5.1 IMPROPER INTEGRALS

5.1.1. Basic ideas of improper integral

Consider the integral

$$\int_a^b f(x)dx \quad \cdots \quad (1)$$

This integral is called an improper integral or an infinite integral when either a or b or both are infinite or $f(x)$ is unbounded in $a \le x \le b$.

It can be proved that if a function $f(x)$ is unbounded in [a, b], then there exists at least one point c in [a, b] such that in the neighbourhood of c, $f(x)$ in not bounded. This point c is called point of *infinite discontinuity* of the function $f(x)$ or the point of singularity of the integral (1).

Illustrations.

(i) The integrals

$$\int_0^{\infty} \frac{dx}{x^2}, \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

are improper integrals

(ii) The integrals $\int_0^1 \frac{dx}{x}$ is improper as $\frac{1}{x} \to \infty$ at $x \to 0$

(iii) The integral $\int_0^2 \frac{dx}{\sqrt{2-x}}$ is improper as $\frac{1}{\sqrt{2-x}} \to \infty$ at $x \to 2$

(iv) The integral $\int_{-3}^{3} \frac{dx}{x(1+x)}$ is improper as $\frac{1}{x(1+x)} \to \infty$ at $x \to 0, -1$ and $0, -1$ lie between -3 and 3.

There are two types of improper integrals. The illustration (i) are the examples of first type improper integral. So the general form of first type improper integral is

$$\int_{-\infty}^{\infty} f(x)dx$$

The illustration (ii) (iii) and (iv) are examples of second type improper integral. Thus the integral $\int_a^b f(x)$ is called second type improper integral when $f(x)$ is not bounded at $x = c, a \le c \le b$.

5.1.2. Evaluation of improper integrals of first type

Type I. Let the function $f(x)$ be bounded and integrable in $a \le x \le X$ for all $X > a$. Then the improper integral $\int_a^\infty f(x)dx$ is defined as

$\lim_{X\to\infty} \int_a^X f(x)dx$, provided the limit exist.

Hence, $\int_a^\infty f(x)dx = \lim_{X\to\infty} \int_a^X f(x)dx$

Illustration.

Consider the improper integral $\int_0^\infty \frac{dx}{1+x^2}$.

Then $\int_0^\infty \frac{dx}{1+x^2}$

$= \lim_{X\to\infty} \int_0^X \frac{dx}{1+x^2}$

$= \lim_{X\to\infty} \left[\tan^{-1} x\right]_0^X$

$= \lim_{X\to\infty} \left[\tan^{-1} X - \tan^{-1} 0\right]$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

Type II. Let $f(x)$ be bounded and integrable in $X \le x \le b$ for every $X < b$ and $\lim_{X\to-\infty} \int_X^b f(x)dx$ exists finitely. Then the improper integral

$$\int_{-\infty}^{b} f(x)dx$$

is defined as

$$\int_{-\infty}^{b} f(x)dx = \lim_{X\to-\infty} \int_X^b f(x)dx.$$

Illustration.

Consider the improper integral $\int_{-\infty}^0 e^x dx$

Then $\int_{-\infty}^0 e^x dx$

$= \lim_{X\to-\infty} \int_X^0 e^x dx$

$= \lim_{X\to-\infty} \left[e^x\right]_X^0$

$= \lim_{X\to-\infty} \left[e^0 - e^X\right]$

$= 1 - 0$ $[\because e^X \to 0$ or $X \to -\infty]$

$= 1$

Type III. Consider the improper integral

$$\int_{-\infty}^{\infty} f(x)dx$$

Let c be any number. Then we can write

$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$

$= \lim_{X_1\to-\infty} \int_{X_1}^{c} f(x)dx + \lim_{X_2\to\infty} \int_{c}^{X_2} f(x)dx$, provided limits exist finitely.

Illustration.

Consider the improper integral

$$\int_{-\infty}^{\infty} \frac{x}{x^4+1}\,dx$$

Let c be any number. Then

$$\int_{-\infty}^{\infty} \frac{x}{x^4+1}\,dx$$

$$= \int_{-\infty}^{c} \frac{x}{x^4+1}\,dx + \int_{c}^{\infty} \frac{x}{x^4+1}\,dx$$

$$= \lim_{X_1\to-\infty} \int_{X_1}^{c} \frac{x}{x^4+1}\,dx + \lim_{X_2\to\infty} \int_{c}^{X_2} \frac{x}{x^4+1}\,dx$$

$$= \lim_{X_1\to-\infty} \frac{1}{2}\int_{X_1}^{c} \frac{d(x^2)}{(x^2)^2+1} + \lim_{X_2\to\infty} \frac{1}{2}\int_{c}^{X_2} \frac{d(x^2)}{(x^2)^2+1}$$

$$= \frac{1}{2}\lim_{X_1\to-\infty}\left[\tan^{-1}(x^2)\right]_{X_1}^{c} + \frac{1}{2}\lim_{X_2\to-\infty}\left[\tan^{-1}(x^2)\right]_{c}^{X_2}$$

$$= \frac{1}{2}\lim_{X_1\to-\infty}\left(\tan^{-1}c^2 - \tan^{-1}X_1^2\right) + \frac{1}{2}\lim_{X_2\to-\infty}\left(\tan^{-1}X_2^2 - \tan^{-1}c^2\right)$$

$$= \frac{1}{2}\left(\tan^{-1}c^2 - \frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}c^2\right)$$

$$= 0.$$

5.1.3. Evaluation of improper integrals of second type

Type I. Let a be the only point of infinite discontinuity of the function $f(x)$. Then the improper integral $\int_a^b f(x)dx$ is defined as

$\lim_{\varepsilon\to0+} \int_{a+\varepsilon}^{b} f(x)dx$, provided limit exist.

Hence $\int_a^b f(x)dx = \lim_{\varepsilon\to0+} \int_{a+\varepsilon}^{b} f(x)dx$

Illustration.

Consider the improper integral

$$\int_0^4 \frac{dx}{\sqrt{x}}.$$

Here 0 is the only point of infinite discontinuity

Thus $\int_0^4 \frac{dx}{\sqrt{x}}$

$$= \lim_{\varepsilon\to0+} \int_{0+\varepsilon}^{4} \frac{dx}{\sqrt{x}}$$

$$= \lim_{\varepsilon\to0+}\left[2\sqrt{x}\right]_{\varepsilon}^{4}$$

$$= \lim_{\varepsilon\to0+}\left(2\sqrt{4} - 2\sqrt{\varepsilon}\right)$$

$$= 4 - 0$$

$$= 4.$$

Type II Let b be the only point of infinite discontinuity of the function $f(x)$. Then the improper integral

$$\int_a^b f(x)dx$$

is defined as

$\lim_{\varepsilon\to0+} \int_a^{b-\varepsilon} f(x)dx$, provided limit exist.

Thus $\int_a^b f(x)dx = \lim_{\varepsilon\to0+} \int_a^{b-\varepsilon} f(x)dx.$

Illustration.

