Density of state :- No. of energy state per unit volume per unit energy within the energy interval E to (E+dE).

$$G(E) = \frac{g(E).dE}{V.dE} = \frac{g(E)}{V}$$

$$\Rightarrow \boxed{G(E) = 2\pi\left(\frac{2m}{h^2}\right)^{3/2} . E^{1/2}}$$

- **Calculation of α & β** :-

For thermodynamical identity to include the change in particle no. is →

$dU = dQ - PdV - \mu dN$ [μ = Chemical Potential.
= It is the form of energy that can be absorbed or released during the exchange of particles in a given system.]

$\Rightarrow dU = TdS - PdV - \mu dN$ — ①   [$\because dS = \frac{dQ}{T}$]

$U = U(S, V, N)$

$\Rightarrow dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial V} dV + \frac{\partial U}{\partial N} . dN$ — ②

$T = \frac{\partial U}{\partial S}$ and $\mu = -\frac{\partial U}{\partial N}$

$S \propto W$ [K = Boltzmann's const]

$\Rightarrow S \propto \ln(W_{max})$ $\Rightarrow$ $\boxed{S = K \ln W_{max}}$

$$\therefore \alpha = \frac{\partial \ln W_{max}}{\partial N} = \frac{1}{K} \cdot \frac{\partial S}{\partial N} = \frac{1}{K} \cdot \frac{\partial S}{\partial U} \times \frac{\partial U}{\partial N}$$

$$\Rightarrow \boxed{\alpha = -\frac{\mu}{KT}}$$

$$\therefore \beta = \frac{\partial \ln W_{max}}{\partial U} = \frac{1}{K} \cdot \frac{\partial S}{\partial U}$$

$$\Rightarrow \boxed{\beta = \frac{1}{KT}}$$

29/01/18

HG

⇒ **Classical Mechanics** :- (Goldstein)

1) Problems in Newtonian eqⁿ of motion.
2) Constraints & classification
3) Degrees of freedom → Generalised co-ordinate system → configuration space
4) Principle of virtual work
5) Lagrange's eqⁿ of motion
6) Hamilton eqⁿ of motion

$m . \frac{d^2x}{dt^2} = f_x^{ext}$ [for single particle]

$m . \frac{d^2\vec{r}}{dt^2} = \vec{F}^{ext}$ [" 3D "]

$m_i . \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i^{ext}$ [$i = 1, 2, 3, \ldots, N$]

⇒ **Conservation rules** →

i) conservation of linear momentum

$\frac{d\vec{p}}{dt} = \vec{F}^{ext}$ (→ not const)   If $\vec{F}^{ext} = 0$ ⇒ $\vec{p}$ = conserved

$\frac{\partial \vec{p}}{\partial t} \longrightarrow p$ is const

$\vec{F}^{ext} = \frac{d\vec{p}}{dt}$

$\Rightarrow \vec{r} \times \vec{F}^{ext} = \vec{r} \times \frac{d\vec{p}}{dt}$

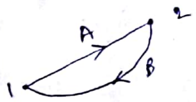
$= \frac{d}{dt}(\vec{r} \times \vec{p}) - \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{=0}$ $\vec{v} \times m\vec{v}$

$\Rightarrow \vec{r} \times \vec{F}^{ext} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right)$ [$\vec{r} \times$ means moment of]

$\Rightarrow \vec{\tau}^{\,ext} = \frac{d\vec{L}}{dt}$

$\vec{\tau}^{\,ext} = 0 \Rightarrow \vec{L} =$ conserved
↓
(angular momentum)

•



$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$

$= \int_1^2 \frac{d\vec{p}}{dt} \cdot \frac{d\vec{r}}{dt} \cdot dt = \int m \cdot \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt$

$= \int_1^2 \frac{d}{dt}\left(\frac{1}{2} m \vec{v} \cdot \vec{v}\right) dt$

$= \left(\frac{1}{2}mv^2\right)_2 - \left(\frac{1}{2}mv^2\right)_1 = T_2 - T_1 \longrightarrow (1)$

[$T \to KE$]

If, $\oint \vec{F} \cdot d\vec{r} = 0$ (conservative force)

$\Rightarrow \int_{1A}^{2} \vec{F} \cdot d\vec{r} + \int_{2B}^{1} \vec{F} \cdot d\vec{r}$

$\Rightarrow \int_{1A}^{2} \vec{F} \cdot d\vec{r} = -\int_{2B}^{1} \vec{F} \cdot d\vec{r} = \int_{1B}^{2} \vec{F} \cdot d\vec{r}$

$\vec{F} \cdot d\vec{r}$ depends only on pts 1 & 2, not on the paths.

$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = V_1(\vec{r}_1) - V_2(\vec{r}_2)$ [$V \to PE$]

$T_2 - T_1 = V_1 - V_2$

$\Rightarrow \boxed{T_1 + V_1 = T_2 + V_2} \Rightarrow$ Total energy is conserved

$\int_1^2 \vec{F} \cdot d\vec{r} = V_1 - V_2 = \int_1^2 -dV = \int -\frac{dV}{dr} \cdot d\vec{r}$

$\Rightarrow \boxed{F = -\frac{dV}{dr}}$

• 3D vector derivative :-

$d f(x,y,z) = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial z} \cdot dz$

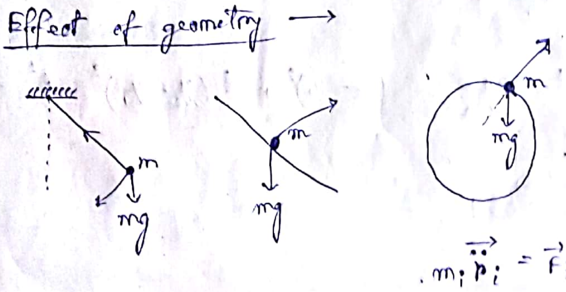
$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$

$= \underbrace{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f}_{\vec{\nabla} f} \cdot \underbrace{\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)}_{d\vec{r}}$

↓
grad V [gradient of V]

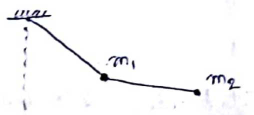
$\Rightarrow \boxed{\vec{F} = -\vec{\nabla} V}$

• (1) Effect of geometry ⟶



$m_i \ddot{\vec{r}}_i = \vec{F}_i^{\,ext} + \vec{R}_i$
↓ force

④ Many body system :-

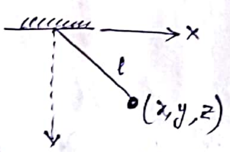


⑤ Accelerated frame of reference / Non-inertial form

12/02/18

- Constraints :-

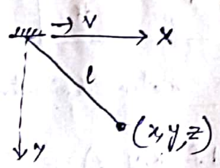
Eg: ①



$z=0$

$x^2+y^2=l^2$

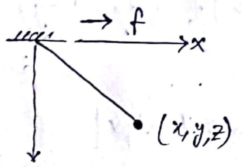
②



$z=0$

$(x-vt)^2+y^2=l^2$

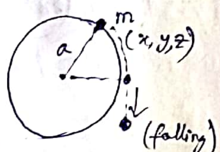
③



$z=0$

$(x-\frac{1}{2}ft^2)^2+y^2=l^2$

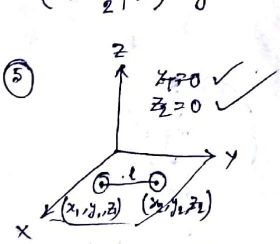
④



$x^2+y^2+z^2=a^2$

$x^2+y^2+z^2 \geq a^2$

⑤



$z_1=0$ ✓ $z_2=0$ ✓ $(x_2-x_1)^2+(y_2-y_1)^2=l^2$ ✓

$(\dot{x}_1\hat{i}+\dot{y}_1\hat{j}) \propto \{(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}\}$

$$\begin{cases} \dot{x}_1 = k(x_2-x_1) \\ \dot{y}_1 = k(y_2-y_1) \end{cases}$$

$$\frac{dx_1}{dy_1} = \frac{x_2-x_1}{y_2-y_1}$$

$\Rightarrow \boxed{(y_2-y_1)dx_1+(x_1-x_2)dy_1=0}$ $\Rightarrow$ Differential constraint eq^n

$(\ )dx_i$; $(\ )dy_i$; $(\ )dz_i$; $(\ )dt$

Constraint eq^{n}s

- Algebraic: $f(x_i,y_i,z_i,t)=0$
- Differential: $X_i dx_i + Y_i dy_i + Z_i dz_i + T_i dt = 0$; func^n of (x_i, y_i, z_i, t)
- (No eq^n)$^\times$

$(x-\frac{1}{2}ft^2)^2+y^2=l^2$

$\Rightarrow f(x,y,z,t)=0$

$x^2+y^2=l^2$

$\Rightarrow \boxed{xdx+ydy=0}$ $\Rightarrow$ valid only for a particular system.

Algebraic constraint eq^n can always be converted into differential constraint eq^n, but the reverse is not always true.

$ydx+xdy=0$

$\Rightarrow d(xy)=0$

$\cancel{\left(\frac{1}{x^2y}\right)}(ydx+x^2dy)=0$

$\Downarrow$

can't be converted to algebraic form by multi-plying $\left(\frac{1}{x^2y}\right)$, because x & y = const may be = 0.

So, for that case the algebraic form is not valid.

Constraint

Holonomic

$\Downarrow$

can be converted into algebraic form / or it is in algebraic form.

Non-holonomic

$\Downarrow$

only diff. form or no eq^n, no algebraic form.

$f_\alpha(x_i, y_i, z_i, t) = 0 \Rightarrow$ the particle is in constraint surface.

$\frac{\partial f}{\partial t} = 0$

Eg:- $x^2+y^2 = 1^2$

$z = 0$

no time involved

$\Downarrow$

Scleronomic constraint eq^n

$\Downarrow$

const. constraint surface.

$\frac{\partial f}{\partial t} \neq 0$

dependent on time

$\Downarrow$

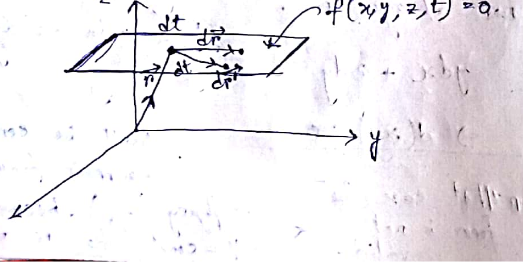
Rheonomic constraint eq^n

$\Downarrow$

constraint surface changes in orientation wrt time.

