} = c_2 e^{-2t} - 2(c_1 + c_2 t)e^{-2t}$

Given $y(0) = 1$ and $\left(\dfrac{dy}{dt}\right) = 0$

$\therefore \quad 1 = (c_1 + c_2.0).1$

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10. $y = 3e^{-ax}\cos \pi x$ 11. $y = e^{x}(-3\cos 2x + 2\sin 2x)$

13. $y = e^{\frac{5x}{2}}\left(2\cos\frac{\sqrt{\pi}}{2}x + \frac{6}{\sqrt{\pi}}\sin\frac{\sqrt{\pi}}{2}x\right)$

14. $y = 2e^{4x} + e^{-3x}$

2.1.4. Rules for finding the particular integral (P.I.) by operator method.

2.1.4.1 The inverse operator.

Definition. $\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants which when operated upon by $f(D)$ gives X.

Thus $f(D)\left\{\frac{1}{f(D)}X\right\} = X.$ ... (7)

Hence $f(D)$ and $\frac{1}{f(D)}$ are inverse operators. Also it is obvious that $\frac{1}{f(D)}X$ satisfies the equation $f(D)y = X$ and is therefore its P.I.

Theorem 3. $\frac{1}{D}X = \int X\,dx.$

Proof. Let $\frac{1}{D}X = y.$

$\therefore$ $D(\frac{1}{D}X) = Dy$ or, $X = \frac{dy}{dx}.$

Hence $y = \int X\,dx.$ ... (8)

Hence proved.

Theorem 4. $\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.$

Proof. Let $\frac{1}{D-a}X = y.$

$\therefore$ $(D-a)\left(\frac{1}{D-a}X\right) = (D-a)\,y$

or, $X = Dy - ay.$

$\therefore$ $\frac{dy}{dx} - ay = X.$ ... (9)

which is a linear equation in y.

$\therefore$ $I.F. = e^{\int(-a)dx} = e^{-ax}.$

So the solution of (9) is given by

$ye^{-ax} = \int Xe^{-ax}dx$

i.e., $y = e^{ax}\int Xe^{-ax}dx.$

$\therefore$ $\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.$ ... (10)

2.1.4.2. General method for finding P. I.

Let $\frac{1}{f(D)} = \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2}$ where A_i $(i = 1, 2)$ are constants.

Then by Theorem 4., we get

$$\frac{1}{f(D)}X = A_1e^{m_1x}\int Xe^{-m_1x}dx + A_2e^{m_2x}\int Xe^{-m_2x}dx. \quad \ldots (11)$$

Illustrative Examples.

Ex. 1. *Evaluate the following :*

(i) $\frac{1}{D+2}e^{x}$. (ii) $\frac{1}{D^2-1}(xe^{2x})$. (iii) $\frac{1}{D^2+1}\sec x$.

(iv) $\frac{1}{D^2}\sin^2 x.$

(i) $\frac{1}{D+2}e^x = e^{-2x}\int e^x . e^{2x}dx = e^{-2x}\int e^{3x}dx$

$= e^{-2x}.\frac{1}{3}e^{3x} = \frac{1}{3}e^x.$

(ii) $\frac{1}{D^2-1}\left(xe^{2x}\right)$

$= \frac{1}{2}\left(\frac{1}{D-1} - \frac{1}{D+1}\right)xe^{2x}$

$= \frac{1}{2}\frac{1}{D-1}(xe^{2x}) - \frac{1}{2}\frac{1}{D+1}(xe^{2x})$

$= \frac{1}{2}e^x\int xe^{2x}.e^{-x}dx - \frac{1}{2}e^{-x}\int xe^{2x}.e^x dx$

$= \frac{1}{2}e^x\int xe^x dx - \frac{1}{2}e^{-x}\int xe^{3x}dx$

$= \frac{1}{2}e^x.e^x(x-1) - \frac{1}{2}e^{-x}.\frac{1}{9}e^{3x}(3x-1)$

$= \frac{1}{2}e^{2x}(x-1) - \frac{1}{18}e^{2x}(3x-1) = \frac{1}{9}e^{2x}(3x-4).$

(iii) $\frac{1}{D^2+1}\sec x = \frac{1}{2i}\left(\frac{1}{D-i} - \frac{1}{D+i}\right)\sec x.$

Now $\frac{1}{D-i}\sec x = e^{ix}\int e^{-ix}\sec x\, dx$

$= e^{ix}\int \frac{\cos x - i\sin x}{\cos x}dx = (\cos x + i\sin x)(x + i\log\cos x)$