Consider the improper integral

$$\int_1^2 \frac{dx}{\sqrt{2-x}}$$

Here 2 is the only singularity of the integral

$$\therefore \int_1^2 \frac{dx}{\sqrt{2-x}} = \lim_{\varepsilon\to 0+} \int_1^{2-\varepsilon} \frac{dx}{\sqrt{2-x}}$$

$$= \lim_{\varepsilon\to 0+}\left[-2\sqrt{2-x}\right]_1^{2-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+}\left[-2\sqrt{\varepsilon}+2\right]$$

$$= 0+2$$

$$= 2$$

Type III. Let both the end points a and b be the only points of infinite discontinuity of the function $f(x)$. We take any point c such that $a<c<b$. Then the improper integral

$$\int_a^b f(x)dx$$

can be written as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

and by type I and type II, we have

$$\int_a^b f(x)dx = \lim_{\varepsilon_1\to 0+} \int_{a+\varepsilon_1}^{c} f(x)dx + \lim_{\varepsilon_2\to 0+} \int_c^{b-\varepsilon_2} f(x)dx,$$

provided both limits exist.

Illustration.

Consider the improper integral

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

Here 0 and 1 are the only points of infinite discontinuity

Thus we can write

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

$$= \int_0^{1/2} \frac{dx}{\sqrt{x(1-x)}} + \int_{1/2}^{1} \frac{dx}{\sqrt{x(1-x)}}$$

$$= \lim_{\varepsilon_1\to 0+} \int_{0+\varepsilon_1}^{1/2} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}} + \lim_{\varepsilon_2\to 0+} \int_{1/2}^{1-\varepsilon_2} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}}$$

$$= \lim_{\varepsilon_1\to 0+}\left[\sin^{-1}\frac{x-1/2}{1/2}\right]_{0+\varepsilon_1}^{1/2} + \lim_{\varepsilon_2\to 0+}\left[\sin^{-1}\frac{x-1/2}{1/2}\right]_{1/2}^{1-\varepsilon_2}$$

$$= \lim_{\varepsilon_1\to 0+}\left[\sin^{-1}(2x-1)\right]_{\varepsilon_1}^{1/2} + \lim_{\varepsilon_2\to 0+}\left[\sin^{-1}(2x-1)\right]_{1/2}^{1-\varepsilon_2}$$

$$= \lim_{\varepsilon_1\to 0+} -\sin^{-1}(2\varepsilon_1-1) + \lim_{\varepsilon_2\to 0+} \sin^{-1}(1-2\varepsilon_2)$$

$$= \sin^{-1}(1)+\sin^{-1}(1) = 2\cdot\frac{\pi}{2} = \pi$$

Type IV. Let c be the only point of infinite discontinuity of the function $f(x)$ in $[a, b]$ so that $a<c<b$. Then we break the integral $\int_a^b f(x)dx$ into two parts

$$\int_a^c f(x)dx \text{ and } \int_c^b f(x)dx$$

Then by type I and type II we have

$$\int_a^b f(x)dx$$

$$= \lim_{\varepsilon_1\to 0+} \int_a^{c-\varepsilon_1} f(x)dx + \lim_{\varepsilon_2\to 0+} \int_{c+\varepsilon_2}^{b} f(x)dx,$$ provided both limits exist

Note. When we take $\varepsilon_1 = \varepsilon_2$, then

$$\int_a^b f(x)dx = \lim_{\varepsilon_1 \to 0+}\left[\int_a^{c-\varepsilon_1} f(x)dx + \int_{c+\varepsilon_1}^{b} f(x)dx\right]$$

which is called the *Cauchy Principal Value* of the integral. Sometimes Cauchy Principal of the integral exist where as the general definition of the integral does not exist.

Illustration. Consider the integral

$$\int_{-1}^{1} \frac{dx}{x^3}$$ [WBUT 2014]

Here $f(x) = \frac{1}{x^3}$ has an infinite discontinuity at $x = 0$

$$\therefore \int_{-1}^{1} \frac{dx}{x^3} = \int_{-1}^{0} \frac{dx}{x^3} + \int_{0}^{1} \frac{dx}{x^3}$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{-1}^{0-\varepsilon_1} \frac{dx}{x^3} + \lim_{\varepsilon_2 \to 0+} \int_{0+\varepsilon_2}^{1} \frac{dx}{x^3}$$

$$= \lim_{\varepsilon_1 \to 0+}\left[-\frac{1}{2x^2}\right]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2 \to 0+}\left[-\frac{1}{2x^2}\right]_{\varepsilon_2}^{1}$$

$$= \lim_{\varepsilon_1 \to 0+}\left(\frac{1}{2} - \frac{1}{2\varepsilon_1^2}\right) + \lim_{\varepsilon_2 \to 0+}\left(-\frac{1}{2} + \frac{1}{2\varepsilon_2^2}\right).$$

As $\lim_{\varepsilon_1 \to 0+} \frac{1}{2\varepsilon_1^2}$ and $\lim_{\varepsilon_2 \to 0+} \frac{1}{2\varepsilon_2^2}$ do not exist, so the given integral

$$\int_{-1}^{1} \frac{dx}{x^3}$$

does not exist.

When $\varepsilon_1 = \varepsilon_2$, then

$$\int_{-1}^{1} \frac{dx}{x^3}$$

$$= \lim_{\varepsilon_1 \to 0+}\left(\frac{1}{2} - \frac{1}{2\varepsilon_1^2}\right) + \lim_{\varepsilon_1 \to 0+}\left(-\frac{1}{2} + \frac{1}{2\varepsilon_1^2}\right)$$

$$= \lim_{\varepsilon_1 \to 0+}\left(\frac{1}{2} - \frac{1}{2\varepsilon_1^2} - \frac{1}{2} + \frac{1}{2\varepsilon_1^2}\right)$$

$$= \lim_{\varepsilon_1 \to 0+} (0)$$

$$= 0.$$

Thus $\int_{-1}^{1} \frac{dx}{x^3}$ exists in the Cauchy Principal value sense but not in the general sense.

Note. When the appropriate limits exist finitely, an improper integral is said to be convergent. In otherwords, when the appropriate limits fail to exist or tend to infinity, an improper integral is said to be non-convergent.

5.1.4. Some standard improper integrals

A. Convergence of $\int_a^{\infty} \frac{dx}{x^n}, (a > 0)$

Here $\int_a^{\infty} \frac{dx}{x^n}$ is an improper integral of first kind.

$$\therefore \int_a^{\infty} \frac{dx}{x^n} = \lim_{X \to \infty} \int_a^{X} \frac{dx}{x^n}$$

$$= \lim_{X \to \infty}\left[\frac{x^{-n+1}}{-n+1}\right]_a^X$$

$$= \lim_{X\to\infty} \frac{1}{n-1}\left(\frac{1}{a^{n-1}} - \frac{1}{X^{n-1}}\right)$$

$$= \begin{cases} \dfrac{1}{(n-1)a^{n-1}} & \text{when } n > 1 \\ \infty & \text{when } n < 1 \end{cases}$$

But, when $n = 1$,

$$\int_a^\infty \frac{dx}{x^n} = \int_a^\infty \frac{dx}{x}$$

$$= \lim_{X\to\infty} \int_a^X \frac{dx}{x}$$

$$= \lim_{X\to\infty} [\log x]_a^X$$

$$= \lim_{X\to\infty} \log\frac{X}{a} = \infty$$

Hence the given improper integral in convergent when $n > 1$.

