5.3 INVERSE LAPLACE TRANSFORMS

5.3.1 Definition of Inverse Laplace Transform.

If $L\{F(t)\} = f(s)$ then we say the function $F(t)$ is inverse Laplace transform of $f(s)$ and we write

$$L^{-1}\{f(s)\} = F(t).$$

For example, since $L\{e^{-3t}\} = \frac{1}{s+3}$ therefore $L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$.

Illustration. **(i)** *Prove that* $L^{-1}\left(\frac{1}{s}\right) = 1$.

First prove $L(1) = \frac{1}{s}$ (which is shown in Art 5.2.2)

$$\therefore L^{-1}\left(\frac{1}{s}\right) = 1$$

(ii) *Prove that* $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$.

First prove $L(t^n) = \frac{n!}{s^{n+1}}$ (which is shown in Art 5.2.2)

$$\therefore L\left(\frac{t^n}{n!}\right) = \frac{1}{n!}L(t^n) = \frac{1}{n!}\cdot\frac{n!}{s^{n+1}} = \frac{1}{s^{n+1}}$$

$$\therefore L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$$

(iii) *Show that* (a) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(b) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

(c) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$.

Hints: All of these follow from the examples of Art 5.2.2. Therefore we can make the following table

Formula of Inverse L. T.

Sl. No.	Function $f(s)$	$L^{-1}\{f(s)\} = F(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
4.	$\frac{1}{s-a}$	e^{at}
5.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
6.	$\frac{s}{s^2+a^2}$	$\cos at$
7.	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
8.	$\frac{s}{s^2-a^2}$	$\cosh at$

Note: In fact Inverse Laplace Transform may not be unique e.g., the Laplace Transform of two functions $F(t) = e^{-3t}$ and

$$G(t) = 0, \quad t = 1$$
$$= e^{-3t}, \quad t \neq 1$$

are same, which is $\frac{1}{s+3}$ i.e., to say $L^{-1}\left(\frac{1}{s+3}\right) = F(t)$ or $G(t)$.

5.3.2. Lerch's Theorem. (On uniqueness of inverse)

If the function $F(t)$ is sectionally continuous in $[0, N]$ for each positive integer N and if there exists a real constant $M > 0$ such that for all $t > N$, $|F(t)| < Me^{vt}$ for some v, then $L^{-1}\{f(s)\} = F(t)$ is unique.

Proof. Beyond the scope of the book.

Note: In the subsequent articles we assume the uniqueness the inverse Laplace Transform.

5.3.3 Linear property of Inverse Laplace Transform

Theorem.

If $L\{F_1(t)\} = f_1(s)$ and $L\{F_2(t)\} = f_2(s)$, then

$$L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\}$$

where c_1 and c_2 are any constants.

Proof. Since $L\{F_1(t)\} = f_1(s), L\{F_2(t)\} = f_2(s)$, therefore

$$F_1(t) = L^{-1}\{f_1(s)\},\ F_2(t) = L^{-1}\{f_2(s)\}.$$

By the linear property of Laplace Transform,

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$$
$$= c_1 f_1(s) + c_2 f_2(s)$$
$$\therefore L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 F_1(t) + c_2 F_2(t)$$
$$= c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\}$$

Illustration : $L^{-1}\left\{\dfrac{4}{s-2} - \dfrac{3s}{s^2+16} + \dfrac{5}{s^2+4}\right\}$

$$= 4L^{-1}\left\{\frac{1}{s-2}\right\} - 3L^{-1}\left\{\frac{s}{s^2+16}\right\} + 5L^{-1}\left\{\frac{1}{s^2+4}\right\}$$
$$= 4e^{2t} - 3\cos 4t + \tfrac{5}{2}\sin 2t$$

5.3.4. Shifting property of Inverse Laplace Transform

Theorem 1. (First shifting property)

$$L^{-1}\{f(s-a)\} = e^{at} L^{-1}\{f(s)\}.$$

Proof. If $L^{-1}\{f(s)\} = F(t)$, we know from the first shifting property of Laplace transform,

$$L\{e^{at} F(t)\} = f(s-a)$$

$$\therefore L^{-1}\{f(s-a)\} = e^{at} F(t) = e^{at} L^{-1}\{f(s)\}$$

Illustrative Examples

(i) $L^{-1}\left(\dfrac{6s-4}{s^2-4s+20}\right) = L^{-1}\left\{\dfrac{6(s-2)}{(s-2)^2+16} + \dfrac{8}{(s-2)^2+16}\right\}$

$$= 6L^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + 8L^{-1}\left\{\frac{1}{(s-2)^2+16}\right\} \quad \ldots (1)$$

$$= 6e^{2t} L^{-1}\left(\frac{s}{s^2+16}\right) + 8e^{2t} L^{-1}\left(\frac{1}{s^2+16}\right)$$

$$= 6e^{2t}\cos 4t + 8e^{2t}\frac{\sin 4t}{4}, \text{ using formulae}$$

$$= 2e^{2t}(3\cos 4t + \sin 4t)$$

(ii) $L^{-1}\left(\dfrac{4s+12}{s^2+8s+16}\right) = L^{-1}\left\{\dfrac{4}{s+4} - \dfrac{4}{(s+4)^2}\right\}$

$$= 4L^{-1}\left(\frac{1}{s+4}\right) - 4L^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$

$$= 4e^{-4t} - 4e^{-4t} L^{-1}\left(\frac{1}{s^2}\right)$$

$$= 4e^{-4t} - 4te^{-4t} = 4(1-t)e^{-4t}$$

Theorem 2. (Second shifting preoperty)

If $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\{e^{-as} f(s)\} = F(t-a),\quad t > a$

$$= 0,\quad t < a$$

Proof. Since $L^{-1}\{f(s)\} = F(t)$, so $L\{F(t)\} = f(s)$.

Let $\quad G(t) = F(t-a),\ t > a$

$$= 0,\ t < a$$

Therfore, by Second shifting property of Laplace transform,

$$L\{G(t)\} = e^{-as} f(s) \qquad \therefore L^{-1}\{e^{-as} f(s)\} = G(t)$$

i.e., $L^{-1}\{e^{-as}f(s)\} = F(t-a), \quad t > a$

$= 0, \quad t < a$

Illustration. ***(i)*** *To find* $L^{-1}\left\{\frac{8e^{-3s}}{s^2+4}\right\}$

We have, $L^{-1}\left(\frac{8}{s^2+4}\right) = 8L^{-1}\left(\frac{1}{s^2+2^2}\right) = 8.\frac{\sin 2t}{2} = 4\text{si}$

Then, applying Second shifting propety, we get

$$L^{-1}\left\{\frac{8}{s^2+4}.e^{-3s}\right\} = 4\sin 2(t-3), \quad t > 3$$

$$= 0, \quad t < 3$$

$$\therefore L^{-1}\left\{\frac{8e^{-3s}}{s^2+4}\right\} = 4\sin(2t-6), \quad t > 3$$

$$= 0, \quad t < 3$$

(ii) *To evaluate* $L^{-1}\left\{\frac{(s+1)e^{-\pi s}}{s^2+s+1}\right\}$

We have, $L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = L^{-1}\left\{\frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$

$$= L^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right.}\right.$$

$$= e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}t}{2} + \frac{1}{2}e^{-\frac{1}{2}t}\frac{\sin\frac{\sqrt{3}t}{2}}{\frac{\sqrt{3}}{2}}$$ **[by first shifti**

$$= e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2}\right)$$

Then, by second shifting property, we get

$$L^{-1}\left\{e^{-\pi s}.\frac{s+1}{s^2+s+1}\right\} = e^{-\frac{1}{2}(t-\pi)}\left\{\cos\frac{\sqrt{3}}{2}(t-\pi) + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-\pi)\right\}, t > \pi$$

$$= 0, t < \pi$$

5.3.5. Change of scale property of Inverse Laplace Transform.

Theorem. If $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\{f(as)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$, a is a constant. **[*W.B.U.Tech.2002*]**

Proof. Since $L^{-1}\{f(s)\} = F(t)$, so $L\{F(t)\} = f(s)$

Now, $L\left\{\frac{1}{a}F\left(\frac{t}{a}\right)\right\} = \frac{1}{a}L\left\{F\left(\frac{t}{a}\right)\right\}$

$$= \frac{1}{a}.\frac{1}{\frac{1}{a}}f\left(\frac{s}{\frac{1}{a}}\right)$$ **[By change of scale property of L.T.]**

$$= f(as) \qquad \therefore L^{-1}\{f(as)\} = \frac{1}{a}F\left(\frac{t}{a}\right)$$

Illustration. *To find* $L^{-1}\left\{\frac{3s}{4s^2+16}\right\}$

We have, $L^{-1}\left\{\frac{s}{s^2+4^2}\right\} = \cos 4t$

By change of scale property, $L^{-1}\left\{\frac{2s}{(2s)^2+4^2}\right\} = \frac{1}{2}\cos 4\left(\frac{t}{2}\right)$

or, $L^{-1}\left\{\frac{2s}{4s^2+16}\right\} = \frac{1}{2}\cos 2t$

or, $\frac{2}{3}L^{-1}\left\{\frac{3s}{4s^2+16}\right\} = \frac{1}{2}\cos 2t$

$$\therefore \quad L^{-1}\left\{\frac{3s}{4s^2+16}\right\} = \frac{3}{4}\cos 2t$$

5.3.6. Inverse Laplace Transform on Derivatives.

Theorem 1. (On 1st order derivative) $L^{-1}\{f'(s)\} = -t\,L^{-1}\{f(s)\}$,

where $f'(s) = \frac{d}{ds}\{f(s)\}$.