14/02/18

- Single particle holonomic system :-



$f(x,y,z,t) = 0$

$$df = \underbrace{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz} + \frac{\partial f}{\partial t}dt$$

$$= \vec{\nabla} f \cdot d\vec{r} + \frac{\partial f}{\partial t}dt = 0$$

Case - I $\gg$ $\frac{\partial f}{\partial t} = 0$.

$\vec{\nabla} f \cdot d\vec{r} = 0$

$\Rightarrow \vec{\nabla} f \perp \underset{\text{surface}}{d\vec{r}}$

$\vec{R} \perp d\vec{r}$

$\vec{R} \propto \vec{\nabla} f$

$\Rightarrow \boxed{\vec{R} = \lambda \vec{\nabla} f}$

Case - II $\gg$ $\frac{\partial f}{\partial t} \neq 0$

$\vec{\nabla} f \cdot d\vec{r} + \frac{\partial f}{\partial t}dt = 0$

$\vec{\nabla} f \cdot d\vec{r}' + \frac{\partial f}{\partial t}dt = 0$

$\vec{\nabla} f \cdot \underbrace{(d\vec{r} - d\vec{r}')}_{\text{virtual displacement}} = 0$

$\vec{\nabla} f \cdot \underbrace{\delta\vec{r}}_{\text{surface}} = 0$

$\vec{\nabla} f \perp \delta\vec{r}$, $\vec{R} \perp \delta\vec{r}$

$\vec{R} \propto \vec{\nabla} f$

$\Rightarrow \vec{R} = \lambda \vec{\nabla} f$

Now,

$m_i \ddot{\vec{r}}_i = \vec{F}_i^{ext} + \vec{R}_i$

$\Rightarrow \boxed{m_i \ddot{\vec{r}}_i = \vec{F}_i^{ext} + \lambda_\alpha \vec{\nabla}_i f_\alpha} \Rightarrow$ Lagrange's eq^n of 1st kind.

Get rid of $\vec{R}_i$

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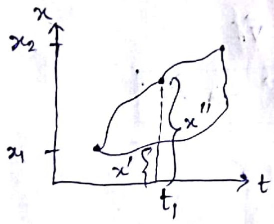
$$\sum_{i=1}^{N} \vec{R_i} \cdot \delta\vec{r_i} = 0 \longrightarrow \text{Principle of virtual work}$$

from $\rightarrow$ $m_i \ddot{\vec{r_i}} = \vec{F_i}^{ext} + \vec{R_i}$

$$\sum_{i=1}^{N} (m_i \ddot{\vec{r_i}} - \vec{F_i}^{ext}) \cdot \delta\vec{r_i} = \sum_{i=1}^{N} \vec{R_i} \cdot \delta\vec{r_i}$$

$$\Rightarrow \boxed{\sum_i (m_i \ddot{\vec{r_i}} - \vec{F_i}^{ext}) \cdot \delta\vec{r_i} = 0} \Rightarrow \text{First fundamental form}$$



$(x'' - x') = \delta x$
$\downarrow$
virtual displacement

$\Rightarrow$ $M dx + N dy = 0$

$\left.\begin{matrix} M=0 \\ N=0 \end{matrix}\right\}$ when x, y are completely individual of each other.

$T_1 \delta r_1 + T_2 \delta r_2 + \dots = 0$

$T_1 = 0$, $T_2 = 0$

$\vec{r_i} \longrightarrow q_j \longrightarrow$ Generalized co-ordinate system

$\sum_j (A_j) \delta q_j = 0$

- **Degrees of Freedom** :-

no.
\# of variable $>$ \# of constraint eq^n
$\downarrow$
dynamic system

n $=$ n $\longrightarrow$ static system

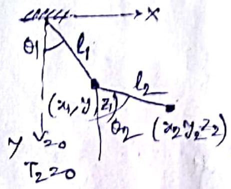
dof (f) = \# of variables − \# of cond eq^n

$$\boxed{f = 3N - K}$$

Generalised co-ordinate
$q = \theta$

$z = 0$

$x^2 + y^2 = l^2$ $\quad f = (3 \times 1) - 2 = 1$



$T_2 = 0$

$z_1 = 0$
$z_2 = 0$

$x_1^2 + y_1^2 = l_1^2$

$\Rightarrow (x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$

x_1, y_1, x_2, y_2

θ_1, θ_2

$f = 3 \times 2 - 4 = 2$

19/02/18

$$\sum_{i=1}^{N} (m_i \ddot{\vec{r_i}} - \vec{F_i}^{ext}) \cdot \delta\vec{r_i} = 0$$
$\downarrow$
$$\sum_{j=1}^{f} (\quad) \cdot \delta q_j = 0$$

$q_j \rightarrow$ Generalised co-ord.
$j = 1, 2, 3, \dots, f$
$\uparrow$
DOF

Holonomic system

$z = 0$

$x^2 + y^2 = l^2$

$q_1 = x$ | $q_2 = y$ | $q_1 = \theta$

$q_j = q_j(x_i, y_i, z_i, t)$

$= q_j(\vec{r_i}, t)$

$\vec{r_i} = \vec{r_i}(q_j, t)$

$$\delta\vec{r_i} = \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \cdot \delta q_j + \underbrace{\frac{\partial \vec{r_i}}{\partial t} \delta t}_{\to 0}$$

$\Rightarrow$ because no time is taken for virtual displacement.

$$\Rightarrow \delta\vec{r_i} = \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \underbrace{\delta q_j}_{\text{Gen. displacement}}$$

Again, velocity of i-th particle $\longrightarrow$

$$\vec{v_i} = \frac{d\vec{r_i}}{dt}$$

$$= \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \cdot \frac{dq_j}{dt} + \frac{\partial \vec{r_i}}{\partial t} \cdot \cancel{\frac{dt}{dt}}$$

$$\vec{v_i} = \dot{\vec{r_i}} = \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \cdot \underbrace{\dot{q}_j}_{\text{Gen. vel.}} + \frac{\partial \vec{r_i}}{\partial t} \longrightarrow (2)$$

Again,

$$\sum_{i=1}^{N} (m_i \ddot{\vec{r_i}} - \vec{F_i}) \cdot \delta\vec{r_i} = 0$$

[As all the forces are ext.]

$$\Rightarrow \sum_{i=1}^{N} m_i \ddot{\vec{r_i}} \cdot \delta\vec{r_i} - \sum_{i=1}^{N} \vec{F_i} \cdot \delta\vec{r_i} = 0$$

$\Rightarrow$ Term 1 $-$ Term 2 $= 0$

$$\text{Term 2} = \sum_i \vec{F_i} \cdot \delta\vec{r_i}$$

$$= \sum_i \vec{F_i} \cdot \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j$$

$$= \sum_{j=1}^{f} \underbrace{\left\{ \sum_i \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q} \right\}}_{Q_j} \cdot \delta q_j \longrightarrow (1)$$

where, $Q_j = \sum_{i=1}^{N} F_i \cdot \left(\frac{\partial \vec{r_i}}{\partial q_j} \right)$

$\therefore$ from (1) $\longrightarrow$

$$\text{Term 2} = \sum_{j=1}^{f} Q_j \cdot \delta q_j$$

Now,

$$\text{Term 1} = \sum_{i=1}^{N} m_i \ddot{\vec{r_i}} \cdot \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j$$

$$= \sum_{i,j} \left\{ m_i \ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \right\} \cdot \delta q_j$$

$$= \sum_{i,j} \left[\frac{d}{dt} \left\{ m_i \dot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \right\} - \left\{ m_i \dot{\vec{r_i}} \cdot \frac{d}{dt} \left(\frac{\partial \vec{r_i}}{\partial q_j} \right) \right\} \right] \delta q_j$$

$$\left[\because \ddot{x} y = \frac{d}{dt}(\dot{x} y) - \dot{x} \frac{dy}{dt} \right]$$

$$= \sum_{i,j} \left[\frac{d}{dt} \left\{ m_i \vec{v_i} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \right\} - \left\{ m_i \vec{v_i} \cdot \underline{\frac{d}{dt} \left(\frac{\partial \vec{r_i}}{\partial q_j} \right)} \right\} \right] \delta q_j$$

from eq^n (2) $\longrightarrow$

$$\vec{v_i} = \sum \frac{\partial \vec{r_i}}{\partial q_j} \cdot \dot{q}_j + \frac{\partial \vec{r_i}}{\partial t}$$

$$\vec{r_i} = \vec{r_i}(q_j, t)$$

$$\therefore \frac{\partial \vec{r_i}}{\partial q_j} = \left(\frac{\partial \vec{r_i}}{\partial q_j} \right)(q_j, t) \longrightarrow \neq \dot{q}_j$$

$$\frac{\partial \vec{r_i}}{\partial t} \longrightarrow \neq \dot{q}_j$$

$$\left[\because F = A\dot{x} + B \quad \frac{\partial F}{\partial \dot{x}} = A \right]$$

$$\Rightarrow \frac{\partial \vec{v_i}}{\partial \dot{q}_j} = \frac{\partial \vec{r_i}}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{d}{dt}\{f(q_j, t)\} \qquad [\because \vec{r}_i = \vec{r}_i(q_j, t)]$$

$$= \sum_{k=1}^{f} \frac{\partial f}{\partial q_k}\cdot \dot{q}_k + \frac{\partial f}{\partial t} \qquad \left[\text{If we write, } f(q_j, t) = \frac{\partial \vec{r}_i}{\partial q_j}\right]$$

$$= \frac{\partial}{\partial q_j}\underbrace{\left[\sum_{k=1}^{f} \frac{\partial \vec{r}_i}{\partial q_k}\cdot \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right]}$$

$$= \frac{\partial \vec{v}_i}{\partial q_j}$$

$$\therefore m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i\right)$$

$$= \frac{\partial T_i}{\partial \dot{q}_j} \qquad \left[T_i = \frac{1}{2} m_i v_i^2\right]$$

$$= \sum_{i,j}\left[\frac{d}{dt}\left\{\frac{\partial T_i}{\partial \dot{q}_j}\right\} - \left\{\frac{\partial T_i}{\partial q_j}\right\}\right]\delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left\{\frac{\partial \sum_i T_i}{\partial \dot{q}_j}\right\} - \left\{\frac{\partial \sum_i T_i}{\partial q_j}\right\}\right]\delta q_j$$

$$= \sum_{j=1}^{f} \underbrace{\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right\}}_{\Downarrow\ B}\delta q_j = 0 \qquad \left[\sum_{i=1}^{N} T_i = T\right]$$

20/01/18

$$\sum_j B_j \,\delta q_j = 0 \qquad , B_1 = 0,\ B_2 = 0,\ B_n = 0.$$

$$g_j = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \quad ✓$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = g_j \qquad \text{—— (1)}$$

Lagrange eq^n of motion of 2nd kind.