Illustration :

(i) The improper integral $\int_1^\infty \frac{dx}{\sqrt{x}}$ is not convergent, as $n = \frac{1}{2} < 1$

(ii) The improper integral $\int_2^\infty \frac{dx}{x^3}$ is convergent, as $n = 3 > 1$

(iii) The improper integral $\int_{1/3}^\infty \frac{dx}{x}$ is not convergent, as $n = 1$

B. Convergence of $\int_a^b \frac{dx}{(x-a)^n}, (n > 0)$

Here $x = a$ is the only point of infinite discontinuity of

$$f(x) = \frac{1}{(x-a)^n}$$

$$\therefore \int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n}$$

$$= \lim_{\varepsilon\to 0+} \left[\frac{(x-a)^{-n+1}}{-n+1}\right]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon\to 0+} \frac{1}{n-1}\left\{\frac{1}{\varepsilon^{n-1}} - \frac{1}{(b-a)^{n-1}}\right\}$$

$$= \begin{cases} -\dfrac{1}{(n-1)(b-a)^{n-1}} & \text{when } n < 1 \\ \infty & \text{when } n > 1 \end{cases}$$

Again, when $n = 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\varepsilon\to 0+} [\log(x-a)]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon\to 0+} \log\left(\frac{b-a}{\varepsilon}\right) = \infty$$

Thus the improper integral integral $\int_a^b \frac{dx}{(x-a)^n} (n > 0)$ is convergent only when $n < 1$.

Illustration :

(i) The improper integral $\int_2^3 \frac{dx}{(x-2)^{3/2}}$ is not convergent, as $n = \frac{3}{2} > 1$.

(ii) The improper integral $\int_1^2 \frac{dx}{\sqrt{x-1}}$ is convergent, as $n = \frac{1}{2} < 1$

(iii) The improper integral $\int_{-3}^3 \frac{dx}{x+3}$ is not convergent, as $n = 1$.

C. Convergence of $\int_a^b \frac{dx}{(b-x)^n}$, $(n > 0)$

Here $x = b$ in the only point of infinite discontinuity of $\frac{1}{(b-x)^n}$.

$$\therefore \quad \int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} \frac{dx}{(b-x)^n}$$

$$= \lim_{\varepsilon \to 0+} \left[-\frac{(b-x)^{-n+1}}{-n+1} \right]_a^{b-\varepsilon}$$

$$= \lim_{\varepsilon \to 0+} \frac{1}{n-1} \left[\frac{1}{\varepsilon^{n-1}} - \frac{1}{(b-a)^{n-1}} \right]$$

$$= \begin{cases} -\dfrac{1}{(n-1)(b-a)^{n-1}} & \text{when } n < 1 \\ \infty & \text{when } n > 1 \end{cases}$$

Again, when $n = 1$,

$$\int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} \frac{dx}{b-x}$$

$$= \lim_{\varepsilon \to 0+} [-\log(b-x)]_a^{b-\varepsilon}$$

$$= \lim_{\varepsilon \to 0+} \log\left(\frac{b-a}{\varepsilon}\right) = \infty$$

Thus the given improper integral $\int_a^b \frac{dx}{(b-x)^n}$, $(n > 0)$ is convergent only when $n < 1$.

Illustration :

(i) The improper integral $\int_1^2 \frac{dx}{\sqrt{2-x}}$ is convergent.

(ii) The improper integral $\int_{-1}^1 \frac{dx}{(1-x)^2}$ is not convergent.

(iii) The improper integral $\int_0^1 \frac{dx}{1-x}$ is not convergent.

5.1.5. Illustrative Examples

Ex. 1. *Examine the convergene of the improper integral*

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

If possible, evaluate the integral.

Here $x = 1$ is the only point of infinite discontinuity of $f(x) = \frac{1}{\sqrt{1-x^2}}$ in [0, 1].

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{\varepsilon \to 0+} \left[\sin^{-1} x\right]_0^{1-\varepsilon}$$

$$= \sin^{-1}(1-\varepsilon)$$

$$= \sin^{-1}(1)$$

$$= \pi/2$$

Thus the given improper integral converges with the value $\frac{\pi}{2}$.

Ex.2. *Examine the convergence of the improper integral*

$$\int_0^2 \frac{dx}{x(2-x)}$$

[WBUT 2012]

Let $f(x) = \frac{1}{x(2-x)}$

So $x = 0, 2$ are the infinite discontinuity of $f(x)$. Thus we can write the integral as

$$\int_0^2 \frac{dx}{x(2-x)}$$

$$= \int_0^1 \frac{dx}{x(2-x)} + \int_1^2 \frac{dx}{x(2-x)}$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{0+\varepsilon_1}^{1} \frac{1}{2}\left(\frac{1}{x} + \frac{1}{2-x}\right)dx + \lim_{\varepsilon_2 \to 0+} \int_1^{2-\varepsilon_2} \frac{1}{2}\left(\frac{1}{x} + \frac{1}{2-x}\right)dx$$

$$= \frac{1}{2} \lim_{\varepsilon_1 \to 0+} \left[\log|x| - \log|2-x|\right]_{\varepsilon_1}^{1} + \frac{1}{2} \lim_{\varepsilon_2 \to 0+} \left[\log(x) - \log(2-x)\right]_1^{2-\varepsilon_2}$$

$$= \frac{1}{2} \lim_{\varepsilon_1 \to 0+} \left\{-\log\left|\frac{\varepsilon_1}{2-\varepsilon_1}\right|\right\} + \frac{1}{2} \lim_{\varepsilon_2 \to 0+} \log\left|\frac{2-\varepsilon_2}{\varepsilon_2}\right|$$

Since $\lim_{\varepsilon_1 \to 0+} \log\left|\frac{\varepsilon_1}{2-\varepsilon_1}\right|$ and $\lim_{\varepsilon_2 \to 0+} \log\left|\frac{2-\varepsilon_2}{\varepsilon_2}\right|$

do not exist, so the given improper integral is not convergent.

Ex. 3. *Show that the improper integral*

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$$

is convergent. Hence find its value. [WBUT 2011]

The given integral can be written as

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$$

$$= \lim_{X \to \infty} \int_0^X \frac{dx}{(1+x)\sqrt{x}}$$

$$= \lim_{X \to \infty} \int_0^{\sqrt{X}} \frac{2udu}{(1+u^2)u}, \quad \text{putting } x = u^2 \text{ i.e. } dx = 2udu$$

$$= 2\lim_{X \to \infty} \left[\tan^{-1} u\right]_0^{\sqrt{X}}$$

$$= 2\lim_{X \to \infty} \left(\tan^{-1}\sqrt{X} - \tan^{-1} 0\right)$$

$$= 2(\pi/2 - 0)$$

$$= \pi.$$

Thus the given intregral is convergent and its value is π.

