2.4 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

2.4.1. Introduction. In this chapter we shall consider differential equations in which there is one independent variable and two or more than two dependent variables. Such equations are called simultaneous linear equations. Here we shall consider simultaneous linear equations with constant coefficients only. To solve such equations completely, we must have as many simultaneous linear equations as the number of dependent variables.

There are two methods for the solution of simultaneous linear differential equations with constant coefficients.

2.4.2. First Method : Symbolic method (*Use of operator D*).

Let the given simultaneous equations be

$$f_1(D)x + f_2(D)y = f(t) \qquad \ldots \quad (1)$$

$$\phi_1(D)x + \phi_2(D)y = g(t) \text{ where } D \equiv \frac{d}{dt} \qquad \ldots \quad (2)$$

in which x and y are functions of t and $f_1(D), f_2(D), \phi_1(D), \phi_2(D)$ are all rational integral functions of D with constant coefficients.

To eliminate y, operating on both sides of (1) by $\phi_2(D)$ and on both sides of (2) by $f_2(D)$ and subtracting, we have

$$[f_1(D)\phi_2(D) - \phi_1(D)f_2(D)]x = \phi_2(D)f(t) - f_2(D)g(t)$$

or, $$F(D)x = \psi(t) \quad \text{(say)} \qquad \ldots \quad (3)$$

which is a linear equation in x and can be solved by the methods already discussed.

Substituting this value of x in either (1) or (2), we get the value of y.

2.4.3. Second method : Method of Differentiation.

Sometimes x or y can be easily eliminated if we differentiate (1) or (2). Then, from the resulting equations after eliminating one dependent variable (x or y), we can solve for the second variable and then the value of the remaining variable can be found.

This method is applied only when the given simultaneous equations are of order one.

Note. In almost all problems we shall use the first method and the 2nd method will be used as necessary.

Illustrative Examples.

Ex.1. *Solve:* $\frac{dx}{dt} - 7x + y = 0$ [*W.B.U.T. 2013, 2007*]

$$\frac{dy}{dt} - 2x - 5y = 0.$$

The given equation can be written as

$$(D-7)x + y = 0 \qquad \ldots \quad \text{(i)}$$

$$-2x + (D-5)y = 0 \text{ where } D \equiv \frac{d}{dt} \qquad \ldots \quad \text{(ii)}$$

Operate on (i) by $(D-5)$ and then subtracting from (ii), we get

$$[(D-5)(D-7) + 2]x = 0,$$

or, $$(D^2 - 12D + 37)x = 0. \qquad \ldots \quad \text{(iii)}$$

Let $x = e^{mt}$ be a trial solution of (iii).

Then the auxiliary equation is $m^2 - 12m + 37 = 0$.

$\therefore \quad m = 6 \pm i.$

Therefore $x = e^{6t}(c_1 \cos t + c_2 \sin t)$

$$\therefore \quad \frac{dx}{dt} = e^{6t}(-c_1 \sin t + c_2 \cos t) + 6e^{6t}(c_1 \cos t + c_2 \sin t).$$

From (i), $y = 7x - Dx$

$$= 7e^{6t}(c_1 \cos t + c_2 \sin t) - e^{6t}(-c_1 \sin t + c_2 \cos t) - 6e^{6t}(c_1 \cos t + c_2 \sin t)$$

$$= e^{6t}\{(c_1 - c_2)\cos t + (c_1 + c_2)\sin t\}.$$

So the required complete solution is

$$x = e^{6t}(c_1 \cos t + c_2 \sin t)$$

$$y = e^{6t}\{(c_1 - c_2)\cos t + (c_1 + c_2)\sin t\},$$

where c_1, c_2 are arbitrary constants.

