Syllabus

NUMERICAL METHODS

Code : M (CS) 301, Contacts : 2L, Credits : 2

Theory

Approximation in numerical computation : Truncation and rounding errors, Fixed and floating-point arithmetic, Propagation of errors. 4

Interpolation : Newton forward / backward interpolation, Lagrange's and Newton's divided difference Interpolation 5

Numerical integration : Trapezoidal rule, Simpson's $\frac{1}{3}$ rd rule, Expressions for corresponding error terms. 3

Numerical solution of a system of linear equations :

Gauss elimination method, Matrix inversion, LU Factorization method Gauss-Seidel iterative method.

Numerical solution of Algebraic equation : 6

Bisection method, Regula-Falsi method, Newton-Raphson method 4

Numerical solution of ordinary differential equation : Euler's method, Runge-Kutta methods, Predictor-Corrector methods and Finite Difference method. 6

Code : (CS) 301,Contacts : 2L, Credits : 1

Practical

1. Assignments on Newton forward / backward, Lagrange's interpolation.
2. Assignments on numerical integration using Trapezoidal rule, Simpson's $\frac{1}{3}$ rd rule, Weddle's rule
3. Assignments on numerical solution of a system of linear equations using Gauss elimination,Gauss-Seidel iterations.
4. Assignments on numerical solution of Algebraic equation by Bisection, Regular-falsi and Newton Raphson methods.
5. Assignments on ordinary differential equation : Taylor series, Euler's, Runga-Kutta methods.
6. Introduction to Software Packages : Matlab/ mathematica scilab / Labview

CONTENTS

1 APPROXIMATION IN NUMERICAL COMPUTATION

1.1 Introduction:

Numerical analysis is the subject of study to find the numerical solutions of mathematical problems by computational methods. In this context we shall consider various numerical methods for computing approximate numerical results of different mathematical problems and analyze the errors in the result due to the errors in the given data or the errors in methods or both. In this chapter, we now proceed to understand different type of errors, their estimate and propagation of errors in calculations of different numerical procedure.

1.2. Approximate numbers and significant figures.

(i) Approximate numbers

The numbers like $5, \frac{1}{2}, \frac{1}{25}$, 200 etc are called *exact numbers* because these numbers can be expressed exactly by a finite number of digits. On the other hand the numbers like $\sqrt{3}$, 0, π etc can not be expressed exactly by a finite number of digits. We can write these numbers to a certain degree of accuracy, as $1\cdot7320, 2\cdot7183, 3\cdot1416$ etc which are only approximations to the true values and are called *approximate numbers*. Thus an approximate number is defind as a number approximated to the exact number and there is a slight difference between the exact and approximate numbers.

(ii) Significant digits

The digits which are used to represent a number are called significant digits (figures). Thus 1, 2, 3, ..., 9 are significant digits and 0 is significant digit except if it is used to fix the decimal place or to discard digits or to fill the unknown places. For example, the number 0.002396 has significant digits 2, 3, 9, 6; the zeros used here are not significant because they only fix the decimal places. But, for the number 0.01205, the significant digits are 1, 2, 0, 5. Here first zero after the decimal point is not significant.

1.3. Rounding off numbers.

There are numbers with large number of digits, viz, $\sqrt{2} = 1.414213 \ldots$. For practical computation, it is necessary to cut-off some unwanted digits and retain only the desired such as 1.414 or 1.4142. This process of dropping unnecessary digits is called *rounding off*.

The general rules for rounding off a number to n significant figures are as follows :

Discard all digits to the right of the n^{th} digit and if among these discarded digits the digit in the $(n+1)^{th}$ place is

(i) greater than 5 then the digit in the n^{th} place is increased by 1.

(ii) less than 5 then the digit in the n^{th} place is left unchanged

(iii) exactly 5 then the convention is to leave the nth digit unaltered if it is even and to increase it by 1 if it is odd.

For example, the following numbers are rounded off to four significant figures :

6.02887	becomes	6.029
2.5632	becomes	2.563
79.3998	becomes	79.40
8.42853	becomes	8.428
8.42756	becomes	8.428

1.4. Errors and their computation

Let us start with some simple definitions about error. The difference between the true value of a quantity and the approximate value computed or obtained by measurement is called the *error*. Thus, if x_T and x_A be the true and approximate value of the solution in solving a problem, then the quantity $x_T - x_A$ gives the *error* in x_A. The *absolute error*, E_a invloved in x_A is given by

$$E_a = |x_T - x_A| \quad \ldots \quad (1)$$

The *relative error*, E_r in x_A is defined by

$$E_r = \frac{|x_T - x_A|}{x_T}, \text{ provided } x_T \neq 0 \quad \ldots \quad (2)$$

The percentage error, E_p is 100 times the relative error,

i.e., $E_p = E_r \times 100 = \frac{|x_T - x_A|}{x_T} \times 100$, provided $x_T \neq 0$ ... (3)

As an illustration, suppose that the number 4.6285 be rounded off to 4.628 correct to four significant figures. Then we have

$$x_T = 4.6285, x_A = 4.628$$

So the absolute error is

$$E_a = |4.6285 - 4.628| = 0.0005$$

The relative error is given by

$$E_r = \frac{0.0005}{4.6285} = 1.0804 \times 10^{-4}$$

So the percentage error is

$$E_p = 1.0804 \times 10^{-4} \times 100$$
$$= 1.0804 \times 10^{-2}$$

Often we come across with two other type of errors in numerical computation neglecting a gross mistake. The error which is inherent in a numerical method itself or in the statement of a given problem is called *truncation error*. Another type of error is the *computational error* which arises during arithmetic computation due to the finite representation of numbers.

Truncation errors

The error which arise due to approximation of the result or due to the replacement of an infinite process by a finite one are called *truncation errors*. For exmaple, if a function $f(x)$ be expressed in the form of an infinite series, then for computation, we have to truncate the series at a certain stage causing an errorr, called truncation error.

Rounding errors

This is a one type of computation error which arise due to the process of rounding off the number during the computation.

Such errors are unavoidable in most of the calculation due to the limitations of the computing aids. In desk calculators we can reduce the round off errors by using more significant figures at each step of the computation, at least one more significant data than that of the given data, must be retained and rounding off is to be performed in the last operation.

For example, when a result 2.01536 is rounded off to four decimal places, then

$$x_T = 2.01536$$
$$x_A = 2.0154$$

So, in this case the rounded off error is given by

$$x_T - x_A = 2.01536 - 2.0154$$
$$= -0.00004.$$

1.5. Fixed and floating point arithmetic

The first step in the computation with digital computers is to convert the decimal numbers to another number system with base b (say, binary number system with base 2) understandable to that particular computer and then to store these converted numbers in computer memory, which is a collection of small cells. Each cell can accommodate a *word*, that consists of the same number of characters, the left most being a sign and the others digits. Negative numbers are stored as absolute values plus a sign. The number of characters (sign plus digits) in a word that can be stored in a cell is called the *word length.* The word length varies from one computer to another. Real numbers can be stored in the computer word in two forms :

(i) Fixed point form

(ii) Floating point form

In fixed point form, a n digit number is assumed to have its decimal point at the left-hand end of the word. So all numbers are assumed to be less than 1 in magnitude. The fixed-point number with base b and n digits word length can be written as

$$\pm\left(a_1 b^{-1} + a_2 b^{-2} + \dots + a_n b^{-n}\right) \quad \dots \quad (4)$$

where $0 \le a_i < b, i = 1, 2, \dots n$

To avoid the difficulty, of keeping every number less than 1 in magnitude during computation, most computer use normalised floating point form for a real number. A normalised floating point form of a real number is

$$\pm \cdot\, d_1 d_2 \dots d_n \times b^k \quad \dots \quad (5)$$

where b is the base of the number system used in the computer, $d_1, d_2, \dots, d_n$ are all digits $(d_1 \ne 0)$ in the b-base system and the number k, called the exponent or characteristic, is such that $m \le k \le M$. The values of the numbers m and M vary with the computer. The fractional part $\cdot\, d_1 d_2 \dots d_n$ is called the *mantissa*. In this case, the number is said to have n *significant digits.*

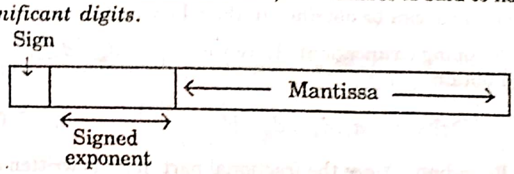

Fig.1

The above figure shows one way that a floating point number could be stored in a word. The first bit is reserved for the sign, the next series of bits for the signed exponent and the last bits for the mantissa.

Note that the mantissa is usually normalized if it has leading zero digits. For example, suppose the quantity $\frac{1}{34} = 0.02941176\dots$ was stored in a floating point base 10 system that allowed only five decimal places to be stored. Thus, $\frac{1}{34}$ would be stored as

$$0.02941 \times 10^0$$

But, in that case, the number can be normalized to remove the leading zero by multiplying the mantissa by 10 and lowering the exponent by 1 to give

$$0.29411 \times 10^{-1}$$

Thus, we retain an additional significant figure when the number is stored.

Floating point representation allows both fractions and very large numbers to be expressed on the computer. However, it has some disadvantage.

For example, floating point numbers take up more space and take longer to process than integer numbers. More significantly, their use introduces a source of error because the matissa holds only a finite number of significant figures. Thus a round off error is introduced. Thus, if a number x has the representation in the form

$$x = \cdot\, d_1 d_2 \ldots d_n d_{n+1} \ldots \times b^k \qquad \ldots \quad (6)$$

then the floating-point number $fl(x)$ in n-digit mantissa standard form can be obtained in the following two ways :

(i) Chopping (truncation) : Here we neglect $d_{n+1}, d_{n+2}, \ldots$ in (6) and obtain

$$fl(x) = \cdot\, d_1, d_2 \ldots . d_n \times b^k \qquad \ldots \quad (7)$$

(ii) Rounding : Here the fractional part in (6) is written as

$$\cdot\, d_1\, d_2 \ldots d_n d_{n+1} + \frac{1}{2} b$$

and the first n digits are taken to write the floating point number.

For example, let us consider the fixed point number $N = 28.516789$ whose floating point form is $\cdot 28516789 \times 10^2$. If the word length of the computer is 5, chopping gives the number $N' = \cdot\, 28156 \times 10^2$, or $28 \cdot 156$ whereas rounding gives the number $N' = \cdot\, 28157 \times 10^2$ or 28.157

The errors due to chopping or rounding are called *storage errors* that occur at the input stage. At the intermediate stages of arithmetic operations,varied amount of errors occur depending on the nature of operations performed which are called *computational errors.*

When two floating point number are added or subtracted, the digits in the number with the smaller exponent must be shifted to align the decimal points. This shifting can lose some of the significant digits of one of the values.

For example, let $x = 387.4$, $y = 0.01234$

$\therefore x - y = 387.38766$ and $x + y = 387.41234$

If this arithmetic operation is done by a hypothetical computer with word length 4, then

$x' = 0.3874 \times 10^3$

and $y' = 0.0000 \times 10^3$ $\left[\because\ y = 0.01234 = 0.00001234 \times 10^3\right]$

Then $x' \pm y' = 0.3874 \times 10^3$

Thus we see that a smaller value has no effect in addition or subtraction and so the error is serious.

Next, let $x = 3780$, $y = .321$

Then $x' = .378 \times 10^4$, $y' = .0000321 \times 10^4$

$\therefore\ x' - y' = .3779679 \times 10^4$

$= .377 \times 10^4$ (chopping)

$= .378 \times 10^4$ (rounding)

Thus shifting to align the decimal points has completely lost the significant digits of the subtrahend.

1.6. Propagation of Errors

Let us now consider the effect of calculations with numbers which involve errors. We first consider arithmetic operations $+, -, \times$ and $\div$. Let ω denote any one of these operations and w^* be the computed version of that operation. Then, if x_A and y_A are the approximations in the calculations containing errors corresponding to the ture values x_T and y_T respectively, we can write

$$x_T = x_A + \varepsilon, \quad y_T = y_A + \varepsilon'$$

where ε, ε' are the corresponding errors. Thus, if $x_A w^* y_A$ is the actually computed number, then for its error, we have

$$x_T \omega y_T - x_A \omega^* y_A = \left(x_T \omega y_T - x_A \omega y_A\right) + \left(x_A \omega y_A - x_A \omega^* y_A\right) \qquad \ldots \quad (8)$$

The first term on the right hand side of (8) within the bracket is known as the *propagated error* while the second term within the bracket is called the *rounding or chopping error.*

We now discuss some particular cases of propogated error

Case (i) For addition and subtraction, we have

$$(x_T \pm y_T)-(x_A \pm y_A)=(x_T - x_A)\pm(y_T - y_A)$$

$$= \varepsilon \pm \varepsilon^1$$

Thus error $(x_A \pm y_A)=$ error $(x_A)\pm$ error (y_A) ... (9)

Case (ii) For multiplication, we have

$$x_T y_T - x_A y_A = x_T y_T - (x_T - \varepsilon)(y_T - \varepsilon')$$

$$= x_T\varepsilon' + y_T\varepsilon - \varepsilon\varepsilon'$$

Thus error $(x_A y_A) = x_T$ error $(y_A) + y_T$ error (x_A) ... (10)

$-$ error (x_A) error (y_A)

So the relative error in $x_A y_A$ is

$$E_r(x_A y_A) = \frac{x_T y_T - x_A y_A}{x_T y_T}$$

$$= \frac{\varepsilon}{x_T} + \frac{\varepsilon'}{y_T} - \frac{\varepsilon}{x_T}\frac{\varepsilon'}{y_T}$$

$$= E_r(x_A) + E_r(y_A) - E_r(x_A)E_r(y_A)$$

If $E_r(x_A); E_r(y_A) << 1$, then

$$\therefore E_r(x_A y_A) \simeq E_r(x_A) + E_r(y_A) \quad \ldots (11)$$

Case (iii) For division, we get by proceeding along the same lines in multiplication,

$$E_r(x_A / y_A) \simeq E_r(x_A) - E_r(y_A), \text{ provided } |E_r(y_A)| << 1 \quad \ldots (12)$$

Notes (i) The relative errors in multiplication or division do not propagate very rapidly

(ii) $E_r(x_A \pm y_A)$ can be quite poor in comparison with $E_r(x_A)$ and $E_r(y_A)$

As an illustration, suppose that $x_T = \pi$, $x_A = 3.1416$ and $y_T = \frac{22}{7}$, $y_A = 3.1429$

Then $x_T - x_A \simeq -7.35 \times 10^{-6}$

$y_T - y_A \simeq -4.29 \times 10^{-5}$

$\therefore\ \varepsilon_r(x_A) = -2.34 \times 10^{-6}$ and $\varepsilon_r(y_A) \simeq -1.36 \times 10^{-5}$

Also $(x_T + y_T) - (x_A + y_A) \simeq -5.02 \times 10^{-5}$

and

$(x_T - y_T) - (x_A - y_A) \simeq 3.55 \times 10^{-5}$

Hence $E_r(x_A + y_A) \simeq -7.99 \times 10^{-6}$

and

$E_r(x_A - y_A) \simeq -2.8 \times 10^{-2}$.

Thus although the error in $(x_A - y_A)$ is quite small, its relative error is not so and is much larger than $E_r(x_A)$ or $E_r(y_A)$. But the relative error in $(x_A + y_A)$ is quite small.

1.7. General formula for errors

Consider the differentiable function $u = f(u_1, u_2, \ldots, u_n)$ of several independent variables $u_1, u_2, \ldots, u_n$ subject to the errors $\Delta u_1, \Delta u_2, \ldots, \Delta u_n$ respectively. Then the errors in $u_i (i = 1, 2, \ldots, n)$ cause an error Δu in u and is given by

$$u + \Delta u = f(u_1 + \Delta u_1, u_2 + \Delta u_2, \ldots, u_n + \Delta u_n)$$

$$= f(u_1, u_2, \ldots, u_n) + \Delta u_1 \frac{\partial f}{\partial u_1} + \Delta u_2 \frac{\partial f}{\partial u_2} + \ldots + \Delta u_n \frac{\partial f}{\partial u_n}$$

$$+ \frac{1}{2}\left[(\Delta u_1)^2 \frac{\partial^2 f}{\partial u_1^2} + (\Delta u_2)^2 \frac{\partial^2 f}{\partial u_2^2} + \ldots + (\Delta u_n)^2 \frac{\partial^2 f}{\partial u_n^2} + 2\Delta u_1 . \Delta u_2 \frac{\partial^2 f}{\partial u_1 \partial u_2} + \ldots\right] + \ldots$$

Noting that the errors $\Delta u_1, \Delta u_2, \ldots, \Delta u_n$ are relatively small, we neglect their squares, products and higher powers. Thus we get

$$u + \Delta u \simeq u + \frac{\partial f}{\partial u_1}\Delta u_1 + \frac{\partial f}{\partial u_2} . \Delta u_2 + \ldots + \frac{\partial f}{\partial u_n}\Delta u_n$$

i.e., $\Delta u \simeq \frac{\partial f}{\partial u_1}\Delta u_1 + \frac{\partial f}{\partial u_2}\Delta u_2 + \ldots + \frac{\partial f}{\partial u_n}\Delta u_n$ ... (13)

This is the *general formula for computing the error* of a function $u = f(u_1, u_2, \ldots, u_n)$.

The relative error of u is given by

$$E_r = \frac{\Delta u}{u} \simeq \frac{\partial f}{\partial u_1}\cdot\frac{\Delta u_1}{u} + \frac{\partial f}{\partial u_2}\cdot\frac{\Delta u_2}{u} + \ldots + \frac{\partial f}{\partial u_n}\cdot\frac{\Delta u_n}{u} \quad \ldots (14)$$

1.7.1. Error in addition of numbers

Let $u = \sum_{i=1}^{n} u_i$, where $u_i (i = 1, 2, \ldots n)$ are n approximate numbers

Then we have

$$\Delta u = \sum_{i=1}^{n}(u_i + \Delta u_i) - \sum_{i=1}^{n} u_i = \sum_{i=1}^{n}\Delta u_i$$

$$\therefore |\Delta u| \le \sum_{i=1}^{n}|\Delta u_i|$$

Thus the absolute error of a sum of approximate numbers is less than on equal to the sum of the absolute errors of these numbers

Also $\left|\frac{\Delta u}{u}\right| \le \left|\frac{\Delta u_1}{u}\right| + \left|\frac{\Delta u_2}{u}\right| + \ldots + \left|\frac{\Delta u_n}{u}\right|$

This gives the maximum relative error of u.

1.7.2. Error in subtraction of numbers

Let $u = u_1 \sim u_2$, where u_1 and u_2 being two approximate numbers.

Then

$$\Delta u = (u_1 + \Delta u_1) \sim (u_2 + \Delta u_2) - (u_1 \sim u_2) = \Delta u_1 \sim \Delta u_2$$

$$\therefore |\Delta u| \le |\Delta u_1| + |\Delta u_2|$$

Thus the absolute error of the difference of two approximate numbers is less than or equal to the sum of the absolute errors of these numbers.

The maximum relative error of u is given by

$$E_r = \left|\frac{\Delta u}{u}\right| \le \left|\frac{\Delta u_1}{u}\right| + \left|\frac{\Delta u_2}{u}\right|$$

1.7.3. Error in multiplication of numbers.

Let $u = u_1 u_2 \ldots u_n$, where $u_i (i = 1, 2, \ldots n)$ being n approximate numbers

Then we have

$$u + \Delta u = (u_1 + \Delta u_1)(u_2 + \Delta u_2)\ldots(u_n + \Delta u_n)$$
$$= u_1 u_2 \ldots u_n + u_2 u_3 \ldots u_n \Delta u_1 + u_1 u_3 u_4 \ldots u_n \Delta u_2 + \ldots + u_1 u_2 \ldots u_{n-1}\Delta u_n + \ldots$$

Since the errors $\Delta u_1, \Delta u_2, \ldots, \Delta u_n$ are relatively small, we neglect their products and, therefore, we have

$$u + \Delta u \simeq u + u_2 u_3 \ldots u_n \Delta u_1 + u_1 u_3 \ldots u_n \Delta u_2 + \ldots + u_1 u_2 \ldots u_{n-1}\Delta u_n$$

so that

$$\frac{\Delta u}{u} \simeq \frac{\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2} + \ldots + \frac{\Delta u_n}{u_n}$$

and hence

$$E_r = \left|\frac{\Delta u}{u}\right| \le \left|\frac{\Delta u_1}{u_1}\right| + \left|\frac{\Delta u_2}{u_2}\right| + \ldots + \left|\frac{\Delta u_n}{u_n}\right|$$

Thus the relative error of the product of n approximate numbers is less than or equal to the sum of the relative errors of these numbers.

1.7.4. Error in division of numbers.

Let $u = \frac{u_1}{u_2}$, u_1 and u_2 being two approximate numbers. Then we get

$$u + \Delta u = \frac{u_1 + \Delta u_1}{u_2 + \Delta u_2} = \frac{u_1 + \Delta u_1}{u_2}\left(1 + \frac{\Delta u_2}{u_2}\right)^{-1}$$
$$= \frac{u_1}{u_2}\left(1 + \frac{\Delta u_1}{u_1}\right)\left(1 - \frac{\Delta u_2}{u_2} + \ldots\right)$$
$$= u\left(1 + \frac{\Delta u_1}{u_1} - \frac{\Delta u_2}{u_2} + \ldots\right)$$

$$\therefore 1+\frac{\Delta u}{u}=1+\frac{\Delta u_1}{u_1}-\frac{\Delta u_2}{u_2}+\ldots$$

Neglecting the products of the relatively small errors Δu_1 and Δu_2, we have

$$\frac{\Delta u}{u}\simeq\frac{\Delta u_1}{u_1}-\frac{\Delta u_2}{u_2}$$

$$\therefore E_r=\left|\frac{\Delta u}{u}\right|\leq\left|\frac{\Delta u_1}{u_1}\right|+\left|\frac{\Delta u_2}{u_2}\right|$$

Hence the relative error of the quotient of two approximate numbers is less than or equal to the sum of the relative errors of the number.

Illustrative Examples

Ex. 1. Round off the numbers 0.004935, 870.91, 100.53, –439.85 to three significant digits.

Solution. 0.00494, 871, 100, –440

Ex. 2. Round off

(a) 123456 to 3 significant figures

(b) 31.520457 to 1 significant figures

[W.B.U.T., CS-312, 2004]

Solution. (a) 12.3×10^4 (b) 3×10^2

Ex.3. Find the significant figures for the number

(i) 0.007501 (ii) 0.07510 (iii) 980.37

(iv) 109.00 (v) 10000

Solution. The significant figures are

(i) 7, 5, 0, 1 (ii) 7, 5, 1 (iii) 9, 8, 0, 3, 7

(iv) 1, 0, 9 (v) 1

Ex. 4. Find the number of significant digits of

(i) 0.1204 (ii) 0.002560

(iii) 3100, (iv) –56.0270

Solution. (i) 4 (ii) 3 (iii) 2 (iv) 5

Ex.5. Round off the following number to 4 decimal places :

(i) 3.567019, 9.77385, 36.00895, 0.00126

(ii) –6.00255, 3.08914×10^2, 0.28997, 100.567

Solution.(i) 3.5670 9.7738 36.0090 0.0013

(ii) –6.0026 3.0891×10^2 0.2900 1.0057×10^2

Ex. 6. If $\pi=3.14$ is used in place of 3.14156 find the absolute and relative errors.

Solution. Here, the true value, $x_T=3.14156$ and approximate value $x_A=3.14$

$$\therefore \text{Absolute error, } E_a=|x_T-x_A|$$
$$=|3.14156-3.14|$$
$$=0.00156$$

and the relative error, $E_r=\dfrac{|x_T-x_A|}{x_T}$

$$=\frac{0.00156}{3.14156}$$
$$=4.966\times10^{-4}$$

Ex. 7. If the value of $e=2.71828$ is replaced by 2.71937, what is the percentage error ?

Solution. Here $x_T=2.71828, x_A=2.71937$

So the required percentage error is

$$E_p=\frac{|x_T-x_A|}{x_T}\times100$$
$$=\frac{|2.71828-2.71937|}{2.71828}\times100$$
$$=4.01\times10^{-2}.$$

Ex. 8. Write down approximate representation of $\frac{2}{3}$ correct upto four significant digits; find the absolute, relative and percentage error.

Solution. Here $\frac{2}{3} = 0.66666.....$

$\therefore\ \frac{2}{3} = 0.6667$, correct upto four significant digits

Let $x_T = \frac{2}{3}$, $x_A = 0.6667$

So the absoluter error is

$$E_a = |x_T - x_A| = \left|\frac{2}{3} - 0.6667\right|$$

$$= \frac{0.0001}{3}$$

$$= 3.3 \times 10^{-5}$$

The relative error is

$$E_r = \frac{E_a}{x_T} = \frac{0.0001}{3} / \frac{2}{3} = 5 \times 10^{-5}$$

$\therefore$ The percentage error is

$$E_p = E_r \times 100$$
$$= 5 \times 10^{-3}.$$

Ex. 9. Find the percentage error in approximating $\frac{4}{3}$ to 1.3333

[W.B.U.T., M(CS)-312, 2010]

Solution. Let $x_T = \frac{4}{3}$, $x_A = 1\cdot 3333$

So the percentage error is

$$\frac{|x_T - x_A|}{x_T} \times 100$$

$$= \frac{\left|\frac{4}{3} - 1.3333\right|}{\frac{4}{3}} \times 100$$

$$= 0.0025$$

Ex. 10. Find the relative error in computations of $x - y$ for $x = 12.05$ and $y = 8.02$ having absolute errors $\Delta x = 0.005$ and $\Delta y = 0.001$

Solution. Let $u = x - y$

$$\therefore\ u = 12.05 - 8.02 = 4.03$$

The maximum possible absolute error is

$$|\Delta u| = |\Delta x| + |\Delta y|$$
$$= 0.005 + 0.001$$
$$= 0.006$$

Hence the relative error in computation of $u = x - y$ is

$$\frac{|\Delta u|}{u} = \frac{0.006}{4.03}$$

$\simeq 0.0015$, correct upto four decimal places.

Ex. 11. Find absolute, relative and percentage error in computation of $f(x) = 3\sin x - 2x^2 - 9$ for $x = 0$ when the error in x is 0.003.

Solution. Here $f(x) = 3\sin x - 2x^2 - 9$

$$\therefore\ \Delta f(x) = \frac{df}{dx} \cdot \Delta x$$

$$= (3\cos x - 4x)\Delta x$$

At $x = 0$, $\Delta x = 0.003$

So the error in computation of $f(x)$ is

$$\Delta f(x) = (3 \times \cos 0 - 4 \times 0)0.003$$
$$= 3 \times 0.003$$
$$= 0.009$$

Thus the absolute error is 0.009

So the relative error is

$$\frac{|\Delta f(x)|}{|f(0)|} = \frac{0.009}{9} = 0.001$$

Hence the percentage error is

$$0.001 \times 100 = 0.1$$

Ex. 12. If $u = \dfrac{x^2 y}{2}$, $\Delta x = 0.01, \Delta y = 0.02$ at $x = 2, y = 1$ compute the maximum absolute and relative errors in evaluating u.

Solution. We have

$$\frac{\partial u}{\partial x} = xy, \frac{\partial u}{\partial y} = \frac{x^2}{2}$$

$$\therefore \Delta u = \frac{\partial u}{\partial x}.\Delta x + \frac{\partial u}{\partial y}.\Delta y$$

$$= xy\Delta x + \frac{x^2}{2}.\Delta y$$

$$\therefore |\Delta u| \le |xy||\Delta x| + \frac{x^2}{2}|\Delta y|$$

$$= 2 \times 0.01 + \frac{4}{2} \times 0.02$$

$$= 0.06$$

So the maximum absolute error in u is 0.06 and hence the maximum relative error of u is given by

$$(E_r)_{max} = \frac{0.06}{u} = \frac{0.06}{\frac{2^2 \times 1}{2}} = 0.03$$

Ex. 13. Obtain the relative error in $u = x_1^n x_2^n x_3^n$ in terms of the relative errors of x_1, x_2, x_3

Solution. We have

$$u = x_1^n x_2^n x_3^n$$

$$\therefore \log u = n \log x_1 + n \log x_2 + n \log x_3$$

$$\therefore \frac{1}{u}\frac{\partial u}{\partial x_1} = \frac{n}{x_1}, \frac{1}{u}\frac{\partial u}{\partial x_2} = \frac{n}{x_2}, \frac{1}{u}\frac{\partial u}{\partial x_3} = \frac{n}{x_3}$$

So the relative error of u is given by

$$E_r = \frac{\Delta u}{u} \approx \frac{\partial u}{\partial x_1}\cdot\frac{\Delta x_1}{u} + \frac{\partial u}{\partial x_2}\cdot\frac{\Delta x_2}{u} + \frac{\partial u}{\partial x_3}\cdot\frac{\Delta x_3}{u}$$

$$= n\frac{\Delta x_1}{x_1} + n\frac{\Delta x_2}{x_2} + n\frac{\Delta x_3}{x_3}$$

As the errors $\Delta x_1, \Delta x_2$ and Δx_3 may be positive or negative, so we take the absolute values of the terms on the right side. Thus we get

$$(E_r)_{max} \le n\left|\frac{\Delta x_1}{x_1}\right| + n\left|\frac{\Delta x_2}{x_2}\right| + n\left|\frac{\Delta x_3}{x_3}\right|$$

Ex. 14. Find an upper limit of the relative error in the measure of $w = \dfrac{x^\alpha y^\beta}{z^\gamma}$

Solution. We have

$$w = \frac{x^\alpha y^\beta}{z^\gamma}$$

$$\therefore \log w = \alpha \log x + \beta \log y - \gamma \log z$$

$$\therefore \frac{1}{w}\frac{\partial w}{\partial x} = \frac{\alpha}{x}, \quad \text{i.e.,} \quad \frac{\partial w}{\partial x} = \frac{\alpha w}{x}$$

Similarly $\dfrac{\partial w}{\partial y} = \dfrac{\beta w}{y}, \dfrac{\partial w}{\partial z} = -\dfrac{\gamma w}{z}$

Now, $\Delta w = \dfrac{\partial w}{\partial x}\cdot\Delta x + \dfrac{\partial w}{\partial z}\cdot\Delta y + \dfrac{\partial w}{\partial z}\cdot\Delta z$

gives

$$\frac{\Delta w}{w} = \frac{\alpha}{x}\Delta x + \frac{\beta}{y}\Delta y - \frac{\gamma}{z}\Delta z$$

Thus the relative error of ω is given by

$$E_r = \left|\frac{\Delta w}{w}\right| \le \alpha\left|\frac{\Delta x}{x}\right| + \beta\left|\frac{\Delta y}{y}\right| + \gamma\left|\frac{\Delta z}{z}\right|$$

Hence the upper limit of the relative error is

$$\alpha\left|\frac{\Delta x}{x}\right|+\beta\left|\frac{\Delta y}{y}\right|+\gamma\left|\frac{\Delta z}{z}\right|.$$

Exercise

I. Short Answer Questions

1. Find the number of significant digit

(i) 120.00 (ii) 12340 (iii) 0.2050

(iv) 20000 (v) –12.970 (vi) 89.7010

2. Round off the following numbers to 3 significant figures :

(i) 0.0063, 2.138, –46.285, 77.75

(ii) 2, 12, 1506, 19928

3. Round off the following numbers :

(i) 170.570 upto four significant figures

(ii) 21753 upto three significant figures

(iii) –79.861 upto three significant figures

(iv) 32056 upto two significant figures

(v) 0.2502 upto one significant figures

4. Round off the following number to 3 decimal places

(i) 2.47235, 0.003568, 42.3085, 9.77345

(ii) 1.2755, 6.4452×10^3, 0.999500, 0.24813

(iii) 98.9268, 0.00282, –72.0506, –0.0056

5. If $\frac{5}{3}$ is approximated to 1.6667, find the absolute error. [W.B.U.T.,CS-312,2007]

6. Find the percentage error in approximate representation of $\frac{7}{6}$ by 1.16

7. If 0.8333 is taken to be an approximate value of $\frac{5}{6}$, find the percentage error

8. Find the percentage error in approximate representation of $\frac{4}{3}$ by 1.33.

9. Find the absolute and relative error when 5.0214 is round off to 3 significant figures.

10. Round off the number 8.03567 to four significant digits and compute the percentage error.

Answers

1. (i) 2, (ii) 4, (iii) 3, (iv) 1, (v) 4, (vi) 5

2. (i) $630\times10^{-5}, 2.14, -46.3, 77.8$

(ii) $200\times10^{-2}, 120\times10^{-1}, 151\times10^{2}, 199\times10^{4}$

3. (i) 170.6 (ii) 21.8×10^3 (iii) –79.9 (iv) 3.2×10^4 (v) 0.3

4. (i) 2.472, 0.004, 42.308 (ii) 1.276, 6.445×10^3, 1.000, 0.248

(iii) 98.927, 0.003, –72.051, –0.006

5. 0.000033 **6.** 0.571 **7.** 0.004 **8.** 0.25

9. 0.0014, 27.88×10^{-5} **10.** $8\cdot036$, $4\cdot1\times10^{-7}$

II. Long Answer Questions

1. Find the absolute, relative and percentage error, if $\frac{1}{3}$ is approximated by 0.333.

2. If $\frac{5}{6}$ is represented by the approximate number 0.8333, compute absolute, relative and percentage errors.

3. If 3.45234 be an approximate value of 3.45678, find the absolute, relative and percentage errors.

4. If $\Delta r = \Delta h = 0.1$, find the absolute and relative errors upto three significant figures in $v = \frac{1}{3}\pi r^2 h$ when $r = 2$ and $h = 3$.

2 Calculus of Finite Differences

2.1 Introduction.

The calculus of finite difference deals with the changes in the values of the dependent variable with the change of the independent variable. Finite difference method is successfully applied in numerical analysis in interpolation, numerical differentiation and integration etc. In this chapter, we introduce and discuss different types of difference operators, their relations, fundamental theorem of difference calculus and their applications.

2.2. Finite differences.

Let $y = f(x)$ be a real-valued function of x defined in an interval $[a,b]$ and its values are known for $(n+1)$ equally spacing points x_i $(i = 0,1,2,\cdots,n)$ such that $x_i = x_0 + ih$ $(i = 0,1,2,\cdots,n)$ where $x_0 = a, x_n = b$ and h is the *space length*. Then x_i $(i = 0,1,2,\cdots,n)$ are called *nodes* and the corresponding values y_i are termed as *entries*.

We now introduce the concept of various type differences in order to find the values of $f(x)$ or its derivative for some intermediate values of x in $[a,b]$.

2.3. Forward Differences.

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \cdots, y_n - y_{n-1}$ for the entries $y_0, y_1, y_2, \cdots, y_{n-1}, y_n$ are called *first forward differences* and are denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \cdots, \Delta y_{n-1}$ respectively. Thus we have

$$\Delta y_i = y_{i+1} - y_i \quad (i = 0,1,2,\cdots,n-1) \qquad \cdots \qquad (1)$$

where Δ is called *forward difference operator*. In general forward difference operator is defined by

$$\Delta f(x) = f(x+h) - f(x). \qquad \cdots \qquad (2)$$

Similarly, the higher order forward differences are define as

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

$$\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i \qquad \ldots \quad (3)$$

$$\ldots \qquad \ldots$$

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$$

where $i = 0, 1, 2, \cdots, n-1$ and r $(1 \le r \le n)$ is a positive integer.

Now $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

Hence, for n^{th} order forward difference, we have

$$\Delta^n y_0 = y_n - {}^n c_1 y_{n-1} + {}^n c_2 y_{n-2} + \ldots + (-1)^n y_0 \qquad \ldots \quad (4)$$

We can calculate the above forward differences very easily with the help of the following tables, called forward difference table:

Table 1 : Forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

As an illustration consider the following difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	0			
		1		
3	1		–4	
		–3		22
5	–2		18	
		15		–35
7	13		–17	
		–2		
9	11			

From the table, we have

$$x_0 = 1, y_0 = 0, \Delta y_0 = 1, \Delta^2 y_0 = -4, \ \Delta^2 y_0 = 22$$

$$x_1 = 3, y_1 = 1, \Delta y_1 = -3, \Delta^2 y_1 = 18, \ \Delta^3 y_1 = -35$$

$$x_2 = 5, y_2 = -2, \Delta y_2 = 15, \Delta^2 y_2 = -17$$

and so on.

2.4. Some properties of Δ.

If a and b be any two constants, then

(i) $\Delta a = 0$

(ii) $\Delta\{af(x)\} = a\Delta f(x)$

(iii) $\Delta\{af(x) \pm bg(x)\} = a\Delta f(x) \pm b\Delta g(x)$

(iv) $\Delta[f(x)g(x)] = f(x)\Delta g(x) + \Delta f(x).g(x+h)$

$$= f(x+h)\Delta g(x) + \Delta f(x) g(x)$$

(v) $\Delta\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$

Proof. (i) Let $f(x) = a$

$\therefore \quad \Delta a = \Delta f(x) = f(x+h) - f(x) = a - a = 0$

(ii) $\Delta\{af(x)\} = af(x+h) - af(x)$

$$= a\{f(x+h) - f(x)\}$$

$$= a\ \Delta f(x).$$

(iii) $\Delta\{af(x) \pm bg(x)\}$

$$= \{af(x+h) \pm bg(x+h)\} - \{af(x) \pm bg(x)\}$$

$$= a\{f(x+h) - f(x)\} \pm b\{g(x+h) - g(x)\}$$

$$= a\Delta f(x) \pm b\Delta g(x)$$

(iv) Left as an exercise.

(v) Left as an exercise.

Example 1. Find $\Delta^2 f(x)$ where $f(x)$ is

(i) x^3 (ii) e^x (iii) $\frac{1}{x}$

Solution : (i) $\Delta f(x) = \Delta x^3$

$$= (x+h)^3 - x^3$$

$$= 3x^2h + 3xh^2 + h^3$$

$$\therefore \Delta^2 f(x) = 3h\Delta(x^2) + 3h^2\Delta x + \Delta h^3$$

$$= 3h\{(x+h)^2 - x^2\} + 3h^2(x+h-x) + 0$$

$$= 3h(2xh + h^2) + 3h^3$$

$$= 6h^2(x+h)$$

(ii) $\Delta f(x) = \Delta(e^x) = e^{x+h} - e^x = e^x(e^h - 1)$

$$\therefore \Delta f(x) = (e^h - 1)\,\Delta(e^x)$$

$$= (e^h - 1).e^x(e^h - 1)$$

$$= e^x(e^h - 1)^2$$

(iii) $\Delta f(x) = \Delta\left(\frac{1}{x}\right) = \frac{1}{x+h} - \frac{1}{x}$

$$= \frac{-h}{x(x+h)}$$

$$\therefore \Delta f(x) = -h\ \Delta\left\{\frac{1}{x(x+h)}\right\}$$

$$= -h\left\{\frac{1}{(x+h)(x+2h)} - \frac{1}{x(x+h)}\right\}$$

$$= \frac{2h^2}{x(x+h)(x+2h)}$$

2.5. Fundamental theorem of difference calculus

Theorem : The n^{th} order difference of a polynomial $P(x)$ of degree n is constant and its $(n+1)^{th}$ order difference varishes.

Proof. Let $P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ where $a_0, a_1, \dots, a_n$ are constant and $a_0 \neq 0$ be a polynomial of degree n. Then since

$$\Delta x^n = (x+h)^n - x^n = \sum_{i=1}^{n}\binom{n}{i}x^{n-i}h^i$$

which is polynomial of degree $n-1$, we have

$$\Delta P(x) = a_0\Delta x^n + a_1\Delta x^{n-1} + \dots + a_{n-1}\Delta x + \Delta a_n$$

$$= a_0nh\,x^{n-1} + b_1x^{n-2} + \dots + b_{n-2}x + b_{n-1}, \text{ say}$$

Then $\Delta P(x)$ is a polynomial of degree $n-1$.

Similary, $\Delta^2 P(x) = a_0nh\ \Delta x^{n-1} + b_1\Delta x^{n-2} + \dots + b_{n-2}\Delta x + \Delta b_{n-1}$

$$= a_0n(n-1)h^2.x^{n-2} + c_1x^{n-3} + \dots + c_{n-3}x + c_{n-2}, \text{ say}$$

which is a polynomial of degree $n-2$.

