— Langrangian & Hamiltonian.

Constraint → The limitations or, geometrical restrictions restriction on the motion of particle or system of particle are generally known as constraints. The force responsible for the constraints are called constraint forces.

Ex –

① Motion of pendulum, constricted to oscillate in the plane containing the string.

② Particle placed on the surface of a solid sphere is restricted, so that it can move only either on the surface or, outside the sphere.

Role of constraints in Dynamics: –

(i) constraints limits the degree of freedom of the system.

(ii) constraint forces leads to relations which may be in the form of equality or inequality.

Classification of constraints –

1. Holonomic and non-Holonomic constraints: –

– If condition of constraints are expressed in terms of equality then holonomic, otherwise non-holonomic

if $\bar{r}_1, \bar{r}_2, \bar{r}_3 \dots, \bar{r}_n$ ~~etc.~~ p.c. of particles

t is time

then $f(\bar{r}_1, \bar{r}_2, \bar{r}_3 \dots, \bar{r}_n, t) = 0$

means holonomic constraints.

otherwise non-" " .

Eg. of holonomic

(i) $(\vec{r}_i - \vec{r}_j)^2 - c_{ij}^2 = 0$

$\vec{r}_i$ & $\vec{r}_j$ position vector of i and j th particle, c_{ij} is the distance between them.

(ii) Point mass of simple pendulum.

$(\vec{r} - \vec{a})^2 = l^2$

$\vec{r}$ = p.v. of point mass

$\vec{a}$ = p.v. of suspension.

l = length of pendulum

$\vec{AB} = \vec{OB} - \vec{OA}$

$\Rightarrow \vec{AB} = \vec{r} - \vec{a}$

$\Rightarrow (\vec{AB})^2 = (\vec{r} - \vec{a})^2$

$\Rightarrow l^2 = (\vec{r} - \vec{a})^2$

Eg. of non holonomic constraints.

(i) motion of particle on the surface of sphere of radius a.

$r^2 - a^2 \geq 0$ (r is distance of particle from centre)

$r^2 - a^2 = 0$ (when particle remains on surface)

$r^2 - a^2 > 0$ (when particle leaves the surface)

(ii) Gas container spherical in shape, radius a.

$\vec{r}$ position vector.

$|r| \leq a$, $r - a \leq 0$

$$\frac{ds}{dt} = a\frac{d\theta}{dt}$$

$$\Rightarrow S - a\theta = \text{const}$$

description of rolling without slipping is holonomic.

2.) Scleronomic & Rheonomic

If eq of ~~motion~~ constraint doesnt contain time explicitly, then they are called scleronomic,
other wise if containing time explicitly rheonomic.

eg. of scleronomic :-
(i) pendulum $(\vec{r} - \vec{a})^2 = l^2$
(ii) bead moving in circular wire $(r^2 = a^2)$

(b) eg of Rheonomic,
(i) bead sliding in a moving wire
(ii) if effective length 'l' of a simple pendulum changes with ~~its~~ temperature at different times t of ~~the~~ a day, then

$$l = f(t) \neq \text{constant}$$

3.) Bilateral and unilateral constraints :-

if constraints are expressed in terms of equations, called bilateral, ~~or~~ in terms of inequalities unilateral.

4.) Conservative and dissipative constraints.

if the constraint forces do not do any work so that total mechanical energy remains constant, then the constraint is conservative.

If on the other hand constraint forces do work and total mechanical energy is not conserved, then the constraint is dissipative.

System	Types of constraints.
1.) A particle sliding down a sphere under gravity.	non-holonomic
2.) A bead moving on a circular wire.	holonomic, scleronomic, bilateral
3.) Rigid body motion	holonomic, scleronomic, bilateral, conservative
4.) Simple pendulum with rigid support	holonomic, scleronomic, bilateral, conservative.
5.) Pendulum with time dependent length	holonomic, rheonomic, bilateral, dissipative.
6.) Deformable body motion.	holonomic, rheonomic, bilateral, dissipative
7.) Hollow sphere filled with gas	holonomic, scleronomic, unilateral conservative
8.) Particle sliding down a ellipsoid.	non-holonomic, rheonomic

Difficulties introduced by constraints:

(1) In ~~force~~ system of particles force on ith particle.

$$\vec{F_i} = \vec{F_i}^e + \sum_{j=1}^{n} \vec{F_{ij}}$$

$\vec{F_{ie}}$ → external force on ith particle

$\vec{F_{ij}}$ = force on ith particle due to jth particle.

by Newton.

$$m_i \frac{d^2\vec{r_i}}{dt^2} = \vec{F_i}^e + \sum_{i=1}^{N} \vec{F_{ij}}$$

This represent N equation. which are connected by

$$f(\vec{r_1}, \vec{r_2}, \vec{r_3}, \ldots \vec{r_n}, t) = 0$$

Thus N-equations are connected and are not all independent.

