1.3 Differential Equation of First Order and Higher Degree

1.3.1. Introduction. A differential equation of first order and n-th degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots + P_{n-1} p + P_n = 0 \quad \ldots \quad (1)$$

where $p = \frac{dy}{dx}$ and $P_1, P_2, \ldots\ldots P_n$ are functions of x and y.

Since it is a differential equation of the first order, its general solution will contain only one arbitrary constant.

In this chapter, we shall discuss three special types of the above equation, in which it is

(i) solvable for p

(ii) solvable for y

(iii) solvable for x.

1.3.2. Equations solvable for p.

Let the equation (1) can be put in the form

$$\{p - f_1(x,y)\}\{p - f_2(x,y)\}\ldots\{p - f_n(x,y)\} = 0$$

which is equivalent to

$$p - f_1(x,y) = 0,\ p - f_2(x,y) = 0, \ldots\ p - f_n(x,y) = 0. \quad \ldots \quad (2)$$

i.e., $\frac{dy}{dx} - f_1(x,y) = 0,\ \frac{dy}{dx} - f_2(x,y) = 0, \ldots, \frac{dy}{dx} - f_n(x,y) = 0.$

Each of these equations is of first order and first degree and can be solved by the methods discussed previous chapter. Let the solution be

$$F_1(x,y,c_1) = 0,\ F_2(x,y,c_2) = 0, \ldots, F_n(x,y,c_n) = 0 \ldots \quad (3)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Hence the general solution of (1) is given by

$$F_1(x,y,c)\, F_2(x,y,c) \ldots F_n(x,y,c) = 0 \quad \ldots \quad (4)$$

where $c_1 = c_2 = \ldots = c_n = c$, is any arbitrary constant. (We should have one arbitrary constant because this is a first order differential equation)

Illustrative Examples.

Ex. 1. *Solve :* $p^2 - p(e^x + e^{-x}) + 1 = 0$.

The equation can be written as $(p - e^x)(p - e^{-x}) = 0$.

i.e., $p - e^x = 0$ or $p - e^{-x} = 0$

$\therefore \quad (p - e^x) = 0$ gives $\dfrac{dy}{dx} = e^x$,

or, $dy = e^x dx$

which on integration gives $y = e^x + c_1$.

Also $p - e^{-x} = 0$ gives $\dfrac{dy}{dx} = e^{-x}$

or, $dy = e^{-x} dx$

which on integration gives $y = -e^{-x} + c_2$

Hence the general solution of the equation is

$(y - e^x - c)(y + e^{-x} - c) = 0$, c is an arbitrary constant.

Ex. 2. *Solve :* $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$.

The given equation can be written as $(px - 2y)(py + 3x) = 0$

i.e., $px - 2y = 0$ or $py + 3x = 0$.

Now $px - 2y = 0$ gives $\dfrac{dy}{dx} = \dfrac{2y}{x}$

or, $\dfrac{dy}{y} - \dfrac{2dx}{x} = 0$

which on integration gives

$\log y - 2\log x = \log c_1$

$\therefore \quad \dfrac{y}{x^2} = c_1$

$\therefore \quad y = c_1 x^2$.

Again $py + 3x = 0$ gives $p = -\dfrac{3x}{y}$ or, $\dfrac{dy}{dx} = -\dfrac{3x}{y}$

or, $ydy = -3xdx$

which on integration gives

$$\frac{y^2}{2} = -\frac{3x^2}{2} + \frac{c_2}{2}$$

$\therefore \quad y^2 + 3x^2 = c_2$.

Hence the general solution of the equation is

$(y - cx^2)(y^2 + 3x^2 - c) = 0$, c is an arbitrary constant.

Ex. 3. *Solve :* $xy\left\{\left(\dfrac{dy}{dx}\right)^2 - 1\right\} = (x^2 - y^2)\dfrac{dy}{dx}$.

The given equation can be written as $xy(p^2 - 1) = (x^2 - y^2)p$.

or, $(xp + y)(yp - x) = 0$

$\therefore \quad xp + y = 0$ or $yp - x = 0$.

Now $xp + y = 0$ gives $x\dfrac{dy}{dx} = -y$

or, $\dfrac{dy}{y} + \dfrac{dx}{x} = 0$

which on integration gives

$\log y + \log x = \log c_1$

$\therefore \quad xy = c_1$

Again $yp - x = 0$ gives $y\dfrac{dy}{dx} - x = 0$

or, $ydy - xdx = 0$

which on integration gives

$$\frac{y^2}{2} - \frac{x^2}{2} = \frac{c_2}{2}$$

$\therefore \quad y^2 - x^2 = c_2$.

