Distribution of more than one dimension

Let E be the random experiment of throwing a die and then tossing a coin. The event space of E is given by

$$S = \{(1,H),(2,H),(3,H)\ldots,(6,H),(1,T),\ldots,(6,T)\}$$

which contains $6\times 2 = 12$ distinct outcomes. A random variable X is defined ~~by~~ on S as follows:

$$X(i,H) = i \text{ for } i=1,2,\ldots,6$$
$$X(i,T) = i \text{ for } i=1,2,\ldots,6$$

Also let another r.v. Y be defined ~~by~~ on S as follows:

$$Y(i,H)=0, \quad Y(i,T)=-1 \quad \text{for } i=1(1)6.$$

Then the above two r.v. can be described as follows:

X denote the no. of the die and

$Y = 0$ if 'head' appears
$= -1$ if "tails" appears.

$(X,Y)(2,H) = (2,0)$ can be described as $(X=2, Y=0)$ which can be read as 'joint occurrence of the events $(X=2)$ and $(Y=0)$'.

In general if S be an event space and if X, Y are random variables defined on S, then the two-dimensional random variables (X,Y) can be defined as a mapping $(X,Y): S \longrightarrow R^2$, where every $w \in S$, for every $\omega \in S$ $(X,Y)(\omega) := (X(\omega), Y(\omega)) \in R^2$.

The range of this ~~spectrum~~ mapping is called the spectrum of the two-dimensional r.v. (X,Y).

- for two real no. a, b, $(X=a, Y=b)$ represents the simultaneous occurance of the events $(X=a)$ & $(Y=b)$.

- $(a < X \leq b,\ c < Y \leq d)$ represents 'simultaneous occurance of the events $a < X \leq b$ and $c < Y \leq d$.

Distribution fn

Let X and Y be two r.v. defined on the same event space S and let P be a given probability fn defined ~~by~~ on a given class of subsets (of S) forming the class Δ of events.

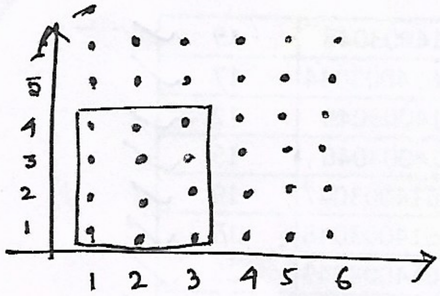
The joint distribution fn of two random variable X, Y or the distribution fn of two dimensional r.v. (X,Y) is a fn $F: R \times R \longrightarrow R$ defined by

$$F(x,y) = P(-\infty < X \leq x,\ -\infty < Y \leq y) \quad \forall (x,y) \in R.$$

For n-dimensional distribution,

$$F(x_1, x_2, .., x_n) = P(-\infty < X_1 \leq x_1, -\infty < X_2 \leq x_2, \dots, -\infty < X_n \leq x_n)$$

$$\forall\ x_1, x_2, .., x_n \in R.$$



$$F(3,4) = P(-\infty < X \leq 3, -\infty < Y \leq 4)$$
$$= \frac{12}{36} = \frac{1}{3}.$$

outcomes of the r. experiment of throwing a pair of dice.

Properties.

1. $F(x,y)$ is monotonically non-decreasing in both the variables x & y.

   If y fixed, $x_2 > x_1$ $\quad F(x_2, y) \geq F(x_1, y)$

   If x fixed, $y_2 > y_1$ $\quad F(x, y_2) \geq F(x, y_1)$.

2. $0 \leq F(x,y) \leq 1$.

