The μ-test for convergence:

Let $f(x)$ be an integrable function when $x \geq a$. Then $F = \int_a^{\infty} f(x)\,dx$ * converges absolutely if

$$\lim_{x \to \infty} x^{\mu} f(x) = \lambda \ , \quad \mu > 1$$

and F diverges if

$$\lim_{x \to \infty} x^{\mu} f(x) = \lambda\,(\neq 0) \text{ or } \pm\infty \ , \quad \mu \leq 1.$$

The Gamma Function:

The 'gamma' function denoted by $\Gamma(x)$ is defined by,

$$\Gamma(x) = \int_0^{\infty} e^{-t} \cdot t^{x-1}\,dt \ , \quad x > 0.$$

Let us now discuss the convergence of $\Gamma(x)$.

We write $f(t) = e^{-t} \cdot t^{x-1}$; $I_1 = \int_0^1 e^{-t} t^{x-1}\,dt$; $I_2 = \int_1^{\infty} e^{-t} \cdot t^{x-1}\,dt$

The part I_1 is proper when $x \geq 1$, improper but absolutely convergent when $0 < x < 1$; for as $t \to 0^+$, $[e^{-t} \cdot t^{x-1} \to \infty$ as $t \to 0^+]$ by μ-test

$$t^{1-x} \cdot f(t) = t^{1-x} \cdot e^{-t} \cdot t^{x-1} = e^{-t} \to 1$$

for $0 < \mu = 1-x < 1$, for $0 < x < 1$.

The part I_2 also converges absolutely for all values of n by μ-test, for as $t \to \infty$

$$t^{r} \cdot f(t) = t^{r} e^{-t} \cdot t^{n-1} = e^{-t} \cdot t^{n+1} \to 0.$$

Thus $\Gamma(x)$ converges for $n > 0$.

* See page 4

** $\Gamma(x)$ is continuous and differentiable in $x>0$.

Relation 1:

$$\int_0^\infty e^{-at}\cdot t^{x-1}\,dt = \frac{\Gamma(x)}{a^x}, \quad x>0.$$

Proof:
Put $at=u$, then

$$\int_\epsilon^B e^{-at}\cdot t^{x-1}\,dt = \int_{a\epsilon}^{aB} e^{-u}\cdot \frac{u^{x-1}}{a^{x-1}}\cdot \frac{du}{a}.$$

As $\epsilon\to 0^+$ and $B\to\infty$

$$\int_0^\infty e^{-at}\cdot t^{x-1}\,dt = \frac{1}{a^x}\int_0^\infty e^{-u}\cdot u^{x-1}\,du = \frac{\Gamma(x)}{a^x}.$$

Relation 2:

$$\Gamma(x+1) = x\,\Gamma(x), \quad x>0.$$

Proof: An integration by parts gives

$$\int_\epsilon^B \overbrace{e^{-t}}^{u}\cdot \overbrace{t^{x-1}\,dt}^{dv} = \left[e^{-t}\cdot \frac{t^x}{x}\right]_\epsilon^B + \frac{1}{x}\int_\epsilon^B e^{-t}\cdot t^x\,dt$$

As $B\to\infty$ and $\epsilon\to 0^+$, the integrand part vanishes at both limits and therefore

$$\int_0^\infty e^{-t}\cdot t^{x-1}\,dt = \frac{1}{x}\int_0^\infty e^{-t}\cdot t^x\,dt$$

i.e. $\Gamma(x) = \frac{1}{x}\Gamma(x+1)$ or, $\Gamma(x+1) = x\,\Gamma(x)$.

Relation 3: $\Gamma(1)=1$.

Proof: ~~Since~~ $\Gamma(1) = \int_0^\infty e^{-t}\,dt = \lim_{B\to\infty}\int_0^B e^{-t}\,dt = \lim_{B\to\infty}(1-e^{-B}) = 1$

Relation 4 :

$\Gamma(n+1) = n!$, n being a positive integer.

