2.2 METHOD OF VARIATION OF PARAMETERS

2.2.1. Method of variation of parameters to find P. I.

In this chapter we shall discuss the alternative method of finding the particular integral (P.I.) of the linear non-homogeneous equation whose complementery funcation (C.F.) is known. In this method, the P.I. is obtained by replacing the arbitrary constants of the C.F. with functions of x.and so the method is known as variation of parameters.

Consider the linear equation of second order with constant coefficients

$$\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y=X. \qquad \ldots \quad (1)$$

Let its C. F. be $y=c_1y_1+c_2y_2$ where y_1 and y_2 are L. I. solutions of the equation $\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y=0.$

$$\therefore \quad \frac{d^2y_1}{dx^2}+P_1\frac{dy_1}{dx}+P_2y_1=0$$

$$\frac{d^2y_2}{dx^2}+P_1\frac{dy_2}{dx}+P_2y_2=0. \qquad \ldots \quad (2)$$

Now, let us assume that P. I. of (1) be

$$y_p=uy_1+vy_2 \qquad \ldots \quad (3)$$

where u and v are unknown functions of x.

Differentiating (3) w.r.t. x, we have

$$y_p'=uy_1'+vy_2'+u'y_1+v'y_2$$

We shall choose u, v in such a manner that

$$u'y_1+v'y_2=0. \qquad \ldots \quad (4)$$

Then $y_p'=uy_1'+vy_2'$. $\qquad \ldots \quad (5)$

$$\therefore \quad y_p''=uy_1''+vy_2''+u'y_1'+v'y_2'. \qquad \ldots \quad (6)$$

Substituting the values of y_p, y'_p, y''_p in (1) and rearranging we get,

$$u(y''_1 + P_1y'_1 + P_2y_1) + v(y''_2 + P_1y'_2 + P_2y_2) + u'y'_1 + v'y'_2 = X.$$

or, $u'y'_1 + v'y'_2 = X$, by (2) ... (7)

Solving (4) & (7), we get

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ X & y'_2 \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}} = \frac{-y_2X}{W} \text{ and } v' = \frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & X \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}} = \frac{y_1X}{W}$$

where $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$ is called the Wronskian of y_1, y_2.

Integrating we get, $u = -\int \frac{y_2X}{W}dx$ ... (8)

$$v = \int \frac{y_1X}{W}dx \quad \text{... (9)}$$

Putting these values of u and v in (3) we obtain a particular integral of (1), namely $y_p = uy_1 + vy_2$.

Illustrative Examples.

Ex. 1. *Apply the method of variation of parameters to solve* $\frac{d^2y}{dx^2} + 4y = \sin 2x$.

[W.B.U.T 2011]

Let $y = e^{mx}$ be a trial solution of $\frac{d^2y}{dx^2} + 4y = 0$.

Then the auxiliary equation is $m^2 + 4 = 0$.

$\therefore \quad m = \pm 2i$.

$\therefore \quad C.F. = c_1 \cos 2x + c_2 \sin 2x$.

Let the particular solution be

$$y_p = u(x)\cos 2x + v(x)\sin 2x.$$

Then $Dy_p = 2(-u\sin 2x + v\cos 2x) + (u'\cos 2x + v'\sin 2x)$

Choose u, v such that

$$u'\cos 2x + v'\sin 2x = 0 \quad \text{... (i)}$$

so that $Dy_p = 2(-u\sin 2x + v\cos 2x)$.

$\therefore \quad D^2y_p = 2(-u'\sin 2x + v'\cos 2x) - 4(u\cos 2x + v\sin 2x)$.

Substituting the values of D^2y_p, y_p in the given equation

$$\frac{d^2y}{dx^2} + 4y = \sin 2x,$$

we get $2(-u'\sin 2x + v'\cos 2x) = \sin 2x$

$$\therefore \quad -u'\sin 2x + v'\cos 2x = \frac{1}{2}\sin 2x. \quad \text{... (ii)}$$

Solving (i) and (ii), we get $u' = -\frac{1}{2}\sin^2 2x = -\frac{1}{4}(1 - \cos 4x)$

$$v' = \frac{1}{2}\sin 2x \cos 2x = \frac{1}{4}\sin 4x.$$

Integrating we get,

$$\therefore \quad u = -\frac{1}{4}(x - \frac{\sin 4x}{4}) \text{ and } v = -\frac{\cos 4x}{16}$$

$$y_p = -\frac{1}{4}x\cos 2x + \frac{1}{16}(\cos 2x \sin 4x - \sin 2x \cos 4x)$$

$$= -\frac{1}{4}x\cos 2x + \frac{1}{16}\sin 2x.$$

So the general solution is

$$y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{16}\sin 2x$$

$$= c_1\cos 2x + c_3\sin 2x - \frac{1}{4}x\cos 2x \text{ where } c_3 = c_2 + \frac{1}{16}$$

where c_1, c_3 are arbitrary constants.