Proceeding in this way, we have

$$\Delta^n P(x) = a_0n(n-1)\dots 2.1h^n$$

$$= a_0n!h^n, \text{ which is constant}$$

Hence $\Delta^{n+1}P(x) = 0$

Example 2. Evaluate $\Delta^3 P(x)$ where $P(x) = 5x^3 - 6x + 11$, taking $h = 2$.

Solution. $\Delta^3 P(x)$

$$= \Delta^3\{5x^3 - 6x + 11\}$$
$$= 5\Delta^3 x^3 - 6\Delta^3 x + \Delta^3 11$$
$$= 5.3!\ 2^3 - 6\times 0 + 0$$
$$= 240$$

2.6. Backward differences :

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1}$ are called the *first backward differences* and denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively.

Thus we have

$$\nabla y_i = y_i - y_{i-1}, \quad i = 1, 2, \ldots, n \quad \ldots \quad (5)$$

where ∇ is called the *backward difference operator*. In general backward differece operator is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

Similarly, the higher order backward differences are defined as

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}$$
$$\nabla^3 y_i = \nabla^2 y_i - \nabla^2 y_{i-1}$$

and so on.

Now $\nabla y_1 = y_1 - y_0$

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

Hence for n^{th} order backward difference, we have

$$\nabla^n y_n = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - \cdots + (-1)^n y_0 \quad \ldots \quad (6)$$

We can calculate the different order backward differences very quickly with the help of the following table, called *backward difference table* :

Table 2 : Back difference table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
x_1	y_1	∇y_1			
x_2	y_2	∇y_2	$\nabla^2 y_2$		
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$	
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$

Example 3. Construct the backward difference table of $y = x^2 + 4$ for $x = 1, 3, 5, 7, 9$ and find the values of $\nabla^2 f(5), \nabla^2 f(7)$ and $\nabla^3 f(9)$.

Solution. The backward difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
1	5			
3	13	8		
5	29	16	8	
7	53	24	8	0
9	85	32	8	0

From the table, we have

$$\nabla^2 f(5) = 8,\ \nabla^2 f(7) = 8, \nabla^3 f(9) = 0$$

Example 4. Show that $\Delta.\nabla \equiv \Delta - \nabla$

[W.B.U.T., CS-312, 2004, 2007]

Solution. We have

$$\Delta f(x) = f(x+h) - f(x)$$

and $\nabla f(x) = f(x) - f(x-h)$

$$\therefore\ \Delta.\nabla f(x)$$
$$= \Delta[f(x) - f(x-h)]$$

$$= \Delta f(x) - \Delta f(x-h)$$
$$= \Delta f(x) - [f(x) - f(x-h)]$$
$$= \Delta f(x) - \nabla f(x)$$
$$= (\Delta - \nabla) f(x)$$
$$\therefore \Delta \cdot \nabla \equiv \Delta - \nabla$$

2.7. Shift Operator

The *Shift operator* is denoted by E and is defind as

$Ef(x) = f(x+h)$, h being the spacing

$\therefore E^2 f(x) = Ef(x+h) = f(x+2h)$

$E^3 f(x) = Ef(x+2h) = f(x+3h)$

In this way, in general, we have

$$E^n f(x) = f(x+nh) \quad \cdots \quad (7)$$

The *inverse shift operator* E^{-1} is defined by

$$E^{-1} f(x) = f(x-h)$$

and in general, we have

$$E^{-n} f(x) = f(x-nh) \quad \cdots \quad (8)$$

Since $\Delta f(x) = f(x+h) - f(x)$, it follows that

$$Ef(x) = f(x+h) = \Delta f(x) + f(x) = (\Delta + 1) f(x)$$

so that

$$E \equiv \Delta + 1$$

i.e. $\Delta \equiv E - 1 \quad \cdots \quad (9)$

Again $\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$

$$= f(x+2h) - 2f(x+h) + f(x)$$
$$= E^2 f(x) - 2Ef(x) + f(x)$$
$$= (E-1)^2 f(x)$$

Hence $\Delta^2 \equiv (E-1)^2 \quad \cdots \quad (10)$

In this way, in general, we have

$$\Delta^n \equiv (E-1)^n \quad \cdots \quad (11)$$

Also $E^{-1} f(x) = f(x-h) = f(x) - \nabla f(x)$

$[\because \nabla f(x) = f(x) - f(x-h)]$

$$= (1 - \nabla) f(x)$$

$$\therefore E^{-1} \equiv 1 - D \quad \cdots \quad (12)$$

Newton-Gregory formula.

We have $f(x+nh) = E^n f(x)$

$$= (1+\Delta)^n f(x)$$

$$= \left\{1 + \binom{n}{1}\Delta + \binom{n}{2}\Delta^2 + \ldots + \binom{n}{n}\Delta^n\right\} f(x)$$

$$= f(x) + \binom{n}{1}\Delta f(x) + \binom{n}{2}\Delta^2 f(x) + \ldots + \binom{n}{n}\Delta^n f(x)$$

$$= \sum_{i=0}^{n} \binom{n}{i} \Delta^i f(x)$$

Thus $f(x+nh) = \sum_{i=0}^{n} \binom{n}{i} \Delta^i f(x) \quad \cdots \quad (13)$

is known as *Newton-Gregory formula*.

Example 5. If $y(0) = -1$, $y(1) = 3$, $y(2) = 8$, $y(3) = 13$ find, $y(6)$

Solution. We construct the following difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	−1			
		4		
1	3		1	
		5		−1
2	8		0	
		5		
3	13			

From Newton-Gregory formula (13), we have

$$f(x+nh)=\sum_{i=0}^{n}\binom{n}{i}\Delta^i f(x)$$

$$\therefore\ f(6)=f(0+6\times 1)=\sum_{i=0}^{6}\binom{6}{i}\Delta^i f(0)$$

$$=\left\{1+\binom{6}{1}\Delta+\binom{6}{2}\Delta^2+\binom{6}{3}\Delta^3+\ldots\right\}f(0)$$

$$=f(0)+6\Delta f(0)+15\Delta^2 f(0)+20\Delta^3 f(0)+\ldots$$

$$=-1+6\times 4+15\times 1+20(-1)$$

$$=18$$

$$\therefore\ y(6)=18$$

16

$\frac{6\cdot 5}{2}=15$

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Example 6. Prove that $E\equiv e^{hD}$

Solution : We have

$$E\,f(x)=f(x+h)$$

$$=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots\text{, using Taylor's expansion}$$

$$=f(x)+hDf(x)+\frac{h^2}{2!}D^2 f(x)+\ldots$$

$$=\left(1+hD+\frac{h^2D^2}{2!}+\ldots\right)f(x)$$

$$=e^{hD}f(x)$$

$$\therefore\ E\equiv e^{hD}$$

Note : From the above result, we have

$$1+\Delta\equiv e^{hD}$$

$$\therefore\ hD=\log(1+\Delta)$$

$$\therefore\ D\equiv\frac{1}{h}\left(\Delta-\frac{1}{2}\Delta^2+\frac{1}{3}\Delta^3-\frac{1}{4}\Delta^4+\ldots\right)$$

2.8. Central difference and Average Operator

The central difference operator is denoted by δ and is defind as

$$\delta f(x)=f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)\qquad\ldots(14)$$

We can write (14) as

$$\delta f(x)=E^{\frac{1}{2}}f(x)-E^{-\frac{1}{2}}f(x)$$

$$=\left(E^{\frac{1}{2}}-E^{-\frac{1}{2}}\right)f(x)$$

$$\therefore\ \delta\equiv E^{\frac{1}{2}}-E^{-\frac{1}{2}}\qquad\ldots(15)$$

The average operator μ is defind as

$$\mu f(x)=\frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]\qquad\ldots(16)$$

$$=\frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)f(x)$$

$$\therefore\ \mu\equiv\frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)\qquad\ldots(17)$$

Example 7. Show that $\Delta-\nabla=\delta^2$

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Solution. From (14), we have

$$\delta\equiv E^{\frac{1}{2}}-E^{-\frac{1}{2}}$$

$$\therefore\ \delta^2\equiv E-2+E^{-1}$$

$$\equiv(1+\Delta)-2+(1-\nabla)\qquad[\because E\equiv 1+\Delta,\ E^{-1}\equiv 1-\nabla]$$

$$\equiv\Delta-\nabla$$

$$\therefore\ \Delta-\nabla\equiv\delta^2$$

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Example 8. Prove that $hD \equiv \sinh^{-1}(\mu\delta)$, where μ is the average operator and D is the differentiation operator.

Solution. We have

$$\mu\delta \equiv \frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)\left(E^{\frac{1}{2}}-E^{-\frac{1}{2}}\right)$$

$$=\frac{1}{2}\left(E-E^{-1}\right)$$

$$=\frac{1}{2}\left(e^{hD}-e^{-hD}\right)\quad\left[\because E\equiv e^{hD}\right]$$

$$=\sinh(hD)$$

This implies

$$hD=\sinh^{-1}(\mu\delta)$$

2.9. Evaluation of missing terms in given datas.

Suppose we are given n values of $y=f(x)$ out of equispaced value of nodes $x_0, x_1, x_2, \ldots, x_n$. Let the unknown value of y be k. Since n values of y are known, we can assume $y=f(x)$ to be a polynomial of degree $n-1$ in x. Then the n^{th} order difference of $f(x)$, i.e., $\Delta^n f(x)=0$, from where we can determine the value of k.

Example 9. Estimate the missing term in the following tables:

x :	0	1	2	3	4
$f(x)$:	1	3	9	–	81

Solution. Since we are given four values of y, so we take $y=f(x)$ to be a polynomial of degree 3 in x so that

$$\Delta^4 f(x)=0$$

i.e., $(E-1)^4 f(x)=0$

i.e., $E^4 f(x)-4E^3 f(x)+6E^2 f(x)-4Ef(x)+f(x)=0$

$$f(x+4)-4f(x+3)+6f(x+2)-4f(x+1)+f(x)=0$$

$$\left[\because E^n f(x)=f(x+nh), h=1\right]$$

Putting $x=0$, we get

$$f(4)-4f(3)+6f(2)-4f(1)+f(0)=0$$

or, $81-4f(3)+6\times9-4\times3+1=0$

$\therefore\ f(3)=31$

2.10. Divided differences.

Let $y=f(x)$ be a real valued function of x defind in a finite interval $[a,b]$ and let $y_i=f(x_i), (i=0,1,2,\ldots n)$ be the functional values corresponding to the distinct nodes $x_i (i=0,1,2,\ldots n)$ not necessarily equispaced. We define divided differences for nodes $x_i (i=0,1,2,\ldots n)$ of $f(x)$ as follows :

The first order divided difference of $f(x)$ for nodes x_0, x_1 is denoted by $f[x_0,x_1]$ and defined by

$$f[x_0,x_1]=\frac{f(x_0)-f(x_1)}{x_0-x_1}$$

$$=\frac{y_0-y_1}{x_0-x_1}=\frac{y_1-y_0}{x_1-x_0}$$

$$=f[x_1,x_0]$$

Similarly, the first order divided difference for nodes x_1, x_2 is

$$f[x_1,x_2]=\frac{f(x_1)-f(x_2)}{x_1-x_2}=f[x_2,x_1]$$

In general, we have

$$f[x_i,x_{i+1}]=\frac{f(x_i)-f(x_{i+1})}{x_i-x_{i+1}}$$

$$=\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}$$

$$=f[x_{i+1},x_i], i=1,2,\ldots n \qquad \ldots (18)$$

The second order divided difference of $f(x)$ for x_0, x_1, x_2 is denoted by $f[x_0, x_1, x_2]$ and defined by

$$f[x_0,x_1,x_2]=\frac{f[x_0,x_1]-f[x_1,x_2]}{x_0-x_2}$$

$$=f[x_2,x_1,x_0] \qquad \cdots \quad (19)$$

In general, the n^{th} order divided difference of $f(x)$ for $x_0, x_1, x_2, \cdots x_n$ is defined by

$$f[x_0,x_1,\cdots,x_n]=\frac{f[x_0,x_1,\ldots\ldots,x_{n-1}]-f[x_1,x_2,\ldots.,x_n]}{x_0-x_n}$$

$$=f[x_n,\ldots\ldots,x_1,x_0] \qquad \cdots \quad (20)$$

We can calculate the above different order divided differences very easily with the help of the following table, called *divided difference table.*

Table 3 : Divided difference table

x	$f(x)$	1st order	2nd order	3rd order	4th order
x_0	$f(x_0)$				
		$f[x_0,x_1]$			
x_1	$f(x_1)$		$f[x_0,x_1,x_2]$		
		$f[x_1,x_2]$		$f[x_0,x_1,x_2,x_3]$	
x_2	$f(x_2)$		$f[x_1,x_2,x_3]$		$f[x_0,x_1,x_2,x_3,x_4]$
		$f[x_2,x_3]$		$f[x_1,x_2,x_3,x_4]$	
x_3	$f(x_3)$		$f[x_2,x_3,x_4]$		
		$f[x_3,x_4]$			
x_4	$f(x_4)$				

Some properties of divided differences

(a) Divided differences are symmetric with respect to their arguments

Proof : This property is evident from (18), (19) and (20). Hence the arguments in a divided difference can be written in an arbitrary order. Thus we can write

$f[x_0,x_1,x_2]=f[x_2,x_0,x_1]=f[x_0,x_2,x_1]$ etc.

(b) Divided difference of a constant is zero.

Proof. Let $f(x)=$ constant $=c$, say

Then $f[x_0,x_1]=\dfrac{f(x_0)-f(x_1)}{x_0-x_1}=\dfrac{c-c}{x_0-x_1}=0$

(c) Divided difference of $k\,f(x)$ where k is constant, is k times the divided difference of $f(x)$.

Proof. Let $g(x)=kf(x)$

Then $g[x_0,x_1]=\dfrac{g(x_0)-g(x_1)}{x_0-x_1}=\dfrac{kf(x_0)-kf(x_1)}{x_0-x_1}$

$$=k\frac{f(x_0)-f(x_1)}{x_0-x_1}$$

$$=kf[x_0,x_1].$$

(d) Divided difference of $f(x)\pm g(x)$ is the sum (or difference) of the corresponding divided difference of $f(x)$ and $g(x)$.

Proof. Let $F(x)=f(x)\pm g(x)$

Then $F[x_0,x_1]=\dfrac{F(x_0)-F(x_1)}{x_0-x_1}$

$$=\frac{\{f(x_0)\pm g(x_0)\}-\{f(x_1)\pm g(x_1)\}}{x_0-x_1}$$

$$=\frac{f(x_0)-f(x_1)}{x_0-x_1}\pm\frac{g(x_0)-g(x_1)}{x_0-x_1}$$

$$=f[x_0,x_1]\pm g[x_0,x_1]$$

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(e) k^{th} order divided difference of x^n is

(i) a polynomial of degree $n-k$ if $k<n$

(ii) a constant if $k=n$

(iii) zero if $k>n$.

Proof. Left as an exercise.

Example 9. If $f(x)=\frac{1}{x}$ whose arguments are x_0, x_1, x_2, prove that

$$f[x_0, x_1, x_2]=\frac{1}{x_0\, x_1\, x_2}$$

Solution. We have

$$f[x_0, x_1]=\frac{f(x_0)-f(x_1)}{x_0-x_1}=\frac{\frac{1}{x_0}-\frac{1}{x_1}}{x_0-x_1}=-\frac{1}{x_0x_1}$$

Similarly

$$f[x_1, x_2]=-\frac{1}{x_1x_2}$$

$$\therefore\ f[x_0, x_1, x_2]=\frac{f[x_0,x_1]-f[x_1,x_2]}{x_0-x_2}$$

$$=\frac{-\frac{1}{x_0x_1}+\frac{1}{x_1x_2}}{x_0-x_2}$$

$$=\frac{1}{x_0x_1x_2}.$$

Example 10. Construct the divided difference table for the following data :

x :	2	4	5	7	8
y :	3	43	138	778	1515

Solution : The divided difference table is

x	y	1st order	2nd order	3rd order	4th order
2	3				
		20			
4	43		25		
		95		10	
5	138		75		1
		320		16	
7	778		139		
		737			
8	1515				

2.11. Propagation of errors in a difference table

The given initial data are affected with round off error which lie between the limits $\pm\frac{1}{2}$ in the last significant figure. We now proceed to find how these initial round off errors affect the successive differeces in a difference table.

Let $y=f(x)$ be a real-valued function of x in $[a, b]$ and $y_i=f(x_i)$ be the exact value of $f(x)$ corresponding to the node x_i. If ε_i be the corresponding round off error, then the entered value of y_i is $y_i-\varepsilon_i$ and the difference table proceeds as follows :

Table 4 : Propagation of error in a difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	$y_0-\varepsilon_0$			
		$\Delta y_0-\varepsilon_1+\varepsilon_0$		
x_1	$y_1-\varepsilon_1$		$\Delta^2 y_0-\varepsilon_2+2\varepsilon_1-\varepsilon_0$	
		$\Delta y_1-\varepsilon_2+\varepsilon_1$		$\Delta^3 y_0-\varepsilon_3+3\varepsilon_2-3\varepsilon_1+\varepsilon_0$
x_2	$y_2-\varepsilon_2$		$\Delta^2 y_1-\varepsilon_3+2\varepsilon_2-\varepsilon_1$	
		$\Delta y_2-\varepsilon_3+\varepsilon_2$		
x_3	$y_3-\varepsilon_3$			

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Now, $\Delta \varepsilon_i = \varepsilon_{i+1} - \varepsilon_i$

$$\Delta^2 \varepsilon_i = \varepsilon_{i+2} - 2\varepsilon_{i+1} + \varepsilon_i$$

and in general,

$$\Delta^r \varepsilon_i = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \varepsilon_{i+r-k}, \quad i = 0, 1, 2, \ldots, n$$

so that $\Delta^r(y_i - \varepsilon_i) = \Delta^r y_i - \Delta^r \varepsilon_i = \Delta^r y_i - \sum_{k=0}^{r} (-1)^k \binom{r}{k} \varepsilon_{i+r-k}$

In the last significant figures, we have

$$\left| \sum_{k=0}^{r} (-1)^k \binom{r}{k} \varepsilon_{i+r-k} \right| \le \sum_{k=0}^{r} \binom{r}{k} |\varepsilon_{i+r-k}|$$

$$\le \frac{1}{2} \sum_{k=0}^{r} \binom{r}{k} = 2^{r-1}$$

Thus the error in the computed value of $\Delta^r y_i$ varies between the limits $\pm 2^{r-1}$ in the last significant figures. In a difference table, this accumulation of errors in successive higher order differences is known as propagation of errors in a difference table :

Now if the step length h is small, then it is seen that the successive differences gradually decrease in significant figures so that the errors gradually increases in the successive differences. This is due to the propagation of errors. Proceeding through the table a stage is reached at which the computed differences are of the same order as their errors and so these differences are unrealistic. This stage is called *noise level*. For this reason, the computation of differences is to be stopped just before one gets differences with only one significant figure.

In the above, we have cosidered only round-off errors. But, sometimes we face another type of error, known as *accidental error*. If there is an accidental error ε, say, presumbly large, in one of the functional values, then this error is propagated through the difference table, as shown in the following table :

Table -5

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0 + \varepsilon$	
x_2	y_2		$\Delta^2 y_1 + \varepsilon$		$\Delta^4 y_0 - 4\varepsilon$
		$\Delta y_2 + \varepsilon$		$\Delta^3 y_1 - 3\varepsilon$	
x_3	$y_3 + \varepsilon$		$\Delta^2 y_2 - 2\varepsilon$		$\Delta^4 y_1 + 6\varepsilon$
		$\Delta y_3 - \varepsilon$		$\Delta^3 y_2 + 3\varepsilon$	
x_4	y_4		$\Delta^2 y_3 + \varepsilon$		$\Delta^2 y_2 - 4\varepsilon$
		Δy_4		$\Delta^3 y_3 - \varepsilon$	
x_5	y_5		$\Delta^2 y_4$		
		Δy_5			
x_6	y_6				

From the above table it is clear that the errors in successive differences grow in size and hence the computed differences behave very irregularly after some stage. The presence of accidental error in the difference table may be diagonosed from this irregular behaviour. Here the propogation of errors are found to be confined within a cone, called diverging cone, from erroneous entry. Also it should be noted that the coefficient of ε in a column, say $\Delta^n y$ are binomial coefficients in the expansion of $(1-x)^n$, e.g. the coefficient of ε in $\Delta^3 y$ are ${}^3C_0, -{}^3C_1, {}^3C_2, -{}^3C_3$. i.e., $1, -3, 3, -1$

So the algebraic sum of the errors in any difference column is zero and the maximum error in the differences is the same horizontal line which corresponds to the erroneous tabular value. All these observation help us to detect an error and make the necessary correction.

Example 11. Use the difference table to locate and correct the error in the following tabulated values :

x	:	0	1	2	3	4
y	:	5	8	10	20	29

Solution : The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	5			
		3		
1	8		−1	
		2		9
2	10		8	
		10		−9
3	20		−1	
		9		
4	29			

From the above table, it is clear that the pattern of $\Delta^2 y$ are quite irregular and irregularity starts arround the horizontal line corresponding to the value of $y = 10$. Also it is observed that the third order differences follow the binomial coefficient pattern and the sum of the 3rd difference is zero. Thus from the third difference pattern we get

$$-3\varepsilon = 9$$

$$\therefore\ \varepsilon = -3$$

So there exist an error -3 in the entry for $x = 2$ and the corresponding true value of y is

$$10 - (-3) = 13.$$

ILLUSTRATIVE EXAMPLES

Ex. 1. Show that $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$

where Δ is the forward difference operator

[*W.B.U.T., CS-312, 2008,2009*]

Solution. We have $\Delta \log f(x) = \log f(x+h) - \log f(x)$, h being the space length

$$= \log \frac{f(x+h)}{f(x)}$$

$$= \log\left\{\frac{\Delta f(x) + f(x)}{f(x)}\right\} \qquad [\because \Delta f(x) = f(x+h) - f(x)]$$

$$= \log\left\{1 + \frac{\Delta f(x)}{f(x)}\right\}$$

Ex. 2. Evaluate

(i) $\Delta\left\{\frac{2^x}{(x+1)!}\right\}$, taking $h = 1$

(ii) $\Delta^2(2x+1)$, taking $h = 1$

(iii) $\Delta^2\left(\frac{1}{x^2+5x+6}\right)$, taking $h = 1$

Solution. (i) $\Delta\left\{\frac{2^x}{(x+1)!}\right\}$

$$= \frac{2^{x+1}}{(x+2)!} - \frac{2^x}{(x+1)!}$$

$$= \frac{2^x}{(x+1)!}\left\{\frac{2}{x+2} - 1\right\}$$

$$= \frac{x\,2^x}{(x+2)!}$$

(ii) $\Delta^2(2x+1)$

$= \{2(x+1)+1\} - (2x+1)$, taking $h = 1$

$= 2$

$\therefore \Delta^2(2x+1) = \Delta(2) = 0$

(iii) $\Delta\left(\dfrac{1}{x^2+5x+6}\right)$

$$= \Delta\left(\frac{1}{x+2} - \frac{1}{x+3}\right)$$

$$= \left(\frac{1}{x+3} - \frac{1}{x+4}\right) - \left(\frac{1}{x+2} - \frac{1}{x+3}\right)$$

$$= \frac{2}{x+3} - \frac{1}{x+4} - \frac{1}{x+2}$$

$$\therefore \Delta^2\left(\frac{1}{x^2+5x+6}\right)$$

$$= \Delta\left(\frac{2}{x+3} - \frac{1}{x+4} - \frac{1}{x+2}\right)$$

$$= \left(\frac{2}{x+4} - \frac{1}{x+5} - \frac{1}{x+3}\right) - \left(\frac{2}{x+3} - \frac{1}{x+4} - \frac{1}{x+2}\right)$$

$$= \frac{3}{x+4} - \frac{1}{x+5} - \frac{3}{x+3} + \frac{1}{x+2}$$

$$= -\frac{3}{(x+3)(x+4)} + \frac{3}{(x+2)(x+5)}$$

$$= \frac{6}{(x+2)(x+3)(x+4)(x+5)}$$

Ex. 3. Show that $\Delta^n e^x = (e-1)^n e^x$

Solution. We have

$\Delta e^x = e^{x+1} - e^x$, taking $h = 1$

$= (e-1)e^x$

$\therefore \Delta^2 e^x = (e-1)\Delta e^x = (e-1)(e-1).e^x = (e-1)^2 e^x$

So the result holds for $n = 1, 2$

Let the given result is true for $n = m$, i.e.,

$\Delta^m e^n = (e-1)^m e^x$

$\therefore \Delta(\Delta^m e^n) = (e-1)^m \Delta e^x$

or, $\Delta^{m+1} e^x = (e-1)^m.(e-1)e^x$

$= (e-1)^{m+1} e^x.$

Hence the result is also true for $n = m+1$, provided it holds for $n = m$. But the result is valid for $n = 1, 2$. Therefore, by the method of induction, the result $\Delta^n e^x = (e-1)^n e^x$ is valid for all values of n.

Ex. 4. Prove that $\left(\dfrac{\Delta^2}{E} e^x\right)\dfrac{Ee^x}{\Delta^2 e^x} = e^x$

Solution. We have

$$\left(\frac{\Delta^2}{E} e^x\right)\frac{Ee^x}{\Delta^2 e^x}$$

$$= (\Delta^2 E^{-1} e^x)\frac{e^{x+h}}{\Delta^2 e^x}$$

$$= (\Delta^2 e^{x-h})\frac{e^{x+h}}{\Delta^2 e^x}$$

$$= e^{-h}\Delta^2 e^x \frac{e^x.e^h}{\Delta^2 e^x}$$

$$= e^{-h}.e^x.e^h$$

$= e^x$, h being the space length.

Ex. 5. Show that $\Delta\left(\dfrac{f_m}{g_m}\right) = \dfrac{\Delta f_m.g_m - f_m \Delta g_m}{g_m g_{m+1}}$

Solution. $\Delta\left(\dfrac{f_m}{g_m}\right) = \dfrac{f_{m+1}}{g_{m+1}} - \dfrac{f_m}{g_m}$

$$= \frac{f_{m+1}g_m - f_m g_{m+1}}{g_m g_{m+1}}$$

$$= \frac{(f_{m+1} - f_m)g_m - f_m(g_{m+1} - g_m)}{g_m g_{m+1}}$$

$$= \frac{g_m \Delta f_m - f_m \Delta g_m}{g_m g_{m+1}}$$

Ex. 6. If $f(x) = u(x)\,v(x)$, show that

$$f(x_0, x_1) = u(x_0)v(x_0, x_1) + u(x_0, x_1)v(x_1).$$

Solution. Here $f(x) = u(x)v(x)$

$$\therefore f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= \frac{u(x_0)v(x_0) - u(x_1)v(x_1)}{x_0 - x_1}$$

$$= \frac{u(x_0)\{v(x_0) - v(x_1)\} + v(x_1)\{u(x_0) - u(x_1)\}}{x_0 - x_1}$$

$$= u(x_0)\frac{v(x_0) - v(x_1)}{x_0 - x_1} + v(x_1)\frac{u(x_0) - u(x_1)}{x_0 - x_1}$$

$$= u(x_0)v(x_0, x_1) + v(x_1)u(x_0, x_1).$$

Ex. 7. Prove that $\delta \equiv \Delta E^{-\frac{1}{2}}$ and hence prove that

$$E \equiv \left(\frac{\Delta}{\delta}\right)^2$$

Solution. We have

$$\Delta E^{-\frac{1}{2}} f(x)$$

$$= \Delta f\left(x - \frac{h}{2}\right), \text{ h being the spacing}$$

$$= f\left(x - \frac{h}{2} + h\right) - f\left(x - \frac{h}{2}\right)$$

$$= f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$$= \delta f(x)$$

$$\therefore \Delta E^{-\frac{1}{2}} = \delta$$

$$\therefore E^{-\frac{1}{2}} \equiv \frac{\delta}{\Delta}$$

$$\therefore E \equiv \left(\frac{\Delta}{\delta}\right)^2$$

Ex. 8. Evaluate $\left(\frac{\Delta^2}{E}\right)x^4$, spacing being one

Solution. $\left(\frac{\Delta^2}{E}\right)x^4$

$$= \left\{\frac{(E-1)^2}{E}\right\}x^4$$

$$= \left(\frac{E^2 - 2E + 1}{E}\right)x^4$$

$$= \left(E - 2 + E^{-1}\right)x^4$$

$$= Ex^4 - 2x^4 + E^{-1}x^4$$

$$= (x+1)^4 - 2x^4 + (x-1)^4$$

$$\left[\because h = 1, Ef(x) = f(x+h), E^{-1}f(x) = f(x-h)\right]$$

$$= 12x^2 + 2$$

Ex. 9. Taking $h = 2$ what will be the value of

$$\Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right]$$

Solution. Since $(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)$ is a polynomial of degree 10 with leading term abcd x^{10}, so

$$\Delta^{10}\left\{(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right\}$$

$$= \Delta^{10}\{abcdx^{10} + \text{other terms of degree} \le 9\}$$

$$= abcd\,\Delta^{10}(x^{10}) + 0 \quad = abcd\,h^{10} \times 10!$$

$$= abcd \times 2^{10} \times 10!$$

Ex.10. If f_i is the value of $f(x)$ at $x = x_i$ where $x_i = x_0 + ih$, $(i = 1, 2, 3, ...)$ and $h > 0$, prove that

$$f_i = E^i f_0 = \sum_{j=0}^{i} \binom{i}{j} \Delta^j f_0 \quad (j < i)$$

Solution. We have

$$f_i = f(x_i)$$
$$= f(x_0 + ih)$$
$$= E^i f(x_0) = E^i f_0$$
$$= (1+\Delta)^i f_0$$
$$= \left\{1 + \binom{i}{1}\Delta + \binom{i}{2}\Delta^2 + ... + \binom{i}{i}\Delta^i\right\} f_0$$
$$= 1 + \binom{i}{1}\Delta f_0 + \binom{i}{2}\Delta^2 f_0 + ... + \binom{i}{i}\Delta^i f_0$$
$$= \sum_{j=0}^{i} \binom{i}{j} \Delta^j f_0$$

Ex.11. Find y_7, given $y_0 = 0, y_1 = 7, y_2 = 26, y_3 = 63, y_4 = 124$

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		7			
1	7		12		
		19		6	
2	26		18		0
		37		6	
3	63		24		
		61			
4	124				

$f(x+nh)$

Here $x_0 = 0, h = 1, n = 7$

$$\therefore y_7 = f(0 + 7.1) = y_0 + \binom{7}{1}\Delta y_0 + \binom{7}{2}\Delta^2 y_0 + \binom{7}{3}\Delta^3 y_0 + ...$$

$$= 0 + 7\times7 + 21\times12 + 35\times6 + 35\times0 = 511$$

Example 12 By constructing a difference table, find the sixth term of the series 8, 12, 19, 29, 42, ...

[W.B.U.T., CS-312, 2004]

Solution : Let the sixth term of the series be p. Then we construct the following difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	8				
		4			
2	12		3		
		7		0	
3	19		3		0
		10		0	
4	29		3		$p-58$
		13		$p-58$	
5	42		$p-55$		
		$p-42$			
6	p				

Since 5 entries are given, $\Delta^4 y$ must be constant

$\therefore p - 58 = 0$

$\therefore p = 58$

$\therefore$ So the sixth term of the series is 58

Ex. 13. Show that $\Delta^{n+1} f(x_0) \approx h^{n+1} f^{n+1}(x_0)$ if h is very small

Solution : We have

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$$

$$\text{or, } \lim_{h\to 0} \frac{\Delta f(x_0)}{h} = f'(x_0)$$

$$\Rightarrow \Delta f(x_0) \approx h f'(x_0)$$

$$\text{Again, } \lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0)}{h} = f''(x_0)$$

or, $\lim_{h\to 0}\frac{\Delta f'(x_0)}{h}=f''(x_0)$

$\Rightarrow \Delta f'(x_0)\simeq hf''(x_0)\Rightarrow \Delta(hf'(x_0))\simeq h^2f''(x_0)$

$\Rightarrow \Delta(\Delta f(x_0))\simeq h^2f''(x_0)\Rightarrow \Delta^2 f(x_0)\simeq h^2f''(x_0)$

Repeating the process, we have by induction

$$\Delta^{n+1}f(x_0)\simeq h^{n+1}f^{n+1}(x_0).$$

Ex. 14. Given $u_0+u_8=1.9243$, $u_1+u_7=1.9590$,

$u_2+u_6=1.9823$, $u_3+u_5=1.9956$. Find u_4

Solution : As we are given 8 values of $u(x)$, so $u(x)$ is a polynomial of degree 7, so that

$\Delta^8 u(x)=0$

and hence in particular,

$\Delta^8 u_0=0$

i.e., $(E-1)^8u_0=0$

i.e., $(E^8-8E^7+28E^6-56E^5+70E^4-56E^3+28E^2-8E+1)u_0=0$

i.e., $u_8-8u_7+28u_6-56u_5+70u_4-56u_3+28u_2-8u_1+u_0=0$

i.e., $(u_0+u_8)-8(u_1+u_7)+28(u_2+u_6)-56(u_3+u_5)+70u_4=0$

i.e., $1.9243-8\times 1.9590+28\times 1.9823-56\times 1.9956+70u_4=0$

i.e. $u_4=\frac{69\cdot 9969}{70}$

$\therefore u_4=0.999955$

Ex. 15. Prove that $f(4)=f(3)+\Delta f(2)+\Delta^2 f(1)+\Delta^3 f(1)$

[W.B.U.T., C.S.-312, 2008]

Solution. We have $\Delta f(3)=f(4)-f(3)$

$\therefore f(4)=f(3)+\Delta f(3)$

$=f(3)+\Delta\{f(2)+\Delta f(2)\}$ $\quad [\because \Delta f(2)=f(3)-f(2)]$

$=f(3)+\Delta f(2)+\Delta^2 f(2)$

$=f(3)+\Delta f(2)+\Delta^2\{f(1)+\Delta f(1)\}$ $\quad [\because \Delta f(1)=f(2)-f(1)]$

$=f(3)+\Delta f(2)+\Delta^2 f(1)+\Delta^3 f(1)$

Thus $f(4)=f(3)+\Delta f(2)+\Delta^2 f(1)+\Delta^3 f(1)$

Ex. 16. Show that $\nabla y_{n+1}=h\left[1+\frac{1}{2}\nabla+\frac{5}{12}\nabla^2+\ldots\right]Dy_n$ where D is the differential operator.

Solution : We have

$\nabla y_{n+1}=y_{n+1}-y_n=(E-1)y_n=(e^{hD}-1)y_n$ $\quad [\because E\equiv e^{hD}]$

$=\left(hD+\frac{h^2D^2}{2!}+\frac{h^3D^3}{3!}+\ldots\right)y_n$

$=h\left(1+\frac{1}{2}hD+\frac{1}{6}h^2D^2+\ldots\right)Dy_n$

Also, $E\equiv e^{hD}$ gives

$e^{-hD}\equiv E^{-1}=1-\nabla$ so that

$-hD=\log_e(1-\nabla)$

$=-\nabla-\frac{1}{2}\nabla^2-\frac{1}{3}\nabla^3\ldots$

$\therefore hD=\nabla+\frac{1}{2}\nabla^2+\frac{1}{3}\nabla^3+\ldots$

Hence $\nabla y_{n+1}=h\left[1+\frac{1}{2}\left(\nabla+\frac{1}{2}\nabla^2+\frac{1}{3}\nabla^3+\ldots\right)\right.$

$\left.+\frac{1}{6}\left(\nabla+\frac{1}{2}\nabla^2+\frac{1}{3}\nabla^3+\ldots\right)^2+\ldots\right]Dy_n$

$=h\left[1+\frac{1}{2}\nabla+\frac{5}{12}\nabla^2+\ldots\right]Dy_n$

Example 17. Show that $\Delta^r y_k=\nabla^r y_{k+r}$

[W.B.U.T., C.S-312, 2006]

Solution. $\nabla^r y_{k+r}$ $\quad [\because \nabla \equiv 1-E^{-1}]$

$$=\left(1-E^{-1}\right)^r y_{k+r}$$

$$=\left(\frac{E-1}{E}\right)^r y_{k+r}$$

$$=(E-1)^r\left(E^{-r}y_{k+r}\right)$$

$$=(E-1)^r y_{k+r-r}$$

$$=(E-1)^r y_k$$

$$=\Delta^r y_k \qquad [\because \Delta \equiv E-1]$$

Ex. 18. Show that if Δ operates on n, then

$$\Delta\binom{n}{x+1}=\binom{n}{x} \text{ and hence } \sum_{n=1}^{N}\binom{n}{x}=\binom{n+1}{x+1}-\binom{1}{x+1}$$

Solution. $\Delta\binom{n}{x+1}$

$$=\binom{n+1}{x+1}-\binom{n}{x+1}$$

$$=\frac{(n+1)!}{(n-x)!(x+1)!}-\frac{n!}{(n-x-1)!(x+1)!}$$

$$=\frac{n!}{(n-x-1)!(x+1)!}\left(\frac{n+1}{n-x}-1\right)$$

$$=\frac{n!}{(n-x-1)!(x+1)!}\,\frac{x+1}{n-x}$$

$$=\frac{n!}{(n-x)!x!}$$

$$=\binom{n}{x}$$

$$\therefore \sum_{n=1}^{N}\binom{n}{x}=\sum_{n=1}^{N}\Delta\binom{n}{x+1}$$

$$=\left\{\binom{2}{x+1}-\binom{1}{x+1}\right\}+\left\{\binom{3}{x+1}-\binom{2}{x+1}\right\}+\ldots+\left\{\binom{N+1}{x+1}-\binom{N}{x+1}\right\}$$

$$=\binom{N+1}{x+1}-\binom{1}{x+1}$$

Ex. 19. Show that $\Delta^m\left(\frac{1}{x}\right)=\frac{(-1)^m m!h^m}{x(x+h)(x+2h)\ldots(x+mh)}$

[W.B.U.T., C.S-312, 2007]

Solution : We have $\Delta\left(\frac{1}{x}\right)=\frac{1}{x+h}-\frac{1}{x}=\frac{-h}{x(x+h)}=\frac{(-1)1!h}{x(x+h)}$

$$\Delta^2\left(\frac{1}{x}\right)=-h\left[\frac{1}{(x+h)(x+2h)}-\frac{1}{x(x+h)}\right]$$

$$=\frac{2h^2}{x(x+h)(x+2h)}$$

$$=\frac{(-1)^2 2!h^2}{x(x+h)(x+2h)}.$$

Thus the result is true for $m=1,2$. We suppose that the result is valid for $m=r$

$$\therefore \Delta^r\left(\frac{1}{x}\right)=\frac{(-1)^r r!h^r}{x(x+h)(x+2h)\ldots(x+rh)}$$

Then $\Delta^{r+1}\left(\frac{1}{x}\right)$

$$=(-1)^r r!h^r\left[\frac{1}{(x+h)(x+2h)\ldots.(x+\overline{r+1}h)}-\frac{1}{x(x+h)\ldots(x+rh)}\right]$$

$$=\frac{(-1)^r r!h^r}{(x+h)(x+2h)\ldots(x+rh)}\left[\frac{1}{x+\overline{r+1}h}-\frac{1}{x}\right]$$

$$=\frac{(-1)^{r+1}(r+1)!h^{r+1}}{x(x+h)(x+2h)\ldots(x+\overline{r+1}h)}$$

Hence the given result holds for $m=r+1$, provided it holds for $m=r$. But the result is ture for $m=1,2$. Hence by induction the result holds for any value of m.

