Classical Mechanics

1. Problems in Newtonian ~~Rational~~ Mechanics.
2. Constraints and classification
3. Degrees of freedom, generalized coordinate system, configuration space.
4. Principle of virtual work.
5. Lagrange's Equation of motion.
6. Hamiltonian Equation of motion (Phase space)

$$\frac{d\vec{p}}{dt} = \vec{F}_{ext}$$

If $\vec{F}_{ext} = 0$, $\frac{d\vec{p}}{dt} = 0$.

$\therefore \vec{p} = \text{constant}$.

$$\vec{F}_{ext} = \frac{d\vec{p}}{dt}$$

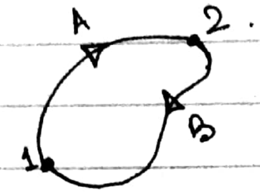
$$\vec{r} \times \vec{F}_{ext} = \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \frac{d}{dt}(\vec{r} \times \vec{p}) - \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{v \times m\vec{v}}$$

$$= \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$\therefore \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

If $\tau_{ext} = 0$,
$\vec{L}$ is conserved.



$$\boxed{\oint \vec{F}\cdot d\vec{r} = 0}$$

Basic condition for Energy to be conserved.

$$\int_{1A}^{2} \vec{F}\cdot d\vec{r} + \int_{2B}^{1} \vec{F}\cdot d\vec{r} = 0$$

$$\int_{1A}^{2} \vec{F}\cdot d\vec{r} = -\int_{2B}^{1} \vec{F}\cdot d\vec{r}$$

$$\Rightarrow \int_{1A}^{2} \vec{F}\cdot d\vec{r} = \int_{1B}^{2} \vec{F}\cdot d\vec{r}$$

$$W_{12} = \int_{1}^{2} \vec{F}\cdot d\vec{r} = \int_{1}^{2} \frac{d\vec{p}}{dt}\cdot\frac{d\vec{r}}{dt}\cdot dt$$

$$= \int_{1}^{2} m\frac{dv}{dt}\cdot\vec{v}\,dt$$

$$= \int_{1}^{2} \frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right)dt$$

$$= \left(\frac{1}{2}mv^2\right)_2 - \left(\frac{1}{2}mv^2\right)_1$$

$$= T_2 - T_1$$

$$W_{12} = \int_1^2 \vec{F}\cdot d\vec{r} = V(\vec{r}_1) - V(\vec{r}_2) = \int_1^2 -dV.$$

$$\Rightarrow T_2 - T_1 = V_1 - V_2.$$

$$\Rightarrow T_1 + V_1 = T_2 + V_2.$$

$$\Rightarrow T + V = E = \text{Conserved}.$$

$$df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$\therefore df = \vec{\nabla} f \cdot d\vec{r}$$

$$\therefore \boxed{\vec{F} = -\vec{\nabla} V}$$

Problems associated with Newton's Equation of Motion.

$$m_i \ddot{\vec{r}}_i = \vec{F}_i^{ext}$$

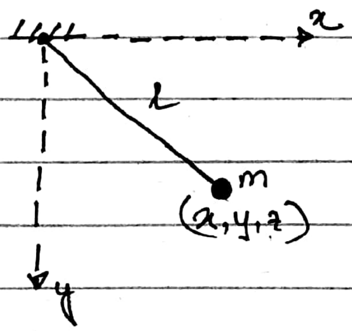
1. Effect of Geometry on the motion of particle.
2. Many body systems.
3. Non-inertial frames of reference

Constrained Motion.

Constrained Equations → Equations involving x_i, y_i, z_i, t.

Examples

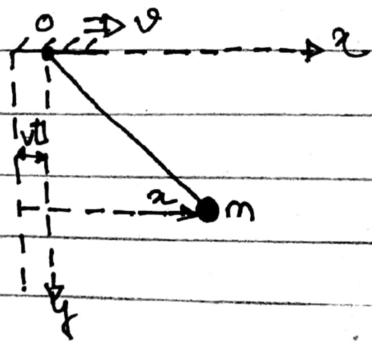
① Simple Pendulum.



