Chapter - 5

Damped Vibration

Whenever a body is displaced slightly from its position of stable equilibrium and left itself, it tends to come back to its original position due to restoring force acting on it. The f quency of these vibrations depends on the mass, shape, size and the elastic properties of the bo and is known as *natural frequency*.

The body should vibrate with this natural frequency for an indefinite period of time wi constant amplitude. Such an ideal vibration is known as free vibration. But practically the fr vibration diminishes gradually in amplitude with time and finally the body comes to rest at its fix position. The decay of vibration is due to the fact that it is resisted by frictional forces – external internal – known as damping. To overcome the frictional resistance, the vibrating body must e pend energy continuously. Hence, the body looses its energy, so that the amplitude of oscillatic diminishes with time. Such vibrations of decaying amplitude are referred to as resisted or dampe vibrations. For instance, in the case of motion of a simple pendulum, the bob finally comes int rest due to the loss of energy caused by friction at the supports and resistance of air. So this is a example of damped vibration.

5.1 DIFFERENT CAUSES OF DAMPING IN NATURE

Different complex physical processes are responsible for damping in a vibration. A bod oscillating in a gaseous or liquid medium experience viscous force during its motion which caus dissipation of energy in terms of sound and heat energy (viscous damping). Oscillation of a solic body over another solid body experiences force of friction between them, which acts as damping force of vibration. In some oscillating solid bodies the damping forces arise due to imperfect elasticity or internal friction of the material.

5.2 NATURE OF DAMPING FORCE

For viscous damping it is a common practice to approximate the damping in a system by an equivalent viscous damping which is a function of the velocity. The direction of damping force is opposite to the direction of the velocity.

5.3 DAMPING FORCE IS ONLY VELOCITY DEPENDENT

Suppose a body is undergoing SHM which is represented by

$$x = a \sin \omega t$$

It is acted upon by a small damping force F. This force is assumed to be so small that the oscillation remains almost unchanged under the action of this force and we can safely use the above expression of displacement.

For the sake of generality, we are considering that F changes with displacement (x), velocity (dx/dt) and acceleration (d^2x/dt^2). Therefore, F is a function of x, dx/dt and d^2x/dt^2.

$$F = A + Bx + C\frac{dx}{dt} + D\frac{d^2x}{dt^2}$$

$\therefore$ The work done by F in displacing the body by a small amount dx is given by

$$dW = Fdx = F\frac{dx}{dt}\,dt = Fvdt$$

Therefore, work done per cycle

$$W = \int_0^T dW, \text{ where } T \text{ is time period of vibration.}$$

$$\therefore W = \int_0^T [A + Bx + C\frac{dx}{dt} + D\frac{d^2x}{dt^2}]\ (dx/dt)dt$$

Now,

$$x = a\sin\omega t$$

or

$$\frac{dx}{dt} = a\omega\cos\omega t$$

and

$$\frac{d^2x}{dt^2} = -a\omega^2\sin\omega t$$

$$\therefore W = \int_0^T A\frac{dx}{dt}dt + \int_0^T Bx\frac{dx}{dt}dt + \int_0^T C\left(\frac{dx}{dt}\right)^2 dt + \int_0^T D\left(\frac{d^2x}{dt^2}\right)\left(\frac{dx}{dt}\right)dt$$

$$= \int_0^T Aa\omega\cos\omega t dt + \int_0^T Ba^2\omega\sin\omega t\cos\omega t + \int_0^T Ca^2\omega^2\cos^2\omega t - \int_0^T Da^2\omega^3\sin\omega t\cos\omega t$$

$$= \int_o^T Aa\omega\cos\omega t + \int_0^T\left(B - D\omega^2\right)a^2\omega\sin\omega t\cos\omega t + \int_0^T Ca^2\omega^2\cos^2\omega t$$

$$= 0 + 0 + \frac{1}{2}Ca^2\omega^2T = \frac{1}{2}Ca^2\omega^2T$$

It is clear from the above deduction that only the term dependent on velocity survives ultimately and all other terms do not contribute anything in the calculation of work done. So the damping force can be taken as dependent only on velocity and independent of displacement and acceleration.