3. $P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$

   $= F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$

   where, $x_2 > x_1$, $y_2 > y_1$

4. $F(\infty, \infty) = 1$ $\quad F(-\infty, y) = 0 = F(x, -\infty)$

5. $F(a+0, c) = F(a, c)$

   $F(a, c+0) = F(a, c)$

A real valued fn $F(x,y)$ of two real variables x & y is a possible distribution fn corresponding to a two dimensional r.v. with respect to a suitable probability space if

i) $F(x,y)$ is m.i in both the variables and continuous to the right with respect two both the variables.

ii) $F(-\infty, y) = F(x, -\infty) = 0$, $F(\infty, \infty) = 1$

iii) for every pair of points (x_1, y_1) and (x_2, y_2) with $x_1 < x_2$ & $y_1 < y_2$

$$F(x_2, y_2) + F(x_1, y_1) - F(x_2, y_1) - F(x_1, y_2) \geq 0.$$

Marginal distribution

Given the distribution of a two-dimensional random variable (X, Y), we now discuss how to

Marginal Distribution

$F(x,y) \to$ Joint distrubution,
Indevidual distributions of X & Y is called the marginal distributions of X & Y.

$$F_X(x) = P(-\infty < X \le x).$$
$$F_Y(y) = P(-\infty < Y \le y).$$

$$F(x,\infty) = P(-\infty < X \le x, -\infty < Y < \infty)$$
$$= P\{(-\infty < X \le x) \cap S\} = P(-\infty < X \le x)$$
$$= F_X(x)$$

$$F(\infty, y) = F_Y(y)$$

Independent r.v.

Two r.v. X & Y defined on the same event space S, are said to be independent if for any x, y the events $(-\infty < X \le x)$ & $(-\infty < Y \le y)$ are independent

ie, $P(-\infty < X \le x, -\infty < Y \le y) = P(-\infty < X \le x)\, P(-\infty < Y \le y)$ $\forall\ x,y \in \mathbb{R}$.

or, $F(x,y) = F_X(x)\, F_Y(y)$

Th: A necessary ~~cond~~ cond^n & sufficient cond^n for two r.v. X and Y to be independent is that, the joint distribution fn $F(x,y)$ can be written as

$$F(x,y) = \varphi(x)\, \psi(y).$$

$\varphi(x)$ is a fn of x only
$\psi(x)$ is a fn of y only.

If independent
$$F(x,y) = F_X(x) F_Y(y)$$
$$= \varphi(x)\psi(y)$$

If $F(x,y) = \varphi(x)\psi(y)$
$$F(\infty,y)\, F(x,\infty) = \varphi(x)\psi(y)\,\varphi(\infty)\psi(\infty)$$
$$\varphi(\infty)\psi(\infty) = 1 = F(\infty,\infty)$$

$\therefore F_X(x) F_Y(y) = \varphi(x)\psi(y)$

~~Mixed Tensor of order two~~

~~A set A^i_j of n^2~~

Discrete r.v. in two dimension

If the spectrum of (X,Y) is at most countable.

Probability mass

Let $F(x,y)$ be the distribution f^n of a two-dimensional r.v. (X,Y). Consider a mass ~~f^n~~ distribution over the entire xy-plane, the total mass being unity. Let the distribution of such a unit mass, which may vary from point to point, be such that the total mass distributed over the region $\{(x,y): -\infty < x \le a, -\infty < y \le b\}$ is equal to $F(a,b)$. Then the prob. of the event $(-\infty < X \le a, -\infty < Y \le b)$ can be interpreted as the mass distributed over the region $\{(x,y): -\infty < x \le a, -\infty < y \le b\}$. Then such a mass distributed over the entire plane is called the probability mass of the two-dimensional variable (X,Y).

$$f_{ij} = P(X = x_i, Y = y_j)$$

The necessary cond^n to be satisfied by f_{ij} is

$$\sum_{(i,j)\in B}\sum f_{ij} = 1 \quad , \quad B = \{(i,j) : (x_i, y_j) \in S_1\}$$

Marginal distribution

Let (X, Y) be a two-dimensional discrete random variable which takes pairs of values (x_i, y_j) $(i, j = 0, \pm 1, \pm 2, \ldots)$

Now the event $(X = x_i)$ can materialise when any one of the following mutually exclusive events happens:

$$\ldots (X = x_i, Y = y_{-2}), (X = x_i, Y = y_{-1}), (X = x_i, Y = y_0) \ldots$$