Ex. 4. *Evaluate the integral* $\int_{-\infty}^{\infty}\frac{dx}{x^2+1}$

The given improper integral can be written as

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+1}=\int_{-\infty}^{a}\frac{dx}{x^2+1}+\int_{a}^{\infty}\frac{dx}{x^2+1}$$

$$=\lim_{X_1\to-\infty}\int_{X_1}^{a}\frac{dx}{x^2+1}+\lim_{X_2\to-\infty}\int_{a}^{X_2}\frac{dx}{x^2+1}$$

$$=\lim_{X_1\to-\infty}\left[\tan^{-1}x\right]_{X_1}^{a}+\lim_{X_2\to-\infty}\left[\tan^{-1}x\right]_{a}^{X_2}$$

$$=\lim_{X_1\to-\infty}\left(\tan^{-1}a-\tan^{-1}X_1\right)+\lim_{X_2\to\infty}\left(\tan^{-1}X_2-\tan^{-1}a\right)$$

$$=\tan^{-1}a-(-\pi/2)+\pi/2-\tan^{-1}a=\pi$$

Ex. 5. *Show that* $\int_{-\infty}^{\infty}xe^{-x^2}dx=0$ [WBUT 2011]

The given improper integral can be written as

$$\int_{-\infty}^{\infty}xe^{-x^2}dx=0$$

$$=\int_{-\infty}^{a}xe^{-x^2}dx+\int_{a}^{\infty}xe^{-x^2}dx$$

$$=\lim_{X_1\to-\infty}\frac{1}{2}\int_{X_1}^{a}e^{-x^2}d(x^2)+\lim_{X_2\to\infty}\frac{1}{2}\int_{a}^{X_2}e^{-x^2}d(x^2)$$

$$=\frac{1}{2}\lim_{X_1\to\infty}\left[-e^{-x^2}\right]_{X_1}^{a}+\frac{1}{2}\lim_{X_2\to\infty}\left[-e^{-x^2}\right]_{a}^{X_2}$$

$$=\frac{1}{2}\lim_{X_1\to-\infty}\left(-e^{-a^2}+e^{-X_1^2}\right)+\frac{1}{2}\lim_{X_2\to\infty}\left(-e^{-X_2^2}+e^{-a^2}\right)$$

$$=\frac{1}{2}\left(-e^{-a^2}+0\right)+\frac{1}{2}\left(0+e^{-a^2}\right)=0.$$

Ex. 6. *Evaluate if possible the improper integral*

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}dx$$

Here $x=1$ is the only singularity of the given integral. So we can write the integral as

$$\therefore\ \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}dx$$

$$=\lim_{\varepsilon\to0+}\int_{-1}^{1-\varepsilon}\sqrt{\frac{1+x}{1-x}}dx$$

$$=\lim_{\varepsilon\to0+}\int_{-1}^{1-\varepsilon}\frac{1+x}{\sqrt{1-x^2}}dx$$

$$=\lim_{\varepsilon\to0+}\left\{\int_{-1}^{1-\varepsilon}\frac{dx}{\sqrt{1-x^2}}+\int_{-1}^{1-\varepsilon}\frac{x}{\sqrt{1-x^2}}dx\right\}$$

$$=\lim_{\varepsilon\to0+}\left\{\left[\sin^{-1}x\right]_{-1}^{1-\varepsilon}+\left[-\sqrt{1-x^2}\right]_{-1}^{1-\varepsilon}\right\}$$

[$\int\frac{x}{\sqrt{1-x^2}}dx=\int\frac{-udu}{u}$, $=-\int dx=u=\sqrt{1-x^2}$

putting $1-x^2=u^2$

$-2xdx=2udu$

i.e., $xdx=-udu$]

$$=\lim_{\varepsilon\to0+}\left\{\sin^{-1}(1-\varepsilon)-\sin^{-1}(-1)\right\}+\lim_{\varepsilon\to0+}\left\{-\sqrt{1-(1-\varepsilon)^2}+0\right\}$$

$$=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)+0$$

$$=\pi$$

Ex. 7. *Examine for convergence*

$$\int_{0}^{\infty}\cos x\,dx$$

By definition,

$$\int_0^\infty \cos x\,dx = \lim_{X\to\infty}\int_0^X \cos x\,dx$$

$$= \lim_{X\to\infty}[\sin x]_0^X$$

$$= \lim_{X\to\infty}(\sin X - 0)$$

$$= \lim_{X\to\infty}\sin X$$

Since $\sin X$ has no limit when $X\to\infty$, the given improper integral does not exist.

Ex. 8. *Examine for convergence*

$$\int_1^\infty \frac{dx}{x^2(x+1)}.$$

By definition $\int_1^\infty \frac{dx}{x^2(x+1)}$

$$= \lim_{X\to\infty}\int_1^X \frac{dx}{x^2(x+1)}$$

$$= \lim_{X\to\infty}\int_1^X\left(\frac{1}{x^2}-\frac{1}{x}+\frac{1}{x+1}\right)dx \qquad \left[\because \frac{1}{x^2(x+1)} = \frac{(x+1)-x}{x^2(x+1)}\right.$$

$$= \lim_{X\to\infty}\left[-\frac{1}{x}-\log x+\log(x+1)\right]_1^X \qquad = \frac{1}{x^2}-\frac{1}{x(x+1)}$$

$$= \lim_{X\to\infty}\left[-\frac{1}{X}-\log X+\log(X+1)+1-\log 2\right] \qquad \left. = \frac{1}{x^2}-\frac{1}{x}-\frac{1}{x+1}\right]$$

$$= \left[-\frac{1}{X}+\log\frac{X+1}{X}+1-\log 2\right] \qquad \left[\lim_{X\to\infty}\log\left(\frac{X+1}{X}\right)\right.$$

$$= -0+0+1-\log 2 \qquad = \lim_{X\to\infty}\log\left(\frac{X+1}{X}\right)$$

$$= 1-\log 2 \qquad \left. = \log(1+0) = 0\right]$$

Thus the given improper integral converges.

Ex. 9. *Test for convergence the improper integral*

$$\int_0^{1/e} \frac{dx}{x(\log x)^2}$$

Let $f(x) = \dfrac{1}{x(\log x)^2}$

Here 0 is the infinite discontinuity of $f(x)$.

By definition

$$\int_0^{1/e}\frac{dx}{x(\log x)^2} = \lim_{\varepsilon\to 0+}\int_\varepsilon^{1/e}\frac{dx}{x(\log x)^2}$$

$$= \lim_{\varepsilon\to 0+}\left[-\frac{1}{\log x}\right]_\varepsilon^{1/e}$$

$$\left[\because \int\frac{dx}{x(\log x)^2} = u\int\frac{dx}{u^2},\ \text{putting } u=\log x,\ i.e.,\ du=\frac{1}{x}dx\right.$$

$$\left. = -\frac{1}{u} = -\frac{1}{\log x}\right]$$

$$= \lim_{\varepsilon\to\infty 0+}\left[-\frac{1}{\log e^{-1}}+\frac{1}{\log\varepsilon}\right]$$

$$= \lim_{\varepsilon\to\infty 0+}\left[1+\frac{1}{\log\varepsilon}\right]$$

$$= 1+0 \qquad [\because As\ \varepsilon\to 0+, \log\varepsilon\to-\infty]$$

$$= 1.$$

Thus the given integral is convergent.

