Laplace Transform

$$\boxed{L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt.}$$

Laplace transform is also a function of s

Formulae

1. $L(1) = \frac{1}{s}$
2. $L(t^n) = \frac{n!}{s^{n+1}}$
3. $L(t) = \frac{1}{s^2}$
4. $L(e^{at}) = \frac{1}{s-a}$
5. $L(\sin at) = \frac{a}{s^2+a^2}$
6. $L(\cos at) = \frac{s}{s^2+a^2}$
7. $L(\sinh at) = \frac{a}{s^2-a^2}$

   or $\sinh z = \frac{e^z - e^{-z}}{2}$
8. $L(\cosh at) = \frac{s}{s^2-a^2}$

   or $\cosh z = \frac{e^z + e^{-z}}{2}$

Linear property

$L\{c_1F_1(t) + c_2F_2(t)\}$
$= c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}.$

Shifting property

1. If $L\{F(t)\} = f(s)$
   then $L\{e^{at}F(t)\} = f(s-a)$
2. If $L\{F(t)\} = f(s)$ and
   $G(T) = F(t-a)$, $t>a$
   $= 0$, $t<a$
   then $L\{G(T)\} = e^{-sa}\cdot f(s)$

Change of scale property

1. If $L\{F(t)\} = f(s)$,
   then $L\{F(at)\} = \frac{1}{a}\cdot f\left(\frac{s}{a}\right)$

$\boxed{\log e = 1}$

$\boxed{\log \infty = \infty}$

$\boxed{\begin{array}{l}\log 0 = -\infty \\ \log 1 = 0\end{array}}$

Laplace transformations on derivatives

1. theorem 1: (on first order derivative)
   If $L\{F(t)\} = f(s)$ and if
   (i) $F(t)$ is continous on $[0,N]$
   (ii) there exist some real no. M and v such that $|F(t)| < Me^{vt}$
   (iii) $F'(t)$ exists and sectionally continous on $[0,N]$, then

$$L\{F'(t)\} = s f(s) - F(0)$$

② Theorem 2 (On double derivation)

If $L\{F(t)\} = f(s)$ and

(i) $F(t)$ and $F'(t)$ are continous on $[0, N]$

(ii) there exist some real no. M and v such that

$|F(t)| < Me^{vt}$ and $|F'(t)| < Me^{vt}$

(iii) $F''(t)$ exists and sectionally continous on $[0, N]$.

then,

$$L\{F''(t)\} = s^2 f(s) - s F(0) - F'(0)$$

③ Theorem 3 (On nth derivative)

If $L\{F(t)\} = f(s)$ and

(i) $F(t), F'(t) \ldots F^{n-1}(t)$ is cont on $[0, N]$.

(ii) $|F(t)|, |F'(t)|, \ldots |F^{n-1}(t)| < Me^{vt}$, for $t > N$

(iii) $F^{(n)}(t)$ exist and sectionally cont on $[0, N]$

then

$$L\{F^{(n)}(t)\} = s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) \ldots - s F^{(n-2)}(0) - F^{(n-1)}(0).$$

Laplace transform on Integrals

If $L\{F(t)\} = f(s)$, then,

$$L\left\{\int_0^t F(u)\,du\right\} = \frac{1}{s} f(s).$$

Multipication by t^n

If $L\{F(t)\} = f(s)$ then,

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s). \quad , \quad n = +\text{ve I}.$$

Division by t

If $L\{F(t)\} = f(s)$, then $L\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} f(u)\,du$, provided $\lim_{t \to 0} \frac{F(t)}{t}$ exist finitely.

