Gamma function :-

6/2/18

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt \,, \quad x > 0.$$

Relation 1 :

$$\boxed{\int_0^\infty e^{-at}\, t^{x-1}\, dt = \frac{\Gamma(x)}{a^x}\,, \quad x>0}$$

$at = u$, $dt = \frac{1}{a} du$.

$$\int_0^\infty e^{-u} \left(\frac{u}{a}\right)^{x-1} \cdot \frac{1}{a}\, du = \frac{1}{a^x}\int_0^\infty e^{-u}\, u^{x-1}\, du$$

$$= \frac{1}{a^x}\,\Gamma(x).$$

Relation 2 :

$$\Gamma(x+1) = x \cdot \Gamma(x).$$

$$\Gamma x = \int_0^\infty \underbrace{e^{-t}}_{u}\, \underbrace{t^{x-1}\, dt}_{du}$$

$$= \left[e^{-t} \frac{t^{x-1+1}}{x}\right]_0^B + \int_0^B \frac{t^{x-1+1}}{x}\, e^{-t}\, dt$$

$$= \lim_{B\to\infty} \left\{ \left[e^{-t}\cdot \frac{t^x}{x}\right]_0^B + \int_0^B e^{-t}\frac{t^x}{x}\, dt \right\}.$$

$$= \lim_{B\to\infty} \left[e^{-B} \frac{B^x}{x}\right] + \frac{1}{x}\int_0^\infty e^{-t}\, t^x\, dt.$$

$$\left(-e^{-t}\right)_0^\infty = \left(-\frac{1}{e^\infty} + \frac{1}{e^0}\right) = 1$$

$$= \frac{1}{x}\,\Gamma(x+1)$$

$$\Rightarrow x\,\Gamma(x) = \Gamma(x+1)$$

Relation 3 :

$$\Gamma(1) = 1.$$

$\Gamma(n+1) = n\Gamma(n)$

where n is a +ve integer.

$\frac{1}{e^9} - \frac{1}{e^{10}} = \frac{1}{8103}$

$$\Gamma(n+1) = n\Gamma(n)$$
$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\Gamma(n-2)$$
$$\vdots$$
$$= n(n-1)(n-2)\cdots 2\cdot 1\,\Gamma(1)$$
$$= n!$$

eg: $\Gamma(8/3) = \Gamma\left(1+\frac{5}{3}\right)$
$$= \frac{5}{3}\Gamma\left(\frac{5}{3}\right) = \frac{5}{3}\Gamma\left(1+\frac{2}{3}\right)$$
$$= \frac{5}{3}\cdot\frac{2}{3}\cdot\Gamma\left(\frac{2}{3}\right).$$

eg:- S.t. $\int_0^\infty e^{-x^2}\cdot x^9\,dx = 12.$

$x^2 = t.$
$2x\,dx = dt.$

$$\Gamma(5) = \frac{1}{2}\int_0^\infty e^{-t}\, t^4\,dt.$$

$$\Gamma(5) = \frac{1}{2}4! = \frac{24}{2} = 12.$$

eg: p.t. $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$

$x^2 = t.\quad 2x\,dx = dt.$

$$\frac{1}{2}\int_0^\infty e^{-t}\cdot\frac{1}{\sqrt{t}}\,dt = \frac{1}{2}\int_0^\infty e^{-t}\,t^{-1/2}\,dt.$$

$$\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\int_0^\infty e^{-t}\,t^{\left(\frac{1}{2}-1\right)}\,dt.$$

$$= \frac{\sqrt{\pi}}{2}\ (\text{from } \beta \text{ func}^n)$$

Beta function :-

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt\ , \quad x,y>0.$$

(improper integral)

(i.e. at $t=0$ or 1 and $x/y=1$ 0^0 comes)

Relation 1 :

$$\beta(x,y) = \beta(y,x).$$

Relation 2 :

$$\beta(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta \cdot \quad x,y>0$$

Relation 3 :

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Relation 4 :

$$\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \pi = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(1)} = \frac{\left(\Gamma(\frac{1}{2})\right)^2}{1}$$

$$\therefore \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

Evaluate :

$$\int_0^{\pi/2} \sin^4 x \cos^4 x\,dx. \qquad \begin{aligned} 2x-1 &= 4 \\ x &= 5/2 \end{aligned}$$

$$\frac{1}{2}\beta\left(5/2, 5/2\right) = \int_0^{\pi/2} \sin^4 x \cos^4 x \,dx.$$

$$= \frac{1}{2}\,\frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{5}{2})}{\Gamma(5)} = \frac{1}{2}\,\frac{\left(\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\right)^2}{24}.$$

$$= \frac{1}{2}\cdot\frac{9\times 1}{4\times 4\times 24}\,\pi = \frac{3\pi}{256}$$

Laplace Transformation

Dt: 13/2/18

$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$; $s>0$. (the name is because it transforms t funⁿ to s funⁿ after integraⁿ)

$F(s)$

Property :-

L.T of Standard functions

1. $L(1) = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$

2. $L(e^{-at}) = \int_0^\infty e^{-st} \cdot e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt$

