Improper Integrals & Beta, Gamma Functions

(1) $I = \int_{-1}^{1} \frac{dx}{x^3}$

$$= \lim_{\varepsilon_1 \to 0^+} \int_{-1}^{\varepsilon_1} \frac{dx}{x^3} + \lim_{\varepsilon_2 \to 0^+} \int_{\varepsilon_2}^{1} \frac{dx}{x^3}$$

$$= \lim_{\varepsilon_1 \to 0^+} \left[-\frac{1}{2x^2}\right]_{-1}^{\varepsilon_1} + \lim_{\varepsilon_2 \to 0^+} \left[-\frac{1}{2x^2}\right]_{\varepsilon_2}^{1}$$

$$= \lim_{\varepsilon_1 \to 0^+} \left[-\frac{1}{2\varepsilon_1^2} + \frac{1}{2}\right] + \lim_{\varepsilon_2 \to 0^+} \left[-\frac{1}{2} + \frac{1}{2\varepsilon_2^2}\right]$$

at this stage, when $\varepsilon_1 \neq \varepsilon_2$, the limit doesn't exist finitely, so the integral doesn't exist.

But if $\varepsilon_1 = \varepsilon_2 = \varepsilon$ (say)

then $I = \lim_{\varepsilon \to 0} \left[-\frac{1}{2\varepsilon^2} + \frac{1}{2\varepsilon^2} - \frac{1}{2} + \frac{1}{2}\right]$

$= 0$

So it exists for $\varepsilon_1 = \varepsilon_2$.

$\therefore$ Integral exists in Cauchy's principal value sense, but not in general sense.

PTO

2) a) $I = \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

$$= \lim_{\varepsilon_1 \to -\infty} \int_{\varepsilon_1}^{0} \frac{dx}{(x+1)^2+1} + \lim_{\varepsilon_2 \to +\infty} \int_{0}^{\varepsilon_2} \frac{dx}{(x+1)^2+1}$$

$$= \lim_{\varepsilon_1 \to -\infty} \left[\tan^{-1}(x+1)\right]_{\varepsilon_1}^{0} + \lim_{\varepsilon_2 \to +\infty} \left[\tan^{-1}(x+1)\right]_{0}^{\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to -\infty} \left(\tan^{-1}1 - \tan^{-1}(\varepsilon_1+1)\right) + \lim_{\varepsilon_2 \to +\infty} \left(\tan^{-1}(\varepsilon_2+1) - \tan^{-1}1\right)$$

$$= \left(\frac{\pi}{4} + \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= 2 \cdot \frac{\pi}{2} = \pi. \text{ (Ans).}$$

b) $I = \int_{-1}^{1} \frac{\sqrt{1+x}}{\sqrt{1-x}}\, dx$

$$= \lim_{\varepsilon \to 0^+} \int_{-1}^{1-\varepsilon} \sqrt{\frac{1+x}{1-x}}\, dx.$$

$$= \lim_{\varepsilon \to 0^+} \int_{-1}^{1-\varepsilon} \frac{1+x}{\sqrt{1-x^2}}\, dx$$

$$= \lim_{\varepsilon \to 0^+} \left[\int_{-1}^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} + \int_{-1}^{1-\varepsilon} \frac{x\,dx}{\sqrt{1-x^2}}\right]$$

$$= \lim_{\varepsilon \to 0^+} \left[\left(\sin^{-1}x\right)_{-1}^{1-\varepsilon} + \left(-\sqrt{1-x^2}\right)_{-1}^{1-\varepsilon}\right]$$

PTO

$$= \lim_{\varepsilon \to 0^+} \left(\sin^{-1}(1-\varepsilon) - \sin^{-1}(-1) \right) + \lim_{\varepsilon \to 0^+} \left(-\sqrt{1-(1-\varepsilon)^2} + 0 \right)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \text{ (Ans)}$$

c) $$I = \int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{\varepsilon \to \infty} \int_1^\varepsilon \frac{dx}{x\sqrt{x^2-1}}$$

let $x = \sec\theta$ | $x = 1 \Rightarrow \theta = 0$

$\Rightarrow dx = \sec\theta \tan\theta \, d\theta$ | $x = \varepsilon \Rightarrow \theta = \sec^{-1}\varepsilon$

$$\therefore I = \lim_{\varepsilon \to \infty} \int_0^{\sec^{-1}\varepsilon} \frac{\sec\theta \tan\theta}{\sec\theta \tan\theta} \, d\theta$$

$$= \lim_{\varepsilon \to \infty} \int_0^{\sec^{-1}\varepsilon} d\theta$$

$$= \lim_{\varepsilon \to \infty} \left(\sec^{-1}\varepsilon \right)$$

$$= \frac{\pi}{2} \text{ (Ans)}$$

PTO

d) $$I = \int_0^{1/e} \frac{dx}{x(\ln x)^2}$$

$$= \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1/e} \frac{dx}{x(\ln x)^2}$$

let $\ln x = z$. $\Rightarrow \frac{dx}{x} = dz$ | $x = 1/e \Rightarrow z = -1$, $x = \varepsilon \Rightarrow z = \ln \varepsilon$

$$\therefore I = \lim_{\varepsilon \to 0^+} \int_{\ln \varepsilon}^{-1} \frac{dz}{z^2}$$

$$= \lim_{\varepsilon \to 0} \left[-\frac{1}{z} \right]_{\ln \varepsilon}^{-1} = \lim_{\varepsilon \to 0} \left[1 + \frac{1}{\ln \varepsilon} \right] =$$

$$= 1 \quad \text{(Ans)}$$

e) $$I = \int_0^{\infty} \frac{dx}{(x+1)(x+2)}$$

$$= \lim_{\varepsilon \to \infty} \int_0^{\varepsilon} \frac{dx}{(x+1)(x+2)}$$

$$= \lim_{\varepsilon \to \infty} \left[\int_0^{\varepsilon} \frac{dx}{x+1} - \int_0^{\varepsilon} \frac{dx}{x+2} \right]$$

$$= \lim_{\varepsilon \to \infty} \left[\ln(x+1) \Big|_0^{\varepsilon} - \ln(x+2) \Big|_0^{\varepsilon} \right]$$

$$= \lim_{\varepsilon \to \infty} \left[\ln \varepsilon + 1 - \ln(\varepsilon + 2) \right]$$

PTO

$$= \lim_{\varepsilon \to \infty} \ln\left(\frac{\varepsilon+1}{\varepsilon+2}\right)$$

$$= \lim_{\varepsilon \to \infty} \ln\left(\frac{1+1/\varepsilon}{1+2/\varepsilon}\right) = \ln 1$$

$$= 0 \quad \text{(Ans)}$$

(c) $$I = \int_0^\infty \frac{x\,dx}{(x^2+a^2)(x^2+b^2)}$$

$$= \lim_{\varepsilon \to \infty} \int_0^\varepsilon \frac{x\,dx}{(x^2+a^2)(x^2+b^2)}.$$

