Superposition of waves

We know that each field of electro-magnetic wave satisfies the scalar 3-D wave equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}\text{.........}\ (A)$$

Since this a linear and homogeneous equation and derivatives appear only to the first power, consequently, if $\psi_1(\boldsymbol{r},t),\psi_2(\boldsymbol{r},t),\dots\dots..,\psi_n(\boldsymbol{r},t)$are individual solutions of the wave equation, then any linear combination of them will also be the solution of the wave equation.

i.e., $\psi(\boldsymbol{r},t)=\sum_{i=1}^{n}C_i\,\psi_i(\boldsymbol{r},t)$ will satisfy equation (A).

Principle of superposition states that the resultant disturbance at any point in a medium is the algebraic sum of the separate constituent waves.

Superposition of two waves having same frequency

A solution of the differential wave equation can be written in the form

$$E(x,t) = E_0 \text{ Sin } [\omega t - (kx + \varepsilon)]$$

where E_0 is the Amplitude of the harmonic disturbance propagating along +ve x-axis, k is the wave vector, ε is the initial phase of the wave and ω is the frequency of the wave**. The phase of the wave is represented by the argument of the *sin* term, i.e [ωt – (kx + ε)]**. Position of the point of interest is denoted by x and t represents the time.
Considering the superposition of two waves moving in the same direction (same k) having same frequency (same ω) but different initial phases and amplitudes, the two superposing waves can be represented by,

$$E_1 = E_{01} \sin(\omega t+\alpha_1) \text{ and } E_2 = E_{02} \sin(\omega t+\alpha_2),$$

Here, the space and time part of the phase of the wave is separated as follows,

$$\alpha(x,\varepsilon) = -(kx+\varepsilon)$$

The resultant wave thus becomes, $E= E_1 + E_2 = E_{01} \sin(\omega t+\alpha_1) + E_{02} \sin(\omega t+\alpha_2)$

$= E_{01}[\sin\omega t \cos\alpha_1 + \cos\omega t \sin\alpha_1] + E_{02} [\sin\omega t \cos\alpha_2 + \cos\omega t \sin\alpha_2]$

$= \sin\omega t\, [E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2] + \cos\omega t\, [E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2]$

$= E_0\cos\alpha \sin\omega t + E_0 \sin\alpha \cos\omega t = E_0\sin(\omega t+\alpha)$ [another wave with same frequency but with different amplitude and phase]

The resultant amplitude is, $E_0^2 = [E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2]^2 + [E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2]^2 =$

$E_{01}^2 \cos^2\alpha_1 + E_{02}^2 \cos^2\alpha_2 + 2E_{01}E_{02} \cos\alpha_1\cos\alpha_2 + E_{01}^2 \sin^2\alpha_1 + E_{02}^2 \sin^2\alpha_2 + 2E_{01}E_{02} \sin\alpha_1\sin\alpha_2 = E_{01}^2 [\cos^2\alpha_1 + \sin^2\alpha_1] + E_{02}^2 [\cos^2\alpha_2 + \sin^2\alpha_2] + 2E_{01}E_{02}[\sin\alpha_1\sin\alpha_2 + \cos\alpha_1\cos\alpha_2]$

$= E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_2 - \alpha_1)$

or, and the resultant phase is, $\tan\alpha = (E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2)/ (E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2)$

If the difference in phase between the two interfering waves E_1 and E_2 is δ,

Then, $\delta = (\alpha_2 - \alpha_1) = -(kx_2 - \varepsilon_2) - [-(kx_1 - \varepsilon_1)] = k(x_1 - x_2) + (\varepsilon_1 - \varepsilon_2)$

or, $\boldsymbol{\delta = (2\pi/\lambda)(x_1 - x_2) + (\varepsilon_1 - \varepsilon_2)}$.

$E = E_0 \sin(\omega t + \alpha)$ Resultant wave of two superposing waves having same frequency ω
$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_2 - \alpha_1)$ Square of Resultant Amplitude
$\tan\alpha = (E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2)/(E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2)$ Resultant Phase is α
$\delta = (2\pi/\lambda)(x_1 - x_2) + (\varepsilon_1 - \varepsilon_2)$ Phase difference between the interfering waves

Coherent Superposition: In order to obtain the maximum or minimum resultant amplitude/intensity at a position, we need to maximize or minimize the **cos *δ*** term, which means for a particular position, **δ should not vary with time such that δ is only dependent on x_1 and x_2**. Since, $(2\pi/\lambda)(x_1 - x_2)$ term will not vary with time for a particular position. **Therefore, the initial phase of the interfering waves either has to be same or should have a constant phase difference**. These types of sources are known as **coherent source**.

If the initial phases are same, $\varepsilon_1 = \varepsilon_2$, then the phase difference will depend only on the path difference of the two interfering waves.

Therefore, the amplitude of the resultant wave at a point varies with time. For an optical wave $\omega \approx 10^{15}$ sec^{-1}, thus the amplitude of the resultant wave changes 10^{15} times per second, but the detection limit of our eye is 1/10 sec. As the intensity of light is proportional to the square of the amplitude of the electric field vector, thus to calculate the intensity of the superposed wave at a particular position we have to consider the time average over the detection period.

Interference by division of wavefront

Young Double Slit Experiment

Thomas Young in 1802 devised an way to produce two coherent sources by dividing the primary wave-front of a source (S) into secondary wave-fronts as if they emanated from two sources (S_1 and S_2 respectively) having a constant phase relationship (i.e. coherent) and thus they will produce a stationary interference pattern.

In the actual experiment, a monochromatic light source illuminates the pinhole S. Light diverging from this pinhole fell on a barrier which contained two closely placed pinholes $S_1$and S_2 and were located equidistant from S. Spherical waves emanating from S_1 and S_2 are coherent and fringes will be observed on the screen GG'. This is an interference effect, because if we cover either S_1 or S_2, the fringe pattern will disappear.

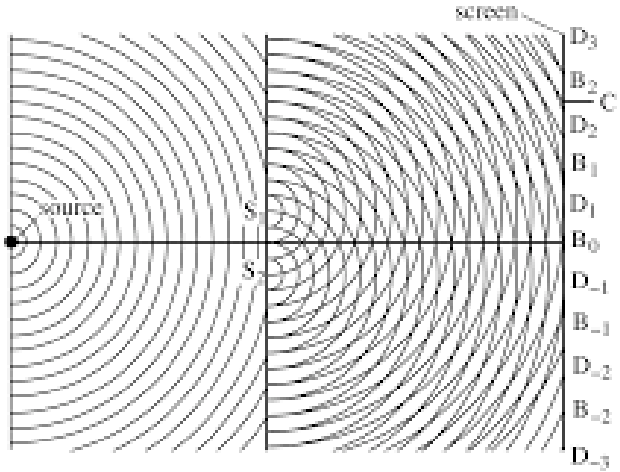

Fig:1 Intereference by divison of wavefront

B_0 is the central bright spot, $B_{\pm n}$ and $D_{\pm n}$ are the n^{th} Bright or Dark Fringes on both side of the central spot respectively. (n =0,1,2…)