$z = 0$

$x^2 + y^2 = l^2$

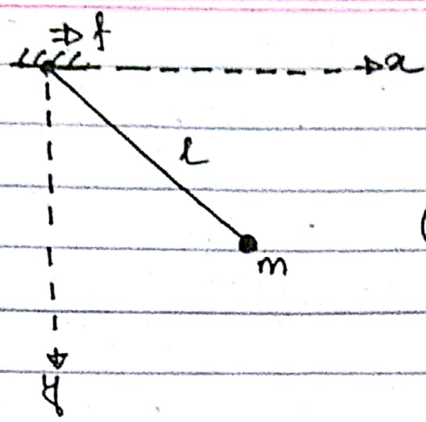
②



$z = 0$

$(x - vt)^2 + y^2 = l^2$

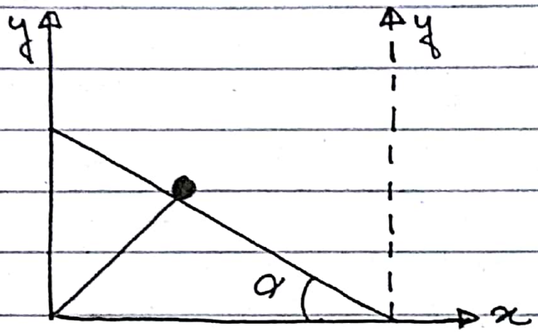
③



$z = 0$

$(x - \frac{1}{2}ft^2)^2 + y^2 = l^2$

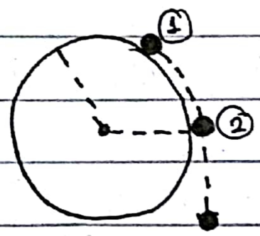
④



$Z = 0$

$\frac{y}{-x} = \tan\alpha$

⑤



① $x^2 + y^2 + z^2 = R^2$

② $x^2 + y^2 + z^2 \geq R^2$

All the above equations have algebric form or not an equation $(x^2 + y^2 + z^2 \geq R^2)$

Some are differential equations $(Xdx + Ydy + Zdz + Tdt = 0)$ having a algebaic form $f(x, y, z, t) = 0$

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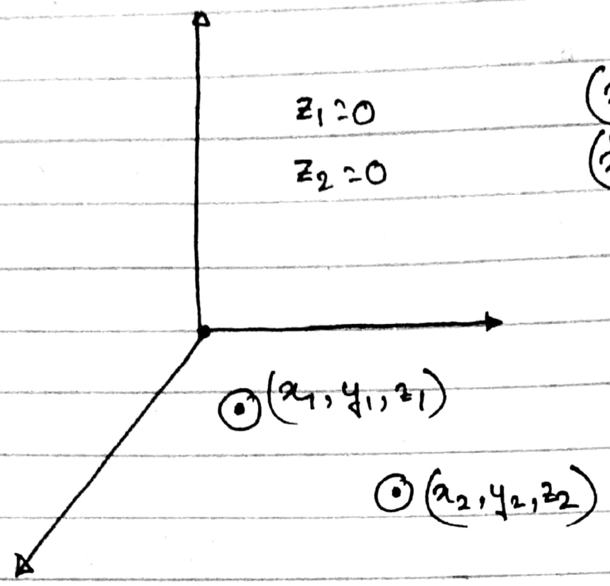
⑥ Bicycle Movement



$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2$$

$$(\dot{x}_1 \hat{i} + \dot{y}_1 \hat{j}) \propto \{(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}\}$$

$$\dot{x}_1 = k(x_2 - x_1)$$

$$\dot{y}_1 = k(y_2 - y_1).$$

$$\therefore \frac{dx_1}{dy_1} = \frac{x_2 - x_1}{y_2 - y_1}.$$

$$(y_2 - y_1)\,dx_1 - (x_2 - x_1)\,dy_1 = 0.$$

Constraint Equations
- Algebraic → $f_\alpha(x_i, y_i, z_i, t) = 0$
- Differential → $X_i dx_i + Y_i dy_i + Z dz_i + T dt$
- No Equations.