$x^2 = z$
$\Rightarrow x\,dx = \frac{dz}{2}$
$x = 0 \Rightarrow z = 0$
$x = \varepsilon \Rightarrow z = \varepsilon^2$

$$= \frac{1}{2} \lim_{\varepsilon \to \infty} \int_0^{\varepsilon^2} \frac{dz}{(z+a^2)(z+b^2)}$$

$$= \frac{1}{2} \lim_{\varepsilon \to \infty} \frac{1}{b^2-a^2} \left[\int_0^{\varepsilon^2} \frac{dz}{z+a^2} - \int_0^{\varepsilon^2} \frac{dz}{z+b^2}\right]$$

$$= \frac{1}{2(b^2-a^2)} \lim_{\varepsilon \to \infty} \left[\int_0^{\varepsilon^2} \frac{dz}{z+a^2} - \int_0^{\varepsilon^2} \frac{dz}{z+b^2}\right]$$

$$= \frac{1}{2(b^2-a^2)} \lim_{\varepsilon \to \infty} \left[\ln(z+a^2)\Big|_0^{\varepsilon^2} - \ln(z+b^2)\Big|_0^{\varepsilon^2}\right]$$

$$= \frac{1}{2(b^2-a^2)} \lim_{\varepsilon \to \infty} \left[\ln\left(\frac{b^2}{a^2} \cdot \frac{\varepsilon^2+a^2}{\varepsilon^2+b^2}\right)\right]$$

$$= \frac{\ln(b^2/a^2)}{2(b^2-a^2)} \quad \text{(Ans)}$$

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PTO

1) $I = \int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$

let $x = \sec\theta$.
$\Rightarrow dx = \sec\theta\tan\theta\, d\theta$.
$\therefore x = 2 \Rightarrow \theta = \frac{\pi}{3}$.
$x = 1+\varepsilon \Rightarrow \theta = \sec^{-1}$

$= \lim_{\varepsilon\to 0^+} \int_{1+\varepsilon}^{2} \frac{dx}{(x+1)\sqrt{x^2-1}}$

$= \lim_{\varepsilon\to 0^+} \int_{\sec^{-1}(1+\varepsilon)}^{\pi/3} \frac{\sec\theta\cancel{\tan\theta}\, d\theta}{(\sec\theta+1)\cancel{\tan\theta}}$

$= \lim_{\varepsilon\to 0^+} \int_{\sec^{-1}(1+\varepsilon)}^{\pi/3} \frac{\sec\theta\, d\theta}{\sec\theta+1}$

$= \lim_{\varepsilon\to 0^+} \int_{\sec^{-1}(1+\varepsilon)}^{\pi/3} \frac{d\theta}{1+\cos\theta} = \lim_{\varepsilon\to 0^+} \int_{\sec^{-1}(1+\varepsilon)}^{\pi/3} \frac{d\theta}{2\cos^2\frac{\theta}{2}}$

$= \frac{1}{2}\lim_{\varepsilon\to 0^+} \int_{\sec^{-1}(1+\varepsilon)}^{\pi/3} \sec^2\frac{\theta}{2}\, d\theta$

$= \lim_{\varepsilon\to 0^+} \tan\frac{\theta}{2} \Big|_{\sec^{-1}(1+\varepsilon)}^{\pi/3}$

$= \lim_{\varepsilon\to 0^+} \left(\tan\frac{\pi}{6} - \tan\frac{\sec^{-1}(1+\varepsilon)}{2}\right)$

$= \frac{1}{\sqrt{3}}$. (Ans).

PTO

$2m-1 = -\frac{1}{2}$
$\Rightarrow m = \frac{1}{4}$
$2n-1 = \frac{1}{2}$
$\Rightarrow n = \frac{3}{4}$

) a)

$$I = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$$

$$= \int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta \, d\theta$$

$$= 2 \times \frac{1}{2} \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

$$= \frac{1}{2} \times \beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{2} \times \frac{\Gamma(1/4) \cdot \Gamma(3/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \times \frac{\Gamma(1/4) \cdot \Gamma(1-1/4)}{\Gamma(1)}$$

$$= \frac{1}{2} \times \frac{\pi / \sin\frac{\pi}{4}}{1}$$

$$= \frac{1}{2} \cdot \frac{\pi}{1/\sqrt{2}} = \frac{\pi}{\sqrt{2}} \text{ (Ans).}$$

)

$$I = \int_0^1 x^3 (1-x^2)^{5/2} dx$$

let $x = \sin\theta$
$\Rightarrow dx = \cos\theta \, d\theta$

$x = 1 \Rightarrow \theta = \pi/2$
$x = 0 \Rightarrow \theta = 0$

$$\therefore\ I = \int_0^{\pi/2} \sin^3\theta \cdot (1-\sin^2\theta)^{5/2} \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^3\theta \cos^6\theta\, d\theta$$

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} \sin^{2\left(\frac{4}{2}\right)-1}\theta \cos^{2\cdot\left(\frac{7}{2}\right)-1}\theta\, d\theta.$$

$$= \frac{1}{2} \times \beta\left(2, \frac{7}{2}\right)$$

$$= \frac{1}{2} \times \frac{\Gamma(2)\cdot\Gamma(7/2)}{\Gamma\left(\frac{11}{2}\right)}$$

$$\cancel{= \frac{1}{2} \times \frac{\Gamma(2)}{\frac{9}{2} \times \frac{7}{2} \times}}$$

$$= \frac{1}{2} \times \frac{\Gamma(2) \times \Gamma(7/2)}{\frac{9}{2} \times \frac{7}{2} \times \Gamma(7/2)}$$

$$= \frac{2}{\frac{63}{2}}$$

$$= \frac{4}{63}.$$

3) c) $I = \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

$= I_1 \times I_2$.

where $I_1 = \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}}$

let $x^2 = \sin\theta$ | $x=0 \Rightarrow \theta = 0$
$\Rightarrow 2x dx = \cos\theta d\theta$ | $x=1 \Rightarrow \theta = \pi/2$.

$\therefore I_1 = \frac{1}{2} \cdot \int_0^{\pi/2} \frac{\sqrt{\sin\theta}\, \cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$

$= \frac{1}{2} \cdot \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta$

$= \frac{1}{2} \cdot \beta\left(\frac{3}{4}, \frac{1}{2}\right)$

$= \frac{1}{2} \cdot \frac{\Gamma(3/4) \cdot \Gamma(1/2)}{\Gamma(5/4)}$

$\therefore I_1 = \frac{1}{2} \cdot \frac{\sqrt{\pi} \cdot \Gamma(3/4)}{\Gamma(5/4)}$

Now, $I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

let $x^2 = \tan\theta$
$\Rightarrow 2x\, dx = \sec^2\theta\, d\theta$.
$\Rightarrow dx = \frac{\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}}$

~~let $x^2 = \tan\theta$~~
~~$\Rightarrow 2x dx = \sec\theta d\theta$~~
~~$\Rightarrow dx = \frac{\cos\theta d\theta}{\sqrt{\sin\theta}}$~~

(9)

PTO

$$\therefore\ I_2 = \frac{1}{2}\int_0^{\pi/4} \frac{\cancel{\sec\theta}\,\sec^2\theta\, d\theta}{\sqrt{\tan\theta}\cdot \cancel{\sec\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{d\theta}{\frac{\sqrt{\sin\theta}}{\sqrt{\cancel{\cos\theta}}}\cdot \cancel{\cos}\theta}$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^{-1/2}\theta\, d\theta .$$

$$= \frac{1}{2}\times 2\int_0^{\pi/2} \sin^{2\cdot\frac{1}{4}-1}\theta\ \cos^{2\cdot\frac{1}{4}-1}\theta\, d\theta .$$

$$= \frac{1}{2}\times \beta\left(\frac{1}{4},\frac{1}{4}\right)$$

$$\therefore\ I_2 = \frac{1}{2}\times \frac{\left(\Gamma(1/4)\right)^2}{\Gamma(1/2)} = \frac{\left(\Gamma(1/4)\right)^2}{2\sqrt{\pi}}$$

$$\therefore\ I = I_1 \times I_2$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(3/4)}{\Gamma(5/4)}\times\frac{\Gamma(1/4)\times\Gamma(1/4)}{2\sqrt{\pi}}$$

$$= \frac{\Gamma(3/4)\cdot\Gamma(1/4)}{2\times 2}\times\frac{\cancel{\Gamma(1/4)}}{\frac{1}{4}\cancel{\Gamma(1/4)}}$$

$$= \frac{\pi}{\cancel{4}\sin\frac{\pi}{4}}\cdot\cancel{4} = \sqrt{2}\,\pi.\ (\text{Ans})$$

~~d) $\beta(m, \frac{1}{2})$~~

~~$\phi = \int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta$.~~

3) d) we know, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$.

