Thus total energy = K.E. + P.E.

$$= \frac{1}{2} m \dot{y}^2 + \frac{1}{2} s y^2 \qquad (9)$$

$$with, y = aSin(wt + \phi)$$

$$\& \; v = \dot{y} = a\omega Cos(wt + \phi)$$

$$So, kinetic energy = \frac{1}{2} m \dot{y}^2 = \frac{1}{2} m a^2 \omega^2 Cos^2 (wt + \phi)$$

$$\text{and P.E.} = \frac{1}{2} s y^2 = \frac{1}{2} m \omega^2 y^2 = \frac{1}{2} m \omega^2 a^2 Sin^2 (\omega t + \phi)$$

Thus total energy = K.E. + P.E.

$$= \frac{1}{2} m \dot{y}^2 + \frac{1}{2} s y^2$$

$$= \frac{1}{2} m \omega^2 a^2 Cos^2 (\omega t + \phi) + \frac{1}{2} m \omega^2 a^2 Sin^2 (\omega t + \phi)$$

$$= \frac{1}{2} m \omega^2 a^2 \qquad (10)$$

Thus the total energy of the harmonic oscillator is a constant and proportional to the square of the amplitude. It is also equal to the maximum K.E. as well as P.E.

Superposition of two linear S.H.Ms(with same frequency):

We often come across physical situations in which a system is subjected simultaneously to two or more simple harmonic oscillations. For e.g. the diaphram of a microphone or our ear membrane may be subjected simultaneously to two or more vibrations. In such cases the resultant motion can be obtained by using the principle of superposition, according to which the resultant displacement is given by the algebraic sum of the displacements caused by the individual sources.

(a) SHMs of same frequency acting along the same direction but having different amplitudes and phases:

Let two SHMs be represented by

$$x_1 = a_1 Sin(\omega t + \phi_1) and \qquad (11i)$$

$$x_2 = a_2 Sin(\omega t + \phi_2) \qquad (11ii)$$

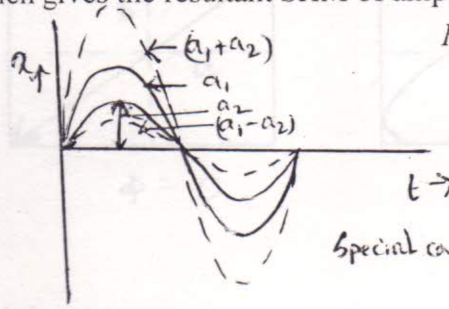
where a_1 and a_2 are the amplitudes , ϕ_1 and ϕ_2 are the initial phase angles of the two two SHMs of same angular frequency.

By the superposition principle the resultant displacement is given by

$$x = x_1 + x_2$$

$$= a_1 \sin(\omega t + \phi_1) + a_2 \sin(\omega t + \phi_2)$$
$$= \sin \omega t (a_1 \cos\phi_1 + a_2 \cos\phi_2) + \cos\omega t (a_1 \sin\phi_1 + a_2 \sin\phi_2)$$
$$= R \sin\omega t \cos\theta + R\cos\omega t \sin\theta$$
$$taken, R\cos\theta = (a_1 \cos\phi_1 + a_2\cos\phi_2)$$
$$R\sin\theta = (a_1\sin\phi_1 + a_2\sin\phi_2)$$
$$So, x = R\sin(\omega t + \theta) \qquad (12)$$

which gives the resultant SHM of amplitude R and initial phase θ where,



$$R^2 = R^2\cos^2\theta + R^2\sin^2\theta = a_1^{\,2} + a_2^{\,2} + a_1 a_2(\sin\phi_1 \sin\phi_1 + \cos\phi_1\cos\phi_2)$$
$$= a_1^{\,2} + a_2^{\,2} + 2a_1 a_2 \cos(\phi_2 - \phi_1) \qquad (13)$$
$$and \tan\theta = \frac{R\sin\theta}{R\cos\theta} = \frac{a_1\sin\phi_1 + a_2\sin\phi_2}{a_1\cos\phi_1 + a_1\cos\phi_1} \qquad (14)$$

Special cases: (i) If $\phi_2 - \phi_1 = 0, 2\pi, 4\pi \ldots$; $R = a_1 + a_2$

(ii) If $\phi_2 - \phi_1 = \pi, 3\pi, 4\pi \ldots$; $R = a_1 - a_2$

(b) Superposition of two SHMs acting at right angles to each other: Lissajous' figures:

When a particle is acted upon simultaneously by two SHMs at right angles to each other, the resultant path traced out by the particle is called Lissajous figures.The nature of the resultant path depends upon:

(i) the amplitude
(ii) the frequencies and
(iii) the phase difference between the two component vibrations.

Oscillations having same frequency:
Let two SHMs of same frequencies acting at right angles to each other be represented by the equations—

$$x = a\sin(\omega t + \phi_1)$$
$$y = b\sin(\omega t + \phi_2) \qquad (15)$$

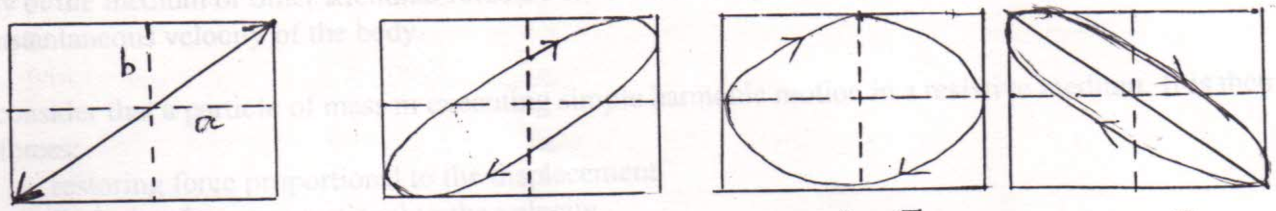
where a is the amplitude of the vibration along x-axis , b is the amplitude of the vibration along the Y-axis. Resulting motion is obtained by eliminating t from ( ) by using ( ).
From ( ) ,