.4 DIFFERENTIAL EQUATION AND ITS SOLUTION

The differential equation of simple harmonic motion is given by

$$m\frac{d^2x}{dt^2} = -sx$$

Here s is known as *restoring force per unit displacement* (as discussed in Chapter-4). In the presence of damping force (proportional to the velocity of the particle), the total force acting on the particle is $-sx - k\frac{dx}{dt}$.

Total f...

Hence the differential equation of damped harmonic motion is

$$m\frac{d^2x}{dt^2} = -sx - k\frac{dx}{dt}$$

Here k is the proportionality constant which is called *damping force per unit velocity*. In above equation, the negative sign in the first term indicates that the direction of restoring force opposite to that of displacement. The negative sign in the second term indicates that the direc of the damping or resistive force is opposite to that of velocity. Rearranging the terms in ab equation we have

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + sx = 0$$

or $$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0$$ ...(5

Here $2b = k/m$ and $\omega^2 = s/m$ ...(5

Let the trial solution be,

$$x = ce^{\lambda t}$$

Here c and λ are arbitrary constants.

Differentiating the trial solution with respect to time t we have,

$$\frac{dx}{dt} = c\lambda e^{\lambda t} \quad \text{and} \quad \frac{d^2x}{dt^2} = c\lambda^2 e^{\lambda t}$$

Putting these relations in equation (5.1) we get,

$$(\lambda^2 + 2b\lambda + \omega^2)ce^{\lambda t} = 0$$

Now, $ce^{\lambda t} \neq 0$

So we have,

$$(\lambda^2 + 2b\lambda + \omega^2) = 0$$ ...(

The roots of the above equations are,

$$\lambda_1 = -b + \sqrt{b^2 - \omega^2} \quad \text{and} \quad \lambda_2 = -b - \sqrt{b^2 - \omega^2}$$ ...(

and two possible solution of equation (5.1) are $x = ce^{\lambda_1 t}$ and $x = ce^{\lambda_2 t}$.

Therefore, the general solution of the equation (5.1) is given by

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$= e^{-bt}\left(c_1 e^{\left(t\sqrt{b^2-\omega^2}\right)} + c_2 e^{\left(-t\sqrt{b^2-\omega^2}\right)}\right)$$ ...(

Here c_1 and c_2 are two arbitrary constants. To evaluate the constants we have to con three different cases depending upon the values of b and ω.

We will discuss three possible cases where $b>\omega$, $b<\omega$ and $b=\omega$. We have to reme that 'b' is the damping factor which is directly proportional to the damping force and ω natural angular frequency of vibration which is directly pro... root of restoring force.

5.4.1 Large Damping (when $b > w$): Overdamped Motion

So $\sqrt{b^2-\omega^2}$ is real. Let $\sqrt{b^2-\omega^2} = \beta$.

$$x = e^{-bt}\left(c_1 e^{\beta t} + c_2 e^{-\beta t}\right) \quad ...(5.6a)$$

and

$$\frac{dx}{dt} = c_1(-be^{-bt}e^{\beta t} + \beta e^{\beta t}e^{-bt}) + c_2(-be^{-bt}e^{-\beta t} - \beta e^{-\beta t}e^{-bt}) \quad ...(5.6b)$$