$\therefore \beta(m, \frac{1}{2}) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta \quad \leftarrow (1)$

& $\beta(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2m-1}\theta \, d\theta$.

let $2\theta = \phi$
$\Rightarrow d\theta = \frac{1}{2}d\phi$
$\theta = 0 \Rightarrow \phi = 0$
$\theta = \pi/2 \Rightarrow \phi = \pi$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2m-1} d\theta.$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1} d\theta.$$

$$= \frac{1}{2} \cdot \frac{1}{2^{2m-2}}\int_0^{\pi} \sin^{2m-1}\phi \, d\phi$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi \, d\phi$$

$$= \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi$$

$\therefore \beta(m,m) = \frac{1}{2^{2m-1}} \beta(m, \frac{1}{2})$ (from (1))

$\therefore \boxed{\beta(m, \frac{1}{2}) = 2^{2m-1}\beta(m,m)}$ (proved)

$\cancel{3)e) \quad I = \int_0^{\ } \sqrt{1-x}}$

$\cancel{\int_0^1 \frac{dx}{\sqrt{1-(1-}}}$

$$3)e) \quad I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

let $x^n = z$ $\quad\Big|\quad$ $x=0 \Rightarrow z=0$, $\quad x=1 \Rightarrow z=1$.

$\Rightarrow n x^{n-1} dx = dz$.

$$\therefore I = \frac{1}{n}\int_0^1 \frac{dz}{z^{1-\frac{1}{n}} \cdot (1-z)^{1/2}}$$

$$= \frac{1}{n}\int_0^1 z^{\frac{1}{n}-1} \cdot (1-z)^{\frac{1}{2}-1}\, dz.$$

$$= \frac{1}{n}\beta\left(\frac{1}{n}, \frac{1}{2}\right)$$

$$= \frac{1}{n} \cdot \frac{\Gamma\left(\frac{1}{n}\right) \cdot \sqrt{\pi}}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$ ~~(proved)~~

$$\therefore I = \frac{\Gamma\left(\frac{1}{n}\right) \cdot \sqrt{\pi}}{n \cdot \Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}$$

(Proved)

4) $I = \int_0^\infty \sqrt{x}\, e^{-x^3}\, dx.$

let $x^3 = Z$. $\Rightarrow x = Z^{1/3}$

$\Rightarrow 3x^2 dx = dZ$. $\Rightarrow \sqrt{x} = Z^{1/6}$

$\Rightarrow dx = \frac{1}{3} \cdot \frac{dZ}{x^2}$

$= \frac{1}{3} \cdot \frac{dZ}{Z^{2/3}}$

$$\therefore\ I = \frac{1}{3}\int_0^\infty Z^{1/6} \cdot e^{-Z} \frac{dZ}{Z^{2/3}}$$

$$= \frac{1}{3}\int_0^\infty Z^{-1/2} e^{-Z} dZ.$$

$$= \frac{1}{3}\int_0^\infty e^{-Z} \cdot Z^{\frac{1}{2}-1} dZ.$$

$$= \frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3} \text{ (proved).}$$

a) $I = \int_0^\infty 55^{-x^2} dx.$

let $55^{-x^2} = Z$.

$\Rightarrow -x^2 \ln 55 = \ln Z$.

$\Rightarrow -x^2 = \frac{\ln Z}{\ln 55}$.

$55^{-x^2} \cdot (-2x) \ln 55\, dx = dZ$.

$dx = \frac{1}{\ln 55} \cdot \frac{dZ}{(-2x) Z}$

PTO

5) a) $I = \int_0^{\infty} 55^{-x^2} dx.$

$= \int_0^{\infty} e^{\ln 55^{-x^2}} dx.$

$= \int_0^{\infty} e^{(\ln 55)(-x^2)} dx.$

$= \frac{1}{2\sqrt{\ln 55}} \int_0^{\infty} e^{-z} z^{-1/2} dz$

$= \frac{1}{2\sqrt{\ln 55}} \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz.$

$= \frac{1}{2\sqrt{\ln 55}} \cdot \Gamma(1/2)$

$= \frac{\sqrt{\pi}}{2\sqrt{\ln 55}}$ (Ans)

let

$x^2 \ln 55 = z.$

$\Rightarrow 2x \ln 55 \, dx = dz.$

$\Rightarrow dx = \frac{dz}{\ln 55 \cdot 2\sqrt{\frac{z}{\ln 55}}}$

$= \frac{z^{-1/2} dz}{2\sqrt{\ln 55}}$

1.b) I

$\therefore I$

$> I_1$

b) $I = \int_0^1 x^4 \ln\left(\frac{1}{x}\right)^3 dx.$

let $\frac{1}{x^3} = z \Rightarrow x = \left(\frac{1}{z}\right)^{1/3}$

$\Rightarrow -\frac{3}{x^4} dx = dz.$

$\Rightarrow dx = -\frac{1}{3} \cdot x^4 dz$

$= -\frac{1}{}$

PTO

5.(b) $I = \int_0^1 x^4 \ln\left(\frac{1}{x}\right)^3 dx.$

$= \int_0^1 -3x^4 . \ln x \, dx.$

let $x = e^{-z}$ | $x = 0 \Rightarrow z = \infty$

$\therefore dx = -e^{-z} dz$ | $x = 1 \Rightarrow z = 0.$

$\therefore I = \int_\infty^0 (3e^{-4z})(-z)(e^{-z} dz)$

$= -3 \int_0^\infty e^{-5z} . z \, dz.$

let $5z = p.$

$\Rightarrow dz = \frac{1}{5} dp.$

$\therefore I = -\frac{3}{25} \int_0^\infty e^{-p} . p \, dp.$

$= -\frac{3}{25} \int_0^\infty e^{-p} . p^{2-1} dp.$

$= -\frac{3}{25} \Gamma(2).$

$= -\frac{3}{25}.$ (Ans).

c) $I_1 = \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \int_0^{\pi/2} \sin^{-\frac{1}{2}} x \, . dx.$

$= \frac{1}{2} \beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{1}{2}\right)$

$= \frac{1}{2} \beta\left(\frac{1}{4}, \frac{1}{2}\right)$

PTO

5).

$$= \frac{1}{2} \frac{\Gamma(\frac{1}{4}) \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}$$

$$I_2 = \int_0^{\pi/2} \sin^{1/2} x \, dx$$

$$= \frac{1}{2} \beta\left(\frac{\frac{1}{2}+1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma(\frac{3}{4}) \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{5}{4})} = \frac{1}{2} \cdot \frac{\Gamma(\frac{1}{2})}{\frac{1}{4}}$$

$$\therefore I_1 \times I_2 = \cancel{\frac{1}{2}} \frac{\Gamma(\cancel{1/4}) \cdot \Gamma(1/2)}{\Gamma(\cancel{3/4})} \cdot \cancel{\frac{1}{2}} \cdot \frac{\Gamma(1/2)}{\cancel{\frac{1}{4}}}$$

$$= \Gamma(1/2) \times \Gamma(1/2)$$

$$= \sqrt{\pi} \times \sqrt{\pi}.$$

$$= \pi. \text{ (Ans)}.$$

d) $$I = \int_0^1 \frac{dx}{(1-x^6)^{1/6}}$$

Let $x^3 = \sin\theta$.

