PART TWO

§ 1. Compton Effect

When a monochromatic beam of X-ray is scattered by a loosely bound electron, the scattered radiation in a particular direction contains two components, one having a lower frequency or longer wavelength called modified radiation and other having same frequency or same wavelength as that of the incident radiation called unmodified radiation . This effect of scattering of radiation is known as Compton effect.
The difference between the wavelengths of the modified and unmodified radiation is called Compton shift.
The angle between the direction of incident and scattered radiation is called `scattering angle'.

§ 2. Experimental arrangement of compton shift

Compton Effect can be studied using experimental arrangement as shown in following fig.

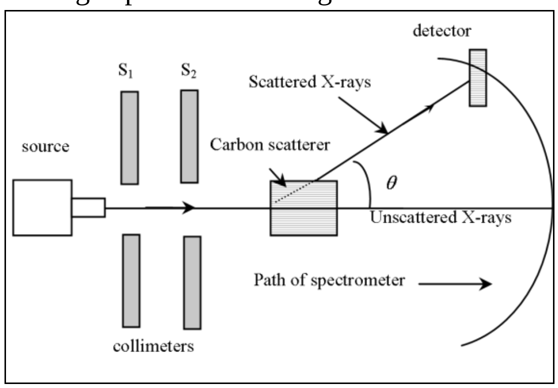


Figure : Experimental setup for compton effect

Collimated, monochromatic X-ray beam after passing through S_1 and S_2 is allowed to fall on carbon scatterer `C'.

1. Carbon block scatters X-rays in different directions.
2. The intensity and wavelength of X- rays scattered through different angles θ were measured fig. shown below. Following graphs show that these peak to peak distance $(\Delta\lambda)$ is found to be greater at higher scattering angles.

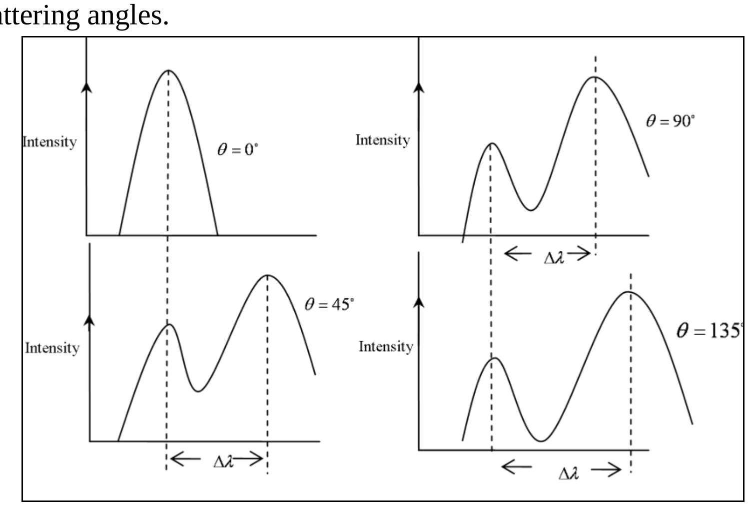


Figure : Experimental result of compton effect

Results

For each value of θ there are distinct intensity peaks.

1. One peak corresponding to the wavelength same as the wavelength of incident wavelength and another with higher value of wavelength.
2. This change of wavelength i.e., Compton shift increases with θ.

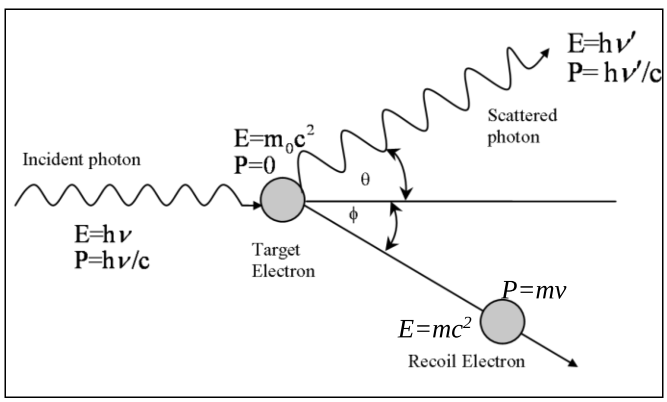
§ 3. Calculation of compton shift

In Compton effect,

1. Compton consider the Einstein quantum theory of radiation. The X-ray of frequency 'ν' consists of large number of photons with energy hν and momentum $\frac{h\nu}{c}$.
2. The collision between the high energy X-ray photon and electron is elastic and relativistic. The energy and momentum carried by scattered X-ray photon and recoil electron is governed by the laws of energy and momentum respectively.

§ 3.1 When X-ray is scattered by a loosely bound electron of the light element

The electron in the scattering block is assumed to be **free** and stationary. Because the energy of the electron (~ 10 eV) is very small as compared to the energy of the incident X-ray photon (~ 10-50 KeV). Thus,



According to the law of conservation of energy we can write $(K.E)_{electron} = h\nu - h\nu'$ …(1)

$$h\nu + m_0c^2 = h\nu' + mc^2 \qquad \rightarrow mc^2 - m_0c^2 = h\nu - h\nu'$$

According to the law of conservation of momentum we can proceed as follows.

Taking momentum **along horizontal** $\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + p_e\cos\phi$

$$\text{or,} \quad \frac{h\nu}{c} - \frac{h\nu'}{c}\cos\theta = p_e\cos\phi \quad \text{.......(2)}$$

again ***along vertical direction*** $0 = \frac{h\nu'}{c}\sin\theta - p_e\sin\phi$

$$\text{or,} \quad \frac{h\nu'}{c}\sin\theta = p_e\sin\phi \quad \text{........(3)}$$

Squaring and adding of (2) and (3) we have $p_e^2 = \left(\frac{h\nu}{c} - \frac{h\nu'}{c}\cos\theta\right)^2 + \left(\frac{h\nu'}{c}\sin\theta\right)^2$

Or, $p_e^2 c^2 = h^2\nu^2 + h^2\nu'^2 - 2h^2\nu\nu'\cos\theta$ ....(4)

From the expression of relativistic energy of electron we have

$E^2 = p_e^2 c^2 + m_0^2 c^4$ or, $\left[(K.E)_{electron} + m_0 c^2\right]^2 = p_e^2 c^2 + m_0^2 c^4$

Putting the value of $(K.E)_{electron}$ from (1) we have $\left[(h\nu - h\nu') + m_0 c^2\right]^2 = p_e^2 c^2 + m_0^2 c^4$

