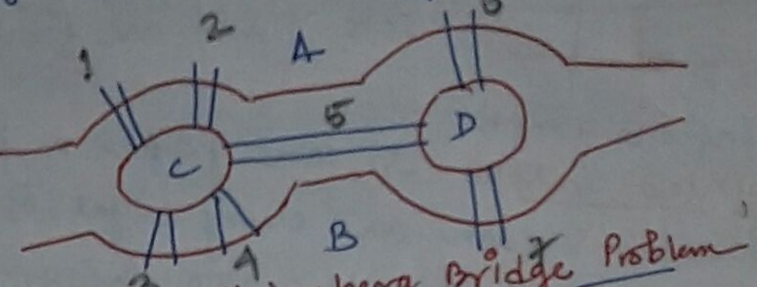
Introduction: Graph theory is an abstract mathematical system. The interest to study this has increased rapidly due to its vast applicability in diff fields like social science, like networking, artificial intelligence, transportation system, operations research and many more.

First Problem On Graph theory:

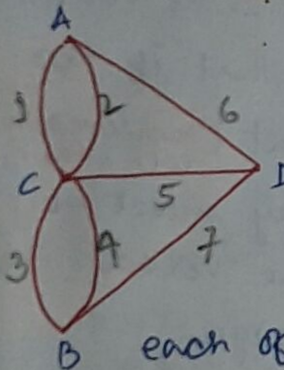


Königsberg Bridge Problem

Graph theory involved while trying finding out the sol^n of the long standing problem known as The Königsberg Bridge problem, solved by Leonard Euler, 1736.

In Soviet Russia there were two islands say C, D formed by the river Pregel. These two islands are also connected to the banks, A, B as well as connected to themselves by 7 different bridges.

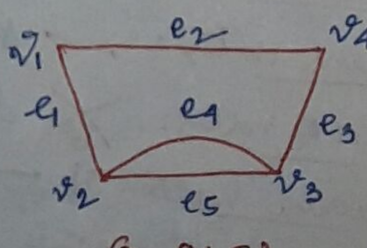
The problem was to start from any of the four lands A, B, C, D and how one has to walk over each of the seven bridges exactly once and has to come back to the starting point. Is that possible =?



This problem was solved by Leonard Euler, by means of transforming the physical problem to a graph and this was the first ever result written in the history of graph theory.

Euler proved that it is not possible to walk over each of the 7 bridges exactly once and return to the starting point.

Def^n of Graph: A graph $G = (V, E, g)$ or, simply $G = (V, E)$ is a mathematical structure consisting of two finite sets V and E, where V is a non-empty set $V = \{v_1, v_2, v_3, ..\}$ are called the vertices and $E = \{e_1, e_2, e_3 ..\}$ are called the edges, and g is called the edge-end point function generally denoted by $g(e_k) = \{v_i, v_j\}$ → which implies e_k is and edge between vertices v_i and v_j.



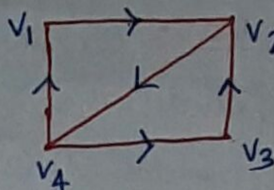
$g(e_1) = \{v_1, v_2\}$
$g(e_2) = \{v_1, v_4\}$
$g(e_3) = \{v_3, v_4\}$
$g(e_4) = \{v_2, v_3\} = g(e_5)$

} Edge Endpoint fn. of G

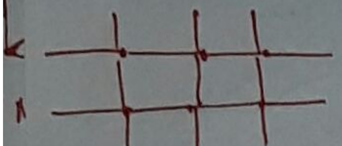
$G = (V, E)$
$V = \{v_1, v_2, v_3, v_4\}$
$E = \{e_1, e_2, e_3, e_4, e_5\}$

Few Definitions:

Digraph: If in a graph G each edge $e = \{v_i, v_j\}$ has a direction from its initial vertex v_i to its terminal vertex v_j.

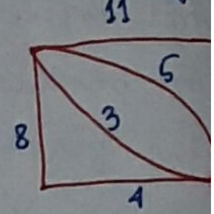


Finite / Infinite Graph: A graph is called finite graph if both its vertex & edge sets V & E are finite, otherwise called infinite graph.



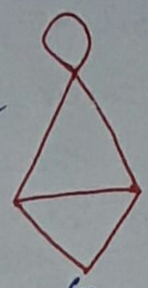
Infinite graph

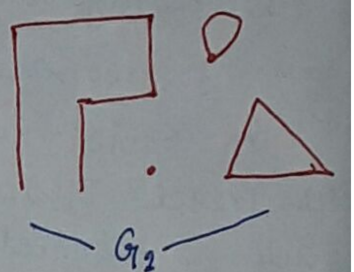
Weighted Graph: In some graph a particular no remains associated to each and every edges of it, these nos are called the weights of the edges and the graph is called weighted.



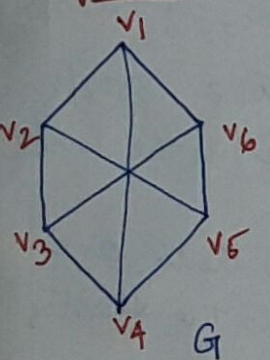
Connected & Disconnected Graph:

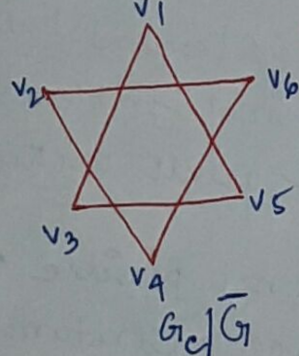
When the all vertices of a certain graph are connected to each other by some edges it is called connected graph, otherwise called disconnected. A disconnected graph may have some components.





Complement of a Graph: The complement of a simple graph G is a (* simple graph – A graph which has no self loop or parallel edges) simple graph $\bar{G}$, which has same vertex set as that of G such that if two vertices (u, v) are adjacent (connected to each other) to each other in G, then they must be not adjacent in $\bar{G}$ & vice versa.





Trivial Graph: A graph with a single vertex but with no edge

Null Graph: A graph " " "

Self Loop: An edge from a vertex to itself.

Parallel Edge: 2/more edges adjacent to same set of pair of vertices.

Proper Edge: An edge neither loop nor parallel edge.

Isolated vertex: • A vertex having no edge incident to it.

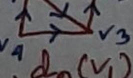
Pendant Vertex: A vertex of degree one.

Degree of Vertex: $d(v) = 4$ The no of edges incident to a vertex.

Adjacent Vertex: If v_1 — v_2 two vertices are connected by an edge.

Incidence: When v_i is an end vertex of e_i, then e_i is called incident to v_i.

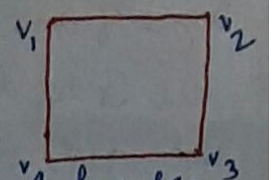
Out-Degree + In-Degree: In digraph the no of edges leaving a vertex is out degree and entering into a vertex is in degree.



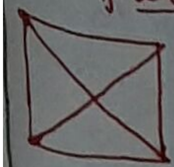
~~d...~~ $d_i(v_1) = 1$, $d_o(v_1) = 2$

CN-1 Pg-2

§ **Regular Graph:** A graph whose all vertices are of equal same degree. $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$.



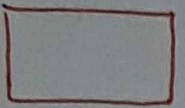
§ **Complete Graph:** A simple graph in which there exists exactly one edge between each pair of distinct vertices is called a complete graph.



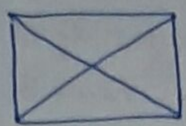
* **Multigraph:** A graph with parallel edges.
* **Pseudograph:** A graph with self loops.

Note:

* A complete graph is always a regular graph as all the vertices are of same degree; But a regular graph is not necessarily be complete.
* If a graph is complete then it must essentially be simple regular as there is exactly one edge between every pair of vertices; But if a graph is simple regular it must not essentially be complete.



simple regular But Not complete



complete & simple regular

Complete graph with n vertex has $^nC_2$ edges

Few Theorems:

1. **Handshaking Th:** The sum of degrees of all the vertices of an undirected graph $G(V,E)$ is twice the sum of the edges in the graph. $\sum_{i=1}^{n} d(v_i) = 2e$

Proof: Suppose, e be an edge of the graph G.
If e is a self loop then it contributes degree 2 to the sum of all the degrees.
But if e is not a self loop, then e must be incident on two vertices say v_1 & v_2. Therefore e contributes degree 1 to v_1 and 1 to v_2.
Hence all the edges contributes $2e$ degree to the sum of degrees of all vertices. $\therefore \sum_{i=1}^{n} d(v_i) = 2e$

2. The no. of vertices of odd degree in a graph is always even.

Proof: Let G be a graph with e edges and n vertices $v_1, v_2, \ldots, v_n$

Now, we know $\sum_{i=1}^{n} d(v_i) = 2e$

Let v_o denotes the vertices with odd degree and v_e denotes the even deg vertices

$\therefore \sum d(v_o) =$ sum of odd deg vertices

$\sum d(v_e) =$ sum of even deg vertices

$\therefore \sum d(v_i) = \sum d(v_o) + \sum d(v_e) = 2e$

as $\sum d(v_e)$ must be an even no as it is sum of all even nos.

Therefore $\sum d(v_o)$ must also be even as rhs is $2e$ an even no.