If the particle, executing damped vibration, starts in a manner such that at $t = 0$, $x = x_o$ and $dx/dt = v_o$. Putting these conditions in equation (5.6) we get

$$x_0 = c_1 + c_2 \quad ...(5.7)$$

and

$$v_0 = \beta(c_1 - c_2) - b(c_1 + c_2) \quad ...(5.8)$$

putting the value of $(c_1 + c_2)$ from equation(5.7) in equation (5.8) we get

$$v_0 = \beta(c_1 - c_2) - bx_0$$

or

$$\frac{v_0 + bx_0}{\beta} = c_1 - c_2 \quad ...(5.9)$$

Adding and subtracting equation (5.7) and (5.9) we get

$$2c_1 = x_0 + \frac{v_0 + bx_0}{\beta} = x_0 + x_0\left(\frac{b + (v_0/x_0)}{\beta}\right)$$

or

$$c_1 = \frac{x_0}{2}\left(1 + \frac{b + (v_0/x_0)}{\sqrt{b^2-\omega^2}}\right) \quad ...(5.10)$$

and

$$2c_2 = x_0 - \frac{v_0 + bx_0}{\beta} = x_0 - x_0\left(\frac{b + (v_0/x_0)}{\beta}\right)$$

or

$$c_2 = \frac{x_0}{2}\left(1 - \frac{b + (v_0/x_0)}{\sqrt{b^2-\omega^2}}\right) \quad ...(5.11)$$

where, $\beta = \sqrt{b^2-\omega^2}$

Thus the general solution becomes

$$x = \frac{x_0}{2}e^{-bt}\left[\left(1 + \frac{b + v_0/x_0}{\sqrt{b^2-\omega^2}}\right)e^{\sqrt{b^2-\omega^2}\,t} + \left(1 - \frac{b + v_0/x_0}{\sqrt{b^2-\omega^2}}\right)e^{-\sqrt{b^2-\omega^2}\,t}\right] \quad ...(5.12)$$

In this case, the value of $(b^2-\omega^2)$ is positive. Therefore, there is no chance of getting any sine or cosine term or any combination of them in the expression of displacement (x). So the motion of the particle is clearly non-oscillatory or aperiodic and hence the particle returns to its equilibrium position from x_0 and there is no oscillation about the equilibrium position ($x = 0$).

The displacement falls off and then asymptotically approaches zero with increasing time. This is called *overdamped* motion

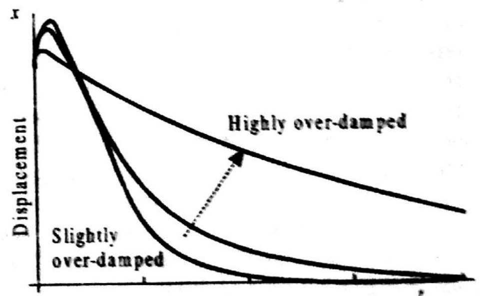


Fig. 5. 1: Displacement-time graph for over damped motion

(Fig. 5.1). It is also called as *dead beat* type motion. This type of motion is found in a aperiodic moving coil galvanometer or a pendulum immersed in a highly viscous liquid.

5.4.2 Critical Damping (when $b = w$)

In this case, the equation (5.3) becomes

$$(\lambda^2 + 2b\lambda + b^2) = 0$$

and the roots of the equations are

$\lambda_1 = -b$ and $\lambda_2 = -b$, i. e., the roots are the same.

Therefore the general solution of the differential equation (equation 5.5) will be of the following form:

$$x = (c_1 + c_2 t)e^{-bt}$$

[For detailed calculation see appendix (6)]

$$\frac{dx}{dt} = -c_1 be^{-bt} - c_2 bte^{-bt} + c_2 e^{-bt}$$

or $$\frac{dx}{dt} = -b(c_1 + c_2 t)e^{-bt} + c_2 e^{-bt}$$

If at $t = 0,\ \ x = x_0\ \ and\ \ \frac{dx}{dt} = v_0$ we have

$$x_0 = c_1 \qquad \text{...(5.13)}$$

and $$v_0 = -bc_1 + c_2 \qquad \text{...(5.14)}$$

Putting the value of c_1 from equation (5.13) in equation (5.14) we get

$$v_0 = -bx_0 + c_2$$

or $$c_2 = v_0 + bx_0 \qquad \text{...(5.15)}$$

Thus the general solution becomes

$$x = (c_1 + c_2 t)e^{-bt}$$

or $$x = (x_0 + tv_0 + btx_0)e^{-bt} = [(1 + bt)x_0 + tv_0]e^{-bt} \qquad \text{...(5.16)}$$