$= (x\cos x - \sin x\log\cos x) + i\,(x\sin x + \cos x\log\cos x).$

Similarly $\frac{1}{D+i}\sec x$

$= (x\cos x - \sin x\log\cos x) - i\,(x\sin x + \cos x\log\cos x)$

$\therefore \quad \frac{1}{D^2+1}\sec x = x\sin x + \cos x\log\sin x.$

(iv) $\frac{1}{D^2}\sin^2 x = \frac{1}{2}\frac{1}{D^2}(1-\cos 2x)$

$= \frac{1}{2}\frac{1}{D}\int(1-\cos 2x)\,dx = \frac{1}{2}\frac{1}{D}(x - \frac{1}{2}\sin 2x)$

$= \frac{1}{2}\int(x - \frac{1}{2}\sin 2x)\,dx = \frac{1}{2}\left(\frac{1}{2}x^2 + \frac{1}{4}\cos 2x\right)$

$= \frac{1}{8}\left(2x^2 + \cos 2x\right).$

2.1.4.3 Short methods for finding P. I. in some special cases.

For the differential equation $f(D)y = X$,

P. I. $= \frac{1}{f(D)}X.$

Let us now consider the following special cases :

Case I. Let $X = e^{ax}$. Then

(i) P.I. $= \frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$ *if* $f(a) \neq 0$ ... (12)

(ii) P.I. $= \frac{1}{f(D)}e^{ax} = x\frac{1}{f'(D)}e^{ax} = x\frac{e^{ax}}{f'(a)}$ if $f(a) = 0$ but $f'(a) \neq 0$ ... (13)

(iii) P.I. $= \frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(D)}e^{ax} = x^2\frac{e^{ax}}{f''(a)}$

if $f(a) = f'(a) = 0$ but $f''(a) \neq 0$ ... (14)

and so on.

Proof. (i) Since $f(D)\,e^{ax} = (D^2 + P_1 D + P_2)e^{ax}$

$$= (a^2 + P_1 a + P_2)e^{ax}$$

therefore $f(D)e^{ax} = f(a)e^{ax}$

or, $\dfrac{1}{f(D)}\{f(D)e^{ax}\} = \dfrac{1}{f(D)}\{f(a)e^{ax}\}$

or, $e^{ax} = f(a)\dfrac{1}{f(D)}e^{ax}$

or, $\dfrac{1}{f(D)}e^{ax} = \dfrac{1}{f(a)}e^{ax}$ if $f(a) \neq 0$.

$\therefore$ $\text{P.I.} = \dfrac{1}{f(D)}e^{ax} = \dfrac{e^{ax}}{f(a)}$ if $f(a) \neq 0$.

(ii) Beyond the scope of the book

(iii) Beyond the scope of the book

Illustrative Example.

Ex. 1. *Find P. I. of the following equations :*

(i) $(D^2+1)\,y = e^{\frac{x}{2}}$. *(ii)* $(D^2 - 3D + 2)\,y = e^{2x}$.

(i) $\text{P. I.} = \dfrac{1}{D^2+1}e^{\frac{x}{2}} = \dfrac{1}{(\frac{1}{2})^2+1}e^{\frac{x}{2}} = \dfrac{4}{5}e^{\frac{x}{2}}$.

(ii) $\text{P. I} = \dfrac{1}{D^2 - 3D + 2}e^{2x} = x\dfrac{e^{2x}}{2D-3}$

$$= x\frac{1}{4-3}.e^{2x} = xe^{2x}.$$

Case II. Let $X = P(x)$ where $P(x)$ is a polynomial of degree m.

Then $\text{P. I.} = \dfrac{1}{f(D)}P(x) = \{f(D)\}^{-1}P(x)$. ... (15)

Expand $\{f(D)\}^{-1}$ in ascending powers of D as far as the term containing D^m, since $D^r(x^m) = 0$ if $r > m$, and then operate on $P(x)$.

Illustrative Example.

Find P. I. of $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = x^2 + 3$.

The equation can be written as

$(D^2 + 4D + 3)\,y = x^2 + 3$. [Here $P(x) = x^2 + 3$.]

$$\therefore \quad P.\,I. = \frac{1}{D^2+4D+3}(x^2+3) = \frac{1}{3}\left(1 + \frac{D^2+4D}{3}\right)^{-1}(x^2+3)$$

$$= \frac{1}{3}\left\{1 - \frac{D^2+4D}{3} + \left(\frac{D^2+4D}{3}\right)^2 - \cdots\right\}(x^2+3)$$

$$= \frac{1}{3}\left(1 - \frac{4D}{3} + \frac{4D^2}{9} - \cdots\right)(x^2+3) \quad \text{[Taken upto } D^2 \text{ term]}$$

$$= \frac{1}{3}\left(x^2 + 3 - \frac{4}{3}\cdot 2x + \frac{4}{9}\cdot 2\right) = \frac{1}{3}\left(x^2 - \frac{8x}{3} + \frac{35}{9}\right).$$

Case III. Let $X = \sin(ax+b)$ or $\cos(ax+b)$.