Hence the general solution of the equation is

$(xy - c)(x^2 - y^2 + c) = 0$, c is an arbitrary constant.

Ex. 4. Solve the following differential equation [WBUT 2006]

$$\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$$

The given equation can be written as

$$p-\frac{1}{p}=\frac{x^2-y^2}{xy} \text{ where } p=\frac{dy}{dx}$$

or, $(p^2-1)xy=p(x^2-y^2)$

or, $p^2xy-xy-px^2+py^2=0$

or, $(py-x)(px+y)=0$

$\therefore$ $py-x=0$ or, $px+y=0$

Now $py-x=0$ gives

$$p=\frac{x}{y}$$

or, $y\,dy-x\,dx=0$

which on integration gives

$$\frac{y^2}{2}-\frac{x^2}{2}=\frac{c_1}{2}$$

$\therefore$ $y^2-x^2-c_1=0$

Again $px+y=0$ gives

or, $\frac{dy}{y}+\frac{dx}{x}=0$

which on integration gives

$$\log y+\log x=\log c_2$$

$\therefore$ $xy-c_2=0$

Thus the required general solution is

$$(y^2-x^2-c)(xy-c)=0$$

where c in an arbitrary constant

EXERCISE-A

Solve the following equations :-

1. $p^2+2py\cot x=y^2$.
2. $p^3-p(x^2+xy+y^2)+x^2y+xy^2=0$.
3. $x^2\left(\frac{dy}{dx}\right)^2+xy\frac{dy}{dx}-6y^2=0$.
4. $p^2+2xp-3x^2=0$.
5. $4y^2p^2+2pxy(3x+1)+3x^3=0$.
6. $p(p+y)=x(x+y)$.
7. $xy(2x+1)\left(\frac{dy}{dx}\right)-y^2\left(\frac{dy}{dx}\right)^2=2x^3$.
8. $x+yp^2=p(1+xy)$.
9. $p^3-\left(y+2x-e^{x-y}\right)p^2+\left(2xy-2xe^{x-y}-ye^{x-y}\right)p+2xye^{x-y}=0$
10. $p^3+2xp^2-y^2p^2-2xy^2p=0$
11. $p^4-(x+2y+1)p^3+(x+2y+2xy)p^2-2xyp=0$
12. $\left(x^2+x\right)p^2+\left(x^2+x-2xy-y\right)p+y^2-xy=0$
13. $x^2p^2+3xyp+2y^2=0$

ANSWERS

1. $y(1\pm\cos x)=c$.
2. $(2y-x^2-c)(y-ce^x)(y+x-1-ce^{-x})=0$.
3. $(x^3y-c)(y-cx^2)=0$.
4. $(2y-x^2-c)(2y+3x^2-c)=0$.

5. $(y^2+x^3-c)(y^2+\frac{1}{2}x^2-c)=0$. 6. $(y-\frac{1}{2}x^2+c)(y+x+ce^{-x}-1)=0$

7. $(y^2-x^2-2c)\times(3y^2-4x^3-6c)=0$.

8. $(2y-x^2-c)(2x-y^2-c)=0$.

9. $(y-ce^x)(y-x^2-c)(e^y+e^x-c)=0$

10. $(y+x^2-c)(y-c)(xy+cy+1)=0$

11. $(y-c)(y-x-c)(2y-x^2-c)(y-ce^{2x})=0$

12. $[y-c(x+1)][y+x\ln cx]=0$ 13. $(xy-c)(x^2y-c)=0$

1.3.3. Equations solvable for y.

In this case, the equation (1) can be put as

$y=f(x,p)$ ... (5)

Differentiating (5) w.r.t x, we get $p=F(x,p,\frac{dp}{dx})$ ... (6)

which is a first order differential equation in p and x.

Suppose the solution of (6) be $\phi(x,p,c)=0$. ... (7)

Eliminating p from (5) and (7), we shall get the required solution.

If p cannot be easily eliminated, then we solve equations (5) and (7) for x and y and obtain $x=\phi_1(p,c)$, $y=\phi_2(p,c)$ as the required solution, where p is kept as a parameter.

Illustrative Examples.

Ex. 1. *Solve :* $y=2px-p^2$.