Or, $(h\nu - h\nu')^2 + m_0^2 c^4 + 2(h\nu - h\nu')m_0 c^2 = p_e^2 c^2 + m_0^2 c^4$

Or, $(h\nu - h\nu')^2 + 2(h\nu - h\nu')m_0 c^2 = p_e^2 c^2$ ....(5)

Equating (4) and (5) we have $h^2\nu^2 + h^2\nu'^2 - 2h^2\nu\nu'\cos\theta = h^2\nu^2 + h^2\nu'^2 - 2h^2\nu\nu' + 2(h\nu - h\nu')m_0 c^2$

Or, $2h^2\nu\nu' - 2h^2\nu\nu'\cos\theta = 2(h\nu - h\nu')m_0 c^2$

or, $2m_0 hc\left(\frac{\nu - \nu'}{\nu\nu'}\right) = \frac{2h^2}{c^2}(1 - \cos\theta)$

or, $\frac{\nu - \nu'}{\nu\nu'}c = \frac{h}{m_0 c}(1 - \cos\theta)$

or, $\frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{m_0 c}(1 - \cos\theta)$

$$\lambda' - \lambda = \frac{b}{m_0 c}(1 - \cos\theta) \quad or, \quad \Delta\lambda = \frac{h}{m_0 c}(1 - \cos\theta)$$

This is the expression for Compton shift.

Here $\Delta\lambda$ is the change in wave length of the incident radiation and it is called as compton's shift.

$$\lambda_c = \frac{h}{m_0 c} = \frac{6.625\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}} = 0.0243\overset{\circ}{A} = 2.43\times10^{-12} \text{ meter}$$

λ_c is a constant which is known as ***Compton wavelength***

§ 3.1.1 Observation from compton shift

1. Compton shift $\Delta\lambda$ is independent of the wavelength of incident radiation
2. Wavelength shift increases with the angle of scattering and it reaches its maximum at $\theta = \pi$
3. For $\theta = 0$; shift i.e., $\Delta\lambda$=0 (minimum shift), we get one peak that means we are getting an unmodified radiation
4. For $\theta = \frac{\pi}{2}$; shift i.e., $\Delta\lambda = \frac{h}{m_0 c}$
5. For $\theta = \pi$; shift i.e., $\Delta\lambda = \frac{2h}{m_0 c}$ (maximum shift)

Comparing (4) and (5) of the above one can understand that as θ increases $\Delta\lambda$ increases. So we are getting one peak which is different from the incident radiation peak that means we are getting ***modified line***.

§ 3.2 When X-ray is scattered by a tightly bound electron or the atom of the light element

If photon collides with a tightly bound electron or the atom of the element, the electron is not detach from the atom because the energy of the electron is comparible to the energy of the incident X-ray photon. So, the whole atom is involved in the collision process and mass of the electron m_0 in the equation $\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)$ is replaced by M, mass of the atom which is several thousand times greater than the mass of electron.

For carbon scatterer $M = 12\times 1837\times m_0$. Now $\Delta\lambda = \frac{h}{Mc}(1-\cos\theta) \rightarrow \Delta\lambda = \frac{h}{12\times 1837\times m_0 c}(1-\cos\theta)$

$$\Delta\lambda = \left(\frac{1}{12\times 1837}\right)\lambda_c(1-\cos\theta) = 1.1\times 10^{-16}(1-\cos\theta) \text{ meter}$$

As compared to the Compton wavelength, this is 10^{-4} times.
Such a small change in wavelength cannot be resolved by Bragg's spectrometer and hence such photons are recorded ***without change of wavelength*** for all values of angle of scatterening. This explains the presence of ***unmodified component*** of X-ray in the study of Compton effect.

Thus when collision of photon takes place with free electron this gives rise to ***modified radiation*** and when it collides with bound electron this gives rise to ***unmodified radiation***.
In case of low atomic number elements the number of electrons in them is less and almost all electrons are loosely bound to the nucleus and are considered as free electron. Hence, when radiation is incident on these materials, there is collision between photon and free electron giving rise to ***modified radiation***.
But in case of heavy elements number of electrons is more hence the number of tightly bound electrons is more as compare to loosely bound electrons. Photon collides with these bound electrons give rise to ***unmodified radiation***.
Hence modified component of radiation is more intense in lower atomic number material whereas unmodified component is more intense in higher atomic number materials.

§ 4. Failure of classical theory to explain compton shift

According to classical theory, radiation is considered as electromagnetic waves. When energy is incident on scattering material, electrons of the material oscillate at the same frequency of incident radiation and an ocsillating charge particle (electron) radiates electromagnetic radiation uniformly in all direction with frequency as that of the frequency of incident radiation. Thus classically an electron should radiate uniformly in all directions and scattering radiation should not depend on `θ'.
Thus classical theory fails to explain existence of the modified wavelength and dependence of Compton shift on the scattering angle.

§ 5. Direction of the recoil electron

When radiation of frequency ν is incident on scattering material, scattered radiation consisting of modified and unmodified radiation is obtained and electron undergoes recoil.
Let ϕ be the angle between direction of recoil electron with the direction of incident radiation.
θ be the angle between scattered photon and direction of incident radiation.
If we apply law of conservation of momentum to this collision along and perpendicular direction we get,

$$0=\frac{h\nu'}{c}\sin\theta - mv\sin\phi \qquad \left(Along \perp^{r}\ direction\right)$$

$$\sin\phi = h\nu'\sin\theta \qquad \ldots(21)$$

and $$\frac{h\nu}{c}+0=\frac{h\nu'}{c}\cos\theta + mv\cos\phi \qquad \left(Along\ horizontal\ direction\right)$$

$$mvc\cos\phi = h\nu - h\nu'\cos\theta \qquad \ldots(22)$$

Dividing (21) by (22), we get,

$$\frac{\sin\phi}{\cos\phi}=\tan\phi=\frac{h\nu'\sin\theta}{h\nu-h\nu'\cos\theta}$$

$$\tan\phi=\frac{\frac{hc}{\lambda'}\sin\theta}{\frac{hc}{\lambda}-\frac{hc}{\lambda'}\cos\theta}=\frac{\sin\theta}{\frac{\lambda'}{\lambda}-\cos\theta}$$

Or, $$\tan\phi=\frac{\sin\theta}{\frac{h\nu}{m_0c^2}\left(1-\cos\theta\right)+\left(1-\cos\theta\right)}$$

Or, $$\tan\phi=\frac{\sin\theta}{\left(\frac{\frac{h}{m_0c}\left(1-\cos\theta\right)+\lambda}{\lambda}\right)-\cos\theta}$$

Or, $$\tan\phi=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\left(\frac{h\nu}{m_0c^2}+1\right)\left(1-\cos\theta\right)}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\left(\frac{h\nu}{m_0c^2}+1\right)\left(2\sin^2\frac{\theta}{2}\right)}=\frac{\cot\frac{\theta}{2}}{\left(1+\frac{h\nu}{m_0c^2}\right)}$$

Thus, $$\boxed{\tan\phi=\frac{\cot\frac{\theta}{2}}{\left(1+\frac{h\nu}{m_0c^2}\right)}}$$

Note:

When $\theta=0$ so that $\tan\phi=\infty$ i.e. $\phi=\frac{\pi}{2}$

When $\theta=\pi$ so that $\tan\phi=0$ i.e. $\phi=0$

This is the maximum value of the angle at which recoil electrons are ejected.