Alternatively. Let the particular solution be

$$y_p = u(x)\cos 2x + v(x)\sin 2x.$$

Then $u(x) = -\int \frac{y_2 X}{W} dx$ by (8).

$v(x) = \int \frac{y_1 X}{W} dx$ by (9)

Hear $y_1 = \cos 2x, y_2 = \sin 2x, X = \sin 2x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2$$

$$\therefore \quad u = -\int \frac{\sin 2x \times \sin 2x}{2} dx = -\frac{1}{4}\int (1 - \cos 4x)\, dx$$

$$= -\frac{1}{4}\left(x - \frac{\sin 4x}{4}\right)$$

$$\therefore \quad v = \int \frac{\cos 2x . \sin 2x}{2} dx = \frac{1}{4}\int \sin 4x dx = -\frac{1}{16}\cos 4x.$$

$$\therefore \quad y_p = -\frac{1}{4} x \cos 2x + \frac{1}{16}(\cos 2x \sin 4x - \sin 2x \cos 4x)$$

$$= -\frac{1}{4} x \cos 2x + \frac{1}{16}\sin 2x.$$

Ex. 2. *Solve by the method of variation of parameters the equation*

$$\frac{d^2y}{dx^2} + y = \sec^3 x \tan x$$

[W.B.U.T 2007, 2012]

Let $y = e^{mx}$ be a trial solution of

$$\frac{d^2y}{dx^2} + y = 0$$

$\therefore$ The auxiliary equation is $m^2 + 1 = 0$

$\therefore$ $m = \pm i$

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$

Let the particular solution be

$$y_p = u(x)\cos x + v(x)\sin x$$

Then $u(x) = -\int \frac{y_2 X}{W} dx$.

$$v(x) = \int \frac{y_1 X}{W} dx$$

Here $y_1 = \cos x . y_2 = \sin x, x = \sec^3 x \tan x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x - 1\sin^2 x = 1$$

$$\therefore \quad u(x) = -\int \frac{\sin x \sec^3 x \tan x}{1} dx$$

$$= -\int \tan^2 x \sec^2 x \, dx$$

$$= -\int z^2 dz \quad [\text{ Putting } \tan x = z\ ]$$

$$= -\frac{z^3}{3}$$

$$= -\frac{\tan^3 x}{3}$$

$$v(x) = \int \frac{\cos x \sec^3 x \tan x}{1} dx$$

$$= \int \tan x \sec^2 x \, dx$$

$$= \int z dz \quad [\text{putting } \tan x = z]$$

$= \frac{z^2}{2}$

$= \frac{\tan^2 x}{2}$

$\therefore \quad y_p = -\frac{\tan^3 x}{3} \cdot \cos x + \frac{\tan^2 x}{2} \cdot \sin x$

$= -\frac{1}{3}\tan^2 x \cdot \sin x + \frac{1}{2}\tan^2 x \sin x$

$= -\frac{1}{6}\tan^2 x \sin x$

Thus the general solution is

$$y = c_1 \cos x + c_2 \sin x + \tfrac{1}{6}\tan^2 x \sin x$$

Ex. 3. *Solve by the method of variation of parameters*

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax. \quad (a \neq 0)$$

[W.B.U.T 2010]

Let $y = e^{mx}$ be a trial solution of $\frac{d^2y}{dx^2} + a^2 y = 0$.

Then the auxiliary equation is $m^2 + a^2 = 0$.

$\therefore \quad m = \pm ai.$

$\therefore \quad$ C. F. $= c_1 \cos ax + c_2 \sin ax$.

Let the particular solution be

$$y_p = u(x)\cos ax + v(x)\sin ax.$$

Then $Dy_p = a\,(-u\sin ax + v\cos ax) + (u'\cos ax + v'\sin ax)$

Choose u, v such that

$$u'\cos ax + v'\sin ax = 0.$$

so that $Dy_p = a\,(-u\sin ax + v\cos ax)$. $\quad \ldots$ (i)

$\therefore \; D^2 y_p = a\,(-u'\sin ax + v'\cos ax) - a^2(u\cos ax + v\sin ax)$.

Substituting the values of $D^2 y_p, y_p$ in the given equation

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax,$$

we get $a\,(-u'\sin ax + v'\cos ax) = \sec ax$.

$\therefore \quad -u'\sin ax + v'\cos ax = \frac{1}{a}\sec ax. \quad \ldots$ (ii)

Solving (i) and (ii), we get $u' = -\frac{1}{a}\tan ax, v' = \frac{1}{a}$.