Proof. Let $L^{-1}\{f(s)\} = F(t)$, therefore $L\{F(t)\} = f(s)$

$$\therefore L\{t\,F(t)\} = -\frac{d}{ds}f(s) = -f'(s)$$

or, $-L\{tF(t)\} = f'(s)$ or, $L\{-tF(t)\} = f'(s)$

$$\therefore L^{-1}\{f'(s)\} = -t\,F(t)$$

Illustrative Example. *Find* $L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$

We have, $L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{\sin 3t}{3}$

Now, $L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+9}\right)\right\} = -tL^{-1}\left(\frac{1}{s^2+9}\right) = -\frac{t\sin 3t}{3}$

or, $L^{-1}\left\{\frac{-2s}{(s^2+9)^2}\right\} = -\frac{t\sin 3t}{3}$

or, $-2L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = -\frac{t\sin 3t}{3}$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = \frac{t\sin 3t}{6}$$

Theorem 2. (On n^{th} order derivative) If $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\{f^{(n)}(s)\} = (-1)^n t^n F(t)$, where $f^{(n)}(s) = \frac{d^n}{ds^n}f(s)$.

Proof. Follows from Theorem 1 above.

Illustrative Example. *Find* $L^{-1}\left\{\log\frac{s+3}{s+2}\right\}$

Let $f(s) = \log\frac{s+3}{s+2} = \log(s+3) - \log(s+2)$

$$\therefore f'(s) = \frac{1}{s+3} - \frac{1}{s+2}$$

$$\therefore L^{-1}\{f'(s)\} = L^{-1}\left(\frac{1}{s+3}\right) - L^{-1}\left(\frac{1}{s+2}\right) = e^{-3t} - e^{-2t}$$

or, $-tL^{-1}\{f(s)\} = e^{-3t} - e^{-2t}$,

or, $L^{-1}\{f(s)\} = \frac{e^{-2t} - e^{-3t}}{t}$

$$\therefore L^{-1}\left\{\log\frac{s+3}{s+2}\right\} = \frac{e^{-2t} - e^{-3t}}{t}$$

5.3.7. Multiplication by s^n.

Theorem 1. If $L^{-1}\{f(s)\} = F(t)$ and $F(0) = 0$, then

$$L^{-1}\{sf(s)\} = F'(t).$$

Proof. Since $L^{-1}\{f(s)\} = F(t)$, therefore $L\{F(t)\} = f(s)$

$\therefore L\{F'(t)\} = sf(s) - F(0)$, by the property of L.T. on derivative

$= sf(s) \quad [\because F(0) = 0]$

$$\therefore L^{-1}\{sf(s)\} = F'(t)$$

Theorem 2. If $L^{-1}\{f(s)\} = F(t)$ and $F(0) = F'(0) = \cdots = F^{(n-1)}(0) = 0$, then $L^{-1}\{s^n f(s)\} = F^{(n)}(t)$.

Proof. Follows from above Theorem 1.

Illustrative Example. *Find* (i) $L^{-1}\left\{\frac{s}{(s+1)^5}\right\}$

(ii) $L^{-1}\left\{\frac{s^2}{(s+1)^5}\right\}$

[W.B.U.Tech.2006]

(i) We know $L^{-1}\left(\frac{1}{s^5}\right)=\frac{t^4}{4!}$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)^5}\right\}=e^{-t}\frac{t^4}{4!}=F(t), \text{ say [by 1st shifting ...]}$$

$$\therefore \text{ By Theorem 1, } L^{-1}\left\{\frac{s}{(s+1)^5}\right\}=\frac{d}{dt}\left(\frac{e^{-t}t^4}{4!}\right)=\frac{1}{4!}\left\{e^{-t}4t^3-\ldots\right\}$$

$$=\frac{e^{-t}t^3}{24}(4-t)$$

(ii) First do part (i)

Then we see $F'(t)=\frac{e^{t}t^3}{24}(4-t)=0 \text{ at } t=0$

Therefore, by Theorem (2)

$$L^{-1}\left\{\frac{s^2}{(s+1)^5}\right\}=F''(t)=\frac{d}{dt}\left\{\frac{e^{-t}t^3}{24}(4-t)\right\}$$

$$=\frac{t^2e^{-t}}{24}\left(12-8t+t^2\right)$$

5.3.8. Division by s.

Theorem. If $L^{-1}\{f(s)\}=F(t)$, then $L^{-1}\left\{\frac{f(s)}{s}\right\}=\int_0^t F(u)du$

Proof. Let $\phi(t)=\int_0^t F(u)du$ $\quad\therefore \phi'(t)=F(t)$ and $\phi(0)=0$

Therefore. $L\{\phi'(t)\}=sL\{\phi(t)\}-\phi(0)$

or, $L\{F(t)\}=sL\{\phi(t)\}$ or, $f(s)=sL\{\phi(t)\}$ $\left[\because L^{-1}\{f(s)\}=F(t)\right]$

or, $L\{\phi(t)\}=\frac{f(s)}{s}$ or, $\phi(t)=L^{-1}\left\{\frac{f(s)}{s}\right\}$

$$\therefore L^{-1}\left\{\frac{f(s)}{s}\right\}=\int_0^t F(u)du$$

Illustrative Example. (*i*) Find $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$

We know $L^{-1}\left(\frac{1}{s^2+1}\right)=\sin t$ $\quad\therefore L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}=\int_0^t \sin u\, du=-[\cos u]_0^t$

$$=1-\cos t$$

$$\therefore L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}=\int_0^t(1-\cos u)\,du=[u-\sin u]_0^t=t-\sin t$$

(*ii*) *Prove* $L^{-1}\left\{\frac{1}{s}\log\frac{s+3}{s+2}\right\}=\int_0^t\frac{e^{-2u}-e^{-3u}}{u}du$

Let $f(s)=\log\frac{s+3}{s+2}$

In an Illustration in Art 13.6, we found

$$L^{-1}\{f(s)\}=\frac{e^{-2t}-e^{-3t}}{t}=F(t), \text{ say}$$

$$\therefore L^{-1}\left\{\frac{1}{s}f(s)\right\}=\int_0^t F(u)\,du=\int_0^t\frac{e^{-2u}-e^{-3u}}{u}du$$

5.3.9. Inverse Laplace Transform of Integrals.

Theorem. If $L^{-1}\{f(s)\}=F(t)$, then $L^{-1}\left\{\int_s^\infty f(u)du\right\}=\frac{F(t)}{t}$.

Proof. Since $L^{-1}\{f(s)\} = F(t)$, so $L\{F(t)\} = f(s)$

Using $\therefore L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$, using the theorem of art [...]

$$L^{-1}\left\{\int_s^\infty f(u)\,du\right\} = \frac{F(t)}{t}.$$

5.3.10. Convolution property of Inverse La[place] Transform.

Definition of Convolution of two functions.

Let $F(t)$ and $G(t)$ be two integrable function. The[ir] Convolution is denoted by $F*G$ and define[d by]

$$F*G = \int_0^t F(u)G(t-u)\,du$$

Theorem 1. $F*G = G*F$

Proof. $F*G = \int_0^t F(u)G(t-u)\,du$

$$= -\int_t^0 F(t-v)G(v)\,dv, \quad \left[\begin{array}{l}\text{putting } t-u=v\\ \therefore du=-dv\end{array}\right]$$

$$= \int_0^t F(t-v)G(v)\,dv = \int_0^t G(u)F(t-u)\,du = G*F$$

Theorem 2. (Convolution theorem) If $L^{-1}\{f(s)\} = F(t)$, $L^{-1}\{g(s)\} = G(t)$ then $L^{-1}\{f(s)g(s)\} = F*G$. [W.B.U.T...]

Proof. beyond the scope of the book.

Illustration. (*i*) *To prove* $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t\sin at}{2a}$

[W.B.U.Tech.200...]

We have $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = F(t)$, say

and $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a} = G(t)$, say

Now using Convolution Theorem, we get

$$L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right) = F*G$$

i.e., $$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t F(u)G(t-u)\,du = \int_0^t \cos au \frac{\sin a(t-u)}{a}\,du$$

$$= \frac{1}{2a}\int_0^t 2\cos au\,\sin a(t-u)\,du = \frac{1}{2a}\int_0^t \{\sin at - \sin(2au-at)\}\,du$$

$$= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin 2at}{4a}\right) - \frac{\cos at}{a}\left(\frac{1-\cos 2at}{4a}\right) = \frac{t\sin at}{2a}$$

(*ii*) *Find* $L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$.

Since $L^{-1}\left(\frac{1}{s^2}\right) = t$, so $L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = te^{-2t} = F(t)$, say

Also $L^{-1}\left(\frac{1}{s-2}\right) = e^{2t} = G(t)$, say

Now using Convolution Theorem, we get $L^{-1}\left\{\frac{1}{(s+2)^2}\cdot\frac{1}{s-2}\right\}$

$$= F*G = \int_0^t F(u)G(t-u)\,du = \int_0^t ue^{-2u}e^{2(t-u)}\,du$$

$$= e^{2t}\int_0^t ue^{-4u}\,du = e^{2t}\left\{\left[\frac{ue^{-4u}}{-4}\right]_0^t + \frac{1}{4}\int_0^t e^{-4u}\,du\right\}$$

$$= e^{2t}\left\{\frac{te^{-4t}}{-4} - \frac{1}{16}\left(e^{-4t}-1\right)\right\} = \frac{1}{16}\left(e^{2t} - e^{-2t} - 4te^{-2t}\right)$$

5.3.11. Method of partial fraction to find Laplace transform.

Inverse Laplace Transform can also be evaluated method of partial fraction. This is shown by the fo examples.