(2) Unknown constraint forces introduce complications in deriving the solution of the problem. As for example: the force of constraint which the wire exerts on a moving bead on it not known initially and thus difficulty towards the solution of the problem.

How to overcome these?

1st problem → (1) Introduction of generalised coordinate

(2) $K \rightarrow$ No of eq of constraint

N - particle system.

no of generalised coordinate

$$\boxed{f = 3N - K}$$

↓ degree of freedom

thus N-eq represented by f independent eq.

2nd difficulty (for unknown constraint)

→ Recast the mechanics in such a way that forces of constraint disappear.
by D'Alembert's concept of virtual displacement & virtual work.

Degree of freedom.

1st def ~~there~~ number of independent ways of executing motion without violating effect of constraints is called degree of freedom of the dynamic system.

2nd def degree of freedom of a dynamic system is defined as the number of coordinates necessary to specify the position (or configuration) of a dynamical system.

K.E in 1D, 2D, 3D

$\frac{1}{2} m \dot{x}^2$, $\frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$, $\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

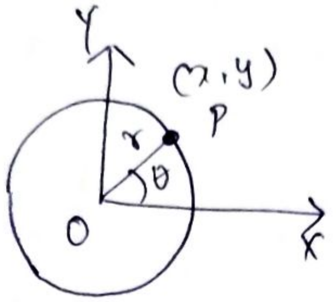
degree of freedom may also defined as no. of squared term occurring in the expression for KE of the system.

Generalised Coordinates

Set of 'independent coordinates sufficient in number to specify unambiguously the system configuration is called Generalised Coordinates.

denoted by $q_1, q_2, q_3 \cdots, q_f$

f - degree of freedom.



$x = r\cos\theta$

$y = r\sin\theta$

$q_1 = \theta$

for sphere $q_1 = \theta$, $q_2 = \phi$

Transformation equations:

~~x_1, x_2, x~~ x_i, y_i, z_i are coordinates of ith element with respect to an origin.

if generalised coordinates $q_1, q_2, q_3 \ldots, q_f, t$.

$$x_i = x_i(q_1, q_2, q_3, \cdots, q_f, t)$$
$$y_i = y_i(q_1, q_2, q_3, \cdots, q_f, t)$$
$$z_i = z_i(q_1, q_2, q_3, \cdots, q_f, t)$$

→ transformation eqn.

$$\vec{r}_i = \hat{i}x + \hat{j}y + \hat{k}z$$

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3 \cdots, q_f, t)$$

Necessary and sufficient cond :-

$$J\left(\frac{q_1, q_2, \ldots, q_f}{x_1, y_1, \ldots, z_f}\right) = \frac{\partial(q_1, q_2 \cdots q_f)}{\partial(x_1, y_1, \cdots, z_f)}$$

$$= \begin{vmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} & \cdots & \frac{\partial q_f}{\partial x_1} \\ \frac{\partial q_1}{\partial x_2} & \frac{\partial q_2}{\partial x_2} & \cdots & \frac{\partial q_f}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial q_1}{\partial z_f} & \frac{\partial q_2}{\partial z_f} & \cdots & \frac{\partial q_f}{\partial z_f} \end{vmatrix} \neq 0$$

Configuration Space:

Motion of a system of N-particles requires 3N independent coordinates for its description.

If no of constraints of that system is represented by K eq.

then degree of freedom

$$f = 3N - K.$$

Hence N particles can be considered as that of a single particle in f - dimensional space.

Generalised Motion:

1. Generalised displacement:

$$\vec{r_i} = \vec{r_i}(q_1, q_2, \ldots, q_f, t)$$

$$\delta \vec{r_i} = \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j \quad (\because \delta t = 0)$$

~~2. Generalised velocity.~~

2. Generalised velocity.

$$\vec{r_i} = \vec{r_i}(q_1, q_2, \ldots, q_f, t)$$

$$\dot{\vec{r_i}} = \sum_{j=1}^{f} \frac{\partial \vec{r}}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r_i}}{\partial t}$$

if $\vec{r_i}$ doesnot depend on t explicitly

then $\dot{\vec{r_i}} = \sum_{j=1}^{f} \frac{\partial \vec{r_i}}{\partial q_j} \dot{q}_j$, $\frac{\partial r_i}{\partial t} = 0$