$$\frac{y}{b} = \sin(wt + \phi_1 + \phi_2 - \phi_1)$$
$$= \sin(wt + \phi_1 + \phi) where \phi = (\phi_2 - \phi_1) the phase diff. bet^n the two SHMs$$
$$s = \sin(\omega t + \phi_1)\cos\phi + \cos(\omega t + \phi_1)\sin\phi$$
$$= \frac{x}{a}\cos\phi + \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \sin\phi$$
$$\Rightarrow \left(\frac{x}{a}\cos\phi - \frac{y}{b}\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2\phi$$

$$\Rightarrow \left(\frac{x^2}{a^2}\cos^2\phi + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi\right) = \left(1-\frac{x^2}{a^2}\right)\sin^2\phi$$

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\phi\right)=\sin^2\phi \qquad (1k)$$

It represents the general equation of an ellipse bounded within a rectangle of sides 2a and 2b .



$\phi = 0$ $\qquad \phi = \pi/4 \qquad \phi = \pi/2 \qquad \phi = \frac{3\pi}{4}$

ecial cases:

Case I

$\phi_1 = \phi_2 \, i.e. \, \phi = \phi_2 - \phi_1 = 0,$

If $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}=0$

$\Rightarrow \left(\frac{x}{a}-\frac{y}{b}\right)=0 \Rightarrow y=\frac{b}{a}x$

Is a pair of coincident line with inclination $\theta = \tan^{-1}(b/a)$ with the x-axis.

CaseII

$\phi=\pi, (22) \Rightarrow \left(\frac{x}{a}+\frac{y}{b}\right)^2=0$

$\Rightarrow y=-\frac{b}{a}x$

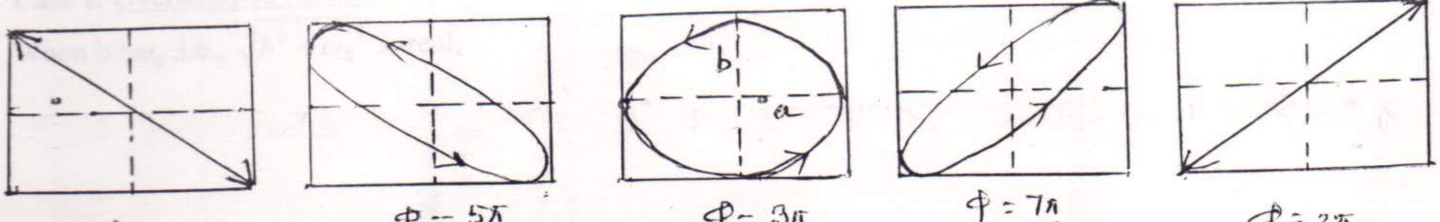
again a pair of coincident st. line but with inclination $\tan^{-1}\left(-b/a\right)$

CaseIII

$\phi = \frac{\pi}{2}$

$equ^n (22) \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

epresents an ellipse with semi axes a & b along the co-ordinate axis. If in addition a=b , then $x^2+y^2=a^2$ epresents a circle.



$\phi = \pi \qquad \phi = \frac{5\pi}{4} \qquad \phi = \frac{3\pi}{2} \qquad \phi = \frac{7\pi}{4} \qquad \phi = 2\pi$

mped vibration

So far we have discussed that the total energy of the harmonic oscillator remains conserved . The oscillator continues forever with a constant amplitude and constant frequency that is determined by the inertia and elastic properties of the system called natural frequency. Such a simple harmonic vibration is only an ideal situation and called free vibration. However, in real practice the energy of the oscillator gradually decreases .The amplitude thereby decreases with time and the oscillator eventually come to rest due to some damping force acting on the system either in the form of viscosity of the medium or other frictional forces. For small velocity we may take the damping force as proportional to the instantaneous velocity of the body.

Let us consider that a particle of mass m executing simple harmonic motion in a resistive medium. It is then subjected to two forces:

(i) restoring force proportional to the displacement.

(ii) Damping force proportional to the velocity

both are directed opposite to the direction of displacement.

The equation of motion of such a particle is,

$$m\frac{d^2y}{dt^2}=-sy-k\frac{dy}{dt}$$ where s is the restoring force per unit displacement and k is the damping force per unit velocity.

$$\Rightarrow \frac{d^2y}{dt^2}+\frac{k}{m}\frac{dy}{dt}+\frac{s}{m}y=0$$

$$\Rightarrow \frac{d^2y}{dt^2}+2b\frac{dy}{dt}+\omega_0^2 y=0 \qquad (17)$$

$$where, \frac{k}{m}=2b; \frac{s}{m}=\omega_0^2$$

Equation (17) is the differential equation for damped simple harmonic vibration.

Here, damping is characterized by the factor b ,which has the dimension of frequency and ω_0 represents the natural frequency of the oscillator.