Illustrative Example. (*i*) *Find* $L^{-1}\left\{\frac{2s^2-4}{(s-3)(s^2-s-2)}\right\}$

We have $\frac{2s^2-4}{(s-3)(s^2-s-2)} = \frac{2s^2-4}{(s-3)(s+1)(s-2)}$

Let $\frac{2s^2-4}{(s-3)(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$

$\therefore 2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$

Now putting s = 2, 3, -1, we get $A = -\frac{1}{6}, B = -\frac{4}{3}, C =$

$$\therefore \frac{2s^2-4}{(s-3)(s^2-s-2)} = \frac{-\frac{1}{6}}{(s+1)} + \frac{-\frac{4}{3}}{(s-2)} + \frac{\frac{7}{2}}{(s-3)}$$

$$\therefore L^{-1}\left\{\frac{2s^2-4}{(s-3)(s^2-s-2)}\right\}$$

$$= -\frac{1}{6}L^{-1}\left(\frac{1}{s+1}\right) - \frac{4}{3}L^{-1}\left(\frac{1}{s-2}\right) + \frac{7}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$= -\frac{e^{-t}}{6} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

(ii) *Evaluate* $L^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$, [by 1st shifting property]

We have $\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$

$$\therefore 5s^2-15s-11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$$

Putting s = -1, 2, 0, 1, we get

$$A = -\frac{1}{3}, D = -7$$

$$4B - 2C = -\frac{20}{3}$$

$$2B - 2C = -\frac{22}{3}$$

$\therefore$ Solving we get, $A = -\frac{1}{3}, B = \frac{1}{3}, C = 4, D = -7$

$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^3} = -\frac{1}{3}.\frac{1}{s+1} + \frac{1}{3}.\frac{1}{s-2} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

$$L^{-1}\left\{\frac{5s^2-15s-11}{(s+1)^2(s-2)^3}\right\} = -\frac{1}{3}L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{3}L^{-1}\left(\frac{1}{s-2}\right)$$

$$+ 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\} - 7L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}L^{-1}\left(\frac{1}{s^2}\right) - 7e^{2t}L^{-1}\left(\frac{1}{s^3}\right)$, using first shifting property

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4te^{2t} - \frac{7}{2}e^{2t}t^2$$

(iii) *Evaluate* $L^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right\}$.

We have $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}=\dfrac{As+B}{s^2+2s+5}+\dfrac{C\ldots}{s^2+\ldots}$

$\therefore s^2+2s+3=(As+B)(s^2+2s+2)+(Cs+D)(s^2\ldots$

Equating the coefficient of s^3, s^2, s and constant te...

$$A+C=0$$

$$2A+B+2C+D=1$$

$$2A+2B+5C+2D=2$$

$$2B+5D=3$$

Solving we get $A=0,\ B=\dfrac{2}{3},\ C=0,\ D=\dfrac{1}{3}$

Thus $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}=\dfrac{2}{3}\cdot\dfrac{1}{s^2+2s+5}+\dfrac{1}{3}\cdot\dfrac{1}{s^2+2s\ldots}$

$$\therefore L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right\}$$

$$=\frac{2}{3}L^{-1}\left\{\frac{1}{s^2+2s+5}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{s^2+2s+2}\right\}$$

$$=\frac{2}{3}L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$=\frac{2}{3}e^{-t}L^{-1}\left(\frac{1}{s^2+2^2}\right)+\frac{1}{3}e^{-t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$=\frac{2}{3}\cdot\frac{1}{2}e^{-t}\sin 2t+\frac{1}{3}e^{-t}\sin t$$

$$=\frac{1}{3}e^{-t}(\sin 2t+\sin t)$$

.3.12. Miscellaneous Examples

Ex 1. *Evaluate* $L^{-1}\left\{\dfrac{5s+10}{9s^2-16}\right\}$.

$$L^{-1}\left\{\frac{5s+10}{9s^2-16}\right\}=5L^{-1}\left\{\frac{s+2}{9s^2-16}\right\}$$

$$=5\left\{L^{-1}\left(\frac{s}{9s^2-16}\right)+2L^{-1}\left(\frac{1}{9s^2-16}\right)\right\}$$

$$=\frac{5}{9}L^{-1}\left\{\frac{s}{s^2-\left(\frac{4}{3}\right)^2}\right\}+\frac{10}{9}L^{-1}\left\{\frac{1}{s^2-\left(\frac{4}{3}\right)^2}\right\}$$

$$=\frac{5}{9}\cosh\frac{4}{3}t+\frac{10}{9}\frac{\sinh\frac{4}{3}t}{\frac{4}{3}}=\frac{5}{9}\cosh\frac{4}{3}t+\frac{5}{6}\sinh\frac{4}{3}t$$

Ex 2. *Evaluate* $L^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$ *in terms of exponential function.*

$$L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}=L^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-4}\right\}$$

$$=3L^{-1}\left\{\frac{s-1}{(s-1)^2-4}\right\}+10L^{-1}\left\{\frac{1}{(s-1)^2-4}\right\}$$

$$=3e^{t}L^{-1}\left(\frac{s}{s^2-4}\right)+10e^{t}L^{-1}\left(\frac{1}{s^2-4}\right)$$

$$=3e^{t}\cosh 2t+10e^{t}\frac{\sinh 2t}{2}$$

$$=3e^{t}\,\frac{e^{2t}+e^{-2t}}{2}+5e^{t}\,\frac{e^{2t}-e^{-2t}}{2}=4e^{3t}-e^{-t}$$

Ex 3. *Evaluate* $L^{-1}\left\{\dfrac{4s+3}{(s-4)^2(s+3)}\right\}$ **[WBUT 2003]**

Let $\dfrac{4s+3}{(s-4)^2(s+3)} = \dfrac{A}{s-4} + \dfrac{B}{(s-4)^2} + \dfrac{C}{s+3}$

$\therefore\ 4s+3 = A(s-4)(s+3) + B(s+3) + c(s-4)^2$

Putting $s = -3, 4, 0$ we get

$$A = \tfrac{1}{7},\ B = 3,\ C = -\tfrac{1}{7}$$

$$\therefore\ \frac{4s+5}{(s-4)^2(s+3)} = \frac{1}{7}\frac{1}{s-4} + \frac{3}{(s-4)^2} - \frac{1}{7}\frac{1}{s+3}$$

$$\therefore\ L^{-1}\left\{\frac{4s+5}{(s-4)^2(s+3)}\right\}$$

$$= \frac{1}{7}L^{-1}\left(\frac{1}{s-4}\right) + 3L^{-1}\left(\frac{1}{(s-4)^2}\right) - \frac{1}{7}L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{7}e^{4t} + 3te^{4t} - \frac{1}{7}e^{-3t}$$

Ex 4. *Find* $L^{-1}\left\{\dfrac{s}{(s+1)^5}\right\}$.

$$L^{-1}\left\{\frac{s}{(s+1)^5}\right\} = L^{-1}\left\{\frac{s+1-1}{(s+1)^5}\right\} = L^{-1}\left\{\frac{1}{(s+1)^4} - \frac{1}{(s+1)^5}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+1)^4}\right\} - L^{-1}\left\{\frac{1}{(s+1)^5}\right\}$$

$$= e^{-t}L^{-1}\left(\frac{1}{s^4}\right) - e^{-t}L^{-1}\left(\frac{1}{s^5}\right) \quad \text{[by 1st shiftin Prope}$$

$$= e^{-t}\frac{t^3}{3!} - e^{-t}\frac{t^4}{4!} = \frac{t^3e^{-t}}{24}(4-t)$$

x 5. *Evaluate* $L^{-1}\left(\dfrac{s+1}{s^2+6s+25}\right)$.

$$L^{-1}\left(\frac{s+1}{s^2+6s+25}\right) = L^{-1}\left\{\frac{(s+3)-2}{(s+3)^2+16}\right\}$$

$$= L^{-1}\left\{\frac{s+3}{(s+3)^2+4^2}\right\} - 2L^{-1}\left\{\frac{1}{(s+3)^2+4^2}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{s}{s^2+4^2}\right\} - 2e^{-3t}L^{-1}\left\{\frac{1}{s^2+4^2}\right\} = e^{-3t}\left(\cos 4t - \frac{1}{2}\sin 4t\right)$$

x 6. *Evaluate* $L^{-1}\left(\dfrac{s}{s^4+s^2+1}\right)$.

$$L^{-1}\left(\frac{s}{s^4+s^2+1}\right) = L^{-1}\left\{\frac{s}{(s^2+1)^2 - s^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{2}\cdot\frac{(s^2+s+1)-(s^2-s+1)}{(s^2+s+1)(s^2-s+1)}\right\}$$

$$= \frac{1}{2}\left\{L^{-1}\left(\frac{1}{s^2-s+1}\right) - L^{-1}\left(\frac{1}{s^2+s+1}\right)\right\}$$

$$= \frac{1}{2}\left[L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= \frac{1}{2}e^{\frac{1}{2}t}L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - \frac{1}{2}e^{-\frac{1}{2}t}L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{2}e^{\frac{1}{2}t}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}} - \frac{1}{2}e^{-\frac{1}{2}t}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right)$$

Ex 7. *Evaluate* $L^{-1}\left(\dfrac{e^{-3s}}{s^2-2s+5}\right)$.

