Q1) PT $\int_{-1}^{1} \frac{dx}{x^3}$ exists in Cauchy Principle value sense but not in general sense.

→ For general sense,

0 is a pt of discontinuity hence using the rule

$$\lim_{\delta_1 \to 0^+} \int_{-1}^{0-\delta_1} \frac{dx}{x^3} + \lim_{\delta_2 \to 0^+} \int_{0+\delta_2}^{1} \frac{dx}{x^3}$$

$$\lim_{\delta_1 \to 0^+} -\left[\frac{1}{2x^2}\right]_{-1}^{\delta_1} + \lim_{\delta_2 \to 0^+} -\left[\frac{1}{2x^2}\right]_{\delta_2}^{1}$$

$$\lim_{\delta_1 \to 0^+} \frac{-1}{2\delta_1^2} + \lim_{\delta_2 \to 0^+} \frac{1}{2\delta_2^2}$$

Which clearly does not exist.

For cauchy sense, $\delta_1 = \delta_2 = \delta$

$$\lim_{\delta \to 0^+} -\frac{1}{2\delta^2} + \frac{1}{2\delta^2} = 0 \text{ (finite)}$$

② Evaluate (if the integral exists)

a) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

Let d be an arbitary pt in $(-\infty, \infty)$

$$= \lim_{X \to -\infty} \int_{X}^{d} \frac{dx}{x^2+2x+2} + \lim_{Y \to \infty} \int_{d}^{Y} \frac{dx}{x^2+2x+2}$$

$$= \lim_{X \to -\infty} \int_{X}^{d} \frac{1}{(x+1)^2+1} + \lim_{Y \to \infty} \int_{d}^{Y} \frac{1}{(x+1)^2+1}$$

$$= \lim_{X \to -\infty} \left[\tan^{-1}(1+x)\right]_{X}^{d} + \lim_{Y \to \infty} \left[\tan^{-1}(1+x)\right]_{d}^{Y}$$

$$= \lim_{X \to -\infty} \left[\tan^{-1}(1+d) - \tan^{-1}(X+1)\right] + \lim_{Y \to \infty} \left[\tan^{-1}(1+Y) - \tan^{-1}(1+d)\right]$$

$$= \tan^{-1}(d+1) - \left(-\frac{\pi}{2}\right) + \cancel{\tan^{-1}} \left(\frac{\pi}{2}\right) - \tan^{-1}(d+1)$$

$$= \pi$$

(b) $\int_{-1}^{1} \frac{\sqrt{1+x}}{\sqrt{1-x}}\, dx$

$\rightarrow$ $\lim_{\delta \to 0^+} \int_{-1}^{1-\delta} \sqrt{\frac{1+x}{1-x}}\, dx$

let $x = \cos t$

$dx = -\sin t \, dt$

$\therefore \lim_{\delta \to 0^+} \int_{-1}^{1-\delta} \sqrt{\frac{2\cos^2 t/2}{2\sin^2 t/2}}\, (-\sin t)\, dt$

$\lim_{\delta \to 0^+} -2\int_{-1}^{1-\delta} \cos^2(t/2)\, dt = \lim_{\delta \to 0^+} -\cancel{2}\int_{-1}^{1-\delta} \frac{(1+\cos t)}{\cancel{2}}\, dt$

$\lim_{\delta \to 0^+} -\left[t + \sin t\right]_{-1}^{1-\delta} = -\lim_{\delta \to 0^+}\left[\cos^{-1}x + \sqrt{1-x^2}\right]_{-1}^{1-\delta}$

$-\lim_{\delta \to 0^+}\left[\cos^{-1}(1-\delta) + \sqrt{1-(1-\delta)^2} - \cos^{-1}(-1) - \sqrt{1-1}\right]$

$= \quad -\pi =$

(c) $\int_{0}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

$\lim_{\delta \to 0^+} \int_{1+\delta}^{d} \frac{dx}{x\sqrt{x^2+1}} + \lim_{X \to \infty} \int_{d}^{X} \frac{dx}{x\sqrt{x^2-1}}$

$\lim_{\delta \to 0^+} \left[\sec^{-1}x\right]_{1+\delta}^{d} + \lim_{X \to \infty} \left[\sec^{-1}x\right]_{d}^{X}$

$\lim_{\delta \to 0^+} \left[\sec^{-1}d - \sec^{-1}(1+\delta)\right] + \lim_{X \to \infty}\left[\sec^{-1}X - \sec^{-1}d\right]$

$\sec^{-1}d - 0 + \frac{\pi}{2} - \sec^{-1}d$

$\pi/2$

(d) $\int_{0}^{1/e} \frac{dx}{x(\log x)^2}$

$\rightarrow$ $\lim_{\delta \to 0^+} \int_{\delta}^{1/e} \frac{dx}{x(\log x)^2}$

let $\log x = t$

$\frac{1}{x} dx = dt$

$\lim_{\delta \to 0^+} \int_{\delta}^{1/e} \frac{1}{t^2}\, dt = \lim_{\delta \to 0^+} \left[\frac{-1}{\log x}\right]_{\delta}^{1/e}$