Conservative system →

$$\vec{F}_i = -\vec{\nabla}_i \cdot V(q_j) \qquad , \quad V \neq V(\dot{q}_j)$$

$$= -\frac{\partial V}{\partial \vec{r}_i}$$

$$g_j = \sum_{i=1}^{N} -\frac{\partial V}{\partial \vec{r}_i}\cdot\frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$g_j = -\frac{\partial V}{\partial q_j}$$

from (1) →

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 \qquad \left[\because V \neq V(\dot{q}_j),\ \frac{\partial V}{\partial \dot{q}_j} = 0\right]$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right) = 0}$$

$L = T - V$

↑

Lagrangian of the system.

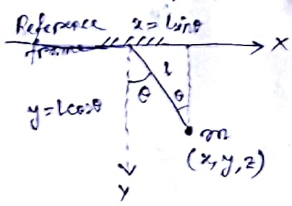
T = kinetic energy

$$L = \underset{\uparrow\ q_j,\ \dot{q}_j,\ t}{T} - \underset{\nwarrow\ q_j}{V} = L(q_j, \dot{q}_j, t)$$

[$\dot{q}_j$ = generalised velocity]

q_j = generalised co-ordinate

Eg. ① : Simple pendulum



$N = 1$

$$\left.\begin{array}{l} z = 0 \\ x^2 + y^2 = l^2 \end{array}\right] k = 2$$

$f = 3\times 1 - 2 = 1$

Generalised co-ordinate, $q_1 = \theta$

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$= \frac{1}{2} m l^2 \dot{\theta}^2$

$x = l\sin\theta$

$\dot{x} = l\cos\theta . \dot{\theta}$

$y = l\cos\theta$

$\dot{y} = - l\sin\theta . \dot{\theta}$

because direction is below ref. frame

~~$V = \ominus mgl(1-\cos\theta)$~~

$V = - mgl\cos\theta$

$L = T - V$

$= \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$

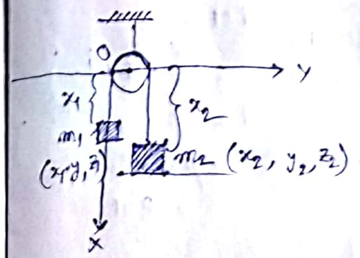
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$, $\frac{\partial L}{\partial \theta} = - mgl\sin\theta$

$\cancel{ml^2}\ddot{\theta} + \frac{\cancel{m}g\cancel{l}}{\cancel{ml^2}}\sin\theta = 0$

$\Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0} \Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$

- Atwood machine :-



$$\left.\begin{array}{l} z_1 = 0 \\ z_2 = 0 \\ y_1 = 0 \\ y_2 = 2a \\ x_1 + x_2 + \pi a = l \end{array}\right\} k = 5$$

$N = 2$

$f = 3\times 2 - 5 = 1$

$q_1 = x_1$

$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \cancel{\dot{y}_1^2}^{\,0} + \cancel{\dot{z}_1^2}^{\,0}) + \frac{1}{2} m_2 (\dot{x}_2^2 + \cancel{\dot{y}_2^2}^{\,0} + \cancel{\dot{z}_2^2}^{\,0})$

$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (-\dot{x}_1)^2$

$\left[\because x_2 = \underset{\substack{\downarrow \\ \text{const} \\ \downarrow \\ (L - \pi a)}}{A} - x_1\right]$

$= \frac{1}{2}(m_1 + m_2)\dot{x}_1^2$

$V = - m_1 g x_1 - m_2 g x_2$

$= - m_1 g x_1 - m_2 g (A - x_1)$

$= -(m_1 - m_2) g x_1 - B$

$L = T - V$

$= \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + (m_1 - m_2) g x_1 + B$

$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$

$\Rightarrow \frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1$

$\frac{\partial L}{\partial x_1} = (m_1 - m_2) g$

$(m_1 + m_2)\ddot{x}_1 = (m_1 - m_2) g$

$$\boxed{\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2} g}$$

$\Rightarrow L = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 + \frac{1}{2} m_3 \dot{q}_3^2 - m_1 m_2 \cos q_1 \frac{m_1}{m_2} \sin^{-1} q_3$

$L \neq L(q_\alpha) \rightarrow$ [Cyclic/Ignorable co-ordinate]

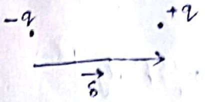
$\frac{\partial L}{\partial q_\alpha} = 0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0$

Generalised momentum (P_α) $\leftarrow \frac{\partial L}{\partial \dot{q}_\alpha}$ = conserved

DOF = 3, $\{\dot{q}_1, \dot{q}_2, \dot{q}_3\}$

(Dimension of configuration space = DOF.)

21/01/18 Physics (PC)

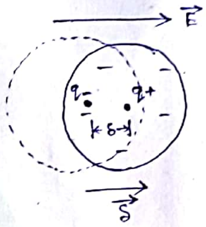


individual dipole moment

$\vec{p} = q\vec{\delta}$

- **Polarization :-**

1) Atomic Polarization $\longrightarrow$ induced dipole
2) Ionic Polarization
3) Oriental Polarization $\longrightarrow$ Permanent dipole

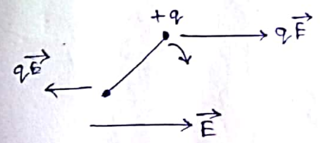


$\vec{p} = q.\vec{\delta}$

If $\vec{E} \uparrow$ Then $\delta \uparrow$

$\therefore p \propto E$

$\Rightarrow p = \alpha E$ [α = polarizibility]



- **Polarization ($\vec{P}$) :-** Dipole moment per volume

$N \longrightarrow$ no. of dipole moment/unit volume

$\vec{P} = N\vec{p} \Rightarrow N.\alpha.E \quad [\because p = \alpha E]$

- **Linear dielectrics :-**

$P = \chi_e \varepsilon_0 E \quad$ [ε_0 = permittivity]

[After inserting dielectric E not E_0]

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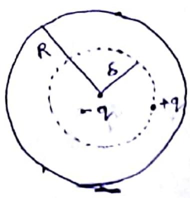
$\cancel{\chi_e}\ \cancel{E} = \quad p = N\alpha E$

$\Rightarrow \chi_e \varepsilon_0 = N\alpha$

$\Rightarrow \boxed{\chi_e = \dfrac{N\alpha}{\varepsilon_0}}$ $\quad [\chi_e = \text{Susceptibility}]$



$$\oint_S \vec{E}\cdot d\vec{a} = \frac{q_{enclosed}}{\varepsilon_0}$$

$$\Rightarrow E\oint_S da \longrightarrow E\cdot 4\pi \delta^2 = \frac{\rho}{\varepsilon_0}\frac{4}{3}\pi\delta^3$$

$$\Rightarrow \rho = \frac{q}{\frac{4}{3}\pi R^3} = \frac{q}{V_a}$$

$$\Rightarrow E = \frac{\rho\delta}{3\varepsilon_0}$$

$$\Rightarrow E = \frac{q\delta}{3\varepsilon_0 V} \quad \Rightarrow E = \frac{1}{3\varepsilon_0 V}p$$

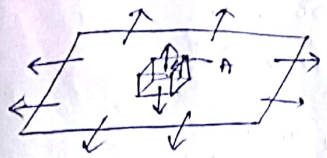
$$\Rightarrow p = (3\varepsilon_0 V)E$$

$$\Rightarrow p = \alpha E$$

$$\Rightarrow \boxed{\alpha = 3\varepsilon_0 V} \quad (\text{Imp})$$

31/...

$\Rightarrow$

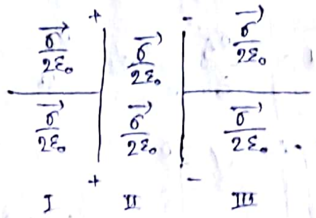


$$\oint_S \vec{E}\cdot d\vec{a} = \frac{q_{encl}}{\varepsilon_0}$$

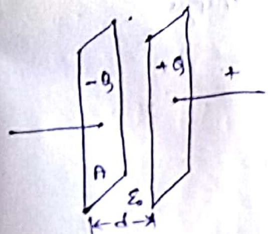
$$\Rightarrow E\int dA \longrightarrow EA$$

$$EA + EA = \frac{1}{\varepsilon_0}\sigma A$$

$$\Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$



$$\frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

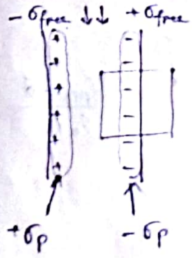


$$E_0 = \frac{\sigma}{\varepsilon_0} \qquad \Rightarrow \sigma = \frac{q}{A}$$

$$\Rightarrow E_0 = \frac{q_{free}}{\varepsilon_0 A} = \frac{\sigma_{free}}{\varepsilon_0}$$

$$\Rightarrow V = -\int_R^P \vec{E}\cdot d\vec{r} \qquad \Rightarrow V_0 = \frac{\sigma_{free}\cdot d}{\varepsilon_0\cdot A}$$

$$\Rightarrow C_0 = \frac{Q_{free}}{V_0} = \frac{\varepsilon_0 A}{d}$$



$$E = \frac{\sigma_{free} - \sigma_p}{\varepsilon_0}$$

$\sigma_p \equiv P \quad [P = \text{Polarization}]$

$$\Rightarrow E = \frac{\sigma_{free} - \chi_e\varepsilon_0 E}{\varepsilon_0}$$

$$\Rightarrow \varepsilon_0 E + \chi_e\varepsilon_0 E = \sigma_{free}$$

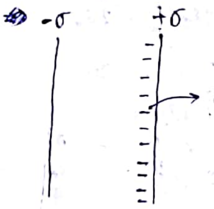
$$\Rightarrow \varepsilon_0 E(1+\chi_e) = \sigma_{free}$$

$$\Rightarrow E = \frac{\sigma_{free}}{\varepsilon_0(1+\chi_e)}$$

$\chi_e > 0$,
$\varepsilon_0(1+\chi_e) > \varepsilon_0$

$1+\chi_e = \varepsilon_r \longrightarrow$ Dielectric const.