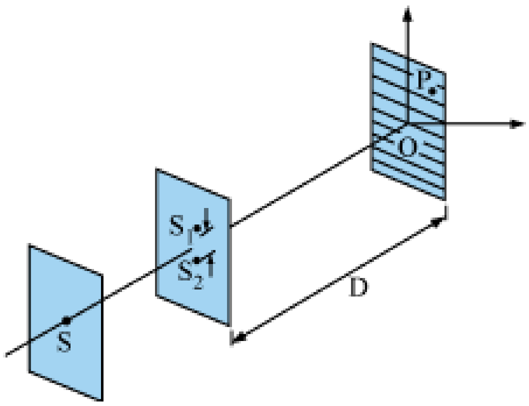

Fig:2 Set up of Young' Double Slit experiment

From the source S, two coherent sources S_1 and S_2 are produced by division of wavefront, which in turn will give interfernce pattern on the screen kept at distance D.

Intensity distribution

Let us assume that $\mathbf{E_1}$and $\mathbf{E_2}$ be the electric fields at point P due the sources S_1 and S_2 respectively. If S_1P and S_2P are very large in comparison to S_1S_2, then $\mathbf{E_1}$and $\mathbf{E_2}$will be in same direction for practical purpose. We will consider the frequency and the initial phases of the two fields to be equal. Therefore, we can write,

$\mathbf{E_1} = \mathbf{i}\, E_{01}\cos[(2\pi/\lambda)S_1P - \omega t]$ and $\mathbf{E_2} = \mathbf{i}\, E_{02}\cos[(2\pi/\lambda)S_2P - \omega t]$, where $\mathbf{i}$ is the direction of the electric field. Due to the superposition of $\mathbf{E_1}$and $\mathbf{E_2}$, the resultant field at P is

$$\mathbf{E} = \mathbf{E_1} + \mathbf{E_2} = \mathbf{i}\, E_{01}\cos[(2\pi/\lambda)S_1P - \omega t] + \mathbf{i}\, E_{02}\cos[(2\pi/\lambda)S_2P - \omega t]$$

Since intensity is proportional to the square of amplitude and if we consider the proportionality constant to be a, the intensity at P will be $I = aE^2$.

$$I = \langle a\, [E_{01}\cos[(2\pi/\lambda)S_1P - \omega t] + E_{02}\cos[(2\pi/\lambda)S_2P - \omega t]]^2 \rangle$$

$= \langle a[\,[E_{01}\cos[(2\pi/\lambda)S_1P - \omega t]]^2 + [E_{02}\cos[(2\pi/\lambda)S_2P - \omega t]]^2$

$+ 2E_{01}E_{02}\cos[(2\pi/\lambda)S_1P - \omega t]\cos[(2\pi/\lambda)S_2P - \omega t]\,]\rangle$

[Since, $2\cos A \cos B = \cos(A+B) + \cos(A-B)$]

Therefore, $I = \langle a\,[E_{01}^2\cos^2[(2\pi/\lambda)S_1P - \omega t] + E_{02}^2\cos^2[(2\pi/\lambda)S_2P - \omega t]$

$+ E_{01}E_{02}\,[\cos[(2\pi/\lambda)(S_2P - S_1P)] + \cos[(2\pi/\lambda)(S_2P + S_1P) - 2\omega t]]\rangle$

The time average over a time period (T) of a time varying function is defined by,

$< f(t) >= \frac{1}{T}\int_0^T f(t)dt.$

Thus, $< cos^2 \left[\left(\frac{2\pi}{\lambda}\right)S_1\text{P} - \omega t\right] > = \frac{1}{T}\int_0^T cos^2\left[\left(\frac{2\pi}{\lambda}\right)S_1\text{P} - \omega t\right]dt = \frac{1}{T}\int_0^T cos^2[\vartheta - \omega t]\,dt$

$= \frac{1}{T}\int_0^T \frac{(1+cos2[\vartheta - \omega t])}{2}dt = \frac{1}{2T}\int_0^T dt + \frac{1}{2T}\int_0^T \cos 2(\vartheta - \omega t)dt = \frac{1}{2} + \frac{1}{2T}\left(\frac{1}{-2\omega}\right)\sin 2(\vartheta - \omega t)|_0^T$

$$= \frac{1}{2} + \frac{1}{2T}\left(\frac{1}{-2\omega}\right)[\sin 2(\vartheta - \omega T) - \sin 2\vartheta] = \frac{1}{2} + \frac{1}{2T}\left(\frac{1}{-2\omega}\right)[\sin 2(\vartheta - 2\pi) - \sin 2\vartheta]$$

$= \frac{1}{2} + \frac{1}{2T}\left(\frac{1}{-2\omega}\right)[\sin 2\vartheta - \sin 2\vartheta] = \frac{1}{2}$ [as for a particular point/position S_1P is constant, thus $(2\pi/\lambda)S_1P$ is also constant (denoted by θ)].

Here, T is the time period of optical wave, which is of the order of 10^{-15} second. Thus, within the eye detection limit, τ (1/10 th of a second), 10^{14} time periods will be covered for an optical wave, resulting the term

$$\frac{1}{\tau}\int_0^\tau \cos 2(\vartheta - \omega t)dt = 0\,,\, therefore, < cos^2\left[\left(\frac{2\pi}{\lambda}\right)S_1\text{P} - \omega t\right] > = \frac{1}{2}$$

Similar arguments will show that $< cos^2\left[\left(\frac{2\pi}{\lambda}\right)S_2\text{P} - \omega t\right] > = \frac{1}{2}$

$\left\langle \cos\left[\left(\frac{2\pi}{\lambda}\right)(S_2\text{P} + S_1\text{P}) - 2\omega t\right]\right\rangle = \frac{1}{2T}\int_0^T \cos\left[\left(\frac{2\pi}{\lambda}\right)(S_2\text{P} + S_1\text{P}) - 2\omega t\right]dt =$
$\frac{1}{2T}\int_0^T \cos[\beta - 2\omega t]dt$

$= \frac{1}{2T}\left(\frac{1}{-2\omega}\right)\sin(\beta - 2\omega t)|_0^T = \frac{1}{2T}\left(\frac{1}{-2\omega}\right)[\sin(\beta - 4\pi) - \sin(\beta)]$

$$= \frac{1}{2T}\left(\frac{1}{-2\omega}\right)[\sin\beta - \sin\beta] = 0$$

[as for a particular point/position $S_1P + S_2P$ is constant, thus $(2\pi/\lambda)(S_1P+S_2P)$ is also constant (denoted by β)].