§ 6. Kinetic Energy of the recoil electron interms of scattering angle

Kinetic energy of recoil electron is given by

$$\left(m-m_0\right)c^2=\left(h\nu-h\nu'\right)$$

$$E_r=\frac{hc}{\lambda}-\frac{hc}{\lambda'}$$

where $h\nu \rightarrow$ Energy of incident photon.

$h\nu^{'} \rightarrow$ Energy of scattered photon.

$E_r \rightarrow$ Kinetic energy of the electron

So, $E_r = \frac{hc}{\lambda} - \frac{hc}{\lambda + \Delta\lambda}$ where $\Delta\ \lambda$ is the Compton shift

$$E_r = hc\left[\frac{\lambda + \Delta\lambda - \lambda}{\lambda(\lambda + \Delta\lambda)}\right]$$

$$E_r = hc\left[\frac{\frac{h}{m_0 c}(1-\cos\theta)}{\left(\lambda + \frac{h}{m_0 c}(1-\cos\theta)\right)\lambda}\right]$$

$$E_r = \frac{hc}{\lambda}\left[\frac{(1-\cos\theta)\frac{h}{m_0 c}}{\lambda\left(1 + \frac{h\nu}{m_0 c^2}(1-\cos\theta)\right)}\right]$$

$$E_r = \frac{(h\nu)\left(\frac{h\nu}{m_0 c^2}\right)(1-\cos\theta)}{1 + \frac{h\nu}{m_0 c^2}(1-\cos\theta)}$$

Hence $E_r = \frac{h\nu\left[\frac{h\nu}{m_0 c^2}(1-\cos\theta)\right]}{\left[1 + \frac{h\nu}{m_0 c^2}(1-\cos\theta)\right]} \Rightarrow$ K.E for recoil electron in terms of scattering angle.

Note:

Compton effect cannot be observed with visible light but can be observed due to X-rays

Let us consider a visible light has a wavelength of 5000 Å. Then energy of that visible light can be estimated out as

$$\frac{hc}{\lambda} = \frac{6.63\times10^{-34}\times3\times10^{8}}{5000\times10^{-10}\times1.6\times10^{-19}} \approx 2.5\,\text{eV}$$

The binding energy of an electron inside an atom is more than 10eV. So when visible light will fall on a target, it cannot liberate electrons of the scatterer. Whereas the energy of a x-ray beam or gamma ray is well above 10eV which can easily liberate electron from a metal surface. So we cannot observe Compton effect with visible light.

§ 7. Pair production

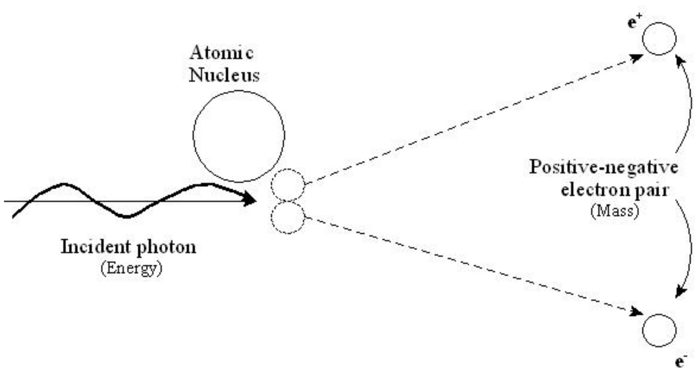
Pair production is a phenomenon of nature where energy is converted to mass. It provides a conceptual framework for how our internal world gets translated into the physical reality we experience. The phenomenon of pair production can be view two different ways. One way is as a particle and antiparticle and the other is as a particle and a hole.

In this phenomenon a wave of energy or a photon (a packet of energy) interacts with a heavy nucleus to form an electron - positron pair. Pair production is observed to occur in nature when a photon or an energy wave packet, of greater than 1.02 million electron volts passes near the electric field of a large atom such as lead, uranium or other heavy material with a large number of protons (around an atomic number of 80 or 90).

In the pair production process, as diagramed in the figure below labeled "Pair Production - Energy Conversion to Mass," the photon is literally split into an electron and its anti-particle, called a positron.

No conservation principles are violated when an electron-positron pair is created near an atomic nucleus. The sum of the charges of the electron (-e) and of the positron (+e) is zero, as is the charge of the photon.

Both have a rest mass energy equivalent of 0.511 million electron volts. Pair production requires photon energy of at least 1.02 MeV.



If the original packet of energy is greater than 1.02 million electron volts, any energy above 1.02 million electron volts is split between the kinetic energy of motion of the two particles. The corresponding maximum photon wavelength is 1.2 pm (electromagnetic waves with such wavelength is called gamma rays). The nucleus absorbs only a negligible fraction of the photon energy because of its enormous mass.

$$h\nu = E_{-} + E_{+} \rightarrow h\nu = \left(K.E + m_0c^2\right)_{-} + \left(K.E + m_0c^2\right)_{+}$$

Positron is produced with a slightly greater kinetic energy than the electron because the coulomb interaction of the pair with the +ve charged nucleus leads to a acceleration of the positron and deceleration of electron

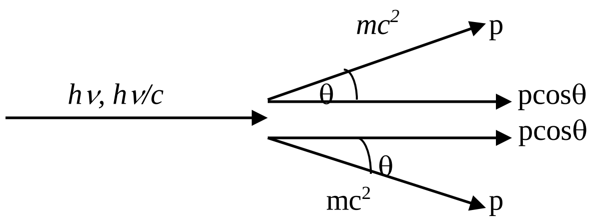
And the ***linear momentum*** is conserved with the help of the nucleus which carries away enough photon momentum for the process to occur.