Constraints

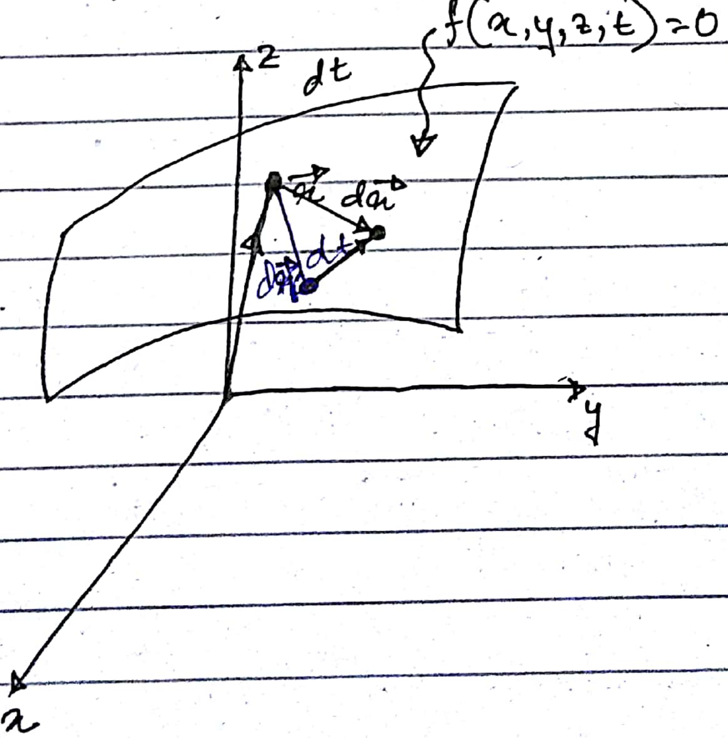
- Holonomic (Algebraic form) $f(x, y, z, t) = 0$
  - $\frac{\partial f}{\partial t} = 0$ Scleronomic
  - $\frac{\partial f}{\partial t} \neq 0$ Rheonomic
- Non-Holonomic (Differential form or No equation)

$$m_i \ddot{\vec{r}}_i = \vec{F}_{i\,ext} + R_i$$

An Equation for R_i:-

Single particle, Holonomic System.

$$f(x, y, z, t) = 0$$



$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt$$

$$\boxed{\vec{\nabla} f \cdot d\vec{r} + \frac{\partial f}{\partial t} \cdot dt = 0}$$

Case I $\quad \boxed{\frac{\partial f}{\partial t} = 0}$

$\Rightarrow \vec{\nabla} f \cdot d\vec{r} = 0$

$\Rightarrow \vec{\nabla} f \perp d\vec{r}$

$\vec{R} \perp$ surface.

$\therefore \vec{\nabla} f \propto \vec{R}$

$\therefore \vec{R} = \lambda \vec{\nabla} f \qquad \lambda \rightarrow$ Unknown

Case II $\quad \boxed{\frac{\partial f}{\partial t} \neq 0}$

$$\vec{\nabla} f \cdot d\vec{r} + \frac{\partial f}{\partial t} dt = 0$$

$$\vec{\nabla} f \cdot d\vec{r}' + \frac{\partial f}{\partial t} \cdot dt = 0$$

$$\vec{\nabla} f \cdot (d\vec{r} - d\vec{r}') = 0.$$

$$\vec{\nabla} f \cdot \delta\vec{r} = 0$$

$\delta\vec{r} = d\vec{r} - d\vec{r}'$ (On the surface).

$\downarrow$

Virtual Displacement.

$\therefore \vec{\nabla} f \perp \delta\vec{r}$, $\Rightarrow \vec{R} \perp \delta\vec{r}$

$\therefore \vec{R} = \lambda \vec{\nabla} f$

$$\Rightarrow \boxed{m_i \ddot{\vec{r}_i} = \vec{F}_{iext} + \lambda_\alpha \vec{\nabla}_i f_\alpha}$$

Lagrange Equation of 1st kind.

$$m_i \ddot{\vec{r}_i} = \vec{F}_{iext} + \vec{R}_i \rightarrow (m_i \ddot{\vec{r}_i} - \vec{F}_{iext}) \cdot \delta\vec{r}_i = \vec{R}_i \cdot \delta\vec{r}_i$$

$$\vec{R}_i = -\lambda_\alpha \vec{\nabla}_i f_\alpha$$

Get rid of $\vec{R}_i$

$$\rightarrow \boxed{\sum_{i=1}^{N} (m_i \ddot{\vec{r}_i} - \vec{F}_{iext}) \cdot \delta\vec{r}_i = 0.}$$

First Fundamental form.

$$\vec{R}_i \perp \delta\vec{r}_i$$

$$\boxed{\sum_{i=1}^{N} \vec{R}_i \cdot \delta\vec{r}_i = 0}$$

Principle of Virtual Work.