Ex. 20. Show that

(a) $u_{n+x}=u_n+\binom{x}{1}\Delta u_{n-1}+\binom{x+1}{2}\Delta^2 u_{n-2}+\binom{x+2}{3}\Delta^3 u_{n-3}+\ldots$

(b) $\Delta^n u_{x-n}=u_x-\binom{n}{1}u_{x-1}+\binom{n}{2}u_{x-2}-\binom{n}{3}u_{x-3}+\ldots$

Solution. (a) u_{n+x}

$$=E^x u_n=\left(\frac{1}{E}\right)^{-x}u_n$$

$$=\left(\frac{E-\Delta}{E}\right)^{-x}u_n \quad [\because 1+\Delta=E]$$

$$=\left(1-\Delta E^{-1}\right)^{-x}u_n$$

$$=\left[1+x\,\Delta E^{-1}+\frac{x(x+1)}{2!}\Delta^2E^{-2}+\frac{x(x+1)(x+2)}{3!}\Delta^3E^{-3}+\ldots\right]u_n$$

$$=u_n+\binom{x}{1}\Delta E^{-1}u_n+\binom{x+1}{2}\Delta^2E^{-2}u_n+\binom{x+2}{3}\Delta^3E^{-3}u_n+\ldots$$

$$=u_n+\binom{x}{1}\Delta u_{n-1}+\binom{x+1}{2}\Delta^2u_{n-2}+\binom{x+2}{3}\Delta^3u_{n-3}+\ldots$$

(b) $\Delta^n u_{x-n}$

$$=\Delta^n E^{-n}u_x$$

$$=\left(\Delta E^{-1}\right)^n u_x$$

$$=\left\{(E-1)E^{-1}\right\}^n u_x \qquad [\because E\equiv 1+\Delta]$$

$$=\left(1-E^{-1}\right)^n u_x$$

$$=\left[1-\binom{n}{1}E^{-1}+\binom{n}{2}E^{-2}-\binom{n}{3}E^{-3}+\ldots\right]u_x$$

$$=u_x-\binom{n}{1}E^{-1}u_x+\binom{n}{2}E^{-2}u_x-\binom{n}{3}E^{-3}u_x+\ldots$$

$$=u_x-\binom{n}{1}u_{x-1}+\binom{n}{2}u_{x-2}-\binom{n}{3}u_{x-3}+\ldots$$

Ex. 21 Prove that

$$U_x=U_{x-1}+\Delta U_{x-2}+\Delta^2U_{x-3}+\ldots\Delta^{n-1}U_{x-n}+\Delta^nU_{x-n}$$

[W.B.U.T., C.S-312, 2006]

Solution : $U_x-\Delta^nU_{x-n}$

$$=U_x-\Delta^nE^{-n}U_x$$

$$=\left(1-\Delta^nE^{-n}\right)U_x$$

$$=\frac{E^n-\Delta^n}{E^n}U_x$$

$$=\frac{1}{E^n}\left(\frac{E^n-\Delta^n}{E-\Delta}\right)U_n \qquad [\because E\equiv 1+\Delta\Rightarrow E-\Delta\equiv 1]$$

$$=E^{-n}\left[E^{n-1}+\Delta E^{n-2}+\Delta^2E^{n-3}+\ldots+\Delta^{n-1}\right]U_x$$

$$=\left[E^{-1}+\Delta E^{-2}+\Delta^2E^{-3}+\ldots+\Delta^{n-1}E^{-n}\right]U_x$$

$$=U_{x-1}+\Delta U_{x-2}+\Delta^2U_{x-3}+\ldots+\Delta^{n-1}U_{x-n}$$

$$\therefore\ U_x=U_{x-1}+\Delta U_{x-2}+\Delta^2U_{x-3}+\ldots+\Delta^{n-1}U_{x-n}+\Delta^nU_{x-n}$$

Ex. 22. Prove that $f[x_0, x_1, x_2] = 1$ if $f(x) = x^2$ where x_0, x_1, x_2 are distinct.

Solution. We have $f(x) = x^2$

$$\therefore f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= \frac{x_0^2 - x_1^2}{x_0 - x_1}$$

$$= x_0 + x_1$$

Similarly $f[x_1, x_2] = x_1 + x_2$

$$\therefore f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2}$$

$$= \frac{(x_0 + x_1) - (x_1 + x_2)}{x_0 - x_2}$$

$$= \frac{x_0 - x_2}{x_0 - x_2} = 1$$

Ex. 23. If $f(x) = \frac{1}{x^2}$ whose arguments are a, b, c, d in this order, prove that

$$f[a, b, c, d] = -\frac{abc + bcd + acd + abd}{a^2b^2c^2d^2}$$

Solution. Here $f(x) = \frac{1}{x^2}$

$$\therefore f[a, b] = \frac{f(a) - f(b)}{a - b} = \frac{\frac{1}{a^2} - \frac{1}{b^2}}{a - b} = -\frac{a + b}{a^2b^2}$$

Similarly

$$f[b, c] = -\frac{b + c}{b^2c^2}$$

$$\therefore f[a, b, c] = \frac{f[a, b] - f[b, c]}{a - c}$$

$$= \frac{-\frac{a+b}{a^2b^2} + \frac{b+c}{b^2c^2}}{a - c}$$

$$= \frac{ab + bc + ca}{a^2b^2c^2}$$

Similarly $f[b, c, d] = \frac{bc + cd + bd}{b^2c^2d^2}$

Thus $f[a, b, c, d]$

$$= \frac{f[a, b, c] - f[b, c, d]}{a - d}$$

$$= \frac{\frac{ab + bc + ca}{a^2b^2c^2} - \frac{bc + cd + bd}{b^2c^2d^2}}{a - d}$$

$$= -\frac{abc + bcd + acd + abd}{a^2b^2c^2d^2}$$

Ex. 24 Find the missing value in the following table :

x	:	2	4	6	8	10
y	:	5.6	8.6	13.9	–	35.6

Solution. Since we are given four values of y, so we take $y = f(x)$ to be a polynomial of degree 3 in x so that

$\Delta^4 f(x) = 0$

i.e., $(E - 1)^4 f(x) = 0$

i.e., $E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4Ef(x) + f(x) = 0$

i.e., $f(x + 8) - 4f(x + 6) + 6f(x + 4) - 4f(x + 2) + f(x) = 0$

$$[\because E^n f(x) = f(x + nh), h = 2]$$

Putting $x = 2$, we get

$f(10) - 4f(8) + 6f(6) - f(4) + f(2) = 0$

or, $35.6 - 4f(8) + 6 \times 13.9 - 4 \times 8.6 + 5.6 = 0$

$\therefore f(8) = 22.55$

Ex. 25. Find the missing term in the following table:

x	:	0	1	2	3	4	5
y	:	0	–	8	15	–	35

[W.B.U.T., C.S-312, 2007]

Solution : Since we are given four values, therefore, we take $f(x)$ to be a polynomial of degree 3 in x so that

$\Delta^3 f(x) = 0$ for all values of x

$\therefore (E-1)^3 f(x) = 0$

i.e., $E^3 f(x) - 3E^2 f(x) + 3Ef(x) - f(x) = 0$

i.e., $f(x+3) - 3f(x+2) + 3f(x+1) - f(x) = 0$ ... (1)

Putting $x = 0$, we get

$f(3) - 3f(2) + 3f(1) - f(0) = 0$

or, $15 - 3\times 8 + 3f(1) - 0 = 0$

or, $3f(1) = 9$

$\therefore f(1) = 3$

Again putting $x = 1$, we get

$f(4) - 3f(3) + 3f(2) - f(1) = 0$

or, $f(4) - 3\times 15 + 3\times 8 - 3 = 0$

$\therefore f(4) = 24$

Example 26. If the 3rd order differences of $f(x)$ be constant and $f(-1) = -1, f(0) = 0, f(1) = 1, f(2) = 8$ and $f(3) = 27$ find $f(4)$ using difference table.

Solution. Let $f(4) = a$

We now construct the following difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
–1	–1			
		1		
0	0		0	
		1		6
1	1		6	
		7		6
2	8		12	
		19		$a-58$
3	27		$a-46$	
		$a-27$		
4	a			

Since 3rd order difference of $f(x)$ is constants, so we must have

$a - 58 = 6$

$\therefore a = 64$

$\therefore f(4) = 64$

Example 27. If $f(x)$ is a polynomial of degree 3 and $f(0) = -1$; $f(1) = 5$ $f(2) = 13$, $f(3) = 36$, $f(4) = 69$ where $f(2)$ is not correct, find the error in $f(2)$.

Solution. Let the correct value of $f(2)$ be $13+\varepsilon$

Then we construct the following difference table:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	–1			
		6		
1	5		$2+\varepsilon$	
		$8+\varepsilon$		$13-3\varepsilon$
2	$13+\varepsilon$		$15-2\varepsilon$	
		$23-\varepsilon$		$-5+3\varepsilon$
3	36		$10+\varepsilon$	
		33		
4	69			

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Since $f(x)$ is a polynomial of degree 3, so $\Delta^3 f(x)$ must be constant

$\therefore\ 13-3\varepsilon=-5+3\varepsilon$

i.e. $6\varepsilon=18$

$\therefore\ \varepsilon=3$

So the error in $f(2)$ is 3 and correct value of $f(2)$ is $13+3=16$

Exercise

I. Short Answer Questions

1. Prove that

(i) $(1+\Delta)(1-\nabla)\equiv 1$

(ii) $\Delta+\nabla\equiv\frac{\Delta}{\nabla}-\frac{\nabla}{\Delta}$

(iii) $\delta\equiv E^{1/2}\nabla$

(iv) $2\mu\delta\equiv\Delta+\nabla$

(v) $1-e^{-hD}\equiv\nabla$

(vi) $\Delta.\nabla=\Delta-\nabla\equiv\delta^2$

2. Prove that

$$D\equiv\frac{1}{h}\log(1+\Delta)\equiv-\frac{1}{h}\log(1-\nabla)$$

3. Show that

$$D\equiv\frac{1}{h}\left[\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\ldots\right]$$

4. Taking $h=1$, find $(\Delta+\nabla)^2$ of the function x^2+x

5. Find $\Delta\left(\frac{x}{x^2-1}\right)$, taking $h=1$

6. Prove the following :

(i) $\left(\frac{\Delta^2}{E}\right)x^3=6xh^2$

(ii) $\Delta^2\cos 2x=-4\sin^2 h\cos 2(x+h)$ [W.B.U.T., CS-312, 2008]

(iii) $\Delta^2\left(x^2+2x+5\right)=2$

(iv) $\Delta(x+\cos x)=\pi-2\cos x$, taking $h=\pi$

(v) $\Delta\tan^{-1}\left(\frac{n-1}{h}\right)=\tan^{-1}\frac{1}{2n^2}$

7. Evaluate :

(i) $\Delta^2\left(ax^2+bx+c\right)$ [W.B.U.T., CS-312, 2007]

(ii) $\Delta^2\left(e^{3x+5}\right)$ (iii) $(\Delta-\nabla)x^2$

(iv) $\Delta\left(\frac{2^x}{x!}\right)$ (v) $\Delta\left(\frac{x}{x^2+7x+12}\right)$

8. Evaluate

(i) $\Delta^3\{(1-x)(1-2x)(1-3x)\}$

(ii) $\Delta^{10}\left\{(1-x)(1-x^2)(1-3x^3)(1-4x^4)\right\}$

9. Show that $\frac{\Delta^2 x^3}{Ex^3}=\frac{6}{(1+x)^2}$, taking $h=1$

10. Given $f(0)=580, f(1)=556, f(2)=520$ and $f(4)=385$, find $f(3)$.

11. If $y_1=1, y_2=3, y_3=7, y_4=13$ and $y_5=21$, find y_0

12. If $f(x)$ is a polynomial of degree 2 and $f(1)=3, f(2)=7, f(3)=13, f(4)=21$, find $f(5)$ using difference table

Answers

4. 8

7. (i) $2ah^2$ (ii) $\left(e^3-1\right)^2e^{3x+5}$ (iii) $2h^2$

(iv) $\frac{2^x(1-x)}{(x+1)!}$ (v) $\frac{4}{x+5}-\frac{7}{x+4}+\frac{3}{x+3}$

8. (i) 36 (ii) $24\times 10!$

11. 1 **10.** 465

II. LONG ANSWER QUESTIONS

1. Evaluate

(i) $\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right)$ [W.B.U.T.,CS-312, 2009]

(ii) $\Delta\left(\frac{x^2}{\sin^2 2x}\right)$, taking $h=1$

2. Show that

(i) $\Delta\left\{\frac{1}{f(x)}\right\} = -\frac{\Delta f(x)}{f(x)f(x+1)}$

(ii) $\Delta^n \alpha^{ax+b} = (\alpha^{ah}-1)\alpha^{ax+b}$

3. If D stands for the differential operator $\frac{d}{dx}$, prove that

(i) $D \equiv \frac{1}{h}\left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \ldots\right)$

(ii) $D \equiv \frac{1}{h}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\right)$

4. If $y(0)=1, y(1)=4, y(2)=10, y(3)=22$; find $y(5)$.

5. Show that $f(E)e^x = e^x f(e)$ where $f(e)$ is a polynomial in E taking units as the interval of differencing

6. Find the missing term/terms in following tables :

(i)

x:	1	2	3	4	5
y:	7	–	27	40	55

(ii)

x:	1	2	3	4	5	6	7
y:	2	4	8	–	32	64	128

(iii)

x:	0	5	10	15	20	25
y:	6	10	–	17	–	31

7. Given that $y_0+y_8=80, y_1+y_7=10, y_2+y_6=5, y_3+y_5=10$, find y_4

8. Prove for equally speced interpolating point

$$x_i = x_0 + ih \quad (h>0, i=0,1,2,\ldots n)$$

$$\Delta^k y_0 = \sum_{i=0}^{k}(-1)^i\binom{k}{i}y_{k-i}$$

9. Show that

$$\binom{n+1}{1}u_0 + \binom{n+1}{2}\Delta u_0 + \binom{n+1}{3}\Delta^2 u_0 + \ldots + \Delta^n u_0 = \sum_{i=0}^{n} u_i$$

10. Find the first term of the series whose second and subsequent terms are

15, 10, 7, 6, 7, 10, ...

11. Calculate the n^{th} divided difference of $\frac{1}{x}$ based on the points $x_0, x_1, \ldots, x_n$

12. Prove that

(i) $u_0 + \frac{x}{1!}u_1 + \frac{x^2}{2!}u_2 + \ldots$

$$= e^x\left[u_0 + x\,\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \ldots\right]$$

(ii) $u_0 - u_1 + u_2 - \ldots = \frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 - \frac{1}{16}\Delta^3 u_0 + \ldots$

13. If $f(x) = \frac{1}{x}$ whose arguments are a, b, c, d in this order, prove that $f[a,b,c,d] = -\frac{1}{abcd}$

14. If $f(x) = \sin x$, find the value of the divided difference

$$f\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}\right)$$

15. $y = f(x)$ is a polynomial of degree 5 with $y_0 = 0$, $y_1 = 3$, $y_2 = 14$, $y_3 = 45$, $y_4 = 84$, $y_5 = 155$, $y_6 = 258$. It is found that there is an error in the value of y_3. Find the correct value of y_3.
[W.B.U.T., CS-312, 2004]

16. If y is a polynomial of degree 3 and the values are as follows. Locate and correct the wrong value of y.

x	:	0	1	2	3	4	5	6	7
y	:	25	21	18	18	27	45	76	123

Answers

1. (i) $\dfrac{4}{(x+2)(x+3)(x+4)} + \dfrac{6}{(x+3)(x+4)(x+5)}$

(ii) $\dfrac{(2x+1)\sin 2x - 2x^2 \sin 1 \cos(2x+1)}{\sin 2(x+1) \sin 2x}$

4. 76 6. (i) 16 (ii) 17 (iii) 13.25, 22.50 7. 6

10. 22 11. $\dfrac{(-1)^n}{x_0 x_1 \dots x_n}$

16. error = −1; correct value of $y(3) = 19$

III. Multiple Choice Questions

1. $\Delta^3(y_0)$ may be expressed as which of the following terms?
(a) $y_3 - 3y_2 + 3y_1 - y_0$ (b) $y_2 - 2y_1 + y_0$
(c) $y_3 + 3y_2 + 3y_1 + y_0$ (d) none of these
[W.B.U.T., CS-312,2005]

2. $\Delta^2(e^x)$ (taking $h = 1$) is equal to
(a) $(e-1)e^x$ (b) $(e-1)^2 e^{2x}$ (c) $(e-1)^2 e^x$ (d) e^x

3. The value of $\Delta^2(ax^2 + bx + c)$ is
(a) ah^2 (b) $2a$ (c) a (d) $2ah^2$

4. Taking $h = \pi$, $\Delta(x + \cos x)$ is equal to
(a) $\pi + 2\cos x$ (b) $x - \sin x$
(c) $\pi - 2\cos x$ (d) $1 - \sin x$

5. If the interval of differencing is unity and $f(x) = ax^2$, a is a constant which of the following choices is wrong ?
(a) $\Delta f(x) = a(2x+1)$ (b) $\Delta^2 f(x) = 2a$
(c) $\Delta^3 f(x) = 2$ (d) $\Delta^4 f(x) = 0$
[W.B.U.T., CS-312, 2009, 2010]

6. $\Delta(ab^{cx})$ is equal to (taking $h = 1$)
(a) $(b^{cx} - 1)ab^{cx}$ (b) ab^{cx}
(c) ab^{cx-h} (d) $(b^c - 1)^2 ab^{cx}$

7. The value of $\dfrac{\Delta^2}{E}(x^3)$ is
(a) x (b) $6x$ (c) $3x$ (d) x^2

8. Which of the following is true ?
(a) $\Delta^n x^n = (n+1)!$ (b) $\Delta^n x^n = n!$
(c) $\Delta^n x^n = 0$ (d) $\Delta^n x^n = n$
[W.B.U.T., CS-312, 2009]

9. If $f(x) = \dfrac{1}{x^2}$, then the divided differnce $f(a,b)$ is
(a) $\dfrac{a+b}{(ab)^2}$ (b) $-\dfrac{a+b}{(ab)^2}$ (c) $\dfrac{1}{a^2 - b^2}$ (d) $\dfrac{1}{a^2} - \dfrac{1}{b^2}$
[W.B.U.T., CS-312, 2009, 2010]

10. $\Delta^2(ax^2 + bx) =$
(a) $2a$ (b) a (c) $a+b$ (d) $a-b$

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3 INTERPOLATION

3.1 Introduction:

Let $f(x)$ be a function of x defined in the interval $I:(-\infty < x < \infty)$ in which it is assumed to be continuous and continuously differentiable for a sufficient number of time. Suppose the analytical formula for the function $y = f(x)$ is not known, but the values of $f(x)$ are known for $(n+1)$ distinct values of x, say $x_0, x_1, \ldots\ldots, x_n$, called *arguments of nodes* which are entered as $y_0 = f(x_0), y_1 = f(x_1), \ldots y_n = f(x_n)$ and there is no other information available about the function. Our problem is to compute the value of $f(x)$, at least approximately, for a given arguments x in the vicinity of the above given values of the arguments. The process by which we can find the value of $f(x)$ for any other value of x in the interval $[x_0, x_1]$ is called *interpolation*. When x lies slightly outside the interval $[x_0, x_n]$, then the process is called *extrapolation*. [W.B.U.T., CS-312, 2006, 2008, 2009]

Since the analytical form i.e., explicit nature of $f(x)$ is not known, it is required to find a simpler function, say $p(x)$, such that

$$p(x_i) = f(x_i),\ i = 0, 1, 2, \ldots., n \qquad \ldots \quad (1)$$

This function $p(x)$ is known as interpolating function and in generals

$$f(x) \simeq p(x) \qquad \ldots \quad (2)$$

If $p(x)$ is a polynomial, then the process is called *polynomial interpolation* and $p(x)$ is called the *interpolating polynomial*. The justification of replacing a function by a polynomial rests on a theorem due to Weierstrass and is stated below without proof.

Theorem. Let $f(x)$ be a function defined and continuous on $a \le x \le b$. Then for $\varepsilon > 0$, there exist a polynomial $p(x)$ such that

$$|f(x) - p(x)| < \varepsilon,\ a \le x \le b$$

3.2. Error or remainder in polynomial interpolation

In virtue of (2), if we write

$$f(x) = p(x) + E(x) \qquad \ldots \quad (3)$$

then $E(x)$ is the error committed in replacing $f(x)$ by $p(x)$.

Using (1), we have

$$E(x_i) = 0,\ i = 0, 1, 2, \ldots n \qquad \ldots \quad (4)$$

By virtue of (4), let us assume $E(x) = k(x)\psi(x)$ ... (5)

where $\psi(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$ ... (6)

and $k(x)$ is to be determined such that (5) holds for any intermediate value of x, say $x = \alpha$, which is different from $x_i (i = 0, 1, 2, \ldots., n)$

Hence $k(\alpha) = \dfrac{E(\alpha)}{\psi(\alpha)} = \dfrac{f(\alpha) - p(\alpha)}{\psi(\alpha)}$, by (3) ... (7)

Let us construct a function $F(x)$ such that

$$F(x) = f(x) - p(x) - k(\alpha)\psi(x) \qquad \ldots \quad (8)$$

Then $F(x_i) = 0$, $i = 0, 1,, 2, \ldots n$, by (1) and (6)

Also $F(\alpha) = 0$, by (7) ... (9)

Hence $F(x)$ vanishes at $(n+2)$ number of points in the interval I. Then by repeated application of Rolle's theorem, we have

$$F^{n+1}(\xi) = 0 \text{ where } \xi \in I \qquad \ldots \quad (10)$$

Since $p(x)$ is a polynomial of degree not greater than n, so we must have

$$p^{n+1}(x) = 0 \qquad \ldots \quad (11)$$

Also, from (6), we have

$$\psi^{n+1}(x) = (n+1)! \qquad \ldots \quad (12)$$

$\therefore$ Hence (8) gives

$$F^{n+1}(x) = f^{n+1}(x) - 0 - (n+1)!\,k(\alpha)$$

or, $f^{n+1}(\xi) - (n+1)!\,k(\alpha) = 0$, by (10)

$$\therefore\ k(\alpha) = \frac{f^{n+1}(\xi)}{(n+1)!} \quad \ldots \quad (13)$$

$\therefore$ From (7),

$$E(\alpha) = \frac{f^{n+1}(\xi)}{(n+1)!}\psi(\alpha)$$

Since α is an arbitrary value of x, so

$$E(x) = \frac{f^{n+1}(\xi)}{(n+1)!}\psi(x)$$

$$= \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)..(x-x_n),\ \xi \in I \quad (14)$$

This expression gives the error in polynomial interpolation

3.3. Newton's forward interpolation formula.

[W.B.U.T., CS-312, 2002, 2006]

Let $y = f(x)$ be a real valued function of x defined in an interval $[a, b]$ and the $(n+1)$ entries $y_i = f(x_i)(i = 0,1,2,\ldots,n)$ are known for the corresponding $(n+1)$ equispaced arguments $x_i(i = 0,1,2,\ldots,n)$ such that $x_i = x_0 + ih\ (i = 0,1,2,\ldots,n)$ with $x_0 = a, x_n = b$ and h is the space length. Let us now construct a polynomial function $p(x)$ of degree not greater than n such that

$$p(x_i) = f(x_i) = y_i \quad (i = 0,1,2,\ldots,n) \quad \ldots \quad (15)$$

Since $p(x)$ is a polynomial of degree $\le n$, so we assume $p(x)$ as

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$
$$+\ldots+a_n(x-x_0)(x-x_1)..(x-x_{n-1}) \quad \ldots \quad (16)$$

where the coefficients $a_0, a_1, a_2, \ldots, a_n$ are constants to be determined by (15).

Substituting $x = x_0, x_1, x_2, \ldots, x_n$ successively in (16) and using (15) we obtain

$$p(x_0) = a_0$$

i.e., $a_0 = y_0$,

$$p(x_1) = a_0 + a_1(x_1 - x_0)$$

i.e., $y_1 = y_0 + a_1.h$

$$\therefore\ a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{1!h}$$

$$p(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

or, $y_2 = y_0 + \dfrac{y_1 - y_0}{h}2h + a_2.2h.h$

or; $a_2 = \dfrac{y_2 - 2y_1 + y_0}{2h^2} = \dfrac{\Delta^2 y_0}{2!h^2}$

Similarly, we get

$$a_3 = \frac{\Delta^3 y_0}{3!h^3}, \ldots, a_n = \frac{\Delta^n y_0}{n!h^n}$$

Substituting the above values of a_i's $(i = 0,1,2,\ldots,n)$ in (16) we obtain

$$p(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1)$$
$$+\ldots+\frac{\Delta^n y_0}{n!h^n}(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \quad \ldots \quad (17)$$

On introducing the phase $\boxed{s = \dfrac{x - x_0}{h}}$ and noting that

$$x - x_r = (x_0 + sh) - (x_0 + rh) = (s-r)h, r = 0,1,2,\ldots,n-1 \quad (18)$$

we get

$$f(x) \simeq p(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0$$
$$+\ldots+\frac{s(s-1)\ldots(s-n+1)}{n!}\Delta^n y_0 \quad \ldots \quad (19)$$

The formula (17) or (19) is known as *Newton's forward interpolation formula.*

Newton's forward interpolation formula with the remainder or error term $E(x)$ can be written as

$$f(x) = p(x) + E(x)$$

$$= y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \ldots +$$

$$+ \frac{s(s-1)(s-2)\ldots(s-n+1)}{n!}\Delta^n y_0 + E(x), \ldots \quad (20)$$

where the remainder or error is given by

$$E(x) = (x-x_0)(x-x_1)\ldots(x-x_n)\frac{f^{n+1}(\xi)}{(n+1)!}$$

$$= s(s-1)(s-2)\ldots(s-n)\frac{h^{n+1}f^{n+1}(\xi)}{(n+1)!}, \quad \ldots \quad (21)$$

$$\min\{x, x_0, x_1, \ldots x_n\} < \xi < \max\{x, x_0, x_1, x_2, \ldots, x_n\}$$

Note. (i) This formula is used only when the interpolating points are equally spaced.

(ii) The formula is used for interpolating the value of y near the beginning of the set of arguments and for extrapolating the values of y within a short distance backward to the left of y_0.

(iii) For better accuracy, x_0 should be chosen such that $s = \frac{x - x_0}{h}$ is as small as possible.

Example. From the following table, find $f(0.16)$ using Newton's forward interpolation formula :

x	:	0.1	0.2	0.3	0.4
$y = f(x)$	:	1.005	1.020	1.045	1.081

Solution : The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	1.005			
		0.015		
0.2	1.020		0.010	
		0.025		0.001
0.3	1.045		0.011	
		0.036		
0.4	1.081			

To find $f(0.16)$, we put $x = 0.16$, $x_0 = 0.2$, $h = 0.1$ so that

$$s = \frac{x - x_0}{h} = \frac{0.16 - 0.2}{0.1} = -0.4$$

Then using (19), we get

$$f(0.16) \simeq y_0 + s\ \Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \ldots$$

$$= 1.020 + (-0.4)\times 0.025 + \frac{(-0.4)(-0.4-1)}{2!}\times 0.011$$

$$= 1.01308$$

$\therefore\ f(0.16) \simeq 1.013$, correct upto three decimal places.

3.4. Newton's backward interpolation formula.

Let the values of the function $f(x)$ be given for the corresponding $(n+1)$ equispaced arguments $x_i (i = 0, 1, 2, \ldots, n)$, the step length being h, such that $x_i = x_0 + ih\ (i = 0, 1, 2, \ldots, n)$ and $y_i = f(x_i)(i = 0, 1, 2, \ldots, n)$

Then $x_{n-i} - x_n = x_0 + (n-i)h - x_0 - nh$

$$= -ih \quad (i = 0, 1, 2, \ldots, n).$$

Now we consider a polynomial $p(x)$ of degree $\leq n$ which replaces $f(x)$ at the interpolating points $x_i (i = 0, 1, 2, \ldots, n)$, i.e.,

$$p(x_i) = f(x_i) = y_i \quad (i = 0, 1, 2, \ldots, n) \quad \ldots \quad (22)$$

Since $p(x)$ is a polynomial of degree $\le n$, we take

$$p(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + \ldots + a_n(x-x_n)(x-x_{n-1})\ldots(x-x_1) \quad \ldots (23)$$

where the coefficient $a_0, a_1, a_2, \ldots, a_n$ are constants to be determined by (22)

Subtituting $x = x_n, x_{n-1}, x_{n-2}, \ldots, x_0$ successively in (23) and using (22), we obtain

$$p(x_n) = a_0$$

i.e., $a_0 = y_n$,

$$p(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

i.e., $y_{n-1} = y_n + a_1(-h)$

$$\therefore a_1 = \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_n}{h},$$

$$p(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

i.e., $y_{n-2} = y_n + a_1(-2h) + a_2(-2n)(-h)$

leading to

$$a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2!h^2}$$

Proceeding in this way, we get

$$a_3 = \frac{\nabla^3 y_n}{3!h^3}, \ldots, a_n = \frac{\nabla^n y_n}{n!\,h^n}$$

Substituting the above values of a_i's $(i = 0, 1, 2, \ldots, n)$ in (23) we obtain

$$p(x) = y_n + \frac{\nabla y_n}{1!h}(x-x_n) + \frac{\nabla^2 y_n}{2!h^2}(x-x_n)(x-x_{n-1}) + \ldots + \frac{(x-x_n)(x-x_{n-1})\ldots(x-x_1)}{n!\,h^n}\nabla^n y_n \quad \ldots (24)$$

On introduction of the phase $s = \dfrac{x - x_n}{h}$ so that

$s = \dfrac{x_{n} - x}{h}$

$$s + r = \frac{x - x_{n-r}}{h} \quad (r = 0, 1, 2, \ldots, n) \text{ in (24) gives}$$

$$f(x) \simeq p(x) = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \ldots + \frac{s(s+1)(s+2)\ldots(s+n-1)}{n!}\nabla^n y_n \quad \ldots (25)$$

which is known as *Newton's backward interpolation formula.*

Newton's backward interpolation formula with remainder or error term $E(x)$ can be written as

$$f(x) = p(x) + E(x)$$

$$= y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \ldots + \frac{s(s+1)(s+2)\ldots(s+n-1)}{n!}\nabla^n y_n + E(x) \quad \ldots (26)$$

where the remainder or error is given by

$$E(x) = (x-x_n)(x-x_{n-1})\ldots(x-x_1)(x-x_0)\frac{f^{n+1}(\xi)}{(n+1)!}$$

$$= s(s+1)\ldots(s+n)h^{n+1}\frac{f^{n+1}(\xi)}{(n+1)!}, \quad \ldots (27)$$

$$\min\{x, x_0, x_1, \ldots, x_n\} < \xi < \max\{x, x_0, x_1, \ldots, x_n\}$$

Note. (i) The formula is used only when interpolating points are equally spaced.

(ii) The formula is used for interpolating the value of y near the end of the given set of arguments and for extrapolating the value of y within a short distance forward to the right of y_n.

(iii) For better accuracy x_n should be chosen such that $\boxed{s = \dfrac{x - x_n}{h}}$ is as small as possible.

Example. Find $f(2.28)$ from the following table :

x :	2.00	2.10	2.20	2.30
$y = f(x)$:	1.7314	1.7811	1.8219	1.8535

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Solution : The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2.00	1.7314			
		0.0497		
			−0.0089	
2.10	1.7811			−0.0003
		0.0408		
			−0.0092	
2.20	1.8219			
		0.0316		
2.30	1.8535			

To find $f(2.28)$, we put $x = 2.28$, $x_n = 2.30$ $h = 0.10$, so that

$$s = \frac{x - x_n}{h} = -0.2$$

Hence using (25) we obtain

$$f(2.28) \simeq y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \ldots \quad \frac{s(s+1)(s+2)}{3!}\nabla^3 y_n$$

$$= 1.8535 + (-0.2) \times 0.0316 + \frac{(-0.2)(-0.2+1)}{2!} \times (-0.0092)$$

$$+ \frac{(-0.2)(-.02+1)(-0.2+2)}{3!} \times (-0.0003)$$

$$= 1.8464504$$

$\therefore$ $f(2.28) \simeq 1.8464$, correct upto four decimal places.

3.5. Lagrange's interpolation formula.

Let $y = f(x)$ be a function of x, continuous and $(n+1)$ times continuously differentiable in $[a, b]$. Let us divide the interval $[a, b]$ by $(n+1)$ points $a = x_0, x_1, \ldots, x_n = b$ which are not necessarily equispaced and the corresponding entries are $y_i = f(x_i)$ $(i = 0, 1, 2, \ldots, n)$. We now wish to find a polynomial $L_n(x)$ in x of degree n such that

$$L_n(x_i) = f(x_i) = y_i (i = 0, 1, 2, \ldots, n) \quad \ldots \quad (28)$$

Since $L_n(x)$ is a polynomial of degree n, so we may take $L_n(x)$ as

$$L_n(x) = a_0(x - x_1)(x - x_2)\ldots(x - x_n) + a_1(x - x_0)(x - x_2)\ldots(x - x_n) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \quad \ldots \quad (29)$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants to be determined by (28).

Putting $x = x_0$ in (29) and using (28), we get

$$L_n(x_0) = a_0(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)$$

$$\therefore\ a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

Now putting $x = x_1$ in (29) and using (28), we get

$$L_n(x_1) = a_1(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)$$

$$\therefore\ a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}$$

Proceeding in the same way, we have

$$\therefore\ a_n = \frac{y_n}{(x_n - x_0)(x_n - x_2)\ldots(x_n - x_{n-1})}$$

Substituting the above values of a_i's $(i = 0, 1, 2, \ldots, n)$ in (29), we obtain

$$L_n(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} y_1 + \ldots + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} y_n$$

$$= \sum_{i=0}^{n} l_i(x) y_i \quad \ldots \quad (30)$$

where

$$l_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)} \quad (31)$$

is called the *Lagrangian function*.

Now let us set

$$p_{n+1}(x) = (x - x_0)(x - x_1)...(x - x_{i-1})(x - x_i)(x - x_{i+1})...(x - x_n)$$

so that

$$p'_{n+1}(x_i) = (x_i - x_0)(x_i - x_1)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)$$

Thus we may write (31) in the form

$$l_i(x) = \frac{p_{n+1}(x)}{(x - x_i)p'_{n+1}(x_i)}$$

and therefore, from (30), we have

$$f(x) = L_n(x) = \sum_{i=0}^{n} \frac{p_{n+1}(x)}{(x - x_i)p'_{n+1}(x_i)} y_i \quad ... \quad (32)$$

which is called *Lagrange's interpolation formula*.

The remainder or error in Lagrange's interpolation formula is given by

$$E(x) = f(x) - L_n(x)$$

$$= \frac{p_{n+1}(x) f^{n+1}(\xi)}{(n+1)!}, \quad ... \quad (33)$$

$$\min\{x, x_0, x_1, ... x_n\} < \xi < \max\{x, x_0, x_1, ... x_n\}$$

Note. (1) Some advantage of Lagrange's interpolation are given below :

(i) The formula is applicable to both equispaced and unequispaced interpolating points.

(ii) There is no restriction on the order of the interpolating points $x_0, x_1, x_2, ..., x_n$.

(iii) The value of x corresponding to which the value of $y = f(x)$ is to be determined may lie anywhere of the tabulated values i.e., x may lie near the begining, end or middle of the tabulated values.

Note. (2) Some disadvantage of Lagrange's interpolation are given below :

(i) For increase of the degree of the interpolating polynomial by adding new interpolating point, the whole calculation would be made afresh.

(ii) The calculations provide no check whether the functional values used are taken correctly or not.

Example. Find the polynomial of degree ≤ 3 passing through the points $(-1, 1), (0, 1), (1, 1)$ and $(2, -3)$.

Solution. Using Lagrange's interpolation formula, we have

$$L_n(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \cdot y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \cdot y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \cdot y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \cdot y_3$$

$$= \frac{(x - 0)(x - 1)(x - 2)}{(-1 - 0)(-1 - 1)(-1 - 2)} .1 + \frac{(x + 1)(x - 1)(x - 2)}{(0 + 1)(0 - 1)(0 - 2)} .1$$

$$+ \frac{(x + 1)(x - 0)(x - 2)}{(1 + 1)(1 - 0)(1 - 2)} .1 + \frac{(x + 1)(x - 0)(x - 1)}{(2 + 1)(2 - 0)(2 - 1)} (-3)$$

$$= \tfrac{1}{3}(-2x^3 + 2x + 3).$$

Hence the required polynomial is

$$\tfrac{1}{3}(-2x^3 + 2x + 3)$$

3.6. Newton's divided difference interpolation.

The Lagrange's interpolation formula has the disadvantage that whenever a new data is added to an existing set, then the interpolating polynomial has to be completely recomputed. In this section, we describe Newton's general interpolation formula based on divided difference to overcome the above diadvantage.

Let $y = f(x)$ be a real valued function defined in $[a, b]$ and known at $(n+1)$ distinct arguments $x_0, x_1, x_2, ..., x_n$ not in order in any way. We seek a polynomial $p(x)$ of degree not greater than n such that

$$y_i = f(x_i) = p(x_i), i = 0, 1, 2, ..., n \quad ... \quad (34)$$

$$\text{and } f(x) = p(x) + R_{n+1}(x), \quad ... \quad (35)$$

$R_{n+1}(x)$ being the remainder or error in interpolation of $f(x)$.

From the definition of divided difference, we have

$$f[x, x_0] = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f[x, x_0, x_1] = \frac{f[x, x_0] - f[x_0, x_1]}{x - x_1}$$

$$f[x, x_0, x_1, x_2] = \frac{f[x, x_0, x_1] - f[x_0, x_1, x_2]}{x - x_2}$$

... ...

$$f[x, x_0, x_1, x_2, ..., x_n] = \frac{f[x, x_0, x_1, ..., x_{n-1}] - f[x_0, x_1, x_2 \cdots x_n]}{x - x_n}$$

Multiplying the above $(n+1)$ relations successively by $(x-x_0), (x-x_0)(x-x_1), (x-x_0)(x-x_1)(x-x_2), ...(x-x_0)(x-x_1)...(x-x_n)$ and then adding we get the following identity which holds for all values of x except possibly at $x = x_i$ $(i = 0, 1, 2, ... n)$:

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2] + ... + (x - x_0)(x - x_1)...(x - x_{n-1})\, f[x_0, x_1, x_2, ... x_{n-1}] + (x - x_0)(x - x_1)...(x - x_n)\, f[x, x_0, x_1, ..., x_n]$$

$$= p(x) + R_{n+1}(x) \quad ... \quad (36)$$

where

$$p(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + ... + (x - x_0)(x - x_1)(x - x_2)....(x - x_{n-1})f[x_0, x_1, x_2, ..., x_n]$$

and

$$R_{n+1}(x) = (x - x_0)(x - x_1)....(x - x_n)\, f[x, x_0, x_1, ..., x_n]$$

It can be easily verify that

$$f(x_i) = p(x_i) \;\forall i,\; i = 0, 1, 2, ..., n$$

Also, clearly

$$R_{n+1}(x_i) = 0, \text{ for } i = 0, 1, 2, ..., n$$

Thus $p(x)$ is the required interpolating polynomial

i.e.,

$$f(x) \simeq p(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1) \times f[x_0, x_1, x_2] + ... + (x - x_0)(x - x_1)...(x - x_{n-1}) \times f[x_0, x_1, x_2, ..., x_n] \quad ... \quad (37)$$

This formula is known as Newton's divided difference interpolation formula with remainder or error as

$$R_{n+1}(x) = (x - x_0)(x - x_1)...(x - x_n)f[x, x_0, x_1, ..., x_n] ... \quad (38)$$

Example Apply Newton's divided difference formula to find the polynomial of lowest possible degree which satisfies the conditions $f(-1) = 21$, $f(1) = 15$, $f(2) = 12$, $f(3) = 3$.

Solution. Let us first construct the following divided difference table :

x	$f(x)$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
–1	–21			
		18		
1	15		–7	
		–3		1
2	12		–3	
		–9		
3	3			

Using the above table, we have from Newton's divided difference formula,

$$f(x) \simeq -21 + (x+1) \times 18 + (x+1)(x-1)(-7) + (x+1)(x-1)(x-2) \times 1$$

$$= x^3 - 9x^2 + 17x + 6 \cdot$$

ILLUSTRATIVE EXAMPLES

Ex. 1. Given the following table of function $F(x) = \frac{1}{x}$, find $\frac{1}{2.72}$ using the suitable interpolation formula. Find an estimate of the error

x	:	2.7	2.8	2.9
$F(x)$	:	0.3704	0.3571	0.3448

[W.B.U.T., CS-312, 2008]

Solution : The difference table is

x	$F(x)$	$\Delta F(x)$	$\Delta^2 F(x)$
2.7	0.3704		
		-0.0133	
			0.0010
2.8	0.3571		
		-0.0123	
2.9	0.3448		

To find $F(2.72)$, we use Newton's forward difference interpolation formula

$$F(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \ldots \qquad \ldots (1)$$

Here $x_0 = 2.7, h = 0.1$

$$\therefore s = \frac{x - x_0}{h} = \frac{2.72 - 2.7}{0.1} = 0.2$$

$\therefore$ From (1), we get

$$F(2.72) = 0.3704 + 0.2 \times (-0.0133) + \frac{0.2(0.2-1)}{2!} \times 0.0010$$

$$= 0.36766$$

Thus $\frac{1}{2.72} = 0.36766$.