$$\Rightarrow E = \frac{\sigma_{free}}{\varepsilon_0 \cdot \varepsilon_r} = \frac{\sigma_{free}}{\varepsilon} \qquad [\varepsilon = \varepsilon_0 \varepsilon_r]$$



$\sigma_p \rightarrow$ total polarization charge

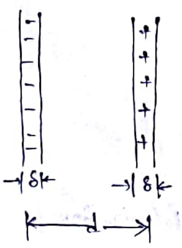
$$\sigma_p = \frac{Q_P/A}{\delta} = \frac{Q_P}{A\delta} = P$$

$$V = Ed = \frac{\sigma_{free} \cdot d}{\varepsilon_0 \varepsilon_r}$$

$$\Rightarrow C = \frac{Q_{free}}{V} = \frac{\cancel{Q_{free}} \cdot \varepsilon_0 \varepsilon_r \cancel{A}}{\cancel{Q_{free}} \cdot d} = \frac{\varepsilon_0 \varepsilon_r A}{d}$$

$$\Rightarrow C_0 = \frac{Q_{free}}{V_0} = \frac{\varepsilon_0 A}{d}$$

04/02/18



$$\frac{N(\cancel{A}\delta) \cdot q}{\cancel{A}} = \sigma_p$$

$$C_0 = \frac{Q_{free}}{V_0}$$

$$Q_{free} = C_0 V_0$$

$$\Rightarrow E_0 = \frac{\sigma_{free}}{\varepsilon_0}$$

$$\Rightarrow V_0 = \frac{\sigma_{free} \cdot d}{\varepsilon_0} = \frac{Q_{free} \cdot d}{\varepsilon_0 A}$$

$$\sigma_p' = Nq\delta = Np = P$$

$$E = \frac{\sigma_{free}}{\varepsilon_0(1+\chi_e)} = \frac{\sigma_{free}}{\varepsilon_0 \varepsilon_r} \qquad 1+\chi_e = \varepsilon_r$$

$$\Rightarrow V = \frac{\sigma_{free} \cdot d}{\varepsilon_0 \varepsilon_r} = \frac{Q_{free} \cdot d}{\varepsilon_0 \varepsilon_r A}$$

$$C = \frac{\cancel{Q_{free}}}{\cancel{Q_{free}} \cdot d} \varepsilon_0 \varepsilon_r A = \frac{\varepsilon_0 \varepsilon_r d}{A} \qquad \Rightarrow C = \frac{Q_{free}}{V}$$

$$C_0 = \frac{Q_{free}}{V_0} = \frac{Q_{free}}{Q_{free} \cdot d} \varepsilon_0 \cdot A = \frac{\varepsilon_0 A}{d}$$

$\varepsilon_r = \infty$ + for conductor

$\varepsilon_r = 1$ + for empty space.

$$E = \frac{\sigma_{free}}{\varepsilon} \qquad \boxed{\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}}$$

$$\vec{D} = \varepsilon_0 \cdot \vec{E} + \vec{P}$$

linear dielectric, $P = \chi_e \varepsilon_0 . E$

$$D = \varepsilon_0 E + P$$

$$= \varepsilon_0 E + \chi_e \cdot \varepsilon_0 . E = \varepsilon_0 \overbrace{(1+\chi_e)}^{\varepsilon_r} E$$

$$= \cancel{\varepsilon_0 (1+\chi_e)} \frac{\sigma_{free}}{\cancel{\varepsilon_0(1+\chi_e)}} =$$

$$\boxed{D = \sigma_{free}}$$

In dielectric material $\rightarrow$

$$E = \frac{\sigma_{free}}{\varepsilon_0(1+\chi_e)}$$

$$D = \varepsilon_0 E + \cancel{P}^{\,0}$$

$$= \varepsilon_0 E = \cancel{\varepsilon_0} \cdot \frac{\sigma_{free}}{\cancel{\varepsilon_0}}$$

$$\boxed{D = \sigma_{free}}$$

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1) Calculate the induced dipole moment per unit volume of gas if it is placed in a electric field of 6000 V/cm. Given $\alpha_{He} = 0.18 \times 10^{-40}$ F.m^2 and gas density 2.6×10^{25} atoms/m^3.

2) For argon, electric polarizibility is 1.45×10^{-40} F.m^2. Find the dielectric const. Given, gas density 1.8 g/cm^3 and atomic mass of argon $= 39.95$ g/mol.

---

1) $\vec{p} = \alpha . E = \cancel{2.6\times 10^{25}}\ 0.18 \times 10^{-40} \times 6000 \times 10^{2}$

$= 1.08 \times 10^{-35}$

$P = N . \vec{p}$

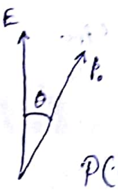
$= 2.6 \times 10^{25} \times 1.08 \times 10^{-35} = \underline{2.808 \times 10^{-10}}$

2) $\chi_e \varepsilon_0 = N\alpha$

$\Rightarrow \chi_e = \dfrac{N\alpha}{\varepsilon_0}$ $\Rightarrow \varepsilon_r = 1 + \chi_e$

$\dfrac{1.8 \text{ g/cm}^3}{39.95 \text{ g/mol}} = 0.04 \text{ mol/cm}^3$

$N = \dfrac{1.8 \times 10^{6} \times 6 \times 10^{23}}{39.95} /\text{m}^3$

• 

$\theta \to 0° \to 180°$

$\theta \to \pi$

$p = \langle p_0 \cos\theta \rangle = p_0 \langle \cos\theta \rangle$

$P(\theta)\, d\theta \to$ Probability of a dipole to allign on an angle b/w θ and $\theta + d\theta$ with E.

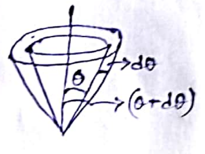
$\boxed{N = N_0 e^{-\varepsilon/kT}}$

↳ Population at energy ε. (Potential energy)

$U = -\vec{p}.\vec{E} = -pE\cos\theta$

$\int_0^{\pi} P(\theta)\, d\theta = 1$

$P(\theta) \sim e$ $\Rightarrow P(\theta) d\theta \sim 2\pi \sin\theta\, d\theta$



$P(\theta) d\theta = 2\pi \sin\theta\, e^{-U/kT}\, d\theta$

$\Rightarrow P(\theta) d\theta = 2\pi \sin\theta \exp(pE\cos\theta/kT) d\theta$

$\Rightarrow \langle \cos\theta \rangle = \dfrac{\int_0^{\pi} P(\theta) \cos\theta\, d\theta}{\int_0^{\pi} P(\theta)\, d\theta}$

sinh cosh cosech

$\langle x \rangle = \dfrac{\int x P(x) dx}{\int P(x) dx}$

$= \dfrac{\sum x_i P(x_i)}{\sum P(x_i)}$.

$\Rightarrow \langle \cos\theta \rangle = \dfrac{\int_0^{\pi} 2\pi \cos\theta \sin\theta . e^{pE\cos\theta/kT}\, d\theta}{\int_0^{\pi} 2\pi . \cos\theta . e^{pE\cos\theta/kT}\, d\theta}$

$x = a\cos\theta$ $\left[\dfrac{p_0 E}{kT} = a, \ \dfrac{p_0 E \cos\theta}{kT} = x \right]$

$\langle \cos\theta \rangle = \dfrac{\int_{-a}^{+a} x e^x dx}{\int_{-a}^{+a} e^x dx} = \dfrac{e^a + e^{-a}}{e^a - e^{-a}} - \dfrac{1}{a}$

$= \coth a - \dfrac{1}{a}$

$= L(a)$.

↳ Langevin f^n.

$$\langle \cos\theta \rangle = \frac{e^a + e^{-a}}{e^a - e^{-a}} - \frac{1}{a} \Rightarrow a = \frac{P_0 E}{KT}$$

07/02/18

$1 + \chi_e = \varepsilon_r$

$\alpha = \alpha_a + \alpha_i + \alpha_0$ [α_a = atomic polarizibility, α_i = ionic ", α_0 = orientational "]

$P = Np$

$= N\alpha E$

Linear dielectrics : $P = \chi_e \varepsilon_0 E$

$N\alpha = \chi_e \varepsilon_0$

$$\alpha = \frac{\varepsilon_0}{N}\chi_e = \frac{\varepsilon_0}{N}(\varepsilon_r - 1)$$

$$\frac{\varepsilon_0}{N}(\varepsilon_r - 1) = \alpha_a + \alpha_i + \alpha_0$$

$$\Rightarrow \varepsilon_r - 1 = \frac{N}{\varepsilon_0}(\alpha_a + \alpha_i + \alpha_0)$$

$p = \langle p_0 \cos\theta \rangle$

$= p_0 \langle \cos\theta \rangle$

$$\langle \cos\theta \rangle = \frac{\int_0^\pi P(\theta)\cos\theta\, d\theta}{\int_0^\pi P(\theta)\, d\theta}$$

$x = a\cos\theta$

$$\Rightarrow a = \frac{p_0 E}{KT}$$

$a \gg 1 \Rightarrow \frac{p_0 E}{KT} \gg 1$

$\Rightarrow p_0 E \gg KT$

$\Rightarrow E \gg \frac{KT}{P_0}$ $E\uparrow$

$\boxed{L(a) \sim 1}$

$\langle \cos\theta \rangle = 1$

$e^a + e^{-a}$

$$\approx 2\left(1 + \frac{a^2}{2!}\right)$$

$$L(a) = \frac{1 + \frac{a^2}{2!}}{a + \frac{a^3}{3!}} - \frac{1}{a}$$

$$= \frac{1}{a}\left[\frac{1 + \frac{a^2}{2}}{1 + \frac{a^2}{6}} - 1\right]$$

$$\Rightarrow L(a) = \frac{1}{a}\cdot\frac{1 + \frac{a^2}{2} - 1 - \frac{a^2}{6}}{1 + \frac{a^2}{6}} = \frac{1}{a}\cdot\frac{\frac{a^2}{3}}{1 + \frac{a^2}{6}} = \frac{a/3}{1 + \frac{a^2}{6}}$$