Proof: Combining relations (2) and (3), when n is a positive integer,

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 3\cdot 2\cdot 1\cdot \Gamma(1)$$
$$= n!$$

Relation 5 :

Show that $\Gamma(x) > \frac{1}{e}\int_0^1 t^{x-1}\,dt = \frac{1}{ex}$, $x > 0$.

Hence deduce that $\Gamma(0+) = \lim_{x\to 0^+} \Gamma(x) = \infty$.

Proof:

Since the integrand of $\Gamma(x) = \int_0^\infty e^{-t}\cdot t^{x-1}\,dt$, $x>0$ is positive, we have

$$\Gamma(x) > e^{-1}\int_0^1 t^{x-1}\,dt = e^{-1}\left[\frac{t^x}{x}\right]_0^1 = \frac{1}{ex}, \quad x>0.$$

and therefore, when $x \to 0^+$, $\Gamma(x) \to \infty$.

Examples :

1. (i) $\Gamma(4) = 3\cdot 2\cdot 1\cdot \Gamma(1) = 3\cdot 2\cdot 1\cdot 1 = 6$.

   (ii) $\Gamma(5/4) = \Gamma\left(\frac{1}{4}+1\right) = \frac{1}{4}\Gamma(1/4)$

   (iii) $\Gamma(8/3) = \Gamma(5/3+1) = 5/3\,\Gamma(5/3) = 5/3\cdot 2/3\cdot \Gamma(2/3)$.

2. Show that $\int_0^\infty e^{-x^2}\cdot x^9\,dx = 12$.

Soln. We put $x^2 = z$. i.e. $x = \sqrt{z}$.

i.e. $dx = \frac{1}{2}\frac{dz}{\sqrt{z}}$.

$$\therefore \int_0^\infty e^{-x^2} \cdot x^9 \, dx = \int_0^\infty e^{-z} \cdot z^{9/2} \cdot \frac{1}{2} \cdot \frac{dz}{\sqrt{z}}$$

$$= \frac{1}{2} \int_0^\infty e^{-z} \cdot z^4 \, dz = \frac{1}{2} \cdot \Gamma(5) = 12.$$

3. Prove that $\displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

<u>Proof:</u> Let $u = x^2$

$\therefore du = 2x\,dx$. i,e. $dx = \frac{1}{2} \cdot \frac{1}{x} du = \frac{1}{2} \cdot \frac{1}{\sqrt{u}} du$.

$$\therefore \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-u} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{u}} \, du.$$

$$= \frac{1}{2} \int_0^\infty e^{-u} \cdot u^{1/2 - 1} \, du$$

$$= \frac{1}{2} \Gamma(1/2).$$

$$= \frac{\sqrt{\pi}}{2} \quad \left[\text{Since } \Gamma(1/2) = \sqrt{\pi}\right].$$