The forward **momentum** of the original photon is also preserved in that both the electron and positron go off at an angle such that their total momentum and that of the recoiling nucleus equals the momentum of the original photon. It is important to note the both mass-energy and momentum are conserved in this interaction. What this means is that when observed from outside the process, everything that existed before the interaction was retained and exists afterwards but only in a different form. In the big picture nothing has changed. In the segments of the process there is a radical transformation and from mass point of view, **something is created from nothing**. There is creation of mass from seemingly nothingness.

The positron is the anti-particle of the electron and will be **annihilated** if it combines with an electron producing energy such that mass is converted back into energy. Since all electrons are equal, any electron can annihilate the positron.

Example Problem

Show that pair production cannot occur in empty space.



from conservation of energy--------- $h\nu = 2mc^2$

from conservation of momentum $\dfrac{h\nu}{c} = 2p\cos\theta$ or, $h\nu = 2pc\cos\theta$ or, $h\nu = 2mv\cos\theta$

or, $h\nu = 2mc^2\left(\dfrac{v}{c}\right)\cos\theta$ Since, $\left(v/c\right) < 1$; $\cos\theta \leq 1$ so, $h\nu < 2mc^2$

But conservation of energy requires that hν = $2mC^2$. Hence it is impossible for pair production to conserve both energy and momentum unless some other object is involved in the process to carry away part of the initial photon momentum.

ASSIGNMENT PROBLEM

1. Prove that it is impossible for a photon to give up all its energy and momentum to a free electron, so that photoelectric effect can take place only when photons strikes bound electrons.
2. A beam of X-rays is scattered by free electrons. At 45° from the beam direction the scattered X rays have a wavelength of 0.022 Å. What is the wavelength of the X rays in the direct beam? [**Ans: 0.015Å**]
3. An X rays photon whose initial frequency was 1.5 $\times 10^{19}$ sec^{-1} emerges from a collision with an electron with a frequency of 1.2 $\times 10^{19}$ sec^{-1}. How much kinetic was imparted to the electron? [**Ans: 12.43KeV**]
4. An x rays photon of initial frequency 3 $\times 10^{19}$ sec^{-1} collides with an electron and is scattered through 90°. Find its new frequency. [**Ans: $2.42\times10^{19}$$sec^{-1}$**]
5. Find the energy of an X- ray photon which can impart a maximum energy of 50 KeV to an electron. [**Ans: 0.141MeV**]
6. A monochromatic X-ray beam whose wavelength is 0.558Å is scattered through 46°. Find the wavelength of the scattered beam. [**Ans: 0.565Å**]
7. A 1,000 Watt radio transmitter operates at a frequency of 880 KHz. How many photons per second does it emit? [**Ans: 1.71×10^{30} photons/sec**]
8. Light from the sun arrives at the earth, an average of 1.5×10^{11} m away, at the rate of about 1,400 watts/m^2 of area perpendicular to the direction of the light. Assume that sunlight is monochromatic with a frequency of 5.0×10^{14} Hz. (a) how many photons fall per second on each square meter of the earth's surface directly facing the sun? (b) What is the power output of the sun in watts, and how many photons per second does it emits? (c) How many photons per m^3 are there near the earth?

[**Ans: 4.2×10^{21}photons/m^2; 4.0×10^{26}W,1.2×10^{45}photons/s; 5.6×10^{13} photons/m^3**]

9. Consider an X-ray beam, with λ=1.00Å, and also a γ-ray beam from a Cs^{137} sample, with $\lambda=1.88\times10^{-2}$ Å. If the radiation scattered from free electrons is viewed at 90° to the incident beam (a) what is the Compton wavelength shift in each case? (b) What kinetic energy is given to a recoiling electron in each case? (c) What percentage of the incident photon energy is lost in the collision in each case?

   [**Ans: 0.0242Å in each case, 0.295KeV & 378KeV, 2.4% & 57%]**

10. Show that maximum kinetic energy of recoiling electron is $K_{max} = \frac{2m_0 c^2 \lambda_C^2}{\lambda^2+2\lambda\lambda_C}$, where m_0, rest mass of electron, λ, wave length of the incident X ray beam and λ_C, the Compton wavelength.
11. Show that $\frac{\Delta E}{E}$, the fractional change in photon energy in the Compton effect, equal $(h\nu'/m_0C^2)(1-\cos\theta)$.
12. What fractional increase in wavelength leads a 75% loss of photon energy in a Compton collision?

   [**Ans: 300%]**

13. Through what angle must a 0.20 MeV photon be scattered by a free electron so that it loses 10% of its energy? [**Ans : 44.19°**]
14. What is the maximum possible kinetic energy of a recoiling Compton electron in terms of the incident photon energy $h\nu$ and the electron's rest mass energy m_0C^2?
15. A particular pair is produced such that the positron is at rest and the electron has a kinetic energy of 1.0 MeV moving in the direction of flight of the pair-producing photon. (a) Neglecting the energy transferred to the nucleus of the nearby atom, find the energy of the incident photon. (b) What percentage of the photon's momentum is transferred to the nucleus? [**Ans: 2.02 MeV, 29.7%**]
16. Assume that an electron-positron pair is formed by a photon having the threshold energy for the process. (a) Calculate the momentum transferred to the nucleus in the process. (b) Assume the nucleus to be that of a lead atom and compute the kinetic energy of the recoil nucleus. Are we justified in neglecting this energy compared to the threshold energy assumed above? [**Ans: 5.44×10^{-22} kg-m/Sec, 2.71 eV, yes**]

PART THREE

§ 1. Wave Particle Duality

The phenomenon like interference, and diffraction and polarization can only be explained only by treating light as a wave whereas the phenomenon like photoelectric effect Compton effect and pair production its particle nature. Therefore, it is worth while to assume that light behaves like wave and particle both depending on the situation. This termed as dual nature of light.

These two different natures are not found simultaneously.

According to Planck's quantum theory of radiation, energy of photon can be represented as

$$h\nu = \frac{hc}{\lambda} \qquad \dots(1)$$

According to energy momentum relation for a photon $E = pc$ (as photon has zero rest mass) …(2)

From relation (1) and (2) we get, $\frac{hc}{\lambda} = pc \Rightarrow \lambda = \frac{h}{p}$

The above expression represents the relation of the particle (p) and wave (λ) nature of a photon.

Louis de-Broglie in 1924 extended the wave particle dualism of radiation to all physical particles. According to him, this wave-particle duality is not only valid for photon particle, it is also valid for any moving particle. " **a moving particle behaves in certain ways as though it has a wave nature**". A wave is associated with a moving particle.

