2.3 CAUCHY–EULER EQUATION

2.3.1. Cauchy-Euler Homogeneous Linear Equations.

An equation of the form

$$x^2\frac{d^2y}{dx^2}+P_1x\frac{dy}{dx}-P_2y=X \qquad \ldots (1)$$

i.e., $(x^2D^2+P_1xD+P_2)\,y=X \qquad \ldots (2)$

where P_1, P_2 are constants and X is a function of x alone, is called Cauchy's homogeneous linear equation.

The above equation can be reduced to linear differential equation with constant coefficients by the substitution $x=e^z$

or, $z=\log x$

$\therefore \quad \frac{dz}{dx}=\frac{1}{x}.$

Now $\frac{dy}{dx}=\frac{dy}{dz}.\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}$

$\therefore \quad x\frac{dy}{dx}=\frac{dy}{dz}.$

$\therefore \quad xDy=D'y$ where $D'\equiv\frac{d}{dz}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}.\frac{dz}{dx}$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}.$$

$$x^2\frac{d^2y}{dx^2}=-\frac{dy}{dz}+\frac{d^2y}{dz^2}$$

$$x^2D^2y=D'(D'-1)\,y. \quad \left(\because D'\equiv\frac{d}{dz}\right)$$

Substituting these values in equation (1) or (2), we get a linear differential equation with constant coefficients which can be solved by the methods already discussed.

Note : An equation of the form

$$(a+bx)^2\frac{d^2y}{dx}+P_1(a+bx)\frac{dy}{dx}+P_2y=X$$

where P_1, P_2 are constants and X is a function of x alone, is called Legendre's homogeneous linear equation. The equation can be reduced to the homogeneous linear form by putting $a+bx=e^z$ and then can be solved by the method indicated in the above article.

Illustrative Examples.

Ex. 1. *Solve :* $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\log x\sin(\log x).$

[W.B.U.T. 2002, 2012]

The given equation can be written as

$$(x^2D^2+xD+1)\,y=\log x\sin(\log x) \qquad \ldots (i)$$

Put $x=e^z$, i.e., $z=\log x$

so that $xDy=D'y$, $D'\equiv\frac{d}{dz}$

$x^2D^2y=D'(D'-1)\,y.$

Thus the equation (i) reduces to $[D'(D'-1)+D'+1]\,y=z\sin z$

i.e., $(D'^2+1)\,y=z\sin z. \qquad \ldots (ii)$

Let $y=e^{mz}$ be a trial solution of $(D'^2+1)\,y=0$.

Then the auxiliary equation is $m^2+1=0$.

$\therefore \quad m=\pm i.$

$\therefore \quad$ C. F. $=c_1\cos z+c_2\sin z.$

Now, P. I. $=\frac{1}{D'^2+1}z\sin z$

$$= \left\{ z - \frac{1}{D'^2+1}.2D' \right\} \frac{1}{D'^2+1} \sin z,$$

$$= \left\{ z - \frac{1}{D'^2+1}.2D' \right\} (-) \frac{z}{2} \cos z$$

$$\left[\because \frac{1}{D'^2+1} \sin z = z \frac{1}{2D'} \sin z = -\frac{z}{2} \cos z \right]$$

$$= -\frac{1}{2} z^2 \cos z + \frac{1}{D'^2+1} D' (z \cos z)$$

$$= -\frac{1}{2} z^2 \cos z + \frac{1}{D'^2+1} (\cos z - z \sin z)$$

$$= -\frac{1}{2} z^2 \cos z + \frac{1}{D'^2+1} \cos z - \frac{1}{D'^2+1} (z \sin z)$$

$$= -\frac{1}{2} z^2 \cos z + z. \frac{1}{2D'} \cos z - \text{P. I.}$$

$$\therefore \quad 2 \times \text{P. I.} = -\frac{1}{2} z^2 \cos z + \frac{1}{2} z \sin z$$

$$\therefore \quad \text{P. I.} = -\frac{1}{4} z^2 \cos z + \frac{1}{4} z \sin z.$$

So the general solution of (ii) is

$$y = c_1 \cos z + c_2 \sin z - \frac{1}{4} z^2 \cos z + \frac{1}{4} z \sin z.$$

Therefore the general solution of the given equation is

$$y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{1}{4} (\log x)^2 \cos(\log x) + \frac{1}{4} \log x \sin(\log x)$$

where c_1, c_2 are arbitrary constants.