$\lim_{\delta \to 0} \left[-\frac{1}{\log \frac{1}{2}} + \frac{1}{\log \delta} \right]$

$1 + 0 = 1$

(3) Show that :

(a) $\int_0^{\pi/2} \sqrt{\cot \theta}\, d\theta = \pi/\sqrt{2}$

$\rightarrow \frac{1}{2}\int_0^{\pi/2} 2 \cos^{1/2}\theta \sin^{-1/2}\theta \, d\theta$

= Using beta-gamma relation

$$B\left(\tfrac{1}{4}, \tfrac{3}{4}\right) = \frac{\frac{1}{2}\, \Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{\Gamma(1)} = \frac{1}{2} \Gamma\left(\tfrac{1}{4}\right) \Gamma\left(1 - \tfrac{1}{4}\right)$$

$[\Gamma(m)\Gamma(1-m) = \pi \operatorname{cosec} m\pi]$

$$\therefore = \frac{1}{2} \pi \operatorname{cosec} \frac{\pi}{4} = \frac{1}{2} \pi \sqrt{2} = \frac{\pi}{\sqrt{2}} = \text{RHS}$$

(b) $\int_0^1 x^3 (1-x^2)^{5/2}\, dx$

let $x = \sin\theta$

$dx = \cos\theta \, d\theta$

$\rightarrow \int_0^1 \sin^3\theta \cos^5\theta (\cos\theta\, d\theta) = \int_0^1 \sin^3\theta \cos^6\theta \, d\theta$

$\rightarrow \frac{1}{2}\int_0^1 2 \sin^3\theta \cos^6\theta \, d\theta$

$$\rightarrow \frac{\frac{1}{2}\Gamma\left(\frac{3+1}{2}\right) \Gamma\left(\frac{6+1}{2}\right)}{\Gamma\left(\frac{3+6+2}{2}\right)} = \frac{1}{2} \frac{\Gamma(2)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}$$

$$\rightarrow \frac{1}{2} \frac{(1\times1)\, \Gamma\left(1+\frac{5}{2}\right)}{\Gamma\left(1+\frac{9}{2}\right)} = \frac{\frac{1}{2}\, \cancel{\frac{5}{2}} \times \cancel{\frac{3}{2}} \times \cancel{\frac{1}{2}} \times \cancel{\sqrt{\pi}}}{\frac{9}{2} \times \frac{7}{2} \times \cancel{\frac{5}{2}} \times \cancel{\frac{3}{2}} \times \cancel{\frac{1}{2}} \times \cancel{\sqrt{\pi}}}$$

$$= \frac{2}{63}$$

(c) $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx = \frac{\pi}{4\sqrt{2}}$

→ $I_1 + I_2$

⇒ let $x^2 = \sin t$ in I

$2x\,dx = \cos t\,dt$

$$I_1 = \int_0^{\pi/2} \frac{\sin t}{\cancel{\cos t}} \times \frac{\cancel{\cos t}}{2\sin^{1/2} t}\,dt = \frac{1}{2}\int_0^{\pi/2} \sin^{1/2} t\,dt$$

$$I_1 = \frac{1}{4}\,\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)} = \frac{1}{4}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$I_1 = \frac{\sqrt{\pi}}{\cancel{4}}\,\frac{\Gamma(3/4)}{\frac{1}{\cancel{4}}\Gamma(1/4)} = \sqrt{\pi}\,\frac{\Gamma(3/4)}{\Gamma(1/4)}$$

$$I_2 = \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx$$

let $x^2 = \tan\theta$.

$2x\,dx = \sec^2\theta\,d\theta$

$$I_2 = \int_0^{\pi/4} \frac{1}{\cancel{\sec\theta}}\,\frac{\sec^2\theta}{2\tan^{1/2}\theta}\,d\theta = \frac{1}{4}\int_0^{\pi/4} 2\sin^{-1/2}\theta\cos^{1/2}\theta\,d\theta$$

$f(2a-x) = f(x)$ where $I = \int_0^{2a} f(x)$

$$\therefore I_2 = \frac{1}{8}\int_0^{\pi/2} \sin^{-1/2}\theta\cos^{1/2}\theta\,d\theta$$

$$= \frac{1}{8}\,\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)}$$

$$= \frac{1}{8}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1/2)}$$

$= \cancel{\frac{1}{8}\sqrt{\pi}}$

$$I_1 \times I_2 = \cancel{\sqrt{\pi}}\,\frac{\Gamma(3/4)}{\cancel{\Gamma(1/4)}} \times \frac{1}{8}\,\frac{\cancel{\Gamma(1/4)}\,\Gamma(1/4)}{\cancel{\Gamma(1/2)}}$$

$$= \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) \qquad \Gamma(m)\Gamma(1-m) = \pi\operatorname{cosec}\pi m$$

$$= \frac{1}{8}\,\pi\operatorname{cosec}\left(\frac{1}{4}\pi\right) = \frac{\pi}{8\sin\frac{\pi}{4}} = \frac{\pi}{8}\times\sqrt{2} = \frac{\pi}{4\sqrt{2}}$$