Therefore, Intensity at a point $I = \frac{aE_{01}^2}{2} + \frac{aE_{02}^2}{2} + 2E_{01}E_{02}[\cos\left[\left(\frac{2\pi}{\lambda}\right)(S_2\text{P} - S_1\text{P})\right]]$

Since, (S_2P-S_1P) is the path difference between the two interfering waves, therefore [$(2\pi/\lambda)(S_2P-S_1P)$] is the corresponding phase difference δ.

Thus, $\boldsymbol{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta}$, where
$\boldsymbol{I_1}$ ***and*** $\boldsymbol{I_2}$ ***are intensities of the constituent waves*** .

Coherent Superposition

a) Condition of Constructive Interference: for cos $\delta = +1$, $I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$

Thus, $\delta = 2n\pi$, where n = 0, 1, 2,…….

If we define path difference (S_2P - S_1P) = Δ, then,

$(2\pi/\lambda)\Delta = 2n\pi$ gives, $\Delta = 2n\lambda/2$, where n = 0,1,2…..

b) Condition of Destructive Interference: for cos $\delta = -1$, $I_{max} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$

Thus, $\delta = (2n+1)\pi$, where n = 0, 1, 2,…….

If we define path difference (S_2P - S_1P) = Δ, then,

$(2\pi/\lambda)\Delta = (2n+1)\pi$ gives, $\Delta = (2n+1)\lambda/2$, where n = 0,1,2…..

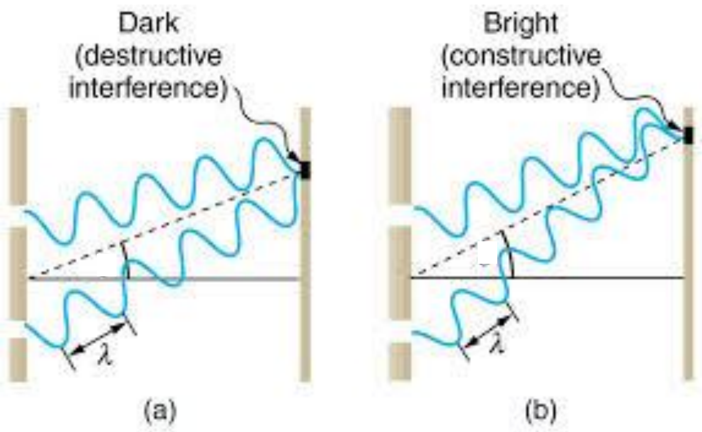

Fig 3: Schematic Representation of Constructive and Destructive Interference

c) If S_2P andS_1P are extremely large compared to S_1S_2, then we can consider,

$I_1 \approx I_2 \approx I_0$ (say), therefore, the resultant intensity

$$\mathbf{I = 2I_0 + 2I_0\cos\delta = 2I_0\,(1+\cos\delta) = 4I_0\cos^2}\,\boldsymbol{(\delta/2)}$$

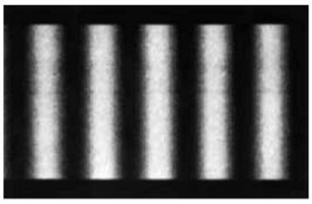

Fig 4: Fringe Pattern in Young's Double Slit Experiment

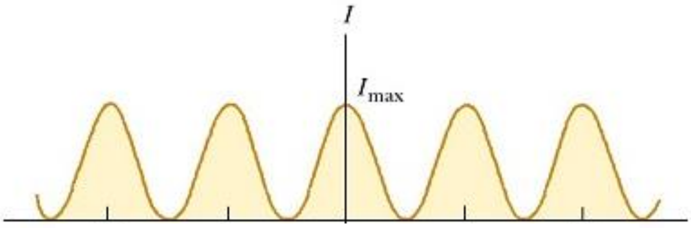

Fig 5: Intensity Distribution when the intensity of interfering waves are equal, $4I_0 \cos^2(\delta/2)$

<u>Incoherent superposition</u>: In this case the phase of the two interfering waves will change rapidly with time (10^8 times per second, as in reality nearly 10^8 wave trains are emitted per second from a source and different wave trains have random phases.), the phase difference between the two waves is now a function of time and as we will measure the intensity at a point, we have to consider the time average over the detection period $\langle \cos \delta(t) \rangle = 0$

Therefore, the average intensity in interference is $I = I_1 + I_2$ **which is same as that of before interference. Thus, energy is globally conserved in interference and redistributed among bright and dark fringes.**

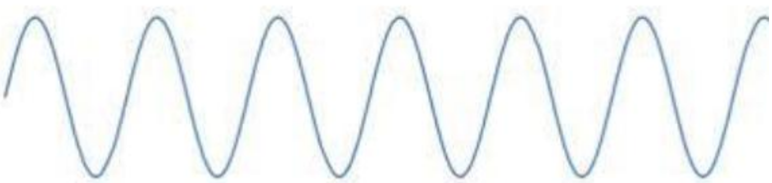

Fig 6: Representation of Coherent waves

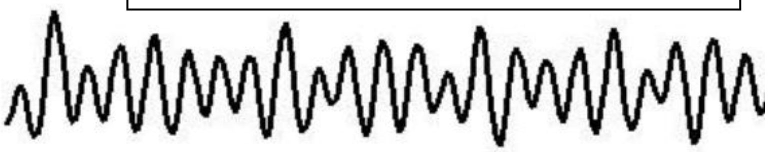

Fig 7: Representation of Incoherent waves

In terms of	Conditions of	
	Constructive Interference	**Destructive Interference**
Phase Difference (δ)	**$\delta = 2n\pi$, n = 0, 1,2,…….**	**$\delta = (2n+1)\pi$, n = 0,1,2…..**
Path Difference (Δ)	**$\Delta = 2n\lambda/2$, n = 0,1,2…..**	**$\Delta = (2n+1)\lambda/2$, n = 0,1,2…..**

Type of Superposition	Expression of Intensity		Inference
	Constructive Interference	**Destructive Interference**	
Coherent ($I_1 \neq I_2$)	$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$	$I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$	**Amplitudes added up**
Coherent ($I_1 \approx I_2 \approx I_0$)	**$4I_0$**	**Zero**	**Amplitudes added up**
Incoherent	**No interference pattern, uniform illumination $I = I_1 + I_2$**		**Intensities added up**

Conservation of energy in Interference

When two monochromatic waves are interfering, they will produce alternative bright (constructive interference) and dark fringes (destructive interference). In order to check the energy conservation, we have to consider the full fringe system. At a particular position on the screen, one will observe either a bright or a dark fringe depending on the path difference/phase difference at that point. But this will vary from one position to the other position. Therefore, if we take the average over all phases, we will obtain the mean intensity after interference.

Since, $\frac{1}{2\pi}\int_0^{2\pi} \cos\delta \, d\delta = 0$, therefore, the average intensity after interference is $\mathbf{I = I_1 + I_2}$, which equal to the intensity before interference. Thus, energy is globally conserved in interference. Or, in other words, in interference phenomena energy is redistributed within the bright and the dark fringes.