Displacement equation:

Let, $y=Ae^{\alpha t}$ be the trial solution of the equation ~~(2)~~. (17)

$$\left(\alpha^2+2b\alpha+\omega_0^2\right)Ae^{\alpha t}=0$$

Thus (7) implies, $\Rightarrow \left(\alpha^2+2b\alpha+\omega_0^2\right)=0 \, as \, e^{\alpha t}\neq 0 \,\&\, A\neq 0$

$$\Rightarrow \alpha=-b\pm\sqrt{b^2-\omega_0^2}$$

Hence, the general solution of equation (2) can be written as

$$y=A_1e^{\left(-b+\sqrt{b^2-\omega_0^2}\right)t}+A_2e^{\left(-b-\sqrt{b^2-\omega_0^2}\right)t} \qquad (18)$$

A_1 and A_2 to be determined from boundary conditions .The nature of motion depends upon the relative values of b and ω_0 i.e., whether $\sqrt{b^2-\omega_0^2}$ is positive, zero or negative.

Case I: Overdamped motion

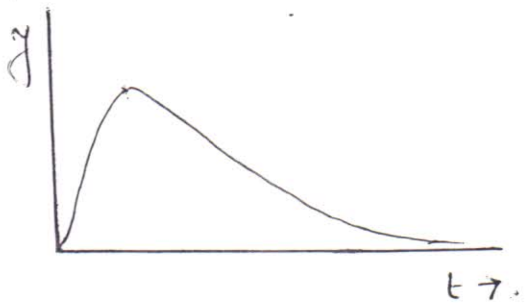
When $b>\omega_0$,i.e., $\sqrt{b^2-\omega_0^2}$ is real, Let, $\sqrt{b^2-\omega_0{}^2} = \text{p}$

$y = e^{-bt}\left(A_1 e^{pt} + A_2 e^{-pt}\right)$

If at t = 0, y = o then ( ) implies $A_1 = -A_2$

$= e^{-bt}\left(A_1 \dfrac{e^{pt} - e^{-pt}}{2}\right)$

$= e^{-bt}\left[A_1 Sinhpt\right]$ (19)



If at t = 0, $\dot{y}$ = o then , $A_1 = \dfrac{v_0}{\sqrt{b^2-\omega_0{}^2}}$,hence

$$y = \frac{v_0}{\sqrt{b^2-\omega_0{}^2}} e^{-bt}\, Sinhpt$$

The motion is non-oscillatory, aperiodic or dead beat type.

he motion is illustrated in fig.( ) . For small t . the term e^{-bt} is very nearly equal to unity. Displacement increases with time as Sinh ($\sqrt{b^2-\omega_0{}^2}\,t$) increases with time .

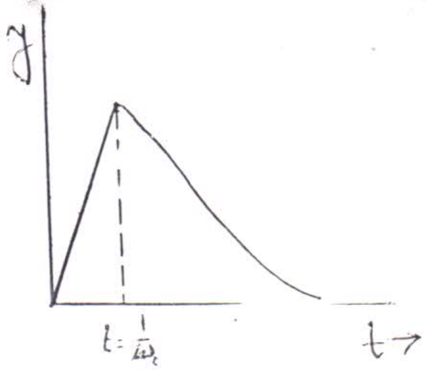
The displacement reaches a maximum value and thereafter the term e^{-bt} dominates and the displacement lecays exponentially with time. This type of motion is found in a dead beat galvanometer or a pendulum mmersed in a highly viscous liquid. In dead beat the damping is large and the pointer reaches slowly in the desired position.(used to measure current)

Case II

Critical damping : $b = \omega_0$ infact $b \to \omega_0$

It is a limiting case of heavy damping when the motion changes from non-oscillatory to oscillatory nature. Putting $b = \omega_0$ in equation ( ) we get,

$$y = (C_1 + C_2 t)e^{-bt}$$
$$= (C_1 + C_2 t)e^{-\omega_0 t}$$



If at t = 0, y = 0 and $\dfrac{dy}{dt} = v_0$

$C_1 = 0, hence, y = C_2 t e^{-\omega_0 t}$

$\& \dfrac{dy}{dt} = C_2 e^{-\omega_0 t} + C_2 t(-\omega_0)e^{-\omega_0 t}$

$\Rightarrow v_0 = C_2 ; thus, y = v_0 t e^{-\omega_0 t}$ (20)

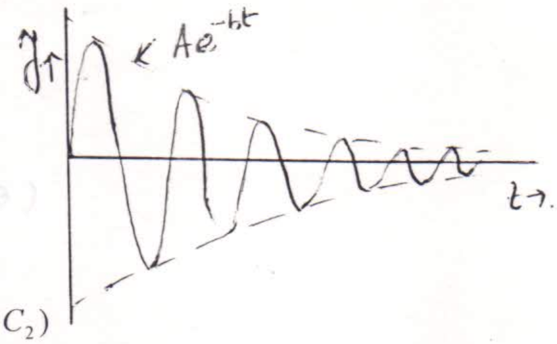
At first for low value of t, the displacement increases almost linearly with time and then becomes maximum at $t = \dfrac{1}{\omega_0}$ when dy/dt = 0 Thereafter the displacement decays exponentially and ultimately becomes zero. The motion is here aperiodic ,critically damped. The decrease in amplitude here is faster than in the case when b> ω_0 .

inter ammeter and galvanometer are the examples of critically damped motion where the pointer reaches quickly to the desired position.

case III : Small damping $b<\omega_0$, i.e., $\sqrt{b^2-\omega_0^2}$ is an imaginary quantity.