The given equation is $y=2px-p^2$ ... (i)

Differentiating (i) w.r.t. x,

$$p=2p+2x\frac{dp}{dx}-2p\frac{dp}{dx}$$

or, $$p+2(x-p)\frac{dp}{dx}=0$$

or, $$p\frac{dx}{dp}+2(x-p)=0$$

$$\therefore \frac{dx}{dp}+\frac{2}{p}x=2 \qquad \text{... (ii)}$$

which is a linear equation in x.

$$\therefore I.F.=e^{\int\frac{2}{p}dp}=e^{2\log p}=p^2$$

Multiplying both sides of (ii) by p^2, we get

$$\frac{d}{dp}(x.p^2)=2p^2$$

or, $$d(xp^2)=2p^2dp$$

which on integration gives $xp^2=\frac{2p^3}{3}+c$.

$$\therefore x=\frac{2}{3}p+cp^{-2} \qquad \text{... (iii)}$$

Putting this value of x in (i), we get

$$y=2p(\frac{2}{3}p+cp^{-2})-p^2=\frac{1}{3}p^2+2cp^{-1}$$

So the required general solution is

$$\left.\begin{aligned} x&=\frac{2}{3}p+cp^{-2}\\ y&=\frac{1}{3}p^2+2cp^{-1}\end{aligned}\right\}, \text{ where } p \text{ is the parameter.}$$

Ex. 2. *Solve :* $y=2px+p^4x^2$.

The given equation is $y=2px+p^4x^2$ ... (i)

Differentiating (i) w.r.t. x,

$$p=2p+2x\frac{dp}{dx}+2p^4x+4p^3x^2\frac{dp}{dx}$$

or, $4p^3x^2\dfrac{dp}{dx}+2x\dfrac{dp}{dx}+2p^4x+p=0$

or, $(p+2x\dfrac{dp}{dx})(1+2p^3x)=0.$

Discarding the factor $(1+2p^3x)$, we have

$$p+2x\frac{dp}{dx}=0.$$

or, $2\dfrac{dp}{p}+\dfrac{dx}{x}=0$

which on integration gives

$$2\log p+\log x=\log c.$$

$\therefore\ p^2x=c.$

$\therefore\ p^2=\dfrac{c}{x}.$ ... (ii)

Eliminating p from (i) and (ii) we get

$$y=2\left(\pm\sqrt{\frac{c}{x}}\right)x+c^2$$

or, $(y-c^2)^2=4\dfrac{c}{x}.x^2$

$\therefore\ (y-c^2)^2=4cx$, which is the required general solution.

Ex.3. *Solve :* $e^y-p^3-p=0.$

The given equation can be written as

$$y=\log(p^3+p)\qquad\ldots(i)$$

Differentiating (i) w.r.t. x,

$$p=\frac{1}{p^3+p}.(3p^2+1)\frac{dp}{dx}$$

or, $\dfrac{dp}{dx}=\dfrac{p^2(p^2+1)}{3p^2+1}$

or, $dx=\dfrac{3p^2+1}{p^2(p^2+1)}dp$

or, $dx=\left(\dfrac{2}{1+p^2}+\dfrac{1}{p^2}\right)dp$

which on integration gives

$$x=2\tan^{-1}p-\frac{1}{p}+c.$$

Thus the required solution is

$$\left.\begin{aligned}x&=2\tan^{-1}p-\frac{1}{p}+c\\ y&=\log(p^3+p)\end{aligned}\right\}\text{, where } p \text{ is the parameter.}$$

Ex. 4. *Solve :* $y=(p+p^2)x+p^{-1}$ **[WBUT 2003]**

We have $y=(p+p^2)x+p^{-1}$ ... (i)

Differentiating (i) w.r.t x,

$$p=p+p^2+x\left(\frac{dp}{dx}+2p\frac{dp}{dx}\right)-\frac{1}{p^2}\frac{dp}{dx}$$

or, $\dfrac{dx}{dp}+x\left(\dfrac{1+2p}{p^2}\right)=\dfrac{1}{p^4}$ ... (ii)

which is a linear equation in x

$\therefore$ I.F $=e^{\int\left(\frac{1+2p}{p^2}\right)dp}$

$=e^{-\frac{1}{p}+2\log p}$

$=p^2e^{-\frac{1}{p}}$

Multiplying both sides of (ii) by I.F. and then integrating we get

$$xp^2 e^{-\frac{1}{p}} = \int \frac{1}{p^4}\cdot p^2 \cdot e^{-\frac{1}{p}}\, dp$$

$$= \int \frac{1}{p^2}\, e^{-\frac{1}{p}}\, dp$$

$$= \int e^{-\frac{1}{p}}\, d\left(-\frac{1}{p}\right)$$

$$= e^{-\frac{1}{p}} + c$$

$\therefore$ $xp^2 = 1 + ce^{\frac{1}{p}}$

$\therefore$ $$x = \frac{1+ce^{\frac{1}{p}}}{p^2} \qquad \ldots \text{(iii)}$$

$\therefore$ From (i),

$$y = \left(p+p^2\right)\cdot\frac{1+ce^{\frac{1}{p}}}{p^2} + \frac{1}{p}$$

$\therefore$ $$y = \left(1+\frac{1}{p}\right)\left(1+ce^{\frac{1}{p}}\right) + \frac{1}{p} \qquad \ldots \text{(iv)}$$

Then (iii) and (iv) together gives the required solution.