Therefore no. of odd degree vertices must be even.

3. Degree of any vertex in a simple graph with n vertices can't exceed $n-1$.

Pf: Let G be a simple graph and v be any vertex of G.

Let $d(v) = k$, therefore k is the no of edges incident to v.

Now since G is a simple graph so it must be free from self loops or parallel edges. Therefore other ends of these k edges must be distinct other than v.

Therefore there exists atleast k no. of vertices in G apart from v.

$\therefore n \geq k+1 \therefore k \leq n-1$. ie, $d(v) \leq n-1$ Hence the proofs

Few Problems:

1. How many vertices are necessary to construct a graph with exactly 12 edges where each vertex are of deg. 3.

Sol: $\sum_{i=1}^{n} d(v_i) = 2e$, Let there are n vertices

Here all $d(v_i) = 3$ and $e = 12$

$\therefore 3 \times n = 2 \times 12 \Rightarrow n = 8$ Ans

2. G be a non-digraph with 12 edges. If G has 6 vertices each of deg 3 and remaining vertices have deg less than 3. Find the minimum no. of vertices G may have?

Sol: Let G has n vertices say $v_1, v_2, \ldots, v_6, v_7, \ldots, v_n$

Let $d(v_1) = d(v_2) = \ldots = d(v_6) = 3$ rest $v_7, \ldots, v_n$ are of deg less than 3

and no of edges $e = 12$ $\therefore \sum_{i=1}^{n} d(v_i) = 2 \times e = 2 \times 12 = 24 \longrightarrow (1)$

again $\sum_{i=1}^{n} d(v_i) = d(v_1) + d(v_2) + \ldots + d(v_6) + d(v_7) + \ldots + d(v_n)$

$< 3 \times 6 + 3(n-6)$ if suppose rest $(n-6)$ vertices has also deg 3 $\longrightarrow (2)$

$\therefore (1) \& (2) \Rightarrow 24 < 18 + 3n - 18 \Rightarrow 3n > 24 \Rightarrow n > 8$

So, Therefore the graph can have minimum $(8+1) = 9$ vertices.

3. Is it possible to have a group of 9 people at a party such that each of them ~~have~~ have friendship with exactly 5 others in the party.

Sol: Suppose consider the relation of the members of the party as a graph

Let 9 people are 9 vertices say $v_1, v_2, \ldots, v_9$.

Now let us define the friendship between any two person as an edge ie, suppose v_1 & v_2 are friend then there must be an edge say e_1 betn them.

Now if each of 9 persons in the party must be friend to exactly 5 pers. this means degree of every vertex is 5.

But we know that the no of odd deg vertex in a graph is always even

But 9 is not even. So it is not possible.

1. If a simple-regular graph has n vertices and 24 edges. Find all possible values of n. $\boxed{CN-2}$ $\boxed{Pg-1}$

Sol: Since the graph is simple and regular, i.e., the degree of every vertex is same

Let k be the degree of every vertex. i.e, $d(v_i) = k$

Therefore $\sum_{i=1}^{n} d(v_i) = n \times k \longrightarrow (1)$

Again the graph has 24 edges So, $\sum_{i=1}^{n} d(v_i) = 2 \times 24 \longrightarrow (2)$

$\therefore \quad nk = 2 \times 24 \Rightarrow n = \frac{48}{k} \longrightarrow (2)$

We know the max. no of edges in a simple-regular graph is $\frac{n(n-1)}{2}$

i.e, $e \le \frac{n(n-1)}{2} \Rightarrow 24 \le \frac{n(n-1)}{2} \Rightarrow n(n-1) \geq 48 \longrightarrow (3)$

from (2) & (3) Let $k = 1 \Rightarrow n = 48$ — satisfies (3)

$k = 2 \Rightarrow n = 24$ satisfies (3)

$k = 3 \Rightarrow n = 16$ satisfies (3)

$k = 4 \Rightarrow n = 12$ satisfies (3)

$k = 5 \Rightarrow n =$ non-integer (impossible for vertex can't be fraction)

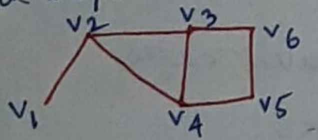
$k = 6 \Rightarrow n = 8$ satisfies (3)

$k = 7 \Rightarrow n =$ non-integer

$k = 8 \Rightarrow n = 6$. does not satisfy (3) as $6 \times 5 = 30 \ngeq 48$

So, all possible values of n are 48, 24, 16, 12, 8.

* Degree Sequence: If the degrees of the vertices of a graph G are listed in a sequence S, then it is called degree sequence of G.



$\{1, 3, 3, 3, 2, 2\}$
$v_1, v_2, v_3, v_4, v_5, v_6$

Prob 4. Is it possible to draw a graph with 4 edges, 4 vertices with deg sequence $\{1, 2, 3, 4\}$

Sol: Since the degree sequence is $\{1, 2, 3, 4\}$

i.e, sum of all the degrees is $\sum_{i=1}^{4} d(v_i) = 1+2+3+4 = 10$

So, no of edges will be $2e = 10 \Rightarrow e = 5$, But it is given that the graph has 4 edges. So, it is not possible to draw such graph.

* Fundamental no of a graph:

Let G be a graph with n vertices, e edges and k components. Then n, e, k are called the fundamental no of G. as these are independent.

* Rank & Nullity:

Rank is defined as $(n-k) = r \geq 0$

Nullity is defined as $(e-n+k) = e - r \geq 0$

* $n \geq k$

* $e \geq n-k$

i.e, $e \geq r$

Few more theorem

Th 4 Show that the number of edges in a simple graph with n vertices can't exceed $\frac{n(n-1)}{2}$.

Proof: Let G be a simple graph with n vertices say $v_1, v_2, \dots, v_n$ and e no of edges.

We know $\sum_{i=1}^{n} d(v_i) = 2e$

$\Rightarrow d(v_1) + d(v_2) + \dots + d(v_n) = 2e$

Since G is a simple graph, therefore $d(v_i) \leq n-1 \quad i=1,2,\dots,n$

$\therefore \quad d(v_1) + d(v_2) + \dots + d(v_n) = 2e$

$(n-1) + (n-1) + \dots + (n-1) \geq 2e$

or, $n(n-1) \geq 2e$

$\Rightarrow e \leq \frac{n(n-1)}{2}$

Th.5 Show that a complete graph with n vertices consists of $\frac{n(n-1)}{2}$ edges

Pf. Since the graph is complete, so it must be a simple graph. ie, no loop / ||el edge

Therefore every vertex has exactly (n-1) adjacent vertices.

ie, deg. of every vertex is exactly (n-1)

Since there are n vertices

So, sum of all degrees of vertices $\sum d(v_i) = n(n-1)$

As $\sum_{i=1}^{n} d(v_i) = 2e = n(n-1)$

$\therefore e = \frac{n(n-1)}{2}$

Th. 6 The maximum no of edges in a simple connected graph with n vertices is $\frac{n(n-1)}{2}$.

Pf. Let G be a simple, connected graph with n vertices.

Since the graph is simple so, it has no loops or parallel edges.

Therefore there exists only one edge between any two vertices.

So, maximum no of edges in G is ${}^nC_2 = \frac{n(n-1)}{2}$

Th.7 Prove that a simple graph with $(n \geq 2)$ vertices has atleast two vertices of same degree

Pf. Let us consider the vertices $v_1, v_2, \dots, v_n$ in G.

Since G is simple, so it has no loop or paralled edge.

Let $d(v_i)$ = no. of edges incident to v_i

clearly $0 \leq d(v_i) \leq n-1$, $i=1,2,\dots n$

Now we start with contradiction, say no two vertices in G have same degree.

therefore $d(v_i)$ has values all distinct from the integers $0, 1, 2, \dots (n-1)$

So, let one vertex v_1 has degree $n-1$ ie, $d(v_1) = n-1$

This mean this vertex is connected to every other vertex of the graph.

So there should not be any vertex of degree 0. So, it's a contradiction.

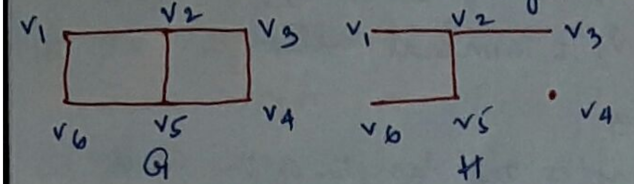
Hence there must exist atleast two vertices of same degree.

What is Subgraph?

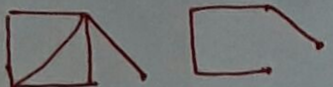
Let G and H be two graphs. Then H is called a subgraph of G if

$V(H) \subseteq V(G)$ i.e, if vertex set of H is a subset of the vertex set of G

$E(H) \subseteq E(G)$ and edge set of H is a subset of the edge set of G.

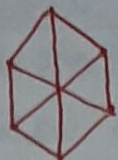


Spanning Subgraph: A subgraph H of G is called a spanning subgraph of G if $V(H) = V(G)$ i.e, all vertices of G are present in H.



Complement of a Graph: The complement of a simple graph G is another simple graph $\bar{G}$ which has same vertex set as of G with a condition that if two vertices u, v are adjacent in G then they must be non-adjacent in $\bar{G}$ and vice-versa.