Ex. 2. *Solve :* $\frac{dy}{dx}+2y-3z=x$

$$\frac{dz}{dx}+2z-3y=e^{2x}.$$

The given equation can be written as

$$(D+2)y-3z=x \qquad \ldots \text{(i)}$$

$$-3y+(D+2)z=e^{2x},\ where\ D\equiv\frac{d}{dx} \qquad \ldots \text{(ii)}$$

Operate on (i) by $(D+2)$ and multiply the equation (ii) by 3 and then add the result, we get

$$(D+2)^2y-9y=(D+2)x+3e^{2x}$$

or, $(D^2+4D-5)y=3e^{2x}+2x+1 \qquad \ldots \text{(iii)}$

Let $y=e^{mx}$ be a trial solution of $(D^2+4D-5)y=0.$

Then the auxiliary equation is $m^2+4m-5=0.$

$\therefore \quad m=-5,1.$

$\therefore \quad$ C. F. $=c_1e^{-5x}+c_2e^{x}.$

Now P. I. $=\frac{1}{D^2+4D-5}(3e^{2x}+2x+1)$

$$=\frac{1}{D^2+4D-5}3e^{2x}+\frac{1}{D^2+4D-5}2x+\frac{1}{D^2+4D-5}e^{0x}$$

$$=\frac{3e^{2x}}{7}-\frac{1}{5}\left(1-\frac{D^2+4D}{5}\right)^{-1}2x-\frac{1}{5}$$

$$=\frac{3}{7}e^{2x}-\frac{2}{5}\left(1+\frac{D^2+4D}{5}+\cdots\right)x-\frac{1}{5}$$

$$=\frac{3}{7}e^{2x}-\frac{2}{5}\left(x+\frac{4}{5}\right)-\frac{1}{5}$$

$$=\frac{3}{7}e^{2x}-\frac{2x}{5}-\frac{13}{25}$$

Therefore $y=c_1e^{-5x}+c_2e^{x}+\frac{3}{7}e^{2x}-\frac{2}{5}x-\frac{13}{25}.$

$$\therefore \quad \frac{dy}{dx}=-5c_1e^{-5x}+c_2e^{x}+\frac{6}{7}e^{2x}-\frac{2}{5}.$$

$\therefore \quad$ From (i),

$$3z=(D+2)y-x=Dy+2y-x$$

$$=-5c_1e^{-5x}+c_2e^{x}+\frac{6}{7}e^{2x}-\frac{2}{5}$$

$$+2c_1e^{-5x}+2c_2e^{x}+\frac{6}{7}e^{2x}-\frac{4}{5}x-\frac{26}{25}-x$$

$$\therefore \quad z=-c_1e^{-5x}+c_2e^{x}+\frac{4}{7}e^{2x}-\frac{3}{5}x-\frac{12}{25}.$$

So the required complete solution is

$$y=c_1e^{-5x}+c_2e^{x}+\frac{3}{7}e^{2x}-\frac{2}{5}x-\frac{13}{25}$$

$$z=-c_1e^{-5x}+c_2e^{x}+\frac{4}{7}e^{2x}-\frac{3}{5}x-\frac{12}{25}$$

where c_1, c_2 are arbitrary constants.

Ex. 3. *Solve :* $\frac{dx}{dt}+5x+y=e^{t},\quad \frac{dy}{dt}-x+3y=e^{2t}.$

From the first equation, we have $y=e^{t}-5x-\frac{dx}{dt}. \qquad \ldots \text{(i)}$

Putting (i) in the second equation, we get

$$e^{t}-5\frac{dx}{dt}-\frac{d^2x}{dt^2}-x+3(e^{t}-5x-\frac{dx}{dt})=e^{2t}$$

or, $\frac{d^2x}{dt^2}+8\frac{dx}{dt}+16x=4e^{t}-e^{2t}. \qquad \ldots \text{(ii)}$

Let $x = e^{mt}$ be a trial solution of $\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 16x = 0.$