Ex. 10. *Examine the convergence of*

$$\int_1^2 \frac{x\,dx}{\sqrt{2-x}}.$$ [WBUT 2015]

Let $f(x)=\dfrac{x}{\sqrt{2-x}}$.

$\therefore$ $x=2$ is the point of infinite discontinuity.

Thus $\displaystyle\int_1^2 \frac{x\,dx}{\sqrt{2-x}} = \lim_{\varepsilon\to 0+}\int_1^{2-\varepsilon}\frac{x\,dx}{\sqrt{2-x}}$

$$\left[\because \int \frac{x\,dx}{\sqrt{2-x}} = \int \frac{(2-u^2)(-2u)\,du}{u},\right.$$

Putting $2-x=u^2$. $\therefore$ $dx=-2u\,du$

$$= \int (u^2-2)\,du = 2\left(\frac{u^3}{3}-2u\right)$$

$$\left.= \frac{2}{3}(2-x)^{3/2}+4(2-x)^{1/2}\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\tfrac{2}{3}(2-x)^{3/2}+4(2-x)^{1/2}\right]_1^{2-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+}\left[\frac{2}{3}\varepsilon^{3/2}+4\varepsilon^{1/2}-\frac{2}{3}-4\right]$$

$$= \frac{2}{3}.0+4.0-\frac{14}{3} = -4\tfrac{2}{3}.$$

Thus the given improper integral is convergent.

5.1.6. Gamma function.

The improper integral

$$\int_0^\infty e^{-x}x^{n-1}\,dx$$

is a function of n and denoted by $\Gamma(n)$, is known as Gamma function. This improper integral is convergent for $n>0$.

$$\therefore\ \Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx,\quad (n>0) \qquad \ldots (1)$$

Properties of Gamma Function.

(i) $\Gamma(1)=1$

Proof. Putting $n=1$ in (1), we get

$$\Gamma(1) = \int_0^\infty e^{-x}x^0\,dx$$

$$= \lim_{X\to\infty}\int_0^X e^{-x}dx$$

$$= \lim_{X\to\infty}\left[-e^{-x}\right]_0^X$$

$$= \lim_{X\to\infty}\left(1-e^{-X}\right)$$

$$= 1-0 = 1.$$

(ii) $\Gamma(n+1)=n\Gamma(n)$ [W.B.U.T. 2013]

Proof. From (1), we have

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^n\,dx$$

$$= \lim_{X\to\infty}\int_0^X e^{-x}x^n\,dx$$

$$= \lim_{X\to\infty}\left\{\left[-x^n e^{-x}\right]_0^X - \int_0^X\left(-e^{-x}\right)nx^{n-1}dx\right\}$$

$$= \lim_{X\to\infty}\left\{\left(-X^n e^{-X}+0\right)+n\int_0^X e^{-x}x^{n-1}dx\right\}$$

$$= 0+n\int_0^\infty e^{-x}x^{n-1}dx$$

$$= n\Gamma(n)$$

$$\therefore\ \Gamma(n+1) = n\,\Gamma(n)$$

(iii) $\Gamma(n+1) = n!$ **when** n **is a positive integer** [WBUT 2015]

Proof. We have

$$\Gamma(n+1) = n\Gamma(n)$$
$$= n(n-1)\Gamma(n-1)$$
$$\ldots\ldots\ldots\ldots$$
$$= n(n-1) \ldots\ldots\ldots 1\cdot\Gamma(1)$$
$$= n! \quad [\because \Gamma(1) = 1]$$
$$\therefore \Gamma(n+1) = n!$$

(iv) $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}, 0 < n < 1$ (Duplication formula)

Proof. Beyond the scope of this book.

(v) $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

Proof. We have

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \quad 0 < n < 1$$

Putting $n = \dfrac{1}{2}$, we get

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}}$$

or, $\left\{\Gamma\left(\dfrac{1}{2}\right)\right\}^2 = \pi$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Illustration.

(i) To evaluate $\Gamma\left(\dfrac{11}{2}\right)$, we have

$$\Gamma\left(\frac{11}{2}\right)$$
$$= \Gamma\left(\frac{9}{2}+1\right)$$
$$= \frac{9}{2}\Gamma\left(\frac{9}{2}\right), \text{ by prop. (ii)}$$
$$= \frac{9}{2}\Gamma\left(\frac{7}{2}+1\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\,\Gamma\left(\frac{7}{2}\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\,\Gamma\left(\frac{5}{2}\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)$$
$$= \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$
$$= \frac{945}{32}\sqrt{\pi}$$

(ii) To find $\Gamma(7)$, we have

$\Gamma(7) = 6!$, by prop. (iii)

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 720$

(iii) To show $\Gamma\left(\dfrac{1}{3}\right)\Gamma\left(\dfrac{2}{3}\right) = \dfrac{2\pi}{\sqrt{3}}$, we have

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{3}\right)$$
$$= \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

$= \frac{\pi}{\sin\frac{1}{3}\pi}$, by prop. (iv)

$= \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{\sqrt{3}}$

5.1.7. Beta Function.

The improper integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

is a function of m and n and denoted by $B(m, n)$, is known as beta function. This integral is convergent when $m, n > 0$.

$$\therefore \quad B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m, n > 0 \qquad \ldots (2)$$

Properties of Beta Function.

(i) $B(m, n) = B(n, m)$ (symmetric property)

Proof. Beyond the scope of this book.

(ii) $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

Proof. Beyond the scope of the book.

(iii) $B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$

Proof. Beyond the scope of the book.

Corollary. $\int_0^{\pi/2} \sin^n\theta d\theta = \frac{1}{2}B\left(\frac{n+1}{2}, \frac{1}{2}\right)$

$\int_0^{\pi/2} \cos^n\theta d\theta = \frac{1}{2}B\left(\frac{1}{2}, \frac{n+1}{2}\right)$

$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$ [WBUT 2015]

...f. From prop. (iii), by putting

...$n = \frac{1}{2}$ we get

$\ldots\frac{1}{2}, \frac{1}{2}\Big) = 2\int_0^{\pi/2} \sin^0\theta\cos^0\theta\, d\theta$

$\int_0^{\pi/2} d\theta$

$2\cdot[\theta]_0^{\pi/2} = \pi$

$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$

...Relation between Beta and Gamma Function.

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

...f. Beyond the scope of this book.

...**ark :** This relation helps us to find the value of $B(m, n)$ ... least one of m and n is positive integer. See the following ...ions.

...**ration. (i)** To find $B(4, 2)$, we use the relation beta and ... function.