Problems: Find the complement of the graph



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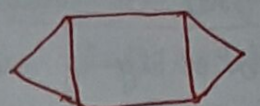
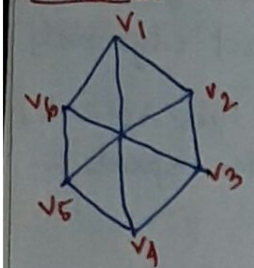
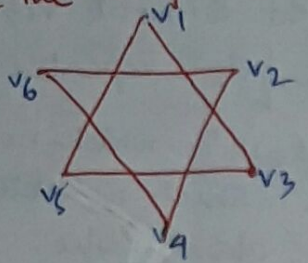
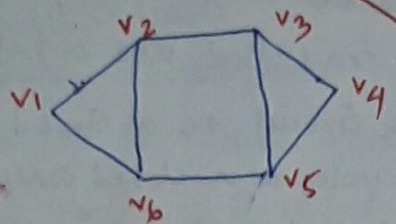
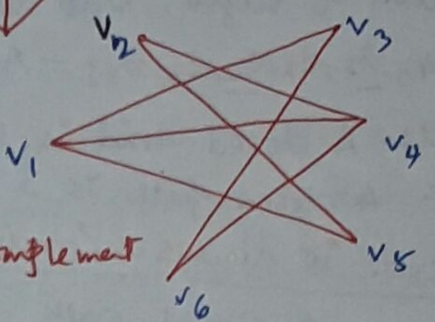










Th: G be a simple graph with n vertices and $\bar{G}$ be its complement for any arbitrary vertex v in G. Show that

$d(v)$ in G + $d(v)$ in $\bar{G}$ = $n-1$

Pf: Let $d(v)$ in G = K. This implies there are K no of adjacent vertices of v in G. (Since G is simple)

This implies by def^n of complement, in $\bar{G}$ v must not be adjacent to these K vertices. In other words v is not connected to $(n-1-K)$ vertices in $\bar{G}$.

So, v must be connected to $(n-1-K)$ vertices in $\bar{G}$

i.e, $d(v)$ in $\bar{G} = (n-1-K)$

$\therefore$ $d(v)$ in G + $d(v)$ in $\bar{G}$ = $\cancel{K} + n-1 - \cancel{K} = n-1$ Proved

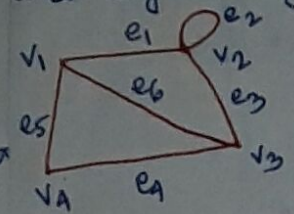
Connected & Disconnected Graph: Done

Component: A disconnected graph is formed by two or more connected graphs. Each of these connected subgraphs are called component graph.

Distance Between 2 Vertices: G be a ~~connected graph~~.

Few Defⁿ Walk, Path, Cycle, Circuit

Walk: A walk in a graph G is a finite and alternating sequence of vertices and edges, begining and ending with vertices.



An example of walk is $\{v_1, e_1, v_2, e_2, v_2, e_3, v_3\}$
Here v_1 is initial vertex & v_3 is terminal vertex.

* Length of the above walk is 3

Length of walk: The no of edges in a walk denotes the length of the walk.

Open & Closed walk: If the initial and terminal vertices of the walk coincide then it is closed otherwise open.

Trail: An open walk with no repeated edges is a trail.

Path: An open walk with no repeated vertices is a path

* A path is always a trail, but converse is not always true.
as obviously if vertices are not repeated in a path, there is no question of repeation of edges. But vertices may be repeated without repeating ~~vertices~~ edges

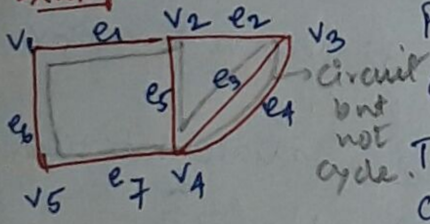
$\{v_1, e_1, \underline{v_2}, e_2, \underline{v_2}, e_3, v_3\}$ – A trail But not a path [v_2 is repeated but edge not]

Circuit: A closed trail is a circuit. or, a closed walk in which no edges repeated

Cycle: A closed path is a cycle. or a closed walk in which no vertices repeated.

* A cycle is always a circuit but converse is not always true.

Example:



A closed walk $\{v_2, e_2, v_3, e_3, v_4, e_4, v_3, e_2, v_2\}$
Path $\{v_2, e_2, v_3, e_3, v_4\}$
Cycle $\{v_2, e_2, v_3, e_3, v_4, e_5, v_2\}$
Trail $\{v_2, e_2, \underline{v_3}, e_3, v_4, e_4, \underline{v_3}\}$
Circuit $\{v_3, e_4, v_4, e_3, v_3\}$ – No edge repeated

So, Trail – No repeated Edge
Path – No repeated Vertex

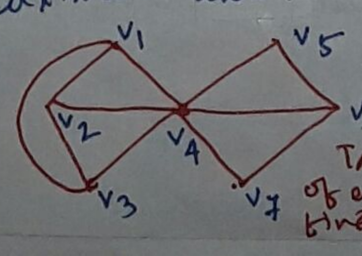
* A circuit where no vertex except ~~first~~ initial & terminal vertices are repeated is a cycle.

* $\{v_2, e_5, v_4, e_3, v_3, e_4, v_4, e_7, v_5, e_6, v_1, e_1, v_2\}$ → A circuit but not a cycle.

Connected & Disconnected Graph: If in a graph G there exists a walk between any two vertices of the graph, then it is connected, otherwise disconnected.

Distance betⁿ 2 vertices: If G be a graph (connected), then the distance betⁿ any two vertices u & v of G is the length of the shortest path betⁿ u & v it is denoted by $d(u, v)$.

Diameter of a connected graph: If G be a connected graph, then the diameter of the graph is the length of the maximum distance between any two vertices u & v. it is denoted by $\text{dim}(G)$



$d(v_2, v_6) = 2$ *
$\text{dim}(G) = 2$
Take the set of distances of every pair of vertex - Then find max among them as dim.

Distance – Shortest path betⁿ u, v
Diameter – ~~Longest path betⁿ~~ u, v

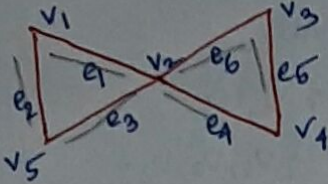
Few Concepts

Eulerian & Hamiltonian Graph:

Euler Circuit: A circuit in a graph is called an Euler Circuit if it contains all the edges of the graph.

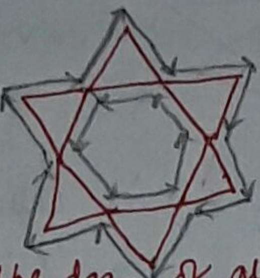
Euler Graph: A graph is called an Euler graph, if it contains an Euler circuit.

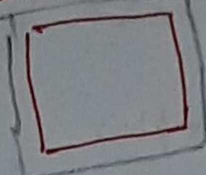
Example:



Eg: { $v_2, e_1, v_1, e_2, v_5, e_3, v_2, e_4, v_4, e_5, v_3, e_6, v_2$ }

Eulerian circuit





* A graph is Eulerian if the deg. of all the vertices are even.

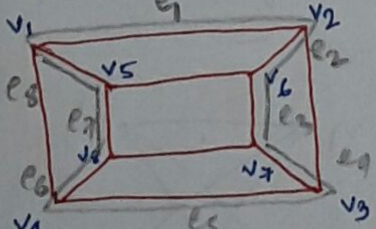
A graph can only be Eulerian if all the vertices are of even degree. If any one of the vertices in the graph is odd, then it can't be an Euler graph as then it can't contain an Euler circuit.

If you need to have an Euler circuit, then you must create a circuit without repeating any edge at the same time you have to cover all the edges of the graph. So, if any vertex is of odd degree then every time you get into the vertex, while coming out of it you must have to repeat an edge.

Hamiltonian Cycle: A cycle in a graph is called Hamiltonian cycle if it contains all the vertices of the graph.

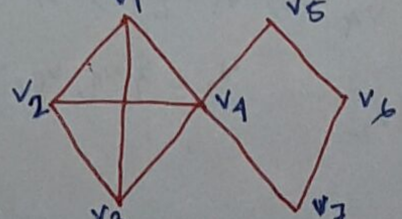
Hamiltonian Graph: A graph is called Hamiltonian if it contains a Hamiltonian cycle.

Example:



Eg: { $v_1, e_1, v_2, e_2, v_6, e_3, v_7, e_4, v_3, e_5, v_4, e_6, v_8, e_7, v_5, e_8, v_1$ }

Hamiltonian cycle



Prob. Find Diameter of the graph

$d(v_2, v_6) = d(v_3, v_6) = 3$

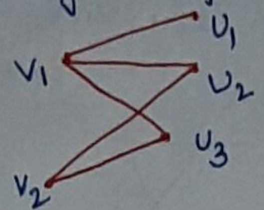
This is the maximum length of all the distances between any two arbitrary distinct pair of vertices.

So, $dim(G) = 3$.

List down the distances then max out of that will be the diameter.