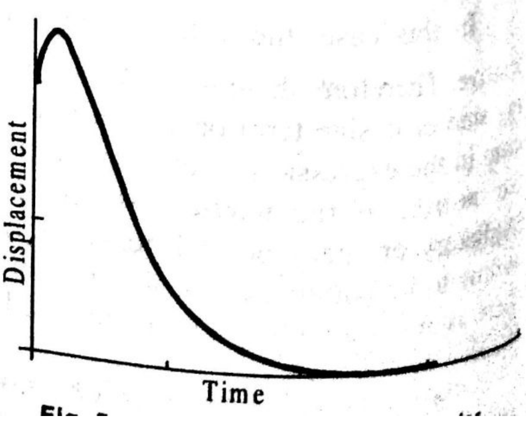
It is the limiting case between dead beat motion and oscillatory motion. For this type of motion, the general equation is given by

$$x = (c_1 + c_2 t)e^{-bt}$$

which shows that the decay of the displacement x is exponential but the change is non-oscillatory. The time taken by the system to return to the initial state depends upon b, the damping factor. In this case, the motion is neither overdamped nor oscillatory and is said to be **critically damped** (Figure 5.2). The property of critical damping is made use of in measuring instruments like ballistic galvanometers.

A comparative study can be made between overdamped and critically damped motions. The solution in the case of overdamped motion is

$$x = e^{-bt}\left(c_1 e^{\beta t} + c_2 e^{-\beta t}\right)$$



and in the case of critically damped motion it is

$$x = (c_1 + c_2 t)e^{-bt}.$$

Let us take the same set of values of the constants c_1 and c_2 for the above equations. For a particular value of damping factor b, we plot the curves shown in Fig. 5.3. It is very much clear from the curve that the rate of fall of displacement is much steeper in the case of critically damped motion than in the case of overdamped motion. In the case of critically damped motion, the particle tends to move to its equilibrium position more rapidly than the overdamped motion.

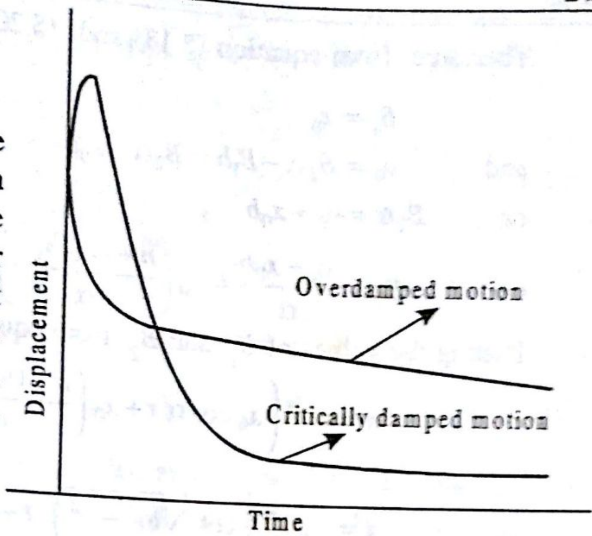


Fig. 5. 3: Comparative study of over damped and critically damped

5.4.3 Small damping (when $b < \omega$): Underdamped Motion

In this case the value of $b^2 - \omega^2$ is negative and hence

$$\sqrt{b^2 - \omega^2} = \sqrt{-(\omega^2 - b^2)} = j\alpha, \text{ where } \alpha = \sqrt{\omega^2 - b^2} \text{ and } j = \sqrt{-1}$$

Now the equation (5.5) can be written as,

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$= e^{-bt}\left(c_1 e^{\left(t\sqrt{b^2-\omega^2}\right)} + c_2 e^{\left(-t\sqrt{b^2-\omega^2}\right)}\right)$$

or

$$x = e^{-bt}\left(c_1 e^{j\alpha t} + c_2 e^{-j\alpha t}\right) \quad \text{...(5.17)}$$

or

$$x = e^{-bt}\left(c_1 \cos\alpha t + jc_1 \sin\alpha t + c_2 \cos\alpha t - jc_2 \sin\alpha t\right)$$