§ 2. de-Broglie hypothesis

According to de-Broglie hypothesis, the motion of a particle is compared by a wave and the wave which is associated with moving particle is called de-Broglie wave or matter wave.

The wavelength of matter wave is given by $\lambda = \dfrac{h}{p} = \dfrac{h}{mv}$

where h -- planck's constant.
p -- momentum of particle of mass `m' and velocity v.

§ 2.1 de-Broglie wavelength interms of kinetic energy

§ 2.1.1 de-Broglie wavelength for non relativistic case

The de-Broglie wavelength associated with material particle of mass `$m$' and moving with velocity `v' is given by $\lambda = \dfrac{h}{mv} = \dfrac{h}{p}$

Now kinetic energy of particle $E_k = \dfrac{1}{2}mv^2 = \dfrac{p^2}{2m} \quad \Rightarrow p = \sqrt{2mE_k}$ or, $\lambda = \dfrac{h}{\sqrt{2mE_k}}$

§ 2.1.2 de-Broglie wavelength for relativistic case

Using the energy momentum relation we have $E^2 = p^2c^2 + m_0^2c^4 \;\rightarrow \left(E_k + m_0c^2\right)^2 = p^2c^2 + m_0^2c^4$

Or, $E_k^2 + 2E_k m_0 c^2 = p^2c^2 \rightarrow p = \dfrac{\sqrt{E_k^2 + 2E_k m_0 c^2}}{c}$

Again from the $\lambda = \dfrac{h}{mv} = \dfrac{h}{p}$, putting the value of $\lambda = \dfrac{h}{mv} = \dfrac{hc}{\sqrt{E_k^2 + 2E_k m_0 c^2}}$

§ 2.2 de-Broglie wavelength associated with charged particle

Let m be the mass of particle, `$q$' be the charge on the particle which is accelerated through `V' v / m. Then, gain in kinetic energy =Loss is potential energy $\dfrac{1}{2}mv^2 = qV \Rightarrow v = \sqrt{\dfrac{2qV}{m}}$

According to de-Broglie hypothesis, $\lambda = \dfrac{h}{mv} = \dfrac{h}{m\sqrt{\dfrac{2qV}{m}}} = \dfrac{h}{\sqrt{2qmV}}$

As we know $h = 6.625\times10^{-34}\ Js$ and for electron , $q = 1.602\times10^{-19}\ C$ and $m = 9.1\times10^{-31}\ kg$. Substituting these value we get, de-Broglie wavelength associated with electron accelerated through `V' volts is given by $\lambda = \dfrac{6.625\times10^{-34}}{\sqrt{2\times1.602\times10^{-19}\times9.1\times10^{-31}V}}$

So, $\lambda = \frac{12.27}{\sqrt{V}} \mathring{A}$

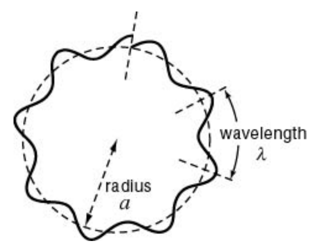
§ 3 Difference between E.M Wave and matter wave

a) Matter waves are produced whether the particle is charged or uncharged one. But electromagnetic waves are produced by the motion of charged particles.

§ 4 de-Broglie hypothesis regarding wave nature of matter leads to Bohr's quantisation for angular momentum

According to Bohr's atomic model, electron revolves round the nucleus in different orbits and only those orbits are possible for which the orbital angular momentum of the electron is equal to an integral multiple of $\frac{h}{2\pi}$ *i.e.,* $mvr = \frac{nh}{2\pi}$

where , h $\rightarrow$ Planck's constant. n $\rightarrow$ Integer ($n = 0, 1, 2, \cdots\cdots$)



On the basis of de-Broglie hypothesis, he proposed an atomic model in which the stationary orbits of the Bohr's model are retained, but he assumed that the electrons in various orbits behaves like wave.

According to him, stationary orbits are those in which orbital circumference is an integral multiple of de-Broglie wavelength λ and stationary orbits for an electron are those which contain the complete waves of electron. Thus $2\pi r = n\lambda$

or, $2\pi r = \frac{nh}{mv}$ or $mvr = \frac{nh}{2\pi} = n\left(\frac{h}{2\pi}\right)$ where , $n = 0, 1, 2, \cdots\cdots$

Which Bohr's quantization condition.

§ 5. Experimental verification of matter wave { Not for Exam}

Davisson-Germer and G.P Thomson' Experiment

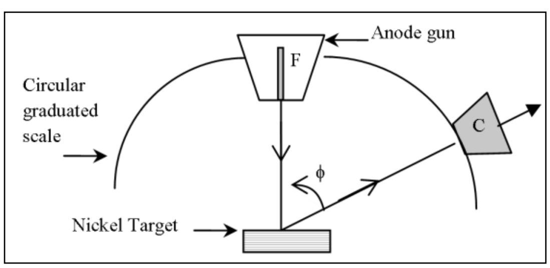
Davisson and Germer gave the experiment, which proves the validity of de-Broglie hypothesis regarding the wave nature of matter, in 1927. They succeeded in measuring the de-Broglie wavelength associated with slow electrons.

Davisson and Germer were studying the reflection of electrons from nickel & tungsten. They found that the reflection shows striking maxima and minima.

Experimental Arrangement

Experimental arrangement used by Davisson and Germer is as shown in fig. below. G is the electron gun in which electrons are produced and obtained in a fine beam. Electron gun consists of tungsten filament F, where electrons are emitted due to thermionic emission.

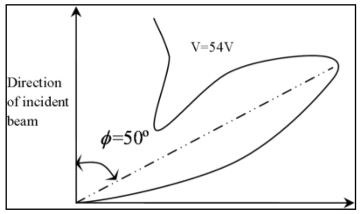
The anode in the gun accelerates these electrons. This accelerated and collimated beam of electrons is then allowed to fall on nickel target.



These electrons acting like waves are diffracted in different directions. The angular distribution is then

Figure 2:

measured by the electron detector C. `C' can be rotated on circular graduated scale between 20° to 90° to receive the reflected electrons.



For a given value of accelerating voltage,the intensity is plotted against θ. The scattering curve shows a peak in a particular direction, C for given value of potential difference. It is observed that the intensity of the scattered beam is maximum at 50° when the accelerating voltage is 54 volts. The results of the experiment are represented in the form of a polar graph as shown above. Theoretical value of $\lambda = \dfrac{h}{\sqrt{2meV}} = \dfrac{12.27}{\sqrt{54}} = 1.67$ Å

Evidently the angle of incidence relative to the crystal plane is $\theta = 65^\circ$. From X-rays measurements the inter planar distance for Ni crystal is found to be 0.909 Å i.e., d value.