So the error is

$$\left(\frac{1}{2.72} - 0.36766\right) = -1.3 \times 10^{-5}$$

Ex. 2. Find the polynomial of the least degree which attains the prescribed values of the given points :

x :	0	1	2	3
y :	3	6	11	18

Hence find y for $x = 1.1$

1,00,000

Solution : The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		
3	18			

Here $x_0 = 0$, $h = 1$ so that $s = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$

$\therefore$ From, Newton's forward difference interpolation formula,

$$y \simeq y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \ldots,$$

we have

$$y \simeq 3 + x \times 3 + \frac{x(x-1)}{2!} \times 2 + 0$$

$$= x^2 + 2x + 3$$

So the required polynomial is

$$y = x^2 + 2x + 3$$

6.41

$$\therefore y(1.1) = (1.1)^2 + 2 \times 1.1 + 3 = \cancel{6.41}\cdot 146$$

Ex. 3. Values of x (in degree) and $\sin x$ are given in the following table :

x (in degree):	15	20	25	30	35	40
$y = f(x)$:	0.2588190	0.3420201	0.4226183	.05	0.5735764	0.6427876

Determine the value of $\sin 38^0$ by Newton's backward difference interpolation formula. [W.B.U.T., CS-312, 2010]

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Solution. The difference table :

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
15	0.2588190				
		0.0832011			
			–0.0026029		
				–0.0006136	
20	0.3420201				–0.0000248
		0.0805982			
			–0.0032165		
				–0.0005888	
25	0.4226183				
		0.0773817			
			–0.0038053		–0.0000289
30	0.5				
				–0.0005599	
		0.0735764			
			–0.0043652		
35	0.5735764				
		0.0692112			
40	0.6427876				

To find $\sin 38^0$, we choose $x_n = 40$

Here $h = 5, x = 38$

$$\therefore s = \frac{x - x_n}{h} = -0.4$$

So the Newton's backward difference formula

$$f(x) = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \ldots$$

gives

$$f(38) \simeq 0.6427876 - 0.4 \times 0.0692112 + \frac{(-0.4)(-0.4+1)}{2!}(-0.0043652)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)}{3!}(-0.005599)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{4!}(-0.0000289)$$

$$= 0.615662777$$

$\therefore$ **$\sin 38 \simeq 0.615663$, correct upto six decimal places.**

Ex. 4. Using approximate formula find $f(0.29)$ from the following table

x :	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$:	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

Solution : First we construct the difference table as given below:

x	y	Δy	$\Delta^2 y$
0.20	1.6596		
		0.0102	
0.22	1.6698		0.0004
		0.0106	
0.24	1.6804		0.0002
		0.0108	
0.26	1.6912		0.0004
		0.0112	
0.28	1.7024		0.0003
		0.0115	
0.30	1.7139		

Here we apply Newton's backward difference interpolation formula for finding $f(0.29)$,

For that we take $x_n = 0.30$ as $x = 0.29$

$$\therefore s = \frac{x - x_n}{h} = \frac{0.29 - 0.30}{0.02} = -0.5$$

Then using Newton's backward formula

$$f(x) = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \ldots,$$

we get

$$f(0.29) \simeq 1.7139 + (-0.5) \times 0.0115 + \frac{(-0.5)(-0.5+1)}{2!} \times 0.0003$$

$$= 1.70777$$

$\simeq 1.708$, correct upto three decimal places.

Ex. 5. The function $y = \sin x$ is tabulated as given below :

x_0 x_1 x_2

x	:	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\sin x$	:	0	0.70711	1.0

y_0 y_1 y_2

Find the value of $\sin\frac{\pi}{3}$ using Lagrange's interpolation formula correct upto 5 places of decimal. [W.B.U.T., C.S-312, 2004]

Solution : Using Lagrange's interpolation formula, we obtain

$$L_n(x) \approx \frac{\left(x-\frac{\pi}{4}\right)\left(x-\frac{\pi}{2}\right)}{\left(0-\frac{\pi}{4}\right)\left(0-\frac{\pi}{2}\right)}\times 0+\frac{(x-0)\left(x-\frac{\pi}{2}\right)}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}\times 0\cdot 70711$$

$$+\frac{(x-0)\left(x-\frac{\pi}{4}\right)}{\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}\times 1$$

$$=-x\left(x-\frac{\pi}{2}\right)\cdot\frac{16\times 0.70711}{\pi^2}+\frac{16x}{\pi^2}\left(x-\frac{\pi}{4}\right)$$

$$\therefore \sin x \simeq \frac{8x}{\pi^2}(-0.41422x+0.45711\pi)$$

$$\therefore \sin\frac{\pi}{3} \simeq \frac{8.\pi}{3\pi^2}\left(-0.41422\frac{\pi}{3}+0.45711\pi\right)$$

$= 0.850764$

$\simeq 0.85076$, correct upto 5 places of decimal.

Ex. 6. Construct Lagrange's interpolation polynomial by using the following data :

0 1 2 3

x	:	40	45	50	55
$y=f(x)$	:	15.22	13.99	12.62	11.13

[W.B.U.T., CS-312,2007]

Solution : Using Lagrange's interpolation formula, we have

$$L_n(x)=\frac{(x-45)(x-50)(x-55)}{(40-45)(40-50)(40-55)}\times 15.22$$

$$+\frac{(x-40)(x-50)(x-55)}{(45-40)(45-50)(45-55)}\times 13.99$$

$$+\frac{(x-40)(x-45)(x-55)}{(50-40)(50-45)(50-55)}\times 12.62$$

$$+\frac{(x-40)(x-45)(x-50)}{(55-40)(55-45)(55-50)}\times 11.13$$

$$=2.7\times 10^{-5}x^3-6.4\times 10^{-3}x^2+15.3\times 10^{-2}x+17.62$$

Ex. 7. Find the polynomial $f(x)$ and hence calculate $f(5.5)$ for the given data :

0 1 2 3 4

x	:	0	2	3	5	7
$f(x)$	:	1	47	97	251	477

[W.B.U.T., CS-312, 2006, 2008]

Solution : Applying Lagrange's interpolation formula, we have

$$L_n(x)=\frac{(x-2)(x-3)(x-5)(x-7)}{(0-2)(0-3)(0-5)(0-7)}\times 1$$

$$+\frac{(x-0)(x-3)(x-5)(x-7)}{(2-0)(2-3)(2-5)(2-7)}\times 47$$

$$+\frac{(x-0)(x-2)(x-5)(x-7)}{(3-0)(3-2)(3-5)(3-7)}\times 97$$

$$+\frac{(x-0)(x-2)(x-3)(x-7)}{(5-0)(5-2)(5-3)(5-7)}\times 251$$

$$+\frac{(x-0)(x-2)(x-3)(x-5)}{(7-0)(7-2)(7-3)(7-5)}\times 477$$

$$=\frac{(x-2)(x-3)(x-5)(x-7)}{210}+\frac{x(x-3)(x-5)(x-7)}{-30}\times 47$$

$$+\frac{x(x-2)(x-5)(x-7)}{24}\times 97+\frac{x(x-2)(x-3)(x-7)}{-60}\times 251$$

$$+\frac{x(x-2)(x-3)(x-5)}{280}\times 477$$

$\therefore\ f(x) = 9x^2 + 5x + 1 \ \therefore\ f(5.5) = 9(5.5)^2 + 5 \times 5.5 + 1 = 300.75$

Ex. 8. Use Newton's divided difference formula to find $f(5)$ from the following data :

x :	0	2	3	4	7	8
$f(x)$:	4	26	58	112	466	668

[W.B.U.T. CS-312, 2009]

Solution : The divided difference table is given below :

x	$f(x)$	1st div.	2nd div.	3rd div.	4th div.
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		21		
		202			
8	668				

Using Newton's divided difference interpolation formula

$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \ldots,$

we get

$f(5) = 4 + (5-0) \times 11 + (5-0)(5-2) \times 7 + (5-0)(5-2)(5-3) \times 1$

$= 194.$

Ex. 9. **Find the equation of the cubic curve which passes through the points $(4,-43), (7,83), (9,327)$ and $(12,1053)$**

Hence find $f(10)$

Solution : Here we use Newton's divided difference formula,

$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \ldots$ ... (1)

The divided difference table is

x	$f(x)$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
4	−43			
		42		
7	83		16	
		122		1
9	327		24	
		242		
12	1053			

From (1), we get

$f(x) = -43 + (x-4) \times 42 + (x-4)(x-7) \times 16 + (x-4)(x-7)(x-9) \times 1$

$= x^3 - 4x^2 - 7x - 15$

$\therefore f(10) = 10^3 - 4 \times 10^2 - 7 \times 10 - 15$

$= 515$

Ex. 10. Using Newton's forward formula compute y_{12} given that $y_{10} = 600$, $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$

Solution. The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	600				
		−88			
20	512		15		
		−73		−35	
30	439		−20		45
		−93		10	
40	346		−10		
		−103			
50	243				

To find y_{12}, we choose $x_0 = 10$, so that

$$s = \frac{x - x_0}{h} = \frac{12-10}{10} = 0.2$$

Then Newton's forward difference interpolation formula

$$y \simeq y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \ldots$$

gives

$$y_{12} \simeq 600 + 0.2\times(-88) + \frac{0.2(0.2-1)}{2!}\times 15 + \frac{0.2(0.2-1)(0.2-2)}{3!}\times(-35)$$

$$+\frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!}\times 45$$

$= 578.008$

$\therefore\ y_{12} \simeq 578$

Exercise

I. Short Answer Questions

1. Fit a polynomial of degree three which takes the following values

x	:	3	4	5	6
y	:	6	24	60	120

Hence find $y(1)$.

2. Find $f(0.3)$ where $f(x) = 5^x$, taking 0 and 1 as interpolating points by the methods of interpolation.

3. Using Newton's forward interpolation formula find the polynomial of degree 3 passing through the points (–1,1), (0,1), (1, 1) and (2, –3).

4. Find the forward interpolation polynomial for the function $f(x)$ where $f(0) = -1, f(1) = 1, f(2) = 1$ and $f(3) = -2$.

5. Find Newton's forward interpolation polynomial of the function $f(x)$ when $f(0) = 1, f(1) = 2, f(2) = 1$ and $f(3) = 10$

6. Using appropriate interpolation formula, find the value of $f(5)$ from the following data :

x	:	3	4	6	8
$f(x)$	:	4.5	13.2	43.7	56.4

7. Find $f(1.02)$ having given

x	:	1.00	1.10	1.20	1.30
$f(x)$	:	0.8415	0.8912	0.9320	0.9636

8. Evaluate $f(1)$ from the following values of x and $f(x)$:

x	:	0	2	4	6
$f(x)$	:	2	6	10	15

9. Using appropriate interpolation formula, find the value of the function $f(x)$ when $x = 7$ from the following data.

x	:	2	4	6	8
$f(x)$	:	15	28	56	89

10. Find Newton's backward difference interpolation polynomial against the tabulated values :

x	:	3	4	5	6
y	:	6	24	60	120

11. Find the value of y when $x = 19$; given

x	:	0	1	20
y	:	0	1	2

12. Compute $f(21)$ using the following datas :

x	:	0	5	10	20
$f(x)$	:	1.0	1.6	3.8	15.4

13. Use Lagrange's interpolation formula to find the value of $f(x)$ for $x = 0$, given the following table :

x	:	–1	–2	2	4
$f(x)$	:	–1	–9	11	69

[W.B.U.T., CS-312, 2007]

14. $f(x)$ is a function defined on [0, 1] having values 0, −1 and 0 at $x = 0, \frac{1}{2}$ and 1. Find the two degree polynomial $\phi(x) \approx f(x)$ such that $\phi(0) = f(0), \phi\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right)$ and $\phi(1) = f(1)$

15. Find the Lagrangian polynomial for the following tabulated value :

x	:	0	1	3
y	:	0	3	1

16. Find Lagrange's interpolation polynomial for the function $f(x)$ when $f(0) = 4, f(1) = 3,\ f(2) = 6$

17. Find Lagrange's interpolation polynomial for the function $f(x) = \sin x\pi$ when $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{1}{2}$. Also compute the value of $\sin \pi/3$.

18. Find the parabola passing through the points (0, 1), (1, 3) and (3, 55) using Newton's divide interpolation formula.

19. Using Newton's divide interpolation formula, find $p(4)$ when $p(1) = 10$ $p(2) = 15$ and $p(5) = 42$.

20. Given

x	:	1	2	5
$f(x)$	:	10	15	42

Find $f(4)$

Answers

1. $x^3 - 3x^2 + 2x, 0$ **2.** 2.2 **3.** $1 - \frac{2}{3}x(x^2 - 1)$

4. $-\frac{1}{6}(x^3 + 3x^2 - 16x + 6)$ **5.** $2x^3 - 7x^2 + 6x + 1$

6. 26.7 **7.** 0.8521 **8.** 4.0625 **9.** 72.5

10. $x^3 - 3x^2 + 2x$ **12.** 17.23 **14.** $\phi(x) = 4x^2 - 4x$

15. $-\frac{4}{3}x^2 + \frac{13}{3}x$ **16.** $2x^2 - 3x + 4$ **17.** $-3x^2 + \frac{7}{2}x$, 0.8333

18. $y = 8x^2 - 6x + 1$ **19.** 31 **20.** 31

II. Long Answer Questions

1. If $y(10) = 35.3, y(15) = 32.4, y(20) = 29.2, y(25) = 26.1,$ $y(30) = 23.2$ and $y(35) = 20.5$, find $y(12)$ using Newton's forward interpolation formula.

[W.B.U.T., CS-312, 2010]

2. Find $f(2.5)$ using Newton's forward difference formula for the given data:

x	:	1	2	3	4	5	6
$y = f(x)$	:	0	1	8	27	64	125

3. A function $y = f(x)$ is given by the following table. Find $f(0.2)$ by a suitable formula.

x	:	0	1	2	3	4	5	6
$y = f(x)$	:	176	185	194	203	212	220	229

4. Find the value of $\sqrt{2}$ correct upto four significant figures from the following table:

x	:	1.9	2.1	2.3	2.5	2.7
$\sqrt{x}$	:	1.3784	1.4491	1.5166	1.5811	1.6432

5. Calculate $f(1.135)$ using suitable formula

x :	1.140	1.145	1.150	1.155	1.160	1.165
$f(x)$:	0.13103	0.13541	0.13976	0.14410	1.14842	0.15272

[W.B.U.T., CS-312, 2007]

6. Compute $y(0.5)$ using the following table:

x :	0	1	2	3	4	5
y :	5.2	8.0	10.4	12.4	14.0	15.2

7. Determine the polynomial of degree 3 from the following table:

x :	0	1	2	3	4	5
y :	–3	–5	–11	–15	–11	–7

8. Find the equation of the cubic curve that passes through the points (0, –5), (1, –10), (2, –9), (3, 4) and (4, 35).
[W.B.U.T., CS-312, 2004]

9. Compute the values of $f(3.5)$ and $f(7.5)$ using Newton's interpolation from the following table:

x :	3	4	5	6	7	8
$f(x)$:	27	64	125	216	343	512

[W.B.U.T., CS-312, 2008]

10. The values of $y = \sin x$ are given below for different values of x. Find the values of y for (i) $x = 32^0$, (ii) $x = 52^0$.

x :	30^0	35^0	40^0	45^0	50^0	55^0
$y = \sin x$:	0.5000	0.5735	0.6428	0.7071	0.7660	0.8192

11. Using appropriate formula find $f(0.29)$ from the following table:

x :	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$:	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

12. The population of a city for five census are given below:

year :	1941	1951	1961	1971	1981	1991
population (in lacs) :	46.52	66.23	81.01	93.70	101.58	120.92

Using suitable formula estimate the population of the city for the year 1985

13. Apply Lagrange's interpolation forrmula to find $f(x)$, if $f(1) = 2, f(2) = 4, f(3) = 8, f(4) = 16$ and $f(7) = 128$.
[W.B.U.T., CS-312, 2006, 2010]

14. Use Lagrange's interpolation formula to fit a polynomial to the following data. Hence find $y(4)$

x	:	–1	0	2	3
y	:	–8	3	1	2

15. Find the fourth degree curve $y = f(x)$ passing through the points (2, 3), (4, 43), (7, 778) and (8, 1515), using Newton's divided difference formula.
[W.B.U.T., CS-312, 2006]

16. Using Newton's divided difference formula to find $y(3.4)$

x :	2.5	2.8	3.0	3.1	3.6
y :	12.1825	16.4446	20.0855	22.1980	36.5982

[W.B.U.T., CS-312, 2007]

17. Using divided difference interpolation formula, compute $f(27)$ from the following data:

x	:	14	17	31	35
$f(x)$	:	68.7	64.0	44.0	39.1

18. Find $f(8)$ using Newton's divided difference formula given that

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

19. Use Newton's divided difference formula to approximate $f(0.05)$ from the following table

x :	0.0	0.2	0.4	0.6	0.8
$f(x)$:	1.0000	1.22140	1.49182	1.82212	2.22554

[W.B.U.T., CS-312, 2002]

20. Find the values of (i) $\log_{10}(11.1)$ and $\log_{10}(17.8)$ from the following table

x :	11	12	13	14	15	16	17
$\log_{10}^{x}$:	1.0414	1.0792	1.1139	1.1461	1.1761	1.2041	1.2304

Answers

1. 8.8345 2. 3.4 3. 177.67 4. 1.414 5. 6.65
7

8. x^3-5x^2+2x-3 10. 0.5299, 0.7888 11. 1.708

12. 107.03 14. $\frac{1}{6}(7x^3-31x^2+28x+18)$, 13.66

15. $13x^3-124x^2+400x-405$ 17. 49.3 18. 448

20. 1.0453, 1.2504

III. Multiple Choice Questions

1. In Newton's forward interpolation, the interval should be

(a) equally spaced (b) not equally spaced

(c) may be equally spaced (d) both (a) and (b)

[W.B.U.T., CS-312, 2008]

2. Newton's forward interpolation formula is used to interpolate

(a) near end (b) near central position

(c) near beginning (d) none of these

3. The coefficient of Newton's forward difference interpolation formula are

(a) $\dfrac{s(s-1)\dots(s-n+1)}{n!}$ (b) $\dfrac{s(s+1)\dots(s+n-1)}{n!}$

(c) $\dfrac{s(s-1)\dots(s-n+1)}{(n-1)!}$ (d) none of these

[where $s=\dfrac{x-x_0}{h}$]

4. In Newton's forward difference interpolation, the value of $s=\dfrac{x-x_0}{h}$ lies between

(a) 1 and 2 (b) −1 and 1 (c) 0 and ∞ (d) 0 and 1

5. Newton's backward interpolation formula is used to interpolate

(a) near end (b) near central position

(c) near the beginning (d) none of these

6. The restriction on the interpolating points for Newton's forward and backward formulae is

(a) should not be so large

(b) should be in arithmetic progression

(c) should be in geometric progression

(d) should be in positive

7. The coefficient of Newton's backward difference interpolation formula are

(a) $\dfrac{u(u-1)\dots(u-n+1)}{n!}$ (b) $\dfrac{u(u+1)\dots(u+n-1)}{n!}$

(c) $\dfrac{u(u-1)\dots(u-n+1)}{(n-1)!}$ (d) none of these

[where $u=\dfrac{x-x_0}{h}$]

8. In Newton's backward difference interpolation formula, the value of $s=\dfrac{x-x_n}{h}$ should lie between

(a) 0 and 1 (b) 0 and ∞

(c) greater than 1 (d) no restriction

9. The coefficient in Newton's forward and backward difference formula are

(a) value of the point of interpolation

(b) value of the common difference of the values of x

(c) value of x

(d) value of y

10. If $f(3)=4, f(4)=13$ and $f(6)=43$, then $f(5)$ is equal to

(a) 20 (b) 26 (c) 25 (d) 39

11. If $f(0)=12, f(3)=6$ and $f(4)=8$, then the linear interpolation function $f(x)$ is

(a) $x^2-3x+12$ (b) x^2-5x (c) x^3-x^2-5x (d) $x^2-5x+12$

[W.B.U.T., CS-312, 2010]

4 NUMERICAL INTEGRATION

4.1 Introduction:

In this chapter we derive and analyse numerical methods to evaluate definite integrals of the form

$$I = \int_a^b f(x)dx$$

for any finite interval $[a,b]$ by replacing the function $f(x)$ with a suitable polynomial $p(x)$ such that $\int_a^b p(x)dx$ is taken to be an approximation of the integral I. The approximation of I is usually known as *numerical integration or quadrature.*

[W.B.U.T., CS-312, 2006]

Let $y = f(x)$ be a real valued function defined in $[a, b]$ such that the values of $f(x)$ are known for $x = x_i$ $(i = 0, 1, 2, \ldots n)$ whose all x_i lies in $[a, b]$ and $y_i = f(x_i)$, $(i = 0, 1, 2, \ldots n)$. Also let $p(x)$ be the interpolating polynomial of degree at most n such that

$$p(x_i) = f(x_i) = y_i \qquad \ldots (1)$$

Thus $p(x) \simeq f(x)$ and so

$$I = \int_a^b f(x)dx \simeq \int_a^b p(x)dx \qquad \ldots (2)$$

Then the expression

$$E(x) = \int_a^b f(x)dx - \int_a^b p(x)dx \qquad \ldots (3)$$

is known as the error of integration.

4.2. The Important Concepts.

(a) Degree of precision.

A quadrature formula is said to have a degree of precision m (m being a positive integer) if it is exact i.e. the error is zero for an arbitrary polynomial of degree $m \le n$ but there exists a

polynomial of degree $m+1$ for which it is not exact i.e. the error is not zero.

(b) Composite rule.

Sometimes it is more convenient to divide the interval $[a, b]$, $(a = a_0 < a_1 < a_2 < \ldots < a_{m-1} < a_m = b)$ of integration into finite number of sub-intervals, say $m(< n)$, $[a_{i-1}, a_i]$, $(i = 1, 2, \ldots m)$. Then we apply a quadrature formula separately to each of these sub-intervals and add the results. The formula thus obtained is called *composite rule* corresponding to that *quadrature formula.*

4.3. A general quadrature formula.

Consider the definite integral

$$I = \int_a^b f(x)dx \qquad \ldots (4)$$

and divide the interval $[a, b]$ of integration into n equal sub-intervals such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ and $x_i = x_0 + i\,h$ $(i = 0, 1, 2, \ldots n)$; h is the space length.

Also let the function $f(x)$ be known at the nodes x_i, i.e., the values $y_i = f(x_i), i = 0, 1, 2, \ldots n$ are given. Then by putting $s = \frac{x - x_o}{h}$ i.e. $x = x_0 + s\,h$ in (4), we get

$$I = \int_{x_0}^{x_0+nh} f(x)dx$$

$$= h\int_0^n f(x_0 + sh)ds$$

$$= h\int_0^n E^s f(x_0)ds \quad [\because E^n f(x) = f(x + nh)]$$

$$= h\int_0^n (1+\Delta)^s y_0\, ds \qquad [\because E = 1 + \Delta]$$

$$= h\int_0^n \left[1 + s\Delta + \frac{s(s-1)}{2!}\Delta^2 + \frac{s(s-1)(s-2)}{3!}\Delta^3 + \ldots\right] y_0\, ds$$

so that

$$I = \int_{x_0}^{x_n} f(x)dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{2n^3 - 3n^2}{12}\Delta^2 y_0 + \frac{n^4 - 4n^3 + 4n^2}{24}\Delta^3 y_0 + \dots\right] \quad \dots (5)$$

The formula (5) is known as general integration formula when the interval of integration is divided into n equal sub-intervals.

We can derive some integration formulae from (5) as particular cases by putting $n = 1, 2, 3, \dots$

4.4. Trapezoidal rule.

Putting $n = 1$ in (5), we obtain

$$\int_{x_0}^{x_1} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{1}{2}h(y_0 + y_1) \quad \dots (6)$$

This is called Trapezoidal rule for numerical integration for two points.

Similarly we have

$$\int_{x_1}^{x_2} f(x)dx \approx \frac{1}{2}h(y_1 + y_2)$$

... ...

$$\int_{x_{n-1}}^{x_n} f(x)dx \approx \frac{1}{2}h(y_{n-1} + y_n)$$

Adding the above integrals, we obtain

$$\boxed{\int_{x_0}^{x_n} f(x)dx \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})]} \quad \dots (7)$$

which is known as composite Trapezoidal rule.

Error in Trapezoidal rule

The error committed in (6) is given by

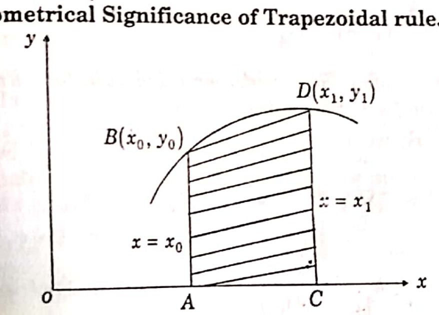
$$E = \int_{x_0}^{x_1} f(x)dx - \frac{1}{2}h(y_0 + y_1)$$

$$= -\frac{1}{2}h^3 f''(x_0) \quad \dots (8)$$

The total error committed in composite Trapezoidal rule (7) is

$$E_T \approx -\frac{1}{2}h^3 n f''(\xi), x_0 < \xi < x_n \quad \dots (9)$$

Geometrical Significance of Trapezoidal rule.



The geometrical significance of Trapezoidal rule lies on the fact that the curve $y = f(x)$ is replaced by straight lines joining the points $B(x_0, y_0)$ and $D(x_1, y_1)$. The area bounded by the curve $y = f(x)$, the ordinates $x = x_0, x = x_1$ and the x-axis is approximately equal to the area of the trapezium $BACD$. For this reason, Trapezoidal rule is also known as *trapezium rule*.

Note. (i) The Trapezoidal rule can be applied for any (even or odd) number of equal sub-intervals.

(ii) In Trapezoidal rule, the error involves second order derivatives of the function $f(x)$ and so it gives the exact value of the integral if $f(x)$ is constant or a first degree polynomial. Hence the degree of precision of this formula is one.

4.5. Simpson's $\frac{1}{3}$ rd rule. [W.B.U.T, CS-312, 2002]

Putting $n = 2$ in (5) we obtain

$\int_{x_0}^{x_2} f(x)dx \simeq 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$, neglecting the third and heigher order differences.

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2) \quad \dots \quad (10)$$

This is called *Simpson's one-third rule for numerical integration for three points.*

Similarly we have

$$\int_{x_2}^{x_4} f(x)dx \simeq \frac{h}{3}(y_2 + 4y_3 + y_4)$$

... ...

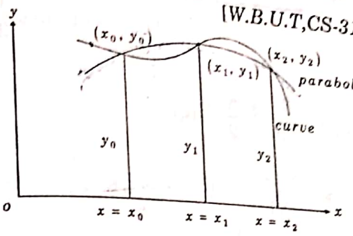
$$\int_{x_{n-2}}^{x_n} f(x)dx \simeq \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})] \quad \dots \quad (11)$$

which is known as *composite Simpson's* $\frac{1}{3}$ *rd rule*

Geometrical Significance of Simpson's $\frac{1}{3}$ rule. [W.B.U.T, CS-312, 2008, 2009]



We have seen that Simpson's one third rule is given by

$$\int_{x_0}^{x_2} f(x)dx \simeq \frac{h}{3}(y_0 + 4y_1 + y_2)$$

The integral $\int_{x_0}^{x_2} f(x)dx$ represents geometrically the area of the region bounced by the curve $y = f(x)$, x-axis and the ordinates $x = x_0, x = x_2$. On the otherhand $\frac{h}{3}(y_0 + 4y_1 + y_2)$ represents geometrically the area of the region bounded by a parabola passing through the points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2), x-axis and the lines $x = x_0, x = x_2$. Thus the area bounded by the curve $y = f(x), x = x_0, x = x_2$ and $y = 0$ is approximated to the area bounded by the second degree, parabola passing through the points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2), $y = 0$ and the lines $x = x_0, x = x_2$.

Error in Simpson's $\frac{1}{3}$ rd rule.

[W.B.U.T, CS-312, 2002, 2006]

The error committed in formula (10) is given by

$$E_s = \int_{x_0}^{x_2} f(x)dx - \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$\simeq -\frac{1}{90}h^5 f^{iv}(x_0) \quad \dots \quad (12)$$

which is the error in the interval $[x_0, x_2]$.

The total error committed in composite Simpson's $\frac{1}{3}$ rd rule (11) is given by

$$E_s^c \simeq -\frac{1}{90}h^5 \cdot \frac{n}{2} f^{iv}(\xi)$$

$$= -\frac{b-a}{180}h^4 f^{iv}(\xi), \quad x_0 < \xi < x_n \qquad \dots \quad (13)$$

Note. (i) The Simpson's $\frac{1}{3}$ rd rule is applicable only if the number of sub-intervals is even.

(ii) Since the error involves fourth order derivatives of $f(x)$, the Simpson's one third rule yields an exact value of the integral if $f(x)$ is a polynomial of degree less than or equal to three. Hence the degree of precision of this formula is three.

4.6. Weddle's rule.

Putting $n = 6$ in (5), we obtain

$$\int_{x_0}^{x_6} f(x)dx = 6h\left[y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{41}{20}\Delta^4 y_0 + \frac{11}{20}\Delta^5 y_0 + \frac{41}{840}\Delta^6 y_0\right]$$

$$= h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{3}{10}\Delta^6 y_0\right] - \frac{h}{140}\Delta^6 y_0$$

If we now choose h in such a way that the sixth order difference are very small, then we may neglect the small term $\frac{h}{140}\Delta^6 y_0$. Thus we have

$$\int_{x_0}^{x_6} f(x)dx \simeq \frac{3h}{10}[20y_0 + 60(y_1 - y_0) + 90(y_2 - 2y_1 + y_0)$$

$$+ 80(y_3 - 3y_2 + 3y_1 - y_0) + 41(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)$$

$$+ 11(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)$$

$$+ (y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0)$$

so that

$$\int_{x_0}^{x_6} f(x)dx \simeq \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6] \qquad \dots \quad (14)$$

This is called Weddle's rule for numerical integration.

Similarly we have

$$\int_{x_6}^{x_{12}} f(x)dx \simeq \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

$$\dots \qquad \dots \qquad \dots$$

$$\int_{x_{n-6}}^{x_n} f(x)dx \simeq \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

Adding the above integrals, we get

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{3h}{10}[y_0 + y_n + (y_2 + y_4 + y_8 + \dots + y_{n-4})$$

$$+ 5(y_1 + y_5 + y_7 + \dots + y_{n-1}) + 6(y_3 + y_9 + y_{15} + \dots + y_{n-3}) \qquad \dots \quad (15)$$

$$+ 2(y_6 + y_{12} + y_{18} + \dots + y_{n-6})]$$

which is known as composite Weddle's rule.

Error in Weddle's rule

The error committed in formula (!4) is given by

$$E_w = \int_{x_0}^{x_6} f(x)dx - \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$\simeq -\frac{h^7}{140} f^{vi}(x_0), \qquad \dots \quad (16)$$

which is the error in the interval $[x_0, x_n]$

The total error committed in composite Weddle's rule (15) is given by

$$E_w^c \simeq -\frac{nh^7}{840} f^{vi}(\xi)$$

$$= -\frac{(b-a)h^6}{840} f^{vi}(\xi) \quad x_0 < \xi < x_n \quad \cdots \quad (17)$$

Note. (i) The Weddle's rule is applicable only when the number of sub-intervals is multiple of six.

(ii) Since, the error involves sixth order derivatives of $f(x)$, the Weddle's rule yields an exact value of the integral if $f(x)$ is a polynomial of degree less than or equal to five. Hence the degree of precision of this formula is five.

Example.1. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (i) Trapezoidal

(ii) Simpson's one-third and [*W.B.U.T.,CS-312, 2006*]

(iii) Weddle's rules, taking $n=6$. Hence compute an approximate value of π in each case.

Solution. Let $f(x) = \frac{1}{1+x^2}$

Since $n = 6$, so $h = \frac{1-0}{6} = \frac{1}{6}$

The tabulated values of $f(x)$ for different values of x are given below :

x	:	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$y = f(x)$	:	1	0.973	0.9	0.8	0.6923	0.5902	0.5

(i) We have by composite Trapezodial rule,

$$\int_0^1 \frac{dx}{1+x^2} \simeq \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

$$= \frac{1}{2\times 6}[1 + 0.5 + 2(0.973 + 0.9 + 0.8 + 0.6923 + 0.5902)]$$

$$= 0.7842$$

Now, since $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1}(1) = \frac{\pi}{4}$,

so we have

$$\frac{\pi}{4} \simeq 0.7842$$

leading to $\pi \simeq 3.1270$

(i) The composite Simpson's one third rule gives

$$\int_0^1 \frac{dx}{1+x^2} \simeq \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$$

$$= \frac{1}{3\times 6}[1 + 0.5 + 4(0.973 + 0.8 + 0.5902) + 2(0.9 + 0.6923)]$$

$$\simeq 0.7854$$

and hence $\frac{\pi}{4} \simeq 3.1416$ giving

$$\pi \simeq 3.1416$$

(iii) The composite Weddle's rule yields

$$\int_0^1 \frac{dx}{1+x^2} \simeq \frac{3h}{10}[y_0 + y_n + (y_2 + y_4 + \ldots) + 5(y_1 + y_5 + \ldots)$$

$$+ 6(y_3 + y_9 + \ldots) + 2(y_6 + y_{12} + \ldots)]$$

$$= \frac{3}{10\times 6}[1 + 0.5 + (0.9 + 0.6923)$$

$$+ 5(0.973 + 0.5902) + 6\times 0.8]$$

$$\simeq 0.7854$$

and therefore $\frac{\pi}{4} \simeq 0.7854$ so that

$$\pi \simeq 3.1417$$

4.7. A quadrature formula based on Lagrange's interpolation formula.

Let $y = f(x)$ be a real valued function defined and continuous in $[a,b]$ at $n+1$ nodes x_i $(i = 0,1,2,\ldots,n)$. Also let $p(x)$ be the interpolating polynomial of degree less than or equal to n such that $f(x_i) = p(x_i), (i = 0,1,2,\ldots,n)$. Then we have

$$f(x) \simeq p(x)$$

so that

$$\int_a^b f(x)\,dx \simeq \int_a^b p(x)\,dx \qquad \ldots \qquad (18)$$

Now, if $p(x)$ is the Lagrange's polynomial i.e., if

$$p(x) \simeq L_n(x) = \sum_{i=0}^{n} \frac{p_{n+1}(x)}{(x-x_i)p'_{n+1}(x_i)} y_i \qquad \ldots \qquad (19)$$

where $p_{n+1}(x) = (x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)$,

then

$$\int_a^b f(x)dx \simeq \int_a^b \left[\sum_{i=0}^{n} \frac{p_{n+1}(x)}{(x-x_i)p'_{n+1}(x_i)} y_i\right] dx.$$

$$= \sum_{i=0}^{n} y_i H_i^{(n)}, \text{ say} \qquad \ldots \qquad (20)$$

$$\text{where } H_i^{(n)} = \int_a^b \frac{p_{n+1}(x)}{(x-x_i)p'_{n+1}(x_i)} dx \qquad \ldots \qquad (21)$$

The formula (20) is called the quadrature formula based on Lagrange's interpolation formulae.

4.8. Newton- Cote's closed type formula.

Suppose that the interpolating points $x_i (i = 0,1,2,\ldots,n)$ are equispaced of step length $h\left(=\frac{b-a}{n}\right)$ so that $x_i = x_0 + ih$ $(h > 0, i = 0,1,2,\ldots,n)$ and $x_0 = a, x_n = b$. Then from (20) we have

$$\int_{x_0}^{x_n} f(x)dx \simeq \sum_{i=0}^{n} y_i H_i^{(n)} \qquad \ldots \qquad (22)$$

$$\text{Now, } H_i^{(n)} = \int_{x_0}^{x_n} \frac{p_{n+1}(x)}{(x-x_i)p'_{n+1}(x_i)} dx$$

$$= \int_{x_0}^{x_0+nh} \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} dx$$

$$= \int_0^n (-1)^{n-i} \frac{h}{i!(n-1)!} \frac{s(s-1)(s-2)\ldots(s-n)}{s-i} ds$$

[putting $x = x_0 + hs$ so that $x - x_i = (s-i)h$]

$$= \frac{b-a}{n} \frac{(-1)^{n-i}}{i!(n-i)!} \int_0^n \frac{s(s-1)(s-2)\ldots(s-n)}{s-i} ds \quad \left[\because h = \frac{b-a}{n}\right]$$

$$= (b-a)C_i^{(n)}, \text{ say} \qquad \ldots \qquad (23)$$

$$\text{where } C_i^{(n)} = \frac{1}{n} \frac{(-1)^{n-i}}{i!(n-i)!} \int_0^n \frac{s(s-1)(s-2)\ldots(s-n)}{s-i} ds,$$

$$i = 0,1,2,\ldots,n \qquad \ldots \qquad (24)$$

are called *Cote's coefficients*.

Thus from (22), we have

$$\int_a^b f(x)dx \simeq (b-a)\sum_{i=0}^{n} y_i C_i^{(n)} \qquad \ldots \qquad (25)$$

The formulae (25) is called Newton-Cote's integration formula.

Some properties of Cote's coefficients

1. $\sum_{i=0}^{n} C_i^{(n)} = 1$

Proof. Putting $f(x)=1$ in (25) we get

$$\int_a^b f(x)dx = (b-a)\sum_{i=0}^{n} C_i^{(n)}$$

or, $b-a = (b-a)\sum_{i=0}^{n} C_i^{(n)}$

$$\therefore \sum_{i=0}^{n} C_i^{(n)} = 1 \quad \ldots \quad (26)$$

2. $C_i^{(n)} = C_{n-i}^{(n)} \quad \ldots \quad (27)$

Proof: From (24) we have

$$C_i^{(n)} = \frac{1}{n}\frac{(-)^{n-i}}{i!(n-i)!}\int_0^n \frac{s(s-1)(s-2)\ldots(s-n)}{s-i}ds$$

$$= \frac{1}{n}\frac{(-1)^{2n-i}}{i!(n-i)!}\int_0^n \frac{s'(s'-1)(s'-2)\ldots(s'-n)}{s'-(n-i)}ds',$$

putting $s=n-s'$

$$= \frac{1}{n}\frac{(-1)^{i}}{i!(n-i)!}\int_0^n \frac{s'(s'-1)\ldots(s'-n)}{s'-(n-i)}ds'$$

$$= C_{n-i}^{(n)}$$

$$\therefore C_i^{(n)} = C_{n-i}^{(n)}$$

3. Few values of $C_{n-i}^{(n)}$

(i) For $n=1$, we have from (26) and (27)

$$C_0^{(1)} + C_1^{(1)} = 1 \quad \text{and} \quad C_0^{(1)} = C_1^{(1)}$$

leading to $C_0^{(1)} = C_1^{(1)} = \frac{1}{2}$

(ii) For $n=2$, (26) and (27) give

$$C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1 \quad \text{and} \quad C_0^{(2)} = C_2^{(2)} \quad \ldots \quad (28)$$

Also from (24),

$$C_0^{(2)} = \frac{(-1)^{2-0}}{2.0!(2-0)!}\int_0^2 \frac{s(s-1)(s-2)}{s}ds$$

$$= \frac{1}{6}$$

$$\therefore \quad C_0^{(2)} = C_2^{(2)} = \frac{1}{6}$$

$\therefore$ from (28),

$$C_1^{(2)} = \frac{2}{3}$$

(iii) when $n=3$, we get from (26) and (27),

$$C_0^{(3)} + C_1^{(3)} + C_2^{(3)} = 1, \quad C_0^{(3)} = C_3^{(3)}, \quad C_1^{(3)} = C_2^{(3)} \quad \ldots \quad (29)$$

Also one gets from (24)

$$C_0^{(3)} = \frac{(-1)^3}{3\cdot 0!(3-0)!}\int_0^3 \frac{s(s-1)(s-2)(s-3)}{s}ds$$

$$= \frac{1}{8}$$

Thus (29) yields

$$C_0^{(3)} = C_3^{(3)} = \frac{1}{8}, \quad C_1^{(3)} = C_2^{(3)} = \frac{3}{8}$$

Proceeding in this way, we can find out the values of $C_i^{(n)}$ for other values of n.