Hence $P(X = x_i) = \sum_{j=-\infty}^{\infty} P(X = x_i, Y = y_j) = \sum_{j=-\infty}^{\infty} f_{ij} = f_{i\cdot}$

where, $f_{i\cdot} = \sum_{j=-\infty}^{\infty} f_{ij} = f_{x_i}$

$$F_X(x) = F(x, \infty) = \sum_{x_i \le x} \sum_{j=-\infty}^{\infty} f_{ij} = \sum_{x_i \le x} f_{i\cdot}$$

$F_X(x)$ is a step fn. having steps of height $f_{i\cdot}$ at x_i $(i = 0, \pm 1, \pm 2, \ldots)$.

Similarly,

$$f_{\cdot j} = P(Y = y_j) = \sum_{i=-\infty}^{\infty} f_{ij} = f_{y_j}$$

Marginal distribution fn.

$$F_Y(y) = F(\infty, y) = \sum_{i=-\infty}^{\infty} \sum_{y_j \le y} f_{ij} = \sum_{y_j \le y} \sum_{i=-\infty}^{\infty} f_{ij}$$

$$= \sum_{y_j \le y} f_{\cdot j}$$

Th: A necessary and suff. condn for two discrete r.v. X & Y to be independent is that

$$f_{ij} = f_{i\cdot}\, f_{\cdot j} = f_{x_i} f_{y_j}.$$

Ex:

Consider the r.E. of throwing a pair of dice. Let X denote the no. of sixes and Y denote the no. of fives that turn up. Find the joint p.m.f of (X, Y) and marginal p.m.f of X & Y. Find $P(X+Y \geq 2)$, & $P(Y=0 \mid X=1)$. Are they independent?

An: Here the spectra of X & Y are given by

x_i $(i=0,1,2)$ and $y_j = j$ $(j=0,1,2)$

respectively and the spectrum of the two-dimensional r.v. (X, Y) is given by

$(x_i, y_j) = (i, j)$ $(i=0,1,2;\ j=0,1,2)$.

with $P(X=x_i, Y=y_j) > 0$

The p.m.f of (X, Y) is given by the following table:

X \ Y	0	1	2	$P(X=x_i)$
0	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{1}{36}$	$\frac{25}{36} = f_{x_0}$
1	$\frac{8}{36}$	$\frac{2}{36}$	0	$\frac{10}{36}$ f_{x_1}
2	$\frac{1}{36}$	0	0	$\frac{1}{36} = f_{x_2}$
$P(Y=y_j)$	$\frac{25}{36}$ f_{y_0}	$\frac{10}{36}$ f_{y_1}	$\frac{1}{36}$ f_{y_2}	1

(6,6) (5,5)

$10+8+2=20$

$p_{ij} = P(X=i, Y=j)$.

$p_{00} = \frac{16}{36}$, $p_{01} = p_{10} = \frac{8}{36}$, $p_{02} = p_{20} = \frac{1}{36}$, $p_{11} = \frac{2}{36}$.

From the row sum of the above table, the marginal distribution of X is

The spectrum of X is given by $x_i = i$ $(i=0,1,2)$ and p.m.f. is given by

$p_{x_i} = P(X=x_i)$, $p_{x0} = \frac{25}{36}$, $p_{x1} = \frac{10}{36}$, $p_{x2} = \frac{1}{36}$.

Similarly, the spectrum of Y is given by $y_j = j$ $(j=0,1,2)$ and p.m.f of Y is given by

$p_{y_j} = P(Y=j)$, $(j=0,1,2)$

$p_{y_0} = \frac{25}{36}$, $p_{y1} = \frac{10}{36}$, $p_{y2} = \frac{1}{36}$.

$P(X+Y \geq 2) = p_{11} + p_{02} + p_{20} = \frac{2}{36} + \frac{1}{36} + \frac{1}{36} = \frac{4}{36} = \frac{1}{9}$.