Ex. 2. *Solve :* $x^2 \dfrac{d^2 y}{dx^2} + 4x \dfrac{dy}{dx} + 2y = \log x.$

The given equation can be written as

$$(x^2 D^2 + 4xD + 2)\, y = \log x. \qquad \ldots \quad \text{(i)}$$

Put $x = e^z$, i.e., $z = \log x$.

Then $xD = D'$, $\quad x^2 D^2 = D'(D'-1)$, $D' \equiv \dfrac{d}{dz}$

So (i) reduces to $[D'(D'-1) + 4D' + 2]\, y = z$

or, $\left(D'^2 + 3D' + 2\right) y = z. \qquad \ldots \quad$ (ii)

Let $y = e^{mz}$ be a trial solution of $(D'^2 + 3D' + 2)\, y = 0.$

Then the auxiliary equation is $m^2 + 3m + 2 = 0.$

or, $(m+2)(m+1) = 0.$

$\therefore \quad m = -1, -2$

$\therefore \quad$ C. F. $= c_1 e^{-z} + c_2 e^{-2z}.$

$$\text{Now, P. I.} = \frac{1}{D'^2 + 3D' + 2} z = \frac{1}{2} \left(1 + \frac{D'^2 + 3D'}{2} \right)^{-1} z$$

$$= \frac{1}{2} \left(1 - \frac{D'^2 + 3D'}{2} \right) z = \frac{1}{2} \left(z - \frac{3}{2} \right).$$

So the general solution of (ii) is

$$y = c_1 e^{-z} + c_2 e^{-2z} + \frac{1}{2} \left(z - \frac{3}{2} \right).$$

Therefore the general solution of the given equation is

$$y = c_1 x^{-1} + c_2 x^{-2} + \frac{1}{2} \left(\log x - \frac{3}{2} \right),$$

where c_1, c_2 are arbitrary constants.

Ex. 3. *Solve* $: (x^2 D^2 - xD - 3)\, y = x^2 \log x.$

[*W.B.U.T. 2015, 2014,2008*]

Let $x = e^z$, i.e., $z = \log x$.

Then $xD = D'$, $x^2 D^2 = D'(D'-1)$, $D' \equiv \dfrac{d}{dz}$

So the given equation reduces to

$$[D'(D'-1) - D' - 3]\, y = z e^{2z}$$

or, $\left(D'^2 - 2D' - 3\right) y = z e^{2z}. \qquad \ldots \quad$ (i)

Let $y = e^{mz}$ be a trial solution of $\left(D'^2 - 2D' - 3\right) y = 0$.

Then the auxiliary equation is $m^2 - 2m - 3 = 0$.

or, $(m-3)(m+1) = 0$.

$\therefore \quad m = -1, 3$

$\therefore \quad$ C. F. $= c_1 e^{-z} + c_2 e^{3z}$.

Now, P. I. $= \dfrac{1}{D'^2 - 2D' - 3} ze^{2z}$

$$= e^{2z} \frac{1}{(D'+2)^2 - 2(D'+2) - 3} . z$$

$$= e^{2z} \frac{1}{D'^2 + 2D' - 3} . z$$

$$= -\frac{1}{3} e^{2z} \left(1 - \frac{D'^2 + 2D'}{3}\right)^{-1} z$$

$$= -\frac{1}{3} e^{2z} \left(1 + \frac{D'^2 + 2D'}{3} + \cdots\right) z$$

$$= -\frac{1}{3} e^{2z} \left(z + \frac{2}{3}\right) = -\frac{1}{9} e^{2z} (3z + 2).$$

So the general solution of (i) is

$$y = c_1 e^{-z} + c_2 e^{3z} - \frac{1}{9} e^{2z} (3z + 2).$$

Therefore the general solution of the given equation is

$$y = c_1 x^{-1} + c_2 x^3 - \frac{1}{9} x^2 (3 \log x + 2),$$

where c_1, c_2 are arbitrary constants.