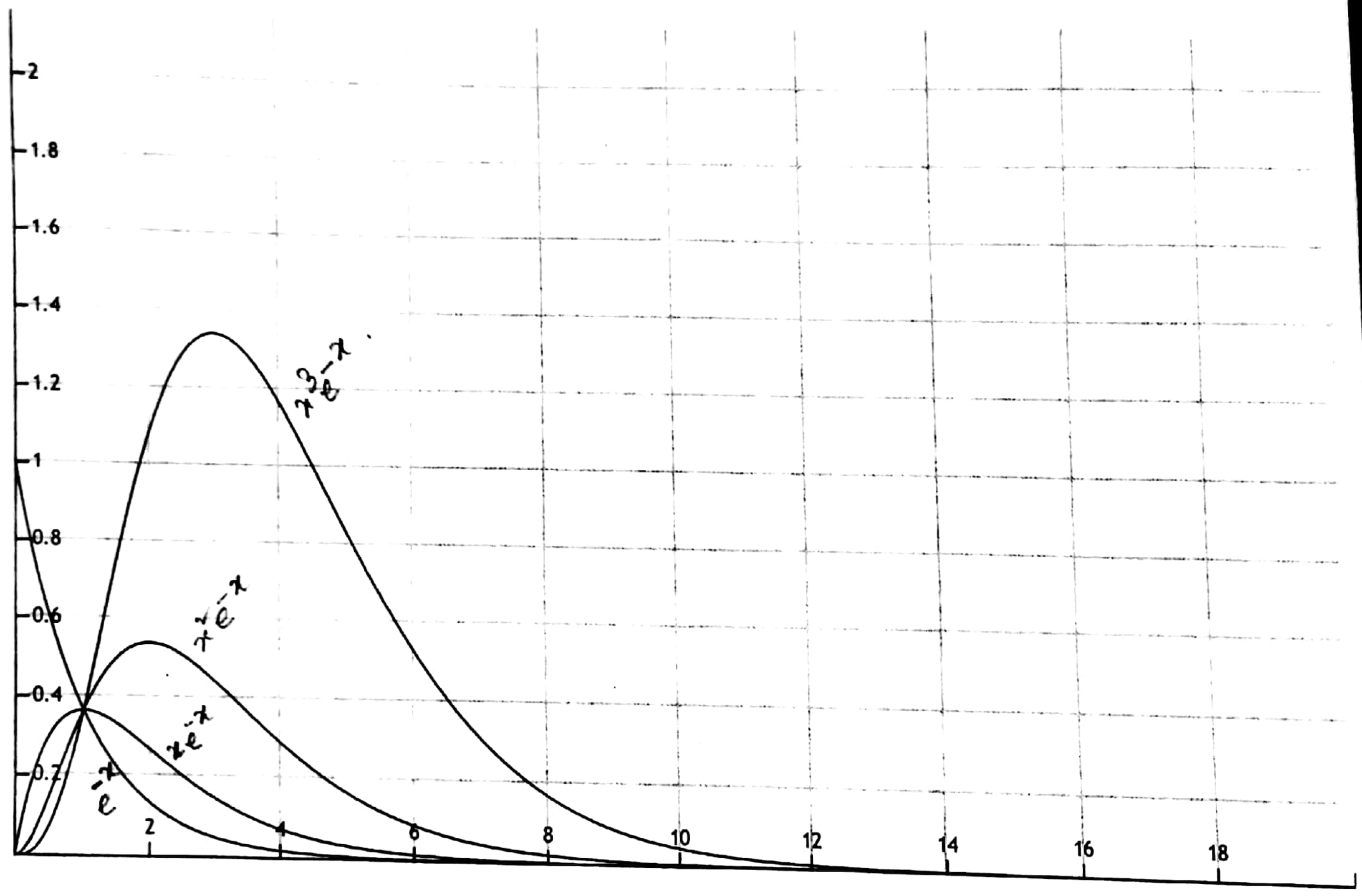
* Let $f(x)$ be an integrable function in the arbitrary interval $(a+\epsilon, b)$ where $0 < \epsilon < b-a$. Then

$F = \int_a^b f(x)\,dx$, converges absolutely if

$$\lim_{x \to a^+} (x-a)^\mu f(x) = \lambda \quad \text{for } 0 < \mu < 1$$

and F diverges if

$$\lim_{x \to a^+} (x-a)^\mu f(x) = \lambda\,(\neq 0) \text{ or } \pm\infty \quad \text{for } \mu \geq 1.$$