$a \ll 1$

$$\Rightarrow L(a) = \frac{a}{3}$$

$$= \frac{p_0 E}{3KT}$$

$p = p_0 \langle \cos\theta \rangle$

$= p_0 L(a)$

$$= p_0 \times \frac{p_0 E}{3KT}$$

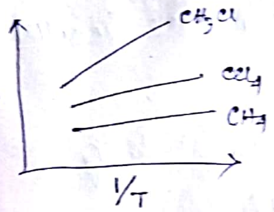
$$p = \frac{p_0^2}{3KT} E$$

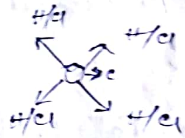
Comparing with $p = \alpha E$

$$\alpha_0 = \frac{p_0^2}{3K}\cdot\frac{1}{T}$$

$$\therefore \varepsilon_r - 1 = \frac{N}{\varepsilon_0}\left(\alpha_a + \alpha_i + \frac{p_0^2}{3K}\cdot\frac{1}{T}\right)$$

$\Rightarrow$





$\Rightarrow$ A parallel plate capacitor of area 4×5 cm^2 is filled with media ($\varepsilon_r = 6$), the distance between the plate is 1 mm, the capacitor is connected initially (throughout) 100 V battery ⟶

(a) Calculate free charge on plate

(b) the capacitance (c) polarized charge surface density.

$$V_0 = \frac{\sigma_f d}{\varepsilon_0 A}$$ ← 1×10^{-3} m, ← $4\times5\times10^{-4}$ m^2, ↑ 100V

$\Rightarrow q_f = ?$

$$C = \frac{q_f}{V} = \frac{q_f \varepsilon_0 \varepsilon_r A}{q_f d}$$

$$= \frac{\varepsilon_0 \varepsilon_r A}{d}$$ (ε_r = 6, d = 1×10^{-3} m)

$A = 4\times5\times10^{-4}$ m^2

$\sigma_p = P = \chi_e \epsilon_0 E = (\epsilon_r - 1)\epsilon_0 E$

$V_0 = 100\,V$ $\quad V \neq 100\,V$

$E = V.d = \frac{V_0}{\epsilon_r}.d$

14/03/18

Magnetic Properties

Paramagnetic $\quad \chi_m > 0$, odd no. of total $e^{\ominus}$

Diamagnetic $\quad \chi_m < 0$, even no. " " "

$q(\vec{v} \times \vec{B})$

Ampere's Circuital Law

$$\oint_l \vec{B}.\vec{dl} = \mu_0 . I_{enclosed}$$

line integral

$\Updownarrow$

Current density

$$\int_s \nabla \times \vec{B}.\, d\vec{a} = \mu_0 . \int_s \vec{J}.\, d\vec{a}$$

surface

$\Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J}$

[$\vec{M} \rightarrow$ Magnetic dipole Moment/vol $\downarrow$ Magnetization]

[$\vec{J_B} = \nabla \times \vec{M}$

$(\vec{K_B} = \vec{M} \times \hat{n})$]

$\Rightarrow \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_{free}$

$\Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J}$

$= \mu_0 (\vec{J}_{free} + \vec{J_B})$

$\Rightarrow \nabla \times \frac{\vec{B}}{\mu_0} = \frac{\cancel{\mu_0}}{\cancel{\mu_0}}(\vec{J}_{free} + \nabla \times \vec{M})$

$\Rightarrow \nabla \times \vec{H} = \vec{J}_{free}$

$\Rightarrow \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

$P = \chi_e \epsilon_0 E \Rightarrow P \propto E \quad P \propto D$

$M = \chi_m H$

$M = \chi_m H$

$\Rightarrow H = \frac{1}{\mu_0} B - M$

$\Rightarrow B = \mu_0 (H + M) = \mu_0 (H + \chi_m H)$

$\Rightarrow B = \mu_0 (1 + \chi_m) H \quad [\mu = \mu_0 (1 + \chi_m)]$

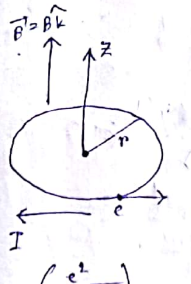
$= \mu H$

$B = \mu_0 (1 + \chi_m) H$

$= \frac{\mu_0 (1 + \chi_m)}{\chi_m} . M$

$\Rightarrow B = \frac{\mu}{\chi_m} M$

$\Rightarrow$



$I = \frac{e}{T} = \frac{eV_0}{2\pi r}$

$T = \frac{2\pi r}{V_0}$, $\quad V \simeq V_0$

$\mu_0 = I.\pi r^2 = \frac{eV_0}{2\cancel{\pi} r}.\cancel{\pi} r^2 = \frac{eV_0 r}{2}$

$\vec{\mu_0} = -\frac{eV_0 r}{2}.\hat{k}$

$\left(\frac{e^2}{4\pi\epsilon_0 r^2}\right)$

$F_0 = \frac{m v_0^2}{r}$

$\Rightarrow F_0 + evB = \frac{mv^2}{r}$

$\Rightarrow \vec{\mu'} = -\frac{evr}{2}.\hat{k}$

$\Delta\mu = \vec{\mu'} - \vec{\mu_0}$

$= -\frac{er}{2}(v - v_0).\hat{k}$

$e(\vec{v}\times\vec{B})$

$= eVB \sin 90^\circ = eVB$

$eVB = \frac{m}{r}\left(v^2 - v_0^2\right)$

$\Rightarrow eVB = \frac{m}{r}(v+v_0)(v-v_0)$

$\Rightarrow eV_0B = \frac{m}{r} 2V_0 \Delta V$

$\Rightarrow \Delta V = \frac{eBr}{2m}$

$P = NP$

$M = N\Delta\mu$

$\Rightarrow \Delta\mu = -\frac{e^2r^2}{4m}\vec{B}$

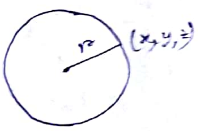
$\Rightarrow M = -\frac{Ne^2r^2}{4m}\cdot\vec{B}$

$\Rightarrow \Delta\mu = -\frac{er}{2}\Delta v\,\hat{k}$

$= -\frac{er}{2}\cdot\frac{erB}{2m}\hat{k}$

$= \frac{e^2r^2}{4m}(-B\hat{k})$

$\langle\vec{M}\rangle = -\frac{Nze^2}{4m}\langle r^2\rangle\vec{B}$



$r^2 = x^2+y^2+z^2$

$\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle + \langle z^2\rangle$

$= 3\langle x^2\rangle$

$r^2 = x^2+y^2$

$\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle$

$= 2\langle x^2\rangle$

$\langle x^2\rangle = \frac{\langle r^2\rangle}{3}$

$\Rightarrow \langle x^2\rangle = \frac{\langle r^2\rangle}{2}$

$\frac{\langle r^2\rangle}{2} = \frac{\langle R^2\rangle}{3} \Rightarrow \langle r^2\rangle = \frac{2}{3}\langle R^2\rangle$

$\vec{M} = -\frac{Nze^2\langle R^2\rangle}{6m}\cdot\vec{B}$

$\boxed{\chi_m = -\frac{Nze^2\langle R^2\rangle\mu_0}{6m}}$

$\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0(1+\chi_m)\vec{H}$

$\chi_m \ll 1$

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$\vec{M} = \chi_m\vec{H}$

$\vec{B} = \mu_0(\vec{M}+\vec{H})$

$\chi_m > 0$ for paramagnetic

< 0 " diamagnetic

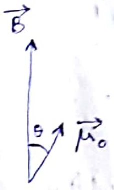
for para/diamagnetic → $\chi_m \sim 10^{-5}$, $\chi_m \ll 1$

$\vec{B} = \mu_0(1+\chi_m)\vec{H} \simeq \mu_0\vec{H}$

for diamagnetic → $\chi_m = -\frac{Nze^2\langle R^2\rangle\mu_0}{6m}$

For Paramagnetic →

$\vec{\mu_0}$



$M = N\langle\mu\rangle$

$\langle\mu\rangle = \langle\mu_0\cos\theta\rangle$

$= \mu_0\langle\cos\theta\rangle$

$\Rightarrow \langle\cos\theta\rangle = \frac{\int_0^{2\pi} P(\theta)\cos\theta\,d\theta}{\int_0^{2\pi}\cos\theta\,d\theta} = L(a)$ ↗ $a \equiv \frac{\mu_m B}{k_BT}$

$\Rightarrow P(\theta) \sim e^{-U/k_BT}$

$U = -\vec{\mu_m}\cdot\vec{B}$

$= -\mu_m\cdot B\cos\theta$

$P(\theta)d\theta \sim 2\pi\sin\theta\,d\theta$

$\Rightarrow P(\theta)d\theta = 2\pi\sin\theta\cdot e^{\frac{\mu_m B\cos\theta}{k_BT}}d\theta$

$\Rightarrow \langle\cos\theta\rangle = \frac{2\pi\int_0^{\pi}\cos\theta\cdot\sin\theta\, e^{\mu_m B\cos\theta/k_BT}d\theta}{2\pi\int_0^{\pi}\sin\theta\cdot e^{\mu_m\cos\theta/k_BT}d\theta}$

$= \frac{\int_0^a x\cdot e^x dx}{\int_0^a e^x dx} = \frac{e^a+e^{-a}}{e^a-e^{-a}}$

$\mu = \mu_m^\circ \langle \cos\theta \rangle$

$= \mu_m^\circ L(a)$

$\langle \mu \rangle = \mu_m^\circ \times \frac{\mu_m^\circ B}{3K_BT}$

$= \frac{\mu_m^{\circ 2} B}{3K_BT}$

Close to 0K →

$\chi_m = -1$, $\vec{B} = 0$ superconductivity.