EXERCISE-B

Solve the following equations :-

1. $y = 3x + \log p$.
2. $y + px = x^4 p^2$.
3. $y = 2px - p^2$.
4. $x - yp = ap^2$.
5. $16x^2 + 2p^2 y - p^3 x = 0$.

6. $xp^2 = \tan(y - 2px)$.
7. $y = p\sin p + \cos p$.
8. $x^2p^4 + 2xp - y = 0$.
9. $3x^4p^2 - xp - y = 0$
10. $p\tan p - y + \log\cos p = 0$
11. $2x + p^2 - y + px = 0$
12. $y = x + a\tan^{-1}\left(\frac{dy}{dx}\right)$
13. $y = px + p^2 x$
14. $y = 2px + 4xp^2$
15. $y = x\left(p + p^3\right)$
16. $xp^2 - 2yp + ax = 0$

ANSWERS

1. $y = 3x + \log\left(\frac{3}{1-ce^{3x}}\right)$.
2. $xy + c = c^2 x$.
3. $\left.\begin{aligned} x &= \frac{c}{p^2} + \frac{2}{3}p \\ y &= \frac{2c}{p} + \frac{1}{3}p^2 \end{aligned}\right\}$, p is the parameter.
4. $\left.\begin{aligned} x &= \frac{p}{\sqrt{1-p^2}}\left(c + a\sin^{-1}p\right) \\ y &= \frac{1}{\sqrt{1-p^2}}\left(c + a\sin^{-1}p\right) - ap \end{aligned}\right\}$, p is the parameter.

5. $16+2c^2y-c^3x^2=0.$

6. $y=2\sqrt{cx}+\tan^{-1}c.$

7. $\left.\begin{array}{l} x=\sin p+c \\ y=p\sin p+\cos p \end{array}\right\}$, p is the parameter.

8. $y=2\sqrt{cx}+c^2.$

9. $xy=c(3cx-1)$

10. $x=\tan p+c$; $y=p\tan p+\log\cos p$, p is the parameter.

11. $x=2(2-p)+ce^{-\frac{p}{2}}$; $y=8-p^2+(2+p)ce^{-\frac{p}{2}}$

12. $x+c=\frac{a}{2}\left\{\log\frac{p-1}{\sqrt{p^2+1}}-\tan^{-1}p\right\}$ with the given relation.

13. $x=\frac{e^{yp}}{cp^2}$, $y=\frac{e^{yp}}{cp^2}(p+p^2)$

14. $(y-4c)^2=4cx$

15. $x=\frac{1}{cp^3}e^{\frac{1}{2p^2}}$, $y=\frac{1}{cp^2}(1+p^2)e^{\frac{1}{2p^2}}$

16. $2y=cx^2+\frac{a}{c}$

1.3.4. Equations solvable for x.

In this case, equation (1) can be put as $x=f(y,p)$. . . . (8)

Differentiating (8) w.r.t. y, we get $\frac{dx}{dy}=F\left(y,p,\frac{dp}{dy}\right)$

or, $\frac{1}{p}=F\left(y,p,\frac{dp}{dy}\right)$ $\left[\because p=\frac{dy}{dx}\right]$ . . . (9)

which is a first order differential equation in p and y.

Suppose the solution of (9) be $\phi(y,p,c)=0$ . . . (10)

Eliminating p between (8) and (10), we shall get the required solution.

If p cannot be easily eliminated, then we solve equations (8) and (10) for x and y and obtain $x=\phi_1(p,c)$, $y=\phi_2(p,c)$, as the required solution, where p is kept as the parameter.

Illustrative Examples.

Ex. 1. *Solve :* $y^2\log y=xyp+p^2$.