2
1.8
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18
$x^3 e^{-x}$
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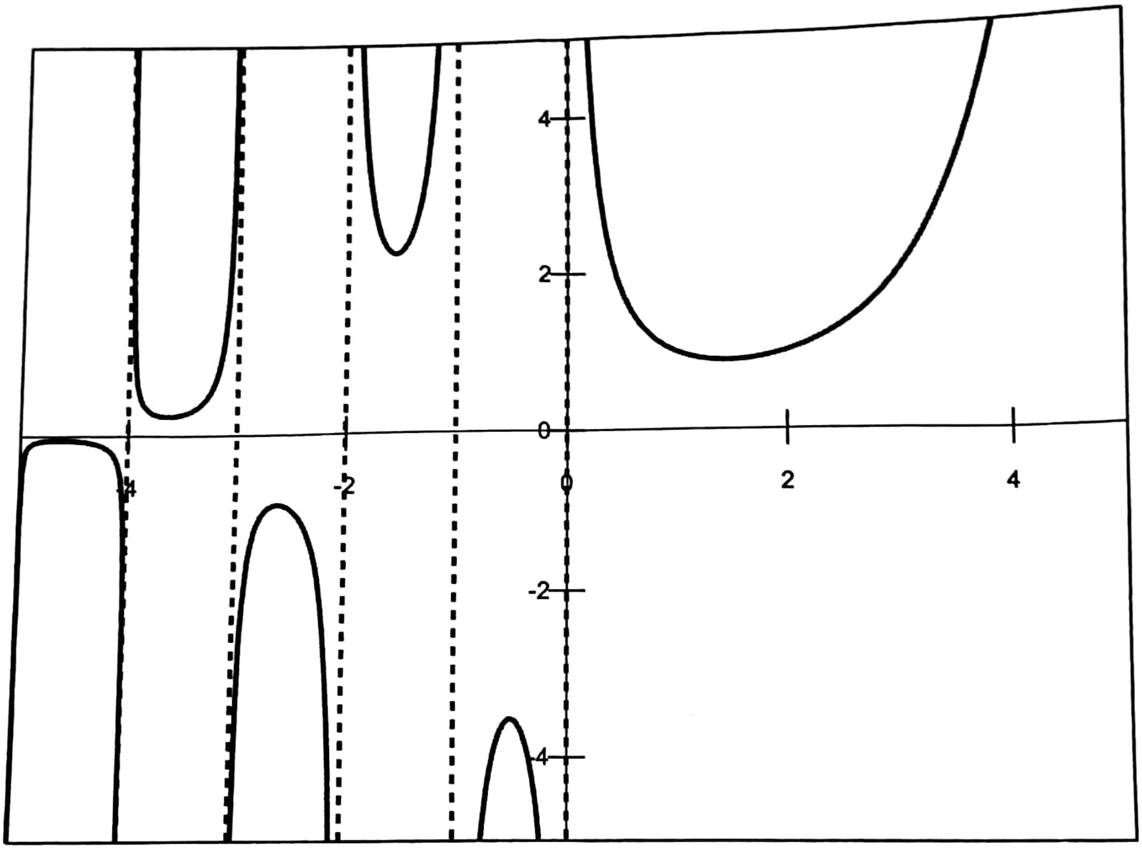
$x e^{-x}$
e^{-x}

Gamma function



The Beta Function:

The beta function denoted by $B(x,y)$ is defined for positive values of x and y by the integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt\,; \quad x,y>0.$$

* $B(x,y)$ is continuous for $x>0, y>0$.

• **Relation 1:** $B(x,y) = B(y,x)$

Proof: Hint: In the definition of $B(x,y)$, put $t=1-u$.

• **Relation 2:** $$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt = \int_0^\infty \frac{t^{y-1}}{(1+t)^{x+y}}\,dt\,; \quad x,y>0.$$

Proof: Hint: Put $t = \frac{1}{1+u}$.

• **Relation 3:**

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta\,; \quad x,y>0.$$

Proof: Hint: Put $t=\sin^2\theta$.

• **Relation 4:** $B(\frac{1}{2},\frac{1}{2}) = \pi$

Proof: Hint: Put $x=\frac{1}{2}=y$.

* Relation between Beta and Gamma Function

Relation 5: $$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,; \quad x,y>0.$$

Proof: Sorry, beyond the scope of this discussion, if interested meet me in my cabin.

Relation 6: $\Gamma(1/2) = \sqrt{\pi}$.