$\ldots(4, 2) = \frac{\Gamma(4)\Gamma(2)}{\Gamma(4+2)} = \frac{3!1!}{5!}$

$= \frac{3\cdot2\cdot1\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = \frac{1}{20}.$

(ii) To evaluate $\int_0^{\pi/2} \cos^6 \theta \, d\theta$,

we use property (iii)

$\therefore \int_0^{\pi/2} \cos^6 \theta \, d\theta$

$$= \frac{1}{2} B\left(\frac{1}{2}, \frac{6+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{7}{2}\right)}$$

$$= \frac{1}{2} \frac{\sqrt{\pi}\cdot\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)}{\Gamma(4)} = \frac{1}{2} \frac{\sqrt{\pi}\cdot\frac{5}{2}\cdot\Gamma\left(\frac{5}{2}\right)}{\Gamma(4)}$$

$$= \frac{1}{2} \frac{\sqrt{\pi}\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{3!}$$

$$= \frac{1}{2} \frac{\pi\cdot\frac{15}{8}}{3\cdot 2\cdot 1}$$

$$= \frac{5\pi}{32}$$

(iii) To evaluate $\int_0^{\pi/2} \sin^7 \theta \cos^4 \theta d\theta$,

we use prop. (iii)

$\therefore \int_0^{\pi/2} \sin^7 \theta \cos^5 \theta d\theta$

$$= \int_0^{\pi/2} \sin^{2\cdot 4-1} \theta \cos^{2\cdot 3-1} \theta d\theta$$

$$= \frac{1}{2} B(4, 3) \qquad [\because \text{ here } m = 4, \; n = 3]$$

$$= \frac{1}{2} \frac{\Gamma(4)\Gamma(3)}{\Gamma(4+3)} = \frac{1}{2} \cdot \frac{3!2!}{6!}$$

$$= \frac{1}{2} \frac{3.2.1.2.1}{6.5.4.3.2.1} = \frac{1}{120}$$

5.1.9. Illustrative Example

Ex. 1. *Show that* $\int_0^{\infty} e^{-4x} x^{\frac{3}{2}} dx = \frac{3}{128}\sqrt{\pi}$ [WBUT 2012]

Solution : $\int_0^{\infty} e^{-4x} x^{\frac{3}{2}} dx$

$$= \int_0^{\infty} e^{-z} \frac{z^{3/2}}{4^{3/2}} \frac{dz}{4}, \text{ putting } 4x = z \text{ i.e., } dx = \frac{dz}{4}$$

$$= \frac{1}{32} \int_0^{\infty} e^{-z} z^{3/2} dz$$

$$= \frac{1}{32} \int_0^{\infty} e^{-z} z^{5/2-1} dz$$

$$= \frac{1}{32} \Gamma(5/2)$$

$$= \frac{1}{32} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{128}\sqrt{\pi}$$

Ex. 2. *Show that* $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx = \frac{8}{315}$. [WBUT 2014]

Solution : $\int_0^{\pi/2} \sin x^4 \cos^5 x\, dx$

$$= \int_0^{\pi/2} \sin^{2\cdot\frac{5}{2}-1} \cos^{2\cdot 3-1} x\, dx$$

$$= \frac{1}{2} B\left(\frac{5}{2}, 3\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{5}{2}\right)2.1}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\Gamma\left(\frac{5}{2}\right)} = \frac{8}{315}.$$

Ex. 3. *Show that* $\int_a^b (x-a)^3 (b-x)^2 dx = \frac{(b-a)^6}{60}$.

Solution : Let $x = a\cos^2\theta + b\sin^2\theta$

$\therefore\ dx = (b-a)\cdot 2\sin\theta\cos\theta\, d\theta$

Now, $x - a = a\cos^2\theta + b\sin^2\theta - a = b\sin^2\theta - a(1-\cos^2\theta)$

$= b\sin^2\theta - a\sin^2\theta$

$= (b-a)\sin^2\theta$

and $\quad b - x = (b-a)\cos^2\theta$

When $\quad x = a,\ \ \theta = 0$

$\qquad\quad x = b,\ \ \theta = \pi/2$

$$\therefore \int_a^b (x-a)^3(b-x)^2 dx = 2(b-a)^6 \int_0^{\pi/2} \sin^7\theta\cos^5\theta\, d\theta$$

$$= 2(b-a)^6 \int_0^{\pi/2} \sin^{2.4-1}\theta\cos^{2.3-1}\theta\, d\theta$$

$$= 2(b-a)^6 \frac{1}{2} B(4, 3)$$

$$= (b-a)^6 \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)}$$

$$= (b-a)^6 \frac{\Gamma(4)2\cdot 1}{6.5.4\Gamma(4)}$$

$$= \frac{(b-a)^6}{60}.$$

Ex. 4. *Show that* $\int_0^{\pi/2} \sin^p x dx \times \int_0^{\pi/2} \sin^{p+1} x\, dx = \frac{\pi}{2(p+1)}$.

$$\int_0^{\pi/2} \sin^p x\, dx \times \int_0^{\pi/2} \sin^{p+1} x\, dx$$

$$= \frac{1}{2}B\left(\frac{p+1}{2}, \frac{1}{2}\right)\cdot\frac{1}{2}\cdot B\left(\frac{p+1+1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}\cdot\frac{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+3}{2}\right)}$$

$$= \frac{1}{4}\left(\sqrt{\pi}\right)^2 \frac{\Gamma\left(\frac{p+1}{2}\right)}{\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{\pi}{2(p+1)}$$

Ex. 5. *Prove that* $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. [WBUT 2017]

Let $u = x^2$

$\therefore\ du = 2x\,dx$ i.e., $dx = \frac{1}{2}\frac{du}{x} = \frac{1}{2}\frac{du}{\sqrt{u}}$

$$\therefore \int_0^\infty e^{-x^2}dx$$

$$= \int_0^\infty e^{-u}\frac{1}{2}\frac{du}{\sqrt{u}}$$

$$= \frac{1}{2}\int_0^\infty e^{-u}u^{\frac{1}{2}-1}du$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2}.$$

Ex. 6. *Show that* $\int_0^{\pi/2}\sqrt{\tan x}\,dx = \frac{\pi}{\sqrt{2}}$ [WBUT 2014, 2016]

$$\int_0^{\pi/2}\sqrt{\tan x}\,dx$$

$$= \int_0^{\pi/2}\sin^{\frac{1}{2}}x\cos^{-\frac{1}{2}}x\,dx$$

$$= \int_0^{\pi/2}\sin^{2\cdot\frac{3}{4}-1}x\cos^{2\cdot\frac{1}{4}-1}x\,dx$$

$$= \frac{1}{2}B\left(\frac{3}{4},\frac{1}{4}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}} \qquad \left[\because\ \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}\right]$$

$$= \frac{\pi}{\sqrt{2}}.$$

Ex. 7. *Show that* $(p+q)B(p+1,q) = pB(p,q)$

Solution : $(p+q)B(p+1,q)$

$$= (p+q)\frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+1+q)}$$

$$= (p+q)\frac{p\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)}$$

$$= p\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = pB(p,q)$$

Ex. 8. *Evaluate* $\int_0^1\frac{x\,dx}{\sqrt{1-x^5}}$.