— x —

if $q_j = q_j(\vec{r_1}, \vec{r_2}, \ldots \vec{r_f}, t)$

$$dq_j = \sum_{i=1}^{f} \frac{\partial q_j}{\partial \vec{r_i}} d\vec{r_i} + \frac{\partial q_j}{\partial t} dt$$

$$\dot{q}_j = \sum_{i=1}^{f} \frac{\partial q_j}{\partial \vec{r_i}} \dot{\vec{r_i}} + \frac{\partial q_j}{\partial t}$$

if q_j does not depend on time explicitly then

$$\dot{q}_j = \sum_{i=1}^{f} \frac{\partial q_j}{\partial \vec{r_i}} \dot{\vec{r_i}}$$

$$\vec{r}_i = \sum_{j=1}^{f} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

$$\ddot{\vec{r}}_i = \frac{d}{dt}\left[\sum_{j=1}^{f} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\right]$$

$$= \sum_{j=1}^{f} \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\dot{q}_j + \sum_{j=1}^{f} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j + \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial t}\right)$$

$$= \sum_{j=1}^{f} \frac{\partial \dot{\vec{r}}_i}{\partial q_j}\dot{q}_j + \sum_{j=1}^{f} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j + \frac{\partial \dot{\vec{r}}_i}{\partial t}$$

$$= \sum_{j=1}^{f} \frac{\partial}{\partial q_j}\left(\sum_{k=1}^{f} \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)\dot{q}_j + \sum_{j=1}^{f} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j$$

$$+ \frac{\partial}{\partial t}\left(\sum_{k=1}^{f} \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)$$

$$\left[\because \dot{\vec{r}}_i = \sum_{j=1}^{f} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\right]$$

$$= \sum_{j=1}^{f}\sum_{k=1}^{f} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k\dot{q}_j + \sum_{j=1}^{f} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}\dot{q}_j +$$

$$\sum_{j=1}^{f} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j + \sum_{k=1}^{f} \frac{\partial^2 \vec{r}_i}{\partial q_k \partial t}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial t^2}$$

$$\ddot{\vec{r}}_i = \sum_{j=1}^{f} \frac{\partial \vec{r}_i}{\partial q_j}\ddot{q}_j + \sum_{j=1}^{f}\sum_{k=1}^{f} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_j\dot{q}_k +$$

$$2\sum_{j=1}^{f} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}\dot{q}_j + \frac{\partial^2 \vec{r}_i}{\partial t^2}.$$

Limitations of Newton's Laws of Motion :-

① Newton's laws are valid only in inertial frame.

② A complete knowledge of forces acting on a system is needed.

③ Newton's laws valid in cartesian frame but more inconvenient to use in other coordinate system.

④ Eq of motion involve vector quantities like force, momentum etc which introduce complexity in solving the problem.

⑤ In case of constraint motion, the determination of all reaction force is difficult in Newtonian method.

Principle of Virtual Work:

The displacement $\delta\vec{r}_i$, which does not involve time interval dt, is called virtual displacement.

~~Principle of Virtual~~ Work:

A system of particles will be in equilibrium if and only if the total virtual work done by the actual or applied forces ~~on~~ in any arbitrary infinitesimal virtual displacement is zero.

if in equilibrium $\vec{F}_i = 0$

$$\delta W = \vec{F}_i \cdot \delta\vec{r}_i = 0$$

$$W = \sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i = 0$$

$$\vec{F}_i = \vec{F}_i^{a} + \vec{f}_i$$

actual force

force of constraint.

$$\therefore \sum_{i=1}^{N} (\vec{F}_i^{a} + \vec{f}_i) \cdot \delta\vec{r}_i = 0$$

or $$\sum_{i=1}^{N} \vec{F}_i^{a} \cdot \delta\vec{r}_i + \sum_{i=1}^{N} \vec{f}_i \cdot \delta\vec{r}_i = 0 \quad \text{— (I)}$$

if $\delta\vec{r}_i$ is ⊥r to $\vec{f}_i$

then $\sum_{i=1}^{N} \vec{f}_i \cdot \delta\vec{r}_i = 0$

his happens when body moves over a spherical surface when constraint force is $\perp r$ to the surface, $\vec{f_i}$ is $\perp r$ to δr_i. (This doesnt hold for frictional forces).

for statistical equilibrium:-

$$\sum_{i}^{N} \vec{F_i}^a \, \delta \vec{r_i} = 0$$

2 advantages :-

1. Nature of constraint force need not be known.
2. to determine the condition of equilibrium, a single equation is sufficient which can be solved easily.

2.7 D'Alembert's Principle

Statement. *A system of particles moves in such a way that the total virtual work done by the effective forces is zero.*

If $\vec{p}_i$ be the momentum of the ith particle of the system, then according to Newton's second law of motion the force $\vec{F}_i$ acting on it is

$$\vec{F}_i = \frac{d\vec{p}_i}{dt} = \dot{\vec{p}}_i \text{ or, } \vec{F}_i - \dot{\vec{p}}_i = 0. \tag{2.26}$$

Eqn. (2.26) represents that a moving system of particles can be considered to be in *equilibrium* under the force

$$(\vec{F}_i - \dot{\vec{p}}_i) = \vec{F}_i + (-\dot{\vec{p}}_i),$$

i.e., the actual force $\vec{F}_i$ plus an additional force $-\dot{\vec{p}}_i$. Here $-\dot{\vec{p}}_i$ is called the *reversed effective force* on the ith particle.