Ex. 4. *Solve :* $\left(x^2 D^2 - xD + 4\right) y = x \sin \log x$ [W.B.U.T 2005, 2011]

Let $x = e^z$ *i.e.,* $z = \log x$

$$\therefore \quad xD = D', \; x^2 D^2 = D'(D'-1), \; D' \equiv \frac{d}{dz}$$

Then the given equation reduces to

$$(D'(D'-1) - D' + 4) y = e^z \sin z$$

or, $(D'^2 - 2D' + 4) y = e^z \sin z \qquad \ldots$ (i)

Let $y = e^{mz}$ be a trial solution of

$$\left(D'^2 - 2D' + 4\right) y = 0$$

$\therefore \quad$ The auxiliary equation is $m^2 - 2m + 4 = 0$

$$\therefore \quad m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4.1.4}}{2.1}$$

$$= 1 \pm \sqrt{3} i$$

$\therefore \quad$ C.F. $= e^z \left(c_1 \cos \sqrt{3} z + c_2 \sin \sqrt{3} z\right)$

Now, P.I. $= \dfrac{e^z \sin z}{D'^2 - 2D' + 4}$

$$= e^z \frac{\sin z}{(D'+1)^2 - 2(D'+1) + 4}$$

$$= e^z \frac{\sin z}{D'^2 + 3}$$

$$= \tfrac{1}{2} e^z \sin z$$

So the general solution of (i) is

$$y = e^z \left(c_1 \cos \sqrt{3} z + c_2 \sin \sqrt{3} z\right) + \tfrac{1}{2} e^z \sin z$$

Therefore the general solution of the given equation is

$$y = x \left\{c_1 \cos\left(\sqrt{3} \log x\right) + c_2 \sin\left(\sqrt{3} \log x\right)\right\} + \tfrac{1}{2} x \sin \log x$$

where c_1, c_2 are arbitrary constants.

Ex. 5. *Solve :* $(3x+2)^2\dfrac{d^2y}{dx^2}+3(3x+2)\dfrac{dy}{dx}-36y=3x^2+4x+1.$

Put $3x+2=e^z$ i.e., $z=\log(3x+2)$

so that $(3x+2)\dfrac{dy}{dx}=3D'y$,

$(3x+2)^2\dfrac{d^2y}{dx^2}=3^2D'(D'-1)\,y$ where $D'\equiv\dfrac{d}{dz}$

So the given equation reduces to

$$\left[3^2D'(D'-1)+3.3D'-36\right]y=3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1$$

or, $9(D'^2-4)\,y=\dfrac{1}{3}e^{2z}-\dfrac{1}{3}$

or, $(D'^2-4)\,y=\dfrac{1}{27}(e^{2z}-1).$ . . . (i)

Let $y=e^{mz}$ be a trial solution of $(D'^2-4)\,y=0.$

Then the auxiliary equation is $m^2-4=0.$

$\therefore\quad m=\pm 2.$

$\therefore\quad C.F.=c_1e^{2z}+c_2e^{-2z}.$

Now P. I. $=\dfrac{1}{D'^2-4}\dfrac{1}{27}(e^{2z}-1)$

$$=\frac{1}{27}\left(\frac{1}{D'^2-4}e^{2z}-\frac{1}{D'^2-4}e^{0.z}\right)$$

$$=\frac{1}{27}\left[z.\frac{1}{2D'}e^{2z}-\frac{1}{0-4}\right]$$

$$=\frac{1}{27}\left[\frac{z}{2}.\frac{e^{2z}}{2}+\frac{1}{4}\right]$$

$$=\frac{1}{108}\left(ze^{2z}+1\right).$$

So the general solution of (i) is

$$y=c_1e^{2z}+c_2e^{-2z}+\frac{1}{108}(ze^{2z}+1).$$

Therefore the general solution of the given equation is

$$y=c_1(3x+2)^2+c_2(3x+2)^{-2}+\frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right],$$

where c_1, c_2 are arbitrary constants.