$M = N\langle\mu\rangle$

$= N\frac{\mu_m^{\circ 2} B}{3K_BT}$

for paramagnetic →

$B = \mu_0 H$

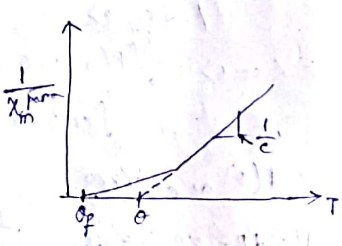
$\Rightarrow M = \frac{N\mu_m^{\circ 2}\mu_0}{3K_BT} H$

$\chi_m^{para} = \frac{N\cdot\mu_m^{\circ 2}\cdot\mu_0}{3K_BT}$

$\chi_m^{para} = \frac{C}{T}$

$\Rightarrow C = \underbrace{\frac{N\mu_m^{\circ 2}\mu_0}{3K_B}}_{const.}$

$\Rightarrow \underbrace{\frac{1}{\chi_m^{para}}}_{y} = \frac{T}{C} = \underbrace{\frac{1}{C}}_{slope}\cdot \underbrace{T}_{X}$



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⇒ The magnetic field (H) intensity in ferrite is 10^6 A/m. If the susceptibility of the material at room temperature is 1.5×10^{-3}, compute the magnetization of the material and the magnetic field induction (Mag. field B) ($\mu_0 = 4\pi \times 10^{-7}$ N/A²)

Soln: $B = \mu_0(M + H)$

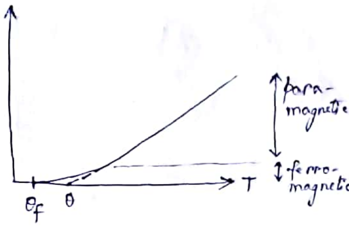
$= \mu_0(1 + \chi_m) H$

$M = \chi_m H$

$\frac{1}{\chi_m^{para}} = \frac{T}{C}$

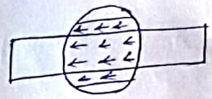
θ → Paramagnetic Curie Temp.

θ_f → Ferromagnetic Curie Temp.



1) What are Weiss considerations?

2) Deduce Curie-Weiss Law.



⇒ Curie-Weiss: $\frac{1}{\chi_m^{para}\cdot f} = \frac{T-\theta}{C}$

$\frac{1}{\chi_m} = \frac{T}{C} \Rightarrow \chi_m = \frac{C}{T}$

$M = \chi_m H = \frac{C}{T}(H + H_m)$ $[H_m = \lambda M]$

$\Rightarrow M = \frac{C}{T}(H + \lambda M)$

$\Rightarrow TM = CH + C\lambda M$

$M(T - C\lambda) = CH$

$\Rightarrow \boxed{M = \frac{C}{T - C\lambda}\cdot H}$

$\chi_m^{para} = \frac{C}{T - C\lambda}$

$= \frac{C}{T - \theta}$

$\Rightarrow \theta = C\lambda$

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- ## Hamilton's eqns of motion $\longrightarrow$

$$L = T - V = L(q_j, \dot{q}_j, t)$$

$$\Big\downarrow \text{transform}$$

$$H = H(q_j, p_j, t) \Longrightarrow \text{Hamiltonian of the system}$$

where, p_j = generalised momentum

$$\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j}}$$

» Hessian matrix :- (for transforming $\dot{q}_j \longrightarrow p_j$).

$$\left| \frac{\partial^2 L}{\partial \dot{q}_j \cdot \partial \dot{q}_k} \right| \neq 0$$

» Defn of Hamiltonian funcn :-

$$\boxed{H(q_j, p_j, t) = \sum_{j=1}^{f} p_j \dot{q}_j - L(q_j, \dot{q}_j, t)} \longrightarrow (1)$$

$\Rightarrow$ d(LHS) = d(RHS)

$$\Downarrow$$

$$\sum_j \frac{\partial H}{\partial q_j} \cdot dq_j + \sum_j \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt$$

$\Rightarrow$ d(RHS) $\Rightarrow$

$$\sum_j p_j \cancel{d\dot{q}_j} + \sum_j \dot{q}_j dp_j - \sum \underbrace{\left\{\frac{\partial L}{\partial q_j}\right\}}_{\dot{p}_j} dq_j - \cancel{\sum \underbrace{\left\{\frac{\partial L}{\partial \dot{q}_j}\right\}}_{p_j} d\dot{q}_j} - \frac{\partial L}{\partial t} \cdot dt$$

from $d(LHS)$ & $d(RHS)$ $\longrightarrow$

$$\underbrace{\boxed{-\dot{p}_j = \frac{\partial H}{\partial q_j} \quad , \quad \dot{q}_j = \frac{\partial H}{\partial p_j}} }_{\text{Hamilton's eq}^n\text{s of motion}} \quad , \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

q_1 & p_1 $\longrightarrow$ canonically conjugate components

>> from (1) $\longrightarrow$

$$H \neq H(q_\alpha)$$

$$\frac{\partial H}{\partial q_\alpha} = 0$$

$$\Rightarrow \dot{p}_\alpha = 0$$

$$\Rightarrow p_\alpha = \text{const.}$$

If a particular generalised co-ordinate is cyclic wrt Hamiltonian of the system, then the corresponding generalized momentum will be a conserved quantity for a particular system.

$$L \neq L(q_\alpha)$$

$$\Rightarrow \frac{\partial L}{\partial q_\alpha} = 0$$

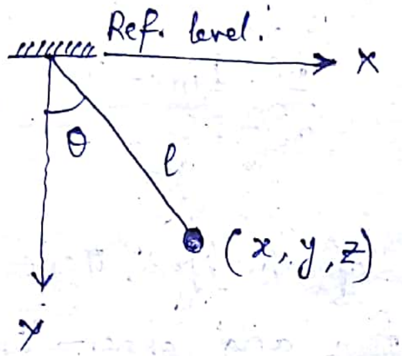
$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0$$

$$\Rightarrow \dot{p}_\alpha = 0$$

$$\Rightarrow \frac{\partial H}{\partial q_\alpha} = 0 \quad \Rightarrow \quad \boxed{H \neq H(q_\alpha)}$$

If a generalized co-ordinate is cyclic in Lagrangian, then that co-ordinate will be cyclic in Hamiltonian too.

⇒ <u>Simple Pendulum</u> :-



$x = l\sin\theta$

$y = l\cos\theta$

Generalised co-ordinate,

$q_1 = \theta$.

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta.$$

$$H(\theta, p_\theta, t) = p_\theta \cdot \dot{\theta} - L \quad \text{--- } ①$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad \dot{\theta} = \frac{p_\theta}{ml^2}$$

∴ <u>from ①</u> ⟶

$$H(\theta, p_\theta, t) = p_\theta \cdot \dot{\theta} - L$$
$$= ml^2\dot{\theta}^2 - \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$$
$$= \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$$
$$= \frac{1}{2} \cdot \frac{\cancel{(ml^2)}}{(ml^2)^{\cancel{2}}} \cdot p_\theta^2 - mgl\cos\theta$$
$$= \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$$

$$\Rightarrow -\dot{p}_\theta = \frac{\partial H}{\partial \theta} \qquad \Rightarrow \boxed{\dot{p}_\theta = -mgl\sin\theta}$$

$$\Rightarrow \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

$$\Rightarrow \boxed{p_\theta = ml^2\dot{\theta}}$$

Replacing, we get ⟶

$$ml^2\ddot{\theta} = -mgl\sin\theta$$

$$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

$\boxed{\text{Phase Space}}$ $\downarrow$ $2f$ Dimension

» In case of Lagrangian eq^n of motion, there configuration of space = f. and generalised co-ordinate = q_j.

But in case of Hamiltonian eq^{n}s of motion, generalised co-ordinate = $\dot{q}_j$ & p_j.
So, instead of config. of space, we write
Phase space = $2f$ Dimension.

19/03/18

BAND THEORY OF SOLIDS

Solids $\longrightarrow$ state of matter
↳ comprised of atoms
↳ nuclei of $e^{\ominus}$s
↓
follows the law of QM.

↗ operator

$$\hat{O}\, f(x) = \alpha\, f(x)$$

Eigen func^n ↑ Eigen value ↑

Schrodinger Eq^n

$$\hat{H}\psi = E\psi$$

Total energy operator.

$$\Rightarrow \frac{d}{dx} \cdot e^{\alpha x} = \alpha \cdot e^{\alpha x}.$$

$$\hat{H} = KE + PE$$

$$= \frac{\hat{p}^2}{2m} + \hat{V}$$

$$= -\frac{\hbar^2}{2m}\nabla^2 + V$$

$$\left[\begin{array}{l} \hat{p} \longrightarrow -i\hbar\nabla \\ \hat{V} \longrightarrow V \end{array}\right]$$

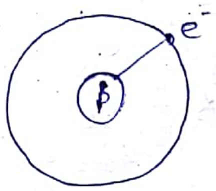
$\frac{d}{dx}$ for 1-D.

$\hat{H}\psi = E\psi$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi}$$

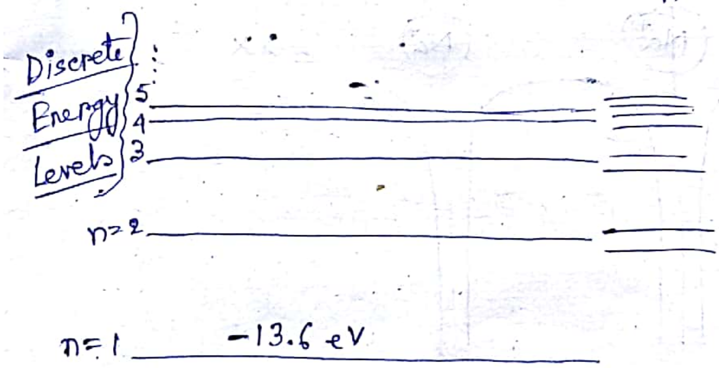
??

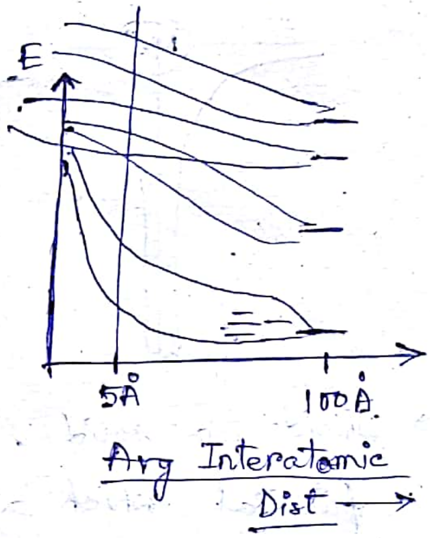
» **H-atom :**



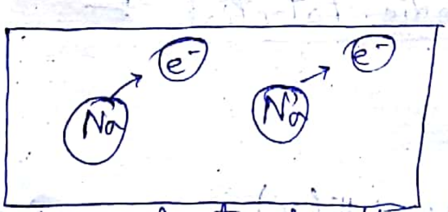
$V(r) = \frac{(+e)(-e)}{r}$

$E_n = -\frac{13.6}{n^2}\,eV$





» $\underset{11}{Na} \longrightarrow \underbrace{1s^2\, 2s^2\, 2p^6}_{\text{core } e^-\text{'s}}\ \underbrace{3s^1}_{\text{valence } e^-\text{'s}}$



→ Total no. of atoms in solid

$[\because N \to 10^{22}]$

$N \longrightarrow Na^+$ atoms $(+Ne)$

$e^- \longrightarrow (N-1)$ no. of $e^\ominus \longrightarrow [-(N-1)e] \simeq (-Ne)$

$\therefore$ Total $V \simeq 0$ on e^-.