$P(Y=0 \mid X=1) = \frac{P(X=1, Y=0)}{P(X=1)} = \frac{p_{10}}{p_{x1}} = \frac{8/36}{10/36} = \frac{8}{10}$
$= 0.8$.

$p_{01} = \frac{8}{36}$.

$p_{x0} = \frac{25}{36}$ $p_{y1} = \frac{10}{36}$.

$\therefore p_{x0} \cdot p_{y1} = \frac{25}{36} \times \frac{10}{36}$

$\therefore p_{01} \neq p_{x0} \cdot p_{y1}$ $\therefore$ not independent.

Ex

The r.v. X & Y have a joint probability mass fn given by

$$P(X=x, Y=y) = \frac{x^2+y^2}{32} \quad \text{for } x=0,1,2,3 \ \& \ y=0,1$$

find the marginal p.m.f. of X & Y. Also find $P(X \le 2, Y \ge 1)$.

X \ Y	0	1	$P(X=x_i)$
0	0	$\frac{1}{32}$	$\frac{1}{32} = p_{x0} = f_X(0)$
1	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{3}{32} = p_{x1} = f_X(1)$
2	$\frac{4}{32}$	$\frac{5}{32}$	$\frac{9}{32} = p_{x2} = f_X(2)$
3	$9/32$	$10/32$	$19/32 = p_{x3} = f_X(3)$
$P(Y=y_j)$	$14/32 = p_{y0} = f_Y(0)$	$18/32 = p_{y1} = f_Y(1)$	1

$$P(X \le 2, Y \ge 1) = \frac{1}{32} + \frac{2}{32} + \frac{5}{32} = \frac{8}{32} = \frac{1}{4}.$$

Ex: A two-dimensional r.v. (X,Y) has the spectrum $(x_i, y_j) = (i, j)$ $(i=1,2,3,\ j=1,2,3)$ and the probability masses p_{ij} are given by

$$p_{ij} = P(X=x_i, Y=y_j) = kij, \text{ when } k \text{ is a constant.}$$

Find a) the value of k, (b) $P(1 \le X \le 3, Y \le 2)$, (c) $P(X \ge 2)$ (d) $P(Y \le 2)$ (e) $P(X=2)$ (f) marginal p.m.f of X & Y.

Are they independent?

$$\sum_{i=1}^{3}\sum_{j=1}^{3} p_{ij} = 1 \Rightarrow k\sum_{i=1}^{3}\sum_{j=1}^{3} ij = 1 \Rightarrow k\sum_{i=1}^{3} 6i = 1$$

$$\Rightarrow k(18+12+6) = 1$$

$$\Rightarrow k = \frac{1}{36}.$$

Continuous Random variables in two dimensions:

$F(x,y) \rightarrow$ Joint distribution fn of X & Y.

The joint distribution is said to be continuous if $F(x,y)$ is continuous $\forall\, x, y$ & $\frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}, \frac{\partial F}{\partial y}, \frac{\partial^2 F}{\partial y^2}, \frac{\partial^2 F}{\partial x \partial y}$, $\frac{\partial^2 F}{\partial y \partial x}$ are all continuous in the whole xy-plane except that there may be a finite no. of curves of discontinuities of these derivatives in any bounded region and for any $a, b, c, d \in R$ $c<d$, $a<b$.

$$\int_c^d \int_a^b \cancel{f(x,y)} \frac{\partial^2 F}{\partial x \partial y} \partial x \partial y \text{ is convergent.}$$

The joint p.d.f. ~~of~~ $f(x,y)$ of r.v. (X,Y) is defined by

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}.$$

Th: $$P(a \leq X \leq b,\ c \leq Y \leq d) = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

Th: $$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\, dx\, dy.$$

Th: $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1.$$

Probability differential

$$P(x < X \leq x+dx,\ y < Y \leq y+dy) = f(x,y)\, dx\, dy$$

Continuous distribution

If Q be a specified subset of the spectrum of the two dimensional random variable (X, Y) then

$$P\{(X,Y) \in Q\} = \iint_Q f(x,y)\,dx\,dy.$$

Marginal Distribution for a continuous Joint distribution

$F(x,y)$ be joint distribution

$F_X(x)$ = marginal distribution of X.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x,y)\,dx\,dy = \int_{-\infty}^{x}\left\{\int_{-\infty}^{\infty} f(x,y)\,dy\right\}dx.$$

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.}$$

marginal p.d.f of X.