$$\sum_{j=1}^{X} (B_j)\, \delta q_j = 0.$$

$$\delta\vec{r}_i \rightarrow q_j$$

Degrees of Freedom

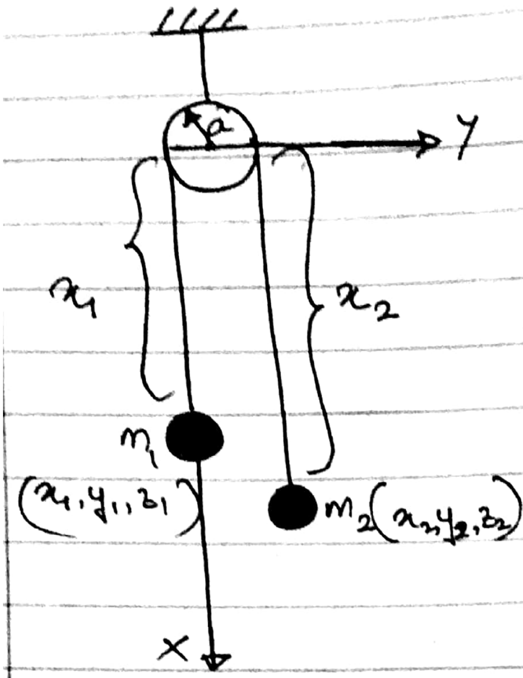
For any dynamic system
# of variables > # of constraint equations.

Degree of Freedom (f) = # of variables − # of constraint equation.

Ex:-

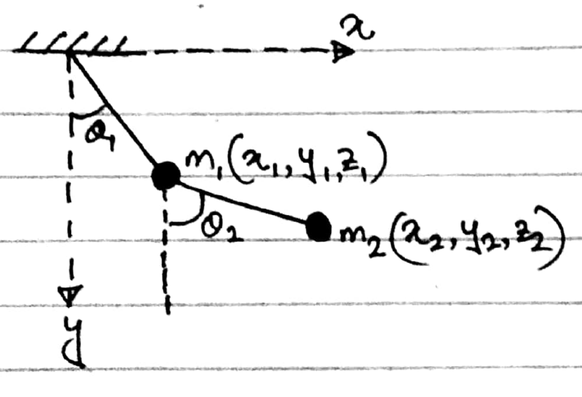
$$\left.\begin{array}{l} x^2+y^2=l^2 \\ z=0 \end{array}\right\} \therefore f = 1.$$

$f = 3N - K.$



$$\left.\begin{aligned} z_1 &= 0 \\ z_2 &= 0 \\ x_1 + x_2 + \pi a &= l \\ y_1 &= 0 \\ y_2 &= 2a \end{aligned}\right\}$$

$f = 3 \times 2 - 5 = 1.$



$$\left.\begin{aligned} z_1 &= 0 \\ z_2 &= 0 \\ x_1^2 + y_1^2 &= l_1^2 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 &= l_2^2 \end{aligned}\right\} K = 4$$

$\therefore f = 3 \times 2 - 4 = 2.$

Generalized Coordinate.

$q_j, \quad j = 1, 2 \ldots f.$

$f(x_i, y_i, z_i, t) = 0$

$q_j = q_j(x_i, y_i, z_i, t)$

$$\left.\begin{aligned} & q_j = q_j(\vec{r_i}, t) \\ & \vec{r_i} = \vec{r_i}(q_j, t) \\ & f(x_i, y_i, z_i, t) = 0 \end{aligned}\right\} \text{Same Holonomic Equation in Different Form.}$$

$$\sum_{i=1}^{N} \left(m_i \ddot{\vec{r_i}} - \vec{F_i}\right) \cdot \delta \vec{r_i} = 0$$

$\uparrow$ Virtual Displacement

$\vec{r_i} = \vec{r_i}(q_j, t)$

Harmonic Constraint Eq^n

$q_j \rightarrow$ Generalized Coordinate

$j = 1, 2, 3 \ldots\ldots f$

$f \uparrow$ DOF

$$\delta \vec{r_i} = \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \partial q_j + \frac{\partial \vec{r_i}}{\partial t} \underbrace{\partial t}_{0}.$$

$$\delta \vec{r_i} = \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \cdot \delta q_j.$$

$$\sum_i m_i \ddot{\vec{r_i}} \cdot \delta \vec{r_i} - \sum_i \vec{F_i} \cdot \delta \vec{r_i} = 0$$

$$\Rightarrow \sum_{j=1}^{f} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] \delta q_j - \sum_{i=1}^{N} \vec{F_i} \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \cdot \delta q_j = 0$$

$$\Rightarrow \sum_j \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} \delta q_j - \sum_j \left\{ \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right\} \delta q_j = 0$$

$$Q_j = \sum_{i=1}^{N} F_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$Q_j \rightarrow$ Generalised Force

$$\sum_{j=1}^{f} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \sum_i \frac{1}{2} m_i v_i^2$$

$$\therefore \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j}$$

Lagrange Equation of Motion of 2nd kind.