Integrating, we get $u = \frac{1}{a^2}\log\cos ax, v = \frac{x}{a}$.

Alternatively

Let the particular solution be

$$y = u(x)\cos ax + v(x)\sin ax$$

where $u(x) = -\int \frac{y_2 X}{W} dx$

$$v(x) = \int \frac{y_1 X}{W} dx$$

Here $y_1 = \cos ax, y_2 = \sin ax, X = \sec ax$

$\therefore \quad W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$

$= a(\cos^2 ax + \sin^2 ax)$

$= a$

$\therefore \quad u(x) = -\int \frac{\sin ax \sec ax}{a} dx$

$= -\frac{1}{a}\int \tan ax\, dx$

$= -\frac{1}{a^2}\log\sec ax$

$= \frac{1}{a^2}\log\cos ax$

and,

$$v(x)=\int\frac{\cos ax\sec ax}{a}dx$$

$$=\frac{1}{a}\int dx$$

$$=\frac{x}{a}$$

$\therefore\quad y_p=\frac{1}{a^2}\cos ax\log\cos ax+\frac{x}{a}\sin ax.$

So the general solution is

$$y=c_1\cos ax+c_2\sin ax+\frac{x}{a}\sin ax+\frac{1}{a^2}\cos ax\log\cos ax,$$

where c_1, c_2 are arbitrary constants.

Ex. 4. *Solve by the method of variation of parameters, the equation*

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\frac{e^x}{1+e^x}.$$

Let $y=e^{mx}$ be a trial solution of $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0.$

Then the auxiliary equation is $m^2-3m+2=0$

or, $(m-2)(m-1)=0.$

$\therefore\quad m=1,2.$

$\therefore\quad$ C. F. $=c_1e^x+c_2e^{2x}.$

Let the particular solution be

$$y_p=u(x)e^x+v(x)e^{2x}.$$

Then $Dy_p=ue^x+2ve^{2x}+(u'e^x+v'e^{2x})$ $\quad\cdots\quad$ (i)

Choose u, v such that

$$u'e^x+v'e^{2x}=0.$$

so that $Dy_p=ue^x+2ve^{2x}.$ $\quad\cdots\quad$ (ii)

$\therefore\quad D^2y_p=ue^x+4ve^{2x}+u'e^x+2v'e^{2x}.$

Substituting the values of D^2y_p, Dy_p, y_p in

$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\frac{e^x}{1+e^x}$, we get

$$ue^x+4ve^{2x}+u'e^x+2v'e^{2x}-3ue^x-6ve^{2x}+2ue^x+2ve^{2x}=\frac{e^x}{1+e^x}.$$

$\therefore\quad u'e^x+2v'e^{2x}=\frac{e^x}{1+e^x}.$ $\quad\cdots\quad$ (iii)

Solving (ii) and (iii), we get $u'=-\frac{1}{1+e^x}$, $v'=\frac{1}{e^x(1+e^x)}.$

Integrating, we get $u=\log(1+e^{-x})$ and $v=-e^{-x}+\log(1+e^{-x}).$

Alternatively :

Let the particular solution be

$$y_p=u(x)e^x+v(x)e^{2x}$$

where $u(x)=-\int\frac{y_2X}{W}dx$

$$v(x)=\int\frac{y_1X}{W}dx$$

Here, $y_1=e^x$, $y_2=e^{2x}$, $X=\frac{e^x}{1+e^x}$

$$\therefore\quad W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$$

$$=\begin{vmatrix}e^x & e^{2x}\\ e^x & 2e^{2x}\end{vmatrix}$$

$$=e^{3x}$$

$\therefore\quad u(x)=-\int\frac{y_2X}{W}dx$

$$=-\int\frac{e^{2x}\frac{e^x}{1+e^x}}{e^{3x}}dx$$

$$= -\int \frac{dx}{1+e^{x}}$$

$$= -\int \frac{e^{-x}}{e^{-x}+1}\,dx$$

$$= \int \frac{dz}{z} \quad \left[\text{putting } e^{-x}+1 = z\right]$$

$$= \log z$$

$$= \log\left(e^{-x}+1\right)$$

and,

$$v(x) = \int \frac{e^{x}\dfrac{e^{x}}{1+e^{x}}}{e^{3x}}\,dx$$

$$= \int \frac{1}{e^{x}\left(1+e^{x}\right)}\,dx$$

$$= \int\left(\frac{1}{e^{x}} - \frac{1}{1+e^{x}}\right)dx$$

$$= \int e^{-x}dx - \int \frac{e^{-x}}{e^{-x}+1}\,dx$$

$$= -e^{-x} + \log\left(1+e^{-x}\right)$$

$\therefore$ from (i), $y_p = e^{x}\log(1+e^{-x}) - e^{x} + e^{2x}\log(1+e^{-x})$

$$= (e^{x}+e^{2x})\log(1+e^{-x}) - e^{x}.$$

So the general solution is

$$y = c_1e^{x} + c_2e^{2x} + (e^{x}+e^{2x})\log(1+e^{-x}) - e^{x}$$

$$= D_1e^{x} + D_2e^{2x} + (e^{x}+e^{2x})\log(1+e^{-x}) - e^{x}$$

where $D_1 = c_1 - 1$, $D_2 = c_2$ are arbitrary constants.