Then the auxiliary equation is $m^2 + 8m + 16 = 0.$

$\therefore \quad m = -4, -4.$

$\therefore \quad$ C. F. $= (c_1 + c_2t)e^{-4t}.$

Now P. I. $= \dfrac{1}{D^2+8D+16}(4e^t - e^{2t})$

$$= \frac{1}{(D+4)^2}4e^t - \frac{1}{(D+4)^2}e^{2t}$$

$$= \frac{4}{25}e^t - \frac{1}{36}e^{2t}.$$

Therefore $x = (c_1 + c_2t)e^{-4t} + \dfrac{4}{25}e^t - \dfrac{1}{36}e^{2t}.$

$\therefore$ From (i), $y = e^t - \dfrac{dx}{dt} - 5x$

$$= e^t + 4(c_1 + c_2t)e^{-4t} - c_2e^{-4t} - \frac{4}{25}e^t + \frac{1}{18}e^{2t}$$

$$-5(c_1 + c_2t)e^{-4t} - \frac{4}{5}e^t + \frac{5}{36}e^{2t}$$

$$= -(c_1 + c_2t + c_2)e^{-4t} + \frac{1}{25}e^t + \frac{7}{36}e^{2t}.$$

So the required complete solution is

$$x = (c_1 + c_2t)e^{-4t} + \frac{4}{25}e^t - \frac{1}{36}c^{2t}$$

$$y = -(c_1 + c_2 + c_2t)e^{-4t} + \frac{1}{25}e^t + \frac{7}{36}e^{2t},$$

where c_1, c_2 are arbitrary constant.

Ex. 4. *Solve :*

$$\frac{dx}{dt} + 3x + y = e^t, \quad \frac{dy}{dt} - x + y = e^{2t}$$

[W.B.U.T. 2004, 2012,2017]

From the first equation, we have

$$y = e^t - 3x - \frac{dx}{dt} \qquad \text{. . . (i)}$$

$$\therefore \quad \frac{dy}{dt} = e^t - 3\frac{dx}{dt} - \frac{d^2x}{dt^2} \qquad \text{. . . (ii)}$$

Substituting these values of y and $\dfrac{dy}{dt}$ in the second equation, we get,

$$e^t - 3\frac{dx}{dt} - \frac{d^2x}{dt^2} - x + e^t - 3x - \frac{dx}{dt} = e^{2t}$$

or, $\quad \dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt^2} + 4x = 2e^t - e^{2t}$

or, $\quad (D^2 + 4D + 4)x = 2e^t - e^{2t}, D \equiv \dfrac{d}{dt}$

$$\therefore \quad (D+2)^2x = 2e^t - e^{2t} \qquad \text{. . . (iii)}$$

Let $x = e^{mt}$ be a trial solution of $(D+2)^2x = 0$

$\therefore$ The auxiliary equation is

$(m+2)^2 = 0$

$\therefore \quad m = -2, -2$

$\therefore \quad$ C.F. $= (c_1 + c_2t)e^{-2t}$

$\therefore \quad$ P.I. $= \dfrac{2e^t - e^{2t}}{(D+2)^2}$

$$= 2\frac{e^t}{(D+2)^2} - \frac{e^{2t}}{(D+2)^2}$$

$$= 2\frac{e^t}{(1+2)^2} - \frac{e^{2t}}{(2+2)^2}$$

$$= \frac{2}{9}e^t - \frac{1}{16}e^{2t}$$

$\therefore \quad x = (c_1 + c_2 t)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}$

$\therefore \quad \frac{dx}{dt} = c_2 e^{-2t} - 2(c_1 + c_2 t)e^{-2t} + \frac{2}{9}e^t - \frac{1}{8}e^{2t}$

$\therefore$ From (i),

$$y = e^t - 3(c_1 + c_2 t)e^{-2t} - \frac{2}{3}e^t + \frac{3}{16}e^{2t}$$

$$-c_2 e^{-2t} + 2(c_1 + c_2 t)e^{-2t} - \frac{2}{9}e^t + \frac{1}{8}e^{2t}$$

$$= -(c_1 + c_2 t)e^{-2t} - c_2 e^{-2t} + \frac{1}{9}e^t + \frac{5}{16}e^{2t}$$

Thus the required solution is

$$x = (c_1 + c_2 t)e^{-2t} + \frac{2}{9}e^t - \frac{1}{16}e^{2t}$$

$$y = (-c_1 + c_2 t)e^{-2t} - c_2 e^{-2t} + \frac{1}{9}e^t - \frac{5}{16}e^{2t}$$