(d) $B(m,\frac{1}{2}) = 2^{2m-1} B(m,m)$

$$LHS = B(m,\tfrac{1}{2}) = \frac{\Gamma(m)\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

$$B(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1}\, d\theta$$

$$= \frac{1}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1}\phi\, d\phi$$

$$= \frac{B(m,\frac{1}{2})}{2^{2m-1}}$$

$$\therefore B(m,\tfrac{1}{2}) = 2^{2m-1} B(m,m).$$

(e) Assume the convergence of the integral, prove that,

$$\int_0^\infty \sqrt{x}\, e^{-x^3} dx = \frac{\sqrt{\pi}}{3}$$

Ans→

Let $x^3 = t$

$3x^2\, dx = dt$

$$\int_0^\infty e^{-t} x^{1/2} \frac{dt}{3x^2} = \int_0^\infty \frac{e^{-t}}{3} x^{-3/2} dt = \int_0^\infty \frac{e^{-t}}{3} t^{-1/2} dt$$

$$\frac{1}{3}\int_0^\infty e^{-t} t^{-1/2}\, dt = \frac{1}{3}\int_0^\infty e^{-t}\, t^{(\frac{1}{2}-1)}\, dt$$

$$= \frac{1}{3}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{3}$$

$$\boxed{\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, \quad n>0}$$

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Evaluate

(4) (a) $\int_0^1 x^4 \log\left(\frac{1}{x}\right)^3 dx$

$\rightarrow 3\int_0^1 x^4 \log\frac{1}{x}\, dx$

let $\log_e \frac{1}{x} = t \Rightarrow \frac{1}{x} = e^t \Rightarrow x = e^{-t}$

$x\left(-\frac{1}{x^2}\right)dx = dt$

$-\frac{1}{x}dx = dt$

$\rightarrow 3\int_0^{\infty} x^4\, t\, (-x)\, dt$

$= -3\int_0^{\infty} t\, x^5\, dt$

$= -3\int_0^{\infty} t\, e^{-5t}\, dt = 3\int_0^1 e^{-5t}\, t\, dt$.

$= -3\int_0^{\infty} e^{-5t}\, t^{(2-1)}\, dt$ $\quad \cancel{\Gamma(an)} = \int_0^{\infty} e^{-at}\, t^{(n-1)}\, dt = \frac{\Gamma(n)}{a^n}$, $n>0$

$= -3\,\frac{\Gamma(2)}{5^2} = \frac{-3}{25}$

(b) $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx$

$\rightarrow \frac{1}{2}\cdot 2\int_0^{\pi/2} \sin^{-1/2} dx \times \frac{1}{2}\cdot 2\int_0^{\pi/2} \sin^{1/2} x\, dx$

$= \frac{1}{2}\cdot \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{2-\frac{1}{2}+0+2}{2}\right)} \cdot \frac{1}{2}\, \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)}$

$= \frac{1}{2}\, \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\cancel{\Gamma\left(\frac{3}{4}\right)}} \cdot \frac{1}{2}\, \frac{\cancel{\Gamma\left(\frac{3}{4}\right)}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}$

$= \frac{1}{2}\cdot\frac{1}{2}\cdot\pi\, \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(1+\frac{1}{4}\right)} = \frac{\pi}{\cancel{4}}\, \frac{\cancel{\Gamma\left(\frac{1}{4}\right)}}{\cancel{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}} = \pi$

(1) * Find the Laplace Transform of the following functions:

(a) $F(t) = (1+te^{-t})^3$

$= 1 + (te^{-t})^3 + 3(te^{-t}) + 3(te^{-t})^2$

$= 1 + t^3e^{-3t} + 3te^{-t} + 3t^2e^{-2t}$

$L\{F(t)\} = L\{1\} + L\{t^3e^{-3t}\} + 3L\{te^{-t}\} + 3L\{t^2e^{-2t}\}$

$= \frac{1}{s} + \frac{6}{(s+3)^4} + \frac{3}{(s+1)^2} + 3\cdot\frac{2}{(s+2)^3}$

(b) $F(t) = [(t^2-3t+2)\sin 3t]$

$L\{F(t)\} = L\{t^2\sin 3t\} - 3L\{t\sin 3t\} + 2L\{\sin 3t\}$

for, $L\{t^2\sin 3t\} =$

$L\{\sin 3t\} = \frac{3}{s^2+9}$

$L\{t^2\sin 3t\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) \quad \left[\because L\{t^n F(t)\} = (-1)^n\frac{d^n}{ds^n}f(s)\right]$

$= 3\frac{d}{ds}\left[\frac{-2s}{(s^2+9)^2}\right]$

$= -6\,\frac{(s^2+9)^2 - s\,2(s^2+9)2s}{(s^2+9)^4}$

$= \frac{18(s^4+6s^2-27)}{(s^2+9)^4}$

similarly, $L\{t\sin 3t\} = (-1)^1\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{3(-2s)}{(s^2+9)^2}$