The given equation can be written as

$$x=\frac{y\log y}{p}-\frac{p}{y} \qquad \text{. . . (i)}$$

Differentiating (i) w.r.t. y,

$$\frac{dx}{dy}=\frac{p(1+\log y)-y\log y\frac{dp}{dy}}{p^2}-\frac{y\frac{dp}{dy}-p}{y^2}$$

or, $\frac{1}{p}=\frac{1}{p}+\frac{1}{p}\log y-\frac{y}{p^2}\log y\frac{dp}{dy}-\frac{1}{y}\frac{dp}{dy}+\frac{p}{y^2}$

or, $\frac{1}{y}\frac{dp}{dy}\left(1+\frac{y^2}{p^2}\log y\right)=\frac{p}{y^2}\left(1+\frac{y^2}{p^2}\log y\right)$

or, $\frac{dp}{dy}=\frac{p}{y}$ [discarding the factor $\left(1+\frac{y^2}{p^2}\log y\right)$]

or, $\frac{dp}{p}=\frac{dy}{y}$

which on integration gives

$\log p=\log y+\log c$

or, $p=cy$. . . . (ii)

Eliminating p from (i) and (ii) we get

$\log y=cx+c^2$, which is the required solution.

Ex. 2. Solve : $ayp^2 + (2x - b)p - y = 0,\ a > 0.$

The given equation can be written as

$$x = \frac{y}{2p} - \frac{ayp}{2} + \frac{1}{2}b. \quad \ldots \text{(i)}$$

Differentiating both sides of (i) w.r.t. y, we get

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{ap}{2} - \frac{ay}{2}\frac{dp}{dy}$$

or, $$\frac{1}{p} + ap + \left(\frac{y}{p^2} + ay\right)\frac{dp}{dy} = 0$$

or, $$a\left(p + y\frac{dp}{dy}\right) + \frac{1}{p^2}\left(p + y\frac{dp}{dy}\right) = 0$$

or, $$\left(a + \frac{1}{p^2}\right)\left(p + y\frac{dp}{dy}\right) = 0$$

$\therefore$ $$p + y\frac{dp}{dy} = 0. \ [\because a + \frac{1}{p^2} \neq 0, \text{since } a > 0.]$$

or, $$\frac{dy}{y} + \frac{dp}{p} = 0$$

which on integration gives

$\log y + \log p = \log c$

$\therefore$ $yp = c.$ $\quad \ldots$ (ii)

Eliminating p from (i) and (ii), we get

$$\frac{ac^2}{y} + (2x - b)\frac{c}{y} - y = 0$$

$\therefore$ $ac^2 + (2x - b)c - y^2 = 0$, which is the required solution.

Ex. 3. *Solve :* $\tan^{-1} p = x - \dfrac{p}{1+p^2}.$

The given equation can be written as

$$x = \tan^{-1} p + \frac{p}{1+p^2}. \quad \ldots \text{(i)}$$

Differentiating (i) w.r.t. y, we get

$$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2) - 2p.p}{(1+p^2)^2}.\frac{dp}{dy}$$

or, $$\frac{1}{p} = \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

or, $$dy = \frac{2p}{(1+p^2)^2}dp$$

which on integration gives

$$y = c - \frac{1}{1+p^2}.$$

So the required solution is

$$\left.\begin{aligned} x &= \tan^{-1} p + \frac{p}{1+p^2} \\ y &= c - \frac{1}{1+p^2} \end{aligned}\right\}, \text{ where } p \text{ is the parameter.}$$

Ex. 4. *Solve :* $y = 2px + yp^2$

The given equation can be written as

$$x = \frac{y}{2p} - \frac{py}{2} \quad \ldots \text{(i)}$$

Differentiating (i) w.v.t. y,

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

or, $$\frac{1}{p} - \frac{1}{2p} + \frac{y}{2p^2}\frac{dp}{dy} + \frac{p}{2} - \frac{y}{2}\frac{dp}{dy} = 0$$

or, $$\left(\frac{1}{p} + p\right) + \frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p} + p\right) = 0$$

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or, $\left(\frac{1+p^2}{p}\right)\left(1+\frac{y}{p}\frac{dp}{dy}\right)=0$

$\therefore \quad 1+\frac{y}{p}\frac{dp}{dy}=0 \quad [\because \; p^2+1\neq 0]$

or, $\frac{dy}{y}+\frac{dp}{p}=0$

which on integration gives

$$\log y+\log p=\log c$$

or, $py=c$

$\therefore \quad p=\frac{c}{y}$

$\therefore$ From (i),

$$x=\frac{y^2}{2c}-\frac{c}{2}$$

$\therefore \quad y^2=2cx+c^2$

which is the required general solution.