<u>Conservative System</u>

$$F = -\vec{\nabla} V$$

$$\vec{F}_i = -\vec{\nabla}_i V(q_j) \qquad \begin{Bmatrix} V = V(q_j) \\ V \neq V(\dot{q}_j) \end{Bmatrix}$$

$$-\vec{\nabla}_i = -\frac{\partial V}{\partial \vec{r}_i}$$

$$\therefore Q_j = \sum_i -\frac{\partial V}{\partial \vec{r}_i} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= -\frac{\partial V}{\partial q_j}.$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}.$$

$$\Rightarrow \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T) - \frac{\partial}{\partial q_j}(T-V) = 0.$$

$$\Rightarrow \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T-V) - \frac{\partial}{\partial q_j}(T-V) = 0.$$

$q_j, \dot{q}_j, t$ $\quad q_j$

Let $T-V = L \rightarrow (q_j, \dot{q}_j, t)$. $\quad \boxed{\because \frac{\partial V}{\partial \dot{q}_j} = 0}$

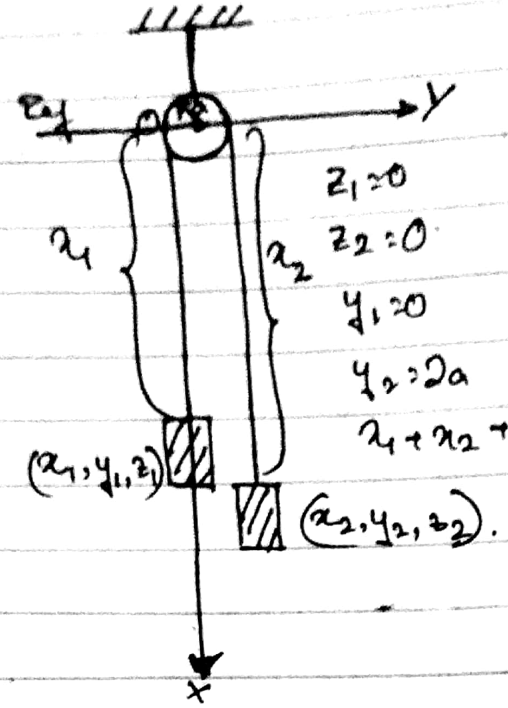
$L \rightarrow$ Lagrangian of the system

$$\Rightarrow \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(L) - \frac{\partial}{\partial q_j}(L) = 0$$

$$\therefore Q = \sum_i \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}.$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}, \quad L = T-V.$$

Atwood Machine



$$z_1 = 0$$
$$z_2 = 0$$
$$y_1 = 0$$
$$y_2 = 2a$$
$$x_1 + x_2 + \pi a = l.$$

$K = 5$

$$f = 3 \times 2 - K$$
$$= 1.$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \cancel{\dot{y}_1^2} + \cancel{\dot{z}_1^2}) + \frac{1}{2} m_2 (\dot{x}_2^2 + \cancel{\dot{y}_2^2} + \cancel{\dot{z}_2^2})$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$[x_2 = (l - \pi a) - x_1.]$$

$$\Rightarrow \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_1^2.$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}_1^2$$

$$V = -m_1 g x_1 - m_2 g x_2.$$
$$= -m_1 g x_1 - m_2 g (A - x_1).$$
$$= -g(m_1 - m_2) x_1 - B.$$

$$\therefore L = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + g(m_1 - m_2)x_1 + B.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0.$$

$$\begin{cases} \dfrac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1 \\[2ex] \dfrac{\partial L}{\partial x_1} = g(m_1 - m_2). \end{cases}$$

$$\Rightarrow (m_1 + m_2)\ddot{x}_1 - g(m_1 - m_2) = 0.$$

$$\Rightarrow \boxed{\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2} \cdot g}$$

$$P_a = \frac{\partial L}{\partial \dot{q}_a} = \text{Conserved}.$$

$$P_j = \frac{\partial L}{\partial \dot{q}_j}$$

$$L = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 + \quad + \sin^2 \dot{q}_3 + m_1 m_3 \cos q_1 \sin^{-1} q_1$$

q_3

$q_j = q_1, q_2, q_3.$ $\quad f \to 3.$

$L \neq L(q_a)$

$q_a \to$ Cyclic Coordinate.

Hamilton's Equation of Motion

$L(q_j, \dot{q}_j, t) \rightarrow$ Possible when the determinant of the Hessian matrix is non zero.