In this case we express the operator function $f(D)$ in function of D^2, say $\phi(D^2)$, or in function of D^2 and D, say $\phi(D^2, D)$.

Then

(i) $\text{P.I.} = \dfrac{1}{f(D)}\sin(ax+b) = \dfrac{1}{\phi(D^2)}\sin(ax+b)$

$$= \frac{\sin(ax+b)}{\phi(-a^2)}, \text{ if } \phi(-a^2) \neq 0. \quad \ldots \quad (16)$$

(ii) $P.I.=\frac{1}{f(D)}sin(ax+b)=\frac{1}{\phi(D^2,D)}sin(ax+b)$

$$=\frac{1}{\phi(-a^2,D)}sin(ax+b), if\ \phi(-a^2,D)\neq 0. \quad \ldots \quad (17)$$

(iii) $P.I.=\frac{1}{f(D)}sin(ax+b)=\frac{\psi(D)}{\phi(D^2)}sin(ax+b)$

$$=\frac{\psi(D)}{\phi(-a^2)}sin(ax+b), if\ \phi(-a^2)\neq 0. \quad \ldots \quad (18)$$

(iv) $\text{P.I.}=\frac{1}{f(D)}\sin(ax+b)=x\frac{1}{f'(D)}\sin(ax+b)$, $\quad \ldots \quad$ (19)

if (i), (ii), (iii) fail

Proof. Beyond the scope of the book.

Illustrative Example

Find P. I. of the following equations

(i) $(D^2+1)y=\sin 2x$. (ii) $(D^2+9)y=cos(3x-1)$.

(i) P.I. $=\frac{1}{D^2+1}\sin 2x=\frac{1}{-2^2+1}\sin 2x=-\frac{1}{3}\sin 2x$. by (16)

(ii) P.I. $=\frac{1}{D^2+9}cos(3x-1)=x\frac{1}{2D}cos(3x-1)$, by(19)

$$=\frac{x\,sin(3x-1)}{6}$$

Case IV. Let $X=e^{ax}V$, where V is any function of x.

Then P. I. $=\frac{1}{f(D)}e^{ax}V=e^{ax}\frac{1}{f(D+a)}V$. $\quad \ldots \quad$ (20)

Proof. Let $U=\frac{1}{f(D+a)}V$

Then $D(e^{ax}U)=e^{ax}DU+ae^{ax}U=e^{ax}(D+a)U$

$$D^2(e^{ax}U)=D\{e^{ax}(D+a)U\}$$

$$=ae^{ax}(D+a)U+e^{ax}D(D+a)U$$

$$=e^{ax}(D+a)^2U.$$

In this way in general, $D^n(e^{ax}U)=e^{ax}(D+a)^nU$.

Hence $f(D)(e^{ax}U)=e^{ax}f(D+a)^nU$.

$$\therefore \quad f(D)\left\{e^{ax}\frac{1}{f(D+a)}V\right\}=e^{ax}V$$

Operating both sides by $\frac{1}{f(D)}$, we get

$$\therefore \quad e^{ax}\frac{1}{f(D+a)}V=\frac{1}{f(D)}e^{ax}V.$$

Hence $\frac{1}{f(D)}e^{ax}V=e^{ax}\frac{1}{f(D+a)}V$.

Illustrative Example.

Find P. I. of $(D^2+1)y=xe^{-2x}$.

P. I. $=\frac{1}{D^2+1}xe^{-2x}=e^{-2x}\frac{1}{(D-2)^2+1}x$

$$=e^{-2x}\frac{1}{D^2-4D+5}x=e^{-2x}.\frac{1}{5}\left(1+\frac{D^2-4D}{5}\right)^{-1}x$$

$$= e^{-2x}.\frac{1}{5}\left(1 - \frac{D^2 - 4D}{5} + \cdots\right).x$$

$$= e^{-2x}.\frac{1}{5}\left(x + \frac{4}{5}\right)$$

$$= \frac{1}{25}(5x + 4)\,e^{-2x}.$$

Case V. Let $X = xV$, V being any function of x.