Ex. 5. *Apply the method of variation of parameters to solve*

$$\frac{d^2y}{dx^2} + 4y = 4\sec^2 2x. \quad \text{[W.B.U.Tech.2006]}$$

Let $y = e^{mx}$ be a trial solution of $\dfrac{d^2y}{dx^2} + 4y = 0$.

Then the auxiliary equation is $m^2 + 4 = 0$.

$\therefore \quad m = \pm 2i.$

$\therefore \quad$ C. F. $= c_1\cos 2x + c_2\sin 2x.$

Let the particular solution be

$$y_p = u(x)\cos 2x + v(x)\sin 2x.$$

Then $Dy_p = 2(-u\sin 2x + v\cos 2x) + (u'\cos 2x + v'\sin 2x)$

Choose u, v such that

$$u'\cos 2x + v'\sin 2x = 0 \quad \ldots \quad \text{(i)}$$

so that $Dy_p = 2(-u\sin 2x + v\cos 2x)$.

$\therefore \quad D^2y_p = 2(-u'\sin 2x + v'\cos 2x) - 4(u\cos 2x + v\sin 2x).$

Substituting the values of D^2y_p, y_p in the given equation

$$\frac{d^2y}{dx^2} + 4y = 4\sec^2 2x,$$

we get $\quad 2(-u'\sin 2x + v'\cos 2x) = 4\sec^2 2x.$

$$\therefore \quad -u'\sin 2x + v'\cos 2x = 2\sec^2 2x. \quad \ldots \quad \text{(ii)}$$

Solving (i) and (ii), we get $u' = -2\tan 2x\sec 2x$, $v' = 2\sec 2x$.

Integrating, we get $u = -\sec 2x$, $v = \log(\sec 2x + \tan 2x)$.

Alternatively :

Let the particular solution be

$$y_p = u(x)\cos 2x + v(x)\sin 2x$$

where $u(x) = -\int \dfrac{y_2X}{W}\,dx$

and $v(x) = -\int \dfrac{y_1X}{W}\,dx$

Here $y_1 = \cos 2x, y_2 = \sin 2x$ and $X = 4\sec^2 2x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2$$

$$\therefore \quad u(x) = -\int \frac{\sin 2x.4\sec^2 2x}{2} dx$$

$$= -2\int \tan 2x \sec 2x \, dx$$

$$= -2\frac{\sec 2x}{2}$$

$$= -\sec 2x$$

and,

$$v(x) = \int \frac{\cos 2x.4\sec^2 2x}{2} dx$$

$$= -2\int \sec 2x dx$$

$$= \log(\sec 2x + \tan 2x)$$

$$\therefore \quad y_p = (-\sec 2x)\cos 2x + \log(\sec 2x + \tan 2x) . \sin 2x$$

$$= -1 + \sin 2x \log(\sec 2x + \tan 2x).$$

So the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x),$$

where c_1, c_2 are arbitrary constants.

Ex. 6. ***Apply the*** *variation of parameters to solve*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 9e^x \qquad \text{[W.B.U.T. 2004, 2016]}$$

Let $y = e^{mx}$ be a trial solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

$\therefore$ The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

or, $(m-1)(m-2) = 0$

$\therefore$ $m = 1, 2$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$

Let $y_p = u(x)e^x + v(x)e^{2x}$ be the particular solution. Then

$$u(x) = -\int \frac{y_2 X}{W} dx$$

$$v(x) = -\int \frac{y_1 X}{W} dx$$

Here $y_1 = e^x$, $y_2 = e^{2x}$, $X = 9e^x$

$$\therefore \quad W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$$

$$\therefore \quad u = -\int \frac{e^{2x} 9e^x}{e^{3x}}$$

$$= -9\int dx$$

$$= -9x$$

$$v = \int \frac{e^x.9e^x}{e^{3x}} dx$$

$$= 9\int e^{-x} dx$$

$$= -9e^{-x}$$

$$\therefore \quad y_p = -9xe^x - 9e^{-x}.e^{2x}$$

$$= -9e^x(x+1)$$

Thus the required general solution is

$$y = c_1 e^x + c_2 e^{2x} - 9e^x(x+1)$$