Ex. 5. *Solve :* $\frac{dx}{dt} = -2y, \ \frac{dy}{dt} = x.$

Let $\frac{dx}{dt} = -2y$ ... (i)

$\frac{dy}{dt} = x.$ ... (ii)

From (ii), $\frac{d^2y}{dt^2} = \frac{dx}{dt}.$

$\therefore \quad \frac{d^2y}{dt^2} = -2y,$ by (i)

or, $\frac{d^2y}{dt^2} + 2y = 0.$ ... (iii)

Let $y = e^{mt}$ be a trial solution of (iii).

Then the auxiliary equation is $m^2 + 2 = 0.$

$\therefore \quad m = \pm\, i\sqrt{2}.$

$\therefore \quad y = c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t.$

$\therefore \quad \frac{dy}{dt} = -\sqrt{2}c_1 \sin\sqrt{2}t + \sqrt{2}c_2 \cos\sqrt{2}t.$

Therefore from (ii) $x = \sqrt{2}(c_2 \cos\sqrt{2}t - c_1 \sin\sqrt{2}t).$

So the required complete solution is

$$x = \sqrt{2}(c_2 \cos\sqrt{2}t - c_1 \sin\sqrt{2}t).$$

$$y = c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t,$$

where c_1, c_2 are arbitrary constants.

Ex. 6. $\frac{dx}{dt} = y + z, \ \frac{dy}{dt} = z + x, \ \frac{dz}{dt} = x + y.$

We have $\frac{dx}{dt} = y + z$ ... (i)

$\frac{dy}{dt} = z + x$ ... (ii)

$\frac{dz}{dt} = x + y$ ... (iii)

From (i), $\frac{d^2x}{dt^2} = \frac{dy}{dt} + \frac{dz}{dt}$.

$= z + x + x + y$, by $(ii), (iii)$

$= 2x + \frac{dx}{dt}$, by (i)

$\therefore \quad \frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0.$ . . . (iv)

Let $x = e^{mt}$ be a trial solution of (iv).

Then the auxiliary equation is $m^2 - m - 2 = 0$.

or, $(m-2)(m+1) = 0$

$\therefore \quad m = 2, -1.$

$\therefore \quad x = c_1e^{2t} + c_2e^{-t}.$

$\therefore \quad \frac{dx}{dt} = 2c_1e^{2t} - c_2e^{-t}.$

$\therefore \quad y + z = 2c_1e^{2t} - c_2e^{-t}$, by (i) . . . (v)

From (ii) and (iii) $\frac{dy}{dt} - \frac{dz}{dt} = z - y$

or, $\frac{d(y-z)}{y-z} = -dt.$

On integration, we get $\log(y-z) = -t + \log c_3$.

$\therefore \quad y - z = c_3e^{-t}.$ . . . (vi)

Solving (v) and (vi), we get $y = c_1e^{2t} - \frac{1}{2}(c_2 - c_3)e^{-t}$

$z = c_1e^{2t} - \frac{1}{2}(c_2 + c_3)e^{-t}$

So the required complete solution is $x = c_1e^{2t} + c_2e^{-t}$

$y = c_1e^{2t} - \frac{1}{2}(c_2 - c_3)e^{-t}, \quad z = c_1e^{2t} - \frac{1}{2}(c_2 + c_3)e^{-t}$,

where c_1, c_2, c_3 are arbitrary constants.