EXERCISE -C

Solve the following equations :

1. $p^3-4xyp+8y^2=0.$
2. $y=2px+y^2p^3.$
3. $x+\frac{p}{\sqrt{1+p^2}}=a.$
4. $x=py-p^2.$

5. $xp^2-2yp+ax=0.$
6. $y=2px+p^2.$
7. $x=y+a\log p.$
8. $p^2-xp+y=0$
9. $xp^2-yp-y=0$
10. $6p^2y^2-y+3px=0$
11. $4y^2+p^3=2xyp$
12. $p^2+y-x=0$
13. $x(1+p^2)=1$
14. $p^2y+2px=y$
15. $x=y+a\log p$

ANSWERS

1. $y=c\,(c-x)^2.$
2. $y^2=2cx+c^3.$
3. $(y+c)^2+(x-a)^2=1.$
4. $x=py-p^2$
   $y=p+(c+\cosh^{-1}p)\times(p^2-1)^{-1/2}.$
5. $2y=cx^2+a/c.$
6. $(3xy+2x^3+c)^2=4\,(x^2+y)^3.$
7. $\left.\begin{array}{l} x=c+a\log\frac{p}{p-1} \\ y=c-a\log(p-1)\end{array}\right\}.$

8. $y = cx - c^2$

9. $x = c(1+p)e^p$, $y = cp^2e^p$

10. $y^3 = 3cx + 6c^2$

11. $2y = c(c-x)^2$

12. $x = c - 2[p + \log(p-1)]$, $y = c - p^2 - 2[p + \log(p-1)]$

13. $y = c + \sqrt{x - x^2} - \tan^{-1}\sqrt{\dfrac{1-x}{x}}$

14. $y^2 = c^2 + 2cx$ 15. $y = c - a\log(p-1)$, $x = c + a\log\left(\dfrac{p}{p-1}\right)$

1.3.5. Clairaut's Equation.

A differential equation of the form $y = px + f(p)$ ... (11)

is known as Clairaut's equation. [W.B.U.T. 2014]

To solve the equation (11), we differentiate both sides of (11) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or, $\dfrac{dp}{dx}\{x + f'(p)\} = 0.$

Either $\dfrac{dp}{dx} = 0$ i.e., $p = c$, a constant. ... (12)

or, $x + f'(p) = 0.$ ... (13)

Eliminating p between (11) and (12), we get

$$y = cx + f(c) \qquad \ldots (14)$$

which is the required general solution of (11)

Again eliminating p between (12) and (13) we get a relation between x and y where no arbitrary constant are involved. This is called *singular solution* of the differential equation (11).

Illustrative Examples.

Ex. 1. *Solve :* $(y - px)(p-1) = p$ *and obtain the singular solution.* [W.B.U.T 2009, 2011]

The given equation can be written as $y - px = \dfrac{p}{p-1}$

i.e., $y = px + \dfrac{p}{p-1}$. ... (i)

Differentiating (i) w.r.t. x, we get

$$p = p.1 + x\frac{dp}{dx} - \frac{1}{(p-1)^2}\frac{dp}{dx}$$

or, $\dfrac{dp}{dx}\left(x - \dfrac{1}{(p-1)^2}\right) = 0.$

Either $\dfrac{dp}{dx} = 0$ i.e., $p = c$, a constant ... (ii)

or, $x - \dfrac{1}{(p-1)^2} = 0$ i.e., $p = \dfrac{1+\sqrt{x}}{\sqrt{x}}$ ... (iii)

Eliminating p between (i) and (ii), we get

$$y = cx + \frac{c}{c-1},$$

which is the required general solution.

Again eliminating p between (i) and (iii), we get

$$y = (1+\sqrt{x})\sqrt{x} + 1 + \sqrt{x}$$

$\therefore \quad (x - y + 1)^2 = 4x$

which is the required singular solution.

Ex. 2. *Obtain the general solution and singular solution of the equation* $y = px + \sqrt{a^2p^2 + b^2}$. [W.B.U.T. 2005, 2013,2017]

The given equation is $y = px + \sqrt{a^2p^2 + b^2}$. ... (i)

Differentiating (i) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{a^2p^2+b^2}} \times a^2 p$$

or, $\frac{dp}{dx}\left(x+\frac{a^2p}{\sqrt{a^2p^2+b^2}}\right)=0$

Either $\frac{dp}{dx}=0$ i.e., $p=c$, a constant. ... (ii)

or, $x+\frac{a^2p}{\sqrt{a^2p^2+b^2}}=0$

$$x=-\frac{a^2p}{\sqrt{a^2p^2+b^2}}. \qquad \text{... (iii)}$$

Eliminating p between (i) and (ii), we get

$y=cx+\sqrt{a^2c^2+b^2}$, which is the required general solution.