Similarly,

$F_Y(y)$ = marginal distribution of Y

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{y} f(x,y)\,dy\,dx = \int_{-\infty}^{y}\left\{\int_{-\infty}^{\infty} f(x,y)\,dx\right\}dy$$

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx}$$

marginal p.d.f. of Y.

122, 126, 129, 208, 117,

Th: The necessary and sufficient condⁿ for two r.v. X & Y to be independent is that the joint density fuⁿ f(x,y) of X & Y can be represented as

$$f(x,y) = f_x(x)\, f_y(y)$$

Proof: If independent, Then

$$F(x,y) = F_x(x)\, F_y(y).$$

where, $F(x,y)$ is the distribution fuⁿ of the two dimensional r.v. (X, Y).

$F_x(x)$ & $F_y(y)$ are marginal distribution of X & Y.

$$\therefore \frac{\partial F}{\partial x} = F'_x(x)\, F_y(y)$$

$$\frac{\partial^2 F}{\partial y\,\partial x} = F'_x(x)\, F'_y(y)$$

But $\dfrac{\partial^2 F}{\partial y\,\partial x} = f(x,y) = f_x(x)\, f_y(y)$. as. $f_x(x) = F'(x)$, $f_y(y) = F'(y)$

If $f(x,y) = f_x(x)\, f_y(y)$

$$\int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy = \int_{-\infty}^{x} f_x(x)\,dx \int_{-\infty}^{y} f_y(y)\,dy$$

$$F(x,y) = F_x(x)\, F_y(y)$$

$\therefore$ X & Y are independent.

Ex:

Raindrops fall at random on a square R with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$ and the distribution of X & Y are given by

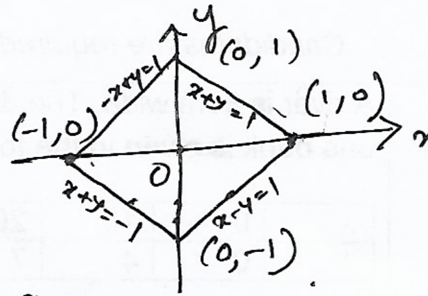
$$f(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y) \in R. \\ 0 & \text{elsewhere.} \end{cases}$$

where R is the region : a square ...... $(0,-1)$.

Determine the marginal distributions of X & Y.

Are the r.v. independent ? Find $P(X \le 1, Y \le 0)$.

$P(-1 \le X \le 1, Y \geq 0)$ & $P(Y \geq 0 \mid -1 \le X \le 1)$.



Solⁿ:

Marginal ~~distribution~~ density fⁿ of X is $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy.$$

Now for $0 < x < 1$ $\quad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_{x-1}^{1-x} \frac{1}{2}\, dy = 1-x.$

$-1 < x \le 0$ $\quad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_{-1-x}^{1+x} \frac{1}{2}\, dy = 1+x.$

$$\therefore f_X(x) = \begin{cases} 1+x & \text{for } -1 < x \le 0 \\ 1-x & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Similarly, $$f_Y(y) = \begin{cases} 1+y & , -1 < y \le 0 \\ 1-y & , 0 < y < 1 \\ 0 & , \text{elsewhere.} \end{cases}$$

$f(x,y) \neq f_X(x)\, f_Y(y)$ hence not independent.

$$P(-1 \le X \le 1, Y \geq 0) = \int_{-1}^{1} \left\{ \int_{0}^{1} f(x,y)\, dy \right\} dx.$$

$$P(Y\geq 0 \mid \cancel{x}\, -1 \leq X \leq 1) = \frac{P(-1\leq X\leq 1,\ Y\geq 0)}{P(-1\leq X\leq 1)}$$

$$= \frac{P(-1\leq X\leq 1,\ Y\geq 0)}{\int_{-1}^{1} f_X(x)\,dx}.$$

Q. The density fu^n of the two dimensional r.v. (X,Y) is given by

$$f(x,y) = \begin{cases} c(2x+5y) & 0\leq x\leq 3,\ 2\leq y\leq 4. \\ 0 & \text{elsewhere,} \end{cases}$$