Now $L^{-1}\left(\dfrac{1}{s^2-2s+5}\right)=L^{-1}\left(\dfrac{1}{(s-1)^2+4}\right)$

$$=e^tL^{-1}\left(\frac{1}{s^2+2^2}\right)=e^t\,\frac{\sin 2t}{2}$$

By applying Second shifting property,

$$L^{-1}\left(\frac{e^{-3s}}{s^2-2s+5}\right)=e^{t-3}\,\frac{\sin 2(t-3)}{2},\ t>3$$

$$=0,\quad t<3$$

$$\therefore\ L^{-1}\left(\frac{e^{-3s}}{s^2-2s+5}\right)=\frac{1}{2}e^{t-3}\sin(2t-6),\quad t>3$$

$$=0,\quad t<3$$

Ex 8. *Evaluate* $L^{-1}\left\{\dfrac{s+1}{(s^2+2s+2)^2}\right\}$.

We see $\dfrac{d}{ds}\left(\dfrac{1}{s^2+2s+2}\right)=\dfrac{-(2s+2)}{(s^2+2s+2)^2}$

Now, $L^{-1}\left(\dfrac{1}{s^2+2s+2}\right)=L^{-1}\left\{\dfrac{1}{(s+1)^2+1}\right\}$

$$=e^{-t}L^{-1}\left(\frac{1}{s^2+1}\right)=e^{-t}\sin t$$

So, by Inverse Laplace Transformation on derivative

$$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+2s+2}\right)\right\}=-t\,L^{-1}\left(\frac{1}{s^2+2s+2}\right)$$

or, $L^{-1}\left\{\dfrac{-(2s+2)}{(s^2+2s+2)^2}\right\}=-t\,e^{-t}\sin t$

or, $-2L^{-1}\left\{\dfrac{s+1}{(s^2+2s+2)^2}\right\}=-t\,e^{-t}\sin t$

$$\therefore\ L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}=\frac{1}{2}t\,e^{-t}\sin t$$

Ex 9. *Evaluate* $L^{-1}\{\tan^{-1}(s+2)\}$.

Let $f(s)=\tan^{-1}(s+2)\qquad \therefore f'(s)=\dfrac{1}{1+(s+2)^2}$

$$\therefore L^{-1}\{f'(s)\}=L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}=e^{-2t}L^{-1}\left(\frac{1}{s^2+1}\right)=e^{-2t}\sin t$$

$$\therefore\ -t\,L^{-1}\{f(s)\}=e^{-2t}\sin t\qquad \therefore L^{-1}\{\tan^{-1}(s+2)\}=-\frac{e^{-2t}\sin t}{t}$$

Ex 10. *Evaluate* $L^{-1}\left\{s\log\dfrac{s}{\sqrt{s^2+1}}+\cot^{-1}s\right\}$.

Let $f(s)=s\log\dfrac{s}{\sqrt{s^2+1}}+\cot^{-1}s$

$$=s\left\{\log s-\frac{1}{2}\log(s^2+1)\right\}+\cot^{-1}s$$

$$\therefore\ f'(s)=\left\{\log s-\frac{1}{2}\log(s^2+1)\right\}+s\left\{\frac{1}{s}-\frac{1}{2}\cdot\frac{2s}{s^2+1}\right\}-\frac{1}{s^2+1}$$

$$=\log s-\frac{1}{2}\log(s^2+1)$$

$$\therefore\ f''(s)=\frac{1}{s}-\frac{1}{2}\cdot\frac{2s}{s^2+1}=\frac{1}{s}-\frac{s}{s^2+1}$$

$$\therefore\ L^{-1}\{f''(s)\}=L^{-1}\left(\frac{1}{s}\right)-L^{-1}\left(\frac{s}{s^2+1}\right)=1-\cos t$$

$$\therefore (-1)^2 t^2 L^{-1}\{f(s)\} = 1-\cos t \quad \text{or,} \quad L^{-1}\{f(s)\} = \frac{1-c}{t^2}$$

Ex 11. *Evaluate* $L^{-1}\left\{\log\left(\frac{s^2-4}{s^3}\right)^{\frac{1}{3}}\right\}$.

$$L^{-1}\left\{\log\left(\frac{s^2-4}{s^3}\right)^{\frac{1}{3}}\right\} = L^{-1}\left\{\frac{1}{3}\log\left(\frac{s^2-4}{s^3}\right)\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\log\left(s^2-4\right) - \log s^3\right\}$$

$$= \frac{1}{3}\left[L^{-1}\left\{\log\left(s^2-4\right)\right\} - 3L^{-1}(\log s)\right]$$

Let $f(s) = \log\left(s^2-4\right)$ $\quad \therefore f'(s) = \frac{2s}{s^2-4}$

$$\therefore L^{-1}\{f'(s)\} = L^{-1}\left(\frac{2s}{s^2-4}\right) = 2L^{-1}\left(\frac{s}{s^2-4}\right) = 2\cosh$$

$$\therefore -tL^{-1}\{f(s)\} = 2\cosh 2t$$

$$\therefore L^{-1}\{f(s)\} = -\frac{2\cosh 2t}{t}$$

$$\therefore L^{-1}\left\{\log\left(s^2-4\right)\right\} = -\frac{2\cosh 2t}{t}$$

Again, let $\phi(s) = \log s$ $\quad \therefore \phi'(s) = \frac{1}{s}$ $\quad \therefore L^{-1}\{\phi'(s)\} = L^{-1}\Big($

or, $-tL^{-1}\{\phi(s)\} = 1$ $\quad$ or, $L^{-1}\{\phi(s)\} = -\frac{1}{t}$

$$\therefore L^{-1}\{\log s\} = -\frac{1}{t}$$

$\therefore$ From (1), we get $L^{-1}\left\{\log\left(\frac{s^2-4}{s^3}\right)^{\frac{1}{3}}\right\} = \frac{1}{3}\left[-\frac{2\cosh 2t}{t}\right.$

$$= \frac{1}{3t}\left(3 - 2\cosh 2t\right)$$

Ex 12. *Evaluate* $L^{-1}\left\{\frac{s^2}{(s+2)^3}\right\}$.

Now $L^{-1}\left\{\frac{1}{(s+2)^3}\right\} = e^{-2t}L^{-1}\left(\frac{1}{s^3}\right) = e^{-2t}\frac{t^2}{2!} = \frac{t^2e^{-2t}}{2} = F(t)$ (say)

Since $F(0) = F'(0) = 0$, so by Theorem 2 of Art 5.2.7,

$$L^{-1}\left\{\frac{s^2}{(s+2)^3}\right\} = \frac{d^2}{dt^2}\left(\frac{t^2e^{-2t}}{2}\right) = e^{-2t}\left(1-4t+2t^2\right)$$

Ex 13. *Evaluate* $L^{-1}\left\{\frac{2s^3+10s^2+8s+40}{s^2\left(s^2+9\right)}\right\}$.

$$L^{-1}\left\{\frac{2s^3+10s^2+8s+40}{s^2\left(s^2+9\right)}\right\} = 2L^{-1}\left\{\frac{s^3}{s^2\left(s^2+9\right)}\right\} + 10L^{-1}\left\{\frac{s^2}{s^2\left(s^2+9\right)}\right\}$$

$$+8L^{-1}\left\{\frac{s}{s^2\left(s^2+9\right)}\right\} + 40L^{-1}\left\{\frac{1}{s^2\left(s^2+9\right)}\right\}$$

$$= 2L^{-1}\left(\frac{s}{s^2+9}\right) + 10L^{-1}\left(\frac{1}{s^2+9}\right)$$

$$+8L^{-1}\left\{\frac{1}{s\left(s^2+9\right)}\right\} + 40L^{-1}\left\{\frac{1}{s^2\left(s^2+9\right)}\right\} \quad \ldots (1)$$

Now $L^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t$ $\quad L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{\sin 3t}{3} = F(t)$

$$\therefore L^{-1}\left\{\frac{1}{s\left(s^2+9\right)}\right\} = \int_0^t F(u)\,du = \int_0^t \frac{\sin 3u}{3}\,du$$

$$= -\frac{1}{9}\left[\cos 3u\right]_0^t = \frac{1}{9}\left(1-\cos 3t\right)$$

$$L^{-1}\left\{\frac{1}{s^2\left(s^2+9\right)}\right\} = \frac{1}{9}L^{-1}\left(\frac{1}{s^2} - \frac{1}{s^2+9}\right)$$

$$= \frac{1}{9}L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{9}L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{9}\left(t - \frac{\sin 3t}{3}\right)$$

$\therefore$ From (1), we get $L^{-1}\left\{\frac{2s^3+10s^2+8s+40}{s^2(s^2+9)}\right\}$

$$= 2\cos 3t + \frac{10}{3}\sin 3t + \frac{8}{9}(1-\cos 3t) + \frac{40}{9}\left(t - \frac{\sin 3t}{3}\right)$$

$$= \frac{10}{9}\cos 3t + \frac{50}{27}\sin 3t + \frac{40}{9}t + \frac{8}{9}$$

Ex 14. *Prove that* $L^{-1}\left\{\frac{1}{s}\log_e\left(1+\frac{1}{s^2}\right)\right\} = 2\int_0^t \frac{1-\cos u}{u}du$.

Let $f(s) = \log\left(1+\frac{1}{s^2}\right)$ $\quad\therefore f'(s) = -\frac{2}{s(s^2+1)}$

$$\therefore L^{-1}\{f'(s)\} = -2L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = -2L^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\}$$

$$= -2\left\{L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2+1}\right)\right\} = -2(1-\cos t)$$

$$\therefore -tL^{-1}\{f(s)\} = -2(1-\cos t)$$

$$\therefore L^{-1}\{f(s)\} = \frac{2(1-\cos t)}{t}$$

$$\therefore L^{-1}\left\{\frac{1}{s}f(s)\right\} = \int_0^t \frac{2(1-\cos u)}{u}du$$

$$\therefore L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\} = 2\int_0^t \frac{1-\cos u}{u}du$$

Ex 15. *Evaluate* $L^{-1}\left\{\tan^{-1}\frac{2}{s^2}\right\}$.