Important Point: If $\mathbf{E_1}$ and $\mathbf{E_2}$ are two interfering waves, then the resultant intensity at any point will be

$$I = a\langle(\boldsymbol{E_1} + \boldsymbol{E_2}).(\boldsymbol{E_1} + \boldsymbol{E_2})\rangle = a\langle|\boldsymbol{E_1}|^2 + |\boldsymbol{E_2}|^2 + 2\boldsymbol{E_1}.\boldsymbol{E_2}\rangle$$

where, ***a*** is the proportionality constant. The $2\boldsymbol{E_1}.\boldsymbol{E_2}$ **term is responsible for interference and thus two perpendicularly plane polarised light will not produce any interference effect even if they are monochromatic and coherent, because in that case,**

$$2\boldsymbol{E_1}.\boldsymbol{E_2} = 2|\boldsymbol{E_1}||\boldsymbol{E_2}| \quad \cos 90^0 = 0$$

Conditions of sustained Interference

1) The two sources should be **coherent** (either having same initial phase or a constant phase difference).
2) The two sources must emit **monochromatic** light (same frequency (ω) and for a given medium same wavelength (λ) because wavelength changes in refraction)
3) The amplitudes of the interfering beams should be equal to get completely dark fringes.
4) The separation between the two sources **d should be small** and the distance between the source and the screen **D should be large** in order to observe distinct fringes.
5) For **constructive interference** (i.e. for maximum intensity) the path difference should be **even multiple of λ/2** and for **destructive interference** (i.e. for minimum intensity) the path difference should be **odd multiple of λ/2**.
6) The two sources **should not be perpendicularly polarized.**

Calculation of Fringe width in Young's Double Slit Experiment

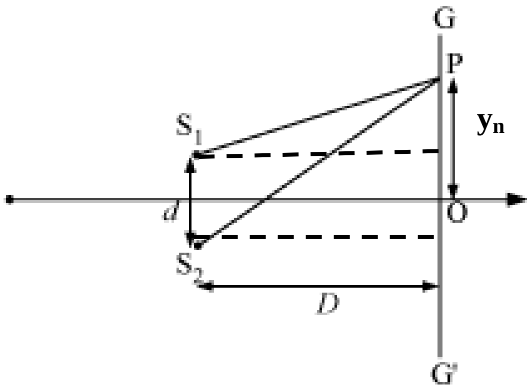

Fig 8: Arrangement of Young's Double Slit Experiment

The path difference (Δ) between the waves emitted from S_1 and S_2 is *($S_1P \sim S_2P$)*, where P is the position of the n^{th} fringe (y_n) on the screen from the central spot O. In order to calculate the fringe width let us first draw two perpendiculars from S_1 and S_2 on the screen, where the foot of the perpendiculars on the screen are A and B respectively. If the distance between the sources (S_1 and S_2) is denoted by *d* and the distance between the screen and the source by *D*, then in practice **D>>>d**, so that $S_1P + S_2P \approx 2D$

Then, $AP = (y_n - d/2)$ and $BP = (y_n + d/2)$

Thus, $(S_1P)^2 = (S_1A)^2 + (AP)^2 = D^2 + (y_n - d/2)^2$ and $(S_2P)^2 = (S_2B)^2 + (BP)^2 = D^2 + (y_n + d/2)^2$

Therefore, $(S_2P)^2 - (S_1P)^2 = D^2 + (y_n + d/2)^2 - D^2 + (y_n - d/2)^2$

or, $(S_2P - S_1P)(S_1P + S_2P) = 2y_n d = 2D\,\Delta$

or, $\mathbf{\Delta = y_n\, d/D}$

Now, the position of the **n^{th} bright fringe** is,

$\Delta = y_n\, d/D = 2n\lambda/2$, where λ is the wavelength of the monochromatic light used.

or, $y_n = nD\lambda/d$, where, $n = 0,1,2,\ldots\ldots\ldots$

Thus, the position of the $(n+1)^{th}$ bright fringe is $y_{n+1} = (n+1)D\lambda/d$

Therefore, the fringe width (distance between two consecutive bright fringes)

$\boldsymbol{\beta = y_{n+1} - y_n = (n+1)D\lambda/d - nD\lambda/d = D\lambda/d}$

Now, the position of the **n^{th} dark fringe** is,

$\Delta = y_n\, d/D = (2n+1)\lambda/2$, where λ is the wavelength of the monochromatic light used.

or, $y_n = (2n+1)D\lambda/2d$, where, $n=0,1,2,\ldots\ldots\ldots$

Thus, the position of the $(n+1)^{th}$ dark fringe is $y_{n+1} =(2n+3)D\lambda/2d$

Therefore, the fringe width (distance between two consecutive dark fringes)

$\boldsymbol{\beta = y_{n+1} - y_n =(2n+3)D\lambda/2d - (2n+1)D\lambda/2d = D\lambda/d}$

Note that the fringe width in this case is independent of the order of the fringe and same for both bright and dark fringe. That is why these fringes are known as fringes of equal width. And the interference pattern (around point O) consists of a series of dark and bright lines perpendicular to plane (fig 4), O being the foot of the perpendicular from the point S on the screen. The central spot is bright because at O, there will be zero path difference.

Fringe width $\beta = D\lambda/d$, independent of the order of bright or dark fringe, fringes of equal width.

Displacement of fringes

One will observe the fringe pattern to be displaced because of the insertion of a thin transparent sheet of thickness ***t*** in front of one of the interfering beam (say S_1).

In this case, the optical path from S_1 to P has been modified. In order to calculate the modified path difference, first consider the time taken for light to reach P from S_1, which is:

$$(S_1P - t)/c + t/v,$$

where c and v are the velocities of light in vacuum and the transparent sheet respectively. Now, $(S_1P - t)/c + t/v = (S_1P - t)/c + \mu t/c$, where μ is the refractive index of the transparent sheet.

Therefore, the modified path difference between the two waves (emitted from S_1 and S_2) at point P on the screen will be = $S_2P - \{S_1P + (\mu-1)t\} = (S_2P - S_1P) - (\mu-1)t = y_n\, d/D - (\mu-1)t$.

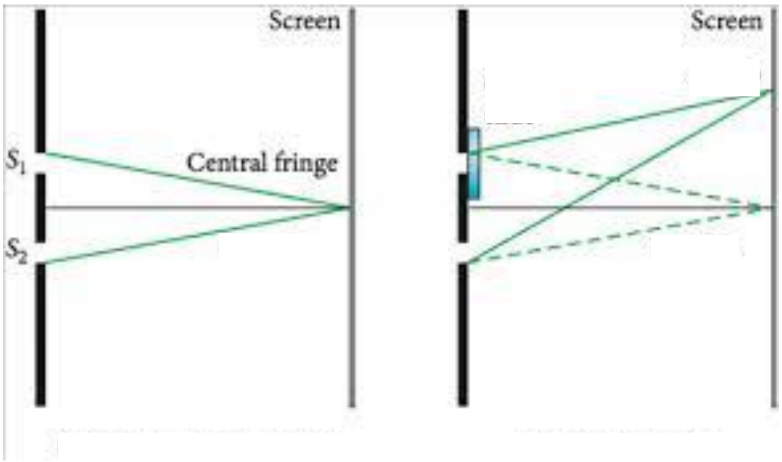

Fig 9: Fringe Shift by inserting a thin transparent sheet

Therefore, the position of the n^{th} bright fringe in this case is,

$y_n\, d/D - (\mu-1)t = 2n\lambda/2$, where λ is the wavelength of the monochromatic light used.

or, $y_n = [n\lambda + (\mu-1)t]D/d$, where, n = 0,1,2,……….