Some deduction from Newton-Cotes formula

1. Trapezoidal rule [W.B.U.T., CS-312, 2009]

For $n=1$, we have for the interval $[a,b]$, $h=b-a$. Then from (25), we get

$$\int_{x_0}^{x_1} f(x)dx = (b-a)\sum_{i=0}^{1} y_i C_i^{(1)}$$

$$= h\left(y_0 C_0^{(1)} + y_1 C_1^{(1)}\right)$$

$$= h\left(y_0 \frac{1}{2} + y_1 \frac{1}{2}\right)$$

$$= \frac{1}{2}h(y_0 + y_1)$$

which is the Trapezoidal rule for numerical integration.

2. Simpson's one-third rule [W.B.U.T., CS-312, 2006]

Putting $n=2, h=\frac{b-a}{2}$ in (25) we obtain for the interval $[a,b]$,

$$\int_{x_0}^{x_2} f(x)dx \simeq (b-a)\sum_{i=0}^{2} y_i C_i^{(2)}$$

$$= 2h\left(y_0 C_0^{(2)} + y_1 C_1^{(2)} + y_2 C_2^{(2)}\right)$$

$$= 2h\left(y_0 \cdot \frac{1}{6} + y_1 \cdot \frac{4}{6} + y_2 \cdot \frac{1}{6}\right),$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

which is Simpson's one-third rule.

ILLUSTRATIVE EXAMPLES

Ex. 1. Evaluate $\int_0^1 \frac{dx}{1+x}$ by Simpson's composite rule taking eleven ordinates and hence find the value of $\log_e^2$ correct upto five significant figures. [W.B.U.T., CS-312, 2004]

Solution. Let $f(x) = \frac{1}{1+x}$

Here $a=0, b=1, n=10$ [$\because$ number of ordinates is 11]

$\therefore h = \frac{1-0}{10} = 0.1$

The tabulated values of x and $f(x)$ are given below:

x :	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x)$:	1	0.9091	0.8333	0.7692	0.7143	0.6667	0.6250	0.5882	0.5556	0.5263	0.5

Using Simpson's one-third rule, we get

$$\int_0^1 \frac{dx}{1+x} \simeq \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\right]$$

$$= \frac{0.1}{3}\left[1 + 0.5 + 4(0.9091 + 0.7692 + 0.6667 + 0.5882 + 0.5263)\right.$$
$$\left. + 2(0.8333 + 0.7143 + 0.6250 + 0.5556)\right]$$

$$= 0.69314667$$

Now $\int_0^1 \frac{dx}{1+x} = \left[\log(1+x)\right]_0^1 = \log 2$

$\therefore \log 2 \simeq 0.69314667$

$\simeq 0.69315$, correct upto five significant figures.

Ex. 2. Use Simpson's one-third rule to evaluate $\int_0^6 \frac{dx}{(1+x)^2}$, taking six equal sub-intervals of $[0, 6]$, correct to three decimal places.

Solution. Let $f(x) = \frac{1}{(1+x)^2}$

Here $x_0 = 0$, $x_n = 6$, $n = 6$ so that $h = \frac{6-0}{6} = 1$

The tabulated values of $f(x)$ for different values of x are given below:

x :	0	1	2	3	4	5	6
y :	1	0.250	0.111	0.062	0.040	0.028	0.020

$\therefore$ By Simpson's $\frac{1}{3}$rd rule, we have

$$\int_0^6 \frac{dx}{(1+x)^2} = \frac{1}{3}[1 + 0.020 + 4(0.250 + 0.062 + 0.028) + 2(0.111 + 0.040)]$$

$= 0.894$, correct upto three decimal places.

Ex. 3. Evaluate $\int_0^1 \cos x\,dx$, correct upto three significant figures using the data:

x :	0	0.2	0.4	0.6	0.8	1.0
$\cos x$:	1	0.9798	0.9199	0.8228	0.6924	0.5340

Solution. As the number of sub-intervals is odd, so we apply Trapezoidal rule to evaluate the integral.

Here $h = 0.2$

Thus using Trapezoidal rule, we get

$$\int_0^1 \cos x\,dx = \frac{0.2}{2}[(1+0.5340) + 2(0.9798 + 0.9199 + 0.8228 + 0.6924)]$$

$= 0.83638$

≈ 0.836, correct upto three significant figures.

Ex. 4. Evaluate $\int xe^x dx$ where the interval $(-1, 0)$ by using Trapezoidal rule taking $n = 6$ [W.B.U.T., CS-312, 2009]

Solution. Let $f(x) = xe^x$

Since $n = 6$, so $h = \frac{0-(-1)}{6} = \frac{1}{6}$

The tabulated values of $f(x)$ for different values of x are given below :

x :	-1	$\frac{-5}{6}$	$\frac{-2}{3}$	$\frac{-1}{2}$	$\frac{-1}{3}$	$\frac{-1}{6}$	0
$f(x)$:	-0.3678	-0.3622	-0.3405	-0.3033	-0.2388	-0.1411	0

Using Trapezoidal rule, we have

$$\int_{-1}^{0} xe^x dx = \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})]$$

$$= \frac{1}{12}[-0.3678 + 0 + 2(-0.3622 - 0.3405 - 0.3033 - 0.2388 - 0.1411)]$$

$= -0.261633$

≈ -0.2616, correct upto four decimal places.

Ex. 5. Find the value of $\log 2^{1/3}$ from $\int_0^1 \frac{x^2}{1+x^3}dx$, using Simpson's $\frac{1}{3}$rd rule with $h = 0.25$ [W.B.U.T., CS-312, 2003, 2008]

Solution. Let $f(x) = \frac{x^2}{1+x^3}$

Here $h = 0.25$

So the tabulated values of x and $f(x)$ are given below:

x :	0	0.25	0.50	0.75	1
$f(x)$:	0	0.06154	0.22222	0.39560	0.50000

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$\therefore$ By Simpson's $\frac{1}{3}$rd rule we get

$$\int_0^1 \frac{x^2}{1+x^3}dx \simeq \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \dots) + 2(y_0 + y_2 + \dots)\right]$$

$$= \frac{0.25}{3}\left[0 + 0.5 + 4(0.06154 + 0.39560) + 2 \times 0.22222\right]$$

$$= 0.23108$$

Now $\int_0^1 \frac{x^2}{1+x^3}dx = \frac{1}{3}\left[\log(1+x^3)\right]_0^1$

$$= \frac{1}{3}\log 2$$

$$= \log 2^{1/3}$$

Hence $\log 2^{1/3} \simeq 0.23108$

Ex 6. Evaluate $\int_0^{0.6} \frac{dx}{\sqrt{1-x^2}}$, using Weddle's rule taking 12 equal sub-intervals.

Solution. Here $f(x) = \frac{1}{\sqrt{1-x^2}}$, $n = 12$

$$\therefore\ h = \frac{0.6-0}{12} = 0.05$$

The tabulated values of $f(x)$ for various values of x are given below:

x :	0	0.05	0.10	0.15	0.20	0.25	0.30
$f(x)$:	1	1.00125	1.00504	1.01144	1.02062	1.03279	1.04838
	0.35	0.40	0.45	0.50	0.55	0.60	
	1.06752	1.09109	1.11978	1.15470	1.19737	1.25	

By using Weddle's rule, we get

$$\int_0^{0.6} \frac{dx}{\sqrt{1-x^2}} \simeq \frac{3\times 0.05}{10}\left[1 + 1.25 + (1.00504 + 1.02062 + 1.09109 + 1.15470) + 5(1.00125 + 1.03279 + 1.06752 + 1.19737) + 6(1.01144 + 1.11978) + 2\times 1.04828\right]$$

$$= 0.64350$$

Ex 7. Using Weddle's rule, evaluate $\int_0^3 \sqrt{x}\,dx$, taking 12 equal sub-intervals.

Solution: Let $y = \sqrt{x}$.

Here $x_0 = 0$, $x_n = 3$, $n = 12$.

$$\therefore\ h = \frac{3-0}{12} = 0.25$$

We now construct the following table:

x :	0	0.25	0.50	0.75	1.00	1.25	1.50
y :	0	0.5	0.707106	0.866025	1	1.118034	1.224745
	1.75	2.00	2.25	2.50	2.75	3.00	
	1.322876	1.414214	1.5	1.581139	1.658312	1.732051	

$\therefore$ By using Weddle's rule we get

$$\int_0^3 \sqrt{x}\,dx \simeq \frac{3\times 0.25}{10}\left[(0 + 1.732051) + (0.707106 + 1 + 1.414214 + 1.581139) + 5(0.5 + 1.118034 + 1.322876 + 1.658312) + 6(0.866025 + 1.5) + 2\times 1.224745\right]$$

$$= 3.005720$$

$\simeq 3.0057$, correct upto four decimal places.

Ex. 8. Evaluate the integral $\int_0^{\pi/2} \sqrt{1 - 0.162\sin^2\theta}\,d\theta$ by using Weddle's rule, taking $n = 6$.

Solution. Let $f(\theta) = \sqrt{1 - 0.162\sin^2\theta}$

Here $\theta_0 = 0, \theta_n = \frac{\pi}{2}, n = 6$

$\therefore h = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12} = 0.26179$, in radian.

The tabulated values of $f(\theta)$ for different values of θ are given below:

θ :	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(\theta)$:	1	0.99455	0.97954	0.95864	0.93728	0.92133	0.91542

By Weddle rule, we get

$$\int_0^{\pi/2} \sqrt{1 - 0.162\sin^2\theta}\, d\theta \simeq \frac{3 \times 0.26179}{10}[1 + 0.91542 + (0.97954 + 0.93728) + 5(0.99455 + 0.92133) + 6 \times 0.95864]$$

$= 1.50504$

Ex. 9. A curve is drawn to pass through the points given by the following table:

x :	1	1.5	2	2.5	3	3.5	4
y :	2	2.4	2.7	2.8	3	2.6	2.1

Estimate by Simpson's one third rule the area bounded by the curve, the x-axis and the lines $x = 1, x = 4$.

Solution. Here $h = 0.5$

So, by using Simpson's one third rule

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + ...) + 2(y_2 + y_4 + ...)],$$

we have the area bounded by the curve, the x-axis and the lines $x = 1, x = 4$ as

$$\frac{0.5}{3}[2 + 2.1 + 4(2.4 + 2.8 + 2.6) + 2(2.7 + 3)]$$

$= 7.78333$

Hence the area bounded by the curve, x-axis, and the lines $x = 1$ and $x = 4$ is 7.78 sq.units, correct upto three decimal places.

Ex. 10. Find from the following table, the area bounded by the curve and x-axis from $x = 7.47$ to $x = 7.52$.

x :	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$:	1.93	1.95	1.98	2.01	2.03	2.06

[W.B.U.T., CS-312, 2010]

Solution. To find the area bounded by the curve and the x-axis from $x = 7.47$ to $x = 7.52$, we evaluate the integral,

$$\int_{7.47}^{7.52} f(x)dx$$

Here the number of sub-interval is odd. So we should use Trapezoidal rule to evaluate the integral. Thus Trapezoidal rule

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + ... y_{n-1})]$$

gives

$$\int_{7.47}^{7.52} f(x)dx \simeq \frac{0.01}{2}[1.93 + 2.06 + 2(1.95 + 1.98 + 2.01 + 2.03)]$$

$= 0.09965$

$\simeq 0.100$, correct upto three decimal places

Ex 11. The following table gives the values of acceleration (f) of a particle in cm/ sec^2 at equal interval of time (t) in sec. Find the velocity of the particle at $t = 2$ secs.

t :	0.0	0.5	1.0	1.5	2.0
f :	0.3989	0.3521	0.2420	0.1295	0.0540

Solution. The velocity (v) at $t = 2$ secs is given by

$$v = \int_0^2 f(t)dt$$

Since the number of sub-intervals is 4, we use Simpson's one-third rule. Hence we get

$$\int_0^2 f(t)\,dt \simeq \frac{0.5}{3}[0.3989+0.0540+4(0.3521+0.1295)+2\times 0.2420)]$$

$$= 0.4772$$

Ex 12. A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines $x=0, x=1$ and a curve through the points with the following co-ordinates:

x	:	0	0.25	0.50	0.75	1
y	:	1	0.9896	0.9589	0.9089	0.8415

Find the volume of the solid of revolution.

Solution. The required volume of the solid is

$$V=\pi\int_0^1 y^2\,dx$$

We now construct the following table:

x	:	0	0.25	0.50	0.75	1
y^2	:	1	0.97931	0.91949	0.82610	0.70812

Here $h=0.25$. So by using Simpson's one-third rule, we have

$$V=\pi\int_0^1 y^2\,dx$$

$$\simeq \pi\times\frac{0.25}{3}[1+0.7081+4(0.97931+0.82610)+2\times 0.91949]$$

$$= 2.82038$$

Exercise

I. Short Answer Questions

1. Integrate numerically $\int_0^4 x^2\,dx$ by Simpson's $\frac{1}{3}$rd rule with 4 sub-intervals.

2. Find the approximate value of $\int\frac{dx}{1+x}$ when the interval is (0, 1) and $h=1/2$. Use Trapezoidal rule. [*W.B.U.T., CS-312, 2009*]

3. Estimate the value of the following integral by Simpson's $\frac{1}{3}$rd rule taking 4 strips :

$$\int_1^3\frac{1}{x}\,dx$$ [*W.B.U.T., CS-312, 2010*]

4. Is it possible to evaluate $\int_1^4 f(x)dx$ by Simpson's one-third rule by using the following table? Give reasons.

x	:	1	1.6	2.2	2.8	3.4	4
$f(x)$	:	0.5	0.1	0.1	0.4	0.9	0.8

5. Simpson's $\frac{1}{3}$rd rule is exact for $\int_a^b x^3 dx$ – comment.

6. Show that the approximate value of $\int_0^1 x\,dx$ calculated by Trapezoidal rule (taking two intervals of equal lengths) is equal to its exact value.

7. Using Trapezoidal rule, evaluate $\int_0^2\frac{dx}{x^2}$ given :

x	:	0	1/2	1	3/2	2
x^2	:	0	1/4	1	9/4	4

8. Find the value of the integral $\int_0^1 e^x dx$ by Trapezoidal rule with $h=0.1$. [*W.B.U.T., CS-312, 2007*]

9. Calculate the area of the function $f(x)=\sin x$ with limits $(0, 90°)$ by Simpson's $\frac{1}{3}$rd rule using 11 ordinates. [*W.B.U.T., CS-312, 2008*]

10. What type of function is used in

(i) Simpson's $\frac{1}{3}$rd rule (ii) Trapezoidal rule

11. Using Simpson's $\frac{1}{3}$rd rule, estimate the area, bounded by the curve $y = f(x)$, x-axis and the ordinates $x = 0$ and $x = 2$ from the following data:

x	:	0.0	0.5	1.0	1.5	2.0
$f(x)$	:	0.3989	0.3521	0.2420	0.2295	0.0540.

12. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ taking 3 ordinates

13. Compute $\int_0^6 f(x)\,dx$ from the following table by Weddle's rule

x :	0	1	2	3	4	5	6
$f(x)$:	0.14	0.16	0.175	0.195	0.215	0.230	0.250

14. Use Simpson's one-third rule to estimate the value of $\log_e^2$ from $\int_0^1 \frac{dx}{1+x}$ where the number of sub-intervals is 3.

15. Calculate $\int_0^6 x(x+1)\,dx$ by (i) Trapezoidal rule

(ii) Simpson's one-third rule dividing the interval into 6 equal parts.

16. Evaluate $\int_0^3 \sqrt{x}\,dx$ using Trapezoidal rule, taking $n = 3$.

Answers

1. 21.33 2. 0.70835

4. no, because the no. of subintervals is odd

7. 2.75 9. 1 10. (i) parabola (ii) linear

11. 0.4772 12. 1.427 13. 1.170 14. 0.6938

15. (i) 91 (ii) 90 16. 3.2802

II. Long Answer Questions

1. Compute $I = \int \frac{x}{\sin x}\,dx$ where the interval is $\left(0, \frac{1}{2}\right)$ using Simpson's rule with $h = 1/4$. [W.B.U.T., CS-312, 2009]

2. Compute $\int_{1.0}^{1.5} (\ln x)^2\,dx$ by taking 10 equal sub-intervals by Trapezoidal rule, correct upto five significant figures.

3. Compute the value of π from the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$$

using Trapezoidal rule with 10 sub-intervals.
[W.B.U.T., CS-312, 2010]

4. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by Simpson's $\frac{1}{3}$rd rule, taking 6 equal sub-intervals. What is the geometrical significance of this rule. [W.B.U.T., CS-312, 2007]

5. Find the approximate value of $\int_0^1 \frac{x}{1+x^2}\,dx$ upto four places of decimal by Simpson's $\frac{1}{3}$rd rule, taking 6 equal sub-intervals of [0, 1] and hence find the approximate value of log2 correct to four places of decimal.

6. Applying Trapezoidal rule, evaluate $\int_0^1 e^{-x^2}\,dx$ with 10 divisions.

7. Evaluate $\int_0^1 \cos x\,dx$ correct upto three decimal places, by using any suitable formula.

8. Evaluate $\int_0^8 \frac{dx}{1+x}$ by

(i) Trapezoidal rule and (ii) Simpson's $\frac{1}{3}$rd rule, taking 8 equal sub-intervals. Hence estimate the value of $\log_e^3$ in each case.

9. Use Simpson's $\frac{1}{3}$rd rule to evalute $\int_0^6 \frac{dx}{(1+x)^2}$, taking six equal sub-intervals of $[0, 6]$.

10. Evaluate approximately by Trapezoidal rule, the integral $\int_0^1 (4x-3x^2)dx$, by taking $n=10$. Compute the exact integral and hence find the absolute and relative error.

11. Compute $\int_{0.2}^{1.0} x^2(1-x)dx$ by taking step length 0.1 by Simpson's $\frac{1}{3}$rd rule, obtaining the result correct to three significant figures.

12. Find the value of $\int_1^5 \log_{10}^x dx$ correct upto three decimal places by Simpson's one-third rule.

13. Find the value of $\int_0^{\pi/2} \sqrt{\sin x}dx$, taking $n=8$, correct to five significant figures, by using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$rd rule.

14. Compute the value of $\int_{1.2}^{1.6} (x+\frac{1}{x})dx$, correct upto five decimal places by using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$rd rule.

15. Evaluate $\int_{-1}^{3} |x| dx$ analytically and numerically by Trapezoidal and Simpson's $\frac{1}{3}$rd rule, taking four equal sub-intervals.

16. Using Weddle's rule, compute $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$.

17. Find Lagrange's interpolation polynomial passing through the set of points

x	:	0	1	2
y	:	4	3	6

Use it to evaluate $\int_0^3 y\, dx$ [*W.B.U.T., CS-312, 2008*]

18. Evaluate $\int_{4.0}^{5.2} \log_e^x dx$ by Weddle's rule.

19. Estimate the area bounded by the curve $y=f(x)$, x-axis, and the ordinates $x=0$ and $x=5$ from the following data.

x	: 0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5
$y=f(x)$	: 1	2.1	3.2	4.0	5.1	6.2	7.2	8.3	9.0	10.2	12

20. A river is 80 ft wide. The depth d in feet at a distance x ft from one bank is given by the following table:

x	:	0	10	20	30	40	50	60	70	80
d	:	0	4	7	9	12	15	14	8	3

Estimate the area of the cross-section of the river.

Answers

1. 1.6213 **2.** 0.030321 **3.** 3.14156

5. 0.3466, 0.6932 **6.** 0.74621

7. 0.839 **8.** (i) 2.273, 1.136 (ii) 2.210, 1.105

9. 0.849 **10.** 0.995, 1, 0.005, 0.005

11. 0.0833 **12.** 1.757 **13.** (i) 1.1703 (ii) 1.1873

14. 0.84794, 0.84768 **15.** 4.666, 5

16. 1.1872 **18.** 1.827847 **19.** 30.9

20. 710sq.ft (approx)

III. Multiple Choice Questions

1. In evaluating $\int_a^b f(x)dx$, the error in Trapezoidal rule is of order

(a) h^2 (b) h^3 (c) h^4 (d) h

[W.B.U.T., CS-312, 2003, 2006, 2008, 2009]

2. In Simpson's $\frac{1}{3}$rd rule for finding $\int_a^b f(x)dx$, $f(x)$ is approximate by

(a) line segment (b) parabola

(c) circular sector (d) part of ellipse

[W.B.U.T., CS-312, 2010]

3. The truncation error in composite Simpson's one third rule is of order

(a) h^3 (b) h^4 (c) h^5 (d) none

[W.B.U.T., CS-312, 2004]

4. In Trapezoidal rule for finding $\int_a^b f(x)dx, f(x)$ is approximated by

(a) line segment (b) parabola

(c) circular sector (d) none

5. In evaluating $\int_a^b f(x)$ the error in Weddle's rule is of order

(a) h^2 (b) h^3 (c) h^6 (d) none

6. Simpson's $\frac{1}{3}$rd rule requires the interval to be divided into an odd number of sub-intervals

(a) True (b) False

[W.B.U.T., CS-312, 2002]

7. Trapezoidal rule can be applied if the number of equal sub-intervals of the interval of integration is

(a) odd (b) even

(c) both (d) none

8. In Trapezoidal rule for evaluating the approximate value of $\int_a^b f(x)dx$, the area given by this integral is approximated by the sum of area of some

(a) rectangle (b) sectorial figure

(c) trapezium (d) none of these

9. Simpson's one third rule is applicable only if the number of sub-interval is even

(a) Ture (b) False [W.B.U.T. CS-312,2008]

10. In Trapezoidal rule for finding the value of $\int_a^b f(x)dx$ there exists no error if $f(x)$ is

(a) parabolic function (b) linear function

(c) logarithemic function (d) none of these

11. The inherent error for Weddle's rule for integration is as (the notation have their usual meanings).

(a) $-\frac{nh^5}{180}f^{iv}(x_0)$ (b) $-\frac{nh^7}{180}f^{vi}(x_0)$

(c) $-\frac{nh^7}{840}f^{vi}(x_0)$ (d) none of these

[W.B.U.T, CS-312, 2008]

12. In Trapezoidal rule for finding the approximate value of $\int_{12}^{24} f(x)dx$, the error is (when number of sub-interval is 12)

(a) $-f''(\xi)$ (b) $-2f''(\xi)$

(c) $f'(\xi)$ (d) none, where $12 < \xi < 24$

13. The degree of precision of Simpson's one third rule is

(a) 1 (b) 2 (c) 3 (d) 5

[W.B.U.T., CS-312, 2007, 2009]

14. The degree of precision of Trapezoidal rule is

(a) 1 (b) 3 (c) 5 (d) 2

15. The degree of precision of Weddle's rule is

(a) 1 (b) 3 (c) 5 (d) 2

16. In Simpson's one third rule for finding the approximate value of $\int_{12}^{24} f(x)dx$, the error is (when the number of sub-interval is 12)

(a) $-\frac{1}{90}f^{iv}(\xi)$ (b) $-\frac{1}{15}f^{iv}(\xi)$

(c) $-\frac{2}{15}f^{iii}(\xi)$ (d) $-\frac{2}{15}f^{ii}(\xi)$

17. The degree of the approxmating polynomial corresponding to Trapezoidal rule and Simpson's $\frac{1}{3}$ rule are respectively

(a) 1.1 (b) 2.1 (c) 10, 2 (d) 2,2

Answers

1.a 2.c 3.b 4.a 5.b 6.ii 7.c 8.c 9.i 10.b
11.c 12. a 13.c 14.a 15.c 16.b 17.c

5 NUMERICAL SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

5.1 Introduction:

System of linear algebraic equations arise in a large number of problems in science and technology. The most common form of the system in n unknowns $x_1, x_2, ..., x_n$ is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ ... \quad ... \quad &... \\ ... \quad ... \quad &... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_n \end{aligned}$$

$$\text{i.e., } \sum_{j=1}^{n} a_{ij}x_j = b_i, \ (i = 1, 2, ..., n) \qquad ... \quad (1)$$

where $a_{ij}(i, j = 1, 2, ...n)$ and $b_i(i = 1, 2, ...n)$ are given numbers. We can also write the equation (1) in the matrix form as

$$AX = b \qquad ... \quad (2)$$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}]_{n \times n},$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} \qquad ... \quad (3)$$

in which it is supposed that the matrix A is non-singular, i.e., det $A \neq 0$ so that the system (2) has a unique solution. The system of equations (1) is said to be homogeneous if all $b_i(i = 1, 2, ..., n)$, are zero; otherwise, the system is called non-homogeneous.

To solve the above system of equations we apply, in general, two methods viz (i) direct method and (ii) indirect or iterative method. In direct method, the solution is obtained after a finite number of steps of elementary arithmetical operations. On the otherhand, in indirect or iterative method, we start with an arbitrary initial approxmation to x and then improve this estimate in an infinite but convergent sequence of steps.

We discuss in this chapter both the above methods in various ways.

Direct methods.

(i) Gauss elimination method

(ii)Matrix inversion method

(iii) LU Factorization method

Indirect or iterative methods

(i) Gauss-Seidel method

5.2. Gauss elimination method.

In this method, the given system of equations is reduced to an equivalent upper triangular system by a systematic elimination procedure from which the unknowns are found by back substitution.

To illustrate the mehtod, we consider the system (1) given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \qquad \ldots \qquad (4) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ \ldots \qquad \ldots \qquad & \ldots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

First suppose that $a_{11} \neq 0$

Multiply the first equation of (4) by $\dfrac{a_{i1}}{a_{11}} (i = 2, 3, \ldots, n)$ and subtract the results from the i-th equation, $(i = 2, 3, \ldots, n)$ and obtain

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \ldots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + \ldots + a_{3n}^{(1)}x_n &= b_3^{(1)} \qquad \ldots \qquad (5) \\ \ldots \qquad \ldots \qquad & \ldots \\ a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \ldots + a_{nn}^{(1)}x_n &= b_n^{(1)} \end{aligned}$$

where

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{1j}a_{i1}}{a_{11}} \qquad \ldots \qquad (6)$$

$$b_i^{(1)} = b_i - \frac{b_1 a_{i1}}{a_{11}} \quad (i, j = 2, 3, \ldots, n)$$

The numbers $\dfrac{a_{i1}}{a_{11}}, (i = 2, 3, \ldots, n)$ are called row multipliers. The first equation of the system (5) contains x_1 while the remaining $(n-1)$ equations are independent of x_1.

Next assume $a_{22}^{(1)} \neq 0$

Multiplying the second equation of (5) by $\dfrac{a_{i2}^{(1)}}{a_{22}^{(1)}}, (i = 3, 4, \ldots, n)$ and subtracting the results from the i-th equation, $(i = 3, 4, \ldots, n)$ we get

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \ldots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{33}^{(2)}x_3 + \ldots + a_{3n}^{(2)}x_n &= b_3^{(2)} \qquad \ldots \qquad (7) \\ \ldots \qquad \ldots \qquad & \ldots \\ a_{n3}^{(2)}x_3 + \ldots + a_{nn}^{(2)}x_n &= b_n^{(2)} \end{aligned}$$

where

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{2j}^{(1)}a_{i2}^{(1)}}{a_{22}^{(1)}},$$

$$b_i^{(2)} = b_i^{(1)} - \frac{b_2^{(1)} a_{i2}^{(1)}}{a_{22}^{(1)}}, (i, j = 3, 4, \dots, n) \quad \dots \quad (8)$$

Here also the numbers $\frac{a_{i2}^{(1)}}{a_{22}^{(1)}}$ are row multipliers.

In the system (7), the last $(n-2)$ equations are independent of x_1 and x_2

Repeating the procedure, we obtain a system of n equations equivalent to an upper triangular system in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \dots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{33}^{(2)}x_3 + \dots + a_{3n}^{(2)}x_n &= b_3^{(2)} \\ \dots \quad \dots \quad \dots & \\ a_{nn}^{(n-1)}x_n &= b_n^{(n-1)} \end{aligned} \quad \dots \quad (9)$$

The coefficients of the leading terms in (9), i.e., the elements $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \dots, a_{nn}^{(n-1)}$ are called *pivotal elements* and the corresponding equations are known as *pivotal equations*. The solutions of the (4) are then obtained from (9) by back substitutions as

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}} \quad \dots \quad (10)$$

$$x_{n-1} = \frac{1}{a_{n-1,n-1}^{(n-2)}} \left[b_{n-1}^{(n-2)} - \frac{a_{n-1,n}^{(n-2)} b_n^{(n-1)}}{a_{nn}^{(n-1)}} \right]$$

etc., provided none of the pivotal element is zero.

The above procedure can also be explained in a more compact form by matrix notation as following :

The augmented matrix is

$$\begin{array}{c} m\,(multiplier) \\ a_{21}/a_{11} \\ a_{31}/a_{11} \\ \vdots \\ a_{n1}/a_{11} \end{array} \left[\begin{array}{cccc:c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & \dots & a_{3n} & b_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array} \right], \text{ provided } a_{11} \neq 0 \quad (11)$$

After the first elimination, we have

$$\begin{array}{c} m\,(multiplier) \\ \\ a_{32}^{(1)}/a_{22}^{(1)} \\ \vdots \\ a_{n2}^{(1)}/a_{22}^{(1)} \end{array} \left[\begin{array}{cccc:c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ & a_{22}^{(1)} & \dots & a_{2n}^{(1)} & b_2^{(1)} \\ & a_{32}^{(1)} & \dots & a_{3n}^{(1)} & b_3^{(1)} \\ & \dots & \dots & \dots & \dots \\ & a_{n2}^{(1)} & \dots & a_{nn}^{(1)} & b_n^{(1)} \end{array} \right], \text{ provided } a_{22}^{(1)} \neq 0 \quad \dots \quad (12)$$

in which $a_{ij}^{(1)}$ and $b_i^{(1)}, (i = 2, 3, \dots, n)$ are given by (6).

The second elimination gives

$$\left[\begin{array}{ccccc:c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} & b_2^{(1)} \\ & & a_{33}^{(2)} & \dots & a_{3n}^{(2)} & b_3^{(2)} \\ & & \dots & \dots & \dots & \\ & & a_{n3}^{(2)} & \dots & a_{nn}^{(2)} & b_n^{(2)} \end{array} \right] \quad \dots \quad (13)$$

Repeating the process for $(n-1)$ times, we obtain the following upper triangular matrix :

$$\left[\begin{array}{ccccc:c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} & b_2^{(1)} \\ & & a_{33}^{(2)} & \dots & a_{3n}^{(2)} & b_3^{(2)} \\ & & \dots & \dots & \dots & \\ & & \dots & \dots & a_{nn}^{(n-1)} & b_n^{(n-1)} \end{array} \right] \quad \dots \quad (14)$$

which is equivalent to the system (9) and hence by back substitution we get the required solutions of the system.

Note. (1) In Gauss elimination method, the total number of multiplications and divisions is $\frac{n^3}{3}+n^2-\frac{n}{3}$ and those of additions and subtractions is $\frac{n^3}{3}+\frac{n^2}{2}-\frac{5}{6}n$

(ii) The method fails if any of the pivotal elements $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \ldots, a_{nn}^{(n-1)}$ is zero. In such cases, we rearrange the equations in such a way that the pivotal elements do not vanish. If it is not at all possible, then the solution of the given system does not exist.

Example.1. Solve the following system of linear equations by Gauss-elimination method.

$$x-2y+9z=8$$
$$3x+y-z=3$$
$$2x-8y+z=-5 \qquad \text{[W.B.U.T., CS-312 2007, 2008]}$$

Solution. In order to eliminate x from the last two equations, we multiply the first equation successively by 3, 2 and subtract the results from the second and third equations respectively. Thus we have

$$7y-28z=-21 \quad \ldots \quad (1)$$
$$-4y-17z=-21 \quad \ldots \quad (2)$$

In the next step, we eliminate y from (2) by multiplying the equation (1) by $\frac{4}{7}$ and add the result from (2) to get

$$-33z=-33.$$

Thus the given system of equations reduces to the following upper triangular form as

$$x-2y+9z=8$$
$$7y-28z=-21$$
$$-33z=-33$$

from which the back substitution leads to the required solution as

$$x=1, y=1, z=1$$

Example.2. Solve the following system of equations by Gauss elimination method :

$$x+2y+z=0$$
$$2x+2y+3z=3$$
$$-x-3y \quad =2$$

Solution. The augmented matrix of the given system of equations is

$$\begin{array}{c} m\,(multiplier) \\ 2 \\ -1 \end{array} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -3 & 0 & 2 \end{array}\right]$$

. eliminate x from 2nd & 3rd eqn
. " y " 3rd eqn.

Using the row operations R_2-2R_1 and R_3+R_1 we get

$$\begin{array}{c} m\,(multiplier) \\ \\ \frac{-1}{2} \end{array} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

Again using the row operation $R_3-\frac{1}{2}R_2$ we have

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Hence the given system of equations is reduced to the upper triangular form given by

$$\left.\begin{array}{l} x+2y+z=0 \\ -2y+z=3 \\ \frac{1}{2}z=\frac{1}{2} \end{array}\right\}$$

$\therefore$ By back substitution, the resulting solutions are

$$x=1, y=-1, z=1.$$

5.3. Matrix inversion method.

For the system (2) viz.

$$AX = b, \quad \cdots \quad (15)$$

we suppose that $\det A \neq 0$ and so A^{-1} exists. Multiplying both side of (15) by A^{-1}, we get

$$X = A^{-1}b \quad \cdots \quad (16)$$

which gives the solution of the given system

Noting that

$$A^{-1} = \frac{adj\ A}{\det A}$$

$$= \frac{(A_{ji})_{n\times n}}{|a_{ij}|}, (i, j = 1, 2, \ldots, n)$$

we have

$$X = \frac{(A_{ji})_{n\times n}\, b}{|a_{ij}|} \quad \cdots \quad (17)$$

where $adj\ A$ is the transpose of the matrix obtained from A by replacing each element a_{ij} of A by its corresponding co-factor A_{ij} $(i, j = 1, 2, \ldots, n)$.

Note. (i) The method fails if the matrix A is singular i.e., $\det A = 0$

(ii) The method is not suitable for $n > 4$, since it involves laborious numerical computation.

Example.3. Solve the following system of equations :

$$x + y + z = 4$$
$$2x - y + 3z = 1$$
$$3x + 2y - z = 1$$

by matrix inversion method.

Solution. The given system of equations can be written as

$$AX = b \quad \cdots \quad (1)$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Now $\det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{vmatrix} = 13 \neq 0$

Hence A is non-singular.

$\because A^{-1}$ exist.

Since $adj\ A = \begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}$, so

$$A^{-1} = \frac{adjA}{\det A} = \frac{1}{13}\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}$$

$\therefore$ From (1), we have

$$X = A^{-1}b$$

which gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}.$$

Thus the required solutions are

$x = -1, y = 3, z = 2$

5.4. LU-factorization method.

This method is also termed as *triangular decomposition* method. The method based on the fact that every square matrix can be expressed as the product of a lower and an upper triangular matrix provided all the principal minors of the given square matrix $A = (a_{ij})_{n \times n}$ are non-singular, i.e.,

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \ldots, \det A \neq 0 \qquad \ldots \quad (18)$$

Further, if the matrix A can be factorized, then it is unique.

Assume that it is possible to decompose the coefficient matrix A of the given system of equation (2) and is expressible as the product of a lower triangular matrix L and an upper triangular matrix U so that

$$A = LU \qquad \ldots \quad (19)$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 & 0 & \ldots & 0 \\ l_{21} & l_{22} & 0 & 0 & \ldots & 0 \\ l_{31} & l_{32} & l_{33} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ l_{n1} & l_{n2} & l_{n3} & l_{n4} & \ldots & l_{nn} \end{bmatrix}, \qquad \ldots \quad (20)$$

$$V = \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} & \ldots & u_{1n} \\ 0 & 1 & u_{23} & u_{24} & \ldots & u_{2n} \\ 0 & 0 & 1 & u_{34} & \ldots & u_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \qquad \ldots \quad (21)$$

Hence the system of equations

$$AX = b$$

become

$$LUX = b \qquad \ldots \quad (22)$$

Putting $UX = Y$ in (22) we get $\qquad \ldots \quad (23)$

$$LY = b \qquad \ldots \quad (24)$$

where $Y = (y_1, y_2, \ldots, y_n)^T$

Thus by forward substitution, the unknowns $y_1, y_2, \ldots, y_n$ are determined from (24) and thereafter the unknowns $x_1, x_2, \ldots, x_n$ are obtained from (23) by backward substitution.

For the sake of clarity and simplicity we now consider a system of three equations with three unknowns viz

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad \ldots \quad (25)$$

Here the coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

can be written uss

$$A = LU$$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Thus we have

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

leading to

$$l_{11} = a_{11}, l_{21} = a_{21}, l_{31} = a_{31}$$

$$l_{11}u_{12} = a_{12}, l_{11}u_{13} = a_{13} \Rightarrow u_{12} = \frac{a_{12}}{l_{11}}, u_{13} = \frac{a_{13}}{l_{11}},$$

$$l_{21}u_{12} + l_{22} = a_{22}, \Rightarrow l_{22} = a_{22} - l_{21}u_{12} = a_{22} - \frac{a_{21}a_{12}}{a_{11}}$$

$$l_{31}u_{12}+l_{32}=a_{32} \Rightarrow l_{32}=a_{32}-l_{31}u_{12}=a_{32}-\frac{a_{31}a_{12}}{a_{11}}$$

$$l_{21}u_{13}+l_{22}u_{23}=a_{23} \Rightarrow u_{23}=\frac{a_{23}-l_{21}u_{13}}{l_{22}}$$

and $l_{31}u_{13}+l_{32}u_{23}+l_{33}=a_{33}$

$$\Rightarrow l_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}.$$

Thus obtained values of $l_{11}, l_{21}, \ldots$ and $u_{12}, u_{13}, \ldots$ gives the matrices L and U.

Example 4. Solve the following system of equations by LU-factorization method.

$$8x_1-3x_2+2x_3=20$$
$$4x_1+11x_2-x_3=33$$
$$6x_1+3x_2+12x_3=36$$

[*W.B.U.T., CS-312, 2004*]

Solution. The given system of equations can be written as

$$AX=b$$

where $A=\begin{bmatrix}8 & -3 & 2\\ 4 & 11 & -1\\ 6 & 3 & 12\end{bmatrix}, b=\begin{bmatrix}20\\ 33\\ 36\end{bmatrix}, X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

Let $A=LU$

LU = A

where

$L=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}$ and $U=\begin{bmatrix}1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{bmatrix}$

Then,

$$\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}\begin{bmatrix}1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}8 & -3 & 2\\ 4 & 11 & -1\\ 6 & 3 & 12\end{bmatrix}$$

leading to

$l_{11}=8, l_{11}u_{12}=-3 \Rightarrow u_{12}=-\frac{3}{8}$

$l_{11}u_{13}=2 \Rightarrow u_{13}=\frac{2}{8}=\frac{1}{4}$

$l_{21}=4, l_{21}u_{12}+l_{22}=11 \Rightarrow l_{22}=11-l_{21}u_{12}$

$\Rightarrow l_{22}=11-4\left(-\frac{3}{8}\right)$

$\Rightarrow l_{22}=\frac{25}{2}$

$l_{21}u_{13}+l_{22}u_{23}=-1$

$\Rightarrow 4.\frac{1}{4}+\frac{25}{2}.u_{23}=-1 \Rightarrow u_{23}=-\frac{4}{25}$

LY = b.