Again from (i) and (iii), we have

$$y=-\frac{a^2p^2}{\sqrt{a^2p^2+b^2}}+\sqrt{a^2p^2+b^2}=\frac{b^2}{\sqrt{a^2p^2+b^2}}$$

$$\therefore \quad \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=\left(-\frac{ap}{\sqrt{a^2p^2+b^2}}\right)^2+\left(\frac{b}{\sqrt{a^2p^2+b^2}}\right)^2$$

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

which is the required singular solution.

Ex. 3. *Find the general and singular solution of the equation*

$$py=p^2(x-b)+a.$$

The given equation can be written as

$$y=px+\frac{a}{p}-bp. \qquad \text{... (i)}$$

Differentiating (i) w.r.t. x, we get

$$p=p+x\frac{dp}{dx}-\frac{a}{p^2}\frac{dp}{dx}-b\frac{dp}{dx}$$

or, $\frac{dp}{dx}\left(x-\frac{a}{p^2}-b\right)=0.$

Either $\frac{dp}{dx}=0$ i.e., $p=c$, a constant ... (ii)

or, $x-\frac{a}{p^2}-b=0.$

i.e., $p=\pm\sqrt{\frac{a}{x-b}}$. ... (iii)

Eliminating p from (i) and (ii), we get

$$cy=c^2(x-b)+a,$$

which is the required general solution.

Again eliminating p between (i) and (iii), we get

$$\pm y\sqrt{\frac{a}{x-b}}=\frac{a}{x-b}.(x-b)+a$$

or, $y^2a=4a^2(x-b)$.

$\therefore \quad y^2=4a(x-b)$,

which is the required singular solution.

Ex.4. *Find the general and singular solution of*

$$y=x\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2 \qquad \text{[W.B.U.T. 2006].}$$

The given equation can be written as

$$y=xp+p^2 \text{ where } p=\frac{dy}{dx} \qquad \text{... (i)}$$

which is in Clairaut's form

Differentiating (i) w.r.t x, we get,

$$p=1.p+x\frac{dp}{dx}+2p\frac{dp}{dx}$$

or, $\frac{dp}{dx}(x+2p)=0$

$\therefore$ Either $\frac{dp}{dx}=0$ i.e., $p=c$

or, $x+2p=0$

$\therefore \quad p=-\dfrac{x}{2}$

Putting $p=c$ in (i) we get,

$$y=cx+c^2$$

which is the required general solution

Again putting $p=-\dfrac{x}{2}$ in(i) we get,

$$y=x\left(-\frac{x}{2}\right)+\left(-\frac{x}{2}\right)^2$$

$$=-\frac{x^2}{4}$$

$\therefore \quad x^2+4y=0$

which is the required singular solution.

Ex.5. *Find the general solution of* $p=\cos(y-px)$ *where*

$p=\dfrac{dy}{dx}$ [W.B.U.T. 2007]

The given equation can be written as

$$y-px=\cos^{-1}p$$

$$\therefore \quad y=px+\cos^{-1}p \qquad \ldots \quad \text{(i)}$$

which is in Clairaut's form

Differentiating (i) w.r.t x, we get,

$$\frac{dy}{dx}=p\cdot 1+x\frac{dp}{dx}-\frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

or, $p=p+\dfrac{dp}{dx}\left(x-\dfrac{1}{\sqrt{1-p^2}}\right)$

$$\therefore \quad \frac{dp}{dx}\left(x-\frac{1}{\sqrt{1-p^2}}\right)=0$$

$\therefore$ Either $\dfrac{dp}{dx}=0$ *i.e.,* $p=c$

or, $x-\dfrac{1}{\sqrt{1-p^2}}=0$

or, $x=\dfrac{1}{\sqrt{1-p^2}}$

Putting $p=c$ in (i) we get,

$$y=cx+\cos^{-1}c$$

which is the required general solution.

Note : We can obtain the singular solution as follows :

we have $x=\dfrac{1}{\sqrt{1-p^2}}$

$\therefore \quad 1-p^2=\dfrac{1}{x^2}$

$\therefore \quad p=\pm\dfrac{\sqrt{x^2-1}}{x}$

Putting these values in (i), we get,

$$y=\pm\frac{\sqrt{x^2-1}}{x}\cdot x+\cos^{-1}\left(\pm\frac{\sqrt{x^2-1}}{x}\right)$$

or, $y\mp\sqrt{x^2-1}=\cos^{-1}\left(\pm\dfrac{\sqrt{x^2-1}}{x}\right)$

or, $\pm\frac{\sqrt{x^2-1}}{x}=\cos\left(y\mp\sqrt{x^2-1}\right)$

or, $x^2-1=x^2\cos^2\left(y\pm\sqrt{x^2-1}\right)$

$$x^2\sin^2\left(y\pm\sqrt{x^2-1}\right)=1$$

which is the sigular solution.