Ex. 7. *Solve :* $\frac{dx}{dt} + y = \sin t,\ \frac{dy}{dt} + x = \cos t,\ x(0) = 2$ *and* $y(0) = 0$

From the first equation, we get,

$$y = \sin t - \frac{dx}{dt} \quad \text{. . . (i)}$$

$\therefore$ From the second equation, we get,

$$\cos t - \frac{d^2x}{dt^2} + x = \cos t$$

or, $$\frac{d^2x}{dt^2} - x = 0 \quad \text{. . . (ii)}$$

Let $x = e^{mt}$ be a trial solution of (ii).

Then the auxiliary equation is

$$m^2 - 1 = 0$$

$\therefore \quad m = \pm 1$

$\therefore \quad x = c_1e^{t} + c_2e^{-t}$

$\therefore \quad \frac{dx}{dt} = c_1e^{t} - c_2e^{-t}$

$\therefore$ From (i),

$$y = \sin t - c_1e^{t} + c_2e^{-t}$$

Given $x(0) = 2,\ y(0) = 0$

$\therefore \quad 2 = c_1 + c_2$

$0 = 0 - c_1 + c_2$

$\therefore \quad c_1 - c_2 = 0$

Solving we get $c_1 = c_2 = 1$

Thus the required solution is

$$x = e^t + e^{-t}$$
$$y = \sin t - e^t + e^{-t}$$

EXERCISE

[I] Short Answer Questions

1. Given $\frac{dx}{dt} - 7x + y = 0$ [W.B.U.T. 2007]

$\frac{dy}{dt} - 2x - 5y = 0$

Express x in terms of t.

2. Given $\frac{dx}{dt} + 2y = 4x, \frac{dy}{dt} - 2y = 5x.$

Express y in terms of t.

3. Given $\frac{dx}{dt} - 5x - 4y = 0, \frac{dy}{dt} + x - y = 0.$

Find y as function of t.

4. From the system of two equations $(D+1)x + 2y = 0;$ $3x + (D+2)y = 0$, find x in terms of z where $D \equiv \frac{d}{dz}$.

5. Show that the integral of the equations $\frac{dx}{dt} = -2y$ and $\frac{dy}{dt} = x$ is given by $x^2 + 2y + 2c = 0$.

6. From the two equations $\frac{dx}{dt} + 2x = y$, $\frac{dy}{dt} + 4x = 3y + 10\cos t$ form the differential equation involving the two variables x and t only.

7. Solve : $\frac{dx}{dt} - y = t, \frac{dy}{dt} + x = 1$

8. Solve : $\frac{dx}{dt} = 2y - 1, \frac{dy}{dt} = 1 + 2x$

Answers

1. $x = e^{6t}(A\cos t + B\sin t)$

2. $y = e^{3t}(A\cos 3t + B\sin 3t)$

3. $y = Ae^{3t} + Bte^{3t}$

4. $x = e^{z} + 2e^{-4z}$

6. $\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 10\cos t$

7. $x = c_1\cos t + c_2\sin t + 2$, $y = -c_1\sin t + c_2\cos t - t$

8. $x = c_1e^{2t} + c_2e^{-2t} - \frac{1}{2}$, $y = c_1e^{2t} - c_2e^{-2t} + \frac{1}{2}$

[II] Long Answer Questions

Solve the following equations :-

1. $\frac{dx}{dt} = 3x + 2y, \frac{dy}{dt} + 5x + 3y = 0.$

2. $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0, \frac{dy}{dt} + 5x + 3y = 0.$

3. $\frac{dx}{dt} + 2x - 3y = t, \frac{dy}{dt} - 3x + 2y = e^{2t}.$

4. $Dx + 4x + 3y = t, Dy + 2x + 5y = e^t.$

5. $\left(\frac{d}{dt} + 2\right)x + 3y = 0$, $3x + \left(\frac{d}{dt} + 2\right)y = 2e^{3t}.$

6. $\frac{dx}{dt} - \frac{dy}{dt} + 2y = \cos 2t$

$\frac{dx}{dt} + \frac{dy}{dt} - 2x = \sin 2t.$

7. $\frac{dx}{dt} - \frac{dy}{dt} - y = -e^t$, $x + \frac{dy}{dt} - y = e^{2t}$