Then P. I. $= \dfrac{1}{f(D)}xV \;= \left\{x - \dfrac{1}{f(D)}f'(D)\right\}\dfrac{1}{f(D)}V.$ . . . (21)

Proof. Let $V_1 = \dfrac{1}{f(D)}V$

Then $D(xV_1) = xDV_1 + V_1$

$$D^2(xV_1) = xD^2V_1 + 2DV_1 = xD^2V_1 + \left(\frac{d}{dD}D^2\right)V_1$$

Hence $f(D)(xV_1) = xf(D)V_1 + \dfrac{d}{dD}f(D)V_1$

$$= xf(D)V_1 + f'(D)V_1.$$

$$\therefore\quad f(D)\left\{x.\frac{1}{f(D)}V\right\} = xV + f'(D).\frac{1}{f(D)}V$$

Thus $x\dfrac{1}{f(D)}V = \dfrac{1}{f(D)}(xV) + f'(D).\dfrac{1}{\{f(D)\}^2}V$

$$\therefore\quad \frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - f'(D).\frac{1}{\{f(D)\}^2}V.$$

$$= \left\{x - \frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}V.$$

Case VI. If $X = x^n.V$, then

$$\text{P. I.} = \left\{x - \frac{1}{f(D)}f'(D)\right\}^n \frac{1}{f(D)}V \qquad \ldots \quad (22)$$

where n is a +ve integer.

Proof. Beyond the scope of the book.

Illustrative Example.

Find P. I. of $(D^2 + 4)y = x\sin x$.

$$\text{P. I.} = \frac{1}{D^2 + 4}(x\sin x)$$

$$= \left\{x - \frac{1}{D^2 + 4}.2D\right\}\frac{1}{D^2 + 4}\sin x$$

$$= \left\{x - \frac{1}{D^2 + 4}.2D\right\}\frac{\sin x}{-1^2 + 4}$$

$$= \frac{1}{3}x\sin x - \frac{2}{3}\frac{1}{D^2 + 4}\cos x$$

$$= \frac{1}{3}x\sin x - \frac{2}{3}\frac{\cos x}{-1^2 + 4}$$

$$= \frac{1}{3}x\sin x - \frac{2}{9}\cos x.$$

Miscellaneous Examples

Ex. 1. *Solve :* $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = x^2$.

The given equation can be written as $(D^2 + 5D + 4)y = x^2$.

Let $y = e^{mx}$ be a trial solution of the homogeneous equation

$$(D^2 + 5D + 4)y = 0.$$

Then the auxiliary equation is $m^2 + 5m + 4 = 0$

or, $(m + 4)(m + 1) = 0$.

$\therefore \quad m = -4, -1$

$\therefore \quad$ C.F. $= c_1e^{-4x} + c_2e^{-x}$, where c_1, c_2 are arbitrary constants.

Now, P. I. $= \dfrac{1}{D^2+5D+4}x^2$

$$= \frac{1}{4}\left(1+\frac{D^2+5D}{4}\right)^{-1}x^2$$

$$= \frac{1}{4}\left\{1-\left(\frac{D^2+5D}{4}\right)+\left(\frac{D^2+5D}{4}\right)^2-\cdots\right\}x^2$$

$$= \frac{1}{4}\left\{1-\frac{D^2+5D}{4}+\frac{25}{16}D^2-\cdots\right\}x^2$$

[we write upto D^2 term since x^2 is a 2 degree polynomial]

$$= \frac{1}{4}\left(x^2-\frac{5}{2}x+\frac{21}{8}\right).$$

So the general solution is

$$y = c_1e^{-4x}+c_2e^{-x}+\frac{1}{4}\left(x^2-\frac{5}{2}x+\frac{21}{8}\right).$$

Ex. 2. *Solve :* $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y = x^2e^{3x}$. [W.B.U.T 2005]

The given equation can be written as $(D^2-5D+6)\,y = x^2e^{3x}$.

Let $y = e^{mx}$ be a trial solution of the homogeneous equation

$(D^2-5D+6)\,y = 0$.

Then the auxiliary equation is $m^2-5m+6 = 0$

or, $(m-3)(m-2) = 0$. $\therefore \quad m = 2, 3$.

$\therefore \quad$ C. F. $= c_1e^{2x}+c_2e^{3x}$, where c_1, c_2 are arbitrary constants.

Now P. I. $= \dfrac{1}{D^2-5D+6}(x^2e^{3x})$

$$= e^{3x}\frac{1}{(D+3)^2-5(D+3)+6}x^2 \quad \text{by (20)}$$

$$= e^{3x}\frac{1}{D^2+D}x^2 = e^{3x}\frac{1}{D}(1+D)^{-1}x^2$$

$$= e^{3x}\frac{1}{D}(1-D+D^2-\cdots)\,x^2$$

$$= e^{3x}\frac{1}{D}(x^2-2x+2)$$

$$= e^{3x}\left(\frac{1}{3}x^3-x^2+2x\right) \quad \left[\frac{1}{D}x^2 = \int x^2dx \text{ etc.}\right]$$

So the general solution is

$$y = c_1e^{2x}+c_2e^{3x}+e^{3x}\left(\frac{1}{3}x^3-x^2+2x\right).$$

Ex. 3. *Solve :* $\dfrac{d^2y}{dx^2}+4y = \operatorname{cosec} 2x$.