$L\{\sin 3t\} = \frac{3}{s^2+9}$

$\therefore L\{F(t)\} = \frac{18(s^4+6s^2-27)}{(s^2+9)^4} - \frac{18s}{(s^2+9)^2} + \frac{6}{s^2+9}$

(c) $F(t) = e^{-3t}(2\cos 5t - 3\sin 5t)$

$L\{F(t)\} = 2L\{e^{-3t}\cos 5t\} - 3L\{e^{-3t}\sin 5t\}$

$= 2 \cdot \frac{(s+3)}{(s+3)^2+25} - 3 \cdot \frac{5}{(s+3)^2+25}$

$= \frac{2(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25} = \frac{2s-9}{(s+3)^2+25}$

(d) $F(t) = e^{-3t}\frac{\sin 2t}{t}$

$\to L\{\sin 2t\} = \frac{2}{s^2+4}$

$L\{e^{-3t}\sin 2t\} = \frac{2}{(s+3)^2+4} = f(s)$

$L\left\{\frac{e^{-3t}\sin 2t}{t}\right\} = \int_s^\infty f(u)\,du$

$= \int_s^\infty \frac{2}{(u+3)^2+4}\,du$

$= 2 \cdot \frac{1}{2}\left[\tan^{-1}\frac{u+3}{2}\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}\frac{s+3}{2}$

$= \frac{\pi}{2} - \tan^{-1}\frac{s+3}{2}$

$= \cot^{-1}\frac{(s+3)}{2}$ //

(e) $F(t) = \int_0^t e^u \frac{\sin u}{u}\,du$

$L\left\{\frac{e^u \sin u}{u}\right\} = f(s)$

$L\left\{\int_0^t \frac{e^u \sin u}{u}\right\} = \frac{1}{s} f(s)$

$\therefore L\left\{\frac{e^u \sin u}{u}\right\} = \int_s^\infty \frac{1}{(s-1)^2+1}\,du$

$= \left[\tan^{-1}(u-1)\right]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}(s-1)$

$= \cot^{-1}(s-1)$

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$\therefore L\left\{ \int_0^t \frac{e^u \sin u}{u} \right\} = \frac{1}{s} \cot^{-1}(s-1)$

(2) Find the Laplace Transform of $\frac{\sin at}{t}$. Hence show that $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$

$\rightarrow$ $F(t) = \frac{\sin at}{t}$

$$L\{F(t)\} = \int_s^\infty \frac{a}{u^2+a^2}\, du = a \cdot \frac{1}{a}\left[\tan^{-1}\frac{u}{a}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

$$L\left\{\frac{\sin t}{t}\right\} = \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{-st} \frac{\sin t}{t} = \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty \frac{\sin t}{t} = \frac{\pi}{2}$$

(3) Evaluate $\int_0^\infty t e^{-2t} \cos t\, dt$

$\rightarrow$ $L\{\cos t\} = \frac{s}{s^2+1}$

$$L\{e^{-2t}\cos t\} = \frac{s+2}{(s+2)^2+1}$$

$$L\{te^{-2t}\cos t\} = (-1)^1 \frac{d}{ds} \frac{(s+2)}{(s+2)^2+1}$$

$$= \frac{(s+2)^2 - 1}{[(s+2)^2+1]^2}$$

$$\int_0^\infty e^{-st}\, t e^{-2t} \cos t = \frac{(s+2)^2-1}{[(s+2)^2+1]^2}$$

$$\int_0^\infty t e^{-2t} \cos t = \frac{3}{25}$$

(3) Find the laplace transform of the Periodic function F(t) given by

$$F(t) = \begin{cases} t, & \text{for } 0 \le t \le c \\ 2c-t, & \text{for } c<t<2c \end{cases}$$

$$\rightarrow L\{F(t)\} = \frac{1}{1-e^{-2cp}} \int_0^{2c} e^{-pt} F(t)\, dt$$

$$= \frac{1}{1-e^{-2cp}} \left[\int_0^c e^{-pt} \cdot t\, dt + \int_c^{2c} e^{-pt} (2c-t)\, dt \right]$$

$$= \frac{1}{1-e^{-2cp}} \left\{ \left[\frac{-t\, e^{-pt}}{p} - \frac{e^{-pt}}{p^2} \right]_0^c + \left[\frac{-2c}{p} \left[e^{-pt}\right]_c^{2c} - \left[-\frac{t e^{-pt}}{p} - \frac{e^{-pt}}{p^2} \right]_c^{2c} \right] \right\}$$

$$= \frac{1}{1-e^{-2cp}} \left[-\frac{c}{p} e^{-pc} - \frac{1}{p^2} e^{-pc} + \frac{1}{p^2} - \frac{2}{p} c e^{-2cp} + \frac{2c}{p} e^{-cp} \right.$$
$$\left. + \left[\frac{2c}{p} e^{-2cp} + \frac{1}{p^2} e^{-2cp} \right] - \left[-\frac{c}{p} e^{-pc} - \frac{1}{p^2} e^{-pc} \right] \right]$$

$$= \frac{1}{1-e^{2cp}} \left(\frac{1-e^{-pc}}{s} \right)^2 = \frac{1-e^{-pc}}{s^2(1+e^{-pc})}$$