Using Bragg's equation $2d\sin\theta = n\lambda$

i.e., $2\times 0.909 sin(65) = 1.\lambda$ *i.e.,* $\lambda = 1.65$Å

The experimental observed results are explained considering that electrons are diffracted by nickel crystal. Diffraction is the phenomenon which can be explained on the basis of wave nature. Thus this experiment gives experimental proof of de-Broglie hypothesis.

6. Nature of the wave associated with the moving particle

According to de-Broglie, a wave is associated with a moving particle and we consider a single wave is associated with a moving particle. So we will check whether the velocity of the single wave train is equal to the particle velocity or not?

6.1 Phase velocity

Let a plane harmonic wave of frequency ν and wavelength λ travelling in positive x-direction is represented as $y = a\sin(\omega t - kx)$, the speed of propagation of this wave will be the speed associated with a point for which phase $(\omega t - kx)$ remains constant i.e $(\omega t - kx)$ = constant

or, $\omega - k\dfrac{dx}{dt} = 0 \quad \rightarrow \dfrac{dx}{dt} = \dfrac{\omega}{k}$ or, $v = \dfrac{\omega}{k}$. We define this velocity as v_p, the phase velocity of the single wave train.

The phase velocity v_p of a monochromatic wave is the velocity with which a crest or trough is propagated in a medium keeping the phase constant. So, $v_p = \dfrac{\omega}{k}$ We further know that

$$E = h\nu = \frac{h}{2\pi}2\pi\nu = \hbar\omega \quad \text{and} \quad p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k \qquad [\text{ where } \hbar = \frac{h}{2\pi}]$$

a) For non-relativistic case:

From the expression of phase velocity we have $v_p = \dfrac{\omega}{k} = \dfrac{E}{p} = \left(\dfrac{\frac{1}{2}mv^2}{mv}\right) = \dfrac{v}{2}$ where v is the particle velocity.

b) For relativistic case:

$$v_p = \frac{\omega}{k} = \frac{E}{p} = \frac{mc^2}{mv} = \frac{c^2}{v}$$

In both non relativistic and relativistic cases we see, ***phase velocity ≠ particle velocity***.
So it is justified that a single wave is not associated with a moving particle.

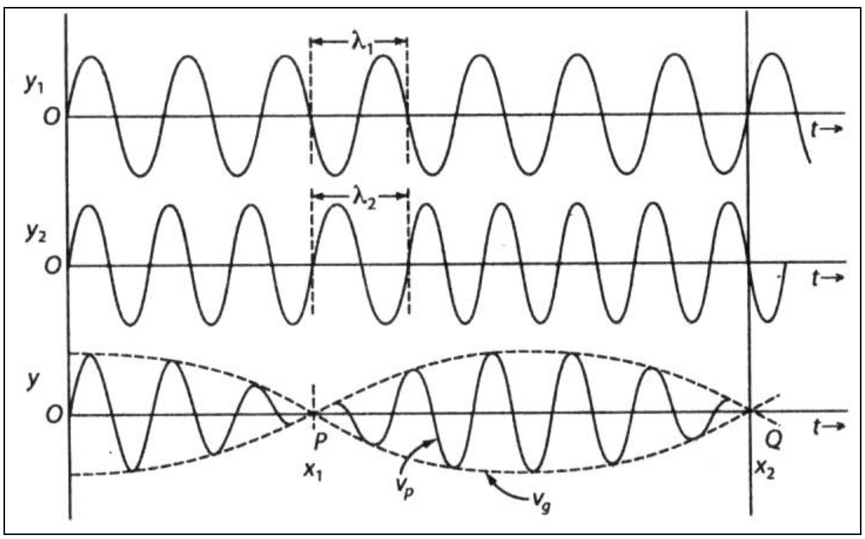
Justification of non existence of single wave associated with a moving particle:

A single wave has infinite extent in space. It has non-zero amplitude over a region of infinite dimension in space. But a material particle can be localized, it must be found within a region of finite dimension in space. So it would have no physical meaning to associate a single wave with a moving particle.

6.2 Group velocity

Now instead of a single wave train we consider a group of waves or wave packet is associated with a moving particle. Similarly we will check the validity of this velocity of this packet of wave associated with a moving particle.

Let us consider two waves of same amplitude, but of slightly different frequencies superpose to form a wave packet.
Let us consider two such waves are represented as $y_1 = a\sin(\omega_1 t - k_1 x)$ and $y_2 = a\sin(\omega_2 t - k_2 x)$
The resultant displacement of the wave packet obtained due to their superposition



$y = y_1 + y_2$ or, $y = a\sin(\omega_1 t - k_1 x) + a\sin(\omega_2 t - k_2 x)$

Or, $y = 2a\sin\left\{\left(\frac{\omega_1+\omega_2}{2}\right)t - \left(\frac{k_1+k_2}{2}\right)x\right\}\cos\left\{\left(\frac{\omega_1-\omega_2}{2}\right)t - \left(\frac{k_1-k_2}{2}\right)\right\}x$

Or, $y = 2a\cos\left\{\left(\frac{\omega_1-\omega_2}{2}\right)t - \left(\frac{k_1-k_2}{2}\right)x\right\}\sin\left\{\left(\frac{\omega_1+\omega_2}{2}\right)t - \left(\frac{k_1+k_2}{2}\right)x\right\}$

Or, $y = 2\cos\left\{\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right\}\sin\left\{\omega' t - k' x\right\}$

Where $\Delta\omega = \omega_1 - \omega_2$, $\Delta k = k_1 - k_2$ and $\omega' = \frac{\omega_1 + \omega_2}{2}$, $k' = \frac{k_1 + k_2}{2}$

So the amplitude of the resultant wave packet will be $A = 2a\cos\left\{\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right\}$

The resultant wave have angular frequency $\omega' = \frac{\omega_1 + \omega_2}{2}$ and wave vector $k' = \frac{k_1 + k_2}{2}$ and it's amplitude varies with cosine term having angular frequency $\frac{\Delta\omega}{2}$ and wave vector $\frac{\Delta k}{2}$.The velocity of the envelope or the modulated wave can be written as

$$v_g = \frac{\frac{\Delta\omega}{2}}{\frac{\Delta k}{2}} \quad or,\ v_g = \frac{\Delta\omega}{\Delta k} \rightarrow v_g = \underset{k\rightarrow 0}{Lt}\frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

We define this velocity as v_g , the group velocity of the wave packet So $v_g = \frac{d\omega}{dk}$

The group velocity is the velocity with which the envelope of the modulated pattern travels in a medium.