We put $x^5 = \sin^2\theta$ i.e., $x = \sin^{\frac{2}{5}}\theta$

$\therefore\ dx = \frac{2}{5}\sin^{-\frac{3}{5}}\theta.\cos\theta\,d\theta$

Now, $\theta = 0$ when $x = 0$

$\theta = \frac{\pi}{2}$ when $x = 1$

$$\therefore \int_0^1\frac{x\,dx}{\sqrt{1-x^5}}$$

$$= \int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{2}{5}}\theta\,\frac{2}{5}\sin^{-\frac{3}{5}}\theta\cos\theta\,d\theta}{\cos\theta} \qquad \left[\because\ \sqrt{1-x^5} = \sqrt{1-\sin^2\theta} = \cos\theta\right]$$

$$= \frac{2}{5}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{5}}\theta \cos^0\theta \, d\theta$$

$$= \frac{2}{5}\int_0^{\frac{\pi}{2}} \sin^{\left(2\cdot\frac{2}{5}-1\right)}\theta \cos^{\left(2\cdot\frac{1}{2}-1\right)}\theta \, d\theta$$

$$= \frac{2}{5}\cdot\frac{1}{2}B\left(\frac{2}{5},\frac{1}{2}\right)$$

$$= \frac{1}{5}\frac{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{5}+\frac{1}{2}\right)}$$

$$= \frac{1}{5}\frac{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4}{5}+1\right)}$$

$$= \frac{1}{5}\frac{\Gamma\left(\frac{2}{5}\right)\sqrt{\pi}}{\frac{4}{5}\Gamma\left(\frac{4}{5}\right)}$$

$$= \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{2}{5}\right)}{\Gamma\left(\frac{4}{5}\right)}$$

Ex. 9. *Show that* $\int_0^{\infty}\frac{dx}{\left(1+x^2\right)^5} = \frac{35\pi}{256}$.

We put $x = \tan\theta$ $\quad\therefore\quad dx = \sec^2\theta \, d\theta$

when $x = 0, \theta = 0$

when $n \to \infty$, $\theta \to \frac{\pi}{2}$

$$\therefore \int_0^{\infty}\frac{dx}{\left(1+x^2\right)^5}$$

$$= \int_0^{\frac{\pi}{2}}\frac{\sec^2\theta \, d\theta}{\left(1+\tan^2\theta\right)^5}$$

$$= \int_0^{\frac{\pi}{2}}\frac{\sec^2\theta \, d\theta}{\sec^{10}\theta}$$

$$= \int_0^{\frac{\pi}{2}}\cos^8\theta \, d\theta$$

$$= \frac{1}{2}B\left(\frac{1}{2},\frac{8+1}{2}\right)$$

$$= \frac{1}{2}B\left(\frac{1}{2},\frac{9}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{9}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{9}{2}\right)}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4.3.2.1\ \Gamma(1)}$$

$$= \frac{35\left(\sqrt{\pi}\right)^2}{256} \quad \left[\because\ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{35\pi}{256}$$

Ex. 10. *Show that* $\int_0^{\infty} e^{-x^2}x^9 \, dx = 12$.

We put $x^2 = z$

i.e., $x = \sqrt{z}$

$\therefore\ dx = \frac{1}{2}\frac{dz}{\sqrt{z}}$

when $x = 0$, $z = 0$

and $x \to \infty$, $z \to \infty$

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$$\therefore \int_0^\infty e^{-x^2} x^9\, dx$$

$$= \int_0^\infty e^{-z} z^{\frac{9}{2}} \frac{1}{2}\frac{dz}{\sqrt{z}}$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^4\, dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{5-1}\, dz$$

$$= \frac{1}{2}\Gamma(5)$$

$$= \frac{1}{2}.4.3.2.1 = 12$$

EXERCISE

[I] SHORT ANSWER QUESTIONS

1. Evaluate the following improper integral :

(i) $\int_0^1 \frac{dx}{x^2}$ (ii) $\int_0^\infty \frac{dx}{x^3}$

(iii) $\int_0^\infty e^{-x} dx$ (iv) $\int_{-\infty}^0 e^{2x} dx$

(v) $\int_{-2}^2 \frac{dx}{x^2}$ (vi) $\int_0^{16} \frac{dx}{\sqrt{x}}$

(vii) $\int_0^\infty x \sin x\, dx$ (viii) $\int_0^1 \frac{dx}{1-x}$

(ix) $\int_0^2 \frac{dx}{(1-x)^2}$ (x) $\int_0^\infty \frac{dx}{x^2+a^2}$ $(a>0)$

(xi) Evaluate : $\int_1^\infty \frac{dx}{x^2}$.

2. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

3. Show that $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$

4. Evaluate (a) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

(b) $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$ [WBUT 2015]

5. Show that $\int_0^{\pi/2} \sin^5\theta\, d\theta = \frac{8}{15}$

6. Show that $\int_0^{\pi/2} \sin^4\theta \cos^4\theta\, d\theta = \frac{3\pi}{256}$

7. Evaluate (a) $\int_0^\infty e^{-x^3} dx$ (b) $\int_0^\infty e^{-x^4} dx$

8. Show that $\int_2^\infty \frac{dx}{x \log x}$ is not convergent

ANSWERS

1. (i) does not exist (ii) does not exist (iii) 1 (iv) $\frac{1}{2}$

(v) does not exist (vi) 8 (vii) does not exist

(viii) does not exist (ix) does not exist (x) $\frac{\pi}{2a}$ (xi) 1

(a) $\sqrt{2}\pi$ (b) $\frac{2\pi}{\sqrt{3}}$ 7. (a) $\Gamma\left(\frac{4}{3}\right)$ (b) $\Gamma\left(\frac{5}{4}\right)$

[II] Long Answer Questions

1. Evaluate the following improper integrals

(i) $\int_{-\infty}^{\infty} xe^{-x^2}dx$ (ii) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

(iii) $\int_0^{\infty} \frac{dx}{4+9x^2}$ (iv) $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$

(v) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ (vi) $\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$

(vii) $\int_0^1 \log x dx$ (viii) $\int_1^{\infty} \frac{dx}{x(1+x)}$

(ix) $\int_0^{\infty} \frac{dx}{x\sqrt{x^2-1}}$ (x) $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$, $a, b>0$

2. Prove that $\int_{-1}^{1} \frac{dx}{x^5}$ exists in Cauchy principal value sense but not in general sense.

3. Assuming that the following integral are convergent, prove that

(i) $\int_0^{\pi/2} \log \tan x \, dx = 0$ (ii) $\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = \frac{\pi}{2} \log \frac{1}{2}$

(iii) $\int_0^{\pi} x \log \sin dx = \frac{\pi^2}{2} \log \frac{1}{2}$

4. Evaluate the following integrals

(i) $\int_0^{\pi/2} \sin^6\theta \cos^5\theta \, d\theta$ (ii) $\int_0^{\infty} x^n e^{-ax} dx$

(iii) $\int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}}$ (iv) $\int_0^{\infty} x^4 e^{-x^4} dx$

(v) $\int_0^{\infty} e^{-x} x^{3/2} dx$ [WBUT 2012] (vi) $\int_0^{\infty} \sqrt{x} e^{-x^3} dx$

(vii) $\int_0^{\infty} e^{-a^2x^2} dx$ (viii) $\int_0^1 x^{3/2}(1-x)^{3/2} dx$

(ix) $\int_0^1 \frac{dx}{(1-x^6)^{1/6}}$ (x) $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$

5. Show that $\int_0^{\infty} 5^{-x^2} dx = \frac{1}{2\sqrt{\log 5}} \cdot \sqrt{\pi}$.

6. Show that $\int_0^{\pi/2} \sqrt{\sin\theta} \, d\theta \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

7. Show that $B(m,n) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$

8. Show that $\int_0^{\infty} e^{-x^4} dx \times \int_0^{\infty} x^2 e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$ [WBUT 2015,2017]