Let $f(s) = \tan^{-1}\frac{2}{s^2}$ $\quad\therefore f'(s) = -\frac{4s}{s^4+4}$

$$= -\frac{4s}{(s^2+2)^2 - (2s)^2}$$

$$= -\frac{(s^2+2s+2)-(s^2-2s+2)}{(s^2+2s+2)(s^2-2s+2)}$$

$$= \frac{1}{s^2+2s+2} - \frac{1}{s^2-2s+2}$$

$$\therefore L^{-1}\{f'(s)\} = L^{-1}\left(\frac{1}{s^2+2s+2}\right) - L^{-1}\left(\frac{1}{s^2-2s+2}\right)$$

$$= L^{-1}\left(\frac{1}{(s+1)^2+1}\right) - L^{-1}\left(\frac{1}{(s-1)^2+1}\right)$$

$$= e^{-t}L^{-1}\left(\frac{1}{s^2+1}\right) - e^{t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= (e^{-t}-e^{t})L^{-1}\left(\frac{1}{s^2+1}\right) = (e^{-t}-e^{t})\sin t$$

$$\therefore -tL^{-1}\{f(s)\} = (e^{-t}-e^{t})\sin t$$

$$\therefore L^{-1}\{f(s)\} = \frac{(e^{-t}-e^{t})\sin t}{-t} \quad \therefore L^{-1}\left\{\tan^{-1}\frac{2}{s^2}\right\} = \frac{(e^{t}-e^{-t})\sin t}{t}$$

Ex 16. *Evaluate* $L^{-1}\left\{\frac{1}{s^3(s+1)}\right\}$.

We know $L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$

$\therefore L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-u}\,du = 1-e^{-t}$, by the Theorem of Art 5.3.8

Again applying same theorem of Art 14.8, we get

$$L^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \int_0^t (1-e^{-u})du = t + e^{-t} - 1$$

$\therefore$ Again by the same theorem we get

$$L^{-1}\left\{\frac{1}{s^3(s+1)}\right\}=\int_0^t\left(u+e^{-u}-1\right)du$$

$$=\frac{t^2}{2}-e^{-t}-t+1$$

Ex 17. *Apply convolution theorem to find inver transform of* $\frac{s}{\left(s^2+9\right)^2}$ **[W.B.U**

We have

$$L^{-1}\left(\frac{s}{s^2+9}\right)=L^{-1}\left(\frac{s}{s^2+3^2}\right)=\cos 3t=F(t),\ say$$

$$L^{-1}\left(\frac{1}{s^2+9}\right)=L^{-1}\left(\frac{1}{s^2+3^2}\right)=\frac{\sin 3t}{3}=\cos(t),\ sa$$

$\therefore$ By convolution theorem,

$$L^{-1}\left\{\frac{s}{\left(s^2+9\right)^2}\right\}$$

$$=L^{-1}\left\{\frac{s}{s^2+9}\cdot\frac{1}{s^2+9}\right\}$$

$$=F*G$$

$$=\int_0^t F(u)\cos(t-u)\,du$$

$$=\int_0^t \cos 3u\frac{\sin 3(t-u)}{3}du$$

$$=\tfrac{1}{6}\int_0^t 2\cos 3u\sin(3t-3u)\,du$$

$$=\tfrac{1}{6}\int_0^t\{\sin 3t+\sin(6u-3t)\}\,du$$

$$=\tfrac{1}{6}\left[\sin 3t\cdot u-\tfrac{1}{6}\cos(6u-3t)\right]_{u=0}^{t}$$

$$=\tfrac{1}{6}\left(\sin 3t\cdot t-\tfrac{1}{6}\cos 3t+\tfrac{1}{6}3t\right)$$

$$=\frac{t\sin 3t}{6}$$

Ex 18. *Evaluate* $L^{-1}\left\{\frac{s}{\left(s^2+4\right)^3}\right\}$.

We know $L^{-1}\left(\frac{s}{s^2+4}\right)=\cos 2t=F(t)$, say

and $L^{-1}\left(\frac{1}{s^2+4}\right)=\frac{1}{2}\sin 2t=G(t)$, say

So, by Convolution Theorem,

$$L^{-1}\left(\frac{s}{s^2+4}\cdot\frac{1}{s^2+4}\right)=F*G\ =\int_0^t F(u)G(t-u)\,du$$

$$=\int_0^t\cos 2u.\frac{1}{2}\sin 2(t-u)\,du$$

$$=\frac{1}{4}\int_0^t\sin 2t\,du-\frac{1}{4}\int_0^t\sin(4u-2t)\,du$$

$$=\frac{t\sin 2t}{4}+\frac{1}{16}(\cos 2t-\cos 2t)$$

$$\therefore\ L^{-1}\left\{\frac{s}{\left(s^2+4\right)^2}\right\}=\frac{1}{4}t\sin 2t$$

Again, by Convolution Theorem,

$$L^{-1}\left\{\frac{s}{(s^2+4)^2}\cdot\frac{1}{s^2+4}\right\}=\int_0^t\frac{u\sin 2u}{4}\cdot\frac{1}{2}\sin 2(t-u)du$$

$$=\frac{1}{16}\int_0^t u\{\cos(4u-2t)-\cos 2t\}du$$

$$=\frac{1}{16}\int_0^t u\cos(4u-2t)du-\frac{\cos 2t}{16}\int_0^t u\,du$$

$$=\frac{1}{16}\left\{\left[\frac{u\sin(4u-2t)}{4}\right]_0^t-\frac{1}{4}\int_0^t\sin(4u-2t)du\right\}-\frac{\cos 2}{16}\Big]^t_0$$

$$=\frac{1}{16}\left\{\frac{1}{4}t\sin 2t+\frac{1}{16}[\cos(4u-2t)]_0^t\right\}-\frac{t^2\cos 2t}{32}$$

$$=\frac{1}{64}t\sin 2t+\frac{1}{16}.0-\frac{1}{32}t^2\cos 2t=\frac{1}{64}t(\sin 2t-2t\cos$$

Ex 19. *Apply convolution theorem to evaluate*

$$L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\}$$ [WBUT 20]

We have

$$=L^{-1}\left(\frac{1}{s^2+2s+5}\right)$$

$$=L^{-1}\left(\frac{1}{(s+1)^2+4}\right)$$

$=e^{-t}L^{-1}\left(\frac{1}{s^2+2^2}\right)$, by first shifting property

$$=e^{-t}\frac{\sin 2t}{2}$$

$$=\frac{e^{-t}\sin 2t}{2}$$

$=F(t)$, *say*

$\therefore$ By convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\}$$

$$=L^{-1}\left\{\frac{1}{s^2+2s+5}\cdot\frac{1}{s^2+2s+5}\right\}$$

$$=F*F$$

$$=\int_0^t F(u)F(t-u)du$$

$$=\int_0^t\frac{e^{-u}\sin 2u}{2}\cdot\frac{e^{-(t-u)}\sin 2(t-u)}{2}du$$

$$=\frac{1}{4}\int_0^t e^{-t}\sin 2u\sin(2t-2u)du$$

$$=\frac{e^{-t}}{8}\int_0^t 2\sin 2u\sin(2t-2u)du$$

$$=\frac{e^{-t}}{8}\int_0^t\{\cos(4u-2t)-\cos 2t\}du$$

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$$= \frac{e^{-t}}{8}\left[\frac{\sin(4u-2t)}{4}\right]_{u=0}^{t} - \frac{e^{-t}\cos 2t}{8}[u]_0^t$$

$$= \frac{e^{-t}}{8}\left(\frac{\sin 2t}{4}+\frac{\sin 2t}{4}\right) - \frac{e^{-t}\cos 2t}{8}\cdot t$$

$$= \frac{e^{-t}}{16}(\sin 2t - 2t\cos 2t)$$

Ex 20. *Evalute* $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$ [W.B.U.T. 20

Now $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

$$= \frac{1}{a^2-b^2}\left(\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right)$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$$

$$= \frac{1}{a^2-b^2}\left\{a^2 L^{-1}\left(\frac{1}{s^2+a^2}\right) - b^2 L^{-1}\left(\frac{1}{s^2+b^2}\right)\right\}$$

$$= \frac{1}{a^2-b^2}\left\{a^2\cdot\frac{\sin at}{a} - b^2\frac{\sin bt}{b}\right\}$$

$$= \frac{1}{a^2-b^2}(a\sin at - b\sin bt)$$

Ex 21. *Use convolution theorem to find* $L^{-1}\left\{\frac{1}{(s-2)(s}\right.$

We have

$$L^{-1}\left(\frac{1}{s-2}\right) = e^{2t} = F(t), \text{ say}$$

and $L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = \cos(t)$, say

$\therefore$ By convolution theorem,

$$L^{-1}\left\{\frac{1}{(s^2-2)}\cdot\frac{1}{(s^2+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s-2}\right\}$$

$$= G * F$$

$$= \int_0^t G(u)F(t-u)\,du$$

$$= \int_0^t \sin u\, e^{2(t-u)}\,du$$

$$= e^{2t}\int_0^t e^{-2u}\sin u\,du$$

$$= e^{2t}\cdot I \qquad \text{where } I = \int_0^t e^{-2u}\sin u\,du$$

$$= \left[-e^{-2u}\cos u\right]_0^t - \int_0^t \frac{e^{-2u}}{-2}(-\cos u)\,du$$

$$= -e^{-2t}\cos t + 1 - \tfrac{1}{2}\int_0^t e^{-2u}\cos u\,du$$

$$= -e^{-2t}\cos t + 1 - \tfrac{1}{2}\left\{\left[e^{-2u}\sin u\right]_0^t - \int_0^t \frac{e^{-2u}}{-2}\sin u\,du\right\}$$

$$= -e^{-2t}\cos t - 1 - \tfrac{1}{2}\left\{e^{-2t}\sin t - \tfrac{1}{2}\int_0^t e^{-2u}\sin u\,du\right\}$$

$$= -e^{-2t}\cos t - 1 - \tfrac{1}{2}e^{-2t}\sin t - \tfrac{1}{4}I$$

$\therefore \quad I\left(1-\tfrac{1}{4}\right) = e^{-2t}\cos t - \tfrac{1}{2}e^{-2t}\sin t - 1$