§ Bipartite Graph:

A graph $G = \{V, E\}$ is called bipartite if its vertex set V can be decomposed into two disjoint subsets V_1 & V_2 such that each edge in G connects a vertex of V_1 and another vertex of V_2, but there exist no edge that connects betn any two pair of vertices belonging to the same set V_1 or V_2. Eg:



$V = V_1 \cup V_2$

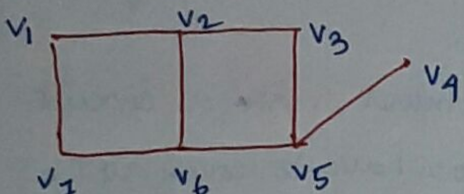
$V = \{v_1, v_2\}$

$U = \{u_1, u_2, u_3\}$

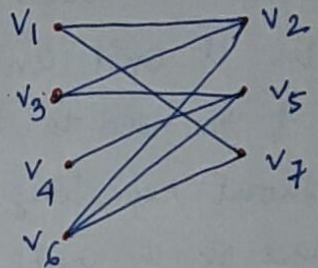
Complete Bipartite graph:

A bipartite graph $G = \{V_1, V_2, E\}$ is called a complete bipartite graph if there exists an edge between every pair of vertices of V_1 & V_2.

Prob: Check the following graph is bipartite?

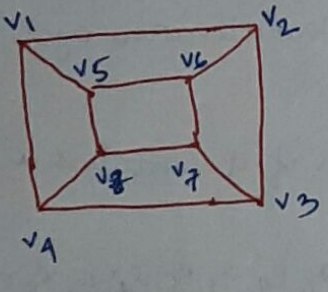


Try to partite the set of the vertices $V = \{v_1, v_2, \ldots, v_7\}$ into disjoint sets V_1 & V_2 such that it can be bipartite



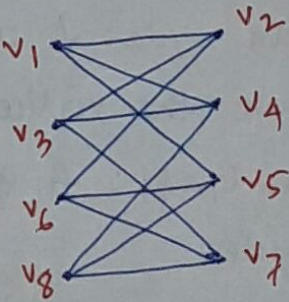
So, it is bipartite, the partite sets are

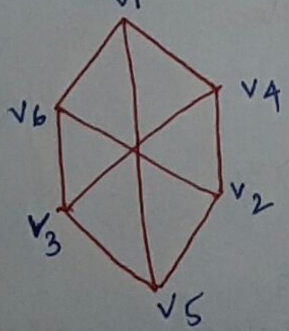
$V_1 = \{v_1, v_3, v_4, v_6\}$ & $V_2 = \{v_2, v_5, v_7\}$



$\Rightarrow V_1 = \{v_1, v_3, v_6, v_8\}$

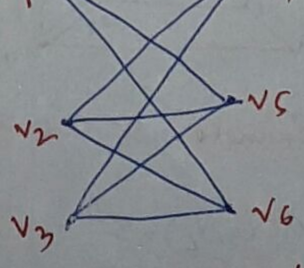
$V_2 = \{v_2, v_4, v_5, v_7\}$





$\Rightarrow V_1 = \{v_1, v_2, v_3\}$

$V_2 = \{v_4, v_5, v_6\}$



Example of complete Bipartite graph.

Th: A bipartite graph can't contain a cycle of odd length.

Proof: Let G be a bipartite graph $G(V, E)$ where $V = V_1 \cup V_2$

(*) → P.T.O.

① If G be a bipartite graph with 22 vertices with partite sets U & V, where $|U| = 12$. Suppose every vertex of U has degree 3, while every vertex of V has degree either 2 or 4. How many vertices of G have degree 2?

$\boxed{\text{5 marks}}$

Sol^n $|U| = 12$ $\therefore |W| = 22 - 12 = 10$

Let $u_i \in U$ & $w_i \in W$

$$\sum_{i=1}^{12} d(u_i) + \sum_{j=1}^{10} d(w_j) = 2e$$ [by Handshaking Theorem]

Since every vertex in U is 3.

So, $\sum d(u_i) = 3 \times 12 = 36$

Now this is the total no of edges in the graph

So, no of edges in the graph $e = 36$

Now let W has x vertices of degree 2

So, $(10 - x)$ vertices of degree 4

So, By handshaking theorem

$$12 \times 3 + x \times 2 + (10 - x) \times 4 = 2 \times 36$$

or, $36 + 2x + 40 - 4x = 72$

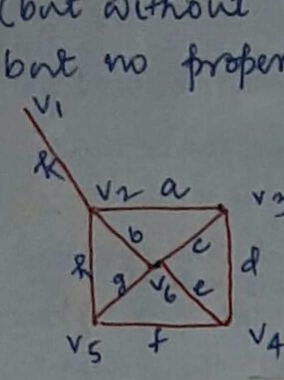
or, $x = 2$

Therefore G has 2 vertices of degree 2.

Few more definitions

Cut Set: (Set of Edges)

(min no of edges)
In a connected graph G, a cut set is a set of edges whose removal (but without removing the end points) from G leaves G disconnected, but no proper subset of that set will do the same.



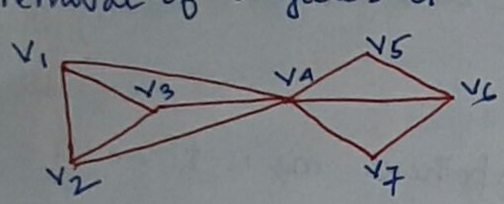
$\{k\} \rightarrow$ cut set, smallest

$\{a, b, g, f\} \rightarrow$ cut set

But $\{a, b, h, f\} \rightarrow$ Not a cut set as a proper subset of this set $\{a, b, h\}$ is already a cutset.

Cut Vertex: (A vertex)

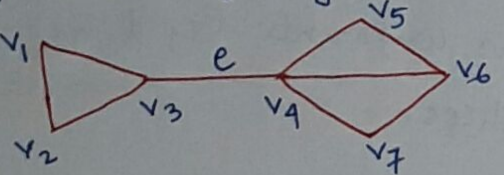
A vertex v of a connected graph G is said to be a cut-vertex if removal of v yields G as disconnected graph.



v_4 is a cut vertex, as removal of v_4 yields the graph to be disconnected into two components.

Cut Edge: (An Edge)

An edge e of a connected graph G is said to be a cut edge if removal of e yield G as disconnected.



Here e is a cut edge.

* What will be the least cardinality of a cut set in a Graph.

This answer is the least cardinality of a cut set in a graph is equal to the least deg. of the vertex (of any vertex) in Graph.

(*) here V_1 & V_2 are disjoint sets.

Now any edge $e \in E$ must connect a vertex from V_1 to V_2

So, let $C = \{v_1, e_1, v_2, e_2, \ldots, v_n, e_n, v_1\}$ be a cycle of n length, we need to prove n is even.

Now if $v_1 \in V_1$ then $v_2 \in V_2$ as v_1 & v_2 can't belong to the same set. Now if $v_2 \in V_2$ then $v_3 \in V_1$ as v_2 & v_3 can't belong to the same set.

Now if e_n be the edge betn v_n & v_1 as $v_1 \in V_1$ so, $v_n \in V_2$. So, n must be even. $\therefore$ the length is even.

Th.9 The minimum number of edges in a connected graph with n vertices is $n-1$

Pf: We will prove the theorem by method of mathematical induction on the no of edges. Let m be the no of edges, where n is the number of vertices

So, we need to prove that $m \geq n-1$ —— (1)

Case-I when $m=0$, the graph consists of only one isolated vertex i.e, $n=1$.

Case-II when $m=1$, the graph consists of only one edge, therefore $n=2$. thus (1) follows. i.e, $m = 1 \geq 2-1$

Since for initial values of m, the result holds good, so let the result is true for any arbitrary no of edges m, i.e, $m \geq n-1$, where no of vertices in the graph, with this hypothesis if we can show it for a graph with no of edges $m+1$ & with n no. of vertices, then we are done.

So, let G be a graph with $(m+1)$ edges. [we need to show $m+1 \geq n-1$]

let e be any edge out of that. Now $G \setminus e$ will be a subgraph of G say G'

Now G' has m edges and n vertices.

Here two cases may arise

Case-I G' is connected, If so, then by our hypothesis $m \geq n-1$ as the result holds good for m edges

i.e, $m \geq n-1 \Rightarrow m+1 \geq n-1+1 = n > n-1$

i.e, $m+1 > n-1$

Case-II G' is disconnected, If so, let G' has two components G_1 & G_2 with m_1 & m_2 edges. and with n_1 & n_2 vertices.

i.e, $m_1 + m_2 = m$ & $n_1 + n_2 = n$

Now since m_1 & m_2 are less than m ($m_1 < m$, $m_2 < m$) so the result is true for both G_1 & G_2

i.e, $m_1 \geq n_1 - 1$ & $m_2 \geq n_2 - 1 \Rightarrow m_1 + m_2 \geq (n_1 + n_2) - 2$

i.e, $m \geq n-2$

i.e, $m+1 \geq n-1$ Hence the result

Th 10. The no of edges in a simple connected graph with n vertices can't exceed $(n-1)$

Sol: Let G be a simple graph with n vertices. Therefore G does not have any self loops or parallel edges.

So, if we consider any vertex v, then this vertex can be connected maximum with rest $(n-1)$ vertices. Therefore the no of edges in a simple connected graph can't exceed $(n-1)$.