Proof: From relations 4 and 5, we can write

$$B(1/2, 1/2) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} = \pi.$$

$$\Rightarrow \Gamma(1/2) = \sqrt{\pi}, \text{ since } \Gamma(1) = 1.$$

Relation 7:

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \frac{1}{2} \frac{\Gamma(m+1/2)\Gamma(n+1/2)}{\Gamma\left(\frac{m+n+2}{2}\right)}; m,n>-$$

Proof: In relation 3, put $2x-1 = m$, $2y-1 = n$.

Relation 8: $\Gamma(x) = 2\int_0^\infty e^{-t^2} t^{2x-1} \, dt$, $x > 0$.

Put: $t = u^2$ in the definition of $\Gamma(x)$.

Relation 9:

$$\int_0^\infty e^{-t^2} dt = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}. ; \int_{-\infty}^\infty e^{-t^2} dt = \sqrt{\pi}.$$

Put $x = 1/2$ in relation 8.

Relation 10:

$$\int_0^{\pi/2} \sin^m x \, dx = \int_0^{\pi/2} \cos^m x \, dx = \sqrt{\pi}/2 \cdot \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}, m > -1$$

Put $n = 0$, in relation 7.

Duplication Formula:

Prove that $\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\,\Gamma(x)\,\Gamma(x+1/2)$, $x > 0$.

Proof: We have from relation 5, for $x, \cancel{y} > 0$

$$B(x, \cancel{y}x) = \frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)} = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2x-1}\theta \, d\theta \quad \text{—(1)}$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2x-1} d\theta = \frac{2}{2^{2x-1}}\int_0^{\pi/2} \sin^{2x-1} 2\theta \, d\theta$$

$$= \frac{1}{2^{2x-1}}\int_0^{\pi} \sin^{2x-1}\phi \, d\phi \quad ; \text{if we put } 2\theta = \phi$$

$$= \frac{2}{2^{2x-1}}\int_0^{\pi/2} \sin^{2x-1}\phi \, d\phi \quad ; \text{since } \sin^{2x-1}(\pi-\phi) = \sin^{2x-1}\phi \quad \text{—(2)}$$

Now put $y = 1/2$ in (1) relations (5) and (3)

$$\frac{\Gamma(x)\Gamma(1/2)}{\Gamma(x+1/2)} = 2\int_0^{\pi/2} \sin^{2x-1}\theta \, d\theta \quad \text{—(3)}$$

From, (2) and (3)

$$\frac{\Gamma(x)\Gamma(x)}{\Gamma(2x)} = \frac{1}{2^{2x-1}} \cdot \frac{\Gamma(x).\Gamma(1/2)}{\Gamma(x+1/2)} = \frac{\sqrt{\pi}}{2^{2x-1}} \cdot \frac{\Gamma(x)}{\Gamma(x+1/2)}. \quad \square$$

This formula is known as Legendre Duplication formula, which plays an important role in finding the relationship between Reimann Zeta function and gamma function.

Relation 11:

$$\Gamma(m)\,\Gamma(1-m) = \pi \operatorname{cosec} m\pi \,, \quad 0 < m < 1.$$

Example:

1. Evaluate: $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$.

Solⁿ: In $\int_\epsilon^B \sqrt{x} \cdot e^{-x^3} dx$ put $x^3 = z$, then

$$\int_\epsilon^B \sqrt{x} \cdot e^{-x^3} dx = \int_{\epsilon^3}^{B^3} e^{-z} \cdot \frac{1}{3} z^{-1/2} dz$$

On letting $\epsilon \to 0^+$ and $B \to \infty$

$$\int_0^\infty \sqrt{x}\, e^{-x^3} dx = \frac{1}{3} \int_0^\infty e^{-x} \cdot x^{-1/2} dx = \frac{1}{3} \Gamma(1/2) = \frac{\sqrt{\pi}}{3}.$$