$\therefore \quad I = \dfrac{e^{-2t}}{5}\left(e^{2t} - 2\sin t - \cos t\right)$

$\therefore \quad L^{-1}\left\{\dfrac{1}{(s-2)(s^2-1)}\right\}$

$$= e^{2t}\cdot\frac{e^{-2t}}{5}\left(e^{2t} - 2\sin t - \cos t\right)$$

$$= \tfrac{1}{5}\left(e^{2t} - 2\sin t - \cos t\right)$$

Ex 22. *Using convolution theorem verily that*

$$\int_0^t \sin u\cos(t-u)\,du = \frac{t\sin t}{2}$$ [WBUT 2[illegible]]

We have $L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t = F(t)$, *say*

and $L^{-1}\left(\dfrac{s}{s^2+1}\right) = \cos t = G(t)$, *say*

$\therefore$ By convolution theorem

$$L^{-1}\left(\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right)$$

$$= F * G$$

$$= \int_0^t F(u)G(t-u)\,du$$

$$= \int_0^t \sin u\cos(t-u)\,du$$

$\therefore \quad \displaystyle\int_0^t \sin u\cos(t-u)\,du$

$$= L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} \quad \ldots\ldots \quad (1)$$

Now $\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right) = -\dfrac{2s}{(s^2+1)^2}$

Also, since $L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$,

so we have

$$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} = -t\sin t$$

or, $\quad L^{-1}\left\{\dfrac{-2s}{(s^2+1)^2}\right\} = -t\sin t$

or, $\quad -2\,L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = -t\sin t$

$\therefore \quad L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{t\sin t}{2}$

$\therefore$ In virtue of (1), we get

$$\int_0^t \sin u\cos(t-u)\,du = \frac{t\sin t}{2}$$

Ex 23. *Show that* $1*1*1*1.....*1(\text{n times}) = \frac{t^{n-1}}{(n-1)!}$, where ... positive integer.

Let $F(t)=1, G(t)=1$

Then $1*1 = F*G = \int_0^t F(u)G(t-u)du = \int_0^t 1\times 1\, du = t$

Again $1*1*1 = \int_0^t u.1du = \frac{1}{2}t^2$

Again, $1*1*1*1 = \int_0^t \frac{u^2}{2}\times 1du = \frac{t^3}{3!}$

Continuing this process we get $1*1*1*1.....*1(\text{n times})$ ...

Ex 24. *Evaluate* $L^{-1}\left\{\frac{2s^3+2s^2+4s+1}{(s^2+1)(s^2+s+1)}\right\}$.

Let $\frac{2s^3+2s^2+4s+1}{(s^2+1)(s^2+s+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+s+1}$

$\therefore 2s^3+2s^2+4s+1 = (As+B)(s^2+s+1)+(Cs+D)(s^2+$ ...

Equating the coeficient of s^3, s^2, s and constant ter...

$A+C=2$

$A+B+D=2$

$A+B+C=4$

$B+D=1$

Solving these we get $A=1, B=2, C=1, D=-1$

$$\therefore \frac{2s^3+2s^2+4s+1}{(s^2+1)(s^2+s+1)} = \frac{s+2}{s^2+1} + \frac{s-1}{s^2+s+1}$$

$$\therefore L^{-1}\left\{\frac{2s^3+2s^2+4s+1}{(s^2+1)(s^2+s+1)}\right\} = L^{-1}\left(\frac{s+2}{s^2+1}\right) + L^{-1}\left\{\frac{s-1}{s^2+s+1}\right\}$$

$$= L^{-1}\left(\frac{s}{s^2+1}\right) + 2L^{-1}\left(\frac{1}{s^2+1}\right) + L^{-1}\left\{\frac{\left(s+\frac{1}{2}\right)-\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= \cos t + 2\sin t + L^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - \frac{3}{2}L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}.$$

$$= \cos t + 2\sin t + e^{-\frac{t}{2}}\left\{\cos\frac{\sqrt{3}}{2}t - \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right\}.$$

EXERCISE

[I] Short Answer Questions

1. Prove that $L^{-1}\{Af(s)-B\phi(s)\} = AL^{-1}\{f(s)\} - BL^{-1}\{\phi(s)\}$.
2. Prove that $L^{-1}\{f(s-a)\} = e^{at}L^{-1}\{f(s)\}$
3. Using definition of Inverse Laplace Transform obtain the inverse Laplace transform of $\frac{1}{s^2+25}$.
4. Find $L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$
5. Find the inverse Laplace transform of $\frac{s^2-3s+4}{s^3}$.
6. Find the inverse Laplace transform of $\frac{s-2}{s^2-4s+13}$
7. Prove that $L^{-1}\left\{\frac{d}{ds}f(s)\right\} = -tL^{-1}\{f(s)\}$
8. Using the result $L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{\sin 3t}{3}$, find $L^{-1}\left(\frac{s}{(s^2+9)^2}\right)$
9. Using convolution theorem find $L^{-1}\left\{\frac{3}{(s-2)(s+5)}\right\}$.

10. Defi..e Con.volution of two functions.

11. Prove that Convolution of two functions is comm

12. Using the property of Laplace Transform on de

function prove that $L^{-1}\{sf(s)\} = \frac{d}{dt}L^{-1}\{f(s)\}$ if L

$t = 0$.

13. Find the convolution of 1 and $\cos t$.

14. Using convolution theorem, find $L^{-1}\left(\frac{1}{(s-1)^2}\right)$

15. Prove that $f*(g-h) = f*g - f*h$ where * stan lution.

ANSWERS

3. $\frac{1}{5}\sin 5t$ 4. $e^{-\frac{t}{2}}\left(\cos\frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2}\right)$ 5. $1-3t+$

6. $e^{2t}\cos 3t$ 8. $\frac{t\sin 3t}{6}$ 9. $\frac{3}{7}\left(e^{2t} - e^{-5t}\right)$ 13. $\sin$

14. $e^t * e^t = te^t$

[II] LONG ANSWER QUESTIONS

Evaluate :

1. $L^{-1}\left(\frac{1}{s^4}\right)$

2. $L^{-1}\left(\frac{6s}{s^2-16}\right)$

3. $L^{-1}\left(\frac{8-6s}{16s^2+9} - \frac{3+4s}{9s^2-16} + \frac{6}{2s-3}\right)$

4. $L^{-1}\left(\frac{3s-8}{4s^2+2}\right.$

5. $L^{-1}\left(\frac{6s-4}{s^2-4s+20}\right)$

6. $L^{-1}\left(\frac{4s+12}{s^2+8s+}\right.$

7. $L^{-1}\left(\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} - \frac{6s-8}{9+16s^2}\right)$

8. $L^{-1}\left(\frac{3s-}{s^2-4}\right.$

9. $L^{-1}\left(\frac{3s-2}{s^2-4s+20}\right)$

10. $L^{-1}\left(\frac{3s+2}{4s^2+12s+9}\right)$

11. $L^{-1}\left(\frac{1}{(s+a)^n}\right)$

12. $L^{-1}\left(\frac{1}{s^2-6s+10}\right)$

13. $L^{-1}\left\{\frac{s^2}{(s+2)^3}\right\}$

14. $L^{-1}\left\{\frac{3s+1}{(s+1)^4}\right\}$

15. $L^{-1}\left\{\frac{5s-2}{3s^2+4s+8}\right\}$

16. $L^{-1}\left(\frac{1}{s^2-3s+2}\right)$

17. $L^{-1}\left(\frac{e^{-2s}}{s^2}\right)$

18. $L^{-1}\left(\frac{8e^{-3s}}{s^2+4}\right)$

19. $L^{-1}\left\{\frac{se^{-2s}}{s^2+3s+2}\right\}$

20. $L^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\}$

21. $L^{-1}\left\{\frac{se^{-\frac{2s\pi}{3}}}{s^2+9}\right\}$

22. $L^{-1}\left\{\frac{s}{(s^2-16)^2}\right\}$

23. $L^{-1}\left\{\frac{s}{(s^2+\alpha^2)^2}\right\}$

24. $L^{-1}\left\{\log_e\left(1+\frac{1}{s^2}\right)\right\}$

25. $L^{-1}\left\{\frac{s}{(s^2-k^2)^2}\right\}$ if $L^{-1}\left\{\frac{1}{s^2-k^2}\right\} = \frac{\sinh kt}{k}$

26. Evaluate $L^{-1}\left(s^{-n}\right)$ using the result $L^{-1}\left(s^{-1}\right) = 1$. Hence find $L^{-1}\left\{(s-6)^{-n}\right\}$.