Find a) c

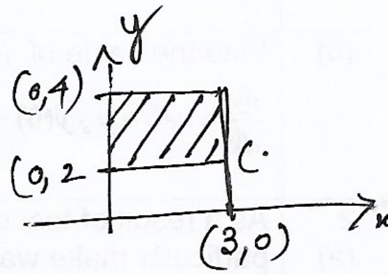
b) marginal density fu^n of X & Y.

c) Joint distribution fu^n $F(x,y)$

d) $P(X+Y\leq 3)$

e) $f_X(x|y)$

f) Are they independent?



a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ $\cancel{\Rightarrow c = \frac{1}{108}}$.

$\Rightarrow \int_0^3 \left\{\int_2^4 c(2x+5y)\,dy\right\}dx = 1 \Rightarrow c = \frac{1}{108}$

b) $f_X(x) = \frac{1}{108}\int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{108}\int_2^4 (2x+5y)\,dy = \frac{4x+30}{108}$

$0\leq x\leq 3.$

$f_Y(y) = \frac{9+15y}{108},\ 2\leq y\leq 4.$

c) $F(x,y) = \int_{-\infty}^{x}\left\{\int_{-\infty}^{y} f(x,y)\,dy\right\}dx$

$= \int_{0}^{x}\left\{\int_{2}^{y} \frac{1}{108}(2x+5y)\,dy\right\}dx$

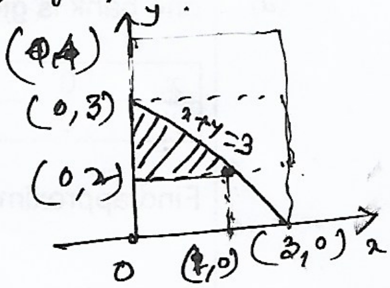
$= \frac{1}{216}\, x\,(y-2)(2x+5y+10)$ $\qquad 0 \le x \le 3,\ 2 \le y \le 4$.

$= \frac{1}{72}(y-2)(5y+16)$ $\qquad x > 3,\ 2 \le y \le 4$.

$= \frac{1}{54}\, x\,(x+15)$ $\qquad 0 \le x \le 3,\ y > 4$.

$= 0$ $\qquad x < 0,\ y < 2$

$= 1$ $\qquad x > 3,\ y > 4$.



d) $P(x+y<3) = \int_{0}^{1}\left\{\int_{2}^{3-x} \frac{1}{108}(2x+5y)\,dy\right\}dx$.

$= \frac{1}{216}\int_{0}^{1}(x^2-26x+25)\,dx = \frac{37}{648}$.

e) $f_X(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{108}(2x+5y)}{\frac{1}{108}(9+15y)} = \frac{2x+5y}{9+15y}$ $\qquad 0 \le x \le 3,\ 2 \le y \le 4$.

$f(x,y) \ne f_X(x)\, f_Y(y)$

not ~~is~~ independent.

Mathematical Expectation -

We consider a two dimensional r.v. (X, Y). Let $g(X,Y)$ be a r.v. where $g: R \times R \to R$ is a continuous fu^n. Then the expectation of $g(X,Y)$ denoted by $E\{g(X,Y)\}$.

Case-I The distribution of (X,Y) is discrete.

$$E(g(X,Y)) = \sum_i \sum_j g(x_i, y_j) f_{ij}$$

Case-II The distribution of (X,Y) is continuous.

$$E(g(x,y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy.$$

Covariance

Let X, Y be two r.v. and Let m_x, m_y exists.