2. $$\int_0^{\pi/2} \sin^4 x \cos^4 x \, dx = \frac{1}{2} \cdot \frac{\Gamma(5/2) \cdot \Gamma(5/2)}{\Gamma(5)} = \frac{1}{2} \cdot \frac{3/2 \cdot 1/2\, \Gamma(1/2) \cdot 3/2 \cdot \frac{1}{2} \Gamma(1/2)}{4 \cdot 3 \cdot 2 \cdot 1 \cdot \Gamma(1)}$$

$$= \frac{3\pi}{256}.$$

3. $$\int_0^1 x^3 (1-x^2)^{5/2} dx = \int_0^{\pi/2} \sin^3\theta \cos^6\theta \, d\theta$$

$$= 1/2 \cdot \frac{\Gamma(2) \cdot \Gamma(7/2)}{\Gamma(11/2)} = 2/63.$$

4. P.T. $$\int_0^1 \sqrt{1-x^4}\, dx = \left\{ \frac{(\Gamma(1/4))^2}{6\sqrt{2\pi}} \right\}$$

Solⁿ: Put $x^4 = u$, then

$$I = 1/4 \int_0^1 u^{-3/4} (1-u)^{1/2} du = \frac{1}{4} B(1/4, 3/2)$$

$$= \frac{1}{4} \frac{\Gamma(1/4)\Gamma(3/2)}{\Gamma(7/4)} = \frac{1}{4} \frac{\Gamma(1/4) \frac{1}{2}\Gamma(1/2)}{3/4\, \Gamma(3/4)}$$

$$= \sqrt{\pi}/6 \, \frac{(\Gamma(1/4))^2}{\Gamma(1/4) \cdot \Gamma(3/4)} = \frac{\{\Gamma(1/4)\}^2}{\pi \operatorname{cosec} \pi/4} = \frac{\{\Gamma(1/4)\}^2}{6\sqrt{2\pi}}.$$ (Rela 11)

Assignment :

1. Show that :

(i) $B(m,n)\,B(m+n,l) = B(n,l)\,B(n+l,m) = B(l,m)\,B(l+m,n)$

(ii) $\displaystyle\int_0^1 x^{m-1}(1-x^p)^{n-1}\,dx = \frac{1}{p}B(m/p, n)$; $m,n,p>0$

(iii) $\displaystyle\int_0^{\pi/2}\sqrt{\tan x}\,dx = \pi/\sqrt{2}$.

(iv) $\displaystyle\int_0^{\pi/2}\sqrt{\cos^7 x\,\sin^9 x}\,dx = \frac{35\sqrt{2}}{4096}\pi$.

(v) $\displaystyle\int_0^{\pi/2}\frac{dx}{\sqrt{\sin x}}\times\int_0^{\pi/2}\sqrt{\sin x}\,dx = \pi$.

(vi) $\displaystyle\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}B(m,n)$; $m,n>0$.

(vii) $\displaystyle\int_0^1 x^{m-1}(\log x)^{n}\,dx = \frac{(-1)^n}{m^{n+1}}\Gamma(n+1)$; $m>0,\ n>-1$.

(viii) $\displaystyle\int_0^1\frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}\,\Gamma(1/4)}{4\cdot\Gamma(3/4)}$.

(ix) $\displaystyle\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx = \frac{1}{9^{1/3}}B(2/3, 1/3)$.

2. Using $\sin\frac{\pi}{a}\cdot\sin\frac{2\pi}{a}\cdots\sin\frac{(a-1)\pi}{a} = \frac{a}{2^{a-1}}$, $a\in\mathbb{Z}^+$ and $a>1$

prove that,

i) $\Gamma(1/a)\cdot\Gamma(2/a)\cdots\Gamma\left(\frac{a-1}{a}\right) = \left\{\frac{(2\pi)^{a-1}}{a}\right\}^{1/2}$.

ii) $\Gamma(1/9)\cdot\Gamma(2/9)\cdots\Gamma(8/9) = \frac{16}{3}\pi^4$.