Let $\sqrt{b^2-\omega_0^2} = j\omega$

Solution of equation ( ) is



$$y = e^{-bt}\left(C_1 e^{j\omega t} + C_2 e^{-j\omega t}\right)$$
$$= e^{-bt}\left[C_1(Cos\omega t + jSin\omega t) + C_2(Cos\omega t - jSin\omega t)\right]$$
$$= e^{-bt}\left[(C_1+C_2)Cos\omega t + j(C_1-C_2)Sin\omega t\right]$$
$$= e^{-bt}\left[(B_1 Cos\omega t + B_2 Sin\omega t\right] where, B_1 = (C_1+C_2)\ \&\ B_2 = j(C_1-C_2)$$
$$putting B_1 = ACos\phi and B_2 = ASin\phi$$
$$y = Ae^{-bt}Cos(\omega t - \phi) \qquad (21)$$

(3) represents a damped oscillatory motion with angular frequency ω and initial phase ϕ and amplitude Ae^{-bt}, diminishes exponentially with time.

At $t=0$, $y=0$ & $\dfrac{dy}{dt} = v_0$

$A\,Cos\phi = 0$, since $A\neq 0, Cos\phi = 0 \Rightarrow \phi = 90^0$

$$\frac{dy}{dt} = -bAe^{-bt}Cos(\omega t-\phi) - Ae^{-bt}\omega Sin(\omega t-\phi)$$
$$\Rightarrow v_0 = -bACos\phi + A\omega = A\omega$$
$$thus, A = \frac{v_0}{\omega} = \frac{v_0}{\sqrt{\omega_0^2-b^2}}$$
$$hence, y = \frac{v_0}{\sqrt{\omega_0^2-b^2}}e^{-bt}Cos[(\sqrt{\omega_0^2-b^2})t-\phi] \qquad (22)$$

The amplitude decreases exponentially with time. The frequency of vibration is slightly less than its natural frequency. The decrease depends on the damping factor. The time period $T = \dfrac{2\pi}{\sqrt{\omega_0^2-b^2}}$ is slightly more than the ideal undamped motion. Fig. ( ) shows the underdamped oscillatory motion.

Ballistic galvanometer is the example where we use to measure not current but charge (transient current) as a sudden throw. Damping is made low and motion is oscillatory.

Different methods for determination of 'b' :

(1) Logarithmic Decrement λ :

It measures the rate at which the amplitude decreases with time. Logarithmic decrement is the natural logarithm of the ratio of two successive amplitudes that are separated by half a period T/2 . Larger amplitude to be in the numerator. We have for damped oscillatory motion ,

$y = Ae^{-bt} Cos(\omega t - \phi)$ where A is the amplitude in absence of damping.

$$A_1 = Ae^{-bt_1}$$

$$A_2 = Ae^{-b(t_1 + \frac{T}{2})}$$

Let $\frac{A_1}{A_2} = \frac{Ae^{-bt_1}}{Ae^{-b(t_1+\frac{T}{2})}} = e^{b\frac{T}{2}} = d$

$$hence, \log_e d = b\frac{T}{2} = \lambda \qquad (23)$$

called logarithmic decrement.

$$\frac{A_1}{A_{n+1}} = \frac{A_1}{A_2} \cdot \frac{A_2}{A_3} \cdots\cdots\cdots \frac{A_n}{A_{n+1}} = d^n$$

$$thus, \lambda = \log_e d = \frac{1}{n}\log_e(\frac{A_1}{A_{n+1}}) \qquad (24)$$

b) Relaxation time : τ

It is defined as the time in which amplitude decays to 1/e of it's initial value.

Let $A_t \rightarrow$ amplitude at any instant of time t.

$A_{t+\tau} \rightarrow$ amplitude after time $t + \tau$ secs when it is 1/e of A_t.

$$\frac{A_{t+\tau}}{A_t} = \frac{Ae^{-b(t+\tau)}}{Ae^{-bt}} = \frac{1}{e} \qquad (25)$$

$$\Rightarrow b = \frac{1}{\tau}$$

We have $\lambda = \frac{bT}{2} = \frac{T}{2\tau}$ is the relation between logarithmic decrement and relaxation time.

© Quality factor:

Quality factor Q is defined as 2π times the ratio between average energy stored to the average energy lost per period .

$$Q = 2\pi . \frac{average\ energy\ stored}{average\ energy\ lost\ in\ a\ period} \qquad (26)$$

Energy of a damped S.H Oscillator:

We have $E = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}sy^2$

Also, $y = Ae^{-bt}Cos(\omega t - \phi)$

$$\frac{dy}{dt} = -bAe^{-bt}Cos(\omega t - \phi) - Ae^{-bt}\omega Sin(\omega t - \phi)$$

$$So, E=\frac{1}{2}m[A^2e^{-2bt}b^2Cos^2(\omega t-\phi)+A^2e^{-2bt}\omega^2Sin^2(\omega t-\phi)+2A^2e^{-2bt}b\omega Cos(\omega t-\phi)Sin(\omega t-\phi)]$$

$$+\frac{1}{2}m\omega_o^{\,2}A^2e^{-2bt}Cos^2(\omega t-\phi)$$

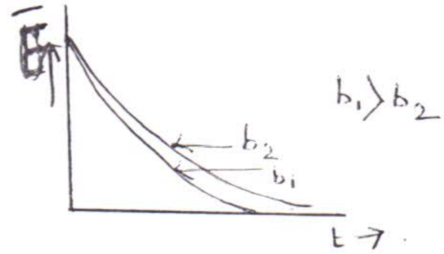
$$=\frac{1}{2}mA^2e^{-2bt}[b^2Cos^2(\omega t-\phi)+\omega^2Sin(\omega t-\phi)+b\omega Sin2(\omega t-\phi)+\omega_0^{\,2}Cos^2(\omega t-\phi)]$$

aking time average value of E over a complete time period,

$$<E>=\frac{1}{T}\int_0^T Edt=\frac{1}{2}mA^2e^{-2bt}[\frac{b^2}{2}+\frac{\omega^2}{2}+\frac{\omega_0^{\,2}}{2}]$$

$$where,\frac{1}{T}\int_0^T Cos^2\omega tdt=\frac{1}{2};\frac{1}{T}\int_0^T Sin^2\omega tdt=\frac{1}{2}\ \&\ \frac{1}{T}\int_0^T Sin2(\omega t-\phi)dt=0$$

nd it was assumed that e^{-2bt} factor almost constant within the time period.



$$<E>=\frac{1}{2}mA^2\omega_0^{\,2}e^{-2bt} \quad (27)$$

$$=E_0e^{-2bt}\ where E_0=\frac{1}{2}mA^2\omega_0^2 s$$

Thus E decreases with time.