Thus, the displacement of the n^{th} bright fringe will be,

$\delta = [n\lambda + (\mu-1)t]D/d - n\lambda\, D/d = (\mu-1)tD/d = (\mu-1)t\, \beta/\lambda$, which is not dependent on the order of the fringe. [The same result can be derived by starting with the n^{th} dark fringe].

Therefore, the whole fringe pattern will be shifted towards the path of thin transparent sheet.

Fringe shift, $\delta = (\mu-1)\, t\, D/d = (\mu-1)t\, \beta/\lambda$,

which is not dependent on the order of the fringe. Experimentally, by measuring the shift of the central spot (O, which is easy to determine), one can find the thickness of an extremely thin transparent sheet.

Numerical :

1) In an experiment using Young's double slits, the distance between the centre of the interference pattern and tenth bright fringe on either side is 3.44cm and the distance between the slits and the screen (D) is 200cm. If the wavelength of light (λ) used is $5.89*10^{-5}$ cm, determine the separation between the slits (d).

Ans. The distance between the nth bright fringes from the centre of interference pattern is

$y_n = D\, n\, \lambda/d$, In our case n=10.

Therefore, separation between the slits, $d = Dn\lambda/y_n$

$= 200*10*5.89*10^{-5}/3.44$ cm = **0.0342 cm**

2) A double slit of 0.5mm separation is illuminated by light of λ=4800Å. How far behind the slits must go to obtain fringes that are 0.1 cm apart?

Ans. We have to find the distance between the slits and the screen, where fringe width β=0.1 cm, separation between the slits d=0.05cm and λ=4800Å.

Therefore, $\beta = D\lambda/d$ gives,

$D=\beta d/\lambda = 0.1*0.05/4800*10^{-8}$ cm = **104.17 cm**

3) In a Young's double slit experiment, the slits are (d) 0.2 mm apart and the screen is 1.5 m (D) away. It is observed that the distance between the central bright fringe and the fourth dark fringe is (y_4) 1.8 cm, Find the wavelength of light.

Ans. The distances of the 4th dark fringe from the central fringe is

y_n= (2n+1)Dλ/2d, here n=4

Therefore, $\lambda = y_n *2*d/(2n+1)*D = 1.8*2*0.02/9*150$ cm = **5333Å**

4) **To do:** In an experiment of Young's double slit, the slits are 2mm apart and are illuminated with a mixture of two wavelengths λ_1=5000Å and λ_2=6000Å. At what minimum distance from the common central bright fringe on a screen, 2m away from the slits, will a bright fringe of one interference pattern coincide with a bright fringe from the other?

Interference by division of amplitude

Interference due to thin parallel film

Let us imagine a thin plate of thickness t and of refractive index μ bounded by two parallel surfaces XY and RS (Fig 10). A ray of light AB incident on the surface XY at an angle ***i*** is partly reflected along BC and partly refracted into the other medium along BD making an angle of refraction ***r***. At D it is again partly reflected along DE and partly refracted out of the medium along DK parallel to AB, as the medium above and below the thin film is same. Similar refractions and reflections occur at G and E.

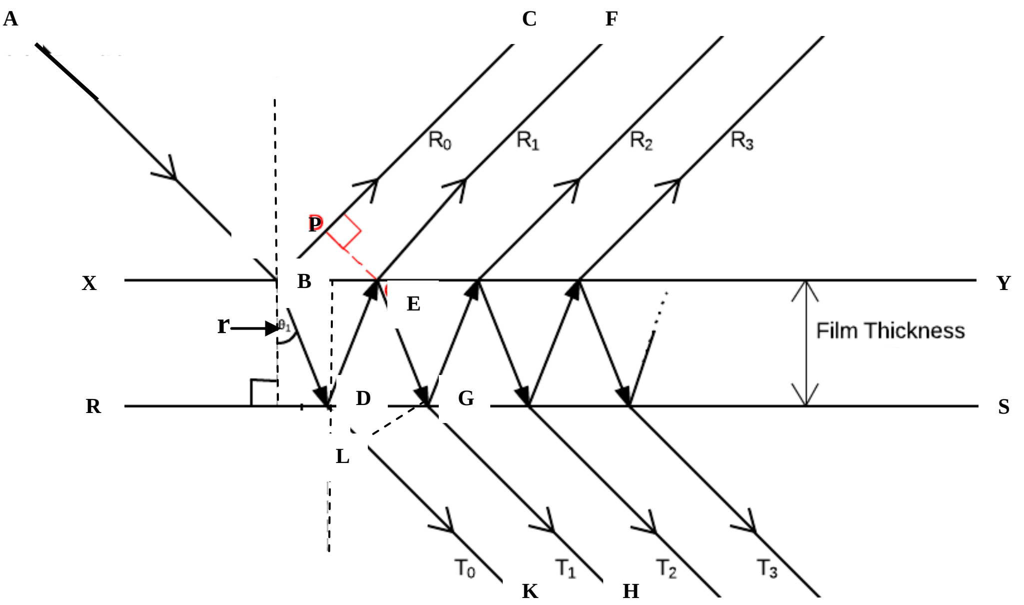

Fig 10: Interference in a Thin Parallel film (Series of Partial reflection and Partial Refractions)

R_0, R_1 .... and T_0, T_1 ...... represents the reflected and the transmitted rays respectively.

a) Interference pattern for reflected light

[**Production of Monochromatic and coherent sources by division of amplitude: In this case the set of reflected rays have same frequency and are coherent since they are derived from the same source (the incident light), but have different amplitudes as the amplitude of the incident light is now divided between the series of partially reflected and refracted light . Same arguments hold for the set of transmitted light too**.]