$\Rightarrow 3x^2 dx = \cos\theta \, d\theta$.

$$\Rightarrow dx = \frac{\cos\theta \cdot d\theta}{3 \cdot (\sin\theta)^{2/3}} = \frac{1}{3} \sin^{-2/3}\theta$$

$$\therefore\ I = \int_0^{\pi/2} \frac{\frac{1}{3} \cdot \sin^{-2/3}\theta \cos\theta \, d\theta}{\cos^{1/3}\theta}$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin^{-2/3}\theta \cos^{2/3}\theta \, d\theta \, .$$

$$= \frac{1}{6} \cdot 2\int_0^{\pi/2} \sin^{-2/3}\theta \cos^{2/3}\theta \, d\theta$$

$$= \frac{1}{6} \cdot \beta\left(\frac{1}{6}, \frac{5}{6}\right)$$

$$= \frac{1}{6} \cdot \frac{\Gamma\left(\frac{1}{6}\right) \cdot \Gamma\left(\frac{5}{6}\right)}{\Gamma(1)}$$

$$= \frac{1}{6} \cdot \frac{\pi}{\sin\frac{\pi}{6}} \qquad \left(\because\ \Gamma(n)\cdot\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right)$$

$$= \frac{1}{6} \cdot \frac{\pi}{1/2}$$

$$= \frac{\pi}{3} \, . \quad \text{(Ans)}$$

B TRANSFORMS
&
INVERSE LAPLACE TRANSFORMS

a) $F(t) = (1 + te^{-t})^3$

$\Rightarrow L(F(t)) = L\{1 + t^3 e^{-3t} + 3te^{-t}(1+te^{-t})\}$

$= L\{1 + t^3 e^{-3t} + 3te^{-t} + 3t^2 e^{-2t}\}$.

$= L(1) + L(t^3 e^{-3t}) + 3L(te^{-t}) + 3L(t^2 e^{-2t})$

$= \frac{1}{s} + \frac{6}{(s+3)^4} + 3\frac{1}{(s+1)^2} + 3 \cdot \frac{2}{(s+2)^3}$

$= \frac{6}{(s+3)^4} + \frac{6}{(s+2)^3} + \frac{3}{(s+1)^2} + \frac{1}{s}$. (Ans).

b) $F(t) = (t^2 - 3t + 2)\sin 3t$.

$\therefore L\{F(t)\} = L\{t^2 \sin 3t - 3t\sin 3t + 2\sin 3t\}$.

$= L(t^2 \sin 3t) - 3L(t\sin 3t) + 2L(\sin 3t)$

$= (-1)^2 \frac{d^2}{ds^2} L(\sin 3t) - 3(-1)\frac{d}{ds} L(\sin 3t) + 2 \cdot \frac{3}{s^2+9}$.

$= \frac{d^2}{ds^2} \cdot \left(\frac{3}{s^2+9}\right) + 3 \cdot \frac{d}{ds} \cdot \left(\frac{3}{s^2+9}\right) + \frac{6}{s^2+9}$

$= \frac{d}{ds}\left(\frac{-3(2s)}{[s^2+9]^2}\right) - \frac{18}{(s^2+9)^2} + \frac{6}{(s^2+9)}$

$= \frac{-6s^2 - 9 + 6s(2s)}{(s^2+9)^4} - \frac{18}{(s^2+9)^2} + \frac{6}{s^2+9}$

PTO

$$= \frac{6s^2-9}{(s^2+9)^4} - \frac{18}{(s^2+9)^2} + \frac{6}{s^2+9} \quad \text{(Ans)}.$$

c) $F(t) = e^{-3t}(2\cos 5t - 3\sin 5t)$

$\Rightarrow L\{F(t)\} = L\{e^{-3t}\,2\cos 5t - 3\cdot e^{-3t}\sin 5t\}$

$= 2L\{e^{-3t}\cdot\cos 5t\} - 3L\{e^{-3t}\sin 5t\}.$

$$= 2\cdot\frac{(s+3)}{(s+3)^2+25} - 3\cdot\frac{5}{(s+3)^2+25}$$

$$= \frac{2s-9}{s^2+6s+34}. \quad \text{(Ans)}$$

d) $F(t) = 7^t$

$L\{F(t)\} = L(7^t)$

$= L\{e^{t\ln 7}\} = L\{e^{t\cdot(\ln 7)}\}.$

$$= \frac{1}{s-(\ln 7)} \quad \text{(Ans)}.$$

e) $F(t) = \frac{\sin^2 t}{t}$,

$\rightarrow$ we have $L(\sin^2 t) = L\left[\frac{1-\cos 2t}{2}\right]$

$$= \frac{1}{2}L(1) - \frac{1}{2}L(\cos 2t)$$

$$= \frac{1}{2s} - \frac{1}{2}\cdot\frac{s}{s^2+4}$$

PTO

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:.(147)}

$$\therefore L(\sin^2 t) = \frac{1}{2s} - \frac{s}{2s^2+8} = f(s).$$

$$\therefore L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty f(u)\,du.$$

$$= \int_s^\infty \frac{1}{2u} - \frac{1}{2}\frac{u}{(u^2+4)}\cdot du.$$

$$= \frac{1}{2}\int_s^\infty \frac{du}{u} - \frac{1}{2}\int_s^\infty \frac{u\,du}{(u^2+4)}$$

$$= \frac{1}{4}\ln(s^2+4) - \frac{1}{2}\ln(s) \quad .(\text{Ans}).$$

f). $F(t) = e^{-3t}.\dfrac{\sin 2t}{t}.$

we have, $L(\sin 2t) = \dfrac{2}{s^2+4} = f(s).$

$$\therefore L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty f(u).du.$$

$$= \int_s^\infty \frac{2\,du}{u^2+4}$$

$$= \frac{2}{2}.\left[\tan^{-1}(\infty) - \tan^{-1}\frac{s}{2}\right].$$

$$= \tan^{-1}\frac{2}{s}.$$

$$\therefore L(F(t)) = \tan^{-1}\frac{2}{(s+3)} \quad .(\text{Ans}).$$

PTO

9) $F(t) = \int_0^t e^{u} \cdot \frac{\sin u}{u}\, du.$

let $f(t) = e^{t} \cdot \frac{\sin t}{t}.$

$\therefore L\{f(t)\} = L\{e^{t} \cdot \frac{\sin t}{t}\}.$

$= \tan^{-1} \frac{1}{s-1}.$

$= g(s).$

$\therefore L\left\{\int_0^t f(u)\, du\right\} = \frac{1}{s} \cdot g(s).$

$= \frac{1}{s} \cdot \tan^{-1} \frac{1}{s-1}$ | Ans

2) $L\left\{\frac{\sin at}{t}\right\}.$

We have $L\{\sin at\} = \frac{a}{s^2+a^2} = f(s).$

$\therefore L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty f(u)\, du$

$= \int_s^\infty \frac{a}{u^2+a^2}\, du.$

$= \frac{a}{a} \cdot \tan^{-1} \frac{u}{a} \Big]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1} \frac{s}{a}$

$= \tan^{-1} \frac{a}{s}.$

L(sint ... $\therefore$ L(sin ... put ... 3) I ... let ...

putting $a=1$,

$$L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}.$$

$$\Rightarrow \int_0^\infty \frac{\sin t}{t}\cdot e^{-st}\,dt = \tan^{-1}\frac{1}{s}.$$

Putting $s=0$,

$$\int_0^\infty \frac{\sin t}{t}\,dt = \tan^{-1}(\infty)$$
$$= \frac{\pi}{2}. \quad \text{(Ans).}$$

3) $$I = \int_0^\infty e^{-2t}\cdot t\cos t\,dt.$$

let $F(t) = e^{-2t}\cdot(t\cos t)$.

now, $L(\cos t) = \frac{s}{s^2+1}$

$$\Rightarrow L(t\cos t) = (-1)\frac{d}{ds}\frac{s}{s^2+1}$$
$$= (-1)\frac{(s^2+1)-s^2}{(s^2+1)^2}$$
$$= \frac{-1}{(s^2+1)^2}.$$

$$\Rightarrow L(e^{-2t}\cdot t\cos t) = \cancel{\frac{-1}{(s^2+4s+5)^2}}$$
$$= \frac{-1}{[(s+2)^2+1]^2}$$