The given equation can be written as $(D^2+4)\,y = \operatorname{cosec} 2x$.

Let $y = e^{mx}$ be a trial solution of the homogeneous equation $(D^2+4)\,y = 0$.

Then the auxiliary equation is $m^2+4 = 0$.

$\therefore \quad m = \pm 2i$.

$\therefore \quad$ C. F. $= c_1\cos 2x + c_2\sin 2x$.

Now, P. I. $= \dfrac{1}{D^2+4}\operatorname{cosec} 2x = \dfrac{1}{4i}\left(\dfrac{1}{D-2i}-\dfrac{1}{D+2i}\right)\operatorname{cosec} 2x$

$\therefore \quad \frac{1}{D-2i} \operatorname{cosec} 2x = e^{2xi} \int e^{-2ix} \operatorname{cosec} 2x \, dx$

$= e^{2xi} \int (\cos 2x - i \sin 2x) \operatorname{cosec} 2x \, dx$

$= e^{2xi} \int (\cot 2x - i) \, dx$

$= (\cos 2x + i \sin 2x) \left(\frac{1}{2} \log \sin 2x - ix \right)$

$= \frac{1}{2} \cos 2x \log \sin 2x + x \sin 2x + i \left(\frac{1}{2} \sin 2x \log \sin 2x - x \cos 2x \right)$

Similarly $\frac{1}{D+2i} \operatorname{cosec} 2x$

$= \frac{1}{2} \cos 2x \log \sin 2x + x \sin 2x - i \left(\frac{1}{2} \sin 2x \log \sin 2x - x \cos 2x \right)$

$\therefore \quad$ P.I. $= \frac{1}{4} (\sin 2x \log \sin 2x - 2x \cos 2x)$.

So the general solution is

$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} (\sin 2x \log \sin 2x - 2x \cos 2x)$.

Ex. 4. *Solve :* $(D^2 - 4)\, y = x \sinh x$.

Let $y = e^{mx}$ be a trial solution of the homogeneous equation $(D^2 - 4)\, y = 0$.

Then the auxiliary equation is $m^2 - 4 = 0$.

$\therefore \quad m = \pm 2$.

$\therefore \quad$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$, where c_1, c_2 are arbitrary constants.

$\therefore \quad$ P.I. $= \frac{1}{D^2 - 4} x \sinh x$

$= \frac{1}{2} \left[\frac{1}{D^2 - 4} x e^x - \frac{1}{D^2 - 4} x e^{-x} \right] \left[\because \sinh x = \frac{e^x - e^{-x}}{2} \right]$

$= \frac{1}{2} \left[e^x \frac{1}{(D+1)^2 - 4} x - e^{-x} \frac{1}{(D-1)^2 - 4} x \right]$

$= \frac{1}{2} \left[e^x \frac{1}{D^2 + 2D - 3} x - e^{-x} \frac{1}{D^2 - 2D - 3} x \right]$

$= \frac{1}{2} \left[-\frac{e^x}{3} \left(1 - \frac{D^2 + 2D}{3} \right)^{-1} x + \frac{e^{-x}}{3} \left(1 - \frac{D^2 - 2D}{3} \right)^{-1} x \right]$

$= -\frac{1}{6} e^x \left(1 + \frac{D^2 + 2D}{3} + \cdots \right) x + \frac{1}{6} e^{-x} \left(1 + \frac{D^2 - 2D}{3} + \cdots \right) x$

$= -\frac{1}{6} e^x \left(x + \frac{2}{3} \right) + \frac{1}{6} e^{-x} \left(x - \frac{2}{3} \right)$

$= -\frac{1}{6} x \left(e^x - e^{-x} \right) - \frac{1}{9} \left(e^x + e^{-x} \right)$

$= -\frac{1}{3} x \sinh x - \frac{2}{9} \cosh x$.

So the general solution is

$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} x \sinh x - \frac{2}{9} \cosh x$.

Ex. 5. *Solve :* $(D^2 + 1)\, y = x \cos x$.

Let $y = e^{mx}$ be a trial solution of the homogeneous equation $(D^2 + 1)\, y = 0$.

Then the auxiliary equation is $m^2+1=0$.

$\therefore \quad m=\pm i$

$\therefore$ C. F. $= c_1\cos x + c_2\sin x$, where c_1, c_2 where are arbitrary constants.