Ex. 6. *Solve :* $\left(x^2D^2-3xD+4\right)y=x$ *given that* $y=0$ *when* $x=1$ *and* $y=e^2$ *when* $x=e$

Let $x=e^z$, *i.e.*, $z=\log x$

$\therefore\quad xD=D'$, $x^2D^2=D'(D'-1)$, $D'\equiv\dfrac{d}{dz}$

So the given equation reduces to

$$\left[D'(D'-1)-3D'+4\right]y=e^z$$

or, $\left(D'^2-4D'+4\right)y=e^z$ . . . (i)

Let $y=e^{mz}$ be a trial solution of

$$\left(D'^2-4D'+4\right)y=0$$

The the auxiliary equation is $m^2-4m+4=0$

or, $(m-2)^2=0$

$\therefore\quad m=2,2$

$\therefore\quad$ C.F. $=(c_1+c_2z)e^{2z}$

Now P.I. $=\dfrac{e^z}{D'-4D'+4}$

$$=\frac{e^z}{(D'-2)^2}$$

$$= \frac{e^z}{(1-2)^2}$$

$$= e^z$$

So the general solution of (i) is

$$y = (c_1 + c_2 z)e^{2z} + e^z$$

$\therefore$ Thus the general solution of the given equation is

$$y = (c_1 + c_2 \log x)x^2 + x$$

Given $y = 0$ when $x = 1$

$\therefore \quad 0 = (c_1 + c_2 . 0) . 1^2 + 1$

$\therefore \quad c_1 = -1$

Also given $y = e^2$ when $x = e$

$\therefore \quad e^2 = (c_1 + c_2 \log e)e^2 + e$

$\quad = (-1 + c_2)e^2 + e$

or, $\quad -1 + c_2 = \frac{e-1}{e}$

$\therefore \quad c_2 = \frac{e-1}{e} + 1 = 2 - \frac{1}{e}$

Hence the required solution is

$$y = \left\{-1 + \left(2 - \frac{1}{e}\right)\log x\right\}x^2 + x$$

$$\therefore \quad y = x - x^2 + \left(2 - \frac{1}{e}\right)\log x$$

EXERCISE

[I] Short Answer Questions

1. Convert the differential equation $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$ to another with constant coefficient.

2. Convert the differential equation $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \sin(\log x)$ to a differential equation with constant coefficient.

3. Convert the differential equation $x^2D^2 + 2xD + 2y = 10\left(x + \frac{1}{x}\right)$ to another form with constant coefficient.

4. Convert the differential equation

$$(1+2x)^2\frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$$

to a form with constant coefficient.

5. Transform the differential equation $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = e^x$ to a form with constant coefficient.

6. Find the general solution of the differential equation

$$(5+2x)\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + \frac{8y}{5+2x} = 0.$$

7. Solve $x\frac{d^2y}{dx^2} = 2\frac{dy}{dx}$

Answers

1. $\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + y = \frac{1}{(1-e^z)^2}$ 2. $\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 5y = e^{2z}\sin z$

3. $\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + 2y = 10(e^z + e^{-z})$ 4. $\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 2e^{2z}$

5. $\frac{d^2y}{dz^2} + \frac{dy}{dz} = e^{z+e^z}$ 6. $y = (5+2x)^2\left\{A(5+2x)^{\sqrt{2}} + B(5+2x)^{-\sqrt{2}}\right\}$

7. $y = \frac{1}{3}Ax^3 + B$

[II] Long Answer Questions

Solve the following equations :

1. $x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$.