Now we will check the validity of the group of waves associated with a moving particle

a) ***For non-relativistic case***:

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp} \quad \text{and } E = \frac{p^2}{2m} \quad \text{so , } \frac{dE}{dp} = \frac{p}{m} \rightarrow v_g = \frac{p}{m} = v$$

So we get $v_g = v$ i.e., the particle velocity is equal to the group velocity in non-relativistic particle.

b) ***For relativistic case***:

Here $E = \sqrt{p^2c^2 + m_0^2c^4}$ so $E^2 = p^2c^2 + m_0^2c^4$ or, $2E\frac{dE}{dp} = 2pc^2$

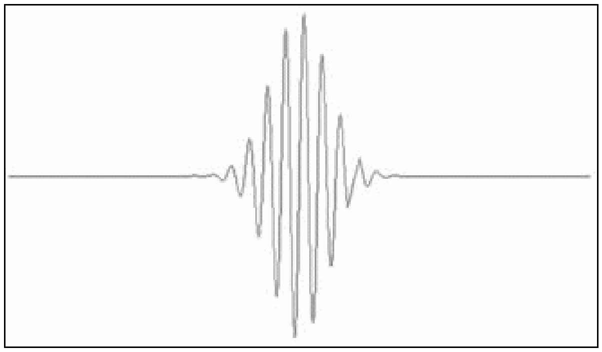
or, $$v_g = \frac{dE}{dp} = \frac{pc^2}{E} = \frac{mvc^2}{mc^2} = v$$

Again, the group velocity is equal to the particle velocity in relativistic particle.

Moreove the wave packet has non zero amplitude in finite dimension in space and the particle must be present in this region. So the concept of wave packet corresponding to particle velocity is justified.

Note:

If the number of superposed waves is increased, the regions with prominent humps will be narrower while the intervening regions of weaker disturbances will become broader. In the limit, if an infinite number of waves of continuously varying frequencies extending over a finite range is superposed we get a single hump within a narrow region with no disturbances at any other point. This is known as wave packet.



6.2.1 Relation between Group velocity and Phase Velocity

Group velocity of the wave packet is given by $v_g = \frac{d\omega}{dK}$

From the relation, $v_p = \frac{\omega}{k}$ we have $dv_p = \frac{d\omega}{k} - \frac{\omega}{k^2}dk$

or, $k\frac{dv_p}{dk} = \frac{d\omega}{dk} - \frac{\omega}{k}$ or, $k\frac{dv_p}{dk} = v_g - v_p$

or, $\boxed{v_g = v_p + k\frac{dv_p}{dk}}$ and as $k = \frac{2\pi}{\lambda}$ $or, \frac{dk}{d\lambda} = -\frac{2\pi}{\lambda^2}$

Hence $k\frac{dv_p}{dk} = \frac{2\pi}{\lambda}\frac{dv_p}{d\lambda}\frac{d\lambda}{dk} = -\lambda\frac{dv_p}{d\lambda}$

So, $$\boxed{v_g = v_p - \lambda\frac{dv_p}{d\lambda}}$$

§ 8 Heisenberg's Uncertainty Relation

a) Position and momentum uncertainty

In reference to the figure of article **6.2** we can write $x_2 - x_1 = \Delta x$. From the figure it is evident that

$\Delta x = \frac{\lambda_m}{2}$ and $\lambda_m = \frac{2\pi}{k_m}$

where λ_m is the wavelength of the modulated wave and k_m is that of the wave vector . Again from the expression of the amplitude of the modulated wave i.e, $A = 2a\cos\left\{\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right\}$ we have

$k_m = \frac{\Delta k}{2}$. So $\lambda_m = \frac{2\pi}{k_m} = \frac{2\pi}{\frac{\Delta k}{2}} = \frac{2\pi\times 2}{\Delta k}$. Putting the value of λ_m in $\Delta x = \frac{\lambda_m}{2}$ we have

$$\Delta x = \frac{2\pi\times 2}{\Delta k\times 2} = \frac{2\pi}{\Delta k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From $p = \hbar k$ we have $\Delta p = \hbar\Delta k \rightarrow \Delta k = \frac{\Delta p}{\hbar}$. Putting this Δk in equation (1) we have $\Delta x\Delta p = h$

But wave packet consists of large number of single wave . Using fourier transform considering these large number of waves it can be shown that $\Delta x \Delta p \geq \hbar$

Statement of uncertainty: *It is impossible to specify accurately and simultaneously the measurement of both momentum and posion of a moving quantum particle. The product of their (positon and momentum) uncertainty is equal to or greater than* $\hbar$. Mathematically $\Delta x \Delta p \geq \hbar$

b) ***Energy and time uncertainty***

For simultaneous measurement of energy and time we can write

$$\Delta E \Delta t \geq \frac{h}{2\pi}$$

where ΔE and Δt uncertainties in energy and time respectively.

§ 8.1 Significance of Heisenberg's uncertainty principle

a) Simultaneous determination of conjugate properties like x and p or E and t, with unlimited accuracy is impossible.
b) Dual nature of wave and particle is inherent like two faces of coin. Hence we can examine one aspect accurately at a time, either wave or particle nature but not both at the same time.

§ 9 Application of Heisenberg's uncertainty principle

Example 1:

By applying uncertainty principle explain non-existence of electrons in the atomic nucleus.

Answer:

We know Max K.E of an electron (β - particle) emitted by a radioactive nuclei $\approx$ 4 MeV

m_0 (rest mass of the electron) $= 9.11 \times 10^{-31}\ Kg$

$2r$ (diameter of nucleus) $\approx 2 \times 10^{-14}\ m$ or, $(\Delta x)_{\max} = 2 \times 10^{-14} m$

From $\Delta x \Delta p \geq \hbar$ or, $\Delta p = \frac{h}{2\pi} \times \frac{1}{\Delta x} = \frac{6.63 \times 10^{-34}}{2 \times 3.14} \times \frac{1}{2 \times 10^{-14}}$

$\therefore \Delta p = 5.278 \times 10^{-21} kgm/Sec$

$\therefore p_{\min} = 5.278 \times 10^{-21} kg\mathrm{m}/sec$

Now $E^2 = p^2c^2 + m_0^2c^4 \approx p^2c^2 = (5.278 \times 10^{-21})^2 \times (3.0 \times 10^8)^2$

Now $E = 5.28 \times 3 \times 10^{-13}\ J = 15.84 \times 10^{-13}\ J$

$E = \frac{1.5 \times 10^{-12}}{1.6 \times 10^{-19}} eV$ Hence $E = 5.875 \times 10^6\ eV = 9.875\ MeV$

Thus, if a free electron exists in the nucleus, it must have energy of the order of 9.0 M*e*V.