Now, P. I. $=\dfrac{1}{D^2+1}x\cos x = \left\{x-\dfrac{1}{D^2+1}.2D\right\}\dfrac{1}{D^2+1}\cos x$

$$=\left\{x-\frac{1}{D^2+1}.2D\right\}\frac{x}{2}\sin x$$

$$\left[\because \frac{1}{D^2+1}\cos x = x\frac{1}{2D}\cos x = \frac{x}{2}\int\cos x\,dx = \frac{x}{2}\sin x\right]$$

$$=\frac{1}{2}x^2\sin x-\frac{1}{D^2+1}D(x\sin x)$$

$$=\frac{1}{2}x^2\sin x-\frac{1}{D^2+1}(\sin x + x\cos x)$$

$$=\frac{1}{2}x^2\sin x-\frac{1}{D^2+1}\sin x-\frac{1}{D^2+1}x\cos x$$

$$=\frac{1}{2}x^2\sin x-\frac{1}{D^2+1}\sin x-\text{P.I.}$$

$$\therefore \quad 2\times\text{P.I.}=\frac{1}{2}x^2\sin x-\frac{x}{2D}\sin x$$

$$=\frac{1}{2}x^2\sin x+\frac{1}{2}x\cos x$$

$$\therefore \quad \text{P. I.}=\frac{1}{4}x^2\sin x+\frac{1}{4}x\cos x.$$

[**Alternatively.** P. I. $=\dfrac{1}{D^2+1}x\cos x = Rl\left\{\dfrac{1}{D^2+1}xe^{ix}\right\}$

$$=Rl\left\{e^{ix}\frac{1}{(D+i)^2+1}x\right\}=Rl\left\{e^{ix}\frac{1}{D^2+2Di}x\right\}$$

$$=Rl\left\{e^{ix}\frac{1}{2Di}\left(1+\frac{D}{2i}\right)^{-1}x\right\}=Rl\left\{e^{ix}\frac{1}{2Di}\left(1-\frac{D}{2i}+\cdots\right)x\right\}$$

$$=Rl\left\{e^{ix}\frac{1}{2Di}\left(x-\frac{1}{2i}\right)\right\}=Rl\left\{\frac{1}{2i}(\cos x+i\sin x)\left(\frac{x^2}{2}-\frac{x}{2i}\right)\right\}$$

$$=Rl\left\{-\frac{1}{4}(\cos x+i\sin x)(x^2 i-x)\right\}=\frac{1}{4}(x^2\sin x+x\cos x).\ ]$$

Hence the general solution is

$$y=c_1\cos x+c_2\sin x+\frac{1}{4}x^2\sin x+\frac{1}{4}x\cos x.$$

Ex. 6. *Solve :* $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y=e^x\cos x$

[W.B.U.T. 2002, 2011, 2016]

The given equation can be written as

$$\left(D^2-5D+6\right)y=e^x\cos x$$

Let $y=e^{mx}$ be a trial solution of the homogeneous equation

$$\left(D^2-5D+6\right)y=0$$

$\therefore$ The auxiliary equation is

$$m^2-5m+6=0$$

or, $\quad (m-2)(m-3)=0$

$\therefore \quad m = 2, 3$

$\therefore \quad C.F. = c_1e^{2x} + c_2e^{3x}$

Now, P. I. $= \dfrac{e^x \cos x}{D^2 - 5D + 6}$

$= e^x \dfrac{\cos x}{(D+1)^2 - 5(D+1) + 6}$

$= e^x \dfrac{\cos x}{D^2 - 3D + 2}$

$= e^x \dfrac{\cos x}{-1^2 - 3D + 2}$

$= e^x \dfrac{\cos x}{-3D + 1}$

$= e^x \dfrac{3D+1}{(1)^2 - (3D)^2} \cos x$

$= e^x \dfrac{3D+1}{1 - 9D^2} \cos x$

$= e^x \dfrac{3D+1}{1 - 9(-1^2)} \cos x$

$= \dfrac{e^x}{10}(-3\sin x + \cos x)$

Thus the general solution of the given equation is

$$y = c_1e^{2x} + c_2e^{3x} + \frac{e^x}{10}(\cos x - 3\sin x)$$

where c_1 and c_2 are constants.