$\downarrow$

p_j

$\therefore H(q_j, p_j, t)$ $\quad$ $H \rightarrow$ Hamiltonian

$$\underbrace{H(q_j, p_j, t)}_{L.H.S.} = \underbrace{\sum_{j=1}^{f} p_j \dot{q}_j - L(q_j, \dot{q}_j, t)}_{R.H.S.}$$

$d(L.H.S.) = d(R.H.S)$

$$d(L.H.S) = \sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt.$$

$$d(R.H.S.) = \cancel{\sum_j p_j d\dot{q}_j} + \sum_j \dot{q}_j dp_j - \sum_j \frac{\partial L}{\partial q_j} \partial q_j - \cancel{\sum_j \frac{\partial L}{\partial \dot{q}_j} \partial \dot{q}_j}$$
$$- \frac{\partial L}{\partial t} dt$$

$$\left[\because \frac{\partial L}{\partial \dot{q}_j} = p_j \ , \ \frac{\partial L}{\partial q_j} = \dot{p}_j\right]$$

By equating coeffecients.

Hamilton's equation of motion
$$\left\{ \begin{array}{l} \dfrac{\partial H}{\partial q_j} = -\dot{p}_j \\ \\ \dfrac{\partial H}{\partial p_j} = \dot{q}_j \end{array} \right.$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

When $H \neq H(q_\alpha)$.

$$\frac{\partial H}{\partial q_\alpha} = 0,$$

$$\Rightarrow -\dot{p}_\alpha = 0$$

$\therefore p_\alpha =$ constant.

$L \neq L(q_\alpha)$

$$\therefore \frac{\partial L}{\partial q_\alpha} = 0.$$

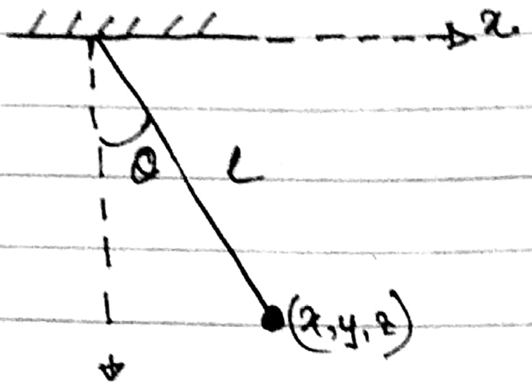
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0.$$

$$\therefore \dot{p}_\alpha = 0$$

$$\therefore -\frac{\partial H}{\partial q_\alpha} = 0, \Rightarrow H \neq H(q_\alpha).$$

$\therefore$ If a coordinate is cyclic in Lagrangian, then it is also cyclic in Hamiltonian.

Simple Pendulum



$$\begin{cases} x^2+y^2=L^2 \\ z=0 \end{cases}$$

$f=1.$

$$L=\frac{1}{2}mL^2\dot{\theta}^2+mgL\cos\theta.$$

$$H=p_1\dot{q}_1-L$$
$$=p_\theta\dot{\theta}-L$$

$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}\quad = m\ell^2\cdot\dot{\theta}^2-\frac{1}{2}mL^2\dot{\theta}^2-mgL\cos\theta.$$

$$=\frac{1}{2}mL^2\dot{\theta}^2-mgL\cos\theta.$$

$$=\frac{1}{2}\cancel{mL^2}\cdot\frac{p_\theta^2}{(mL^2)^{\cancel{2}}}\quad -mgL\cos\theta$$

$$=\frac{1}{2}\frac{p_\theta^2}{mL^2}-mgL\cos\theta.$$

$$\therefore H=\frac{1}{2}\frac{p_\theta^2}{m\ell^2}-mgL\cos\theta.$$

$$-\dot{p}_\theta=\frac{\partial H}{\partial\theta}=mgL\sin\theta.$$

$$\dot{\theta}=\frac{\partial H}{\partial p_\theta}=\frac{p_\theta}{mL^2}$$

$$ml^2\dot{\theta} = p_\theta .$$

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) = -mgl\sin\theta .$$

$$\therefore \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta .}$$

Band Theory of Solids

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}$$

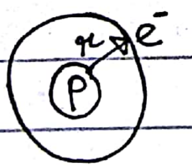
$$\hat{p} = -i\hbar\vec{\nabla} \quad \text{[For 3 Dimensional]}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V$$