PTO

$$\therefore \quad L(F(t)) = \frac{-1}{\{(s+2)^2+1\}^2}$$

$$\Rightarrow \int_0^\infty F(t).\, e^{-st}\, dt = \frac{-1}{\{(s+2)^2+1\}^2}$$

putting $s=0$,

$$\int_0^\infty F(t)\, dt = -\frac{1}{25} \quad \text{(Ans).}$$

4) let $F(t) = e^{-4t}.\, t^2 \sin 2t$.

now, $L(\sin 2t) = \frac{2}{s^2+4}$

$$\therefore \quad L(t^2 \sin 2t) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$$\cancel{= \frac{8s^2-4}{(s^2+4)^2}} = \frac{6s^2-8}{(s^2+4)^3}$$

$$\therefore \quad L(e^{-4t}.\, t^2 \sin 2t) = L\{F(t)\} = \cancel{\frac{8(s+4)^2-4}{[(s+4)^2+4]^2}} \quad \frac{6(s+4)^2-8}{((s+4)^2+4)^3}$$

$$\Rightarrow \int_0^\infty F(t).\, e^{-st}\, dt = \cancel{8(s+4)}\ \frac{6(s+4)^2-8}{[(s+4)^2+4]^3}$$

putting $s=0$,

$$\int_0^\infty e^{-4t}.\, t^2 \sin 2t\, dt = \frac{11}{500} \quad \text{(Proved)}$$

PTO

$$F(t) = \begin{cases} t, & 0 < t < c \\ 2c - t, & c < t < 2\pi \end{cases}$$

$$\therefore L(F(t)) = \frac{1}{1-e^{2\pi s}} \int_0^{2\pi} e^{-st} F(t)\,dt$$

$$= \frac{1}{1-e^{2\pi s}} \left[\int_0^{c} e^{-st} \cdot t\,dt + \int_c^{2\pi} e^{-st}(2c - t)\,dt \right]$$

$$= \frac{1}{1-e^{2\pi s}} \left[\left\{ -\frac{1}{s} e^{-st}\left(t + \frac{1}{s}\right) \right\}_0^{c} + 2c\left(-\frac{1}{s}\right)\left[e^{-st}\right]_c^{2\pi} - \left\{ -\frac{1}{s} e^{-st}\left(t + \frac{1}{s}\right) \right\}_c^{2\pi} \right]$$

$$= \frac{1}{1-e^{2\pi s}} \left[\left\{ -\frac{1}{s} e^{-sc}\left(c + \frac{1}{s}\right) + \frac{1}{s} \right\} - \frac{2c}{s}\left[e^{-2\pi s} - e^{-sc}\right] + \left[\frac{1}{s} e^{-2\pi s}\left(2\pi + \frac{1}{s}\right) - \frac{1}{s} e^{-sc}\left(c + \frac{1}{s}\right) \right] \right]$$

(Ans).

PTO

(6) $F(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$

now, $F(t)$ may be written as:

$$F(t) = 0 + \{(t-1) - 0\} u(t-1) + \{1 - (t-1)\} . u(t-2)$$

or, $F(t) = (t-1) u(t-1) + (2-t) u(t-2)$

$\therefore F(t) = (t-1) u(t-1) - (t-2) u(t-2)$

where $u(t-1) = \begin{cases} 1, & t \geq 1 \\ 0, & t < 1 \end{cases}$

& $u(t-2) = \begin{cases} 1, & t \geq 2 \\ 0, & t < 2 \end{cases}$

$\therefore L(F(t)) = L\{(t-1) u(t-1)\} - L\{(t-2) u(t-2)\}$

$= e^{-s} L(t-1) - e^{-2s} L(t-2)$

$= e^{-s} [1/s^2 - 1/s] - e^{-2s} [1/s^2 - 2/s]$

$= \frac{e^{-s}(1-s)}{s^2} - e^{-2s}\left(\frac{1-2s}{s^2}\right)$

$= \frac{e^{-s} - e^{-2s} - se^{-s} + 2se^{-2s}}{s^2}$ (Ans)

PTO

(1) Let $y = \sin\sqrt{t}$.

$y(0) = 0$

$y'(0) = \lim_{t \to 0} \frac{1}{2} \frac{\cos\sqrt{t}}{\sqrt{t}}$

$= 0$.

$\Rightarrow \frac{dy}{dt} = \frac{\cos\sqrt{t}}{2\sqrt{t}}$.

$$\Rightarrow \frac{d^2y}{dt^2} = \frac{\cancel{2\sqrt{t}}.(-\sin\sqrt{t})\frac{1}{\cancel{2\sqrt{t}}} - \cos\sqrt{t}\,\cancel{2}.\frac{1}{\cancel{2}\sqrt{t}}}{4t}$$

$$= \frac{-\sin\sqrt{t} - \frac{\cos\sqrt{t}}{2\sqrt{t}}}{4t}.$$

$$\Rightarrow 4t\frac{d^2y}{dt^2} + 2.\frac{dy}{dt} + y = 0.$$

Taking L.T on both sides:

$$4L\left(t.\frac{d^2y}{dt^2}\right) + 2L\left(\frac{dy}{dt}\right) + L(y) = 0.$$

$$\Rightarrow 4\left\{-\frac{d}{ds}.[s^2f(s) - sy(0) - y'(0)]\right\} + 2[sf(s) - y(0)] + L(y) = 0.$$

$$\Rightarrow 4\left[-\frac{d}{ds}[s^2L(y)]\right] + 2sL(y) + L(y) = 0.$$

Let $L(y) = P$.

$$\therefore -4\frac{d}{ds}(s^2P) + 2sP + P = 0.$$

$$\cancel{\Rightarrow -4\left(2s\frac{dP}{ds}\right)}$$

PTO

$$\Rightarrow -4\left[2sP + s^2\frac{dP}{ds}\right] + 2sP + P = 0.$$

$$\Rightarrow 4s^2\frac{dP}{ds} - 6sP + P = 0.$$

$$\Rightarrow 4s^2\frac{dP}{ds} = P(6s-1).$$

$$\Rightarrow \int\frac{dP}{P} = \int \frac{6}{4s} - \frac{1}{4s^2}\, ds$$

$$\Rightarrow \ln P = \frac{6}{4}\ln s + \frac{1}{4s} + c.$$

$$\Rightarrow P = e^{\left(\frac{6}{4}\ln s + \frac{1}{4s} + c\right)}$$

$$\therefore P = k s^{6/4} e^{1/4s} \quad \text{(when k=}$$

$$= k s^{3/2} e^{1/4s}$$

$$\Rightarrow L(y) = k s^{3/2} e^{1/4s}$$

or, $$L(\sin\sqrt{t}) = k s^{3/2} e^{1/4s}$$

now, $$\frac{d}{dt}(\sin\sqrt{t}) = \frac{\cos\sqrt{t}}{2\sqrt{t}}.$$

$$\Rightarrow \frac{\cos\sqrt{t}}{\sqrt{t}} = 2\frac{d}{dt}(\sin\sqrt{t})$$

$$\Rightarrow L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = 2L\left(\frac{d}{dt}\sin\sqrt{t}\right).$$

$$= 2(s f(s) - F(0))$$

$$= 2(s \cdot k s^{3/2} e^{1/4s} - 0)$$

$$\therefore L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = 2ks^{5/2}e^{1/4s} \text{ (Ans)}.$$

8) a) let $\frac{4s}{(s-}$

$\Rightarrow 4$

putt

$\therefore L^{-1}\left\{\frac{4s+}{(s-4)}\right.$

let F

G

a)

let $\frac{4s+5}{(s-4)^2(s+3)} = \frac{A}{s-4} + \frac{B}{(s-4)^2} + \frac{C}{(s+3)}$