Ex. 7. *Solve :* $(D^2 - 2D)y = e^x \sin x$ [W.B.U.T. 2007]

Let $y = e^{mx}$ be a trial solution of $(D^2 - 2D)y = 0$

$\therefore$ The auxiliary equation is $m^2 - 2m = 0$

$\therefore \quad m = 0, 2$

$\therefore \quad$ C.F. $= c_1 + c_2e^{2x}$

$\therefore \quad$ P.I. $= \dfrac{e^x \sin x}{D^2 - 2D}$

$= e^x \dfrac{\sin x}{(D+1)^2 - 2(D+1)}$

$= e^x \dfrac{\sin x}{D^2 - 1}$

$= e^x \dfrac{\sin x}{-1^2 - 1}$

$= -\frac{1}{2}e^x \sin x$

Thus the general solution is

$$y = c_1 + c_2e^{2x} - \tfrac{1}{2}e^x \sin x$$

Ex. 8. *Solve :* $(D^2 + 5D - 6)y = \sin 4x \sin x$

Let $y = e^{mx}$ be a trial solution of $(D^2 + 5D - 6) = 0$

$\therefore$ The auxiliary equation is $m^2 + 5m - 6 = 0$

or, $\quad (m+6)(m-1) = 0$

$\therefore \quad m = 1, -6$

$\therefore \quad$ C.F. $= c_1e^x + c_2e^{-6x}$

$$\text{Now P. I.} = \frac{\sin 4x \sin x}{D^2+5D-6}$$

$$= \frac{1}{2}\frac{\sin 4x \sin x}{D^2+5D-6}$$

$$= \frac{1}{2}\frac{\cos 3x - \cos 5x}{D^2+5D-6}$$

$$= \frac{1}{2}\frac{\cos 3x}{D^2+5D-6} - \frac{1}{2}\frac{\cos 5x}{D^2+5D-6}$$

$$= \frac{1}{2}\frac{\cos 3x}{-3^2+5D-6} - \frac{1}{2}\cdot\frac{\cos 5x}{-5^2+5D-6}$$

$$= \frac{1}{10}(D+3)\frac{\cos 3x}{D^2-9} - \frac{1}{2}(5D+31)\frac{\cos 5x}{25D^2-961}$$

$$= \frac{1}{10}(D+3)\frac{\cos 3x}{-3^2-9} - \frac{1}{2}\cdot(5D+31)\frac{\cos 5x}{25(-5^2)-961}$$

$$= \frac{1}{-180}(D+3)\cos 3x + \frac{1}{3172}(5D+31)\cos 5x$$

$$= -\frac{1}{180}(-3\sin 3x + 3\cos 3x) + \frac{1}{3172}$$

$$(-25\sin 5x + 31\cos 5x)$$

$$= \frac{1}{2}\left(\frac{\sin 3x - \cos 3x}{30} + \frac{31\cos 5x - 25\sin 5x}{1586}\right)$$

Thus the general solution is

$$y = c_1e^x + c_2e^{-6x} + \frac{1}{2}\left(\frac{\sin 3x - \cos 3x}{30} + \frac{31\cos 5x - 25\sin 5x}{1586}\right)$$

Ex. 9. *Solve :* $(D^2-2D+1)\,y = xe^x \sin x$.

Let $y = e^{mx}$ be a trial solution of the homogeneous equation $(D^2-2D+1)\,y = 0$.

Then the auxiliary equation is $m^2 - 2m + 1 = 0$

or, $(m-1)^2 = 0$

$\therefore \quad m = 1, 1.$

$\therefore \quad C.F. = (c_1 + c_2x)\,e^x$, where $c_1\ c_2$ are arbitrary constants.

$$\text{Now P. I.} = \frac{1}{D^2-2D+1}xe^x \sin x$$

$$= \frac{1}{(D-1)^2}xe^x \sin x$$

$$= e^x\frac{1}{(D+1-1)^2}x\sin x = e^x\frac{1}{D^2}x\sin x$$

$$= e^x\frac{1}{D}\int x\sin x\,dx = e^x\frac{1}{D}(-x\cos x + \sin x)$$

$$= e^x\int(-x\cos x + \sin x)\,dx$$

$$= e^x(-x\sin x - 2\cos x)$$

$$= -\,e^x(x\sin x + 2\cos x).$$

So the general solution is

$$y = (c_1 + c_2x)e^x - e^x(x\sin x + 2\cos x).$$

Ex. 10. *Solve :* $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = e^{2x}$, *when* $y = 0$ *and* $\dfrac{dy}{dx} = 1$ *for* $x = 0$.

The given equation can be written as $(D^2-4D+4)\,y = e^{2x}$.

Let $y=e^{mx}$ be a trial solution of the homogeneous equation

$$(D^2-4D+4)\,y=0.$$

So the axiliary equation is

$\therefore\quad m^2-4m+4=0$

or, $\quad (m-2)^2=0.$

$\therefore\quad m=2,2.$

$\therefore\quad C.F.=(c_1+c_2x)e^{2x}$, where c_1, c_2 are arbitrary constants.