$l_{31}=6, l_{31}u_{12}+l_{32}=3 \Rightarrow 6\left(-\frac{3}{8}\right)+l_{32}=3$

$\Rightarrow l_{32}=\frac{21}{4}$

$l_{31}u_{13}+l_{32}u_{23}+l_{33}=12$

$\Rightarrow 6.\frac{1}{4}+\frac{21}{4}\left(\frac{-4}{25}\right)+l_{33}=12$

$\Rightarrow l_{33}=\frac{567}{50}$

Hence $L=\begin{bmatrix}8 & 0 & 0\\ 4 & \frac{25}{2} & 0\\ 6 & \frac{21}{4} & \frac{567}{50}\end{bmatrix}, U=\begin{bmatrix}1 & -\frac{3}{8} & \frac{1}{4}\\ 0 & 1 & \frac{-4}{25}\\ 0 & 0 & 1\end{bmatrix}$

Thus the equation $AX=b$ i.e., $LUX=b$ gives

$$LY=b \qquad \ldots (1)$$

where $UX=Y$

i.e., $\begin{bmatrix}1 & -\frac{3}{8} & \frac{1}{4}\\ 0 & 1 & \frac{-4}{25}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} \qquad \ldots (2)$

From (1), we get

$$\begin{bmatrix} 8 & 0 & 0 \\ 4 & \frac{25}{2} & 0 \\ 6 & \frac{21}{4} & \frac{567}{50} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 33 \\ 36 \end{bmatrix}$$

so that

$8y_1 = 20 \Rightarrow y_1 = \frac{5}{2}$

$4y_1 + \frac{25}{2}.y_2 = 33$

$\therefore y_2 = \left(33 - 4.\frac{5}{2}\right)\frac{2}{25} = \frac{46}{25}$

$6y_1 + \frac{21}{4}y_2 + \frac{567}{50}y_3 = 36$

$\Rightarrow y_3 = \frac{50}{567}\left[36 - 15 - \frac{21 \times 23}{50}\right] = 1.$

UX=Y

Then from (2), we have

$$\begin{bmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & \frac{-4}{25} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{46}{25} \\ 1 \end{bmatrix}$$

which gives

$x_3 = 1,$

$x_2 - \frac{4}{25}x_3 = \frac{46}{25}$

$\Rightarrow x_2 = \frac{46}{25} + \frac{4}{25} = 2$

$x_1 - \frac{3}{8}x_2 + \frac{x_3}{4} = \frac{5}{2}$

$\Rightarrow x_1 = \frac{5}{2} + \frac{3}{8}.2 - \frac{1}{4} = 3$

Hence, the required solution is

$$x_1 = 3, x_2 = 2, x_3 = 1.$$

5.5. Gauss-Seidel iteration method.

[*W.B.U.T.,CS-312 2002*]

This method is an improvement of the Gauss-Jacobi method in the sense that the improved values of x_i are used here in each iteration instead of the values of the previous iteration and hence the method is also known as the *method of successive displacements.*

To illustrate the method, we rewrite the system of equations (1) in the following form

$$\begin{aligned} x_1 &= (b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n)/a_{11} \\ x_2 &= (b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n)/a_{22} \quad \ldots \quad (26) \\ \ldots & \quad \ldots \quad \ldots \\ x_n &= (b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{nn-1}x_{n-1})/a_{nn}, \end{aligned}$$

provided $a_{ii} \neq 0, i = 1, 2, \ldots, n$

To solve the equations (26), suppose $x_1^{(0)}, x_2^{(0)}, \ldots x_n^{(0)}$ be the initial approximations (usually $x_i^{(0)}, i = 1$ to n are taken to be zero) of the solutions of (1). We substitute these initial values on the right hand side of the first equation of (26) and get the first approximation of x_1 as

$$x_1^{(1)} = \left(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \ldots - a_{1n}x_n^{(0)}\right)/a_{11}$$

In the second equation of (26), we substitute the improved value $x_1^{(1)}$ and initial values $x_3^{(0)}, x_4^{(0)}, \ldots, x_n^{(0)}$ and obtain the first approximation of x_2 as

$$x_2^{(1)} = \left(b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)} - \ldots - a_{2n}x_n^{(0)}\right)/a_{22}$$

We then substitute in the third equation of (26) the improved values $x_1^{(1)}, x_2^{(1)}$ and the initial values $x_4^{(0)}, x_5^{(0)}, \ldots, x_n^{(0)}$ to obtain the first approximation of x_3 as

$$x_3^{(1)} = \left(b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)} - a_{34}x_4^{(0)} - \ldots - a_{3n}x_n^{(0)}\right)/a_{33}$$

Proceeding in this way, the first approximation of x_n is given by

$$x_n^{(1)} = \left(b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \dots - a_{nn-1}x_{n-1}^{(1)}\right) / a_{nn}$$

Thus at the end of the first stage of iteration, we get the first approximation $x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}$ to the solutions $x_1, x_2, \dots, x_n$.

Now if $x_i^{(k)} (k = 0, 1, 2, \dots)$ be the k^{th} approximation to the solutions $x_i (i = 1, 2, \dots, n)$, then the $(k+1)^{th}$ the approximation $x_i^{(k+1)}$ of $x_i (i = 1, 2, \dots, n)$, are given by

$$x_1^{(k+1)} = \left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \dots - a_{1n}^{(k)}x_n^{(k)}\right) / a_{11}$$

$$x_2^{(k+1)} = \left(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \dots - a_{2n}^{(k)}x_n^{(k)}\right) / a_{22} \quad \dots \quad (27)$$

$$\dots \qquad \dots \qquad \dots$$

$$x_n^{(k+1)} = \left(b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \dots - a_{nn-1}x_{n-1}^{(k+1)}\right) / a_{nn}$$

The process is continued until we get the solutions $x_1, x_2, \dots, x_n$ with sufficient degree of accuracy

The sequence $\left\{x_i^{(k)}\right\}$ generated from (27) can be shown to be convergent to the solution $\left\{x_i^*\right\}$ if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, (i = 1, 2, \dots, n) \quad \dots \quad (28)$$

Hence the Gauss-Seidel iteration method is convergent if the system of equations (1) is *strictly diagonally dominant.*

Note. (1) It may be noted that the strictly diagonally dominant condition may not be necessary in some problems for the convergence of iteration.

(2) The order of convergence of iteration in Gauss-Seidel method is one.

(3) The rate of convergence is faster (roughly twice) than that of Gauss-Jacobi method.

diagonally dominant?

Example 5. Using Gauss-Seidel method find the solution of the following system of linear equations correct upto 2 places of decimal:

$$3x + y + 5z = 13$$

$$5x - 2y + z = 4$$

$$x + 6y - 2z = -1 \qquad \text{[W.B.U.T., CS-312, 2004]}$$

Solution. First we rearrange the given system of equations so that they are diagonally dominant as given below :

$$5x - 2y + z = 4$$

$$x + 6y - 2z = -1$$

$$3x + y + 5z = 13$$

We rewrite the system in the form

$$x = (4 + 2y - z) / 5 \quad \dots \quad (1)$$

$$y = (-1 - x + 2z) / 6 \quad \dots \quad (2)$$

$$z = (13 - 3x - y) / 5 \quad \dots \quad (3)$$

The initial approximations are chosen to be

$$x_1^{(0)} = 0, x_2^{(0)} = 0, x_3^{(0)} = 0$$

First iteration :

Putting $y^{(0)} = 0, z^{(0)} = 0$ in (1), we get $x^{(1)} = 0.8$

Putting $x^{(1)} = 0.8, z^{(0)} = 0$ in (2), we have $y^{(1)} = -0.3$

Putting $x^{(1)} = 0.8, y^{(1)} = -0.3$ in (3) yields $z^{(1)} = 2.18.$

Second iteration :

$$x^{(2)} = \{4 + 2 \times (-0.3) - 2.18\} / 5 = 0.2441$$

$$y^{(2)} = \{-1 - 0.244 + 2 \times 2.18\} / 6 = 0.5192$$

$$z^{(2)} = \{13 - 3 \times 0.244 - 0.519\} / 6 = 2.3497$$

Proceeding as above, the successive iterations are obtained and are shown in the following table :

k	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
0	0	0	0
1	0.8	−0.3	2.18
2	0.2441	0.5192	2.3497
3	0.5377	0.5271	2.1720
4	0.5763	0.4615	2.1628
5	0.552	0.462	2.176
6	0.550	0.467	2.177

Thus the required solutions are

$x = 0.55, y = 0.47, z = 2.18$, correct to two decimal places.

5.6. Computation of Inverse of matrix

Method I

To compute the inverse of a matrix $A = (a_{ij})_{n\times n}$, we determine a matrix $X = (x_{ij})_{n\times n}$ of the same order such that

$$AX = I \qquad \cdots \quad (29)$$

where I is the unit matrix of the same order. So for determination of each element of X, we solve a system of linear equations given by (29). This can be done by a systemetic procedure using Gauss elemination method. We illustrate the technique for a third order matrix

Let us consider the equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is equivalent to the three system of linear equations given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \cdots \quad (30)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then applying Gauss elimination method to each of these system, we get the corresponding column of X, i.e., the inverse of the matrix A^{-1}. But the coefficient matrix of each system of equations are same and so we can solve the three system of equations simultaneously considering the following augmented matrix :

$$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Then employing the same procedure as in Gauss elemination, we can easily solve the three set of the system of equations.

Example.6. Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 4 & -1 \end{bmatrix}$$

Solution. Consider the augmented matrix

$$\left[\begin{array}{ccc:ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 4 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 \\ 0 & 3 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right], \text{(using } R_2 - R_1 \text{ and } R_3 + \tfrac{1}{2}R_1\text{)}$$

$$\sim\left[\begin{array}{ccc:ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 \\ 0 & 0 & \frac{11}{5} & \frac{11}{10} & \frac{-3}{5} & 1 \end{array}\right], \text{(using } R_3 - \frac{3}{5}R_1)$$

Thus we have an equivalent system of three equations given by

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & \frac{11}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ \frac{11}{10} \end{bmatrix} \quad \ldots (1)$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & \frac{11}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -\frac{3}{5} \end{bmatrix} \quad \ldots (2)$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & \frac{11}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \ldots (3)$$

Equation (1) is equivalent to the following system of equations:

$$2x - 2y + 4z = 1$$

$$5y - 2z = -1$$

$$\frac{11}{5}z = \frac{11}{10}$$

Solving by back substitutions, we get

$$x = -\frac{1}{2}, y = 0, z = \frac{1}{2}$$

Similarly solving (2) and (3) we get

$$x = \frac{7}{11}, y = \frac{1}{11}, z = \frac{-3}{11}$$

and $x = \frac{-8}{11}, y = \frac{2}{11}, z = \frac{5}{11}$

Thus $A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{7}{11} & -\frac{8}{11} \\ 0 & \frac{1}{11} & \frac{2}{11} \\ \frac{1}{2} & -\frac{3}{11} & \frac{5}{11} \end{bmatrix}$

Method II. This method is very similar to method I to compute the inverse matrix A^{-1} of the matrix A. Here also we consider the given matrix A with the same order identity matrix simultaneously and convert the matrix A into an identity matrix. As a result, the identity matrix is converted into a matrix which is the inverse of A.

Example 7. Find the inverse of the matrix

$$A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$

[*W.B.U.T., C.S-312, 2007, 2008*]

Solution. Consider the augmented matrix given by

$$\left[\begin{array}{ccc:ccc} 8 & -4 & 0 & 1 & 0 & 0 \\ -4 & 8 & -4 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc:ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{8} & 0 & 0 \\ -4 & 8 & -4 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right], \text{ using } R_1 \to \frac{1}{8}R_1$$

$$\sim\left[\begin{array}{ccc:ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 6 & -4 & \frac{1}{2} & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right], \text{ using } R_2 \to R_2 + 4R_1$$

$$\sim\left[\begin{array}{ccc:ccc} 1 & \frac{-1}{2} & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 1 & \frac{-2}{3} & \frac{1}{12} & \frac{1}{6} & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right], \text{ using } R_2 \to \frac{1}{6}R_2$$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{3} & \frac{1}{6} & \frac{1}{12} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{12} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{16}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right], \text{ using } R_1 \to R_1 + \tfrac{1}{2}R_2$$

$$R_3 \to R_3 + 4R_2$$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{3} & \frac{1}{6} & \frac{1}{12} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{12} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{array}\right], \text{ using } R_3 \to \tfrac{3}{16}R_3$$

$$\sim\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{array}\right], \text{ using } R_1 \to R_1 + \tfrac{1}{3}R_3$$

$$R_2 \to R_2 + \tfrac{2}{3}R_3$$

Hence the required inverse matrix is

$$A^{-1} = \begin{bmatrix} \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Illustrative Examples

Ex.1. Solve the following system of equations by LU factoriztion method

$$2x - 6y + 8z = 24$$
$$5x + 4y - 3z = 2$$
$$3x + y + 2z = 16$$

[W.B.U.T.,CS-312, 2009]

Solution. The given system of equations can be written as

$$AX = b \quad \ldots (1)$$

where $A = \begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$

Also let $A = LU$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Thus

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix}$$

gives $l_{11} = 2, l_{21} = 5, l_{31} = 3$, $u_{12} = \frac{-6}{2} = -3, u_{13} = \frac{8}{2} = 4$

$l_{22} = 4 - 5 \times (-3) = 19$, $l_{32} = 1 - 3 \times (-3) = 10$

$u_{23} = \dfrac{-3 - 5 \times 4}{19} = \frac{-23}{19}$, $l_{33} = 2 - 3 \times 4 - 10 \times \frac{-23}{19} = \frac{40}{19}$

$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 19 & 0 \\ 3 & 10 & \frac{40}{19} \end{bmatrix}, U = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -\frac{23}{19} \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ From (1)

$$AX = b \text{ i.e., } LUX = b \text{ gives}$$
$$LY = b \quad \ldots (2)$$

where $UX = Y$

i.e., $\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & \frac{-23}{19} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \ldots (3)$

From (2), we have,

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 19 & 0 \\ 3 & 10 & \frac{40}{19} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

i.e., $2y_1 = 24$

$$5y_1 + 19y_2 = 2$$

$$3y_1 + 10y_2 + \frac{40}{19}y_3 = 16$$

whose solutions are

$$y_1 = 12, y_2 = -\frac{58}{19}, y_3 = 5$$

Thus from (3) we get

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & \frac{-23}{19} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ \frac{-58}{19} \\ 5 \end{bmatrix}$$

gives $x - 3y + 4z = 12$

$$y - \frac{23}{19}z = -\frac{58}{19}$$

$$z = 5$$

whose solutions by backward substitutions are

$x = 1, y = 3, z = 5$ which are the required solutions.

Ex.2. Find the solutions of the following system of equations by LU-factorization method

$$2x - 3y + 10z = 3$$

$$-x + 4y + 2z = 20$$

$$5x + 2y + z = -12$$

Solution. The given equations can be written as

$$AX = b$$

where $A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$

Let $A = LU$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Then

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

leading to $l_{11} = 2, l_{21} = -1, l_{31} = 5, u_{12} = \frac{-3}{2}$,

$u_{13} = \frac{10}{2} = 5, l_{21}u_{21} + l_{22} = 4 \Rightarrow l_{22} = 5/2$

$l_{31}u_{12} + l_{32} = 2 \Rightarrow l_{32} = \frac{19}{2}$

$l_{21}u_{13} + l_{22}u_{23} = 2 \Rightarrow u_{23} = \frac{14}{5}$

and

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 1$

$\Rightarrow l_{33} = \frac{-253}{5}$

Hence

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \frac{5}{2} & 0 \\ 5 & \frac{19}{2} & \frac{-253}{5} \end{bmatrix}, U = \begin{bmatrix} 1 & \frac{-3}{2} & 5 \\ 0 & 1 & \frac{14}{5} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $AX = b$ i.e., $LUX = b$

gives

$LY = b$ where $UX = Y$

i.e., $\begin{bmatrix} 1 & \frac{-3}{2} & 5 \\ 0 & 1 & \frac{14}{5} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

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Now from the equation $LY = b$, we have

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & \frac{5}{2} & 0 \\ 5 & \frac{19}{2} & \frac{-253}{5} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

so that

$$2y_1 = 3$$

$$-y_1 + \frac{5}{2}y_2 = 20$$

$$5y_1 + \frac{19}{2}y_2 - \frac{253}{5}y_3 = -12$$

whose solutions are

$$y_1 = \frac{3}{2}, y_2 = \frac{43}{5}, y_3 = 2$$

Then from the equation $UX = Y$ we get

$$\begin{bmatrix} 1 & \frac{-3}{2} & 5 \\ 0 & 1 & \frac{14}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{43}{5} \\ 2 \end{bmatrix}$$

gives $x - \frac{3}{2}y + 5z = \frac{3}{2}$

$$y + \frac{14}{5}z = \frac{43}{5}$$

$$z = 2$$

whose solutions by backward substitutions are

$x = -4, y = 3, z = 2$

which are the required solutions.

Ex.3. Solve the system of equations by Gauss-Seidel method :

$$3x + 4y + 15z = 54.8$$

$$x + 12y + z = 39.66$$

$$10x + y - 2z = 7.74$$

[W.B.U.T.,CS-312, 2007, 2008]

Solution. Obviously the coefficient matrix of the given system of equations is not diagonally dominant.We therefore rearrange the given equation as

$$10x + y - 2z = 7.74$$

$$x + 12y + z = 39.66$$

$$3x + 4y + 15z = 54.8.$$

Now we rewrite the system as

$$x = (7.74 - y + 2z)/10$$

$$y = (39.66 - x - z)/12$$

$$z = (54.8 - 3x - 4y)/15$$

Taking $x^{(0)} = y^{(0)} = z^{(0)} = 0$ as the initial approximation, the first approxmations to the solutions are

$$x^{(1)} = (7.74 - 0 + 2 \times 0)/10 = 0.774$$

$$y^{(1)} = (39.66 - 0.774 - 0)/12 = 3.2405$$

$$z^{(1)} = (54.8 - 3 \times 0.774 - 4 \times 3.2405)/25$$

$$= 2.6344$$

Proceeding as above the successive iterations are obtained and are shown in the following table :

k	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
0	0	0	0
1	0.774	3.2405	2.6344
2	0.97683	3.00406	2.65688
3	1.00497	2.99985	2.65238
4	1.00449	3.00026	2.65237

The solutions of the given system of equations are

$x = 1.004$, $y = 3.000$, $z = 2.652$, correct upto 3 decimal places.

Ex.4. Solve the following system of equations by Gauss-Seidel's iteration method

$$10x + 2y + z = 9$$
$$x + 10y - z = -22$$
$$-2x + 3y + 10z = 22$$

[W.B.U.T.,CS-312, 2010]

Solution. Clearly the given system of equations is diagonally dominant. We now rewrite the equations in the form

$$x = (9 - 2y - z)/10$$
$$y = (-22 + z - x)/10$$
$$z = (22 + 2x - 3y)/10$$

Let the initial approximation be $x^{(0)} = y^{(0)} = z^{(0)} = 0$.

Thus the first approximation of the solutions is given by

$$x^{(1)} = (9 - 2\times 0 - 0)/10 = 0.9$$
$$y^{(1)} = (-22 + 0 - 0.9)/10 = -2.29$$
$$z^{(1)} = (22 + 2\times 0.9 + 3\times 2.29)/10 = 3.067$$

Proceeding as above, the successive iterations are obtained and are shown in the following table :

k	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
0	0	0	0
1	0.9	−2.29	3.067
2	1.0513	−1.9984	3.0098
3	0.9987	−1.9989	2.9994
4	0.9998	−2.0000	3.0000
5	1.0000	−2.0000	3.0000

Thus the required solutions are

$$x = 1, y = -2, z = 3$$

Ex.5. Solve the following system of equations by Gauss-elimination method :

$$5x_1 - x_2 = 9$$
$$-x_1 + 5x_2 - x_3 = 4$$
$$-x_2 + 5x_3 = -6$$

[W.B.U.T.,CS-312, 2009]

Solution. Multiplying the first equation by $\frac{1}{5}$ and then adding the result with the second equation we get

$$5x_1 - x_2 = 9$$
$$\frac{24}{5}x_2 - x_3 = \frac{29}{5}$$
$$-x_2 + 5x_3 = -6$$

Then multiplying the second equation by $\frac{5}{24}$ and add the result with the 3rd equation we have

$$5x_1 - x_2 = 9$$
$$\frac{24}{5}x_2 - x_3 = \frac{29}{5}$$
$$\frac{115}{24}x_3 = \frac{-115}{24}$$

$\therefore$ By back substitution, the required solutions are $x_1 = 2,\ x_2 = 1, x_3 = -1$.

Ex.6. Solve the following system of equations by using Gauss-elimination method

$$3x + 2y + 4z = 19$$
$$2x + 7y - 5z = 1$$
$$x - 8y + 9z = 12$$

Solution. We multiply the first equation successively by $\frac{2}{3}, \frac{1}{3}$ and subtract the results from the second and third equations respectively. Thus we have

$$3x+2y+4z = 19$$
$$\frac{17}{3}y-\frac{23}{3}z=-\frac{35}{3}$$
$$-\frac{26}{3}y+\frac{23}{3}z=\frac{17}{3}$$

Next we multiply the second equations by $\frac{26}{17}$ and add the result to the 3rd equation we get

$$3x+2y+4z = 19$$
$$\frac{17}{3}y-\frac{23}{3}z=-\frac{35}{3}$$
$$-\frac{69}{17}z=-\frac{207}{17}$$

∴ By back substitution the resulting solutions are

$$x=1, y=2, z=3$$

Ex.7. Find the inverse of the matrix

$$A=\begin{bmatrix}2&1&1\\3&2&3\\1&4&9\end{bmatrix}$$

Solution. Consider the augmented matrix as

$$\left[\begin{array}{ccc:ccc}2&1&1&1&0&0\\3&2&3&0&1&0\\1&4&9&0&0&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc:ccc}2&1&1&1&0&0\\0&\frac{1}{2}&\frac{3}{2}&-\frac{3}{2}&1&0\\0&\frac{7}{2}&\frac{17}{2}&-\frac{1}{2}&0&1\end{array}\right], \text{ using } \begin{array}{l}R_2\to R_2-\frac{3}{2}R_1\\ R_3\to R_3-\frac{1}{2}R_1\end{array}$$

$$\sim\left[\begin{array}{ccc:ccc}2&1&1&1&0&0\\0&\frac{1}{2}&\frac{3}{2}&-\frac{3}{2}&1&0\\0&0&-2&10&-7&1\end{array}\right], \text{ using } R_3\to R_3-7R_2$$

Thus we have an equivalent system of three equations given by

$$\begin{bmatrix}2&1&1\\0&\frac{1}{2}&\frac{3}{2}\\0&0&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-\frac{3}{2}\\10\end{bmatrix}$$

$$\begin{bmatrix}2&1&1\\0&\frac{1}{2}&\frac{3}{2}\\0&0&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\1\\-7\end{bmatrix}$$

$$\begin{bmatrix}2&1&1\\0&\frac{1}{2}&\frac{3}{2}\\0&0&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Solving these system of equations by back substitution we get the corresponding column of A^{-1}. Thus

$$A^{-1}=\begin{bmatrix}-3&\frac{5}{2}&-\frac{1}{2}\\12&-\frac{17}{2}&\frac{3}{2}\\-5&\frac{7}{2}&\frac{-1}{2}\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}-6&5&-1\\24&-17&3\\-10&7&-1\end{bmatrix}$$

Ex.8. Find the inverse of the matrix

$$\begin{bmatrix}4&1&2\\2&3&-1\\1&-2&2\end{bmatrix}$$

Solution. Consider the augmented matrix given by

$$\left[\begin{array}{ccc:ccc}4&1&2&1&0&0\\2&3&-1&0&1&0\\1&-2&2&0&0&1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right], \text{ operating } R_1 \to \frac{1}{4}R_1$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{5}{2} & -2 & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{9}{4} & \frac{3}{2} & -\frac{1}{4} & 0 & 1 \end{array}\right], \text{ operating } \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & -\frac{4}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & \frac{-9}{4} & \frac{3}{2} & -\frac{1}{4} & 0 & 1 \end{array}\right], \text{ operating } R_2 \to \frac{2}{5}R_2$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{7}{10} & \frac{3}{10} & -\frac{1}{10} & 0 \\ 0 & 1 & -\frac{4}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & -\frac{3}{10} & -\frac{7}{10} & \frac{9}{10} & 1 \end{array}\right], \text{ operating } ,\begin{array}{l} R_1 \to R_1 - \frac{1}{4}R_2 \\ R_3 \to R_3 + \frac{9}{4}R_2 \end{array}$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{7}{10} & \frac{3}{10} & -\frac{1}{10} & 0 \\ 0 & 1 & -\frac{4}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 & \frac{7}{3} & -3 & -\frac{10}{3} \end{array}\right], \text{ operating } R_3 \to -\frac{10}{3}R_3$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{3} & 2 & \frac{7}{3} \\ 0 & 1 & 0 & \frac{5}{3} & -2 & -\frac{8}{3} \\ 0 & 0 & 1 & \frac{7}{3} & -3 & -\frac{10}{3} \end{array}\right], \text{ operating } \begin{array}{l} R_1 \to R_1 - \frac{7}{10}R_3 \\ R_2 \to R_2 + \frac{4}{5}R_3 \end{array}$$

Thus the required inverse matrix is

$$\begin{bmatrix} -\frac{4}{3} & 2 & \frac{7}{3} \\ \frac{5}{3} & -2 & -\frac{8}{3} \\ \frac{7}{3} & -3 & -\frac{10}{3} \end{bmatrix}$$

$$\text{i.e. } \frac{1}{3}\begin{bmatrix} -4 & 6 & 7 \\ 5 & -6 & -8 \\ 7 & -9 & -10 \end{bmatrix}$$

Exercise

I. SHORT ANSWER QUESTIONS

1. What is meant by diagonally dominant matrix ?
2. Express Gauss Seidel method for a system of three linear equations in three unknowns.
3. Does Gauss seidel method always perform better than Gauss-Jacobi method ?
4. How is the solution obtained in Gauss elemination method ?
5. State sufficient condition for convergence of Gauss-Seidel method.
6. Solve
$$4x + 3y = 20.91$$
$$3x - y = 6.94$$
by Gauss elimination method.
7. Solve by Gauss elimination method
$$4.69x + 7.42y = 17.4$$
$$3x + 11.3y = 23.2$$
8. Solve
$$2x + 3y = 2.03$$
$$x + y = 7.8$$
by Gauss-Seidel method
9. Solve
$$3x + 2y = 9.8$$
$$2x + y = 5.5$$
by Gauss elimination method.
10. Solve by LU-factorization method
$$x + 3y = 5$$
$$7x + 2y = -3$$

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11. Find the inverse of the following matrix

(i) $\begin{bmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Answers

6. $x = 3.21, 7 = 2.69$ **7.** $x = 0.796,\ y = 1.84$

8. [Hints : The coefficient matrix is not diagonally dominant and cannot be written in diogonally dominant form by an rearrangement. So the system of equations cannot be solved by Gauss-seidel method]

9. $x = 1.2, y = 3.1$ **10.** $x = -1, y = 2$

11. (i) $\frac{1}{5}\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & -2 \\ 1 & -1 & 2 \end{bmatrix}$

II. Long Answer Questions

1. Solve the following system of equations using Gauss-elemination method

(i) $x+y+z=9$

$2x-3y+4z=13$

$3x+4y+5z=40$

(ii) $8x-7y+4z=32$

$x+5y-3z=28$

$-2x+2y+7z=19$

(iii) $5x-y-z=3$

$-x+10y-2z=7$

$-x-y+10z=8$

(iv) $2x_1-3x_2+10x_3=3$

$-x_1+4x_2+2x_3=20$

$5x_1+2x_2+x_3=-12$

(v) $5x_1-x_2 \qquad =3$

$-x_1+5x_2-x_3=4$

$-x_2+5x_3=-6$ [W.B.U.T.,CS-312, 2009]

2. Solve the following system of equations by matrix inversion method :

(i) $x+2y+3z=7$

$2x+7y+15z=26$

$3x+15y+41z=62$

(ii) $3x+y+2z=3$

$2x-3y-z=-3$

$x+2y+z=4$

(iii) $3x+2y-z+\omega=1$

$x-y-2z+4w=3$

$2x-3y+z-2w=-2$

$5x-2y+3z+2w=0$

3. Solve the following system of equations by LU-factorization method :

(i) $x+3y+z=9$

$x+4y+2z=3$

$x+2y-3z=6$ [W.B.U.T., CS-312, 2010]

(ii) $3x+4y+2z=15$

$5x+2y+z=18$

$2x+3y+2z=10$ [W.B.U.T.,CS-312, 2003]

(iii) $x_1+x_2-x_3=2$

$2x_1+3x_2+5x_3=-3$

$3x_1+2x_2-3x_3=6$ [W.B.U.T.,CS-312, 2006]

(iv) $2x+y+z=3$

$x+3y+z=-2$

$x+y+4z=-6$ [W.B.U.T.,CS-312, 2008]

(v) $5x-y-z=3.245$

$x+4y+z=7.075$

$x+y+3z=8.870$

4. Solve the following system of equations by using Gauss-Seidel method :

(i) $x+y+54z=110$

$27x+6y-z=85$

$6x+15y+2z=72$ [W.B.U.T.,CS-312, 2009]

(ii) $10x+2y+z=9$

$2x+20y-2z=-44$

$-2x+3y+10z=22$ [W.B.U.T.,CS-312, 2010]

(iii) $10x_1-x_2-x_3=13$

$x_1-10x_2+x_3=36$

$x_1+x_2-10x_3=-35$ [W.B.U.T.,CS-312, 2002]

(iv) $9x_1-2x_2+x_3=50$

$x_1+5x_2-3x_3=18$

$-2x_1+2x_2+7x_3=19$ [W.B.U.T., CS-312, 2006]

(v) $10x+y-z=12$

$2x+10y-z=13$

$2x+2y-10z=14$

5. Find the inverse of the following matrix

(i) $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ [W.B.U.T.,CS-312, 2009]

(ii) $\begin{pmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{pmatrix}$ [W.B.U.T.,CS-312, 2004]

(iii) $\begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ [W.B.U.T., CS-312, 2007]

(iv) $\begin{pmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{pmatrix}$

Answers

1. (i) $x=1, y=3, z=5$ (ii) $x=6.15, y=4.31, z=3.24$

(iii) $x=7, y=3, z=1$ (iv) $x_1=-3.86, x_2=2.98, x=1.89$

(v) $x_1=2, x_2=1, x_3=-1$

2. (i) $x=2, y=z=1$ (ii) $x=1, y=2, z=-1$

(iii) $x=\frac{19}{50}, y=\frac{-29}{50}, z=\frac{-51}{50}, w=0$

3. (i) $x=33, y=-9, z=3$ (ii)

(iii) $x=1, y=0, z=-1$ (iv) $x=\frac{27}{17}, y=\frac{-29}{17}, z=\frac{26}{17}$

(v) $x=1.274, y=0.891, z=2.235$

4. (i) $x=2.4255, y=3.5730, z=1.9260$

(ii) $x=1, y=-2, z=3$ (iii)

(iv) $x_1=6.15, x_2=4.32, x_3=3.24$

(v) $x=1, y=3, z=-1$

5. (i) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (iii) $\begin{pmatrix} 2 & -1 & 1 \\ -3 & 2 & -2 \\ 1 & -1 & 2 \end{pmatrix}$ (iv) $\frac{1}{5}\begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

13. A system of equations $AX = b$ where $A = (a_{ij})_{n \times n}$ is said to be diagonally dominant if

(a) $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$ for all i (b) $|a_{ii}| < \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$ for all i

(c) $|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|$ for all i (d) $|a_{ii}| < \sum_{j=1}^{n} |a_{ij}|$ for all i

14. The iterative method is known as

(a) direct method (b) indirect method

(c) none of these

15. The solution of a system of equations is obtained by successive approximation method is known as

(a) direct method (b) indirect method

(c) both (a) and (b) (d) none of these

Answers

1.d	**2.a**	**3.b**	**4.a**	**5.b**	**6.a**	**7.a**	**8.b**	**9.b**	**10.b**
11.a	**12.a**	**13.a**	**14.b**	**15.b**					

6 NUMERICAL SOLUTIONS OF ALGEBRAIC EQUATIONS

6.1. Introduction.

In applied mathematics and engineering we frequently face the problem of finding one or more roots of the equation

$$f(x) = 0 \qquad \cdots \quad (1)$$

where $f(x)$ is, in general, a nonlinear function of the real variable x. But in most cases, it is very difficult to have explicit solutions of the equation (1) and, therefore, we proceed to look for a root of (1) numercially with any specified degree of accuracy. The numerical methods of finding these roots are called *iterative methods.*

The function $f(x)$ may have any one of the following forms:

(i) $f(x)$ is an algebraic or pclynomial function of degree n, say, so that

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

where $a_i (i = 0,1,2,\cdots,n)$ are constants, real or complex, and $a_n \neq 0$. For example, $x^3 - 7x + 1$, $x^{12} + x^5 - 4x + 3$ etc. are algebraic functions. In such cases, the equation $f(x) = 0$ is called *algebraic equation.*

(ii) $f(x)$ is a transcendental function, i.e. $f(x)$ is a function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots to\ \infty,$$

where $a_i, (i = 0,1,2,\cdots)$, are constants, real or complex, and all $a_i \neq 0$. For example, $\sin x + 2x^2 - 1$, $e^x + \log x + 5$ etc. are transcendental functions, Here the equation $f(x) = 0$ is called *transcendental equation.*

Every value α of x for which the function $f(x)$ is zero, i.e., $f(\alpha) = 0$ is called a root or zero of the equation (1). In this chapter we shall discuss different numerical methods to compute the approximate real roots of an algebraic or transcendental equation $f(x) = 0$

To develop the methods we assume that

(i) The function $f(x)$ is continuous and continuously differentiable for a sufficient number of times.

(ii) $f(x)$ has no multiple root, i.e., if α is a real root of $f(x)=0$ then

$$f(\alpha)=0,\ f'(\alpha)\neq 0.$$

Determination of approximate (real) root of (1) by numerical methods to be discussed here, consists, in general, of the following two steps.

(i) Isolating the roots, i.e., finding the smallest possible interval $[a, b]$ containing one and only one root of (1).

(ii) Improving the values of the approximate roots, i.e refinding them to the desired degree of accuracy.

To implement the first step, we use the following theorem of a continuous function :

Theorem 1. If real valued function $f(x)$ is continuous in $[a, b]$ and $f(a), f(b)$ are of opposite signs, then there is at least one real root of $f(x)=0$ in (a, b)

6.2. Iteration Processes.

Let the sequence $\{x_n\}$ of iterates of a root α of the equation

$$f(x)=0$$

is produced by a given method. Then the error ε_n involved at the n th iteration is given by

$$\varepsilon_n=\alpha-x_n. \qquad \ldots \quad (2)$$

If $\varepsilon_n\to 0$ as $n\to\infty$, we say that the iteration converges and the sequence $\{x_n\}$ converges to α. Otherwise, the iteration is divergent and the method of computation fails. Thus our primary task is to find the *condition of convergence of the iteration processes.* The error ε_{n+1} can be expressed in terms of $\varepsilon_n, \varepsilon_{n-1}, \varepsilon_{n-2}, \cdots$ which we call *error equation.*

If we define h_n by $h_n=x_{n+1}-x_n=\varepsilon_n-\varepsilon_{n+1}$, then h_n is an approximation of ε_n if x_{n+1} approximates α. If the iteration converges, then we can find an upper bound for $|\varepsilon_{n+1}|$ in terms of h_n. This is called *estimation of error.*

In case an iterative method converges, we can find two constants $p\geq 1$ and $q>0$ such that

$$\lim_{n\to\infty}\left|\frac{\varepsilon_{n+1}}{\varepsilon_n^p}\right|=q \qquad \ldots \quad (3)$$

Here p is called the *order of convergence* and q is known as *asymptotic error constant.* The iterative method with $p>1$ generally converges rapidly.

The convergence of the iterative method also depends on the initial approximation x_0 of the root α. If this initial approximation is not satisfactory, the iterative method does not converge and then we look for the new computation.

The iterative method is self correct, i.e. if there is an accidental error in the calculations of iteration, the erroneous iterate acts as a new initial approximation leading to a correct result, provided that the error is not large enough for which the method fails.

6.3. Bisection Method.

A. *Basic principle and formula*

The method of bisection is the most simplest iterative method. It is also known as *half-interval* or *Bolzano method.* This method is based on Theorem 1 on the change of sign.

In this method, we first find out a sufficiently small interval $[a_0, b_0]$ containing the required root α of the equation (1). Then $f(a_0)f(b_0)<0$ and $f'(x)$ has the same sign in $[a_0, b_0]$ and so $f(x)$ is strictly monotonic in $[a_0, b_0]$

To generate the sequence $\{x_n\}$ of iterates, we put $x_0=a_0$ or b_0 and $x_1=\frac{1}{2}(a_0+b_0)$ and find $f(x_1)$. If $f(a_0)$ and $f(x_1)$ are of opposite signs, then set $a_1=a_0, b_1=x_1$ so that $[a_1, b_1]=[a_0, x_1]$. On the other hand, if $f(x_1)$ and $f(b_0)$ are of opposite signs then put $a_1=x_1, b_1=b_0$, i.e. $[a_1, b_1]=[x_1, b_0]$. Thus we see that $[a_1, b_1]$ contains the root α in either case.

Next set $x_2 = \frac{1}{2}(a_1 + b_1)$ and repeat the above process till we obtain

$$x_{n+1} = \frac{1}{2}(a_n + b_n) \quad \ldots \quad (4)$$

with desired accuracy with $x_n \to \alpha$ as $n \to \infty$.

B. *Convergence of bisection method*

Suppose the interval $[a_n, b_n]$ contains the root α and $f(a_n)f(b_n) < 0$.

Let $x_{n+1} = \frac{1}{2}(a_n + b_n)$.

If $f(a_n)f(x_{n+1}) < 0$, then set $a_{n+1} = a_n, b_{n+1} = x_{n+1}$.

On the other hand, if $f(x_{n+1})f(b_n) < 0$, then

$$a_{n+1} = x_{n+1}, b_{n+1} = b_n.$$

Thus in any case

$$\alpha \in [a_{n+1}, b_{n+1}],\ f(a_{n+1})f(b_{n+1}) < 0$$

and

$$b_{n+1} - a_{n+1} < \frac{1}{2}(b_n - a_n) < \cdots < \frac{b_0 - a_0}{2^n}.$$

If ε_{n+1} be the error in approximating α by x_{n+1}, then

$$\varepsilon_{n+1} = |\alpha - x_{n+1}| \le b_n - a_n < \frac{b_0 - a_0}{2^n} \to 0 \text{ as } n \to \infty.$$

Thus the iteration converges.

Since $\frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{1}{2}$, so the convergence in bisection method is linear.) finds only 1 root even if there exist more root in interval).

C. *Advantage and disadvantage of bisection method*

Advantage. This method is very simple, as at any stage of iteration the approximate value of the desired root of the

equation $f(x) = 0$ does not depend on the values $f(x_n)$ but on their signs only. Also the method is unconditionally and surely convergent.

Disadvantage. The method is very slow and requires large number of iteration to obtain moderately accurate results and hence it is laborious.

Example.1. Find the root of the equation $x \tan x = 1.28$, that lies in the interval $(0,1)$, correct to four places of decimal, using bisection method. [*W.B.U.T., CS-312,2005*]

Solution. Let $f(x) = x \tan x - 1 \cdot 28$.

$\frac{180 \times 1}{\pi}$

$\therefore f(0) = -1 \cdot 28 < 0, f(1) = 0 \cdot 277408 > 0$.

So a root lies between 0 and 1.

Take $a_0 = 0$, $b_0 = 1$ so that $x_1 = \frac{1}{2}(0+1) = 0.5$. Since $f(0.5) = -1.006849 < 0$ and $f(1) > 0$, the root lies between 0.5 and 1. Thus we have $x_2 = \frac{1}{2}(0.5 + 1) = 0.75$.

Proceeding in this way, we obtain the following table :

No. of iteration (n)	a_n $f(a_n) < 0$	b_n $f(b_n) > 0$	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	0	1	0.5	−1.006849
1	0.5	1	0.75	−0.581303
2	0.75	1	0.875	−0.232256
3	0.875	1	0.9375	−0.003058
4	0.9375	1	0.968750	0.129819
5	0.9375	0.96875	0.953125	0.061675
6	0.9375	0.961675	0.945312	0.028898
7	0.9375	0.945312	0.941406	0.012819
8	0.9375	0.941406	0.939453	0.004856
9	0.9375	0.939453	0.938477	0.000893
10	0.9375	0.938477	0.937988	−0.001084

0.000001

11	0.937988	0.938477	0.938232	−0.000096
12	0.938232	0.938477	0.938354	0.000398
13	0.938232	0.938354	0.938293	0.000151
14	0.938232	0.938293	0.938263	0.000028

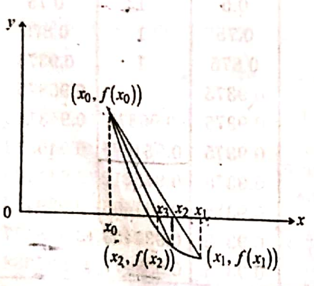
Thus a real root of the given equation is 0.9383 correct to four decimal places.

6.4 Regula-Falsi Method.

A. *Basic principle and formula* [*W.B.U.T., CS-312,2003*]

The regula-falsi method or false position method is also sometimes referred to as the method of *linear interpolation* and it is the oldest method for computing real roots of an equation $f(x)=0$.