$= \lim_{M\to\infty} \frac{1}{(a-s)} \left[e^{(a-s)t}\right]_0^M = \frac{1}{(s-a)}$ (if $s>a$)

3. $L(t^n) = \int_0^\infty e^{-st} t^n\,dt$

$= \cancel{\left(\frac{t^n}{-s} e^{-st} - \int t^n e^{-st} dt\right)}$

$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{1}{s}\,dx$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\,dx$

$= \frac{\Gamma(n+1)}{s^{n+1}}$

4. $L(\sin at) = \int_0^\infty e^{-st} \sin at\,dt$ $\quad st = x$

$= \int_0^\infty e^{-x} \sin\left(\frac{ax}{s}\right) dx$

$= \lim_{M\to\infty} \int_0^M e^{-st} \sin at\,dt$

$= \lim \left[\frac{e^{-st}}{s^2+a^2}(s \sin at - a\cos at)\right]_0^\infty$

$$= \frac{a}{s^2+a^2} \quad (|s| > |a|).$$

5) $L(\cos at) = \frac{s}{s^2+a^2} \quad (|s| > |a|).$

Properties of Laplace Transformation:

1. **First shifting property:**

Let $L(f(t)) = F(s)$, then $L(e^{at}.f(t)) = F(s-a)$

$$L(e^{at}.f(t)) = \int_0^\infty e^{-st}.e^{at}.f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt.$$

$$= F(s-a).$$

eg:-

$$L(e^{2t}\sin 3t) = \frac{a}{(s-2)^2+a^2} = \frac{3}{(s-2)^2+9}.$$

2. **Change of Scale property:**

$$L(f(at)) = \frac{1}{a}F(s/a).$$

$$L(f(at)) = \int_0^\infty e^{-st} f(at)\,dt : \qquad at = z, \quad dt = \frac{1}{a}dz.$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)z} f(z)\,dz.$$

$$= \frac{1}{a}F(s/a).$$

Eg:- $L(\sin 2t) = \frac{2}{s^2+4}$

we know, $L(\sin t) = \frac{1}{s^2+1}$

$$\Rightarrow L(f(2t)) = \frac{1}{2}F(s/2) = \frac{1}{2}\int_0^\infty e^{-\frac{s}{2}t} f(\sin 2t)\,dt$$

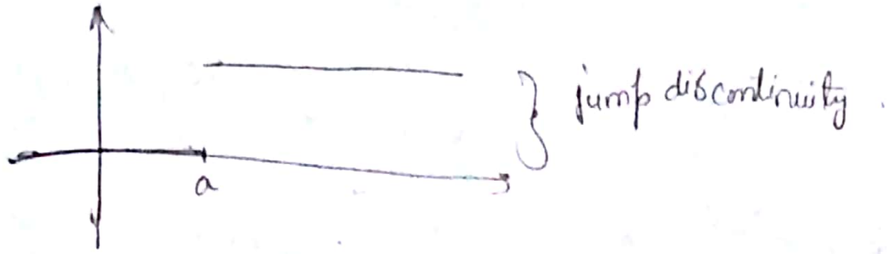
$$= \frac{1}{2}\frac{1}{(s/2)^2+1} = \frac{2}{4+s^2}.$$

2) Second shifting property:

(Heaviside funcⁿ).

$$H(t-a) / u(t-a) := \begin{cases} 1, & t>a \\ 0, & t<a \end{cases} \quad (a>0)$$

unit step funcⁿ



Q) Find $L(H(t-a)) = \int_0^\infty e^{-st} \cdot H(t-a)\, dt$

$$= \int_0^a e^{-st} H(t-a)\, dt + \int_a^\infty e^{-st} H(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st}\, dt$$

$$= \frac{-1}{s}\left(e^{-st}\right)_a^\infty = \frac{-1}{s}\left(0 - \frac{1}{e^{as}}\right) = \frac{e^{-as}}{s}$$

$$= \frac{1}{s} e^{-a\cancel{s}} \qquad = \frac{1}{s} e^{-as}$$

Let $g(t) = f(t-a)$ $t>a$ $\quad a>0$

$\qquad\qquad 0 \;,\; t<a$

then $L(g(t)) = \int_0^\infty e^{-st} g(t)\, dt$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\, dt \qquad \text{Let } t-a=z$$

$$= \lim_{M\to\infty} \int_0^M e^{-s(a+z)} f(z)\, dz$$

$$= \lim_{M\to\infty} e^{-sa} \int_0^M e^{-sz} f(z)\, dz = e^{-sa} L(f(t))$$

$$\Rightarrow L(g(t)) = e^{-sa} L(f(t)) = e^{-as} F(s)$$

eg:- Let $f(t) = \begin{cases} \cos(t-5), & t>5 \\ 0, & t<5 \end{cases}$

$$\Rightarrow L(f(t)) = \int_5^\infty e^{-st} \cos(t-5)\, dt.$$

$$= e^{-5s} L(\cos t).$$

By, S.S.P

$$L(f(t)) = e^{-5s}\left(\frac{s}{s^2+1}\right).$$

Alternatively, the S.S.P can be stated as,

If $L(f(t)) = F(s)$ then,

$$L(H(t-a).f(t-a)) = e^{-as}F(s)$$

$$\left\{ \because g(t) = H(t-a).f(t-a) = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases} \right\}$$