Prob: Find the minimum and maximum no of edges of a simple connected graph with 15 vertices.

Sol: Max-no of edges of a graph (simple, connected) with n vertices is ${}^{n}C_2$

Min-no " " " $n-1$

So, max edge $= {}^{15}C_2$

Min edge $= 14$

Th 11. Show that in a simple graph with n vertices and k components can have maximum $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof: Suppose G be a simple graph with n vertices and k components. Let us name the components as $G_1, G_2, \dots, G_k$ and let the vertices in these k components are $n_1, n_2, \dots, n_k$

$\therefore \sum_{i=1}^{k} n_i = n$, since G is a simple graph, so all the edges in G are proper edge.

So maximum no of edges in every component can be ${}^{n_i}C_2$.

Therefore maximum no of edges in every component can be... if $e_1, e_2, \dots, e_n$ are edges in the components

So, if e be the total edges in G and

So, $e = \sum_{i=1}^{k} e_i$, as $e_i \le {}^{n_i}C_2$

$$\therefore e \le \sum_{i=1}^{k} {}^{n_i}C_2 = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right] = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right] \longrightarrow (1)$$

need to show $e \le \frac{(n-k)(n-k+1)}{2}$

Again, since all the components G_i are simple, so max edge in each component can be max. $n_i - 1$, let for each $e_i = n_i - 1$ $\quad e_i \ge n_i - 1$

$$\therefore e = \sum_{i=1}^{k} e_i = \sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = (n-k) \qquad e \ge (n-k)$$

Squaring bothside, $\left\{\sum_{i=1}^{k}(n_i-1)\right\}^2 = (n-k)^2 = n^2 - 2nk + k^2$

$$\text{or, } \sum_{i=1}^{k}(n_i-1)^2 + \sum_{i=1}^{k}\sum_{\substack{j=1 \\ i\ne j}}^{k}(n_i-1)(n_j-1) = n^2 - 2nk + k^2$$

$$\text{or, } \sum_{i=1}^{k}(n_i-1)^2 \le n^2 - 2nk + k^2 \quad \left[\begin{array}{l}\text{Since all } G_i\text{'s are non-trivial graph} \\ \text{So, } n_i \ge 1 \text{ i.e. } (n_i - 1) \ge 0 \\ \qquad (n_j - 1) \ge 0\end{array}\right]$$

$$\text{or, } \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1^2 \le n^2 - 2nk + k^2$$

$$\text{or, } \sum_{i=1}^{k} n_i^2 \le n^2 - 2nk + k^2 - 2n - k \quad \left[\text{as } \sum_{i=1}^{k} n_i = n \text{ \& } \sum_{i=1}^{k} 1^2 = k\right]$$

$$\text{or, } \sum_{i=1}^{k} n_i^2 \le n^2 - (k-1)(2n-k) \quad [\text{factorize } (k-1)(2n-k) = 2nk - k^2 - 2n + k] \longrightarrow (2)$$

from (1) & (2) $e \le \frac{1}{2}\left[n^2 - (k-1)(2n-k) - n\right] = \frac{1}{2}(n^2 - 2nk + k^2 + 2n - k - n)$

$= \frac{1}{2}(n^2 - nk + n - nk + k^2 - k)$

$= \frac{1}{2}(n-k)(n-k+1)$ Proved

Q.12. Show that minimum no of edges in a simple disconnected graph with n vertices and k components is $(n-k)$

Proof: Suppose G is a simple disconnected graph with n vertices & k components

Let us denote the components as $G_1, G_2, \ldots, G_k$

and also let the vertices as $n_1, n_2, \ldots, n_k$

and also let the edges as $e_1, e_2, \ldots, e_k$

Now since all the components are simple, so, minimum no of edges in each component are $(n_i - 1)$

ie, $e_i = n_i - 1$ $\therefore \sum_{i=1}^{k} e_i = \sum_{i=1}^{k} (n_i - 1) \Rightarrow e = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = (n-k)$

Prob: Find minimum & maximum no of edges of a simple graph having 15 vertices and 4 components.

Soln Here total no of vertices $n = 15$

" " components $K = 4$

So, min no of edges $= n - K = 15 - 4 = 11$

max no of edges $= \frac{(n-K)(n-K+1)}{2} = \frac{(15-4)(15-4+1)}{2} = 66$

Connected Graph (n vertex)	Disconnected graph (n vertex, k components)
* min no of edge $e_{min} = n-1$	* min no of edges $e_{min} = n-k$
* max no of edge $e_{max} = {}^nC_2$	* max no of edges $e_{max} = \frac{(n-k)(n-k+1)}{2}$
$\therefore$ $n-1 \le e_{G_c} \le {}^nC_2$	$n-K \le e_{G_d} \le \frac{(n-K)(n-K+1)}{2}$
$e_{min} \le e_{G_c} \le e_{max}$	$e_{min} \le e_{G_d} \le e_{max}$

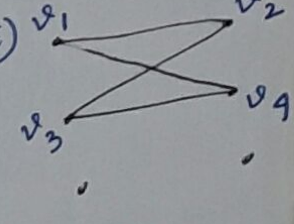
Th 13 If a graph has exactly two vertices of odd degree, then there must exist a path joining these two vertices.

Pf. Let G be a graph having two odd degree vertices. Now either G is connected or disconnected.

Case-I G is connected. So, if G is connected, then therefore there must exist a path betn the two vertices say v_1 & v_2 which are the odd degree vertices. So, the theorem is trivially true.

Case-II G is disconnected. Let G is not connected and has two components say g_1 & g_2 s.t $v_1 \in g_1$ & $v_2 \in g_2$. Then G becomes a graph with exactly two odd degree vertices v_1 & v_2 belonging to two diff components. So, g_1 is a connected graph having only one odd degree vertex so as g_2. But it is a basic contradiction as a graph can't have odd no of odd degree vertices. Therefore G must be a connected graph. So, there must exist a path between v_1 & v_2.

Th. 14 A bipartite graph can't contain a cycle of odd length.



Pf. Let G be a bipartite graph, if G(V, E) then $V = V_1 \cup V_2$ where V_1 & V_2 are two disjoint sets, and each edge of the set E connects between one vertex of V_1 to one vertex of V_2.

Let $c = \{v_1, e_1, v_2, e_2, v_3, e_3, v_4, \ldots v_n, e_n, v_1\}$ be a cycle.

now obviously if $v_1 \in V_1$ then $v_2 \in V_2$ as v_1, v_2 can't belong to the same set. Similarly again if $v_2 \in V_2$ then $v_3 \in V_1$ So, if $v_i \in V_2$ the i must be even & $v_j \in V_1$ then j must be odd

Now since the last edge e_n connects betn v_n & v_1 So, $v_n \in V_2$ So, n is even no.

So, there are n no of edges and it is even.

$C = \{v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, \dots, v_n, e_n, v_1\}$

$v_2, v_4, v_6, \dots, v_n \in V_2$

$v_1, v_3, v_5, \dots, v_{n-1} \in V_1$

There are n edges & n is even.

Dijkstra Algorithm (Determines the shortest path between two given vertices)

We can apply Dijkstra's algorithm to a graph only when the graph is weighted.

A graph is weighted when each of the edges of the graph are assigned with a particular weight. Then shortest path would be a path which bears the smallest possible weight.

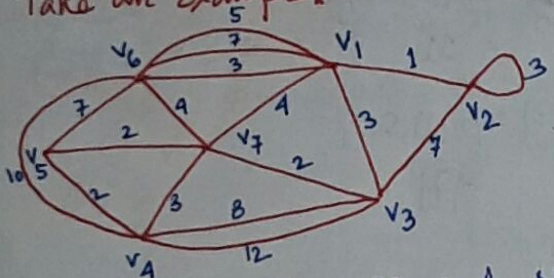
Suppose v_i and v_j be two vertices which are connected by an edge e_{ij}. Then the weight associated to e_{ij} will be denoted by w_{ij}.

The steps of the algorithm:

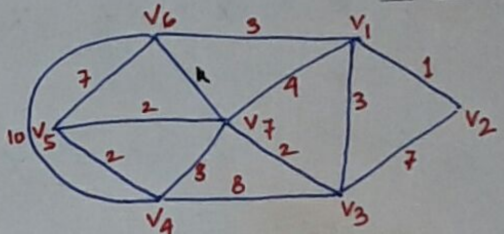
The algorithm is an iterative process which follows the steps mentioned below.

Step-I If the graph is not simple, first turn that to simple, first by discarding all self loops if exists and also discard all parallel edges except which bears the smallest weight among rest of the other.

Take an Example: Find shortest path between v_2 & v_5 in G:



So, after step I The graph turns to ⇒



Step-II Labelling of vertex: A step-II label the vertex as per following

w_{ij} = weight of e_{ij}, {if v_i, v_j are adjacent}

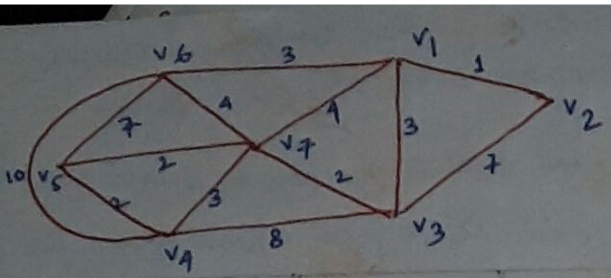
$w_{ij} = \infty$ if v_i, v_j are non-adjacent

$w_{ii} = 0$ as we discard all self loops

Step-III In this step we follow an iterative scheme of re-labelling the vertices. Starting vertex of the path v_2 is labelled permanently to zero. and remaining vertices are labelled to ∞ temporarily. Now once a vertex is permanently labelled it remains unchanged throughout the process.