Path difference between the reflected rays BC and EF will be obtained by dropping perpendicular EP to BC, then,

Path difference = μ(BD+DE) –BP ……. (1)

Now, $\cos r = QD/BD = t/BD = t/BE$

Or, $BD = DE = t/\cos r$ and $\mu(BD+DE) = 2\mu t/\cos r$ …………. (2)

Again, $BP/BE = \sin i$, or $BP = BE \sin i = (BQ+QE) \sin i = 2QE \sin i = 2\mu\, QE \sin r$ ……(3)

Now, $QE/QD = \tan r$, or $QE = t \tan r$ …………. (4)

Therefore, from equation (3), & (4) $BP = 2\mu \sin r\, t \tan r = 2\mu t \sin^2 r / \cos r$ ………… (5)

Thus, equation (1), (2) and (5) gives, Path Difference = $2\mu t/\cos r - 2\mu t \sin^2 r / \cos r$

$= 2\mu t (1- \sin^2 r)/\cos r = 2\mu t \cos^2 r/ \cos r = \mathbf{2\mu t \cos r}$ ………. (6)

According to Stokes' law, since BC suffers reflection at air-medium interface, it undergoes a phase change of π or path difference of $\lambda/2$.

So, the net path difference between the reflected rays BC and EF = $\mathbf{2\mu t \cos r \pm \lambda/2}$.

Therefore, the condition of constructive interference (Bright fringe) is

$2\mu t \cos r \pm \lambda/2 = 2n\lambda/2$

$$\text{or, } 2\mu t \cos r = (2n \pm 1)\lambda/2, \text{ where } n = 0, 1, 2, 3\ldots\ldots\ldots$$

and the condition for destructive interference (dark fringe)is ,

$2\mu t \cos r \pm \lambda/2 = (2n \pm 1)\lambda/2$

$$\text{or, } 2\mu t \cos r = 2\, n\lambda/2 \text{ where } n = 0, 1, 2, 3\ldots\ldots\ldots$$

Interference pattern due to transmitted light

Path difference between DK and GH is μ (DE+EG) – DL = $\mathbf{2\mu t \cos r}$

Therefore, the condition for observing bright fringe is

$$2\mu t \cos r = 2\, n\lambda/2 \text{ where } n = 0, 1, 2, 3\ldots\ldots\ldots$$

and the condition for observing dark fringe is,

$$2\mu t \cos r = (2n \pm 1)\lambda/2, \text{ where } n = 0, 1, 2, 3\ldots\ldots\ldots$$

Thus, the reflected and transmitted fringe patterns are complimentary of each other.

	Reflected Light	Transmitted Light
Condition for constructive interference	**$2\mu t \cos r = (2n \pm 1)\lambda/2$, $n = 0, 1, 2, 3\ldots\ldots\ldots$**	**$2\mu t \cos r = 2\, n\lambda/2$ $n = 0, 1, 2, 3\ldots\ldots\ldots$**
Condition for destructive interference	**$2\mu t \cos r = 2\, n\lambda/2$ $n = 0, 1, 2, 3\ldots\ldots\ldots$**	**$2\mu t \cos r = (2n \pm 1)\lambda/2$, $n = 0, 1, 2, 3\ldots\ldots\ldots$**

Amount of Phase Change	Rarer to Denser Medium	Denser to Rarer Medium
In Reflection	**0**	**π**
In Transmission (or Refraction)	**0**	**0**
Conclusion:	**Only the ray BC which is reflected from the denser (thin Film medium) to the rarer medium (air) suffers a π phase change.**	

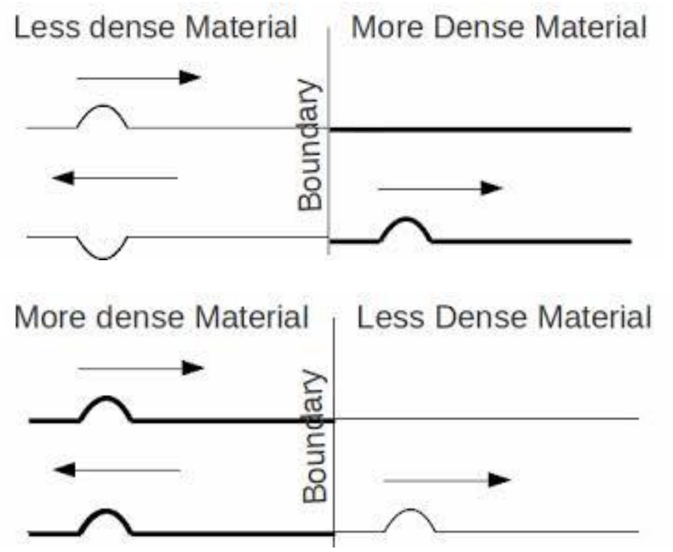

Fig 11: Change of phase in reflection from denser to rarer medium is explained here through a mechanical analogy by considering the interface of a thin and a thick rope which are attached together

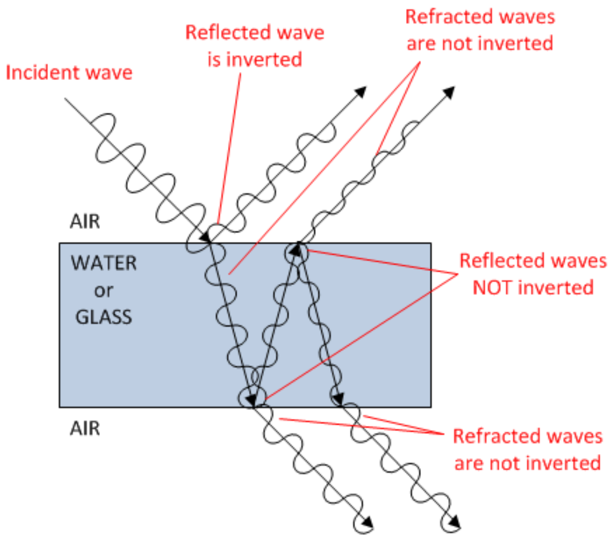

Fig 12 : Interference between reflected and transmitted light in a Thin parallel film

Fringes of equal inclination: When a **thin parallel film** is illuminated with **monochromatic light (μt and λ are constant),** then the **path difference 2μt cos r is only dependent on the angle of refraction (thus, on the angle of incidence).** Now for a particular order of the dark or bright fringe **(n constant), the angle of refraction *r* has to be constant.** Therefore, the fringes are called fringes of equal inclination **and they are the arcs of concentric circles with common centre at the foot of the perpendicular drawn from the eye to the plane of the film.**

Special Cases:

1) **Extremely Thin Film ($t << \lambda$)** : When the thickness of the film is very small compared to the wavelength of light, then **$2\mu t \cos r$ will be a negligible** quantity. Thus, the path difference between **two reflected will be effectively $\lambda/2$** (which corresponds to Destructive Interference) and the film will appear **dark**. With the **transmitted light** the film will appear **bright** as in this special case **the effective path difference is zero** (which corresponds to Constructive Interference).

2) **Colouring effect in thin film:** When a thin film is illuminated with a parallel beam of white light, colouring effect will be observed. In such a case, neglecting the small variation in the angle of refraction for different colours, the light waves corresponding to different colours pursue approximately the same within the film. The optical path difference $2\mu t \cos r$ will be the same to a close approximation for all the colours. Hence the colours whose wavelength satisfy the equation $2\mu t \cos r = (2n)\lambda/2$ will be absent in the reflected beam but they will be present in the transmitted beam. On the other hand, colours whose wavelength satisfies the relation $2\mu t \cos r = (2n \pm 1)\lambda/2$, will be present in the reflected beam but absent in the transmitted beam. Consequently, the film will have a uniform colouration all over but the resultant colour of the film by reflected is exactly complementary to its resultant colour by the transmitted light.