$$\therefore H\psi = E\psi$$

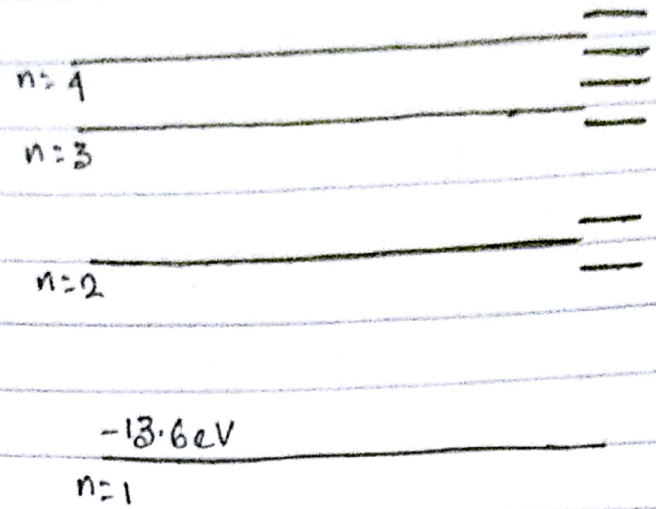
$$\therefore -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

Isolated H-atom



$$V = \frac{e(-e)}{r}$$

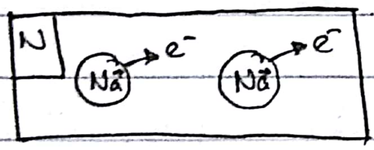
$$\therefore E_n = \frac{-13.6\ eV}{n^2}, \quad n = 1, 2, 3, \ldots\ldots$$



Discrete Energy Levels.

On introduction of another H atom to the original H atom, the energy ~~gets~~ levels gets splitted as shown in the figure. So now every energy level is splitted into two energy levels. On introduction of another atom, ~~the~~ each energy level gets splitted into three energy levels.

$$Na_{11} \rightarrow \underbrace{1s^2\, 2s^2\, 2p^6}_{\text{Core } e^-s}\, \underbrace{3s^1}_{\text{Valence Electrons}}$$



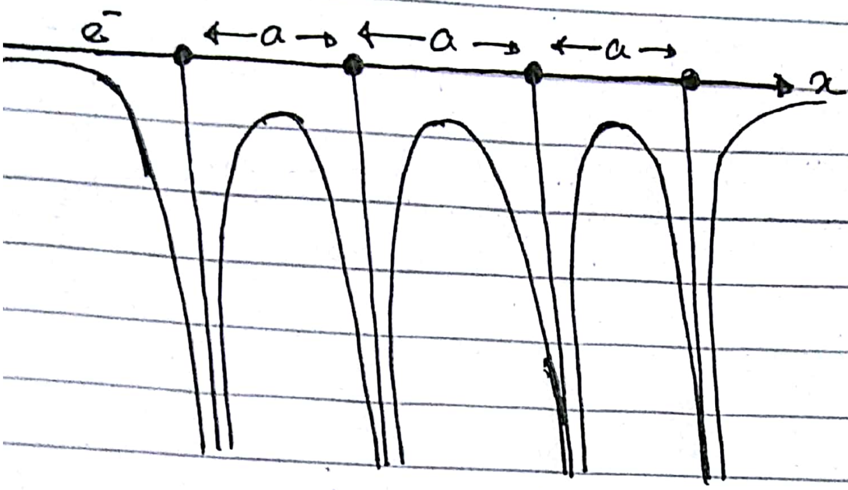
N → +Ne charges

(N−1) → −(N−1)e charges.

∴ $Ne \approx -(N-1)e$

∴ No force is acting on the electron, therefore becomes a free electron.

Normal Free Electron and a free electron inside a solid.



$\boxed{V(x+a) = V(x)}$ → Periodic Potential

Periodicity → a.

<u>Known Equations</u> :—

1. $$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x).$$

2. $$V(x) = V(x+a).$$

1D Lattice

$$\psi(x+a) = e^{ika}\psi(x)$$

$$\psi(x+na) = e^{ikna}\psi(x).$$

Block Theorem:-

This expression is valid only when

$$\psi(x) = e^{ikx}\phi(x)$$

where $\phi(x+a) = \phi(x)$

$$\psi(x) = e^{ikx}\phi(x)$$

$$\psi(x+a) = e^{ik(x+a)}\phi(x+a)$$

$$= e^{ika}\left[e^{ikx}\phi(x)\right]$$

$$\therefore \psi(x+a) = e^{ika}\psi(x)$$

$$K \xleftrightarrow[\text{Relation.}]{\text{Dispersion}} k$$

Kronig - Penney

$$V(x) = \frac{\hbar^2}{m} \Omega \sum_{n=-\infty}^{+\infty} V_n(x)$$

$$V_n(x) = \delta(x - na) = 1, \quad \text{when } x = \pm na.$$

$$= 0, \quad \text{otherwise.}$$

Dirac Comb.