Step-IV Next step onwards, at each subsequent steps a new vertex is labelled permanently and remaining vertices would be temporarily labelled as per the scheme $v_i = \min[\text{Label of } v_i \text{ at preceeding step, permanent label of vertex } v_j + w_{ij}]$

where v_j is the vertex that has been labelled permanently in the preceeding step. This step is now repeated untill we reach to the terminal vertex i.e v_5 as permanently labelled.



	v_1	v_2	v_3	v_4	v_5	v_6	v_7
step-I	∞	$\boxed{0}$	∞	∞	∞	∞	∞
step-II	$\min[\infty, 0+1] = \boxed{1}$	×	$\min[\infty, 0+7] = 7$	$\min[\infty, 0+\infty] = \infty$	$\min[\infty, 0+\infty] = \infty$	$\min[\infty, 0+\infty] = \infty$	$\min[\infty, 0+\infty] = \infty$
step-III	×	×	$\min[7, 1+3] = \boxed{4}$	$\min[\infty, 1+\infty] = \infty$	$\min[\infty, 1+\infty] = \infty$	$\min[\infty, 1+3] = \boxed{4}$ Alternative	$\min[\infty, 1+4] = 5$
step-IV	×	×	×	$\min[\infty, 4+8] = 12$	$\min[\infty, 4+\infty] = \infty$	$\min[4, 4+\infty] = \boxed{4}$	$\min[5, 4+2] = 5$
step-V	×	×	×	$\min[12, 4+10]$ 10 $= 12$	$\min[\infty, 4+7] = 11$	×	$\min[5, 4+4]$ 8 $= \boxed{5}$
step-VI	×	×	×	$\min[12, 5+3] = 8$	$\min[11, 5+2] = \boxed{7}$	×	×
step-VII	×	×	×	$\min[8, 7+2] = \boxed{8}$	×	×	×
step-VIII	×	×	×	×	×	×	×

Step-I In the above table, look at the step where the terminal vertex v_5 is permanently labelled. i.e, in step-VI. Then look straight upright along this column, search where the valued got changed. Its changed immediately up in the step-V. Then in this step-V look row-wise and search for the vertex which got permanently labelled (i.e, inside box). Here it is v_7. Then include v_7 after v_5 in the desired shortest path.

Step-II In Next steps keep repeating the above process untill you reach to the intial vertex of the path i.e, v_2.

the desired shortest path as

Hence after this process we get the

$v_5 - v_7 - v_1 - v_2$. Weight of this path is $2+4+1=7$

Matrix Representation of Graph:

Although diagramatic representation of graph is very convenient for the visual study, but it has limited usefulness as its only possible to draw a graph if the the no. of vertices are reasonably small. Why we prefer the representation:

* The matrix is commonly used to represent graphs for computer processing.
* This matrix representation can readily be applied to the study the structural properties of graphs from algebraic point of view.

Types of Matrix representation

There are mainly two types of matrix representation of graphs.

1. Incidence Matrix
   - for directed graph
   - for undirected graph
2. Adjancency Matrix.
   - for directed graph
   - for undirected graph.

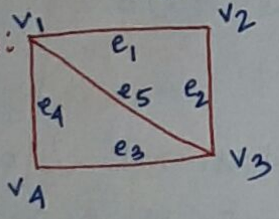
Defn of Incidence Matrix (Undirected Graph)

Let $G=\{V,E\}$ be an undirected graph with no-self-loops having vertex set $V=\{v_1,v_2,..,v_n\}$ and edge set $E=\{e_1,e_2,..,e_m\}$.

Then the incidence matrix of this graph $I=\{a_{ij}\}_{n\times m}$ is an $n\times m$ binary matrix whose n rows corresponds to n vertices and m columns corresponds to m edges, such that

$$a_{ij}=\begin{cases}1 & \text{if edge } e_j \text{ is incident to vertex } v_i\\ 0 & \text{o.w}\end{cases}$$

Example:



$$I(G)=\begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5\\ \hline v_1 & 1 & 0 & 0 & 1 & 1\\ v_2 & 1 & 1 & 0 & 0 & 0\\ v_3 & 0 & 1 & 1 & 0 & 1\\ v_4 & 0 & 0 & 1 & 1 & 0\end{array}_{4\times5}$$

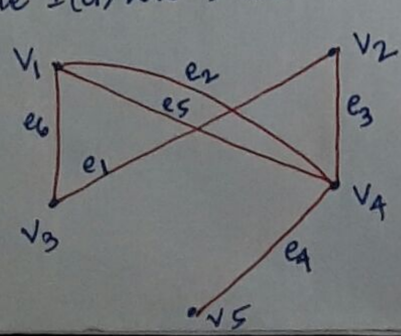
observations:

1. The entries of $I(G)$ is either 1 or 0, ie binary digit
2. The number of 1's in each row is equal to the degree of the corresponding vertex
3. A row with all 0's corresponds to an isolated vertex.
4. Every column of $I(G)$ has exactly 2 1's, as every edge is incident to exactly two vertices.
5. If two columns of $I(G)$ are identical, this imply these two edges are parallel.
6. Permutation of any two rows/columns corresponds to relabelling the vertices & edges.

Prob Draw the graph corresponding to the incidence matrix

$$I(G)=\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6\\ \hline v_1 & 0 & 1 & 0 & 0 & 1 & 1\\ v_2 & 1 & 0 & 1 & 0 & 0 & 0\\ v_3 & 1 & 0 & 0 & 0 & 0 & 1\\ v_4 & 0 & 1 & 1 & 1 & 1 & 0\\ v_5 & 0 & 0 & 0 & 1 & 0 & 0\end{array}$$

Here $I(G)$ has 5 rows & 6 columns. ⇒ there are 5 vertices, 6 edges.



* An observation:
It is not possible that in an Incidence matrix, No column can have all zeros as its entry. that is meaningless.

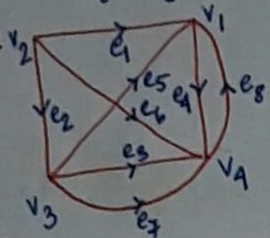
Defn of Incidence Matrix (Directed Graph)

Let $G = \{V, E\}$ be a directed graph with no self loops, having vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$.

Then the incidence matrix of G, $I = \{a_{ij}\}_{n \times m}$ is an $n \times m$ ~~binary~~ matrix whose n rows corresponds to n vertices and m columns corresponds m edges, such that

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is intial vertex of edge } e_j \\ -1 & \text{if } v_i \text{ is terminal " " } \\ 0 & \text{if } e_j \text{ is not incident to } v_i \end{cases}$$

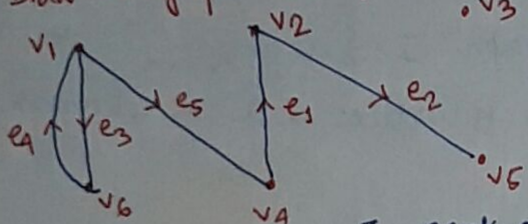
Example:



$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & -1 & 1 \end{bmatrix}_{4 \times 8} \end{array}$$

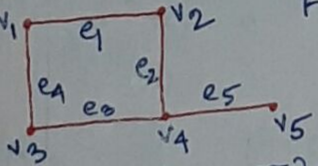
Prob: Draw the graph corresponding to the incidence matrix

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$



Rank of Incidence Matrix: The rank of an incidence matrix of a connected (or digraph) graph having n vertices (ie n rows) is $(n-1)$.

Prob: Verify that the rank of the incidence matrix of the following graph G is one less than the no. of its vertices.



First let us construct the

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

we will perform some elementary row-operations on $I(G)$ to get its rank. We will transform the matrix to row-reduced echelon-form. Then the no. of non zero row will be the rank.

$$I(G) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} R_2' = R_1 - R_2$$

So, rank of $I(G)$ is 4

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} R_4' = R_4 + R_2$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} R_4' = R_3 - R_4$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_5 = R_5 + R_4$$

Incidence matrix of a disconnected graph:

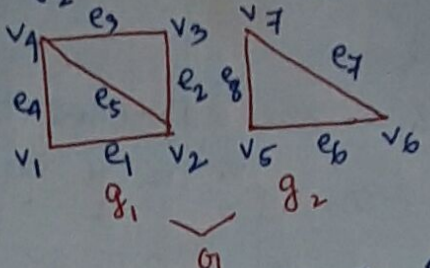
Let G be a disconnected graph with two components g_1 and g_2.

Then the incidence matrix of G is a block diagonal form

$$I(G) = \begin{bmatrix} I(g_1) & 0 \\ 0 & I(g_2) \end{bmatrix}$$

where $I(g_1)$ and $I(g_2)$ are the incident matrices of g_1 and g_2 and 0 represents the null matrices.