Dt: 20/2/18

Property:

Let $L(f(t)) = F(s)$.

then $L(t.f(t)) = -\frac{d}{ds}(F(s))$.

$$F(s) = \int_0^\infty e^{-st}.f(t)\, dt.$$

$$= \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\, dt\right).$$

$$= \int_0^\infty \frac{d}{dt}\left(e^{-st} f(t)\right) dt.$$

$$= -\int_0^\infty e^{-st}(t f(t))\, dt.$$

$$= -L(t f(t))$$

$$\left\{ \begin{array}{l} \frac{d}{dx}\int (x+y)\, dy. \\ = \frac{d}{dx}\, \left(xy\right) = y.. \\ \int \frac{d}{dx}(x+y)\, dy = y. \end{array} \right.$$

$$\frac{d}{dx}\int (x+y)\, dx = x+y.$$

$$\int \left(\frac{d}{dx}(x+y)\right) dx = x.$$

Similarly,

$$L(t^2 f(t)) = (-1)^2 \frac{d^2}{ds^2} F(s)$$

$$L(t^3 f(t)) = (-1)^3 \frac{d^3}{ds^3} F(s) \ldots$$

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s)$$

Q) Find $L(t \sin 3t) = -\frac{d}{ds} F(s)$.

$$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= -3 \times \frac{-2s}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$$

Q) Find $L(te^{2t} \sin 4t)$

$$= -\frac{d}{ds} F(s) \quad \left(\because \int_0^\infty e^{-(s-2)t} \sin 4t\right)$$

$$= -\frac{d}{ds} F(s-2)$$

$$F(s) = \frac{4}{s^2+16} \qquad F(s-2) = \frac{4}{(s-2)^2+16}$$

$$\Rightarrow L(te^{2t} \sin 4t) = -\frac{d}{ds}\left(\frac{4}{(s-2)^2+16}\right)$$

<u>Property</u> :-

If $L(f(t)) = F(s)$.

$$L\left(\frac{1}{t} f(t)\right) = \int_s^\infty F(s)\,ds$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left(\int_0^\infty e^{-st} f(t)\,dt\right) ds$$

$$= \int_0^\infty \left(\int_s^\infty e^{-st} f(t)\,ds\right) dt$$

$$= \int_0^\infty \left[-\frac{e^{-st}}{t}\right]_s^\infty f(t)\,dt$$

$$\Rightarrow \int_s^\infty F(s)\,ds = \int_0^\infty e^{-st} \cdot \frac{1}{t} f(t)\,dt = L\left(\frac{1}{t} f(t)\right)$$

Q.) Evaluate $\int_0^\infty \frac{\sin t}{t}\, dt$

$\int_s^\infty F(s)\, ds$. $L(\sin t) = \frac{1}{1+s^2}$,

$L\left(\frac{1}{t}\sin t\right) = \int_s^\infty \frac{1}{1+s^2}\, ds$.

$\Rightarrow \int_0^\infty e^{-st} \frac{\sin t}{t}\, dt = (\tan^{-1} s)_s^\infty = \frac{\pi}{2} - \tan^{-1} s$.

$\therefore e^{-st} = 1 \Rightarrow s = 0$.

$\therefore$ Ans : $\pi/2$.

Property :-

If $L(f(t)) = F(s)$, then

$$\boxed{L(f'(t)) = s \cdot F(s) - f(0)}$$

Proof :

$L(f'(t)) = L\left(\frac{d}{dt} f(t)\right)$.

$= \int_0^\infty e^{-st} \left(\frac{d}{dt} f(t)\right) dt$.

By parts :

$\left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty \left[-s e^{-st} f(t)\right] dt$

Laplace transformation exists only for "func"s of exponential order.

i.e. $\lim_{t \to \infty} e^{-st} f(t) = 0$. or $|e^{-st} f(t)| \leq M$

$\therefore \left(e^{-st} f(t)\right)_0^\infty = 0 - 1 \cdot f(0)$

and $\int_0^\infty e^{-st} f(t)\, dt = F(s)$.

$\therefore L(f'(t)) = -f(0) + s F(s)$ (proved)

$$L(f''(t)) = L\left(\frac{d^2}{dt^2} f(t)\right)$$

$$= \int_0^\infty e^{-sr} \frac{d^2}{dt^2} f(t)\, dt.$$

$$= \left[e^{-sr} \frac{d}{dt} f(t)\right]_0^\infty + s\int_0^\infty e^{-sr}\left(\frac{d}{dt} f(t)\right) dt.$$

$$= -f'(0) + s\, L(f'(t)).$$

$$= -f'(0) - sf(0) + s^2 F(s).$$

$$\boxed{L(f''(t)) = s^2 F(s) - sf(0) - f'(0)}$$

$$L(f'''(t)) = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0).$$

$$\vdots$$

$$L(f^n(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{n-2}(0) - f^{n-1}(0).$$