But the maximum Kinetic energy of a β-particle (equivalent electron emitted from a radioactive nuclei) is 4 MeV which is much less than theoretical value (discussed above). So a free electron can not be present inside the nucleus.

Example 2

Assume that an electron is inside a nucleus of radius 10^{-15} m. Calculate from the uncertainty principle the minimum kinetic energy of the electron.[Given, $h = 6.63\times10^{-34}\,J\text{-}s$ .Electron mass $= 9.1\times10^{-31}$ kg].

Answer:

According to Heisenberg's uncertainty principle we can write $\Delta x \Delta p_x \geq \frac{\hbar}{2} \geq \frac{h}{4\pi}$;

$$\Delta p_x \geq \frac{6.62\times10^{-34}}{4\times3.14\times2\times10^{-15}} = 0.2635\times10^{-21} \text{ Kg mS}^{-1}$$

Kinetic Energy = $E = \frac{p^2}{2m}$ *hence* $\Delta E = \frac{\Delta p^2}{2m} \geq \frac{\left(0.2635\times10^{-21}\right)^2}{2\times9.1\times10^{-31}}$

Or, $\Delta E \geq \frac{0.0694 \times10^{-42}}{2\times9.1\times10^{-31}} J \geq 0.0038\times10^{-12}\ J$

Or, $\Delta E \geq 38000\times10^{-19}$ J $\geq 38000eV \geq 38KeV$

Example 4:

What is the uncertainty in momentum of a proton when it is inside the nucleus? Is it consistent with binding energy of nuclear constituent?

Answer:

If a proton is inside the nucleus the uncertainty in momentum will be $\Delta p \geq \frac{\hbar}{\Delta x} = \frac{1.054\times10^{-34}}{2\times10^{-14}} = 5.3\times10^{-21} kg\ m/s$

The corresponding kinetic energy $T << m_0 c^2$

Hence $T = \frac{p^2}{2m} = \frac{5.3\times10^{-21}}{2\times1.67\times10^{-27}} = 52KeV$

The binding energy of the nuclei (proton-neutron) are of this order. So it is verified.

ASSIGNMENT PROBLEM

1. An electron and a photon each have a wavelength of 2.0Å. What are their (a) momenta (b) total energies? (c) Compare the kinetic energies of the electron and the photon.

[3.31×10^{-24} kg m/s, E_e =510037.62 eV and E_{photon} =6200eV, $(E_K)_{electron}$=37.62 eV and $(E_K)_{photon}$=6200 eV]

2. A non relativistic particle is moving three times as fast as an electron. The ratio of their de Broglie wavelengths, particle to electron is 1.1813×10^{-4}. Identify the particle. ***[Neutron]***

3. (a) Show that the de-Broglie wavelength of a particle, of charge 'e' rest mass m_0, moving at relativistic speeds is given as a function of the accelerating potential V as

$$\lambda = \frac{h}{\sqrt{2m_0 eV}}\ (1 + \frac{eV}{2m_0 C^2})^{-\frac{1}{2}}$$

4. Find the kinetic energy of an electron whose de-Broglie wavelength is the same as that of a 100-KeV X-ray. ***[9.74 KeV]***

5. An electron and a proton have the same kinetic energy. Compare the wavelengths and the phase and group velocities of their de-Broglie waves. ***[$\lambda_e > \lambda_p$, both the particle have the same phase & group velocity]***

6. An electron and a proton have the same velocity. Compare the wavelengths and the phase and group velocities of their de-Broglie waves. ***[$\lambda_e > \lambda_p$, both the particle have the same phase & group velocity]***

7. An electron and a proton have the same wavelength. Prove that the energy of the electron is greater.

8. Calculate de-Broglie wavelength of a neutron or an electron of energy 1MeV. Compare it with the wavelength of electromagnetic radiation for which the photon has the same energy.
[$\lambda_n = 2.87\times10^{-14}$m, $\lambda_e = 10.11\times10^{-13}$m, $\lambda_{photon} = 12.43\times10^{-13}$m]

9. Show that if a particle moves with velocity 0.707C where C is the velocity of light in vacuum, then its de-Broglie wavelength and Compton wavelength become equal. ***[0.707C=C/√2]***

10. The phase velocity of ripples on a liquid surface is $\sqrt{\frac{2\pi S}{\lambda\rho}}$, where S is the surface tension and ρ the density of the liquid. Find the group velocity of the ripples. ***[$v_g = (3/2)v_p$]***

11. The phase velocity of ocean waves is $\sqrt{g\lambda/2\pi}$, where g is the acceleration of gravity. Find the group velocity of ocean waves. ***[$v_g = ½\ v_p$]***

12. Find the phase and group velocities of the de-Broglie waves of an electron whose speed is 0.900C.

13. Show that the group velocity of a wave is given by $v_g = \frac{dv}{d\left(\frac{1}{\lambda}\right)}$

14. A) show that the phase velocity of the de-Broglie waves of a particle of mass m and de-Broglie wavelength λ is given by $v_p = C\sqrt{1 + (m_0 C\lambda/h)^2}$

b) Compare the phase and group velocities of an electron whose de Broglie wavelength is exactly 1×10^{-13}m. ***[$v_p = 1.01C$, $v_g = 0.99C$]***

14. An electron of mass 9.1×10^{-31} kg has a speed of 1km/sec with an accuracy of 0.05%. Calculate the uncertainty with which the position of the electron can be located. ***[0.232×10^{-3}m]***

15. An electron is confined to a box of length 10^{-9} m. Calculate the minimum uncertainty in its velocity. ***[0.116×10^{6} m/s]***

16. The life time of an excited state of an atom is 10^{-8} sec. Calculate the minimum uncertainty in the determination of energy of the excited state. ***[6.25×10^{-8} eV]***

17. Show that if the uncertainty in the location of a particle is about equal to its de Broglie wavelength, than the uncertainty in its velocity is about equal to one sixth its velocity?

18. The speed of a bullet of mass 50 gm is measured to be 300 m/s with an uncertainty of 0.01%. With what accuracy can we locate the position of the bullet if it is measured simultaneously with its speed? ***[7.07×10^{-32}m]***

19. The angular momentum of the electron in the hydrogen atom is $2\hbar$ with an error of 5 %. Show that its angular position in a circular orbit around the nucleus cannot be specified at all.