9. Prove that $\int_0^{\infty} e^{-\sqrt{5}x^2} dx = \frac{\sqrt{\pi}}{25}$ and hence find $\int_{-\infty}^{\infty} e^{-5x^2} dx$

Answers

1. (i) 0 (ii) $\frac{\pi}{2}$ (iii) $\frac{\pi}{12}$ (iv) $\frac{\pi}{2}$ (v) π (vi) $\frac{\pi}{2}$

(vii) -1 (viii) $\log 2$ (ix) $\frac{\pi}{2}$ (x) $\frac{\pi}{2ab(a+b)}$

4. (i) $\frac{8}{693}$ (ii) $\Gamma(n+1)/a^{n+1}$ (iii) $\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)}$ (iv) $\frac{1}{16}\Gamma\left(\frac{1}{4}\right)$

(v) $\frac{3}{4}\sqrt{\pi}$ (vi) $\frac{\sqrt{\pi}}{3}$ (vii) $\frac{1}{2a}\sqrt{\pi}$ (viii) $\frac{3\pi}{128}$

(ix) $\frac{\pi}{3}$ (x) $\frac{\pi}{\sqrt{2}}$ 9. $\sqrt{\frac{\pi}{5}}$.

[III] MULTIPLE CHOICE QUESTIONS

1. The singularity of the integral $\int_{-1}^{2}\frac{dx}{x(x-1)}$ are

(a) 1, 2 (b) −1, 2

(c) 0, 1 (c) 0, 2 [WBUT 2011]

2. The infinite discontinuity of the integral $\int_{-1}^{1}\sqrt{\frac{1-x}{x}}dx$ is

(a) −1 (b) 1

(c) 0 (d) none

3. The value of the integral $\int_0^{\infty}e^{-2x}dx$ is

(a) 1 (b) −1

(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

4. $\int_{-\infty}^{\infty}xe^{-x^2}dx =$

(a) $2\sqrt{\pi}$ (b) $-2\sqrt{\pi}$

(c) $\frac{\sqrt{\pi}}{2}$ (d) 0

5. The improper integral $\int_1^{\infty}\frac{1}{x}dx$ is convergent. The statement is

(a) True (b) False

6. The integral $\int_{-1}^{1}\frac{dx}{x}$ exists is Cauchy principal value sense. The statement is

(a) True (b) False

7. The improper integral $\int_a^{\infty}\frac{dx}{x^n}\,(a>0)$ converges if and only if

(a) $n<1$ (b) $n=1$

(c) $n>1$ (d) none of these

8. The integral $\int_2^{\infty}\frac{dx}{\sqrt{x}}$ is convergent. The statement is

(a) True (b) False

9. The improper integral $\int_a^b\frac{dx}{(x-a)^n}$ is convergent if

(a) $n<1$ (b) $n=1$

(c) $n>1$ (d) none of these [WBUT 2012]

10. The improper integral $\int_a^b\frac{dx}{(b-x)^n}$ is convergent only for

(a) $n>1$ (b) $n<1$

(c) $n=1$ (d) none of these

11. The improper integral $\int_0^2\frac{dx}{\sqrt{2-x}}$ is convergent.The statement

(a) True (b) False

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12. The improper integral $\int_1^2 \frac{dx}{(x-1)^3}$ is convergent. The statement is

(a) True (b) False

13. The value of the integral $\int_1^\infty \frac{dx}{x^{3/2}}$ is

(a) 2 (b) $\frac{1}{2}$

(c) $-\frac{1}{2}$ (d) -2

14. The value of the improper integral $\int_0^\infty \frac{1}{4+x^2}dx$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) π (d) 0

15. The singularity of the integral $\int_{-1}^{1} \frac{dx}{\sqrt{x(x+1)}}$ are

(a) $-1, 1$ (b) -1

(c) $0, -1$ (d) $0, 1$

16. The value of the $\beta(\frac{1}{2}, \frac{1}{2})$ is

(a) $\sqrt{\pi}$ (b) π

(c) $\frac{\pi}{2}$ (d) none of these [WBUT 20..]

17. The value of $\Gamma\left(\frac{5}{2}\right)=$

(a) $\frac{3}{2}\sqrt{\pi}$ (b) $\frac{3}{4}$

(c) $\frac{3}{4}\sqrt{\pi}$ (d) $\frac{1}{2}\sqrt{\pi}$

18. The value of $\Gamma(6)$ is

(a) 720 (b) 120

(c) b (d) none [WBUT 2011]

19. The value of $\Gamma\left(\frac{1}{2}\right)$ is

(a) $\sqrt{\pi}$ (b) π

(c) $\frac{\sqrt{\pi}}{2}$ (d) none [WBUT 2011]

20. $\Gamma(1)=$

(a) 1 (b) -1

(c) 0 (d) none

21. $\Gamma(2)=\Gamma(1)$

(a) True (b) False

22. If $B(x, y)=5$, then $B(y, x)$ is

(a) 5 (b) -5

(c) 25 (d) none

23. $\Gamma p\ \Gamma(1-p)=\frac{\pi}{\sin p\pi}$ is true for alls value of p

(a) True (b) False

24. The value of $\Gamma(m)\Gamma(1-m)$ is

(a) $\frac{2\pi}{\sin\pi}$ (b) $\frac{3\pi}{\sin m\pi}$

(c) $\frac{\pi}{\sin m\pi}$ (d) none of these [WBUT 2011]

25. $\Gamma(n+1)=n!$ is true for all value of n

(a) yes (b) no

26. The value of the $\int_0^\infty e^{-x}x^{\frac{3}{2}}\,dx$ is

(a) $\frac{3}{4}\sqrt{\pi}$ (b) $\frac{5}{4}\sqrt{\pi}$

(c) $\frac{3}{5}\sqrt{\pi}$ (d) $\frac{1}{4}\sqrt{\pi}$

[WBUT 2011]

27. The value of the $\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)$ is

(a) $\frac{3\sqrt{\pi}}{4}$ (b) $\frac{3}{2}\pi$

(c) $\frac{3\pi}{4}$ (d) none of these

28. $\int_0^\infty e^{-x^2}dx =$

(a) π (b) $\sqrt{\pi}$

(c) $\frac{\sqrt{\pi}}{2}$ (d) $\frac{\pi}{2}$

29. $\frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} =$

(a) $\frac{7}{2}$ (b) $\frac{5}{2}$

(c) $\frac{35}{4}$ (d) none of these

30. The integral $\int_0^\infty e^{-x}x^{n-1}dx$ is convergent when

(a) $n<0$ (b) $n=0$

(c) $n>0$ (d) n is even

31. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent when

(a) $m>0, n>0$ (b) $m<0, n<0$

(c) $m>0, n<0$ (d) $m<0, n>0$

Answers

1. c	2. c	3. c	4. d	5. b	6. b
7. c	8. b	9. a	10. b	11. a	12. b
13. a	14. b	15. c	16. b	17. c	18. b
19. a	20. a	21. a	22. a	23. b	24. c
25. b	26. a	27. c	28. c	29. b	30. c
31. a					

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