Now, power dissipated due to damping force = instantaneous vel. X instantaneous damping force.

$$P=kv^2=kA^2e^{-2bt}[b^2Cos^2(\omega t-\phi)+\omega^2Sin^2(\omega t-\phi)+b\omega Sin2(\omega t-\phi)]$$

Average power dissipated by the damping force:

$$<P>=\frac{1}{T}\int_0^T Pdt=kA^2e^{-2bt}[\frac{b^2}{2}+\frac{\omega^2}{2}]=\frac{1}{2}kA^2\omega_0^2e^{-2bt}=\frac{1}{2}2bmA^2\omega_0^2e^{-2bt}=2bE_0e^{-2bt} \quad (28)$$

Average energy dissipated by the damping force over a complete time period is

$$<P>T=2bE_0e^{-2bt}\frac{2\pi}{\omega}$$

Quality factor.
$$Q=2\pi\frac{E_0e^{-2bt}\omega}{2\pi 2bE_0e^{-2bt}}=\frac{\omega}{2b} \quad (29)$$

$$\Rightarrow Q=\frac{\omega}{2b}$$

Smaller is the damping larger is the Q value i.e., it takes longer time for oscillations to damp out.

Dr. Ajanta Das.

FORCED VIBRATION:

The amplitude of damped vibration decreases exponentially with time and the frequency of the natural oscillations is slightly reduced. Usually the change in frequency is too small to be of any significance. We shall investigate the behavior of a weak damped harmonic oscillator when an external time dependent force is applied to the system to maintain the amplitude of the oscillation. Initially the vibrating system vibrates with its own frequency and then in the long run it starts vibrating with the frequency of the applied force. This vibration is called forced vibration .e.g.-diaphram of microphone is set in vibration by sound wave. The electrical circuit in the radio receiver oscillates because it is linked with the oscillating system (i.e. the transmitter) in the broadcasting station. In all these examples of forced oscillations, the driving system remains practically unaffected by the forced oscillations of the driven system. The driving system only serves as the supplier of the periodic force.

Differential equation for forced vibration:

Let an external harmonic force F $e^{i\omega t}$ act on a mass which when subjected to a restoring force proportional to the displacement and a damping force proportional to velocity. The equation of motion will be

$$m\frac{d^2y}{dt^2} = -sy - k\frac{dy}{dt} + Fe^{i\omega_1 t}$$

$$or, \frac{d^2y}{dt^2} + \frac{k}{m}\frac{dy}{dt} + \frac{s}{m}y = \frac{F}{m}e^{i\omega_1 t}$$

$$or, \frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0 y = fe^{i\omega_1 t}$$

where, $\quad 2b = k/m, \omega_0{}^2 = s/m, f = F/m$

The system initially vibrates with the effective frequency $\omega = \sqrt{\omega_0{}^2 - b^2}$ and ultimately starts to vibrate with frequency ω_1 of the applied force .Therefore we expect the actual motion in this case is the superposition of two oscillations one at frequency w of damped oscillations and other at frequency of the driving force.

Let y_1 be the solution of the equation $\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0 y = 0$

And y_2 be the particular solution of the equation $\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0 y = fe^{i\omega_1 t}$

We know that the solution of the damped oscillation is given by,

$y = Ae^{-bt}Cos(\omega t - \phi)$

The motion corresponding to $y_1$dies out with time exponentially and the motion due to the applied force take over the situation. A tussle ensues between the damping force tending to retered the motion and the driving force tending to help the motion. The motion of the forced oscillator thereby is given by the solution y_1+y_2. This is called the transient state. After sufficient time the damped vibration corresponding to y_1 dies out and the oscillator executes harmonic oscillations with the frequency of the driving force. This is called the steady state.

Let y_2 be the steady state solution and should satisfy the following equ$_n$:

$$\frac{d^2 y_2}{dt^2}+2b\frac{dy_2}{dt}+{\omega_0}^2 y_2=\frac{F}{m}e^{i\omega_1 t}$$

Let the trial solution be, $y_2 = \mathrm{Re}^{iw_1 t}$

$$\Rightarrow \left(-\omega_1^2+i2b\omega_1+\omega_0^2\right)Ae^{i\omega_1 t}=\frac{Fe^{i\omega_1 t}}{m}$$

$$\Rightarrow x_2=\frac{\frac{F}{m}}{\left((\omega_0^2-\omega_1^2)+i2b\omega_1\right)}e^{i\omega_1 t}$$

$$Let, z\cos\delta=(\omega_0^2-\omega_1^2); and z\sin\delta=2b\omega_1$$

$$then, x_2=\frac{\frac{F}{m}}{z\cos\delta+iz\sin\delta}e^{i\omega_1 t}=\frac{\frac{F}{m}}{z}e^{i(\omega_1 t-\delta)}$$

$$where, z=\sqrt{\left((\omega_0^2-\omega_1^2)^2+4b^2{\omega_1}^2\right)}$$

So the solution in the transient state is given by,

$$Y=y_1+y_2=y=Ae^{-bt}Cos(\omega t-\phi)+\frac{\frac{F}{m}}{z}e^{i(\omega_1 t-\delta)}$$