27. $L^{-1}\left\{\log_e\frac{s+2}{s+1}\right\}$

28. $L^{-1}\left\{\log_e\left(1-\frac{1}{s^2}\right)\right\}$

29. $L^{-1}\left\{\cot^{-1}\left(\frac{s-2}{3}\right)\right\}$

30. $L^{-1}\left\{\tan^{-1}\left(\ldots\right.\right.$

31. $L^{-1}\left\{\tan^{-1}(s+1)\right\}$

32. $L^{-1}\left\{\frac{s+\ldots}{(s^2+4s\ldots}\right.$

33. $L^{-1}\left\{\log_e\sqrt{\frac{s-1}{s+1}}\right\}$

34. $L^{-1}\left\{\frac{1}{2}\log_e\left(\ldots\right.\right.$

35. Prove that $L^{-1}\{sf''(s)\}=t^2F'(t)+2tF(t)$ where $F(t)\ldots$

36. Prove that $L^{-1}\left\{s^2\frac{d^2f}{ds^2}\right\}=t^2\frac{d^2F}{dt^2}+4t\frac{dF}{dt}+2F$ where $F(\ldots$

37. If $L\{F(t)\}=f(s)$, prove that $L^{-1}\left\{s^2\frac{df}{ds}+F(0)\right\}=-\ldots$

Evaluate :

38. $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

39. $L^{-1}\left\{\frac{s^2}{(s^2+4)\ldots}\right.$

40. (a) $L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$ [W.B.U.Tech.2003] (b) $L^{-1}\left\{\frac{\ldots}{(s^2+\ldots}\right.$

41. $L^{-1}\left\{\frac{1}{s(s+1)^3}\right\}$

42. $L^{-1}\left\{\frac{1}{s}\log\ldots\right.$

43. $L^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\}$

44. $L^{-1}\left\{\frac{1}{s}\log_e\ldots\right.$

45. $L^{-1}\left\{\frac{s^{-4}}{(s^2+1)}\right\}$

46. $L^{-1}\left\{\frac{(s^2-\ldots}{s^{\ldots}}\right.$

47. $L^{-1}\left\{\log_e\frac{s^2+1}{s^2+s}\right\}$

48. $L^{-1}\left\{\log_e\sqrt{\frac{s^2-3^2}{s^3}}\right\}$

49. $L^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\}$

50. $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

51. $L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\}$

52. $L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$

53. $L^{-1}\left\{\frac{1}{(s^2+1)^3}\right\}$

Evaluate :

54. (a) $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$ [W.B.U.Tech.2005]

(b) $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}$

55. $L^{-1}\left\{\frac{19s+37}{(s+1)(s+3)(s-2)}\right\}$

56. $L^{-1}\left\{\frac{5+s}{(s^2+1)(s+1)}\right\}$

57. $L^{-1}\left\{\frac{s-1}{(s^2+2s+2)(3+s)}\right\}$

58. $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$

59. $L^{-1}\left\{\frac{1}{s^3(s+1)}\right\}$

60. (a) $L^{-1}\left\{\dfrac{1}{s^4-a^4}\right\}$ (b) $L^{-1}\left\{\dfrac{1}{(s^2+1)(s^2+9)}\right\}$ [W.B.U...

(c) $L^{-1}\left\{\dfrac{1}{(s^2+a^2)(s^2+b^2)}\right\}$ [W.B.U...

61. $L^{-1}\left\{\dfrac{18+22s+6s^2}{6+11s+6s^2+s^3}\right\}$

62. $L^{-1}\left\{\dfrac{3(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}\right\}$ **63.** $L^{-1}\left\{\dfrac{6s^2-1...}{(s+1)(...}\right.$

64. $L^{-1}\left\{\dfrac{1}{(s-2)^4(s+3)}\right\}$

65. $L^{-1}\left\{\dfrac{1}{s(s+2)^3}\right\}$ **66.** $L^{-1}\left(\dfrac{s}{s^2-k^2...}\right.$

67. $L^{-1}\left\{\dfrac{5s-2}{s^2(s+2)(s-1)}\right\}$ **68.** $L^{-1}\left\{\dfrac{1+...}{(s+2)^2(...}\right.$

69. $L^{-1}\left\{\dfrac{s^3+16s-24}{s^4+20s^2+64}\right\}$ **70.** $L^{-1}\left\{\dfrac{s^2-3...}{(s+2)(s-3)(s^2...}\right.$

71. $L^{-1}\left\{\dfrac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}\right\}$ **72.** $L^{-1}\left\{\dfrac{s}{(s^2+2s+2)(s^2-...}\right.$

73. $L^{-1}\left(\dfrac{1}{s^3+1}\right)$

74. Find $L^{-1}\left\{\dfrac{s}{(s-2)^5(s+1)}\right\}$ **75.** $L^{-1}\left\{\dfrac{3s^3-3s^2-40s+36}{(s^2-4)^2}\right\}$

ANSWERS

1. $\dfrac{t^3}{6}$ **2.** $6\cosh 4t$ **3.** $3e^{\frac{3t}{2}}-\frac{1}{4}\sinh\frac{4t}{3}-\frac{4}{9}\cosh\frac{4t}{3}+\frac{2}{3}\sin\frac{3t}{4}-\frac{3}{8}\cos\frac{3t}{4}$

4. $\frac{3}{4}\cos\frac{5t}{2}-\frac{4}{5}\sin\frac{5t}{2}$ **5.** $2e^{2t}(3\cos 4t+\sin 4t)$ **6.** $4e^{-4t}(1-t)$

7. $3e^{\frac{3t}{2}}-\dfrac{1}{4}\sinh\dfrac{4t}{3}-\dfrac{4}{9}\cosh\dfrac{4t}{3}+\dfrac{2}{3}\sin\dfrac{3t}{4}-\dfrac{3}{8}\cos\dfrac{3t}{4}$

8. $e^{2t}(3\cos 2t-4\sin 2t)$ **9.** $3e^{2t}\cos 4t+e^{2t}\sin 4t$

10. $\frac{1}{8}e^{-\frac{3t}{2}}(6-5t)$ **11.** $e^{-at}\dfrac{t^{n-1}}{(n-1)!}$ **12.** $e^{3t}\sin t$

13. $e^{-2t}(1-4t+2t^2)$ **14.** $e^{-t}\left(\frac{3t^2}{2}-\frac{t^3}{3}\right)$

15. $\dfrac{e^{-\frac{2t}{3}}}{15}\left\{25\cos\frac{2\sqrt{5}t}{3}-24\sqrt{5}\sin\frac{2\sqrt{5}t}{3}\right\}$ **16.** $e^{2t}-e^{t}$

17. $\begin{matrix} t-2, t>2 \\ 0, t<2 \end{matrix}$ **18.** $4\sin 2(t-3)u(t-3)$ **19.** $\begin{matrix} 2e^{-2t+4}-e^{-t+2}, t>2 \\ 0, t<2 \end{matrix}$

20. $\begin{matrix} \frac{1}{6}(t-5)^3e^{2(t-5)}, t>5 \\ 0, t<5 \end{matrix}$ **21.** $\begin{matrix} \cos 3\left(t-\frac{2\pi}{3}\right), & t>\frac{2\pi}{3} \\ 0 & , t<\frac{2\pi}{3} \end{matrix}$

22. $\dfrac{t}{8}\sinh 4t$ **23.** $\dfrac{t\sin\alpha t}{2\alpha}$ **24.** $\dfrac{2(1-\cos t)}{t}$ **25.** $\dfrac{t}{2k}\sinh kt$

27. $\dfrac{e^{-t}-e^{-2t}}{t}$ **28.** $\frac{2}{t}(1-\cosh t)$ **29.** $\frac{1}{t}e^{2t}\sin 3t$

30. $-\frac{1}{t}e^{2t}\sin 3t$ **31.** $-\frac{1}{t}e^{-t}\sin t$ **32.** $\frac{1}{3}te^{-2t}\sin t$

33. $-\frac{1}{t}\sinh t$ 34. $\frac{1}{t}(1-\cosh at)$ 38. $\frac{t^2}{2}+$

39. $\frac{1}{4}(\sin 2t + 2t\cos t)$

40. (a) $e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$ (b) $e^{-2t}\left(\frac{\sin 2t}{t}-t\sin\right.$

41. $1-e^{-t}\left(\frac{t^2}{2}+t+1\right)$ 42. $\int_0^t \frac{e^{-x}-e^{-2x}}{x}dx$ 43. $\frac{2t}{3}$

44. $\int_0^t \frac{e^{-2u}-e^{-3u}}{u}du$ 45. $\frac{1}{6}t^3+\sin t - t$ 46. $1-t^2+$

47. $\frac{1}{t}\left(1+e^{-t}-2\cos t\right)$ 48. $\frac{1}{t}(1-\cosh 3t)$

49. $e^t - e^{2t}$ 50. $te^{-t}+2e^{-t}+t-2$ 51. $\frac{1}{4}\left(e^t - e\right.$

52. $\frac{1}{2}\left(\sin t - \cos t + e^{-t}\right)$

53. $\frac{1}{8}\left\{\left(3-t^2\right)\sin t - 2t\sin t\right\}$

54. (a) $\frac{t\sin t}{2}$ (b) $\frac{1}{16}(1-t\sin 2t-\cos 2t)$

55. $5e^{2t}-3e^{-t}-2e^{-3t}$

56. $2e^{-t}+3\sin t-2\cos t$

57. $\frac{1}{5}e^{-t}\left(4\cos t-3\sin t\right)-\frac{4}{5}e^{-3t}$ 58. $\frac{1}{2a}\{at\cos at+\sin$