Laplace Transformation of Periodic function

Let $F(t)$ be a periodic function, ~~then~~ of period T,

then $$L\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\,dt.$$

Laplace transformation on unit step function

unit step function

$u(t-a) = 1, \; t>a$
$\quad\quad\;\; = 0, \; t<a.$

(1) Theorem 1.

$$F(t) = F_1(t), \quad t<a$$
$$\quad\;\; = F_2(t), \quad t>a.$$

Then $$F(t) = F_1(t) + \{F_2(t) - F_1(t)\}\, u(t-a).$$

(2) Theorem 2:

$$F(t) = F_1(t), \quad t<a_1$$
$$\quad\;\; = F_2(t), \quad a_1<t<a_2$$
$$\quad\;\; = F_3(t), \quad a_2<t.$$

$$F(t) = F_1(t) + \{F_2(t) - F_1(t)\}\, u(t-a_1) + \{F_3(t) - F_2(t)\}\, u(t-a_2)$$

(3) Theorem 3:

If $u(t-a)$ is a step function, then
$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

(4) Theorem 4:

Let $L\{F(t)\} = f(s)$ and $u(t-a)$ be a step funct.

Then $L\{F(t-a)\, u(t-a)\} = e^{-as} f(s).$
$$= e^{-as} L\{F(t)\}.$$

Inverse Laplace Transforms

$$L\{F(t)\} = f(s)$$

$$\text{then} \quad L^{-1}(f(s)) = F(t)$$

Formula

1. $L^{-1}\{\frac{1}{s}\} = 1$
2. $L^{-1}\{\frac{1}{s^2}\} = t$
3. $L^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{n!}$
4. $L^{-1}\{\frac{1}{s-a}\} = e^{at}$
5. $L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{\sin at}{a}$
6. $L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$
7. $L^{-1}\{\frac{1}{s^2-a^2}\} = \frac{\sinh at}{a}$
8. $L^{-1}\{\frac{s}{s^2-a^2}\} = \cosh at$

Lerch's theorem

$F(t)$ is sectionally continous on $[0,N]$ for each N^+ and if there exists a real constant $M>0$ and such that for all $t>N$, $|F(t)| < Me^{vt}$ for some v, then.

$$L^{-1}\{f(s)\} = F(t) \text{ is unique.}$$

In dis chapter all L^{-1} is unique.

Linear Property

$$L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\}$$

Shifting Property

1. $L^{-1}\{f(s-a)\} = e^{at} L^{-1}\{f(s)\}$ or $L^{-1}\{f(s-a)\} = e^{at}F(t)$

2. If $L^{-1}\{f(s)\} = F(t)$

   then $L^{-1}\{e^{-as}f(s)\} = F(t-a), \quad t>a$

   $= 0, \quad t<a$

Change of scale property

$$L^{-1}\{f(as)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$$

Inverse Laplace Transform on Derivative

(1) $L^{-1}\{f'(s)\} = -t\, L^{-1}\{f(s)\} = -t\, F(t)$

where $f'(s) = \frac{d}{ds}\{f(s)\}$.

(2) $L^{-1}\{f^{n}(s)\} = (-1)^n t^n F(t)$

$= (-1)^n t^n L^{-1}\{f(s)\}$.

Multiplication by s^n

(1) If $L^{-1}\{f(s)\} = F(t)$ and $F(0) = 0$.

then $L^{-1}\{s f(s)\} = F'(t)$.

(2) If $L^{-1}\{f(s)\} = F(t)$ and $F(0) = F'(0) = F^{n-1}(0) = 0$.

then $L^{-1}\{s^n f(s)\} = F^{(n)}(t)$

Division by s

If $L^{-1}\{f(s)\} = F(t)$

then $L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\, du$.

Inverse Laplace Transform on Integrals

If $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\left\{\int_s^\infty f(u)\,du\right\} = \frac{F(t)}{t}$.

Convolution Property of ILT

Let $F(t)$ and $G(t)$ be 2 integrable function.

$F * G = \int_0^t F(u)\, G(t-u)\, du.$

Theorem (1) $F * G = G * F$.

② If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(T)$

then $L^{-1}\{f(s) \cdot g(s)\} = F * G$

$$= \int_0^t F(u)\, G(t-u)\, du$$