(4) Express the following function interms of unit Step Function & then find its Laplace transform:

$$F(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & t>2 \end{cases}$$

$$F(t) = 0 + (t-1) H(t-1) + \{1-(t-1)\} H(t-2)$$

$$L\{F(t)\} = L\{(t-1) H(t-1)\} + L\{(2-t) H(t-2)\}$$

$$= L\{tH(t-1)\} - L\{H(t-1)\} + L\{2H(t-2)\} - L\{tH(t-2)\}$$

$$L\{tH(t-1)\} \Rightarrow L\{H(t-1)\} = \frac{e^{-s}}{s}$$

$$L\{tH(t-1)\} = (-1)^1 \frac{d}{ds}\left(\frac{e^{-s}}{s}\right) = -\left[\frac{-se^{-s} - e^{-s}}{s^2}\right]$$

$$= \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}$$

$$L\{H(t-2)\} = e^{-2s}/s$$

$$2L\{H(t-2)\} = \frac{2e^{-2s}}{s}$$

$$L\{tH(t-2)\} = 2(-1)^1 \frac{d}{ds}\left(\frac{e^{-2s}}{s}\right) = -\left[\frac{\frac{se^{-2s}}{-2} - e^{-2s}}{s^2}\right] = \frac{e^{-2s}}{2s} + \frac{e^{-2s}}{s^2}$$

$L\{F(t)\} = \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$

(5) Find $L(\sin\sqrt{t})$, $(t>0)$ and then obtain the value of $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

→ $\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{\lfloor 3} + \frac{(\sqrt{t})^5}{\lfloor 5} - \frac{(\sqrt{t})^7}{\lfloor 7} + \cdots$

$L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2p^{3/2}}\left[1 - \frac{1}{\lfloor 3}\frac{3}{2p} + \frac{1}{\lfloor 5}\frac{15}{4p^2} - \cdots\right]$

$= \frac{\sqrt{\pi}}{2p^{3/2}}\left[1 - \frac{1}{2^2 p} + \frac{1}{2}\left(\frac{1}{2^2 p}\right)^2 - \cdots\right]$

$= \frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$

$\frac{1}{2} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = p\frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$

(6) If $L\{F(t)\} = f(s)$ then P.T. $L\{t^n(F(t))\} = (-1)^n \frac{d^n}{ds^n} f(s)$, where n is a +ve integer.

→ $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

$\frac{d}{ds} f(s) = \frac{d}{ds}\int_0^\infty e^{-st} F(t)\,dt$

$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st} F(t)\,dt)$

$= \int_0^\infty -t e^{-st} F(t)\,dt$

$= -L\{tF(t)\}$

$L\{t(F(t))\} = -\frac{d}{ds} f(s)$

now, $L\{t^2 F(t)\} = -\frac{d}{ds} L\{t F(t)\}$

$= (-1)^2 \frac{d^2 f(s)}{ds^2}$

∴ Theorem is true for $m=1,2$

Let us assume that that is true for $m=n$

∴ $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

(3) (7) Evaluate the following:

(a) $L^{-1}\left\{\frac{4s+5}{(s-4)^2(s+3)}\right\}$

$$\frac{4s+5}{(s-4)^2(s+3)} = \frac{A}{s-4} + \frac{B}{(s-4)^2} + \frac{C}{(s+3)}$$

$\Rightarrow$ $= A(s-4)(s+3) + B(s+3) + C(s-4)^2$

Putting $s=4$,
$B=3$

Putting $s=-3$
$C = \frac{-7}{49} = -\frac{1}{7}$

Putting $s=1$
we gets $A = \frac{1}{7}$

now,

$$\frac{1}{7} L^{-1}\left\{\frac{1}{s-4}\right\} + 3\, L^{-1}\left\{\frac{1}{(s-4)^2}\right\} - \frac{1}{7} L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\frac{1}{7}e^{4t} + 3te^{4t} - \frac{1}{7}e^{-3t}$$

(b) $L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\}$

Ans $\rightarrow$ $\frac{1}{a} L^{-1}\left\{\frac{a}{(s^2-a^2)}\right\} = \sinh at = F(t)$

$a\, L^{-1}\left\{\frac{d}{ds}(s^2-a^2)^{-1}\right\} = -tF(t)$

$a\, L^{-1}\left\{\frac{-2s}{(s^2-a^2)^2}\right\} = -t\sinh at$

$+2a\, L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\} = t\sinh at$

$$L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\} = \frac{t\sinh at}{2a}$$