(Applying Eulers' theorem $e^{j\theta} = \cos\theta + j\sin\theta$)

or

$$x = e^{-bt}\left((c_1 + c_2)\cos\alpha t + j(c_1 - c_2)\sin\alpha t\right)$$

or

$$x = e^{-bt}\left(B_1 \cos\alpha t + B_2 \sin\alpha t\right) \quad \text{...(5.18)}$$

where

$$B_1 = c_1 + c_2$$

and

$$B_2 = j(c_1 - c_2) \quad \text{...(5.19)}$$

are constants to be evaluated from initial conditions and then,

$$\frac{dx}{dt} = e^{-bt}(-B_1\alpha\sin\alpha t + B_2\alpha\cos\alpha t) + (-be^{-bt})(B_1\cos\alpha t + B_2\sin\alpha t)$$

$$\frac{dx}{dt} = e^{-bt}\left[(B_2\alpha - B_1 b)\cos\alpha t - (B_2 b + B_1\alpha)\sin\alpha t\right] \quad \text{...(5.20)}$$

Equation (5.17) and (5.18) indicates that the motion of the particle is oscillatory and the amplitude of the motion decreases exponentially with time. The variation of amplitude with time is shown in Fig. (5.4).

Let us take a special initial condition to evaluate the constant terms as follows:

At $t = 0$,

$x = x_o$ and $dx/dt = v_o$

Therefore, from equation (5.18) and (5.20), we get

$$B_1 = x_0$$

and $$v_0 = B_2\alpha - B_1 b = B_2\alpha - x_0 b \quad ...(5.21)$$

or $$B_2\alpha = v_0 + x_0 b$$

or $$B_2 = \frac{v_0 + x_0 b}{\alpha} = x_0\left(\frac{b + (v_0 / x_0)}{\alpha}\right) \quad ...(5.22)$$

Putting the values of B_l and B_2 from equation (5.21) and (5.22) in equation (5.18) we get

$$x = e^{-bt}\left(x_0 \cos\alpha\, t + x_0\left(\frac{b+(v_0/x_0)}{\alpha}\right)\sin\alpha\, t\right)$$

or $$x = x_0 e^{-bt}\left(\cos\left(\sqrt{\omega^2 - b^2}\right)\ t + \left(\frac{b+(v_0/x_0)}{\sqrt{\omega^2-b^2}}\right)\sin\left(\sqrt{\omega^2-b^2}\right)\ t\right) \quad ...(5.23)$$

where, $\alpha = \sqrt{\omega^2 - b^2}$

Equation (5.23) can be written as

$$x = Ce^{-bt}\sin\left[\left(\sqrt{\omega^2-b^2}\right)t + \theta\right] \quad ...(5.24)$$

where

$$C\sin\theta = x_0$$

and $$C\cos\theta = x_0\left(\frac{b+(v_0/x_0)}{\sqrt{\omega^2-b^2}}\right)$$

Therefore,

$$C^2 = C^2\left(\sin^2\theta + \cos^2\theta\right) = x_0^{\,2}\left(1 + \frac{b^2 + 2bv_0/x_0 \ + v_0^{\,2}/x_0^{\,2}}{\omega^2 - b^2}\right) = x_0^{\,2}\left(\frac{x_0^{\,2}\omega^2 + 2x_0 b v_0 + v_0^{\,2}}{x_0^{\,2}(\omega^2 - b^2)}\right)$$

$$C = \left(\frac{x_0^{\,2}\omega^2 + 2x_0 b v_0 + v_0^{\,2}}{\omega^2 - b^2}\right)^{\frac{1}{2}}$$

and $$\tan\theta = \frac{C\sin\theta}{C\cos\theta} = \left(\frac{b+(v_0/x_0)}{\sqrt{\omega^2-b^2}}\right)^{-1}$$

If the particle is initially displaced to x_0 and then released, then $v_0 = 0$ and $x = x_0$ at $t = 0$.