Hence valence $e^\ominus$s are free electrons.

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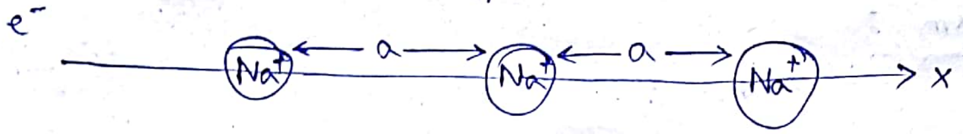
Normal free e^- | free e^- inside solid

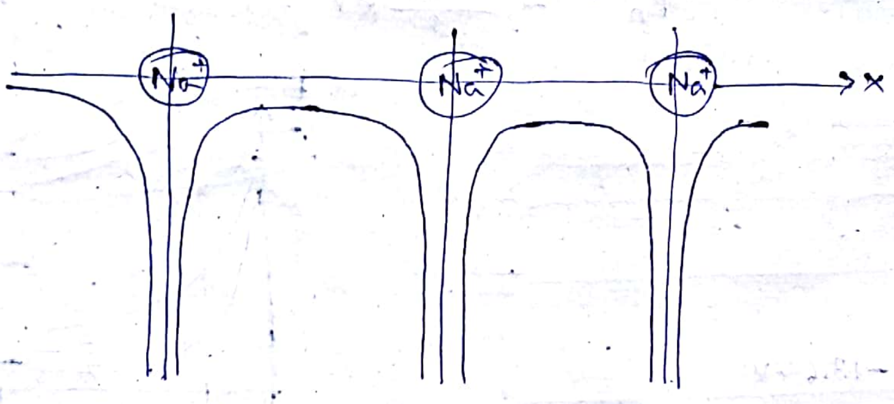
1-D crystal lattice

$a \rightarrow$ lattice const

Periodic potential



Potential func^n, $V(x) = \frac{+ze.(-e)}{x} \propto -\frac{1}{x}$



» Inside a solid, a free $e^\ominus$ moves with periodic potential which a value equal to lattice const.

» for normal $e^\ominus$, $e^\ominus$ moves with zero periodic potential.

- Quantum Mechanics in Periodic Potential :-

» $V(x+a) = V(x)$

» $-\frac{\hbar^2}{2m} \cdot \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$ —— ①

Th. ① :- If $V(x+a) = V(x)$, then $H(x+a) = H(x)$

$x + a = x'$

$H(x) = -\frac{\hbar^2}{2m} \cdot \frac{d^2}{dx^2} + V(x)$

$\frac{d}{dx'} \equiv \frac{d}{dx}$

$\Rightarrow H(x+a) = H(x')$

$V(x+a) \rightarrow V(x)$

$\Rightarrow \frac{d^2}{dx'^2} \equiv \frac{d^2}{dx^2}$

$= -\frac{\hbar^2}{2m} \cdot \frac{d^2}{dx'^2} + V(x')$

$= -\frac{\hbar^2}{2m} \cdot \frac{d^2}{dx^2} + V(x) = H(x)$

Th. ② $H\Psi = E\Psi$

$\Psi(x)$ is an Eigen func^n of $H(x)$ with Eigen value E. Lets define an operator $\hat{T}_a$ such that

$\hat{T}_a f(x) = f(x+a)$.

↳ giving $f(x)$ a translational motion.

$\hat{T}_a \rightarrow$ lattice translational operator.

[$\{\hat{T}_a \Psi(x)\}$ will be an eigen func^n of $H(x)$ with eigen value E.

⇒ We know, $H(x)\,\Psi(x) = E\Psi(x)$

$\Rightarrow \hat{T}_a\{H(x)\,\Psi(x)\} = \hat{T}_a\{E\Psi(x)\}$

$\Rightarrow H(x+a)\,\Psi(x+a) = E.\Psi(x+a)$

$\Rightarrow \underbrace{H(x).\{\hat{T}_a\Psi(x)\} = E\{\hat{T}_a\Psi(x)\}}$ [applying, Th.①]

⇓

$\hat{T}_a\Psi(x) = \Psi(x+a) \quad \alpha\,\Psi(x)$

$\Rightarrow \hat{T}_a\Psi(x) = \Psi(x+a) = \lambda\,\Psi(x)$

$\lambda = ??$

⇒ **Floquet's Th.** :

If in a periodic lattice with $V(x) = V(x+a)$ and $\hat{T}_a\Psi = \lambda\Psi$ then, λ is a complex no. of unit modulus; i.e. $\lambda = e^{i\theta}$

Let $u_1(x)$ & $u_2(x)$ are two independent sol^{n}s of Schrodinger Eq^n.

$$u_1(x+a) = M_{11}\, u_1(x) + M_{12}\, u_2(x)$$
$$u_2(x+a) = M_{21}\, u_1(x) + M_{22}\, u_2(x)$$

$$\Downarrow$$

$$\begin{pmatrix} u_1(x+a) \\ u_2(x+a) \end{pmatrix} = \underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}_{\text{Transfer matrix}} \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$$

$$\psi(x) = A\, u_1(x) + B\, u_2(x)$$ [Any func^n of $\psi(x)$ is a linear func^n of $u_1(x)$ & $u_2(x)$]

$$\Rightarrow \psi(x+a) = A\, u_1(x+a) + B\, u_2(x+a)$$
$$\Rightarrow \lambda \psi(x) = A\left(M_{11} u_1(x) + M_{12} u_2(x)\right) + B\left(M_{21} u_1(x) + M_{22} u_2(x)\right)$$ [$\because \psi(x+a) = \lambda \psi(x)$]

$$\Rightarrow \lambda A\, u_1(x) + \lambda B\, u_2(x) = \left(AM_{11} + BM_{21}\right) u_1(x) + \left(AM_{12} + BM_{22}\right) u_2(x)$$

$$\Rightarrow AM_{11} + BM_{21} = \lambda A$$
$$AM_{12} + BM_{22} = \lambda B$$

$$(M_{11} - \lambda)A + M_{21} B = 0$$
$$M_{12} A + (M_{22} - \lambda) B = 0$$

$$\begin{vmatrix} M_{11} - \lambda & M_{21} \\ M_{12} & M_{22} - \lambda \end{vmatrix} = 0 \quad \begin{matrix} \nearrow \lambda_1 \\ \searrow \lambda_2 \end{matrix}$$

[λ has 2 values so, 2 wave func^n can have]

$$\psi_1(x+a) = \lambda_1 \psi_1(x)$$
$$\psi_2(x+a) = \lambda_2 \psi_2(x)$$

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$$\lambda \to \lambda_1 \Rightarrow \psi_1(x+a) = \lambda_1 \psi_1(x)$$

$$\lambda \to \lambda_2 \Rightarrow \psi_2(x+a) = \lambda_2 \psi_2(x)$$

Define, $W(x) = \psi_1(x)\psi_2'(x) - \psi_2(x)\psi_1'(x)$

$$\Rightarrow W'(x) = \cancel{\psi_1'\psi_2'} + \psi_1\psi_2'' - \cancel{\psi_2'\psi_1'} - \psi_2\psi_1''$$

$$= \psi_1\psi_2'' - \psi_2\psi_1''$$

$$\psi_1'' + \frac{2m}{\hbar^2}\left[E - V(x)\right]\psi_1 = 0 \qquad \times \psi_2 \qquad \to \text{from } ①$$

$$\psi_2'' + \frac{2m}{\hbar^2}\left[E - V(x)\right]\psi_2 = 0 \qquad \times \psi_1$$

$$\psi_2\psi_1'' - \psi_1\psi_2'' = 0$$

$\therefore W'(x) = 0$

$\Rightarrow W(x) =$ const wrt x

$\Rightarrow W(x+a) = W(x)$

$$\therefore W(x) = W(x+a) = \hat{T}_a W(x)$$

$$= \hat{T}_a\left[\psi_1(x)\psi_2'(x) - \psi_2(x)\psi_1'(x)\right]$$

$$= \psi_1(x+a)\psi_2'(x+a) - \psi_2(x+a)\psi_1'(x+a)$$

$$= \lambda_1\psi_1(x)\lambda_2\psi_2'(x) - \lambda_2\psi_2(x)\lambda_1\psi_1'(x)$$

$$= \lambda_1\lambda_2\left[\psi_1\psi_2' - \psi_2\psi_1'\right]$$

$$= \lambda_1\lambda_2 W(x)$$

$$\Rightarrow \boxed{\lambda_1\lambda_2 = 1}$$

~~Let~~ $|\lambda| > 1$ say $\lambda = 5$

$\Rightarrow \psi(x+a) = 5\psi(x)$

$\Rightarrow \psi(x+2a) = 5\psi(x+a) = 5^2\psi(x)$

$\Rightarrow \psi(x+na) = 5^n \psi(x)$

$\therefore$ Prob. of finding e^- at 1st lattice pt $= |\psi\psi^*|$.

" " " " " nth " " $= 5^{2n}|\psi\psi^*|$

$\downarrow$ monotonically increasing!

If $|\lambda|<1$ say $\frac{1}{5}$

$\psi(x+na) = \frac{1}{5^n}\psi(x)$ ×

$\downarrow$ monotonically decreasing.

Hence, $|\lambda| \not> 1$.

$\therefore$ $\boxed{|\lambda| = 1}$

$\lambda = e^{\pm i\theta}$

$\hat{T}_a \psi_1(x) = \psi_1(x+a) = e^{i\theta}\psi_1(x)$

$\hat{T}_a \psi_2(x) = \psi_2(x+a) = e^{-i\theta}\psi_2(x)$.

$\theta \longrightarrow$ angle , $\theta \propto a$.

$\Rightarrow \theta = ka$

$[\theta] \longrightarrow$ dimensionless.

$[a] \longrightarrow$ length, L

$\therefore [k] = \frac{1}{\text{length}}$

$\therefore k =$ Some kind of wave vector.