Prob Find the incidence matrix of



Sol First we construct $I(g_1)$ & $I(g_2)$

$$I(g_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$I(g_2) = \begin{array}{c} \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccc} e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

So,

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Prob Does there exist a graph corresponding to the following incidence matrix

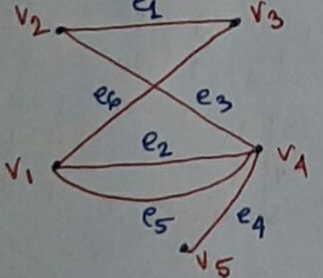
$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Sol It is not possible to have a graph corresponding to the $I(G)$ as, the second column all the entries are zero. which implies for an edge no incident to any vertices which is impossible.

Prob Draw the graph whose incidence matrix is

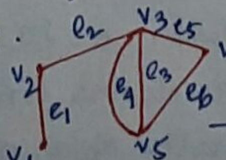
$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Sol



* Permutation of rows / columns of an Incidence Matrix

The interchange of rows/columns of a matrix are known as permutation of rows/columns of matrix.



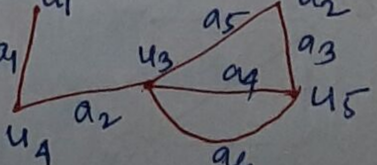
$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

inter. Ch $R_2 \leftrightarrow R_4$

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2' \\ v_3 \\ v_4' \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Again doing $C_3 \leftrightarrow C_6$

$$I(G) = \begin{array}{c} \\ \\ u_1\,v_1 \\ u_2\,v_2 \\ u_3\,v_3 \\ u_4\,v_4 \\ u_5\,v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array} \\ \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array} \Rightarrow$$



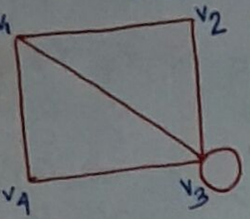
Def^n of Adjacency Matrix (Undirected Graph)

Let $G = \{V, E\}$ be an undirected graph with no parallel edges having vertex set $V = \{v_1, v_2, \dots, v_n\}$.

Then the adjacency matrix of this graph $A = \{a_{ij}\}_{n \times n}$ is an $n \times n$ binary (symmetric) matrix whose both the n-rows & n-columns corresponds to the n vertices, such that

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ \& } v_j \text{ are adjacent to each other} \\ 0 & \text{o.w} \end{cases}$$

Example:



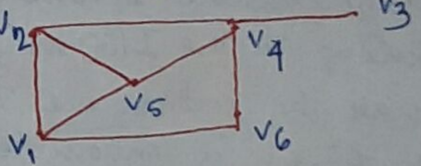
$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4 \times 4} \end{array}$$

observations:

1. The entries of $A(G)$ is either 0 or 1 ie, binary digit.
2. The entries along the principal diagonal should be all zero, if the graph has no self loops.
3. $a_{ii} = 1$ means the vertex v_i has a self loop.
4. The adjacency matrix is always symmetric ie, $a_{ij} = a_{ji}$ for all i & j. but if the graph is undirected.

* An observation: If the adjacency matrix of a graph is not-symmetric, then this implies the graph is directed.

Prob: 1 Determine the adjacency matrix of the graph

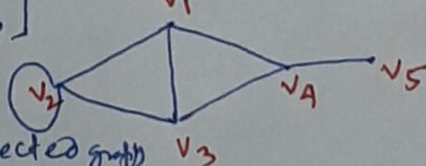


Ans

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Prob-2 Draw the graph whose adjacency matrix is

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



Def^n of Adjacency Matrix (Directed graph)

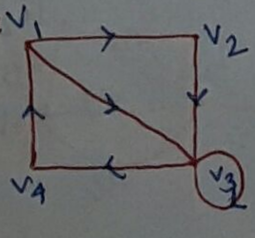
Let $G = \{V, E\}$ be a directed graph with no parallel edges having vertex set $V = \{v_1, v_2, \dots, v_n\}$

Then the adjacency matrix of this graph $A = \{a_{ij}\}_{n \times n}$ is an $n \times n$ binary (assymetric) matrix whose both n-rows & n-columns corresponds to the n vertices such that

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ be the initial vertex of an edge \& } v_j \text{ be the terminal vertex} \\ 0 & \text{o.w} \end{cases}$$

$v_i \rightarrow v_j \quad a_{ij} = 1 \quad a_{ji} = 0$

Example:



$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{4 \times 4} \end{array}$$

Graph Isomorphism:

Two graphs $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ are said to be isomorphic if there exists an one-to-one & onto correspondance between V_1 & V_2 and E_1 & E_2.

Suppose there exists a fn $f: V_1 \to V_2$ such that

(i) f is one-to-one & onto / bijective

(ii) if $\{v_1, v_2\}$ is an edge in E_1, then $\{f(v_1), f(v_2)\}$ is an edge in E_2 where $v_1, v_2 \in V_1$ and $f(v_1), f(v_2) \in V_2$.

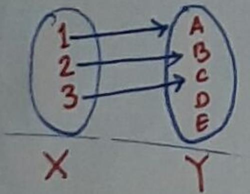
What is mapping?

If $f: X \to Y$ is a rule which applied on every element of the domain set X, then $y = f(x)$ (the value of f when applied on x) is called the image of x under f which will go to the co-domain set Y.

Mapping Types:

Mappings are of two types. one-one & onto or injective & surjective

One-to-one / Injective: A mapping $f: X \to Y$ is called an one-one mapping or injective mapping if for each of the distinct element in X, there exists a distinct f-image in Y.



→ Injective/ one-one
But not surjective/ onto

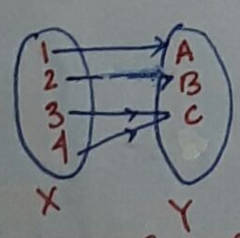
if $x_1 \neq x_2$ in X
$\Rightarrow f(x_1) \neq f(x_2)$ in Y

*each element of X will mapped to each distinct element in Y.

Onto / Surjective: A mapping $f: X \to Y$ is called an onto mapping or surjective mapping if $f(X) = Y$.

$\forall y \in Y \ \exists x \in X$ s.t. $y = f(x)$

Surjective



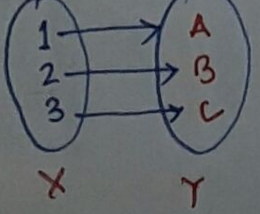
→ Surjective/ Onto
But not injective/ one-one

each element in Y has atleast one pre-image in X.

2 elements may mapped to the one

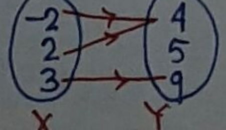
*each element of Y has atleast one (may be more than one) pre-image in X.

One-One & Onto / Bijective: If a mapping is both one-one & onto then it is called bijective mapping. $|X| = |Y|$



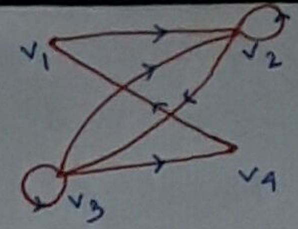
→ Bijective
both injective & surjective

when each element in Y has exactly one pre-image in X.



$f(x) = x^2$ This mapping is neither injective nor surjective.

Prob: Find the adjacency matrix of the digraph

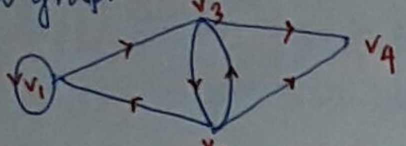


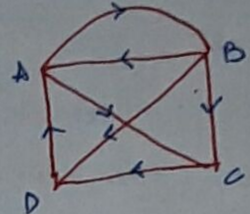
Sol $A(G) = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 1 & 0 \\ v_3 & 0 & 1 & 1 & 1 \\ v_4 & 1 & 0 & 0 & 0 \end{array}_{4\times 4}$

Prob: Find whether there exist any graph corresponds to $A(G) = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 1 & 0 & 1 & 0 \\ v_2 & 1 & 0 & 1 & 1 \\ v_3 & 0 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 0 & 0 \end{array}_{4\times 4}$

If exist then draw

Since the adjacency matrix here is an assymetric so it is a graph which is directed. Lets try to draw it.





Prob Find the adjacency matrix of the following graph

Sol $A(G) = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 0 \\ B & 1 & 0 & 1 & 1 \\ C & 0 & 0 & 0 & 1 \\ D & 1 & 0 & 0 & 0 \end{array}_{4\times 4}$

Def Adjacency matrix of a Disconnected Graph:

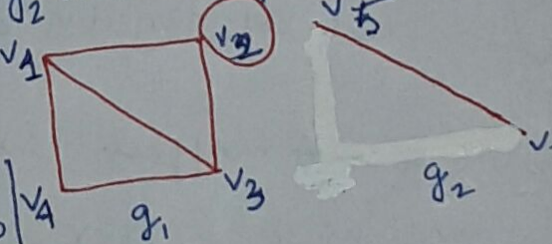
Adjacency matrix of a disconnected graph with two components g_1 and g_2.

Let G be a disconnected graph with two components g_1 and g_2. Then, the adjacency matrix of G is a block diagonal form

$$A(G) = \begin{bmatrix} A(g_1) & 0 \\ 0 & A(g_2) \end{bmatrix}$$

where $A(g_1)$ and $A(g_2)$ are the adjacency matrices of g_1 and g_2 and 0 represents the null matrices.