Then

$$C = \left(\frac{x_0^{\,2}\omega^2}{\omega^2 - b^2}\right)^{\frac{1}{2}} = \frac{x_0\omega_0}{\sqrt{\omega^2-b^2}}$$

And $$\tan\theta = \left(\frac{b}{\sqrt{\omega^2-b^2}}\right)^{-1} = \tan\theta' \text{ (say)}$$

And the solution will be, [using equation (5.24)]

$$x = \left(\frac{x_0\omega_0}{\sqrt{\omega^2-b^2}}\right)e^{-bt}\sin\left(\sqrt{\omega^2-b^2}\ \ \mathrm{t}\ \ +\theta'\right) \quad ...(5.25)$$

If the particle is given an initial velocity v_0 at rest, then $x_0 = 0,\ v = v_0$ at $t = 0$.

Then $$C = \left(\frac{v_0^{\,2}}{\omega^2 - b^2}\right)^{\frac{1}{2}} = \frac{v_0}{\sqrt{\omega^2 - b^2}}$$

and $\tan\theta = 0$ or, $\theta = 0$.

In this case the solution is

$$x = \left(\frac{v_0}{\sqrt{\omega^2 - b^2}}\right) e^{-bt} \sin\left(\sqrt{\omega^2 - b^2}\ t\right) \qquad ...(5.26)$$

Equation (5.24) shows that the motion may be regarded as oscillatory but its amplitude is not constant and being equal to Ce^{-bt}, it decays exponentially with time, the motion can be called as **damped oscillatory** (Fig. 5.4). This is also called **underdamped** motion.

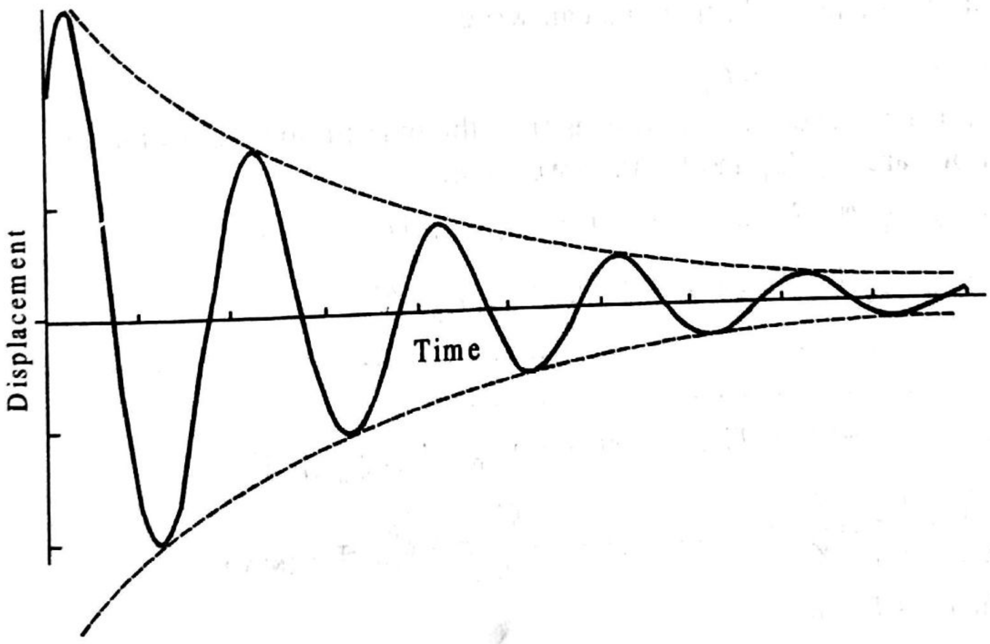


Fig. 5.4: Displacement-time graphs for underdamped motion

5.5 DIFFERENT PARAMETERS FOR MEASURING DAMPING

5.5.1 Decay constant or Relaxation time (Γ):

In the case of damping, the amplitude is time dependent. The amplitude decays exponentially with time in the case of damped oscillatory or underdamped motion. Decay constant or Relaxation time (Γ) is a time in which the amplitude decays to 1/e of its initial value. As the amplitude at any instant is Ae^{-bt}, then Γ is given by,