As discussed above the steady state solution is given by

$Y=\frac{\frac{F}{m}}{z}e^{i(\omega_1 t-\delta)}$; the system vibrating with constant amplitude $\frac{F}{mz}$

And a frequency which is same as that of the impressed force, but lagging in phase by an angle δ. The velocity of the particle is given by,

$$v=\frac{dy}{dt}=\frac{\frac{F}{m}i\omega_1}{\left((\omega_0^2-\omega_1^2)+i2b\omega_1\right)}e^{i\omega_1 t}$$

$$= \frac{\frac{F}{m} i\omega_1}{\left(2b\omega_1 + i(\omega_1^2 - \omega_0^2)\right)} e^{i\omega_1 t} = \frac{\frac{F}{m}\omega_1}{z} e^{i(\omega_1 t - \delta')} where, z = \left(4b^2\omega_1{}^2 + (\omega_1^2 - \omega_0^2)^2\right)^{1/2}$$

$$taking, z\cos\delta' = 2b\omega_1; z\sin\delta' = (\omega_1^2 - \omega_0^2)$$

Phase relations in forced vibrations:

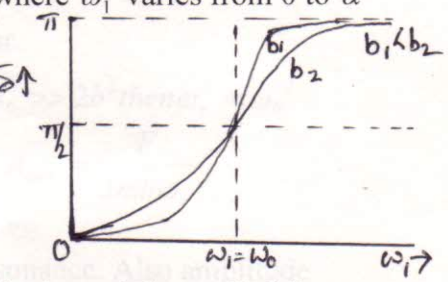
We have displacement at steady state in case of forced vibration is,

$$Y = \frac{\frac{F}{m}}{z} e^{i(\omega_1 t - \delta)} \text{ \& the driving force } F = F_0 e^{i\omega_1 t}$$

Where δ is the phase difference between the displacement and the driving force.

The displacement lags behind the driving force by an angle δ given by,

$$\delta = \tan^{-1}\frac{2b\omega_1}{\omega_0^2 - \omega_1^2} \text{ where } \omega_1 \text{ varies from 0 to } \alpha$$



(i) when $\omega_1 = 0$, $\tan\delta = 0$, hence $\delta = 0$

(ii) $\omega_1 < \omega_0, \tan\delta, is + ve. \Rightarrow 0 < \delta < \pi/2$

(iii) $\omega_1 = \omega_0, \tan\delta, is \infty \Rightarrow \delta = \pi/2$

(iv) $\omega_1 > \omega_0, \tan\delta, is - ve. \Rightarrow \pi/2 < \delta < \pi$

(v) $as\, \omega \to \infty, \delta \to \pi$

We also know that the phase difference between the velocity and the applied force is δ'.,where

$$\tan\delta' = \frac{\omega_1{}^2 - \omega_0{}^2}{2b\omega_1} = -\cot\delta$$

$$\delta = \delta' + \pi/2$$

Thus the displacement and velocity has a phase difference of $\pi/2$.

Amplitude and velocity resonance:

Resonance is the special condition of forced vibration if the frequency of the applied force happens to be equal to the natural frequency of the system then the system vibrates with maximum amplitude Both potential as well as kinetic energy becomes maximum.

<u>Amplitude resonance</u>:

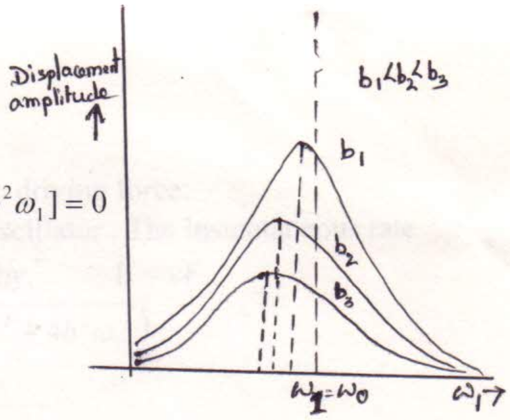
The steady state amplitude of a forced vibrator is given by,

$$= \frac{\frac{F}{m}}{\left((\omega_0^2 - \omega_1^2)^2 + 4b^2\omega_1{}^2\right)^{1/2}}$$

Thus amplitude varies with the frequency of the applied force.

For the amplitude to be maximum ,

$$\text{i.e.,} \frac{d}{d\omega}\ [f_0 / \sqrt{\left((\omega_0^2 - \omega_1^2)^2 + 4b^2\omega_1{}^2\right)}] = 0$$



$$hence, -\frac{1}{2} f_0\left(4b^2\omega_1{}^2 + (\omega_0^2-\omega_1^2)^2\right)^{-3/2}[2(\omega_0^2-\omega_1^2)(-2\omega_1)+8b^2\omega_1]=0$$

$$\Rightarrow 2(\omega_0^2-\omega_1^2)(-2\omega_1)+8b^2\omega_1=0$$

$$\Rightarrow 4\omega_1(\omega_0^2-\omega_1^2)=8b^2\omega_1$$

$$\Rightarrow (\omega_0^2-\omega_1^2)=2b^2$$

$$\Rightarrow \omega_1^2=\omega_0^2-2b^2=\omega_r{}^2(say)$$

Thus the angular frequency at which resonance occurs is slightly less than the natural frequency of the driven system as well as the freq. with which damped oscillation occurs.