3) **No colouring effect in thick film:** Suppose white light fall on a thick parallel film. In this case for a given value of angle of incidence i, due to large thickness of the film, ***t,*** and values of n can be found to satisfy the condition of constructive interference for every colour in the spectrum. Due to this reason, different coloured fringes of equal inclination will be superposed to produce a general white illumination.

4) **Necessity of broad source of light in thin film:** When a thin film is illuminated with light from a point source and it is observed with a lens of small aperture such as the eye, the rays that can enter the eye are confined to a small range of directions. Thus, the field of view is narrow. But if we employ an extended source, light from every point of the source after reflection from the film enters the eye in separate direction, thus the field of view in wide and the interference effects can be observed over the entire film.

Numerical:

1) Find the thickness (t) of a soap film that gives constructive second order interference of reflected red light (λ=7000Å). The refractive index of the film is (μ) 1.33. [Assume parallel beam of incident light is directed at 30^0 to the normal].

Ans. We know that, $2\mu t \cos r = (2n+1)\lambda/2$, where r is the angle of refraction and $\sin i/\sin r = \mu$ [Here $i=30^0$ and n=1]

therefore, $\sin 30^0/\sin r = 1.33$, $\sin r = \sin 30^0/1.33$

or, $r = \sin^{-1}(1/(2*1.33)) = \mathbf{22.08^0}$

$\mathbf{2*1.33*t \cos 22.08^0 = 3 * 7000/2}$ **Å, t = 4260Å**

2) **To do:** Find the minimum thickness (t) of coating with MgF_2 (μ=1.38] required for no reflection at the centre of visible spectrum (λ=5500Å). [Consider normal incidence on thin film].

Wedge Shaped film (Film with varying thickness): Consider a film in the shape of a thin wedge whose sides form a small **wedging angle α** and illuminated with plane monochromatic light waves.

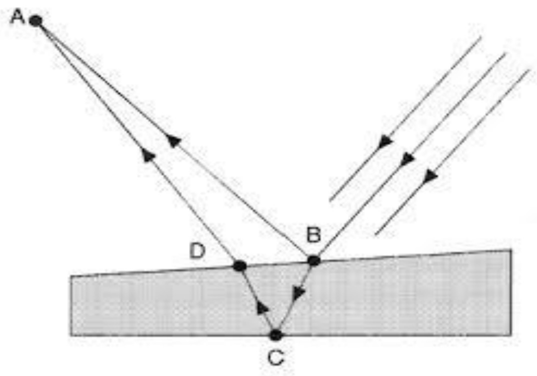

Fig 12: Interference in wedge shaped film, fringes of equal thickness

In this case the path difference will **be $2\mu t \cos(r+\alpha)$** and the conditions of constructive and destructive interference with reflected and transmitted light is given in the following table.

	Reflected Light	Transmitted Light
Condition for constructive interference	**$2\mu t \cos(r+\alpha) = (2n \pm 1)\lambda/2$, n = 0, 1, 2,3…………**	**$2\mu t \cos(r+\alpha) = 2n\lambda/2$ n = 0, 1, 2,3…………**
Condition for destructive interference	**$2\mu t \cos(r+\alpha) = 2n\lambda/2$ n = 0, 1, 2, 3…………**	**$2\mu t \cos(r+\alpha) = (2n \pm 1)\lambda/2$, n = 0, 1, 2, 3…………**

Here, the different order Dark or bright fringes will be obtained as we change the thickness of the film, t and the fringes will be of constant thickness.

Newton's Ring

The rings which are circular interference fringes are observed when a plano-convex lens of large radius of curvature is placed on the plane glass plate. A thin film of air is formed between the curved surface of the lens and the glass plate. The thickness of the air film gradually increases all round this point as we move away from the centre (ie the point of contact of the lens and the glass plate). The air film thus enclosed is wedge-shaped. Therefore, the loci of all points having the same thickness of air are circles. Thus, we will see alternative dark and bright concentric rings using a monochromatic light.

Experimental Set up: A monochromatic source S is placed at the focus of a lens L to obtain a set of parallel rays. A glass plate G is inclined at 45^0(thus resulting in normal incidence of light) towards the air film enclosed by the plano-convex lens of large radius of curvature N and the plane glass plate P. The reflected beam from the air film is viewed with a microscope M.

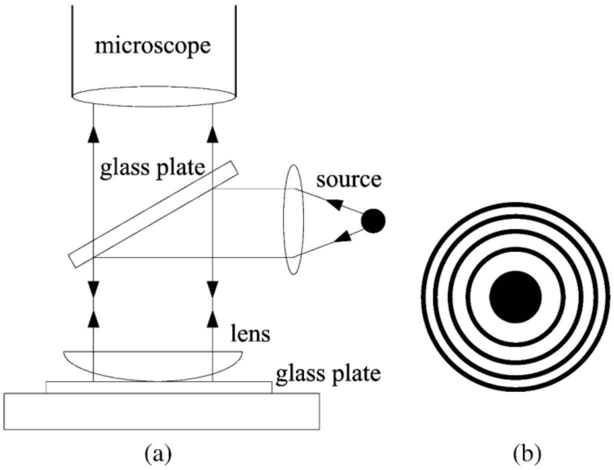

Fig 13: (a) Experimental set up Newton's Ring,

(b) Concentric circular fringes with reflected light

Production of coherent sources in a Newton's rings arrangement: In this case a ray of light incident normally on the apparatus is partially reflected from the convex of the plano-convex lens and partially refracted. The refracted ray is reflected from the top surface of the plane glass plate, placed below the plano-convex lens. These two reflected rays have a constant phase difference depending on the thickness of the air film at the point of reflection. Thus the two rays are coherent.

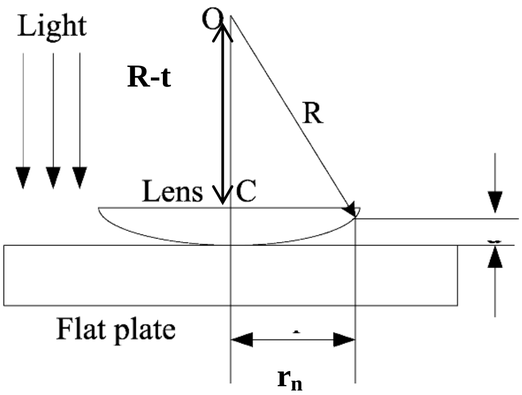

Fig 14: Formation of the wedge shaped film in Newton's ring arrangement between a plano-convex lens and a plane glass plate

If radius of curvature of the plano-convex lens is R, t is the thickness of the air film at that point and r_n is the radius of the n^{th} Newton's ring, then

$R^2 = (R-t)^2 + r_n^2 = R^2 + t^2 – 2Rt + r_n^2$

or, $r_n^2 = 2Rt – t^2 \approx 2Rt$ (as t is very small)

The path difference between two reflected ray will be same as of a wedge-shape film,

i.e. $2\mu t \cos(r+\alpha) \pm \lambda/2$ and for transmitted ray $2\mu t \cos(r+\alpha)$.