To find a real root α of $f(x)=0$, we first choose a sufficiently small interval $[x_0,x_1]$ in which the root α lies. Then $f(x_0)$ and $f(x_1)$ must be of opposite signs so that $f(x_0)f(x_1)<0$ and the graph of $f(x)$ must cross the x-axis between $x=x_0$ and $x=x_1$. Since the interval $[x_0,x_1]$ is sufficiently small, the portion of the curve between $A[x_0,f(x_0)]$ and $B[x_1,f(x_1)]$ can be approximated by a secant line (straight line) and so the intersection of the secant AB with the x-axis gives an approximate value x_2, say, of the root.



The equation of the secant line AB is

$$y-f(x_1)=\frac{f(x_0)-f(x_1)}{x_0-x_1}(x-x_1)$$

Putting $y=0, x=x_2$, we derive

$$x_2=x_1-\frac{f(x_1)}{f(x_1)-f(x_0)}(x_1-x_0) \quad \ldots (5)$$

If $f(x_2)=0$, then x_2 is a root of $f(x)=0$; otherwise, $f(x_2)<0$ or $f(x_2)>0$. If $f(x_0)$ and $f(x_2)$ are of opposite signs the root lies between x_0 and x_2 and in this case we set $x_1=x_0$ and $x_2=x_1$. On the other hand if $f(x_1)$ and $f(x_2)$ are of opposite signs, the root lies between x_1 and x_2 and thus, in either case,

$$f(x_1)f(x_2)<0$$

Hence the next approximation of the root, say x_3 lies between x_1 and x_2 and get

$$x_3=x_2-\frac{f(x_2)}{f(x_2)-f(x_1)}(x_2-x_1) \quad \ldots (6)$$

b = a − f(a) (b − a)

The general formula based on the above process is

$$x_{n+1}=x_n-\frac{f(x_n)}{f(x_n)-f(x_{n-1})}(x_n-x_{n-1}),\ n=1,2,\ldots \quad \ldots (7)$$

This is *regula falsi iteration formula*. The process is repeated until the root is obtained to required degree of accuracy.

B. Convergence of regula falsi method.

Let α be a simple root of the equation $f(x)=0$. Then putting $x_n=\alpha+\varepsilon_n$ in (7), we get

$$\varepsilon_{n+1}=\varepsilon_n-\frac{(\varepsilon_n-\varepsilon_{n-1})f(\alpha+\varepsilon_n)}{f(\alpha+\varepsilon_n)-f(\alpha+\varepsilon_{n-1})}$$

$$= \varepsilon_n - \frac{(\varepsilon_n - \varepsilon_{n-1})\left[\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + \cdots\right]}{\left[\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + \cdots\right] - \left[\varepsilon_{n-1} f'(\alpha) + \frac{1}{2}\varepsilon_{n-1}^2 f''(\alpha) + \cdots\right]}$$

(Expanding f in Taylor's series and noting $f(\alpha) = 0$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + \cdots}{f'(\alpha) + \frac{1}{2}(\varepsilon_n + \varepsilon_{n-1}) f''(\alpha) + \cdots}$$

$$= \varepsilon_n - \varepsilon_n\left[1 + \frac{1}{2}\varepsilon_n \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right] \times \left[1 + \frac{1}{2}(\varepsilon_n + \varepsilon_{n-1})\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$= \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\varepsilon_{n-1}\varepsilon_n + O\left(\varepsilon_n^2\varepsilon_{n-1} + \varepsilon_n\varepsilon_{n-1}^2\right), \quad \ldots (8)$$

so that

$$\varepsilon_{n+1} = C\,\varepsilon_{n-1}\varepsilon_n \quad \ldots (9)$$

where $C = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$ and we have neglect higher power of ε_n

The relation (9) is called the error equation.

To find the order of convergence, we set $\varepsilon_{n+1} = A\varepsilon_n^m$ and $\varepsilon_n = A\varepsilon_{n-1}^m$ where the constants A and m are to be determined. Then the equation (9) gives

$$A\varepsilon_n^m = C A^{-\frac{1}{m}}\varepsilon_n^{\frac{1}{m}}\varepsilon_n = CA^{-\frac{1}{m}}\varepsilon_n^{1+\frac{1}{m}} \quad \ldots (10)$$

$$\Rightarrow m = 1 + \frac{1}{m}$$

$$\Rightarrow m = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

Neglecting the minus sign, we find that the order of the convergence of $\{x_n\}$ is $m = 1\cdot618$. Also from (10) we get

$$A = C^{m/m+1}$$

C. Advantage and disadvantage of regula falsi method. [W.B.U.T., CS-312,2003]

Advantage. The method is very simple and does no required to calculate the derivative of $f(x)$ which is difficult for some problems. Moreover, the method is evidently convergent.

Disadvantage. Sometimes the method is very slow and not suitable for practical computation. Also the initial interval in which the root lies is to be chosen very small.

Example.2. Find a root of the equation $x^3 - 2x - 5 = 0$ by Regula-Falsi method correct upto 4 places of decimal [W.B.U.T. CS-312,2004]

Solultion. Let $f(x) = x^3 - 2x - 5$.

$\therefore f(0) = -5, f(1) = -6, f(2) = -1, f(3) = 16$. So a real root lies between 2 and 3. We choose $x_0 = 2, x_1 = 3$ giving $f(x_0) = -1$ and $f(x_1) = 16$. Then the iteration (7) gives

$$x_2 = 3 - \frac{16}{16 - (-1)}(3 - 2) = 2\cdot05882 \text{ and } f(x_2) = -0\cdot39082$$

Proceeding in this way, the iteration (7) gives the following table :

No, of iteration (n)	x_{n-1} $(f(x_{n-1})<0)$	x_n $(f(x_n)<0)$	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	2	3	−1	16	2.05882	−0.39084
2	2.05882	3	−0.39084	16	2.08126	−0.147244
3	2.08126	3	−0.147244	16	2.08964	−0.054667
4	2.08964	3	−0.054667	16	2.09274	−0.020198
5	2.09274	3	−0.020198	16	2.09388	−0.007491
6	2.09388	3	−0.007491	16	2.09430	−0.002806
7	2.09430	3	−0.002806	16	2.09445	−0.001133
8	2.09445	3	−0.001133	16	2.094451	

Hence 2.0944 is a root of the given equation correct upto four decimal places

6.5. Newton-Raphson method.

A. *Basic principle and formula* [W.B.U.T., CS-312,2002]

Let x_0 be an initial approximation of the desired root α of the equation $f(x)=0$ and $x_1 = x_0 + h$ is the correct root

$$\therefore f(x_1)=0$$

i.e., $f(x_0+h)=0, (\min(x_1,x_0) < h < \max(x_1,x_0))$

i.e., $f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots=0$,

by Taylor's series, expansion

Neglecting the second and higher order terms, we obtain

$$f(x_0)+hf'(x_0)=0,$$

i.e., $h=-\frac{f(x_0)}{f'(x_0)}$

Thus a better approximation of the root α is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \quad \cdots \quad (11)$$

Repeating the above process and replacing x_0 by x_1 we obtain the second approximation of the root as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Proceeding in this way, we get the successive approximations $x_3, x_4, \cdots, x_{n+1}$ where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \cdots \quad (12)$$

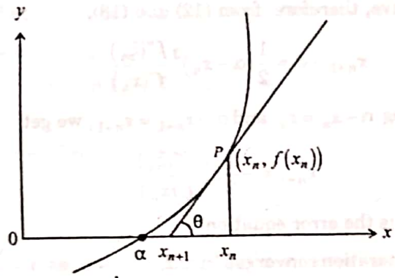
provided $f'(x_n) \neq 0, n = 1,2,\cdots$

The result (11) is known as *Newton-Raphson iteration formula.*

If $[a,b]$ be the initial interval in which the root α of the given equation $f(x)=0$ lies and $f'(x)\neq 0$, then the initial approximation may be started with $x_0 = a$ or b.

B. Geometrical meaning of Newton-Raphson formula

Let the curve $y=f(x)$ cuts the x–axis at the point $x=\alpha$ so that α is a root of the equation $f(x)=0$.



If the tangent to the curve at the point $P(x_n, f(x_n))$ cuts the x–axis at the point $x = x_{n+1}$ and is inclined at an angle θ with the positive direction of the x–axis, then

$$f'(x_n) = \tan\theta = \frac{f(x_n)}{x_n - x_{n+1}}$$

so that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Accordingly, the Newton-Raphson method may also be called the *method of tangents.*

C. *Convergence of Newton-Raphson method.* [W.B.U.T., CS-312,2003,2007]

Let α be a root of the equation $f(x)=0$

$$\therefore f(\alpha)=0$$

i.e., $f(x_n+\alpha-x_n)=0$

i.e., $f(x_n)+(\alpha-x_n)f'(x_n)+\frac{(\alpha-x_n)^2}{2}f''(\xi_n)=0$

$[\min|\alpha,x_n| < \xi_n < \max|\alpha,x_n|]$, by Taylor's theorem

$$\therefore -\frac{f(x_n)}{f'(x_n)} = (\alpha - x_n) + \frac{1}{2}(\alpha - x_n)^2 \frac{f''(\xi_n)}{f'(x_n)} \quad \cdots \quad (13)$$

We have, therefore, from (12) and (13),

$$x_{n+1} - \alpha = \frac{1}{2}(\alpha - x_n)^2 \frac{f''(\xi_n)}{f'(x_n)}$$

Putting $\alpha - x_n = \varepsilon_n$ and $\alpha - x_{n+1} = \varepsilon_{n+1}$, we get

$$\varepsilon_{n+1} = -\frac{1}{2}\varepsilon_n^2 \frac{f''(\xi_n)}{f'(x_n)} \quad \cdots \quad (14)$$

which is the error equation.

If the iteration converges then $x_n, \xi_n \to \alpha$ as $n \to \infty$ so that

$$\boxed{\lim_{n\to\infty} \left|\frac{\varepsilon_{n+1}}{\varepsilon_n^2}\right| = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}} \quad \cdots \quad (15)$$

Rate of convergence

Hence Newton–Raphson method is a second order iteration process. So the convergence is quadratic and the constant asymptotic, error is equal to $\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$

D. *Advantage and disadvantage of Newton–Raphson method.*

Advantage. The rate of convergence of this method is quadratic. So the method converges more rapidly than other numerical method.

Disadvantage. In this method, the initial approximation must be chosen very close to the root; otherwise, the method will fail. Since the method depends on the derivative $f'(x)$, it may not be suitable for a function $f(x)$ whose derivative is difficult to compute. Also the method fails if $f'(x) = 0$ or small in the neighbourhood of the root.

Note. When $f'(x)$ is large in the neighbourhood of the real root, i.e., when the graph of function $y = f(x)$ is nearly vertical, then it crosses the x-axis. In this case, the method is very useful and the correct value of the root can be obtained more rapidly.

Example.3. Find the smallest positive root of the equation $3x^3 - 9x^2 + 8 = 0$ correct to four places decimal, using Newton–Raphson method. [*W.B.U.T., CS-312,2009*]

Solution. Let $f(x) = 3x^3 - 9x^2 + 8$

$\therefore f'(x) = 9x^2 - 18x$.

Then the iteration formula (12) gives

$$x_{n+1} = x_n - \frac{3x_n^3 - 9x_n^2 + 8}{9x_n^2 - 18x_n}$$

$$= \frac{6x_n^3 - 9x_n^2 - 8}{9x_n^2 - 18x_n}, n = 0, 1, 2, \cdots \quad \cdots \quad (1)$$

Now $f(0) = 8$, $f(1) = 2$, $f(2) = -4$. So a positive root lies between 1 and 2. Choose $x_0 = 1$

$\therefore$ From (1)

$$x_1 = \frac{6x_0^3 - 9x_0^2 - 8}{9x_0^2 - 18x_0} = \frac{6\times1^3 - 9\times1^2 - 8}{9\times1^2 - 18\times1} = 1\cdot22222$$

$$x_2 = \frac{6(1\cdot22222)^3 - 9(1\cdot22222)^2 - 8}{9(1\cdot22222)^2 - 18\times1\cdot222222} = 1\cdot22607$$

$$x_3 = \frac{6(1\cdot22607)^3 - 9(1\cdot22607)^2 - 8}{9(1\cdot22607)^2 - 18\times1\cdot22607} = 1\cdot22607$$

Hence positive real root of the given equation correct to four decimal places is 1.2261.

E. *Newton-Raphson method for finding an assigned root of a positive real number.* [W.B.U.T., CS-312,2009]

Suppose we are to find the m^{th} root of a real number R.

So let $x = \sqrt[m]{R}$

$\therefore x^m = R$

i.e., $x^m - R = 0$

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Let $f(x) = x^m - R$

$\therefore f'(x) = mx^{m-1}$.

The Newton–Raphson iteration formula (13) gives

$$x_{n+1} = x_n - \frac{x_n^m - R}{mx_n^{m-1}}$$

$$= \frac{(m-1)x_n^m + R}{mx_n^{m-1}}, n = 0, 1, 2, \cdots \quad \cdots \quad (16)$$

with $|x_{n+1} - x_n| < \varepsilon$, ε being the desired degree of accuracy.

It is obvious from (17) that for finding the square root of any positive number R (where $m = 2$), we have

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right), \ (n = 0,1,2,\cdots).$$

Example.4. Find the cube root of 10 upto 5 significant figures by Newton–Raphson method.

Solution. Let $x = \sqrt[3]{10}$

$\therefore x^3 - 10 = 0$

Let $f(x) = x^3 - 10$.

$\therefore f'(x) = 3x^2$

Now $f(0) = -10, f(1) = -9, f(2) = -2, f(3) = 17$ so that a real root of $f(x) = 0$ lies between 2 and 3. Hence using the iteration formula (16) with $m = 3$, we have

$$x_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}, \ n = 0,1,2,\cdots$$

which gives with $x_0 = 2$

$$x_1 = \frac{2x_0^3 + 10}{3x_0^2} = 2.16666$$

Similarly, $x_2 = \frac{2(2.16666)^3 + 10}{3(2.16666)^2} = 2.15450$.

$x_3 = 2.15443,$

$x_4 = 2.15443$

Hence we have $x = \sqrt[3]{10} \simeq 2.1544$ correct to five significant figures.

ILLUSTRATIVE EXAMPLES

Ex 1. Find a real root of the transcendental equation $x^x + 2x - 2 = 0$, correct upto two decimal places using bisection method.

Solution. Let $f(x) = x^x + 2x - 2$.

Then $f(0\cdot5) = -0\cdot293, f(1) = 1$.

So a real root lies between 0.5 and 1.

Take $a_0 = 0\cdot5, b_0 = 1$ so that $x_1 = \frac{0\cdot5+1}{2} = 0\cdot75$.

Again, since $f(0\cdot75) = 0\cdot3059$, so the root lies between 0.75 and 1. Proceeding in this way, we construct the following table :

No. of iteration(n)	a_n $(f(a_n) < 0)$	b_n $(f(b_n) > 0)$	$x_{n+1} = \frac{1}{2}(a_n + b_n)$	$f(x_{n+1})$
0	0.5	1	0.75	0.3059
1	0.5	0.75	0.625	–0.0045
2	0.625	0.75	0.687	0.1466
3	0.625	0.687	0.656	0.0704
4	0.625	0.656	0.640	0.0325
5	0.625	0.640	0.632	0.0041
6	0.625	0.632	0.628	0.0036
7	0.625	0.628	0.626	

Then the root correct upto two decimal places is 0.63.

Ex 2. Find the smallest positive root of the equation $e^x = 4\sin x$ correct to four decimal places by bisection method.

Solution. The given equation is

$$4\sin x - e^x = 0.$$

Let $f(x) = 4\sin x - e^x$.

Since $f(0) = -1$, $f(1) = 0.64760$, so the smallest positive root lies between 0 and 1. Take $a_0 = 0, b_0 = 1$ so that

$$x_1 = \frac{1}{2}(a_0 + b_0) = 0.5.$$

Since $f(0.5) = 0.26898$, so the root lies between 0 and 0.5. Proceeding in this way, we construct the following table: (shown in the next page)

No. of iteration (n)	a_n $(f(a_n)<0)$	b_n $(f(b_n)>0)$	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	0	1	0.5	0.26898
1	0	0.5	0.25	−0.29440
2	0.25	0.5	0.375	0.01010
3	0.25	0.375	0.33333	−0.08690
4	0.33333	0.375	0.35416	−0.03770
5	0.35416	0.375	0.36458	−0.01360
6	0.36458	0.375	0.36979	−0.00168
7	0.36979	0.375	0.37240	0.00420
8	0.36979	0.37240	0.37109	0.00126
9	0.36979	0.37109	0.37044	−0.00240
10	0.37044	0.37109	0.37077	0.00047
11	0.37044	0.37077	0.37060	0.00011
12	0.37044	0.37060	0.37052	−0.00007
13	0.37052	0.37060	0.37056	

Thus the required smallest positive root is 0.3706 correct to four decimal places.

Ex. 3. Find a root of the equation $x\sin x + \cos x = 0$ using Newton-Raphson method correct upto 5 places of decimal. [*W.B.U.T.,CS-312, 2004*]

Solution. Let $f(x) = x\sin x + \cos x$

$\therefore f'(x) = x\cos x + \sin x - \sin x = x\cos x$

So the Newton-Raphson iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, ...$$

gives

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$$

$$= x_n - \frac{1}{x_n} - \tan x_n \quad ... (1)$$

Now $f(0) = 1, f(1) = 1.38, f(2) = 1.4, f(3) = -0.56$

So a root lies between 2 and 3.

Choose $x_0 = 2.5$, for quick convergence.

Then we have, from (1),

$$x_1 = 2.5 - \frac{1}{2.5} - \tan 2.5$$

$$= 2.847022$$

$$x_2 = 2.847022 - \frac{1}{2.847022} - \tan 2.847022$$

$$= 2.799175$$

Similarly $x_3 = 2.798386$

$x_4 = 2.798386$

Hence a positive real root of the given equation correct to five decimal places is 2.79839.

Ex.4. Find out the root of the following equation using Regula falsi method $x^3 - 5x - 7 = 0$ that lies between 2 and 3, correct to 4 decimal places. [*W.B.U.T.CS-312, 2006, 2009*]

Solution. Let $f(x) = x^3 - 5x - 7$

We choose $x_0 = 2, x_1 = 3$ so that $f(x_0) = -9, f(x_1) = 5$. Then the regula-falsi iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})}(x_n - x_{n-1}), n = 1, 2, \ldots \quad \ldots (1)$$

gives

$$x_2 = 3 - \frac{5}{5-(-9)}(3-2)$$

$$= 2.642857$$

Proceeding in this way, the iteration formula (1) gives the following table :

No of iteration (n)	x_{n-1} $(f(x_{n-1})<0)$	x_n $(f(x_n)<0)$	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	2	3	–9	5	2.642857	–1.754740
2	2.642857	3	–1.754740	5	2.735635	–0.205506
3	2.735635	3	–0.205506	5	2.746072	–0.022474
4	2.746072	3	–0.022474	5	2.747208	–0.002444
5	2.747208	3	–0.002444	5	2.747332	–0.000257
6	2.747332	3	–0.000257	5	2.747345	–0.000027

Hence a real root of the given equation correct to four decimal places is 2.7473.

Ex. 5. Find the root of the equation $xe^x - 3 = 0$ that lies between 1 and 2, correct to 4 significant figure using the method of False position.

Solution. Let $f(x) = xe^x - 3$

Here we choose $x_0 = 1, x_1 = 2$ so that $f(x_0) = -0.2817$,

$f(x_1) = 11.7781$

So the iteration formula for false position method

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})}(x_n - x_{n-1}), \quad n = 1, 2, \ldots.$$

gives

$$x_2 = 2 - \frac{11.7781}{11.7781 - (-0.2817)}(2-1)$$

$$= 1.02336$$

Proceeding in this way, we obtain the following table :

No of iteration (n)	x_{n-1} $(f(x_{n-1})<0)$	x_n $(f(x_n)>0)$	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	1	2	–0.2817	11.7781	1.02336	–0.15247
2	1.02336	2	–0.15247	11.7781	1.03584	–0.08155
3	1.03584	2	–0.08155	11.7781	1.04247	–0.04333
4	1.04247	2	–0.04333	11.7781	1.04598	–0.02295
5	1.04598	2	–0.02295	11.7781	1.04802	–0.01105
6	1.04802	2	–0.01105	11.7781	1.04891	

Thus the required root correct to four significant figure is 1.049.

Ex. 6. Find the root of the equation $e^x = 2x + 1$ correct to 4 places of decimal, using Newton-Raphson method near $x = 1$. [*W.B.U.T., CS-312, 2005*]

Solution. The given equation is

$$e^x = 2x + 1$$

i.e.., $e^x - 2x - 1 = 0$

Let $f(x) = e^x - 2x - 1$

$\therefore f'(x) = e^x - 2$

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So the Newton-Raphson iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \ldots$$

gives

$$x_{n+1} = x_n - \frac{e^{x_n} - 2x_n - 1}{e^{x_n} - 2}$$

$$= \frac{x_n e^{x_n} - e^{x_n} + 1}{e^{x_n} - 2} \quad \ldots \quad (1)$$

For quick convergence, we choose $x_0 = 1.5$

From (1), we get

$$x_1 = \frac{1.5e^{1.5} - e^{1.5} + 1}{e^{1.5} - 2} = 1.305903$$

$$x_2 = \frac{1.305903e^{1.305903} - e^{1.305903}}{e^{1.305903} - 2}$$

$$= 1.259059$$

Similarly $x_3 = 1.25906$

Hence the root of the given equation near $x = 1$ is 1.25906.

Ex. 7. Using Newton - Raphson method, find the value of $\sqrt[4]{12}$.

Solution. Let $x = \sqrt[4]{12}$

$\therefore x^4 = 12$

So let $f(x) = x^4 - 12$

$\therefore f'(x) = 4x^3$

Then the Newton - Raphson iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2,\cdots$$

gives

$$x_{n+1} = x_n - \frac{x_n^4 - 12}{4x_n^3}$$

$$= \frac{3x_n^4 + 12}{4x_n^3}, \quad n = 0,1,2,\cdots$$

Now $f(0) = -12 < 0, f(1) = -11 < 0, f(2) = 4 > 0$

3.2401
2.4817

So a real root of $f(x) = 0$ lies between 1 and 2.

Taking $x_0 = 1.5$ for rapid convergence, we get

$$x_1 = \frac{3x_0^4 + 12}{4x_0^3}$$

$$= \frac{3(1 \cdot 5)^4 + 12}{4(1 \cdot 5)^3}$$

$$= 2.0138$$

$$x_2 = \frac{3(2 \cdot 0138)^4 + 12}{4(2 \cdot 0138)^3}$$

$$= 1.8777$$

Similarly, $x_3 = 1.8614$, $x_4 = 1.8612$.

Hence $\sqrt[4]{12} \simeq 1.861$, correct to four significant figures.

Ex. 8. Using Newton - Raphson method, obtain iteration formula for the reciprocal of a number N and hence find the value of $\frac{1}{22}$, correct to three significant figures.

Solution. Let $\frac{1}{N} = x$

Then $\frac{1}{x} - N = 0$.

Let $f(x) = \frac{1}{x} - N$

$\therefore f'(x) = -\frac{1}{x^2}$

So the Newton - Raphson iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2,\cdots$$

gives

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}$$

$$= x_n + x_n(1 - Nx_n)$$

$$= (2 - Nx_n)x_n, \quad n = 0, 1, 2, \cdots \qquad \cdots (1)$$

which is the required iterative formula.

As the value of $\frac{1}{22}$ lies between $\frac{1}{25}$ and $\frac{1}{20}$.

i.e. between 0.04 and 0.05, so we take $N = 22, x_0 = 0.04$.

Then, from (1),we have

$$x_1 = (2 - 22 \times 0.04)0.04 = 0.0448$$

$$\therefore \quad x_2 = (2 - 22 \times 0.0448)0.0448 = 0.04544.$$

$$x_3 = 0.04545$$

$$x_4 = 0.04545.$$

Thus the required value of $\frac{1}{22}$ is 0.0454, correct to three significant figures.

Ex. 9. Evaluate $\sqrt{12}$ to three places of decimal by Newton-Raphson method [*W.B.U.T., CS-312, 2007*]

Solution. Let $x = \sqrt{12}$

$$\therefore x^2 - 12 = 0$$

Let $f(x) = x^2 - 12$

$\therefore f'(x) = 2x$

So the Newton-Raphson's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots.$$

gives

$$x_{n+1} = x_n - \frac{x_n^2 - 12}{2x_n}$$

$$= \frac{x_n^2 + 12}{2x_n}, \quad n = 0, 1, 2, \ldots.$$

Now $f(3) = -3$, $f(4) = 4$

So a root lies between 3 and 4.

Choose $x_0 = 3$.

$$\therefore x_1 = \frac{3^2 + 12}{2 \times 3} = 3.5$$

$$\therefore x_2 = \frac{(3.5)^2 + 12}{2 \times 3.5} = 3.46421$$

Similarly, $x_3 = 3.46410$

$x_4 = 3.46410$

$\therefore \sqrt{12} = 3.464$, correct upto 3 places of decimal.

Exercise

I. Short Answer Questions

1. Explain bisection me· .od
2. Show that the convergence of bisection method is linear
3. Discuss the advantage and disadvantage of bisection method
4. Find out an expression of the inherent error in Newton-Raphson method. [*W.B.U.T., CS-312, 2003*]
5. Show that Newton-Raphson method has quadratic rate of convergence. [*W.B.U.T.,CS-312, 2003, 2007*]
6. Give a geometrical interpretation of Newton Raphson method
7. Evaluate $\sqrt{12}$ to three places of decimal by Newton-Raphson method. [*W.B.U.T., CS-312, 2003*]
8. Discuss the advantage and disadvantage of Regula-Falsi method. [*W.B.U.T., CS-312, 2003*]
9. Find the positive root of $x^2 - \sin x = 0$ correct upto three significant digits by the method of false position.

10. Interpret Regula-Falsi method geometrically [W.B.U.T., CS-312, 2003]

Answers

7.3.464 9.0.877

II. Long Answer Questions

1. Compute a real root of the following equations by bisection method correct to five significant figures :

(a) $x^3-3x-5=0$

(b) $x^4-x-10=0$

(c) $x^3-9x+1=0$ in $[2, 3]$

(d) $\cos x = xe^x$

(e) $\tan x + x = 0$

(f) $3x-\cos x-1=0$ in $[0,1]$

2. Using the method of bisection to compute a root of $x^3-4x-1=0$ between 2 and 3 upto four significant digits.

3. Find the positive real root of $x^3-x^2-1=0$ using the bisection method of 4 iterations. [W.B.U.T., CS-312, 2010]

4. Using the method of bisection to compute a root of $x^3-x-1=0$ correct upto two significant digits. [W.B.U.T.,CS-312, 2000]

5. Use Regula-Falsi method to evaluate the smallest real root of each of the following equations

(a) $x^3+x^2-1=0$ (b) $x^3-4x+1=0$

(c) $2x^3-3x-6=0$ (d) $xe^x=\cos x$

(e) x^3-3x-5 [W.B.U.T.,CS-312, 2003]

6. Find out the root of the following equation using Regula Falsi method:

$3x-\cos x-1=0$ that lies between 0 and 1 correct to four decimal places) [W.B.U.T., CS-312, 2007,2008]

7. Using Newton-Raphson method, find a real root of the following equations correct to three decimal places

(i) $x^4-x-1=0$ (ii) $3x^2+2x-9=0$

(iii) $2x-3\sin x-5=0$ (iv) $x+e^x=0$

8. Construct a iterative formula to evaluate the following using Newton-Raphson method and hence evaluate

(i) $\sqrt[9]{15}$ (ii) $\sqrt[7]{125}$ (iii) $\sqrt[3]{21}$ (iv) $\sqrt[4]{13}$

9. Prove that Newton-Raphson's iteration formula for $\sqrt{N}$ is

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{N}{x_n}\right).$$ Hence find $\sqrt{21}$.

10. Find the Newton-Raphson iteration formula to find the p^{th} root of positive number N and hence find the cube root of 17.. [W.B.U.T.,CS-312, 2009]

11. Prove that Newton-Raphson's iteration formula for $\frac{1}{N}$ is $x_{n+1}=x_n(2-Nx_n)$. Hence find the value of $\frac{1}{13}$ correct upto two significant figures.

12. Find the root of $e^x-x=0$, near $x=0$ correct to three significant figures. [W.B.U.T.,CS.312, 2002]

Answers

1. (a) 2.278 (b) 1.855 (c) 2.953 (d) 0.5177
   (e) 2.0289 (f) 0.6071

2. 2.687 3. 1.467 4. 1.3

5. (a) 0.7548 (b) 0.254 (iii) 0.732 (d) 0.5177

6. 0.6071 7. (i) 1.221 (ii) 1.430 (iii) 2.883 (iv) –0.567

8. (i) 1.35106 (ii) 1.993 (iii) 2.7589 (iv) 1.8988

10. 2.5113

16. Newton-Raphson's iterative formula for finding the square root of a positive real number R is

(a) $x_{n+1} = x_n + \frac{R}{x_n}$ (b) $x_{n+1} = x_n - \frac{R}{x_n}$

(c) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$ (d) $x_{n+1} = \frac{1}{2}\left(x_n - \frac{R}{x_n}\right)$

17. Regula-Falsi method is

(a) Conditionally convergent (b) linearly convergent

(c) divergent (d) none of these

[W.B.U.T.,CS-312, 2009]

18. Regula Falsi method used for finding the real roots of a numerical equation is

(a) an analytical method (b) graphical method

(c) iterative method (d) none of these

[W.B.U.T.,CS-312, 2004, 2006]

19. Newton-Raphson method even does not fail when $f'(x) = 0$ in the neighbourhood of the real root

(a) True (b) False

20. If $f(a_0)f(b_0) < 0$ and $a_0 < b_0$, then the first approximation of one of the roots of $f(x) = 0$ by Regula-Falsi method is

(a) $\frac{a_0 f(b_0) + b_0 f(a_0)}{f(b_0) + f(a_0)}$ (b) $\frac{a_0 f(a_0) - b_0 f(b_0)}{f(a_0) - f(b_0)}$

(c) $\frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)}$ (d) $\frac{a_0 f(a_0) + b_0 f(b_0)}{f(a_0) + f(b_0)}$

Answers

1.a 2.a 3. b 4.c 5.b 6.c 7. b 8. b,c 9.b 10.c

11.b 12. a 13.c 14.b 15.c 16.c 17.a 18.c 19.b 20.c

7 NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

7.1 Introduction:

Differential equation are involved in many problems in Engineering and Science. In this chapter, we discuss various numerical methods for solving ordinary differential equations. Our aim is to study the solution of the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y_0 = y(x_0) \quad \dots \quad (1)$$

in which $f(x, y)$ is a continuous function of x and y in some domain D of the xy-plane and (x_0, y_0) is a given point in D. The condition $y_0 = f(x_0)$ is known as the initial condition. Sufficient conditions for the existence and uniqueness of the solution of the equation (1) are the well known Lipschitz conditions given by

(i) $f(x, y)$ is defined and continuous in D, the region containing (x_0, y_0)

(ii) there exists a constant L such that

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2| \quad \dots \quad (2)$$

for all $(x, y_1), (x, y_2) \in D$.

We now proceed to consider numerical techniques for solving (1) at a sequence of points $x_i = x_0 + ih$, called the *mesh points*, h being the step length. Let y_i be the approximation to the exact solution $y(x_i)$ of (1). A continuous approximation to y is then obtained by interpolating the data points (x_i, y_i).

7.2. Euler's method.

If h not given, take h = 0.1

We shall now describe a method, known as Euler's method, which gives the solution in the form of a set of tabulated values. In single step method, we determine a function $\phi(x, y; h)$ of x, y and h (the step length) depending on $f(x, y)$ and its derivatives such that

$$y(x+h) = y(x) + h\phi(x, y; h) + O(h^{p+1}), \quad \dots \quad (3)$$

where p is a positive integer, called the order of the method.

A general single-step method of order p can be obtained by expanding $y(x+h)$ by Taylor's theorem as follows:

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \dots + \frac{h^p}{p!}y^{(p)}(x)$$

$$+\frac{h^{p+1}}{(p+1)!}y^{(p+1)}(x+\theta h),\ 0<\theta<1 \quad \dots (4)$$

When $p=1$, we have, from (4)

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x+\theta h),\ 0<\theta<1 \quad \dots (5)$$

so that $\phi(x, y; h) = y'(x) = f(x, y)$ ... (6)

Let $y_n = y(x_n)$ and $y_{n+1} = y(x_{n+1}) = y(x_n+h), (n = 1, 2, \dots)$

Then neglecting the last term in (5) and putting $x = x_n$, we have

$$y_{n+1} = y_n + hy'(x_n)$$

$$= y_n + h\,f(x_n, y_n), \text{ by (6)}$$

Thus we get

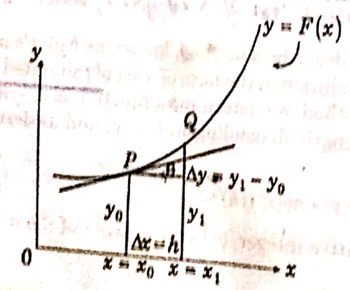
$$y_{n+1} = y_n + h\,f(x_n, y_n),\ n = 0, 1, 2, \dots \quad \dots (7)$$

This is the general *recursion formula for Euler's method* which is a single step method of order 1.

The truncation error of Euler's method is given by

$$\frac{h^2}{2}\cdot y''(x_n+\theta h), 0<\theta<1 \text{ and } n = 1, 2, \dots$$

The geometrical illustration of Euler's method is given below



Let $y = F(x)$ be the solution of (1) and its graph be as shown in the adjoining figure. Since a very small portion of a smooth curve can be thought of as a line segment, so we can write

$$\frac{\Delta y}{\Delta x} = \tan\theta$$

$\therefore$ From the adjacent figure, we have

$$\Delta y = \left(\frac{dy}{dx}\right)_p .\Delta x \text{ and } y_1 = y_0 + \Delta y$$

$$\therefore y_1 = y_0 + \left(\frac{dy}{dx}\right)_p .\Delta x$$

or, $y_1 = y_0 + hf(x_0, y_0)$

which is the approximate value of y for $x = x_1$. On the same lines, the approximate value of y for $x = x_2$ is given by

$$y_2 = y_1 + hf(x_1, y_1)$$

Thus, in general we have

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

Note. A great disadvantage of the method lies in the fact that if h is not small enough then the method yields erroneous result; on the otherhand, if h is taken too small enough then the method becomes very slow.

Example.1. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x = 0$, find y for $x = 0.1$ by Euler's method, correct upto 4 decimal places, taking step length $h = 0.02$.

[*W.B.U.T., CS-312, 2007*]

Solution. Here $f(x, y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$ and $h = 0.02$

$\therefore$ From (7), we get

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$= 1 + 0.02\left(\frac{1-0}{1+0}\right)$$

$\therefore y(0.02) = 1.02$

$$y(0.04) = y_2 = y_1 + h\, f(x_1, y_1)$$

$$= 1.02 + 0.02\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right)$$

$$= 1.039231$$

Similarly $y(0.06) = 1.039231 + 0.02\,\dfrac{1.039231 - 0.04}{1.03923 + 0.04}$

$$= 1.057748$$

$$y(0.08) = 1.057748 + 0.02 \times 0.892641$$

$$= 1.075601$$

$$y(0.10) = 1.075601 + 0.02 \times 0.861544 = 1.092832$$

$\therefore y(0.1) = 1.0928$, correct upto 4 decimal places.

7.3. Modified Euler's Method.

To remove the drawback to some extent, we shall discuss modified Euler's method starting with the initial value y_0 an approximate value for y_1 is computed from the Euler's method as

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0) \qquad \dots \quad (8)$$

Then to get the second approximation for y_1 we replace $f(x_0, y_0)$ in (8) by the average value of $f(x_0, y_0)$ and $f\left(x_1, y_1^{(0)}\right)$.

Thus the second approximation for y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

Similarly, third approximation for y_1 is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

Thus, in general

$$y_n^{(k)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f\left(x_n, y_n^{(k-1)}\right)\right],\ k = 1, 2, 3, \dots$$

is used to approximate y_n $\qquad \dots \quad (9)$

Example.2. Given $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{x^2}$, $y(1) = 1$. Evaluate $y(1.2)$ by modified Euler's method correct upto 4 decimal places.

[W.B.U.T., CS-312, 2003, 2004, 2006]

x_0 y_0

Solution. Here $f(x, y) = \dfrac{1}{x^2} - \dfrac{y}{x}$, $x_0 = 1$, $y_0 = 1$ $\qquad \frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$

Let $h = 0.1$ so that $x_1 = 1 + 0.1 = 1.1$

$$\therefore y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.1 \times (1 - 1) = 1$$

$\therefore$ From (9), we get

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$= 1 + \frac{0.1}{2}\left[(1-1) + \left\{\frac{1}{(1.1)^2} - \frac{1}{1.1}\right\}\right]$$

$$= 1 + 0.05(-0.08264)$$

$$= 0.99587$$

$$\therefore y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 1 + \frac{0.1}{2}\left[(1-1) + \left(\frac{1}{(1.1)^2} - \frac{0.99587}{1.1}\right)\right]$$

$$= 1 + 0.05(-0.078888)$$

$$= 0.99606$$

Similarly $y_1^{(3)} = 1 + 0.05\left[(1-1) + \left(\dfrac{1}{(1.1)^2} - \dfrac{0.99606}{1.1}\right)\right]$

$$= 1 + 0.05(-0.079063)$$

$$= 0.99607$$

Hence $y_1 = y(1.1) \approx 0.9961$

$\therefore x_1 = 1.1, y_1 = 0.9961$

$\therefore f(x_1, y_1) = \frac{1}{(1.1)^2} - \frac{0.9961}{1.1} = -0.079$

$\therefore y_2^{(0)} = y_1 + hf(x_1, y_1)$

$= 0.9961 + 0.1 \times (-0.079)$

$= 0.98819$

$\therefore$ From (9), we have

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$$

$$= 0.9961 + 0.05\left[-0.079 + \frac{1}{(1.2)^2} - \frac{0.98819}{1.2}\right]$$

$= 0.98569$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$$

$$= 0.9961 + 0.05\left[-0.079 + \frac{1}{(1.2)^2} - \frac{0.98569}{1.2}\right]$$

$= 0.98580$

Similarly,

$$y_2^{(3)} = 0.9961 + 0.05\left[-0.079 + \frac{1}{(1.2)^2} - \frac{0.98580}{1.2}\right]$$

$= 0.985797$

Thus $y_2 \approx 0.9858$, correct upto four decimal places

$\therefore y(1.2) \approx 0.9858$

7.4. Runge-Kutta method.

This method is one of the most widely used methods to obtain greater accuracy and most suitable in case when computation of higher order derivatives is complicated. In single step method, it follows from (3) that

$$y(x+h) = y(x) + h\phi(x, y; h) + 0\left(h^{p+1}\right) \quad \ldots \quad (10)$$

When $p = 2$, we get from (8)

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x+\theta h), 0<\theta<1 \ldots \quad (11)$$

so that $\phi(x, y; h) = y'(x) + \frac{h}{2}y''(x)$

$$= f + \frac{h}{2}(f_x + f\, f_y) \quad \ldots \quad (12)$$

In an 2-stage Runge-Kutta method, we set

$$k_1 = hf(x, y)$$

$$k_2 = hf(x + \alpha h, y + \beta k_1)$$

$$k = w_1 k_1 + w_2 k_2 \quad \ldots \quad (13)$$

The constants α, β, w_1 and w_2 are determined so that (12) agree with Taylor's series of order as high as possible.

$\therefore$ From (10), we get

$$y(x+h) = y(x) + k + 0\left(h^3\right) \quad \ldots \quad (14)$$

From (12) and (13), it follows that

$$f + \frac{1}{2}h\left(f_x + ff_y\right) = w_1 f(x, y) + w_2 f(x + \alpha h, y + \beta k_1)$$

$$= \omega_1 f + w_2\left(f + \alpha h f_x + \beta k_1 f_y\right) + 0\left(h^2\right)$$

$$= (\omega_1 + \omega_2)f + h\left(\omega_2 \alpha f_x + \omega_2 \beta f f_y\right) + 0\left(h^2\right) \quad [\because k_1 = hf]$$

which is true for all values of the constant α, β, ω_1 and ω_2 and therefore, for arbitrary f. Thus we have

$$w_1 + w_2 = 1$$

$$w_1 \alpha = w_2 \beta = 1/2 \quad \ldots \quad (15)$$

Any set of values of the constants α, β, w_1, w_2 satisfying (15) gives a one-parameter family of solutions and each of these corresponds to a 2-stage Runge-Kutta method of order 2.