(c) $L^{-1}\left\{\tan^{-1}\frac{2}{s}\right\}$

$\rightarrow$ $f(s) = \tan^{-1} 2/s$

$$f'(s) = \frac{1}{1+(2/s)^2} \cdot \frac{-2}{s^2}$$

$$= \frac{s^2}{(s^2+4)} \cdot \frac{-2}{s^2} = \frac{-2}{s^2+4}$$

$$L^{-1}\{f'(s)\} = L^{-1}\left\{\frac{-2}{s^2+4}\right\}$$

$$= -\sin 2t$$

$$-t\, L^{-1}\{f(s)\} = -\sin 2t$$

$$L^{-1}\{f(s)\} = \frac{\sin 2t}{t}$$

$$L^{-1}\{\tan^{-1}(2/s)\} = \frac{\sin 2t}{t}$$

(d) $L^{-1}\left\{\log\left(1+\frac{a^2}{s^2}\right)\right\}$

$\Rightarrow$ $f(s) = \log\left(1+\frac{a^2}{s^2}\right)$

$$f'(s) = \left(\frac{s^2}{s^2+a^2}\right)\left(\frac{-2a^2}{s^3}\right) = \frac{-2a^2}{s(s^2+a^2)}$$

$$L^{-1}\{f'(s)\} = L^{-1}\left\{\frac{-2a^2}{s(s^2+a^2)}\right\} \quad \text{——— (i)}$$

now, $-2L^{-1}\left\{\frac{a^2}{s^2+a^2}\right\} = -2\sin at$

$$-2L^{-1}\left\{\frac{1}{s}\left(\frac{a^2}{s^2+a^2}\right)\right\} = \int^t -2\sin au\, du$$

$$= -2\left[-\cos au\right]_0^t$$

$$= 2(\cos at - 1) \quad \text{——— (ii)}$$

Putting (ii) in (i)

$$L^{-1}\{f'(s)\} = 2(\cos at - 1)$$

$$-t\, L^{-1}\{f(s)\} = 2(\cos at - 1)$$

$$L^{-1}\{f(s)\} = \frac{2(1-\cos at)}{t}$$

$$L^{-1}\{\log(1+a^2/s^2)\} = \frac{2(1-\cos at)}{t}$$

(3 e) $L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}$

$\Rightarrow f(s) = \log(s+a) - \log(s+b)$

$f'(s) = \frac{1}{s+a} - \frac{1}{s+b}$

$L^{-1}\{f'(s)\} = L^{-1}\left\{\frac{1}{s+a}\right\} - L^{-1}\left\{\frac{1}{s+b}\right\}$

$L^{-1}\{f'(s)\} = e^{-at} - e^{-bt}$

$-t\, L^{-1}\{f(s)\} = e^{-at} - e^{-bt}$

$L^{-1}\{f(s)\} = \frac{e^{-bt} - e^{-at}}{t}$

1) Find the inverse of Laplace Theorem of the following by convolution theorem.

a) $\frac{1}{(s^2+1)(s^2+9)}$

→ Using Convolution Theorem,

$L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{\sin 3t}{3}$

By theorem

$$L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \int_0^t \frac{\sin 3u}{3} \cdot \sin(t-u)\, du$$

$$= \int_0^t \frac{\sin 3u(\sin t \cos u - \cos t \sin u)}{3}\, du$$

$$= \frac{1}{3}\left[\frac{\sin t}{2}\int_0^t 2\sin 3u \cos u\, du - \frac{\cos t}{2}\int_0^t 2\sin 3u \sin u\, du\right]$$

$$= \frac{1}{3}\left[\frac{\sin t}{2}\int_0^t (\sin 4u + \sin 2u)\, du - \frac{\cos t}{2}\int_0^t (\cos 2u - \cos 4u)\, du\right]$$

$$= \frac{1}{3}\left[\frac{\sin t}{2}\left[-\frac{\cos 4u}{4} - \frac{\cos 2u}{2}\right]_0^t - \frac{\cos t}{2}\left[\frac{\sin 2u}{2} - \frac{\sin 4u}{4}\right]_0^t\right]$$

$$= \frac{1}{3}\left[\frac{\sin t}{2}\left(-\frac{\cos 4t}{4} - \frac{\cos 2t}{2} + \frac{1}{4} + \frac{1}{2}\right) - \frac{\cos t}{2}\left(\frac{\sin 2t}{2} - \frac{\sin 4t}{4}\right)\right]$$

$$= \frac{1}{3}\left[\frac{-\sin t \cos 4t}{8} - \frac{\sin t \cos 2t}{4} + \frac{3\sin t}{8} - \frac{\sin 2t \cos t}{4} + \frac{\sin 4t \cos t}{8}\right]$$

(c) $\frac{s}{(s^2+9)^2}$

$\rightarrow \quad L^{-1}\left\{\frac{s}{(s^2+9)}\right\} = \cos 3t$ ——— (i)

$L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{\sin 3t}{3}$ ——— (ii)