$\Rightarrow 4s+5 = A(s-4)(s+3) + B(s+3) + C(s-4)^2$

putting $s=4 \Rightarrow 7B = 21$

$\Rightarrow B = 3.$

$s = -3 \Rightarrow 49C = -7.$

$c = -\frac{1}{7}.$

$s = 0 \Rightarrow 5 = -12A + 9 \cdot 3 - \frac{16}{7}.$

$= -12A + \frac{47}{7}.$

$\Rightarrow 12A = \frac{12}{7}.$

$\Rightarrow A = \frac{1}{7}.$

$\therefore L^{-1}\left\{\frac{4s+5}{(s-4)^2(s+3)}\right\} = \frac{1}{7}L^{-1}\left\{\frac{1}{s-4}\right\} + 3L^{-1}\left\{\frac{1}{(s-4)^2}\right\}$

$- \frac{1}{7}L^{-1}\left\{\frac{1}{s+3}\right\}.$

$= \frac{1}{7}e^{4t} + 3te^{4t} - \frac{1}{7}e^{-3t}.$ (Ans)

)

let $F(s) = \frac{s}{s^2-a^2}$

$G(s) = \frac{1}{s^2-a^2}$

PTO

$\therefore L^{-1}\{F(s)\} = \cosh at = f(t)$.

$L^{-1}\{G(s)\} = \frac{1}{a}\sinh at = g(t)$.

$\therefore L^{-1}\{F(s)\cdot G(s)\} = f(t) * g(t)$ (convolution theorem)

$$= \int_0^t f(u)\cdot g(t-u)\,du.$$

$$= \frac{1}{a}\int_0^t \cosh au\,\sinh(at-au)\,du.$$

$$= \frac{1}{a}\int_0^t \frac{e^{au}+e^{-au}}{2}\cdot\frac{e^{at-au}-e^{au-at}}{2}\,du$$

$$= \frac{1}{4a}\int_0^t (e^{au}+e^{-au})\left(e^{at}/e^{au} - e^{au}/e^{at}\right)du$$

$$= \frac{1}{4a}\int_0^t e^{at} - \frac{e^{2au}}{e^{at}} + e^{at}\cdot e^{-2au} - e^{-at}\,du.$$

$$= \frac{1}{4a}\left[te^{at} - \frac{e^{-at}}{2a}\left[e^{2at}-1\right] + \left(\frac{e^{at}}{-2a}\right)\left[e^{-2at}-1\right] - te^{-at}\right].$$

$$= \frac{te^{at}}{4} - \cancel{\frac{e^{at}}{8a^2}} + \cancel{\frac{e^{-at}}{8a^2}} - \cancel{\frac{e^{-at}}{8a^2}} + \cancel{\frac{e^{at}}{8a^2}} \ldots$$

$$= \frac{t}{2}\left(\frac{e^{at}-e^{-at}}{2}\right)$$

$$= \frac{t}{2}\sinh at. \text{ (Ans)}.$$

PTO

1) $\cancel{L^{-1}\left\{\tan^{-1}\frac{2}{s^2}\right\}}$.

2) $\cancel{L^{-1}\left\{\cot^{-1}\frac{s^2}{2}\right\}}$.

$\cancel{= L^{-1}\left\{\frac{\pi}{2} - \tan^{-1}\frac{s^2}{2}\right\}}$.

c) Let $f(s) = \tan^{-1}\frac{2}{s^2}$.

$$\Rightarrow f'(s) = \frac{2}{1+\frac{4}{s^4}}\left(-\frac{2}{s^3}\right) = \frac{2s^4}{s^4+4}\cdot\left(-\frac{2}{\cancel{s^3}}\right)$$

$$\therefore f'(s) = -\frac{4s}{s^4+4}.$$

$$L^{-1}\{f'(s)\} = L^{-1}\left(-\frac{4s}{s^4+4}\right)$$

$$= L^{-1}\left(-\frac{4s}{(s^2+2)^2-(2s)^2}\right)$$

$$= L^{-1}\left\{-\frac{(s^2+2s+2)-(s^2-2s+2)}{(s^2+2s+2)(s^2-2s+2)}\right)$$

$$= L^{-1}\frac{1}{s^2+2s+2} - L^{-1}\frac{1}{s^2-2s+2}$$

$$= L^{-1}\frac{1}{(s+1)^2+1} - L^{-1}\frac{1}{(s-1)^2+1}$$

$$= e^{-t}L^{-1}\frac{1}{s^2+1} - e^{t}L^{-1}\frac{1}{s^2+1}$$

$$= (e^{-t}-e^{t})\sin t.$$

PTO

$$\Rightarrow -t\, L^{-1}\{f'(s)\} = (e^{-t} - e^{t})\sin t .$$

$$\Rightarrow L^{-1}\{f(s)\} = -\frac{(e^{-t} - e^{t})\sin t}{t}$$

$$\therefore L^{-1}\left\{\tan^{-1}\frac{2}{s^2}\right\} = \frac{\sin t}{t}\,(e^{t} - e^{-t}) . \text{ (Ans)}$$

d) $L^{-1}\left\{\dfrac{e^{4-3s}}{(s+4)^{5/2}}\right\}$

$$= e^4 L^{-1}\left(e^{-3s}\cdot(s+4)^{-5/2}\right).$$

now, $L^{-1}\{(s+4)^{-5/2}\} = \frac{4}{3} e^{-4t} t^{3/2}$

$$\therefore L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^4 L^{-1}\left\{e^{-3s}(s+4)^{-5/2}\right\}$$

$$= \frac{4}{3} e^4 \cdot e^{-4(t+3)}\cdot(t+3)^{3/2}$$

$$= \frac{4}{3} e^{-(4t+8)} (t+3)^{3/2} . \text{ (Ans)}$$

e) $L^{-1}\left\{\ln\left(1+\dfrac{a^2}{s^2}\right)\right\}$

$$= L^{-1}\left\{\ln(s^2+a^2) - 2\ln s\right\}.$$

let $f(s) = \ln(s^2+a^2) - 2\ln s .$

$$\Rightarrow f'(s) = \frac{2s}{s^2+a^2} - \frac{2}{s}$$

PTO

$L^{-1}\{f'(s)\} = 2L^{-1}\frac{s}{s^2+a^2} - 2L^{-1}\frac{1}{s}$

$\Rightarrow -t L^{-1}\{f(s)\} = 2\cos at - 2t.$

$\Rightarrow L^{-1}\{f(s)\} = 2 - \frac{2\cos at}{t}$. (Ans).

) Let $f(s) = \ln\left(\frac{s+a}{s+b}\right)$

$= \ln(s+a) - \ln(s+b)$

$\Rightarrow f'(s) = \frac{1}{s+a} - \frac{1}{s+b}$.

$\Rightarrow L^{-1}\{f'(s)\} = L^{-1}\frac{1}{s+a} - L^{-1}\frac{1}{s+b}$

$= e^{-at} - e^{-bt}$

$\Rightarrow -t L^{-1}\{f(s)\} = e^{-at} - e^{-bt}$.

$\Rightarrow L^{-1}\{f(s)\} = \frac{e^{-bt} - e^{-at}}{t}$. (Ans).

~~g) Let $F(s) = \frac{1}{s^2+4}$,~~

~~$G(s) = s$~~

PTO

g) $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$

$= L^{-1}\left\{\frac{s^2+4-4}{(s^2+4)^2}\right\}$

$= L^{-1}\left\{\frac{1}{s^2+4} - \frac{4}{(s^2+4)^2}\right\}$.

g) Let $F(s) = \frac{s}{s^2+4}$

$G(s) = \frac{s}{s^2+4}$.

$\therefore L^{-1}\{F(s)\} = \cos 2t = f(t)$.

$L^{-1}\{G(s)\} = \cos 2t = g(t)$

now, $L^{-1}\{F(s)\cdot G(s)\} = f(t) * g(t)$

$= \int_0^t \cos 2u \cdot \cos(2t-2u)\, du$.