59. $1-t+\frac{t^2}{2}-e^{-t}$

60. (a) $\frac{1}{2a^3}(\sinh at-\sin at)$ (b) $\frac{1}{24}(3\sin t-\sin 3t)$

(c) $\frac{b\sin at - a\sin bt}{ab\left(b^2-a^2\right)}$

61. $e^{-t}+2e^{-2t}+3e^{-3t}$

62. $e^{-t}\sin t + e^{-t}\sin 2t$

63. $-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4te^{2t}-\frac{7}{2}t^2e^{2t}$

64. $\frac{e^{2t}}{30}\left(t^3-\frac{3}{5}t^2+\frac{6}{25}t-\frac{6}{125}\right)+\frac{e^{-3t}}{625}$

65. $\frac{e^{-2t}}{4}\left(t^2-t-\frac{1}{2}\right)+\frac{1}{8}$

66. $\cosh kt$ 67. $t+e^t+e^{-2t}-2$

68. $\frac{t}{3}\left(-e^{-2t}+e^t\right)$ 69. $\frac{1}{2}\sin 4t+\cos 2t-\sin 2t$

70. $\frac{3}{50}e^{3t}-\frac{1}{25}e^{-2t}+\frac{7}{10}e^{-t}\sin 2t$

71. $\frac{1}{2}\sin t+\frac{1}{2}t\cos t-te^{-t}$ 72. $\frac{1}{2}\sin t\sinh t$

73. $\frac{1}{3}\left\{e^{-t}-e^{\frac{t}{2}}\left(\cos\frac{\sqrt{3}}{2}t-\sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)\right\}$

74. $e^{2t}\left(\frac{t^4}{72}+\frac{t^3}{54}+\frac{t^2}{54}+\frac{t}{81}+\frac{1}{243}\right)-\frac{e^{-1}}{243}$

75. $(5t+3)e^{-2t}-2te^{2t}$

[III] Multiple Choice Questions

1. $L^{-1}\left\{\frac{a}{s^2+a^2}\right\}$ is equal to

(a) $\sin at$ (b) $\cos at$

(c) $\frac{\sin at}{a}$ (d) $\frac{\cos at}{a}$

2. $L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$ is equal to

(a) $\frac{t\sin at}{2a}$ (b) $\frac{t\sin at}{a}$

(c) $\frac{\sin at}{a^2}$ (d) at

[WBU 006]

3. $L^{-1}\left\{\frac{1}{s^2-4}\right\}$ is equal to

(a) $\sinh 2t$ (b) $\frac{\sinh 2t}{2}$

(c) $\cosh 2t$ (d) $\frac{\cosh 2t}{2}$

4. $L^{-1}\left\{\frac{4}{s}\right\}$ is equal to

(a) 4 (b) 1

(c) $4t$ (d) t

5. $L^{-1}\left\{\frac{1}{s+2}\right\} =$

(a) e^{2t} (b) $2e^{2t}$

(c) $2e^{-2t}$ (d) e^{-2t}

6. $L^{-1}\left\{\frac{1}{s(s+1)}\right\}$ is equal to

(a) $1+e^{t}$ (b) $1-e^{t}$

(c) $1+e^{-t}$ (d) $1-e^{-t}$

7. $L^{-1}\left(\frac{3}{s^5}\right)$ is equal to

(a) $\frac{3t^4}{4!}$ (b) $\frac{3t^5}{5!}$

(c) $\frac{3t^4}{5!}$ (d) none

8. $L^{-1}\left\{\frac{1}{s^2+5s+6}\right\}$ is equal to

(a) $e^{-2t}+e^{-3t}$ (b) $e^{2t}+e^{3t}$

(c) $e^{-2t}-e^{-3t}$ (d) $e^{2t}-e^{3t}$

9. $L^{-1}\left(\frac{s}{s^2+5}+\frac{1}{s^2-4}\right)$ is equal to

(a) $\cos 5t+\frac{\sinh 2t}{2}$ (b) $\cos\sqrt{5}t+\frac{\sinh 2t}{2}$

(c) $\cos\sqrt{5}t+\frac{\sinh t}{2}$ (d) none

10. $L^{-1}\left(\frac{4}{s^2-7}+\frac{2}{s^2+7}\right)$ is equal to [WB

(a) $\frac{4\sinh 7t}{7}-\frac{2\sin 7t}{7}$ (b) $\frac{4\cos 7t}{7}+\frac{2\sin 7t}{7}$

(c) $\frac{4\cos\sqrt{7}t}{7}-\frac{2\sin\sqrt{7}t}{7}$ (d) $\frac{4\sinh\sqrt{7}t}{\sqrt{7}}+\frac{2\sin\sqrt{7}t}{\sqrt{7}}$

11. $L^{-1}\left\{\frac{24}{(s-1)^5}\right\}$ is equal to

(a) $\frac{24t^3}{e^t}$ b) $\frac{24t^4}{e^t}$

(c) $\frac{2t^4}{e^t}$ (d) $\frac{t^4}{e^t}$

12. $L^{-1}\left\{\frac{e^{-as}}{s^2+1}\right\}$ is equal to

(a) $\sin t\, u(t-a)$ (b) $\sin t\, u(t+a)$

(c) $u(t-a)$ (d) $-\sin t\, u(t-a)$

13. $L^{-1}\left\{\frac{s}{(s+3)^2+4}\right\}$ is equal to

(a) $\cos 2t - 1\cdot 5\sin 2t$ (b) $e^{-3t}(\cos 2t - 1\cdot 5\sin 2$

(c) e^{-3t} (d) none

14. If $L^{-1}\left(\frac{1}{s^3}\right)=a*a*a$ then $a=$

(a) 3 (b) $\frac{1}{s}$

(c) 1 (d) 0

15. If $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=f(t)*g(t)$ then $f(t)$ and $g(t)$ are respectively

(a) $\cos at, \frac{\sin at}{a}$ (b) $\frac{\cos at}{a}, \frac{\sin at}{a}$

(c) $\frac{\cos at}{a}$, $\sin at$ (d) none of these

16. $L^{-1}\left\{\frac{\cos(3+s)+5}{s^2+6s+9}\right\}$ is equal to

(a) $e^{-3t}L^{-1}\left\{\frac{\cos s+5}{s^2}\right\}$ (b) $e^{-t}L^{-1}\left\{\frac{\cos s+5}{s^2}\right\}$

(c) $e^{-3t}L^{-1}\left\{\frac{\cos 3s+5}{s^2}\right\}$ (d) $L^{-1}\left\{\frac{\cos 3s+5}{s^2}\right\}$

17. $L^{-1}\{f(s)\}=f(t)$, then $L^{-1}\{f(as)\}=$ is equal to

(a) $F\left(\frac{t}{a}\right)$ (b) $\frac{1}{a}F\left(\frac{t}{a}\right)$

(c) $\frac{1}{a}F(t)$ (d) none of these

18. $L^{-1}\left\{\frac{2}{s^2+4}\right\}=\sin 2t$, *then* $L^{-1}\left\{\frac{2}{as^2+4}\right\}=$ is equal to

(a) $\sin\frac{2t}{3}$ (b) $\frac{1}{3}\sin 2t$

(c) $\frac{1}{3}\sin\frac{2t}{3}$ (d) none of these

19. $L^{-1}\left\{\dfrac{1}{s(s^2+7)}\right\}$ is equal to

(a) $\frac{1}{7}(1-\cos\sqrt{7}t)$ (b) $\frac{1}{7}(1-\cos 7t)$

(c) $\dfrac{\cos\sqrt{7}t}{7}$ (d) none

20. If $L^{-1}\{f(s)g(s)\} = \phi * \psi$ then ϕ and ψ are respectively

(a) $L\{f(s)\}, L\{g(s)\}$ (b) $L^{-1}\{f(s)\}, L^{-1}\{g(s)\}$

(c) $f(s)$ and $g(s)$ (d) $L\{f(s)\}$ and $L^{-1}\{g(s)\}$

21. $L^{-1}\{\log(s+3)\}$ is equal to

(a) e^{-3t} (b) $\dfrac{e^{-3t}}{t}$

(c) $-\dfrac{e^{-3t}}{t}$ (d) $\dfrac{e^{3t}}{t}$

22. $L^{-1}\left\{\dfrac{f(s)}{s^2}\right\}$ is equal to

(a) $\int_0^t\int_0^\xi F(u)\,du\,d\xi$ (b) $\dfrac{d^2}{ds^2}F(s)$

(c) $\int_0^t\int_0^\xi f(s)\,ds\,d\xi$ (d) none [where $F(t) = L$

23. The convolution $1*1$ is

(a) 1 (b) 2

(c) t (d) t^2

24. $L^{-1}\left\{\int_s^\infty f(u)\,du\right\}$ is equal to

(a) $\frac{1}{t}L^{-1}\{f(s)\}$ (b) $\frac{1}{t^2}L^{-1}\{f(s)\}$

(c) $tL^{-1}\{f(s)\}$ (d) none

25. $L^{-1}\left\{\int_s^\infty \dfrac{du}{u^2+25}\right\}$ is equal to

(a) $\dfrac{\sin 5t}{5}$ (b) $\dfrac{\sin 5t}{5}$

(c) $\dfrac{\cos 5t}{5t}$ (d) none

26. $L^{-1}\left\{\int_s^\infty \dfrac{du}{u-7}\right\}$ is equal to

(a) $\dfrac{e^{-7t}}{t}$ (b) $\dfrac{e^{7t}}{t^2}$

(c) $\dfrac{e^{7t}}{t}$ (d) e^{-7t}

ANSWERS

1.a	2.d	3.b	4.a	5.d	6.d	7.a	8.c
9.b	10.d	11.d	12.d	13.b	14.c	15.a	16.a
17.b	18.c	19.a	20.b	21.c	22.a	23.c	24.a
25.b	26.c						