But as in Newton's ring arrangement, radius of curvature of the plano-convex lens (R) is large, therefore the wedging angle α is very small, air film gives μ=1 and because of normal incidence angle of refraction r is zero. **Thus, the effective path difference between two reflected light is 2t ±λ/2 and between two transmitted light is 2t**.

[At any point of the curved surface of the plano-convex lens the condition of normal incidence is valid since R is very large.]

Newton's ring with reflected light

The radius of the n^{th} bright ring can be obtained from the condition of constructive interference,

$$\text{is } 2t \pm \lambda/2 = 2n\lambda/2$$

$$\text{or, } r_n^2/R \pm \lambda/2 = 2n\lambda/2$$

$$\text{or, } r_n^2 = (2n\pm1)R\lambda/2$$

$$\text{or, } r_n = [(2n\pm1)R\lambda/2]^{1/2} \text{ where } n = 0, 1, 2, 3\ldots\ldots\ldots\ldots$$

and the radius of the n^{th} dark ring will be,

$$2t \pm \lambda/2 = (2n\pm1)\lambda/2$$

$$\text{or, } r_n^2/R \pm \lambda/2 = (2n\pm1)\lambda/2$$

$$\text{or, } r_n^2 = 2nR\lambda/2$$

$$\text{or, } r_n = [2nR\lambda/2]^{1/2} \text{ where } n = 0, 1, 2, 3\ldots\ldots\ldots\ldots$$

When n=0, the radius of dark ring is zero and the radius of bright ring is $(\lambda R/2)^{1/2}$. Thus, the centre is dark. (Or in other words, at the point of contact t=0 and the path difference between two reflected light is λ/2 thus giving a dark ring).

Newton's ring with transmitted light

The radius of the n^{th} bright ring can be obtained from the condition of constructive interference,

$$\text{is } 2t = 2n\lambda/2$$

$$\text{or, } r_n^2/R = 2n\lambda/2$$

$$\text{or, } r_n^2 = (2n)R\lambda/2$$

$$\text{or, } r_n = [(2n)R\lambda/2]^{1/2} \text{ where } n = 0, 1, 2, 3\ldots\ldots\ldots\ldots$$

and the radius of the n^{th} dark ring will be, $2t = (2n\pm1)\lambda/2$

$$\text{or, } r_n^2/R = (2n\pm1)\lambda/2$$

$$\text{or, } r_n^2 = (2n\pm1)R\lambda/2$$

$$\text{or, } r_n = [(2n\pm1)R\lambda/2]^{1/2} \text{ where } n = 0, 1, 2, 3\ldots\ldots\ldots\ldots$$

When n=0, the radius of bright ring is zero and the radius of dark ring is $(\lambda R/2)^{1/2}$. Thus, the centre is bright. (Or in other words, at the point of contact t=0 and the path difference between two reflected light is zero thus giving a bright ring).

	Reflected Light	**Transmitted Light**
Radius of the n^{th} bright ring	$\mathbf{r_n = [(2n\pm1)R\lambda/2]^{1/2}}$, **n = 0, 1, 2,3……**	$\mathbf{r_n = [(2n)R\lambda/2]^{1/2}}$ **n = 0, 1, 2,3……**
Radius of the n^{th} dark ring	$\mathbf{r_n = [(2n)R\lambda/2]^{1/2}}$ **n = 0, 1, 2, 3……**	$\mathbf{r_n = [(2n\pm1)R\lambda/2]^{1/2}}$, **n = 0, 1, 2, 3……**
	Central ring is Dark	**Central ring is Bright**

Applications of Newton's ring

1) Determination of wavelength of an unknown light

For n^{th} dark ring with reflected light, $r_n^2 = n\lambda R$

$$\text{or, } (2r_n)^2 = 4n\lambda R$$

$$\text{or, } D_n^2 = 4n\lambda R, \text{ [}D_n \text{ is the diameter of the } n^{th} \text{ dark ring]}$$

Thus, the diameter of the $(n+m)^{th}$ dark ring is , $D_{n+m}^2 = 4(n+m)\lambda R$

Thus, $D_{n+m}^2 - D_n^2 = 4(n+m)\lambda R - 4n\lambda R = 4m\lambda R$

or, $\boldsymbol{\lambda = (D_{n+m}^2 - D_n^2)/4mR}$, **where m is the difference in the ring number.**

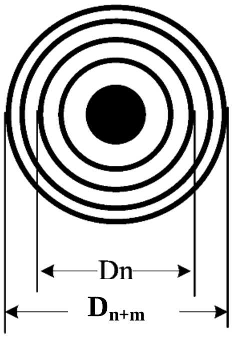

Fig 15: Applications of Newton's ring,

[Diameter of the n^{th} and $(n+m)^{th}$ Newton's rings are indicated here]

2) Determination of refractive index of a liquid

When the air gap between the plano-convex lens and plane glass plate is replaced by a liquid of refractive index μ, we can write the for the n^{th} and $(n+m)^{th}$ dark ring with reflected light as, $D_n^2 = 4n\lambda R/\mu$ and $D_{n+m}^2 = 4(n+m)\lambda R/\mu$

Therefore, $\mu = 4(n+m)\lambda R - 4n\lambda R/ (D_{n+m}^2 - D_n^2)$

$$\text{or, } \boldsymbol{\mu = 4m\lambda R/ (D_{n+m}^2 - D_n^2)}$$

Note : If we replace the plano-convex lens by a biconvex lens, the Newton's ring will be also observed, where the effective radius of curvature can be determined from the following equation
$1/R = (1/R_1 + 1/R_2)$, where R_1 and R_2 are the radii of curvature of the two curved surfaces of the lens respectively.

Numerical

1) A film of r.i (μ) 1.70 is placed between a plane glass plate and equi-convex lens. The focal length of the lens is 1m. Determine the radius of the 10^{th} dark ring in case of reflected light where λ=6000Å.

Ans. The focal length of a equi-convex lens is given by,

$1/f = (\mu-1)2/R$, where R is the radius of curvature.

Here, f =100 cm and μ=1.70.

Therefore, R = (μ-1)*2*f =(1.70-1)*2*100 cm =**140 cm**.

Again, the radius of the nth dark ring is $r_n = (n\lambda R/\mu)^{1/2}$. [Here n=10]

Radius of the 10^{th} dark ring $r_{10} = (10*6000*10^{-8} *140/1.7)^{1/2}$ cm = **0.22 cm**.