Ex.6. *Find general and singular solution of*

$$y=4xp-16y^3p^2$$

Let us make the substitution $x=u,\ y^4=v$

$\therefore\quad du=dx,\ dv=4y^3dy$

$\therefore\quad p=\frac{dy}{dx}=\frac{1}{4y^3}\frac{dv}{du}=\frac{1}{4y^3}q$ where $q=\frac{dv}{du}$

Putting this value of p in the given equation, we get,

$$y=4x\cdot\frac{1}{4y^3}q-16y^3\cdot\frac{1}{16y^6}q^2$$

or, $y^4=xq-q^2$

$$v=uq-q^2\quad\left[\because\ x=u,\ v=y^4\right]\qquad\ldots\text{(i)}$$

which is in Clairaut's form

Differentiating (1) w.r.t.u, we get,

$$\frac{dv}{du}=1.q+u.\frac{dq}{du}-2q\frac{dq}{du}$$

or, $q=q+\frac{dq}{du}(u-2q)=0$

$$\frac{dq}{du}(u-2q)=0$$

Therefore, either $\frac{dq}{du}=0$ *i.e.*, $q=c$

or, $u-2q=0$

$\therefore\quad q=\frac{u}{2}$

Putting $q=c$ in (i) we get,

$$v=uc-c^2$$

$\therefore\quad y^4=cx-c^2$

which is the general solution.

Again putting $q=\frac{u}{2}$ in (i), we get,

$$v=u\cdot\frac{u}{2}-\left(\frac{u}{2}\right)^2$$

or, $v=\frac{u^2}{4}$

and $u^2-4v=0$

$\therefore\quad x^2-4y^4=0$

which is the required singular solution.

Ex.7. *Reduce the equation* $(p-1)e^{3x}+p^3e^{2y}=0$, *into the Clairaut's form and hence obtain the general solution.*

Put $e^x=u,\ e^y=v$

so that $P=\frac{dv}{du}=\frac{dv}{dx}/\frac{du}{dx}=\frac{e^y\frac{dy}{dx}}{e^x}=\frac{v}{u}p$

$\therefore\ p=\frac{u}{v}P$

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Substituting this value of p, e^x, e^y in the given equation we get

$$\left(\frac{u}{v}p-1\right)u^3+\frac{u^3}{v^3}p^3\cdot v^2=0$$

or, $v=uP+P^3$ which is the Clairaut's equation.

Hence its general solution is given by

$$v=cu+c^3$$

i.e., $e^y=ce^x+c^3$.

Ex.8. *Find the general solution and singular solution of* $yp^2-2xp+y=0$ *where* $p=\dfrac{dy}{dx}$

Let us use the substitution $x=u,\ y^2=v$

$\therefore\quad du=dx,\ dv=2ydy$

$$\therefore\quad p=\frac{dy}{dx}=\frac{1}{2y}\frac{dv}{du}$$

$$=\frac{1}{2y}q \text{ where } q=\frac{dv}{du}$$

Putting $p=\dfrac{q}{2y}$ in the given equation, we get,

$$y\cdot\left(\frac{q}{2y}\right)^2-2x\cdot\frac{q}{2y}+y=0$$

or, $q^2-4xq+4y^2=0$

or, $q^2-4uq+4v=0$

$\therefore\quad v=qu-\frac{1}{4}q^2 \qquad \ldots \text{(ii)}$

which is in Clairaut's form

Differentiating (ii) w.r.t. u and using $q=\dfrac{dv}{du}$,

we get

$$q=q.1+u\frac{dq}{du}-\tfrac{1}{4}\cdot 2q\cdot\frac{dq}{du}$$

or, $\dfrac{dq}{du}\left(u-\frac{1}{2}q\right)=0$

$\therefore$ Either $\dfrac{dq}{du}=0$ *i.e.,* $q=c$

or, $u-\frac{1}{2}q=0$

$\therefore\quad q=2u$

Putting $q=c$ in (ii) we get,

$$v=cu-\tfrac{1}{4}c^2$$

i.e., $y^2=cx-\frac{1}{4}c^2$

$\therefore\quad 4y^2=4cx-c^2$

which is the required general solution.

Again putting $q=2u$ in (ii), we get,

$$v=2u.u-\tfrac{1}{4}(2u)^2$$

or $v=u^2$

$\therefore\quad y^2-x^2=0$

which is the required sigular solution.

EXERCISE-D

[I] Short Answer Questions

1. Is the differential equation $y=px+p^5$ reducible ?

2. If $a_0p^3+a_1p^2+a_2p+a_3=0$ can be factorised into