Hence, amplitude at resonance $A_{\max}=\dfrac{F}{2mb\sqrt{\omega_0^2-b^2}}=\dfrac{F}{2mb\omega}$ where,

$\omega=\sqrt{\omega_0{}^2-b^2}$ is the frequency of the damped oscillator.

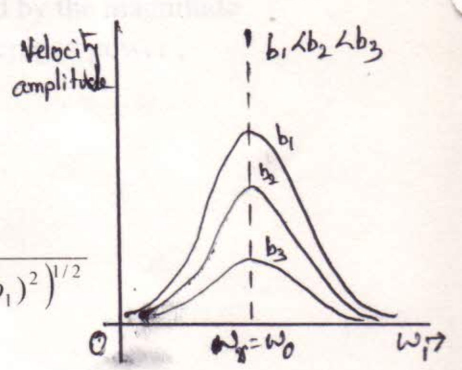
when $\omega_0^2 >> 2b^2 \, then \, \omega_r^2=\omega_0{}^2$

hence, $A_{\max}=\dfrac{F}{2mb\omega_r}$

Small value of damping produces large amplitude at resonance. Also amplitude resonance frequency ω_r shifts towards ω_0, the natural frequency when damping is small.

Velocity resonance:

The energy of the driven system is maximum when the frequency of the driven system without damping is equal to the frequency of the driver. This is known as velocity resonance or energy resonance .



We have,

$$v=\frac{dy}{dt}=\frac{\frac{F}{m}\omega_1}{\left(4b^2\omega_1{}^2+(\omega_1^2-\omega_0^2)^2\right)^{1/2}}e^{i(\omega_2 t-\delta')}$$

$$\Rightarrow v_0=\frac{\frac{F}{m}\omega_1}{\left(4b^2\omega_1{}^2+(\omega_1^2-\omega_0^2)^2\right)^{1/2}}=\frac{\frac{F}{m}}{\left(4b^2+(\omega_1-\omega_0^2/\omega_1)^2\right)^{1/2}}$$

For v_0 to be maximum ,

i.e., $\dfrac{d}{d\omega}[F/m\{4b^2+(\omega_1-\omega_0^2/\omega_1{}^2)^2\}^{-\frac{1}{2}}]=0$

$\Rightarrow \omega_1=\omega_0$

thus velocity resonance frequency is equal to the natural frequency of the system for free oscl.

and $v_{\max}=\dfrac{F}{2bm}$

Power dissipation in forced vibration:

Expression for the power supplied to the oscillator by the driving force:

Let a periodic force $F_0 \cos\omega t$ be applied to a harmonic oscillator . The instantaneous rate of work by the driving force in the steady state is given by, P = vF

$$\text{where,Re(v)} = -\frac{\frac{F_0}{m}\omega_1}{z}\sin(\omega_1 t - \delta) \ \& \quad z = \sqrt{\left((\omega_0^2 - \omega_1^2)^2 + 4b^2\omega_1{}^2\right)}, \quad \delta = \tan^{-1}\frac{2b\omega_1}{(\omega_0^2 - \omega_1^2)}$$

Average power over a complete time period is,

$$\bar{P} = \frac{1}{T}\int_0^T P dt = \frac{F_0{}^2\omega_1}{2mz}\sin\delta = \frac{F_0{}^2\omega_1}{2mz}\frac{2b\omega_1}{z} = \frac{F_0{}^2\omega_1{}^2 b}{mz^2} = mbA^2\omega^2{}_1$$

Also rate of work done by the oscillator against the damping force:

$$\frac{dw}{dt} = (2bm\frac{dy}{dt})\frac{dy}{dt} = 2bm(\frac{dy}{dt})^2 = 2bm\frac{F_0{}^2\omega_1{}^2}{m^2z^2}\sin^2(\omega t - \delta) = mbA^2\omega_1{}^2[1 - \cos 2(\omega t - \delta)]$$

$$hence < \frac{dw}{dt} >= mbA^2\omega_1{}^2$$

Thus the power supplied to oscillator by the driving force is equal to power dissipated by the oscillator in presence of damping..

Sharpness at resonance and quality factor:

The response of the oscillator to the driving force may be gauged by the magnitude of power it extracts from the driving force. We have the time averaged power ,

$$\bar{P} = mb\omega_1{}^2 A^2$$

$$= \frac{mb\omega_1{}^2 f_0{}^2}{(\omega_0^2 - \omega_1^2)^2 + 4b^2\omega_1{}^2} \quad where f_0 = \frac{F_0}{m}$$

$$= \frac{mbf_0{}^2}{(\frac{\omega_0^2}{\omega_1} - \omega_1)^2 + 4b^2}$$

$$\bar{P}\, is \max{}^m \ when, (\frac{\omega_0^2}{\omega_1} - \omega_1)^2 + 4b^2 is \min{}^m,$$

$$i.e., (\frac{\omega_0^2}{\omega_1} - \omega_1)^2 = 0 or \omega_1 = \omega_0$$

putting $\omega_1 = \omega_0$ in the expression for time avg. for power we get,

$$P_{\max} = \frac{mf_0^{\ 2}}{4b}$$

$$thus, \bar{P} = \bar{P}_{\max} = \frac{4b^2\omega^2}{(\omega_0^2 - \omega_1^2)^2 + 4b^2\omega_1^{\ 2}}$$