A possible solution of (15) is

$$w_1 = w_2 = \frac{1}{2}, \alpha = \beta = 1$$

Thus $k_1 = hf(x, y)$

$$k_2 = hf(x + h, y + k_1)$$

$$k = \frac{1}{2}(k_1 + k_2) = \frac{h}{2}[f(x, y) + f(x + h, y + hf(x, y)]$$

$$y(x + h) = y(x) + k + 0(h^3)$$

Hence the iterative formula is

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))],$$

$$n = 0, 1, 2, \ldots \quad \ldots \quad (16)$$

This is know as *Runge-Kutta method of order 2* with truncation error of order h^3

In the similar manner the *Runge-Kutta method of order 4* can be written as

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \quad \ldots \quad (17)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3), \; n = 0, 1, 2, \ldots$$

Here the truncation error is of order h^5

Note. (i) The advantage of this method is that the method is stable and self starting. It is easy to change the step size h for higher order accuracy.

(ii) For this method several evaluations of the first derivative are require and so the method is time consuming. Most disadvantage of this method is that neither the truncation errors nor the estimates of them are obtained in the computation procedure.

Example.3. Use Runge-Kutta method of order 2 to calculate $y(0.2)$ for the equation

$$\frac{dy}{dx} = x + y^2, y(0) = 1$$

Solution. Here $f(x, y) = x + y^2, x_0 = 0, y_0 = 1$

We take $h = 0.1$. Then

$k_1 = hf(x_0, y_0) = 0.1 \times (0 + 1) = 0.1$

$k_2 = hf(x_0 + h, y_0 + k_1) = 0.1 f(0.1, 1.1)$

$= 0.1 \times \{0.1 + (1.1)^2\} = 1.31$

$\therefore$ From iterative formula (16) of Runge-Kutta method of order 2 we get

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 1 + \frac{1}{2}(0.1 + 1.31)$$

$$= 1.1155$$

$\therefore y(0.1) \approx 1.1155$

Thus $x_1 = 0.1, y_1 = 1.1155$

$\therefore k_1 = hf(x_1, y_1) = 0.1 \times \{0.1 + (1.1155)^2\}$

≈ 0.1344

$k_2 = hf(x_1 + h, y_1 + k_1)$

$= 0.1 \times f(0.2, 1.2499)$

≈ 0.1762

$\therefore$ From (16), we get

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

$$= 1.1155 + \frac{1}{2}(0.1344 + 0.1762)$$

$$= 1.2708$$

Hence $y(0.2) \simeq 1.2708$

Example.4. Find $y(1.1)$ using **Runge-Kutta method of fourth order**, given that

$$\frac{dy}{dx} = y^2 + xy,\ y(1) = 1$$ [W.B.U.T., CS-312, 2005]

Solution. Here $f(x,y) = y^2 + xy,\ x_0 = 1, y_0 = 1$

Taking $h = 0.1$, we have

$$k_1 = hf(x_0, y_0) = 0.1(1^2 + 1\times 1) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$= 0.1f(1.05, 1.1)$$

$$= 0.1\{(1.1)^2 + 1.05\times 1.1\}$$

$$= 0.2365$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= 0.1f(1.05, 1.11825)$$

$$= 0.1\{(1.11825)^2 + 1.11825\times 1.05\}$$

$$= 0.2425$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1f(1.1, 1.2425)$$

$$= 0.1\{(1.2425)^2 + 1.1\times 1.2425\}$$

$$= 0.2910556$$

$\therefore$ From iterative formula (17) of Runge-Kutta method of order 4, we get

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.2 + 2\times 0.2365 + 2\times 0.2425 + 0.2910556)$$

$\simeq 1.2415$, correct upto four decimal places.

$\therefore y(0.11) \simeq 1.2415$

7.5. Predictor-Corrector methods.

In order to solve the differential equation (1), by this method, we first obtain the approximate value of $y_{n+1} = y(x_{n+1})$ by predictor formula and then improve this value by means of a corrector formula. It may be noted that the corrector formula is more accurate than the predictor one although it requires a companion of predictor formula and knowledge of the initial set of values $y_0, y_1, \dots y_n$

The simplest formula of this type is Euler's formula and the modified Euler's one is given by

$$y_{n+1}^{(p)} = y_n + hf(x_n, y_n) \quad \dots \quad (18)$$

$$y_{n+1}^{(c)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y_{n+1}^{(p)}\right)\right] \quad \dots \quad (19)$$

The first is an open formula which can be used for predicting y_{n+1} and this value can be used to compute $f(x_{n+1}, y_{n+1})$ to get a corrector formula which can be used in an iterative manner.

Let us now proceed to discuss some multi-step methods which are of much practical use as predictor-corrector methods.

I. Adams-Bashforth method.

We consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

which when integrated over the range $[x_n, x_{n+1}]$ leads to

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x,y)dx \quad \cdots \quad (20)$$

To evaluate the integral on the right hand side of (20), we can replace $f(x,y)$ by a polynomial which interpolates $f(x,y(x))$ at the $(k+1)$ equidistant points $x_{n-k}, x_{n-k+1} \cdots, x_n$; by the Newton's backward difference interpolation formula.

$$P_k(x) = \sum_{j=0}^{k} \binom{s+j-1}{j} \nabla^j f_n, \quad \cdots \quad (21)$$

where $s = \dfrac{x - x_n}{h}$

Then, in virtue of (20) and (21), we get

$$y_{n+1} = y_n + h\sum_{j=0}^{k} \alpha_j \nabla^j f_n \quad (n \geq k) \quad \cdots \quad (22)$$

where $\alpha_j = \int_0^1 \binom{s+j-1}{j} ds, \ (j = 0, 1, 2, \ldots k+1) \quad \cdots \quad (23)$

The formula (22) is known as the $(k+1)$ step Adams-Bashforth formula and is an predictor formula of order $k+1$

A few values of α_j as obtained from (23) are

$$\alpha_0 = 1, \alpha_1 = \tfrac{1}{2}, \alpha_2 = \tfrac{5}{12}, \alpha_3 = \tfrac{3}{8}, \alpha_4 = \tfrac{251}{720}, \ldots$$

Thus, for $k = 3$ we can rewrite (22) in the form

$$y_{n+1} = y_n + h\left[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n\right] \quad \cdots \quad (24)$$

Substituting the expressions of differences given by

$\nabla f_n = f_n - f_{n-1}$, $\nabla^2 f_n = f_n - 2f_{n-1} + f_{n-2}$

$\nabla^3 f_n = f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}$,

we get

$$y_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] \quad \cdots \quad (25)$$

which is known as the 4-*th step Adams-Bashforth formula in ordinate form.*

II. Adams-Moulton method.

This method can be dealt with along the same lines as in Adams-Bashforth method with the exception that instead of the interpolating points $x_{n-k}, x_{n-k+1}, \ldots, x_n$, we consider the points $x_{n-k+1}, x_{n-k+2}, \ldots, x_{n+1}$. Then, as usual integrating the equation

$$\frac{dy}{dx} = f(x,y)$$

over $[x_n, x_{n+1}]$ we get after simplification,

$$y_{n+1} = y_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] \text{ for } k = 3 \quad \ldots \quad (26)$$

This formula is known as the 4-th step Adams-Moulton corrector formula of order 4

Thus another set of predictor-corrector formula which is commonly used is given below:

$$y_{n+1}^{(p)} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37y_{n-2} - 9f_{n-3}] \quad (n > 3) \quad \ldots \quad (27)$$

$$y_{n+1}^{(c)} = y_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] \quad (n > 3) \cdots \quad (28)$$

Example.5. Given that $\dfrac{dy}{dx} = y - \dfrac{2x}{y}, y(0) = 1, y(0.1) = 1.0954$, $y(0.2) = 1.1852, y(0.3) = 1.2649$. Find $y(0.4)$ by Adams-Moulton formula.

Solution. Here $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, h = 0.1$ and $y_0 = 1, y_1 = 1.0954, y_2 = 1.1852, y_3 = 1.2649$

Also $f(x,y) = y - \dfrac{2x}{y}$

Hence we obtain

$f_0 = f(x_0, y_0) = 1, f_1 = f(x_1, y_1) = 0.9128$

$f_2 = f(x_2, y_2) = 0.8451, f_3 = f(x_3, y_3) = 0.7906$

For $n = 3$, (27) and (28) give respectively

$$y_4^p = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0) \quad \dots (1)$$

and

$$y_4^c = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1) \quad \dots (2)$$

Then from (1), we get

$$y_4^p = 1.2649 + \frac{0.1}{24}(55 \times 0.7906 - 59 \times 0.8451 + 37 \times 0.9128 - 9 \times 1)$$

$$\approx 1.3415$$

and hence $\tilde{f}_4 = f(x_4, y_4^p) = 1.3415 - \dfrac{2 \times 0.4}{1.3415} = 0.7452$

$\therefore$ From (2), we get

$$y_4^{c(1)} = 1.2649 + \frac{0.1}{24}(9 \times 0.7452 + 19 \times 0.7906 - 5 \times 0.8451 + 0.9128)$$

$$\approx 1.3416$$

Hence $\tilde{f}_4 = f(x_4, y_4^{c(1)}) = 1.3416 - \dfrac{2 \times 0.4}{1.3416} = 0.7453$

$$\therefore y_4^{c(2)} = 1.2649 + \frac{0.1}{24}(9 \times 0.7453 + 19 \times 0.7906 - 5 \times 0.8451 + 0.9128)$$

$$= 1.3416$$

$\therefore y_4^{c(1)} = y_4^{c(2)} = 1.3416$, correct to four decimal places.

Hence $y(0.4) = 1.3416$

III. Milne's method.

The multistep method due to Milne is obtained by integration over more than one step. Integration of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

over the range $[x_{n-3}, x_{n+1}]$ gives

$$y_{n+1} = y_{n-3} + \int_{x_{n-3}}^{x_{n+1}} f(x, y)dx$$

Evaluating this integral by 3 point Newton-Cotes quadrature rule and neglecting error term, we get

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n) \quad \dots (29)$$

which is the 4-step explicit recursion formula of order 4, known as *Milne's prodictor formula of order 4.*

On the other hand, if we integrate the differential equation $\dfrac{dy}{dx} = f(x, y)$ over the range $[x_{n-1}, x_{n+1}]$, we get

$$y_{n+1} = y_{n-1} + \int_{x_{n-1}}^{x_{n+1}} f(x, y)dx$$

Evaluating this integral by Simpson's one-third rule with interpolating points x_{n-1}, x_n, x_{n+1} and neglecting error term, we obtain

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}) \quad \dots (30)$$

which is the 2-step implicit recursion formula of order 4, called *Milne's corrector formula of order 4.*

Example 6. Compute $y(0.4)$ by Milne's predictor-corrector method from the equation

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$$\frac{dy}{dx} = xy + y^2,$$

given that $y(0) = 1, y(0.1) = 1.1169, y(0.2) = 1.2773, y(0.3) = 1.5040$

Solution. We have $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, h = 0.1$ and $y_0 = 1, y_1 = 1.1169, y_2 = 1.2773, y_3 = 1.5040$

Also $f(x, y) = xy + y^2$

$\therefore f_0 = f(x_0, y_0) = 1, f_1 = f(x_1, y_1) = 1.3591, f_2 = f(x_2, y_2) = 1.8869$

$f_3 = f(x_3, y_3) = 2.7132$

Now putting $n = 3$ in (29) and (30), we get

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \quad \dots (1)$$

$$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$\therefore$ From (1) and (2),

$$y_4^{p(1)} = 1 + \frac{4 \times 0.1}{3}(2 \times 1.3591 - 1.8869 + 2 \times 2.7132)$$

$$= 1.8344$$

and hence $f_4^{p(1)} = f\left(x_4, y_4^{p(1)}\right) = 4.0988$

$$\therefore y_4^{c(1)} = 1.2773 + \frac{0.1}{3}(1.8869 + 4 \times 2.7132 + 4.0988)$$

$$= 1.8386$$

and so $f_4^{c(1)} = f\left(x_4, y_4^{c(1)}\right) = 4.1159$

Hence from (2),

$$y_4^{c(2)} = 1.2773 + \frac{0.1}{3}(1.8869 + 4 \times 2.7132 + 4.1159)$$

$$= 1.8391$$

Hence $f_4^{c(2)} = f\left(x_4, y_4^{c(2)}\right) = 4.1182$

$$\therefore y_4^{c(3)} = 1.2773 + \frac{0.1}{3}(1.8869 + 4 \times 2.7132 + 4.1182)$$

$$= 1.8392$$

$\therefore y_4^{c(2)} = y_4^{c(3)} = 1.839$, correct upto 3 decimal places

Hence $y(0.4) \simeq 1.839$

7.6. Finite difference method.

The finite difference method which in also known as net method is a popular method for solving boundary value problems. In this method, the derivatives are replaced by finite difference relations and then solving the resulting system of equations by a standard procedure. In general, for better accuracy, central differences are preferred to replace the derivatives.

Thus

$$y'(x_i) = \frac{y_{i+1} - y_{i-1}}{2h} \qquad \ldots (31)$$

$$y''(x_i) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \text{ etc.}$$

To solve the boundary value problem, defined by

$$y''(x) + p(x)y' + q(x)y = r(x) \qquad \ldots (32)$$

with the boundary conditions

$$y(x_0) = a$$

$$y(x_n) = b,$$

we divide the interval $[a,b]$ into n equal subintervals of width h so that the end points are $x_0 = a, x_n = b$ and the interior mesh points are

$$x_i = x_0 + ih, i = 1, 2, \ldots n$$

The corresponding values of y at these points are denoted by

$$y_i = y(x_i) = y(x_0 + ih), \; i = 1, 2, \ldots n$$

At the point $x = x_i$ we get from (32),

$$y_i'' + p_i y_i' + q_i y_i = r_i$$

with $y_0 = a, y_n = b, \; i = 1, 2, \ldots n-1$

where $p_i = p(x_i), q_i = q(x_i)$ and $r_i = r(x_i)$

Substituting the expression for y_i' and y_i'' the equation (29) gives

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + p_i \frac{y_{i+1} - y_{i-1}}{2h} + q_i y_i = r_i$$

or, $\left(1 - \frac{h}{2} p_i\right) y_{i-1} + \left(-2 + q_i h^2\right) y_i + \left(1 + \frac{h}{2} p_i\right) y_{i+1} = r_i h^2,$

$$i = 1, 2, \ldots n-1$$

This linear system of equations can be solved by using any of the elimination methods. The solution of this system constitutes an approximate solution of the boundary value problem defined by (32)

Example.7. Solve the equation

$$\frac{d^2y}{dx^2}+y=0$$

with $y(0)=0, y(1)=1$, using finite difference method taking $h=0.25$

Solution. The given equation is approximated as

$$\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}+y_i=0,\ i=1,2,\ldots n-1$$

i.e. $y_{i-1}+\left(h^2-2\right)y_i+y_{i+1}=0,\ i=1,2,3 \quad \ldots (1)$

which, together with the boundary conditions $y_0=0, y_n=1$, comprise a system of $(n+1)$ equations for the $(n+1)$ unknown $y_0, y_1, y_2, \ldots y_n$

Choosing $h=0.25$ i.e. $n=4$, the above system of equations are

$$y_0-1.9375y_1+y_2=0$$
$$y_1-1.9375y_2+y_3=0$$
$$y_2-1.9375y_3+y_4=0$$

where $y_0=0, y_4=1$

Solving the system we get

$$y_1=0.2943, y_2=0.5701, y_3=0.8108$$

i.e., $y(0.25)=0.2943, y(0.5)=0.5701, y(0.75)=0.8108$

ILLUSTRATIVE EXAMPLES

Ex. 1. Find the solution of the differential equation

$$\frac{dy}{dx}=x^2-y,\ y(0)=1$$

for $x=0.3$ taking $h=0.1$ and using Euler's method. Compare the result with the exact solution.

Solution. Here $f(x,y)=x^2-y, x_0=0, y_0=1, h=0.1$ so that $x_i=x_o+ih(i=0,1,2,\ldots)$ gives

$x_1=0.1, x_2=0.2$ etc

Thus the recursion formula

$$y_{n+1}=y_n+hf(x_n,y_n), n=0,1,2,\ldots$$

yields

$$y_1=y_0+hf(x_0,y_0)=1+0.1(0-1)=0.9$$

$$y_2=y_1+hf(x_1,y_1)=0.9+0.1\left\{(0.1)^2-0.9\right\}=0.811$$

$$y_3=y_2+hf(x_2,y_2)=0.811+0.1\left\{(0.2)^2-0.811\right\}=0.7339$$

Hence $y(0.3)\simeq 0.7339$

The given equation can be written as

$$\frac{dy}{dx}+y=x^2$$

which is a linear equation in y.

$$\therefore I.F.=e^{\int 1.dx}=e^x$$

Multiplying both sides of the equation by e^x and then integrating we get

$$ye^x=\int x^2e^xdx+c$$
$$=x^2e^x-2xe^x+2e^x+c$$
$$\therefore y=x^2-2x+2+ce^{-x}$$

Also given $y(0)=1$

$\therefore 1=0-2\times 0+2+c$

$\therefore c=-1$

$\therefore y=x^2-2x+2-e^{-x}$

$\therefore y(0.3)=(0.3)^2-2\times 0.3+2-e^{-0.3}=0.7492$

Hence the error is $0.7492-0.7339=0.0153$

Ex. 2. Using Euler's method, find an approximate value of y at $=0.5$ given that

$$\frac{dy}{dx}=x+y, y(0)=1$$

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Solution. We have $f(x,y)=x+y, x_0=0, y_0=1$

Taking $h=0.1$, we have from recursion formula of Euler's method,

$$y_{n+1}=y_n+hf(x_n,y_n)$$
$$=y_n+0.1(x_n+y_n), n=0,1,2,\ldots$$
$$\therefore y_1=y(0.1)=1+0.1(0+1)=1.10$$
$$y_2=y(0.2)=y_1+0.1(x_1+y_1)$$
$$=1.10+0.1(0.1+1.10)$$
$$=1.22$$
$$y_3=y(0.3)=y_2+0.1(x_2+y_2)$$
$$=1.22+0.1(0.2+1.22)$$
$$=1.36$$
$$y_4=y(0.4)=y_3+0.1(x_3+y_3)$$
$$=1.36+0.1(0.3+1.36)$$
$$=1.53$$
$$y_5=y(0.5)=y_4+0.1(x_4+y_4)$$
$$=1.53+0.1(0.4+1.53)$$
$$=1.72$$

Thus $y(0.5)=1.72$

Ex. 3. Solve the equation

$$5x\frac{dy}{dx}+y^2-2=0; y(4)=1$$

for $y(4.1)$, taking $h=0.1$ and using modified Euler's method.

Solution. The given equation can be written as

$$\frac{dy}{dx}=\frac{2-y^2}{5x}$$

$$\therefore f(x,y)=\frac{2-y^2}{5x}$$

Here $x_0=4$, $y_0=1$, $h=0.1$

So the recursion formula of modified Euler's method gives

$$y_1^{(0)}=y_0+hf(x_0,y_0)$$
$$=1+0.1f(4,1)$$
$$=1+0.1\times 0.05 \quad \left[\because f(4,1)=\frac{2-1}{5\times 4}=0.05\right]$$
$$=1.005$$
$$y_1^{(1)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f\left(x_1,y_1^{(0)}\right)\right]$$
$$=1+\frac{0.1}{2}[f(4,1)+f(4.1,1.005)]$$
$$=1+0.05\left[0.05+\frac{2-(1.005)^2}{5\times 4.1}\right]$$
$$=1.0049$$

Similarly, $y_1^{(2)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f\left(x_1,y_1^{(1)}\right)\right]$

$$=1+0.05[f(4,1)+f(4.1,1.0049)]$$
$$=1+0.05\left[0.05+\frac{2-(1.0049)^2}{5\times 4.1}\right]$$
$$=1.0049$$

$\therefore y(4.1)\simeq 1.005$, correct upto three decimal places

Ex. 4. Use Runge-Kutta method of order two to find $y(0.2)$ and $y(0.4)$ given that

$$y\frac{dy}{dx}=y^2-x, y(0)=2, \text{ taking } h=0.2$$

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Solution. The given equation can be written as

$$\frac{dy}{dx} = \frac{y^2 - x}{y}$$

$\therefore$ Here $f(x,y) = \frac{y^2 - x}{y}, x_0 = 0, y_0 = 2, h = 0.2$

$\therefore$ By Runge-Kutta method of order 2, we have

$y(x_0 + h) = y_0 + k$

where $k = \frac{1}{2}(k_1 + k_2)$,

$k_1 = hf(x_0, y_0) = 0.2 \times \frac{2^2 - 0}{2} = 0.4$

$k_2 = hf(x_0 + h, y_0 + k_1)$

$= 0.2\, f(0.2, 2.4)$

$= 0.2 \times \frac{(2.4)^2 - 0.2}{2.4}$

$= 0.46333$

Thus $y(0 + 0.2) = 2 + \frac{1}{2}(0.4 + 0.46333)$

$= 2.43166$

$\therefore y(0.2) \simeq 2.43166$

To compute $y(0.4)$ we have $x_1 = 0.2, y_1 = 2.43166$

$\therefore y(x_1 + h) = y(x_1) + k$

where $k = \frac{1}{2}(k_1 + k_2)$,

$y_1 = 2.43166$

$k_1 = hf(x_1, y_1) = 0.2f(0.2, 0.432) = -0.00633$

$k_2 = hf(x_1 + h, y_1 + k_1)$

$= 0.2f(0.4, 0.42533)$

$= -0.10302$

$\therefore y(0.2 + 0.2) = 2.43166 + \frac{1}{2}(-0.00633 - 0.10302)$

$\therefore y(0.4) = 2.37698$

Hence $y(0.2) = 2.432, y(0.4) = 2.377$ correct three decimal places.

Ex. 5. Find the value of $y(0.4)$ using Runge-Kutta method of fourth order with $h = 0.2$, given that

$$\frac{dy}{dx} = \sqrt{x^2 + y}, y(0) = 0.8$$

Solution. Here $f(x,y) = \sqrt{x^2 + y}, x_0 = 0, y_0 = 0.8, h = 0.2$

$\therefore k_1 = hf(x_0, y_0) = 0.2f(0, 0.8) = 0.2\sqrt{0^2 + 0.8} = 0.17889$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 0.88944)$

$= 0.2\sqrt{(0.1)^2 + 0.88944} = 0.18968$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1, 0.89484)$

$= 0.2\sqrt{(0.1)^2 + 0.89484}$

$= 0.19025$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 0.99025)$

$= 0.2\sqrt{(0.2)^2 + 0.99025}$

$= 0.20300$

$\therefore y_1 = y(x_0 + h)$

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$$= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.8 + \frac{1}{6}(0.17889 + 12 \times 0.18968 + 2 \times 0.19025 + 0.20300)$$

$$= 0.99029$$

$\therefore y(0.2) = 0.99029$

To compute $y(0.4)$, we have $x_1 = 0.2, y_1 = 0.99029, h = 0.2$

$\therefore k_1 = hf(x_1, y_1) = 0.2f(0.2, 0.99029) = 0.20301$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f(0.3, 1.09180) = 0.21742$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2f(0.3, 1.09901) = 0.21808$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.20838) = 0.23396$$

$\therefore y_2 = y(x_1 + h)$

$$= y_1 + \frac{1}{6}(k_2 + 2k_2 + 2k_3 + k_4)$$

$$= 0.99029 + \frac{1}{6}(0.20301 + 2 \times 0.21742 + 2 \times 0.21808 + 0.23396)$$

$$= 1.20832$$

$\therefore y(0.4) \simeq 1.2083$, correct upto four decimal places.

Ex. 6 Solve initial value problem

$$10\frac{dy}{dx} = x^2 + y^2, y(0) = 1$$

for $x = 0.1, 0.2$ by using Runge-Kutta fourth order method and find the solution correct upto 4 places of decimal.

[*W.B.U.T., CS-312, 2004*]

Solution. Here $f(x, y) = \dfrac{x^2 + y^2}{10}, x_0 = 0, y_0 = 1$

Let $h = 0.1$

$\therefore$ By fourth order Runge-Kutta method,

$$y(x_0 + h) = y_0 + k$$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

$$k_1 = hf(x_0, y_0) = 0.1\left(\frac{0+1}{10}\right) = 0.01$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1f(0.05, 1.005)$$

$$= 0.1 \times \frac{(0.05)^2 + (1.005)^2}{10} = 0.010125$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05, 1.0050625)$$

$$= 0.1 \times \frac{(0.05)^2 + (1.0050625)^2}{10}$$

$= 0.000025$ 0.010126506

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.000025)$$

$$= 0.1 \times \frac{(0.1)^2 + (1.000025)^2}{10} = 0.0101$$

$$\therefore k = \frac{1}{6}(0.01 + 2 \times 0.010125 + 2 \times 0.000025 + 0.0101)$$

$$= 0.00673$$

$\therefore y(x_0 + h) = y(0.1) = 1 + 0.00673 \simeq 1.0067$

For $y(0.2)$, we have $x_1 = 0.1, y_1 = 1.0067$

$$\therefore k_1 = hf(x_1, y_1) = 0.1 \times \frac{(0.1)^2 + (1.0067)^2}{10} = 0.010235$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f(0.15, 1.0118175)$$

$$= 0.1 \times \frac{(0.15)^2 + (1.0118175)^2}{10}$$

$$= 0.01046$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f(0.15, 1.01193)$$

$$= 0.1 \times \frac{(0.15)^2 + (1.01193)^2}{10}$$

$$= 0.010465$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1f(0.2, 1.017165)$$

$$= 0.1 \times \frac{(0.2)^2 + (1.017165)^2}{10}$$

$$= 0.010746$$

$$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.010235 + 2 \times 0.01046 + 2 \times 0.010465 + 0.010746)$$

$$= 0.01047$$

$$\therefore y(x_1 + h) = y(0.2) = 1.0067 + 0.01047$$

$$\simeq 1.0172$$

Ex. 7. Compute $y(0.8)$ by Adams-Moulton predictor-corrector method from

$$\frac{dy}{dx} = 1 + y^2, \; y(0) = 0$$

given $y(0.2) = 0.2027$, $y(0.4) = 0.4228$, $y(0.6) = 0.6842$

Solution. Here $f(x, y) = 1 + y^2, x_0 = 0, x_1 = 0.2, x_2 = 0.4$

$x_3 = 0.6$ and $y_0 = 0, y_1 = 0.2027, y_2 = 0.4228$, $y_3 = 0.6842$

Hence $f_0 = f(x_0, y_0) = 1, f_1 = f(x_1, y_1) = 1.0411, f_2 = 1.1788,$

$f_3 = 1.4680$

Now fourth order Adam's Bashforth formula is

$$y_4^{(p)} = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0) \quad \ldots \quad (1)$$

and Adam's Moulton formula is

$$y_4^{(c)} = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1) \quad \ldots \quad (2)$$

$\therefore$ From (1),

$$y_4^{(p)} = 0.6842 + \frac{0.2}{24}(80.3960 - 69.5468 + 38.5203 - 9)$$

$$= 1.0235$$

$$\therefore f_4 = f\left(x_4, y_4^{(p)}\right) = 2 \cdot 0475$$

$\therefore$ From (2),

$$y_4^{c(1)} = 0.6842 + \frac{0.2}{24}(18.4275 + 27.8945 - 5.8938 + 1.0411)$$

$$= 1.0298$$

$$\therefore f_4 = f\left(x_4, y_4^{c(1)}\right) = 2 \cdot 0604$$

$\therefore$ From (2),

$$y_4^{c(2)} = 0.6842 + \frac{0.2}{24}(18.5440 + 27.8945 - 5.8938 + 1.0411)$$

$$= 1.0308$$

$$\therefore \tilde{f}_4 = f\left(x_4, y_4^{c(2)}\right) = 2.0624$$

$\therefore$ From (2), $y_4^{c(3)} = 1.0309$

$$\therefore \tilde{f}_4 = f\left(x_4, y_4^{c(3)}\right) = 2.0628$$

$\therefore$ From (2), $y_4^{c(4)} = 1.0309$

Hence $y_4^{c(3)} = y_4^{c(4)} = 1.0309$, correct upto four decimal places

$\therefore y(0.8) \simeq 1.0309$

Ex. 8. Apply Milne's method to find the solutions of the differential equation

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

at $x = 0.08$, given $y(0) = 1, y(0.02) = 1.02, y(0.04) = 1.0392$ $y(0.06) = 1.0577$

Solution. Here $x_0 = 0, x_1 = 0.02, x_2 = 0.04, x_3 = 0.06$ and $y_0 = 1, y_1 = 1.02, y_2 = 1.0392$, $y_3 = 1.0577$

Also, $f(x, y) = \dfrac{y-x}{y+x}$

$$\therefore f_1 = f(x_1, y_1) = \frac{1.02 - 0.02}{1.02 + 0.02} = 0.9615$$

Similarly, $f_2 = f(x_2, y_2) = 0.9259$

$$f_3 = f(x_3, y_3) = 0.8926$$

Now Milne's predictor formula of order 4 is

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \quad \dots \quad (1)$$

and the corrector formula of order 4 is

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \quad \dots \quad (2)$$

$\therefore$ From (1),

$$y_4^{(p)} = 1 + \frac{4 \times 0.02}{3}(2 \times 0.9615 - 0.9259 + 2 \times 0.8926)$$

$$\simeq 1.0742$$

$$\therefore \tilde{f}_4 = f\left(x_4, y_4^{(p)}\right) = 0.8614$$

$\therefore$ From (2),

$$y_4^{c(1)} = 1.0392 + \frac{0.02}{3}(0.9259 + 4 \times 0.8926 + 0.8614)$$

$$\simeq 1.0749$$

$$\therefore \tilde{f}_4 = f\left(x_4, y_4^{c(1)}\right) = 0.8615$$

$\therefore$ From (2),

$$y_4^{c(2)} = 1.0749$$

$\therefore y_4^{c(1)} = y_4^{c(2)} = 1.0749$, correct upto four decimal places.

$\therefore y_4 = 1.0749$ i.e. $y(0.08) = 1.0749$

Ex. 9. Using the method of finite difference find the solution of the boundary value problem

$$x^2 y'' + xy' = 1; \; y(1) = 0, \; y(1.4) = 0.0566$$

Solution. The finite difference form of the given equation is

$$x_i^2 \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + x_i \frac{y_{i+1} - y_{i-1}}{2h} = 1$$

i.e. $\left(2x_i^2 - hx_i\right)y_{i-1} - 4x_i^2 y_i + \left(2x_i^2 + hx_i\right)y_{i+1} = 2h^2$

$$i = 1, 2, \dots n-1$$

with the boundary conditions $y_0 = 0, y_n = 0.0566$

Taking $h = 0.1$ i.e. $n = 4$, the above system becomes

$$2.31y_0 - 4.84y_1 + 2.53y_2 = 0.02$$

$$2.76y_1 - 5.76y_2 + 3y_3 = 0.02$$

$$3.25y_2 - 6.76y_3 + 3.51y_4 = 0.02$$

where $y_0 = 0, y_n = 0.0566$

Solving the system we get

$$y_1 = 0.0046, y_2 = 0.0167, y_3 = 0.0345$$

Hence $y(1.1) = 0.0046$, $y(1.2) = 0.0167$, $y(1.3) = 0.0345$

Ex. 10. Using finite difference method, solve the following BVP:

$$\frac{d^2y}{dx^2} + y + 1 = 0$$

with $y(0) = 0, y(1) = 0$

Solution. The finite difference form of the given equation is

$$\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}+y_i+1=0$$

i.e. $y_{i-1}+(h^2-2)y_i+y_{i+1}=-h^2,\ i=1,2,\ldots n-1$

with the boundary conditions at $x_0=0$ and at $x_n=1$ i.e. $y_0=0, y_n=0$

Taking $h=0.25$ i.e. $n=4$, the above system becomes

$$y_0-1.9375y_1+y_2=-0.0625$$

$$y_1-1.9375y_2+y_3=-0.0625$$

$$y_2-1.9375y_3+y_4=-0.0625$$

where $y_0=0, y_4=0$

Solving the system we obtain,

$$y_1=0.10468, y_2=0.14031, y_3=0.10468$$

Hence $y(0.25)=0.10468, y(0.5)=0.14031$

and $y(0.75)=0.10468$

Exercise

I. Short Answer Questions

1. Describe Euler's method to find the solution of the differential equation

$$\frac{dy}{dx}=f(x,y) \text{ with } y(x_0)=y_0$$

Interpret the error involved in this method geometrically.

[W.B.U.T., CS-312, 2003]

2. Evaluate $y(0.1)$ by Euler's method, given

$$\frac{dy}{dx}=3x+y^2, y(0)=1$$

3. Solve $\frac{dy}{dx}=xy, y(1)=1$

by Euler's method to compute $y(1.1)$ and $y(1.2)$

4. Find $y(0.10)$ and $y(0.15)$ by Euler's method from the differential equation

$$\frac{dy}{dx}=x^2+y^2, \text{ with } y(0)=0$$

correct to four decimal places, taking step length $h=0.05$

[W.B.U.T., CS-312, 2007]

5. Solve by using Euler's method the following differential equation for $x=1$ by taking $h=0.2$.

$$\frac{dy}{dx}=xy, y=1 \text{ when } x=0.$$

[W.B.U.T., CS-312, 2008, 2009]

6. What is the truncation error in the Runge-Kutta method of order n? In what way is it related to that of Taylor's series method? [W.B.U.T., CS-312, 2006]

7. Evaluate $y(0.02)$ given

$$\frac{dy}{dx}=x^2+y, y(0)=1$$

by modified Euler's method

8. Using Runge-Kutta method with $h=0.1$, find $y(0.1)$ given

$$\frac{dy}{dx}=x+y, y(0)=1$$

9. Using Runge-Kutta method with $h=0.2$ find $y(1)$ given

$$\frac{dy}{dx}=y-x, y(0)=1.5$$

Answers

2. 1.1272 **3.** 1.111, 1.412 **4.** $1.25\times10^{-4}, 6.25\times10^{-4}$

5. 1.4593 **7.** 1.0202 **8.** 1.11034 **9.** 3.36

II. LONG ANSWER QUESTIONS

1. Solve numerically the differential equation

$$\frac{dy}{dx} = \frac{1}{2}\left(y^3 - \frac{y}{x}\right)$$

using Euler's method at $x = 1.6$, given that $y = 1$ when $x = 1$. Also find the exact value at $x = 1.6$

2. Solve the following equation by Euler's method

(a) $\frac{dy}{dx} = 1 + x^2 y, y(0) = 0.5$ at $x = 0.1$

(b) $\frac{dy}{dx} = 3x + y^2, y(0) = 1$ at $x = 0.1$

3. Solve numerically the differential equation

$$\frac{dy}{dx} = 1 - 2xy$$

provided that $y = 0$ at $x = 0$ using Euler's method in $[0, 0.6]$ taking $h = 0.2$

4. Solve the equation

$$\frac{dy}{dx} = \frac{x^2}{1+y^2}$$

with the initial condition $y(0) = 0$ by Euler's method to obtain y for $x = 0.25, 0.5$

5. Compute $y(0\cdot3)$ from $\frac{dy}{dx} = 1 + xy$ by modified Euler's method, given that

$$y(0) = 2$$

6. Using modified Euler's method, solve the following

(i) $\frac{dy}{dx} = x + \sqrt{y}, y(0) = 1$ at $x = 0.6$

(ii) $\frac{dy}{dx} = 1 - y, y(0) = 0$ at $x = 0.2$

7. Given $\frac{dy}{dx} = y - x, y(0) = 2$. Find $y(0.1)$ and $y(0.2)$ using second order Runge-Kutta method.

8. Using Runge-Kutta method with $h = 0.1$, find $y(0.2)$, $y(0.4)$ given

(i) $\frac{dy}{dx} = x - y^2, y(0) = 1$

(ii) $\frac{dy}{dx} = -xy, y(0) = 1$

(iii) $\frac{dy}{dx} = 1 + y^2, y(0) = 0$ [*W.B.U.T., CS-312, 2010*]

9. Find the values of $y(0.1), y(0.2)$ and $y(0.3)$, using Runge-Kutta method of the fourth order, given that

$$\frac{dy}{dx} = xy + y^2, y(0) = 1$$ [*W.B.U.T., CS-312, 2009*]

10. Solve the equation

$\frac{dy}{dx} = \frac{1}{x+y}, y(0) = 1$, for $y(0.1)$ and $y(0.2)$, using Runge-Kutta method of the fourth order. [*W.B.U.T., CS-312, 2006*]

11. Compute $y(0.2)$ from the equation

$$\frac{dy}{dx} = x + y, y(0) = 1$$

taking step length $h = 0.1$ by 4th order Runge-Kutta method correct to three decimal places. [*W.B.U.T., CS-312, 2007*]

12. Solve by Milne's predictor-corrector method

$\frac{dy}{dx} = \frac{2y}{x}$ at $x = 2$, given that $y(1) = 2, y(1.25) = 3.13$, $y(1.5) = 4.5$ and $y(1.75) = 6.13$

13. Apply Milne's method to find $y(0.8)$ for the equation $\frac{dy}{dx} = x + y^2$, given that $y(0) = 0, y(0.2) = 0.02, y(0.4) = 0.0805$, $y(0.6) = 0.1839$

14. Find $y(4.4)$ using Adam's Bashforth method from

$$\frac{dy}{dx}=\frac{2-y^2}{5x},\ y(4)=1,\ y(4.1)=1.0049,\ y(4.2)=1.0097,$$

$y(4.3)=1.0143$

15. Using Adam's Bashforth method, determine $y(1.4)$ given that $\frac{dy}{dx}=x^2(1+y), y(1)=1$ where the starting values are to be obtained from Runge-Kutta method.

16. Solve the following B.V.P using finite difference method:

(a) $x^2y''+xy'^2-1=0;\ y(1)=0, y(1.4)=0.0566$

(b) $xy''+y'=1$ with $y(1)=1$ and $y(1.4)=1.736$

17. Solve $y''-64y+10=0, y(0)=y(1)=0$

by finite difference method and compute the value of $y(0.5)$

Answers

1. 1.0859, 1.0659 2. (a) 0.100025 (b) 1.127

3. $y(0.2)=0.1948, y(0.4)=0.3599, y(0.6)=0.4748$

4. 0.0052, 0.0416 5. 2.4019

6. (i) 1.88278 (ii) 0.181408 7. 2.2050, 0.2421

8. (i) 0.851, 0.780 (ii) 0.9802, 0.9231 (iii) 0.2027, 0.4228

9. 1.1138, 1.2689, 1.4856 10. 1.0914, 1.1696

11. 1.2205 12. 8 13. 0.3364 14. 1.0187 15. 2.3840

16. (a) $y(1.1)=0.0046, y(1.2)=0.0167, y(1.3)=0.0345$

(b) $y(1.1)=1.193, y(1.2)=1.560, y(1.3)=1.378$

17. $y(0.5)=0.1470$

III. Multiple Choice Questions

1. The recursion formula of Euler's method is

(a) $y_{n+1}=y_n+hf(x_n,y_{n-1})$ (b) $y_{n+1}=y_n+hf(x_n,y_{n+1})$

(c) $y_{n+1}=y_n+hf(x_n,y_n)$ (d) none

2. Error in the 4-th order Runge-Kutta method is of

(a) $0(h^3)$ (b) $0(h^2)$ (c) $0(h^4)$ (d) $0(h^5)$

3. Runge-Kutta formula has a truncation error, which is of the order of

(a) h^2 (b) h^4 (c) h^5 (d) none

[*W.B.U.T., CS-312, 2004, 2006, 2010*]

4. Error in the 2nd order Runge-Kutta method is of

(a) $0(h^3)$ (b) $0(h^2)$ (c) $0(h^4)$ (d) $0(h^5)$

5. The truncation error of Euler's method is

(a) $0(h)$ (b) $0(h^3)$ (c) $0(h^4)$ (d) $0(h^2)$

6. The predictor-corrector method is

(a) Euler's method

(b) 4-th order Runge-Kutta method

(c) Taylor's series method

(d) Modified Euler's method

7. Runge-Kutta method is used to solve

(a) an algebraic equation

(b) a first order ordinary differential equation

(c) a first order partial differential equation

(d) none of these