Now, P. I. $=\dfrac{1}{D^2-4D+4}e^{2x}$

$=x\dfrac{1}{2D-4}e^{2x}$

$=x^2\dfrac{1}{2}e^{2x}$

$=\dfrac{1}{2}x^2e^{2x}.$

So the general solution is

$y=(c_1+c_2x)\,e^{2x}+\dfrac{1}{2}x^2e^{2x}$

$\therefore\quad \dfrac{dy}{dx}=c_2e^{2x}+2(c_1+c_2x)e^{2x}+xe^{2x}+x^2e^{2x}$

Given that $y=0, \dfrac{dy}{dx}=1$ for $x=0.$

$\therefore\quad c_1=0$ and $2c_1+c_2=1$

$\therefore\quad c_2=1.$

Hence the required particular solution is

$y=xe^{2x}+\dfrac{1}{2}x^2e^{2x}\;=\dfrac{1}{2}x(2+x)e^{2x}.$

Ex. 11. *Solve :* $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y=e^x$. if $y=3$ and $\dfrac{dy}{dx}=3$ when $x=0$

The given equation can be written as

$$\left(D^2-3D+2\right)y=e^x$$

Let $y=e^{mn}$ be a trial solution of

$$\left(D^2-3D+2\right)y=0$$

$\therefore\quad$ The auxiliary equation is

$m^2-3m+2=0$

or, $\quad (m-2)(m-1)=0$

$\therefore\quad m=1,2$

$\therefore\quad$ C.F. $=c_1e^x+c_2e^{2x}$

Now, P. I. $=\dfrac{e^x}{D^2-3D+2}$

$=e^x\dfrac{1}{(D+1)^2-3(D+1)+2}$

$=e^x\dfrac{1}{D^2-D}$

$=e^x\dfrac{1}{D(D-1)}$

$=e^x\left(\dfrac{1}{D-1}-\dfrac{1}{D}\right)$

$$= e^x \left(\frac{e^{0x}}{D-1} - \int dx \right)$$

$$= e^x \left(\frac{1}{0-1} - x \right)$$

$$= -e^x (1+x)$$

So the general solution is

$$y = c_1 e^x + c_2 e^{2x} - e^x (1+x)$$

$$\therefore \quad \frac{dy}{dx} = c_1 e^x + c_2 e^{2x} - e^x (1+x) - e^x$$

Given $y = 3, \frac{dy}{dx} = 3$ when $x = 0$

$\therefore \quad 3 = c_1 + c_2 - 1$

$\therefore \quad c_1 + c_2 = 4$

and $3 = c_1 + 2c_2 - 1 - 1$

$\therefore \quad c_1 + 2c_2 = 5$

Solving we get,

$c_1 = 3, \ c_2 = 1$

Thus the required particular solution is

$$y = 3e^x + e^{2x} - e^x (1+x)$$

$$\therefore \quad y = (2-x)e^x + e^{2x}$$

Ex. 12. *Solve :* $(D^2+9)y = 4\cos\left(x + \frac{\pi}{3}\right)$, given $y(0) = 0$

$y\left(\frac{\pi}{6}\right) = 2$

Let $y = e^{mx}$ be a trial solution of $(D^2+9)y = 0$

$\therefore$ The auxiliary equation is

$m^2 + 9 = 0$

$\therefore \quad m = \pm 3i$

$\therefore \quad$ C.F. $= c_1 \cos 3x + c_2 \sin 3x$

Now, P.I. $= \dfrac{4\cos\left(x+\frac{\pi}{3}\right)}{D^2+9}$

$$= \frac{4\cos\left(x+\frac{\pi}{3}\right)}{-1^2+9}$$

$$= \tfrac{1}{2}\cos\left(x+\frac{\pi}{3}\right)$$

So the general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x + \tfrac{1}{2}\cos\left(x+\frac{\pi}{3}\right)$$

Given $y(0) = 0$

$\therefore \quad 0 = c_1 + c_2.0 + \tfrac{1}{2}\cos\frac{\pi}{3}$

$\therefore \quad c_1 = -\tfrac{1}{4}$

Also given $y\left(\frac{\pi}{6}\right) = 2$

$$\therefore \quad 2 = c_1 \cos 3.\frac{\pi}{6} + c_2 \sin 3.\frac{\pi}{6} + \tfrac{1}{2}\cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right)$$

$$= c_1.0 + c_2.1 + \tfrac{1}{2}.0$$

$\therefore \quad c_2 = 2$

Thus the required particular solution is

$$y = -\tfrac{1}{4}\cos 3x + 2\sin 3x + \tfrac{1}{2}\cos\left(x+\frac{\pi}{3}\right)$$