$m_x = E(X)$ $\quad$ $m_y = E(Y)$. Similarly. $m_y = \int_{-\infty}^{\infty} y f_Y(y)\, dy$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left\{ \int_{-\infty}^{\infty} f(x,y)\, dy \right\} dx$$

Measure of 'tendency of having linear relationship between X & Y,' is given by $E\{(X - m_x)(Y - m_y)\} = Cov(X,Y)$

Covariance for bivariate distribution.

Correlation Coeff.

$Cov(X, Y)$ is not dimensionless.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \qquad \begin{array}{l} \sigma_X > 0 \ (\text{s.d. of } X) \\ \sigma_Y > 0 \ (\text{s.d. of } Y). \end{array}$$

Ex:

Let the joint distribution of X & Y be given by p.d.f.

$$f(x,y) = \begin{cases} x+y & \text{if } 0<x<1, \ 0<y<1 \\ 0 & \text{elsewhere.} \end{cases}$$

find i) $E(XY)$ ii) $E(X+Y)$ iii) $E(X^2)$, iv) $Cov(X,Y)$

$$E(XY) = \int_0^1 \int_0^1 xy f(x,y)\, dx\, dy = \int_0^1\int_0^1 xy(x+y)\, dx\, dy = \frac{1}{3}.$$

$$E(X+Y) = \int_0^1 \int_0^1 (x+y) f(x,y)\, dx\, dy = \int_0^1 \int_0^1 (x+y)^2\, dx\, dy$$

$$= 7/6.$$

$$E(X^2) = \int_0^1 \int_0^1 x^2 f(x,y)\, dx\, dy = \int_0^1 x^2 \left\{ \int_0^1 (x+y)\, dy \right\} dx.$$

$$Cov(X,Y) = E(X - m_X)(Y - m_Y) = \int_0^1\int_0^1 (x-m_x)(y-m_y) f(x,y)\, dx\, dy$$

Ex: Two balls are drawn without replacement from an urn containing three balls numbered 1, 2, 3. Let X be the r.v. denoting the no. on the first ball drawn and Y the larger of the two nos. drawn. Find $Cov(X,Y)$

Sol^n Spectrum of X & Y are respectively $\{1,2,3\}$ & $\{2,3\}$.

$X \backslash Y$	2	3	$P(X=x_i)$
1	$\frac{1}{6}$	$\frac{1}{6}$	$2/6$
2	$\frac{1}{3}\cdot\frac{1}{2}$	$\frac{1}{3}\cdot\frac{1}{2}$	$2/6$
3	0	$\frac{1}{3}\cdot\frac{2}{2}$	$2/6$
$P(Y=y_j)$	$2/6$	$4/6$	1

$\frac{1}{3}\ \frac{2}{3}\ \frac{1}{2}$

2 1

3

3 1

32

$$m_x = E(X) = 1\cdot\frac{1}{3} + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{3} = 2.$$

$$m_y = E(Y) = 2\times\frac{1}{3} + 3\times 2/3 = 8/3.$$

$$E(XY) = 1\cdot 2\cdot\frac{1}{6} + 1\cdot 3\cdot\frac{1}{6} + 2\cdot 2\cdot\frac{1}{6} + 2\cdot 3\cdot\frac{1}{6} + 3\cdot 2\cdot 0 + 3\cdot 3\cdot\frac{2}{6} = \frac{11}{2}.$$

$$\therefore \operatorname{Cov}(X,Y) = E(XY) - m_x m_y = \frac{11}{2} - \frac{2\times 8}{3} = \frac{1}{6}.$$

Properties:

1. $\operatorname{Cov}(X,Y) = \operatorname{Cov}(Y,X)$
2. $\rho(X,Y) = \rho(Y,X)$ if $\sigma_x > 0,\ \sigma_y > 0$.
3. $-1 \le \rho \le 1$,
4. uncorrelated r.v.

Two r.v. X, Y are said to be uncorrelated if and only if $\operatorname{cov}(X,Y) = 0$

$\therefore \rho(X,Y) = 0$.

If X, Y are independent then $\rho(X,Y) = 0$

But the converse is not true.