$$At half power points,$$

$$i.e., \bar{P} = \bar{P}_{\max}/2,$$

$$equ^n (5) implies,$$

$$(\omega_0^2 - \omega_1^2)^2 = 4b^2\omega_1$$

$$\Rightarrow (\omega_0^2 - \omega_1^2) = \pm 2b\omega_1$$

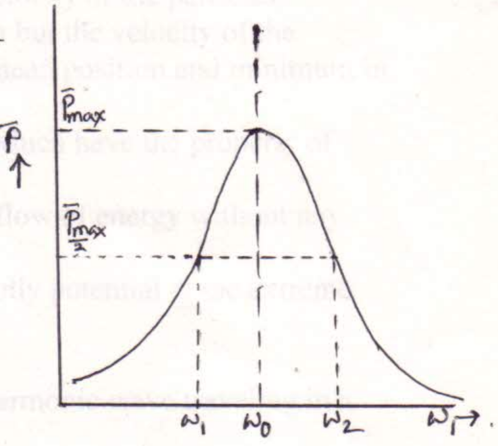
$$\Rightarrow \omega_1^2 \pm 2b\omega_1 - \omega_0^2 = 0$$

$$\Rightarrow \omega_1 = \mp b \pm \sqrt{b^2 + \omega_0^2}$$

$$since -ve freq. is not possible,$$

$$\omega_1 = -b + \sqrt{b^2 + \omega_0^2}$$

$$\omega_2 = b + \sqrt{b^2 + \omega_0^2}$$



The frequency interval between two half power points is often called the bandwidth. Thus bandwidth

$$\Delta\omega = \omega_2 - \omega_1 = 2b$$

As the frequency of the driving force varies on either side of the resonant frequency ω_0 ,the avg. power also falls from its maximum value at $\omega_1 = \omega_0$ and the width at which power falls to half of its maximum value depends on damping factor. The less is b less will be the bandwidth and sharper will be the resonance .We find that as the frequency of the driving force differs from the resonant frequency ω_0 the response of the system diminishes on either side of ω_0. The rate of fall of response with departure from the equality between driving frequency and resonant frequency gives a measure of the sharpness of resonance. Quantitatively the sharpness of resonance is measured in terms of Quality factor which is defined as the ratio between the resonant frequency and the bandwidth and is denoted by Q.

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{\omega_2 - \omega_2} = \frac{\omega_0}{2b}$$

Wave

Progressive wave equation and its differential form:-

Wave is a form of disturbance which travels through medium due to the repeated periodic vibration of the particles of the medium about their mean positions, the disturbance being handed over from one particle to the next particle.

When the wave travels away from the source & the disturbance is of simple harmonic in character then it is called harmonic progressive wave.

Characteristics of wave motion:

Vibrations and Waves

(A) SIMPLE HARMONIC MOTION:

1. Define S.H.M., write down the differential equation and solve it.
2. Displacement of a moving particle at any instant t is given by y = a coswt +b sinwt, show that the motion is S.H.M.
3. What are the dimension and unit of stiffness constant and damping constant?
4. Name the periodic motion which is not oscillatory.
5. Show that for a particle executing S.H.M. the average value of K.E. and P.E. is the same and each is equal to half the total energy.
6. Derive the equation for S.H.M. from energy consideration.
7. Derive the differential equation of S.H.M. for an electrical circuit. Can it be realized in practice. What will be the freq. of oscillation?
8. What are lissajous figures? How will you trace graphically the Lissajous figures when (i) the periods are equal and phase differences are 0, $\pi/4$, $\pi/2$, π and ~~(ii) the periods are in the ratio of 2:1 for phase difference zero and $\pi/2$~~.

(B) DAMPED HARMONIC MOTION:

9. What is damped vibration? Write down the differential equation of motion and solve it. Explain over damped, under damped and critically damped motions.
10. Explain the damped vibration for electrical oscillator.
11. Derive an expression for loss of energy per cycle in damped oscillatory motion.
12. Show that the energy of damped vibrations decreases exponentially with time.
13. Define logarithmic decrement (λ) and establish a relation with damping co-efficient
14. Define relaxation time (τ) of damped oscillatory system. Derive the expression for τ of a mechanical and an electrical system. What is the relation between τ and λ.
15. What is quality factor of a damped oscillator? Derive an oscillator for mechanical and electrical oscillator.

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16. A body in damped SHM is acted on by a periodic force. Write down the differential equation of motion and solve it for transient and steady state. Establish the velocity expression.
17. Derive the mechanical impedance of forced vibrations and explain the physical significance of it.
18. What is resonance? Explain amplitude and velocity resonance. Discuss the behavior of displacement and velocity versus driving force frequency for forced vibrations. What is the power factor at velocity resonance?
19. Calculate the average power in forced vibration and band width.
20. What is sharpness of resonance and explain the effect of damping on sharpness of resonance?
21. Show that the power supplied by the forced oscillator is equal to the average power dissipated by the system.
22. Derive the quality factor of an oscillator in terms of resonance absorption band width.
23. Explain electrical oscillator and calculate Q value.
24. Show that for a forced vibration the total energy of the vibrating system is not constant and that (i) $\dfrac{average P.E.}{average K.E.} = \dfrac{\omega_0^2}{\omega^2}$ (ii) $(K.E.)_{max} = \dfrac{mF^2}{2Z_m^2}$

$$\text{(iii)}\ (P.E.)_{max} = \left(\frac{mF^2}{2Z_m^2}\right)\left(\frac{\omega_0^2}{\omega^2}\right) \qquad \text{(iv)}\ \begin{aligned} E_{max} &= (P.E.)_{max}\ if \omega_0 > \omega \\ &= (K.E.)_{max}\ if \omega_0 < \omega \end{aligned}$$