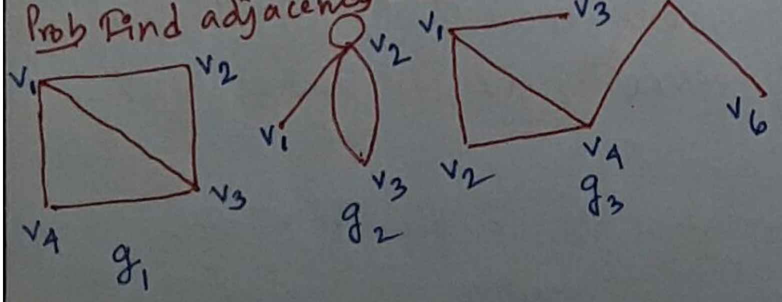
Prob: Find the adjacency matrix of



$A(g_1) = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 1 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}$ $A(g_2) = \begin{array}{c|cc} & v_5 & v_6 \\ \hline v_5 & 0 & 1 \\ v_6 & 1 & 0 \end{array}$

$\therefore A(G) = \begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 1 & 1 & 0 & 0 \\ v_2 & 1 & 1 & 1 & 0 & 0 & 0 \\ v_3 & 1 & 1 & 0 & 1 & 0 & 0 \\ v_4 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 1 \\ v_6 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}_{6\times 6}$

Prob Find adjacency matrices of



$A(g_1) = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}_{4\times 4}$

$A(g_2) = \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_1 & 0 & 1 & 0 \\ v_2 & 1 & 1 & 2 \\ v_3 & 0 & 2 & 0 \end{array}$

Therefore it is not possible to have adj mat of this graph as it has parallel edge

$A(g_3) = \begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 1 & 1 & 0 & 0 \\ v_2 & 1 & 0 & 0 & 1 & 0 & 0 \\ v_3 & 1 & 0 & 0 & 0 & 0 & 0 \\ v_4 & 1 & 1 & 0 & 0 & 1 & 0 \\ v_5 & 0 & 0 & 0 & 1 & 0 & 1 \\ v_6 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$

§ Graph Isomorphism:

Two graphs $G_1 = \{V_1, E_1\}$ & $G_2 = \{V_2, E_2\}$ are said to be isomorphic if there exists a bijective mapping between V_1 & V_2 as well as E_1 & E_2

Let $f: V_1 \to V_2$ be the mapping, Then

(i) f must be bijective i.e, one-one & onto

(ii) If $\{v_1, v_2\}$ be an edge in E_1 then $\{f(v_1), f(v_2)\}$ be an edge in E_2

Isomorphic Invariant:

The property shared by graphs G_1 and G_2 which are isomorphic is called isomorphic invariant. They are as follows:

(i) $|V_1| = |V_2|$ i.e, number of vertices in G_1 and G_2 are same.

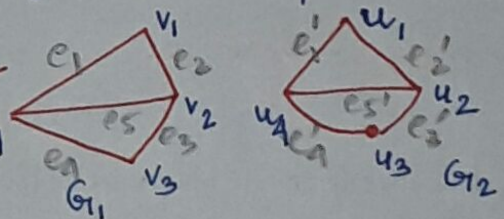
(ii) $|E_1| = |E_2|$ i.e, number of edges in G_1 and G_2 are same.

(iii) $deg(v_1) = deg(f(v_1))$, i.e, the degree sequence preserved.

(iv) if $\{v_1, v_2\}$ is a self loop, Then $\{f(v_1), f(v_2)\}$ is also a self loop.

Th: Two graphs G_1 and G_2 are isomorphic iff their incidence matrices $I(G_1)$ & $I(G_2)$ can be obtained by performing of row/column permutation of each other.

Prob - Show that following graphs are isomorphic



$G_1 = \{V_1, E_1\}$ $G_2 = \{V_2, E_2\}$

Isomorphic Invariance

$|V_1| = 4 = |V_2|$, $|E_1| = 5 = |E_2|$

Now we need to indentify a bijective mapping $f: V_1 \to V_2$ such that degree sequence remains preserved.

$f(v_1) = u_1$, $f(v_2) = u_2$, $f(v_3) = u_3$, $f(v_4) = u_4$

So, $deg(v_1) = deg(f(v_1)) = deg(u_1)$

$deg(v_2) = deg(f(v_2)) = deg(u_2)$

$deg(v_3) = deg(f(v_3)) = deg(u_3)$

$deg(v_4) = deg(f(v_4)) = deg(u_4)$

Let us now see the edges are invariant

$e_1 = \{v_1, v_4\} = \{u_1, u_4\} = e_1'$

$e_2 = \{v_1, v_2\} = \{u_1, u_2\} = e_2'$

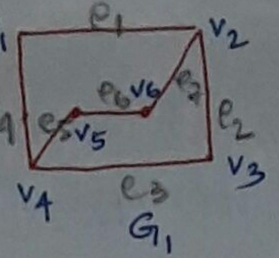
$e_3 = \{v_2, v_3\} = \{u_2, u_3\} = e_3'$

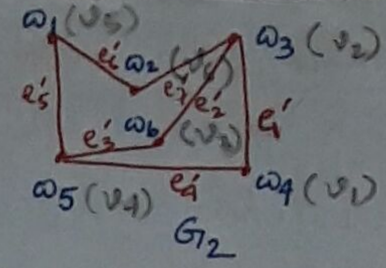
$e_4 = \{v_3, v_4\} = \{u_3, u_4\} = e_4'$

$e_5 = \{v_2, v_4\} = \{u_2, u_4\} = e_5'$

So These two graphs are isomorphic.

Prob(2) Show that G_1 & G_2 are isomorphic.





$G_1 = \{V_1, E_1\}$ $G_2 = \{V_2, E_2\}$

Isomorphic Invariance:

$|V_1| = 6 = |V_2|$, $|E_1| = 7 = |E_2|$

Now we need to introduce a bijective mapping $f: V_1 \to V_2$ such that degree sequence preserved.

$f(v_1) = \omega_4$, $f(v_2) = \omega_3$, $f(v_3) = \omega_6$, $f(v_4) = \omega_5$, $f(v_5) = \omega_1$, $f(v_6) = \omega_2$

So, $\deg(v_1) = \deg(f(v_1)) = \deg(\omega_4)$

$\deg(v_2) = \deg(f(v_2)) = \deg(\omega_3)$

$\deg(v_3) = \deg(f(v_3)) = \deg(\omega_6)$

$\deg(v_4) = \deg(f(v_4)) = \deg(\omega_5)$

$\deg(v_5) = \deg(f(v_5)) = \deg(\omega_1)$

$\deg(v_6) = \deg(f(v_6)) = \deg(\omega_2)$

Now see whether edges are invariant

$e_1 = \{v_1, v_2\} = \{\omega_4, \omega_3\} = e_1'$

$e_2 = \{v_2, v_3\} = \{\omega_3, \omega_6\} = e_2'$

$e_3 = \{v_3, v_4\} = \{\omega_6, \omega_5\} = e_3'$

$e_4 = \{v_1, v_4\} = \{\omega_4, \omega_5\} = e_4'$

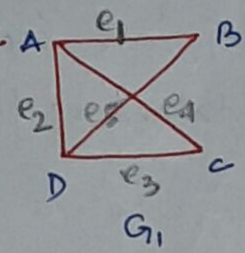
$e_5 = \{v_4, v_5\} = \{\omega_5, \omega_1\} = e_5'$

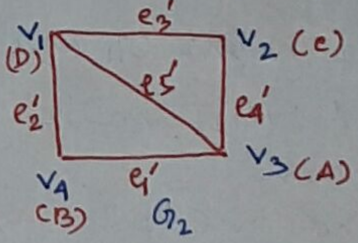
$e_6 = \{v_5, v_6\} = \{\omega_1, \omega_2\} = e_6'$

$e_7 = \{v_6, v_2\} = \{\omega_2, \omega_3\} = e_7'$

Hence G_1 & G_2 are isomorphic.

Prob(3) Show that G_1 & G_2 are isomorphic





$G_1 = \{V_1, E_1\}$ $G_2 = \{V_2, E_2\}$

Isomorphic Invariance:

$|V_1| = 4 = |V_2|$ $|E_1| = 5 = |E_2|$

Now we need to find a bijective mapping $f: V_1 \to V_2$ such that deg seq. preserved

$f(A) = v_3$, $f(B) = v_4$, $f(C) = v_2$, $f(D) = v_1$

So, $\deg(A) = \deg(f(A)) = \deg(v_3)$

$\deg(B) = \deg(f(B)) = \deg(v_4)$

$\deg(C) = \deg(f(C)) = \deg(v_2)$

$\deg(D) = \deg(f(D)) = \deg(v_1)$

for edge invariance

$e_1 = \{A, B\} = \{v_3, v_4\} = e_1'$

$e_2 = \{A, D\} = \{v_3, v_1\} = e_2'$

$e_3 = \{D, C\} = \{v_1, v_2\} = e_3'$

$e_4 = \{A, C\} = \{v_3, v_2\} = e_4'$

$e_5 = \{B, D\} = \{v_4, v_1\} = e_5'$

So, G_1 & G_2 are isomorphic.