$\therefore$ $\boxed{\psi(x+a) = e^{ika}\psi(x)}$

$\therefore$ $\boxed{\psi(x+na) = e^{ikna}\psi(x)}$

BLOCH'S THEOREM

are valid when $\psi(x) = e^{ikx}\cdot\phi(x)$

***

Bloch Theorem :- (in 1D).

If the potential $V(x)$ is periodic with periodicity of the lattice (a), then the solution $\psi(x)$ of the wave eq^n, $H\psi = E\psi$ with $\longrightarrow$

$H = -\frac{\hbar^2}{2m}\cdot\frac{d^2}{dx^2} + V(x)$

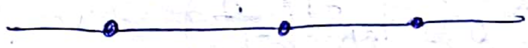
is of the form, $\underline{\psi(x) = e^{ikx}\,\phi(x)}$, for a given E where $\phi(x)$ is a periodic funcⁿ,

$$\phi(x+a) = \phi(x).$$

$$\left\{\begin{aligned} &\psi(x) = e^{ikx}\,\phi(x) \\ \Rightarrow\ &\psi(x+a) = e^{ik(x+a)}\,\phi(x+a) \\ \Rightarrow\ &\psi(x+a) = e^{ika}\left[e^{ikx}\cdot\phi(x)\right] \\ &\qquad = e^{iKa}\cdot\psi(x). \end{aligned}\right.$$

[K = Bloch wave vector]
↓
cap 'K'

$\psi(x) \longrightarrow$ Bloch wave eqⁿ.



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$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - V(x)\right)\psi = 0$$

$\psi \sim e^{\pm ikx} \longrightarrow$ small 'k': wave vector for free particle.

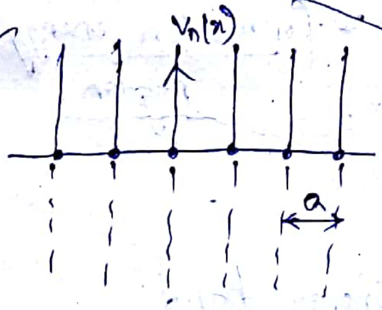
- Relation b/w K & k :- (Dispersion Relⁿ)

Cronig - Penney Model :

$$V(x) = \frac{\hbar^2}{m}\Omega \sum_{n=-\infty}^{+\infty} V_n(x)$$

[n → lattice points]

$$V_n(x) = \delta(x - na) = 1 \quad \text{when } x = \pm na$$
$$= 0 \quad \text{otherwise.}$$



Dirac Comb potential

⇓ we have to from here, write dispersion relⁿ.

Dispersion Relⁿ :-

$$\cos Ka = \cos ka + \frac{\Omega}{k}\sin ka$$

$$|\cos Ka| \le 1$$

$$\left|1.\cos ka + \left(\frac{\Omega}{k}\right)\sin ka\right| \le 1$$

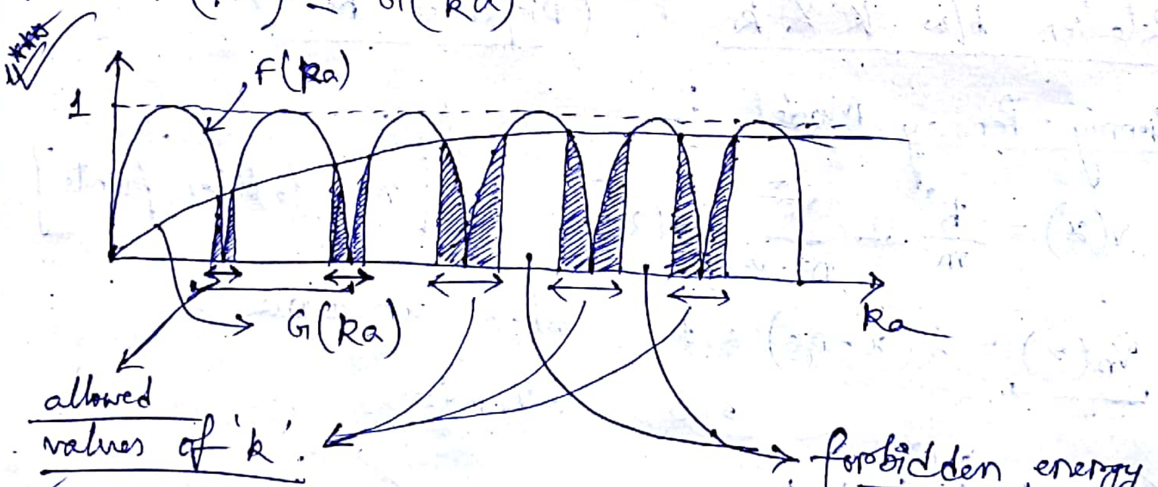
$$\Rightarrow \left|\varepsilon\cos\sigma\cos ka + \varepsilon\sin\sigma\sin ka\right| \le 1$$

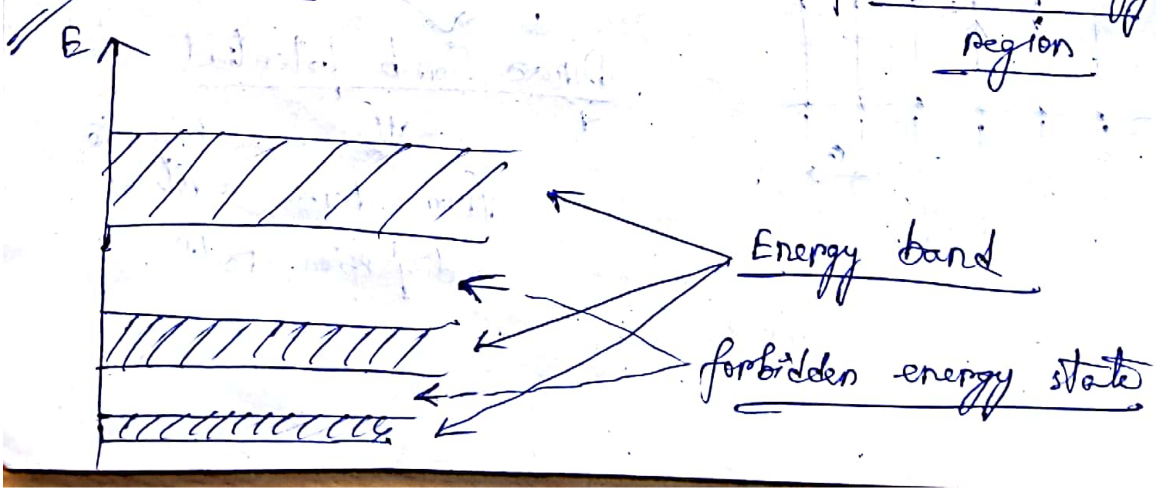
Let,
$$\left.\begin{aligned} 1 &= \varepsilon\cos\sigma \\ \frac{\Omega}{k} &= \varepsilon\sin\sigma \end{aligned}\right\} \Rightarrow \left|\varepsilon\cdot\cos(ka-\sigma)\right| \le 1$$

$$\varepsilon = \sqrt{1+\frac{\Omega^2}{k^2}} \quad , \quad \sigma = \tan^{-1}\left(\frac{\Omega}{k}\right)$$

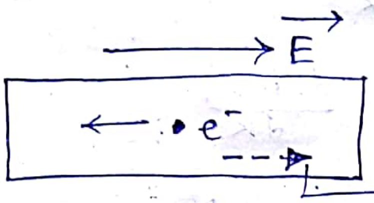
$$\Rightarrow \left|\cos\left\{ka - \tan^{-1}\left(\frac{\Omega a}{ka}\right)\right\}\right| \le \frac{1}{\left|\sqrt{1+\frac{\Omega^2 a^2}{k^2 a^2}}\right|}$$

$$\Rightarrow F(ka) \le G(ka)$$





- Effective mass :



can move in this dir^n also. (∵ quantum particles)

$\vec{F} = m\vec{a}$

$\Rightarrow -e\vec{E} = m\vec{a}$

1) e^- is a quantum mech. particle.

2) In quantum mech. there is no concept of mass. It is a wave eq^n.

$$E(k) = \cancel{E_0}^{\,0} + \cancel{\frac{\partial E}{\partial k}}^{\,0}\Big| k + \frac{\partial^2 E}{\partial k^2}\Big| \frac{k^2}{2!} + \cdots$$

(Taylor series expansion)

Energy of particle is a function of 'k'.

$\Rightarrow E(-k) = E(k)$

[∵ Energy of a particle is dir^n independent]

$$E(k) = \frac{k^2}{2}\left(\frac{\partial^2 E}{\partial k^2}\right) \quad \text{——} \ (1)$$

for a normal free e^- ⟶

$$E = \frac{\hbar^2 k^2}{2m} \quad \text{——} \ (2)$$

$$\Rightarrow \frac{\hbar^2}{2m} = \frac{1}{2}\left(\frac{\partial^2 E}{\partial k^2}\right)$$

$$\Rightarrow \boxed{m^* = \frac{\hbar^2}{\left(\frac{d^2E}{dk^2}\right)}}$$

[m^* ⟶ Effective mass]

11/04/18.

~~Group~~ Group velocity, $v_g = \frac{d\omega}{dk}$

$\downarrow$ (Velocity of the particle in the particular band.)

$$= \frac{1}{\hbar} \cdot \frac{d(\hbar\omega)}{dk}$$

$$= \frac{1}{\hbar} \cdot \frac{dE(k)}{dk}$$

$\left[\hbar\omega = \frac{h}{2\pi} \cdot \omega = h\nu = E \right]$

$\frac{dE(k)}{dk} \longrightarrow$ odd func^n of k.

$v_g(-k) = -v_g(k)$

$p = m \cdot v_g(k)$ [$p \to$ linear momentum]

$= (-m) \cdot \{-v_g(k)\}$

$= \underbrace{(-m)}_{\text{Hole}} \cdot \{v_g(-k)\}$

- Prob. current density :-

$j = n e v_g(k)$

$= n(-e)\{-v_g(k)\}$

$= n\underbrace{(-e)}_{\text{Hole}}\{v_g(-k)\}$

Dispersion rel^n given.

$$E = E_0 + \alpha \cos(ka)$$

i) $m^* = ?$

ii) identify the particle.

iii) max and min value of Energy. [(coska) value]

iv) group velocity

Derive exp. of m^*.