$= \int_0^t \cos 2u \cdot (\cos 2t \cos 2u + \sin 2t \sin 2u)\, du$.

$= \int_0^t (\cos 2t) \cos^2 2u\, du + \int_0^t (\sin 2t) \sin 2u \cos 2u\, du$.

$= \cos 2t \int_0^t \cos^2 2u\, du + \frac{\sin 2t}{2} \int_0^t \sin 4u\, du$.

$= \cos 2t \int_0^t \frac{1+\cos 4u}{2}\, du + \frac{\sin 2t}{2} \int_0^t \sin 4u\, du$.

$= \frac{\cos 2t \sin 4t}{8} + \frac{t \cos 2t}{2} - \frac{\sin 2t \cos 4t}{8} + \frac{\sin 2t}{8}$

$= \frac{\sin 2t}{4} + \frac{t \cos 2t}{2}$. (Ans)

(16)

PTO

$L^{-1}\{\cot^{-1}s\} = L^{-1}\left\{\tan^{-1}\frac{1}{s}\right\}$

$= L^{-1}\left\{\frac{\pi}{2} - \tan^{-1}s\right\}$

$= L^{-1}\left\{\int_s^\infty \frac{du}{u^2+1}\right\}$

$\cancel{\therefore L^{-1}\{\cot^{-1}s\} = \frac{\sin t}{t}}$.

i) let $f(s) = s\ln\frac{s}{\sqrt{s^2+1}} + \cot^{-1}s$.

$\Rightarrow f(s) = s\left[\ln s - \frac{1}{2}\ln(s^2+1)\right] + \cot^{-1}s$.

$\Rightarrow f'(s) = \ln s + 1 - \frac{1}{2}\ln(s^2+1) - \frac{s(2s)}{2(s^2+1)} - \frac{1}{(s^2+1)}$

$\Rightarrow f'(s) = \ln s - \frac{1}{2}\ln(s^2+1)$

$\Rightarrow f''(s) = \frac{1}{s} - \frac{2s}{\cancel{2}(s^2+1)}$

$\Rightarrow f''(s) = \frac{1}{s} - \frac{s}{s^2+1}$

$\Rightarrow L^{-1}\{f''(s)\} = 1 - \cos t$.

$\Rightarrow t^2 L^{-1}\{f(s)\} = 1 - \cos t$.

$\Rightarrow L^{-1}\{f(s)\} = \frac{1-\cos t}{t^2}$. (Ans)

in 2t

8

PTO

i) let $f(s) = \ln \left\{ \left(\frac{s^2-4}{s^3} \right)^{1/3} \right\}$.

$= \frac{1}{3} \ln \left\{ \frac{s^2-4}{s^3} \right\}$.

$= \frac{1}{3} \left[\ln \{ s^2-4 \} - 3 \ln s \right]$.

$\therefore f'(s) = \frac{1}{3} \left[\frac{2s}{s^2-4} - \frac{3}{s} \right]$.

$\therefore L^{-1}\{f'(s)\} = \frac{2}{3} L^{-1} \frac{1}{s^2-4} - 3 L^{-1} \frac{1}{s}$.

$= \frac{2}{3} \times \frac{1}{2} \sinh 2t - \frac{3}{2}$.

$= \frac{\sinh 2t}{3} - \frac{3}{2}$.

$\Rightarrow -t\, L^{-1}\{f(s)\} = \frac{\sinh 2t}{3} - \frac{3}{2}$.

$\Rightarrow L^{-1}\{f(s)\} = \frac{3}{2t} - \frac{\sinh 2t}{3t}$. (Ans)

9) a) Let $F(s) = \frac{1}{s^2+1}$ & $G(s) = \frac{1}{s^2+9}$.

$L^{-1}\{F(s)\} = \sin t = f(t)$.

$L^{-1}\{G(s)\} = \frac{1}{3} \sin 3t = g(t)$.

PTO

$\therefore L^{-1}\{F(s)\cdot G(s)\} = f(t) * g(t)$ (conv. theory)

$$= \int_0^t f(u)\cdot g(t-u)\,du.$$

$$= \frac{1}{3}\int_0^t \sin u \cdot \sin(3t-3u)\,du.$$

$$= \frac{1}{6}\int_0^t 2\sin u \sin(3t-3u)\,du.$$

$$= \frac{1}{6}\int_0^t \cos(4u-3t) - \cos(3t-2u)\,du.$$

$$= \frac{1}{6}\int_0^t \cos(4u-3t)\,du - \frac{1}{6}\int_0^t \cos(3t-2u)\,du.$$

$$= \frac{1}{24}\sin(4u-3t)\Big)_0^t + \frac{1}{12}\sin(3t-2u)\Big)_0^t$$

(Ans)

$$= \frac{1}{24}[\sin t + \sin 3t] + \frac{1}{12}[\sin t - \sin 3t].$$

$$= \frac{\sin t}{4} - \frac{\sin 3t}{24}. \text{ (Ans)}$$

1) b) Let $F(s) = \frac{s}{s^2+9}$ & $G(s) = \frac{1}{s^2+9}$.

$\therefore L^{-1}\{F(s)\} = \cos 3t = f(t).$

$L^{-1}\{G(s)\} = \frac{1}{3}\sin 3t = g(t)$

PTO

$$\therefore L^{-1}\{F(s)\cdot G(s)\} = f(t) * g(t)$$

$$= \frac{1}{3}\int_0^t \cos 3u \sin(3t-3u)\,du.$$

$$= \frac{1}{3}\int_0^t \cos 3u\,(\sin 3t \cos 3u - \cos 3t \sin 3u)\,du.$$

$$= \frac{1}{3}\int_0^t (\sin 3t)\cos^2 3u\,du - \frac{1}{3}\int_0^t (\cos 3t)\sin 3u \cos 3u\,du$$

$$= \frac{\sin 3t}{3}\int_0^t \cos^2 3u\,du - \frac{\cos 3t}{6}\int_0^t \sin 6u\,du.$$

$$= \frac{\sin 3t}{3}\left[\frac{t}{2} + \frac{1}{2}\int_0^t \cos 6u\,du\right] + \frac{\cos 3t(\cos 6t - 1)}{36}$$

$$= \frac{\sin 3t}{3}\left[\frac{t}{2} + \frac{\sin 6t}{12}\right] + \frac{\cos 3t \cos 6t}{36} - \frac{\cos 3t}{36},$$

$$= \frac{t \sin 3t}{6} + \frac{\sin 3t \sin 6t + \cos 3t \cos 6t}{36} - \frac{\cos 3t}{36}$$

$$= \frac{t \sin 3t}{6} + \cancel{\frac{\cos 3t}{36}} - \cancel{\frac{\cos 3t}{36}}$$

$$= \frac{t \sin 3t}{36}. \quad (\text{Ans}).$$

P.T.O

17a) $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = 4e^{2t}$

$\Rightarrow L\left(\dfrac{d^2y}{dt^2}\right) - 3L\left(\dfrac{dy}{dt}\right) + 2L(y) = 4L(e^{2t})$

$\Rightarrow (s^2 L(y) - s y'(0) - y(0)) - 3(\cancel{y(0)}\ sL(y) - y(0))$
$+ 2L(y) = 4 \cdot \dfrac{1}{s-2}$