From Convolution theorem

$$L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = \int_0^t \frac{\cos 3u \, \sin 3(t-u)}{3} \, du$$

$$= \frac{1}{6}\int_0^t 2\cos 3u \, \sin 3(t-u) \, du$$

$$= \frac{1}{6}\int_0^t [\sin 3t + \sin(3t-6u)] \, du$$

$$= \frac{1}{6}\left\{ [u\sin 3t]_0^t + \left[\frac{\cos(3t-6u)}{6}\right]_0^t \right\}$$

$$= \frac{1}{6}\left[t\sin 3t + \frac{\cos(-3t)}{6} - \frac{\cos 3t}{6} \right]$$

$$= \frac{t\sin 3t}{6}$$

(9) Solve the following Differential Eqn using Laplace Transform :

(a) $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4e^{2t}, \quad y(0) = -3, \; y'(0) = 5$

$\rightarrow \quad y'' - 3y' + 2y = 4e^{2t}$

$L\{y'' - 3y' + 2y\} = L\{4e^{2t}\}$

$L\{y''\} - 3L\{y'\} + 2L\{y\} = \frac{4}{p-2}$

$p^2 y(p) - 3p - 5 - 3py(p) - 9 + 2y(p) = \frac{4}{p-2}$

$y(p)(p^2 - 3p + 2) = \frac{4}{p-2} + 14 - 3p$

$$y(p) = \frac{-3p^2 + 20p - 24}{(p-2)^2(p-1)}$$

$$\frac{-3p^2+20p-24}{(p-2)^2(p-1)} = \frac{A}{p-2} + \frac{B}{(p-2)^2} + \frac{C}{(p-1)}$$

$$= A(p-2)(p-1) + B(p-1) + C(p-2)$$

putting $p=2$,

$-3(4) + 40 - 24 = B$

$\therefore B = 4$

Putting $p=1$

$-3 + 20 - 24 = -C$

$C = 7$

Putting $p=0$,

$A = 4$

$$L^{-1}\{y(p)\} = y(t) = L^{-1}\left\{\frac{4}{p-2} + \frac{4}{(p-2)^2} - \frac{7}{p-1}\right\}$$

$$= 4e^{2t} + 4e^{2t}t - 7e^{t}$$

b) $\frac{d^2y}{dt^2} + 9y = 1$, $y(0) = 1$, $y(\pi/2) = -1$.

$\rightarrow$ $y'' + 9y = 1$

$L\{y''\} + 9L\{y\} = L\{1\}$

$p^2 y(p) - \cancel{p} - y'(0) + 9y(p) = \frac{1}{p}$

let $y'(0) = K$

$p^2 y(p) - p - K + 9y(p) = \frac{1}{p}$

$y(p)(p^2+9) = p + K + \frac{1}{p}$

$$y(p) = \frac{p + K + \frac{1}{p}}{p^2+9} = \frac{p}{p^2+9} + \frac{K}{p^2+9} + \frac{1}{p(p^2+9)}$$

$$L^{-1}\{y(p)\} = y(t) = L^{-1}\left\{\frac{p}{p^2+9}\right\} + L^{-1}\left\{\frac{K}{p^2+9}\right\} + L^{-1}\left\{\frac{1}{p(p^2+9)}\right\}$$

$$= \cos 3t + K\frac{\sin 3t}{3} + L^{-1}\left\{\frac{1}{p(p^2+9)}\right\}$$

$$L^{-1}\left\{\frac{1}{p(p^2+9)}\right\} = \int_0^t \frac{\sin 3u}{3}\,du = \left[\frac{-\cos 3u}{9}\right]_0^t = \frac{1}{9}(1-\cos 3t)$$

$$y(t) = \cos 3t + K\frac{\sin 3t}{3} + \frac{1}{9}(1-\cos 3t)$$

at $t=\frac{\pi}{2}$, $y(\frac{\pi}{2})=-1$

$$-1 = 0 - \frac{k}{3} + \frac{1}{9}$$

$$\frac{1}{3k} = \frac{1/10}{9}$$

$$k = \frac{10}{3}$$

$\rightarrow$ $\therefore y(t) = \cos 3t + \frac{10}{9}\sin 3t + \frac{1}{3}(1-\cos t)$

(c) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = t\cos t$, $y(0) = y'(0) = 0$