Dispersion Relation

$$\cos Ka = \cos ka + \frac{\Omega}{k} \sin ka$$

$$|\cos Ka| \leq 1$$

$$\left| \cos ka + \frac{\Omega}{k} \sin ka \right| \leq 1$$

$$1 = \xi \cos \sigma$$

$$\frac{\Omega}{k} = \xi \sin \sigma$$

$$\Rightarrow \left| \xi \cos(ka - \sigma) \right| \leq 1.$$

$$\Rightarrow \left| \cos(ka - \sigma) \right| \leq 1/|\xi|$$

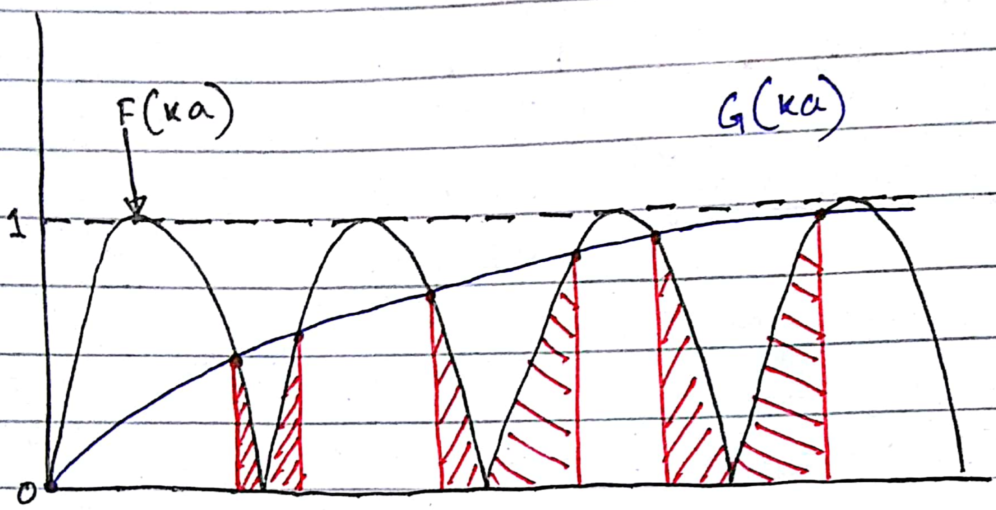
$$\xi = \sqrt{1 + \frac{\Omega^2}{k^2}}$$

$$\sigma = \tan^{-1} \frac{\Omega}{k}.$$

$$\left|\cos\left\{ka - \tan^{-1}\left(\frac{\Omega}{k}\right)\right\}\right| \leq \frac{1}{\left|\sqrt{1+\left(\frac{\Omega}{k}\right)^2}\right|}$$

$$\left|\cos\left\{ka - \tan^{-1}\left(\frac{\Omega a}{ka}\right)\right\}\right| \leq \frac{1}{\sqrt{1+\left(\frac{\Omega a}{ka}\right)^2}}$$

$$F(ka) \leq G(ka)$$



$$E(k) = \cancel{E_0}^{\,0} + \cancel{\frac{\partial E}{\partial k}}^{\,0}\Big|_k + \frac{\partial^2 E}{\partial k^2}\Big|\frac{k^2}{2!} + \cdots$$

$$E(-k) = E(k).$$

$$E(k) = \frac{k^2}{2}\frac{\partial^2 E}{\partial k^2} \quad \rightarrow (1).$$

$$E = \frac{\hbar^2 k^2}{2m} \quad \rightarrow (2).$$

$$\therefore \frac{1}{2}\frac{d^2 E}{dk^2} = \frac{\hbar^2}{2m}.$$

$$\therefore \boxed{m^* = \frac{\hbar^2}{d^2E/dk^2}}$$

$$v_g = \frac{d\omega}{dk} = \frac{1}{\hbar}\frac{d(\hbar\omega)}{dk} = \frac{1}{\hbar}\frac{dE}{dk}.$$

$$\therefore v_g = v_g(k).$$

$$v_g(-k) = -v_g(k).$$

$$p = mv_g(k) = (-m)\{-v_g(k)\}$$
$$= (-m)\{v_g(-k)\}.$$

Hole.

$$j = nev_g(k) = n(-e)\{-v_g(k)\}$$
$$= n(-e)\{v_g(-k)\}$$