$\Rightarrow [s^2 L(y) - 5s + 3] - 3[sL(y) + 3] + 2L(y) = \dfrac{4}{s-2}$

$\Rightarrow (s^2 - 3s + 2)L(y) - 5s - 9 = \dfrac{4}{s-2}$

$\Rightarrow (s^2 - 3s + 2)L(y) = \dfrac{4 + (5s+9)(s-2)}{s-2}$

$= \dfrac{4 + 5s^2 - s - 18}{s-2}$

$= \dfrac{5s^2 - s - 14}{s-2}$

$\Rightarrow L(y) = \dfrac{5s^2 - s - 14}{(s-2)(s^2 - 3s + 2)}$

$\Rightarrow y = L^{-1} \dfrac{5s^2 - s - 14}{(s-2)(s^2 - 3s + 2)}$

$= L^{-1} \dfrac{5s^2 - s - 14}{(s-2)^2 (s-1)}$

Let $\dfrac{5s^2 - s - 14}{(s-2)^2 (s-1)} = \dfrac{A}{(s-2)^2} + \dfrac{B}{(s-1)}$

PTO

$\therefore \; 5s^2 - s - 14 = (s-1)A + (s-2)^2 B.$

at $s=1 \Rightarrow -10 = B$

at $s=2 \Rightarrow 4 = A.$

$$\therefore L^{-1}\left\{\frac{5s^2-s-14}{(s-2)^2(s-1)}\right\} = L^{-1}\frac{4}{(s-2)^2} - L^{-1}\frac{10}{s-1}$$

$$= 4L^{-1}\frac{1}{(s-2)^2} - 10L^{-1}\frac{1}{s-1}$$

$$\Rightarrow \quad y = 4.e^{2t}.t - 10e^{t}. \quad \text{(Ans)}.$$

b) $\dfrac{d^2y}{dt^2} + 9y = 1.$

$$\Rightarrow L\left(\frac{d^2y}{dt^2}\right) + 9L(y) = L(1)$$

$$\Rightarrow [s^2L(y) - sy'(0) - y(0)] + 9L(y) = \frac{1}{2}.$$

$$\Rightarrow (s^2+9)L(y) = \frac{1}{2}. \qquad \cancel{(\because y_0 = y'_0 = 0)}$$

$$\Rightarrow L(y) = \frac{1}{2(s^2+9)},$$

$$\Rightarrow y = \frac{1}{2}L^{-1}\frac{3}{s^2+9}$$

$$= \frac{1}{2} \times \frac{1}{3}\sin 3t.$$

$$\Rightarrow y = \frac{1}{2}\sin 3t. \quad \text{(Ans)}.$$

P.T.O.

10) (c) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = t\cos t$

$$\Rightarrow L\left(\frac{d^2y}{dt^2}\right) - 2L\left(\frac{dy}{dt}\right) - 3L(y) = L(t\cos t)$$

$$= (-1)\frac{d}{ds}\cdot\frac{s}{s^2+1}$$

$$= (-1)\frac{(s^2+1) - 2s^2}{(s^2+1)^2}$$

$$= \frac{s^2-1}{(s^2+1)^2}$$

$$\Rightarrow [s^2L(y) - sy'(0) - y(0)] - 2[sL(y) - y(0)] - 3L(y) = \frac{s^2-1}{(s^2+1)^2}$$

$$\Rightarrow (s^2+2s-3)L(y) = \frac{s^2-1}{(s^2+1)^2}$$

$$\Rightarrow L(y) = \frac{s^2-1}{(s^2+1)^2(s^2+2s-3)}$$

$$y = L^{-1}\frac{s^2-1}{(s^2+1)^2(s+1)(s-3)}$$

$$= L^{-1}\left\{\frac{s-1}{(s^2+1)^2(s-3)}\right\}$$

Let $\frac{s-1}{(s^2+1)^2(s-3)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{(s^2+1)^2}$

putting $s = 0, 1, -1, 2, 3$

$A = \frac{1}{50}$, $B = -\frac{1}{50}$, $C = -\frac{3}{50}$, $D = -\frac{1}{5}$, $E = \frac{2}{5}$.

PTO

$$\therefore y = \frac{1}{50} L^{-1}\frac{1}{s-3} - \frac{1}{50} L^{-1}\frac{s}{s^2+1} - \frac{3}{50} L^{-1}\frac{1}{s^2+1}$$

$$- \frac{1}{5} L^{-1}\frac{s}{(s^2+1)^2} + \frac{2}{5} L^{-1}\frac{1}{(s^2+1)^2}$$

$$= \frac{e^{3t}}{50} - \frac{\cos t}{50} - \frac{3\sin t}{50} - \frac{1}{5} L^{-1}\frac{s}{(s^2+1)^2}$$

$$+ \frac{2}{5} L^{-1}\frac{1}{(s^2+1)^2}$$

now, $L^{-1}\frac{1}{s^2+1} = \sin t$.

$\Rightarrow L^{-1}\frac{d}{ds}\frac{1}{s^2+1} = -t\sin t$.

$\Rightarrow L^{-1}\frac{-2s}{(s^2+1)^2} = -t\sin t$.

$\Rightarrow L^{-1}\frac{s}{(s^2+1)^2} = \frac{1}{2} t\sin t$.

$\therefore L^{-1}\frac{1}{(s^2+1)^2} = \int_0^t \frac{1}{2} t\sin t \, dt$

$= \frac{1}{2} (\sin t - t\cos t)$.

$$\therefore y = \frac{e^{3t}}{50} - \frac{\cos t}{50}(1+10t) + \frac{\sin t}{50}(7-5t).$$

(Ans)

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PTO

d) $y''(t) + y(t) = 8\cos t$.

$\Rightarrow L(y''(t)) + L(y(t)) = 8L(\cos t)$

$= \dfrac{8 \cdot s}{s^2+1}$

$\Rightarrow \left[s^2 L(y) - s y'(0) - y(0)\right] + L(y) = \dfrac{8s}{s^2+1}$

$\Rightarrow (s^2+1)L(y) + s - 1 = \dfrac{8}{s^2+1}$.

$\Rightarrow$ ~~$L(y) =$~~ $(s^2+1)L(y) = \dfrac{8}{s^2+1} + 1 - s$

$= \dfrac{8 + s^2 + 1 - s^3 - s}{s^2+1}$

$= \dfrac{-s^3 + s^2 - s + 9}{s^2+1}$

$L(y) = \dfrac{-s^3+s^2-s+9}{(s^2+1)^2}$

$\Rightarrow y = L^{-1} \dfrac{-s^3+s^2-s+9}{(s^2+1)^2}$ $\cancel{= L^{-1} \dfrac{-s(s^2+1) + (s^2+9)}{(s^2+1)^2}}$

$\cancel{= L^{-1}\dfrac{s}{s^2+1} + L^{-1}\dfrac{s^2+9}{(s^2+1)^2}}$

$\cancel{\dfrac{-s^3+s^2-s+9}{(s^2+1)^2} \neq \dfrac{As+B}{s^2+1} + \dfrac{BC}{(s^2+1)^2}}$

$\cancel{(s^2+1)A + B}$

$\therefore$ $\cancel{-s^3+s^2-s+9 =}$ $\cancel{A+B = 9}$

$\cancel{A = 1, B \Rightarrow B = 8}$

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$\therefore$

PTO

$\therefore \cancel{-s^3+s^2-s+9 = (s^2+1)(As+B)+C \cdots}$

$\therefore \cancel{A=-1, \quad B+C=1,}$

or, $y = L^{-1} \dfrac{-s^3+s^2-s+9}{(s^2+1)^2}$

$= L^{-1} \dfrac{(s^2+9) - s(s^2+1)}{(s^2+1)^2} = L^{-1} \dfrac{s^2+9}{(s^2+1)^2} - L^{-1} \dfrac{s}{s^2+1}$

$= L^{-1} \dfrac{1}{s^2+1} + 8L^{-1} \dfrac{1}{(s^2+1)^2} - L^{-1} \dfrac{s}{s^2+1}$

$\therefore y = \sin t + 8L^{-1} \dfrac{1}{(s^2+1)^2} - \cos t$

$= (\sin t - \cos t) + 4(\sin t - t\cos t)$

$\therefore y = 5\sin t - \cos t\,(4t+1)$

(Ans)