$\rightarrow$ $y'' - 2y' - 3y = t\cos t$

$L\{y''\} - 2L\{y'\} - 3L\{y\} - L\{t\cos t\}$

$p^2 y(p) \cancel{-} - 2py(p) = -\frac{d}{dp}\left(\frac{p}{p^2+1}\right) = \frac{p^2-1}{(p^2+1)^2}$

$$y(p)\left[p^2-2p-3\right] = \frac{p^2-1}{(p^2+1)^2}$$

$$y(p) = \frac{(p-1)(p+1)}{(p^2+1)^2(p^2-2p-3)} = \frac{(p-1)\cancel{(p+1)}}{(p^2+1)^2\cancel{(p+1)}(p-3)}$$

$$y(p) = \frac{(p-1)}{(p^2+1)^2(p-3)}$$

$$L^{-1}\{y(p)\} = y(t) = L^{-1}\left\{\frac{(p-1)}{(p^2+1)^2(p-3)}\right\} = L^{-1}\left\{\frac{p-3+2}{(p-3)(p^2+1)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} + 2L^{-1}\left\{\frac{1}{(p-3)(p^2+1)^2}\right\}$$

for $L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\}$

$\rightarrow$ Using convolution theorem,

$L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin \cancel{t}$

$$L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} = \int_0^t \sin u \sin(t-u)\,du = \frac{1}{2}\int_0^t 2\sin u \sin(t-u)\,du$$

$$\cancel{= \frac{1}{2}\int_0^t \{\sin (\ldots) + \sin(\ldots)\}\,du}$$

$$= \frac{1}{2}\int_0^t \{\cos(t-2u) - \cos t\}\,du$$

$$= \frac{1}{2}\left[\frac{\sin(t-2u)}{-2} - \sin t\right]_0^t$$

$$= \frac{\sin t}{2} - \frac{t\cos t}{2}$$

$$\left\{\frac{2}{(p^2+1)^2(p-3)}\right\}$$

$$L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} = \frac{\sin t}{2} - \frac{t\cos 2t}{2}$$

$$L^{-1}\left\{\frac{2}{(p^2+1)^2}\right\} = \sin t - t\cos 2t$$

$$L^{-1}\left\{\frac{1}{p-3}\right\} = e^{3t}$$

Using Convolution Theorem,

$$L^{-1}\left\{\frac{2}{(p-3)(p^2+1)^2}\right\} = \int_0^t (\sin u - u\cos u)\, e^{3(t-u)}\, du$$

$$= e^{3t}\left[\int_0^t \sin u\, e^{-3u}\, du - \int_0^t u\cos u\, e^{-3u}\, du\right]$$

$$= e^{3t}\left[\frac{-3}{10}\sin t\, e^{-3t} - \frac{1}{10}(\cos t\, e^{-3t} - 1) - \frac{9}{8}\left[\frac{-2}{27}(1-e^{-3t}) + \frac{1}{9}e^{-3t} t\sin t + \left(\frac{-1}{3}\right)t\cos t\, e^{-3t}\right]\right]$$

$$= e^{3t}\left[\frac{-3}{10}\sin t\, e^{-3t} - \frac{\cos t\, e^{-3t}}{10} + \frac{1}{10} + \frac{1}{12}(e^{-3t}-1) - \frac{1}{8}e^{-3t} t\sin t + \frac{3}{8}t\cos t\, e^{-3t}\right]$$

$$= \frac{-3}{10}\sin t - \frac{\cos t}{10} + \frac{e^{3t}}{10} + \frac{1}{12}(1-e^{3t}) - \frac{1}{8}t\sin t + \frac{3}{8}t\cos t$$

Putting the values in eq ①,

$$y(t) = \frac{\sin t}{2} - \frac{t\cos t}{2} - \frac{3}{10}\sin t - \frac{\cos t}{10} + \frac{e^{3t}}{10} + \frac{1}{12}(1-e^{3t}) - \frac{1}{8}t\sin t + \frac{3}{8}t\cos t$$

$$= \frac{\sin t}{5} - \frac{t\cos t}{8} - \frac{\cos t}{10} + \frac{e^{3t}}{10} + \frac{1}{12}(1-e^{3t}) - \frac{1}{8}t\sin t$$

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(7) $y''(t) + y(t) = 8\cos t$, $y(0) = 1$, $y'(0) = -1$

$\rightarrow L\{y''(t)\} + L\{y(t)\} = L\{8\cos t\}$

$p^2 y(p) - p + 1 + y(p) = \frac{8p}{p^2+1}$

$y(p) = \frac{p}{p^2+1} - \frac{1}{p^2+1} + \frac{8p}{(p^2+1)^2}$

$L^{-1}\{y(p)\} = L^{-1}\left\{\frac{p}{p^2+1}\right\} - L^{-1}\left\{\frac{1}{p^2+1}\right\} + 8L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\}$

$= \cos t - \sin t + 8L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\}$

now, $L^{-1}\left\{\frac{8p}{(p^2+1)^2}\right\}$

$L\left\{\frac{1}{p^2+1}\right\} = \sin t$

$L\left\{\frac{8p}{p^2+1}\right\} = 8\cos t$

From convolution theorem,

$L^{-1}\left\{\frac{8p}{(p^2+1)^2}\right\} = 8\int_0^t \sin u \cos(t-u)\, du$

$= 4\int_0^t 2\sin u \cos(t-u)\, du$

$= 4\int_0^t [\sin t + \sin(2u-t)]\, du$

$= 4\left[-\cos t - \frac{\cos(2u-t)}{2}\right]_0^t$

$= 4t\sin t$

$\therefore y(t) = 4t\sin t + \cos t - \sin t$