$$\frac{Ae^{-b(t+\Gamma)}}{Ae^{-bt}} = 1$$

In the absence of any dissipative force, a vibrating body will continue its motion fore. But, practically the natural vibration will die out due to damping. However, the system can maintained in its state of vibration if an external periodic force is applied. Let us consider a perio force of constant frequency and amplitude acting on a system. Initially the system tends to vibr with its own natural frequency whether the external applied force tries to impose its own freque on the system. Under this type of situation, the motion of the vibrating body will be ceased due the act of damping force and in the steady state, the system will ultimately vibrate with the frequen of the applied periodic force. Such vibrations are called forced vibration. Here we are assumi that the forced system, by its reaction, does not modify the forcing system.

The common examples are vibration of a loudspeaker cone, a gramophone sound box, stretched wire under tension actuated by a tuning fork, the periodic motion of the piston in engine, etc.

6.1 DIFFERENTIAL EQUATION AND ITS SOLUTION

Let an external periodic force $F = F_0 \sin pt$ (where F_0 is the amplitude and p is the angul frequency) acts on a particle of mass m under the action of a restoring force proportional to displacement (x) and a damping force proportional to its instantaneous velocity (dx/dt). Then the equation of motion can be written as,

$$m\frac{d^2x}{dt^2} = -sx - k\frac{dx}{dt} + F_0 \sin pt \qquad ...(6.1)$$

where s and k are stiffness constant and damping constant respectively.

Therefore,

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = f \sin pt, \qquad ...(6.2)$$

where $2b = k/m$, $\omega^2 = s/m$ and $f = F_0/m$

In order to find out the solution of this equation we have to find out complimentary function (C.F.) and particular integral (P.I.).

In order to calculate C.F., we have to equate the L.H.S. of the equation (6.2) to zero.

Therefore,

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0 \qquad ...(6.3)$$

whose solution can be calculated as follows:

Let $x = ce^{\lambda t}$ be a trial solution of the above equation, where c and λ are arbitrary constants. Then differentiating the trial solution with respect to time t we have,

$$\frac{dx}{dt} = c\lambda e^{\lambda t}$$

and $$\frac{d^2x}{dt^2} = c\lambda^2 e^{\lambda t}$$

Putting these relations in equation (6.3), we get

$$(\lambda^2 + 2b\lambda + \omega^2)ce^{\lambda t} = 0$$

Since, $ce^{\lambda t} \neq 0$

$$\lambda^2 + 2b\lambda + \omega^2 = 0 \qquad ...(6.4)$$

The roots of the above equations are

$$\lambda_1 = -b + \sqrt{b^2 - \omega^2} \quad \text{and} \quad \lambda_2 = -b - \sqrt{b^2 - \omega^2}$$

and two possible solutions of equation (6.3) are

$$x = ce^{\lambda_1 t} \quad \text{and} \quad x = ce^{\lambda_2 t} \qquad ...(6.5)$$

Therefore the general solution of the equation (6.3) is given by

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

or, $$x = e^{-bt}\left[c_1 e^{\left(\sqrt{b^2-\omega^2}\right)t} + c_2 e^{-\left(\sqrt{b^2-\omega^2}\right)t}\right] \qquad ...(6.6)$$

Here c_1 and c_2 are two arbitrary constants. From the discussion of damped harmonic oscillation in Chapter 5, we know that, in case of low damping, the solution (let it be denoted by becomes

$$x_1 = A_1 e^{-bt} \sin\left(\sqrt{\omega^2 - b^2}\ t + \theta\right) \qquad ...(6.7)$$

where A_1 and θ are constants.

In order to calculate the P.I., let us take the trial solution as

$$x = A\sin(pt - \alpha) \qquad ...(6.8)$$

keeping in mind that the system will ultimately vibrate with the angular frequency p of the applied external periodic force.

Thus we have,

$$\frac{dx}{dt} = Ap\cos(pt - \alpha)$$

$$\frac{d^2x}{dt^2} = -Ap^2\sin(pt - \alpha)$$

Substituting these values in equation of motion (6.2) we get

$$-Ap^2\sin(pt-\alpha) + 2bAp\cos(pt-\alpha) + \omega^2 A\sin(pt-\alpha) = f\sin pt$$

Or

$$A(\omega^2 - p^2)\sin(pt-\alpha) + 2bAp\cos(pt-\alpha) = f\sin(pt - \alpha + \alpha)$$

Or

$$A(\omega^2 - p^2)\sin(pt-\alpha) + 2bAp\cos(pt-\alpha)$$

$$= f\sin(pt-\alpha)\cos\alpha + f\cos(pt-\alpha)\sin\alpha \qquad ...(6.9)$$

The above equation is true for all values of t.

Therefore, the co-efficient of sin $(pt-\alpha)$ and cos $(pt-\alpha)$ can be equated from the both sides of equation (6.9).

Then we get,

$$f\cos\alpha = A(\omega^2 - p^2) \quad \text{...(6.10)}$$

$$f\sin\alpha = 2bAp \quad \text{...(6.10)}$$

Squaring and adding we get,

$$f^2\cos^2\alpha + f^2\sin^2\alpha = f^2\left(\sin^2\alpha + \cos^2\alpha\right) = f^2 = A^2(\omega^2 - p^2)^2 + 4b^2A^2p^2$$

$$= A^2\left[\left(\omega^2 - p^2\right)^2 + \left(4b^2p^2\right)\right]$$

$$\therefore A^2 = \frac{f^2}{\left[\left(\omega^2 - p^2\right)^2 + \left(4b^2p^2\right)\right]}$$

or

$$A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2p^2}} \quad \text{...(6.11)}$$

Also we can write,

$$\tan\alpha = \frac{f\sin\alpha}{f\cos\alpha} = \frac{2bAp}{A(\omega^2 - p^2)} = \frac{2bp}{(\omega^2 - p^2)}$$

or

$$\alpha = \tan^{-1}\frac{2bp}{(\omega^2 - p^2)} \quad \text{...(6.12)}$$

So, the P.I. part of the solution will be equal to $x_2 = A\sin(pt - \alpha)$ where A and α are given by equations (6.11) and (6.12).

Therefore the complete solution of equation will be,

$$x = \text{C.F.} + \text{P.I.}$$

or

$$x = A_1e^{-bt}\sin\left(\sqrt{\omega^2 - b^2}\ t - \theta\right) + \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2p^2}}\sin(pt - \alpha) \quad \text{...(6.13)}$$

The first part of the right hand side of the equation (6.13) represents the damped vibrations of frequency $\left(\sqrt{\omega^2 - b^2}\right)/2\pi$ (say v^*) and the second part represents the forced vibrations of frequency $p/2\pi$ (say v), same as the frequency of the applied (external) periodic force.

At the initial stage of motion, a tussle exists between the damping force (tends to retard the motion) and the driving force (tends to continue the motion). The damped oscillator tries to oscillate at its own frequency v^* and the driving force causes the system to oscillate with frequency. But the contribution of damping becomes negligible very soon as the amplitude diminishes exponentially with time. If damping is very small, the natural vibrations will persist for a longer time. The resultant vibration x at any instant is the sum of natural vibration represented by C.F. and the forced sustained vibration represented by P.I. This is known as transient state.

After a sufficiently long time, when C.F. becomes negligible, we can write

$$x = \frac{f}{\sqrt{4b^2p^2 + (\omega^2 - p^2)^2}}\sin(pt - \alpha) \quad \text{...(6.14)}$$

which represents the sustained forced vibration. This is known as steady state.