Now from the principle of virtual work,

$$\sum_i \left(\vec{F}_i - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0. \tag{2.27}$$

If the forces of constraint are present, then

$$\vec{F}_i = \vec{F}_i^a + \vec{f}_i. \tag{2.28}$$

Using eqn. (2.26) in eqn. (2.27), we get

$$\sum_{i=1}^{N} \left(\vec{F}_i^a - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i + \sum_{i=1}^{N} \vec{f}_i \cdot \delta \vec{r}_i. \tag{2.29}$$

If the virtual work done by the constraint forces is zero, i.e.,

$$\sum_{i=1}^{N} \vec{f}_i \cdot \delta \vec{r}_i = 0$$

then

$$\sum_{i=1}^{N} \left(\vec{F}_i^a - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0. \tag{2.30}$$

This is *D'Alembert's principle.*

Dropping the superscript a, we have

$$\sum_{i=1}^{N} \left(\vec{F}_i - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0. \tag{2.31}$$

This principle is valid for all *rheonomic* and *scleronomic systems* that are either *holonomic* or *non-holonomic.*

2.8 Derivation of Lagrange's Equations of Motion

Lagrange's equation of motion may be derived in various methods. Here we propose to derive it using *D'Alembert's principle.*

According to the coordinate transformation equation, the position vectors of the particles

$$\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_i, \cdots$$

are expressed as the functions of generalised coordinates $q_1, q_2, \cdots, q_j, \cdots, q_f$ and time t in the form,

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \cdots, q_j, \cdots, q_f; t). \tag{2.32}$$

If $\vec{F}_i$ be the *external force* acting on the ith particle of mass m_i, then according to D'Alembert's principle,

$$\sum_{i=1}^{N}\left(\vec{F}_i - \dot{\vec{p}}_i\right)\cdot\delta\vec{r}_i = 0, \tag{2.33}$$

where $\dot{\vec{p}}_i$ is the *inertial force* for the ith particle and $\delta\vec{r}_i$ is the *virtual displacement* of the ith particle due to $\vec{F}_i$.

$$\therefore \sum_{i=1}^{N}\left(\vec{F}_i - m_i\ddot{\vec{r}}_i\right)\cdot\delta\vec{r}_i = 0. \tag{2.34}$$

Differentiating eqn. (2.32) partially with respect to t, we get

$$\begin{aligned}\dot{\vec{r}}_i &= \frac{d\vec{r}_i}{dt}\\ &= \frac{\partial\vec{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial\vec{r}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_j + \cdots + \frac{\partial\vec{r}_i}{\partial q_f}\dot{q}_f + \frac{\partial\vec{r}_i}{\partial t}\\ &= \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}.\end{aligned} \tag{2.35}$$

Again, the virtual displacement $\partial\vec{r}_i$ in terms of *generalised coordinates* is given by

$$\delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j. \tag{2.36}$$

This equation does not involve the variation of time. Now, eqn. (2.34) becomes

$$\sum_i\left(\vec{F}_i - m_i\ddot{\vec{r}}_i\right)\cdot\sum_j\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = 0$$

$$\text{or,}\quad \sum_i\sum_j \vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = \sum_i\sum_j m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j. \tag{2.37}$$

Again,

$$\because \quad \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) = \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} + \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right),$$

$$\therefore \quad \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right). \tag{2.38}$$

Using eqn. (2.38) in eqn. (2.37), we get

$$\sum_i \sum_j \left(\vec{F} \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) \delta q_j = \sum_i \sum_j m_i \left[\frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right] \delta q_j. \tag{2.39}$$

Also, since

$$\dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}, \quad \therefore \quad \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \tag{2.40}$$

$$\begin{aligned}
\text{and} \quad \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) &= \sum_k \frac{\partial}{\partial q_k}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\frac{dq_k}{dt} + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\frac{dt}{dt} \\
&= \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial t \partial q_j} \\
&= \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} \\
&= \frac{\partial}{\partial q_j}\left[\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right] \\
&= \frac{\partial}{\partial q_j}\left(\frac{d\vec{r}_i}{dt}\right) = \frac{\partial \vec{v}_i}{\partial q_j} \quad \left[\because \ \vec{v}_i = \frac{d\vec{r}_i}{dt}\right]
\end{aligned} \tag{2.41}$$

Substituting eqns. (2.40) and (2.41) in eqn. (2.39), we get

$$\sum_i \sum_j \left(\vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) \delta q_j = \sum_i \sum_j \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j}\right] \delta q_j$$

$$\left[\because \ \frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right].$$

$$= \sum_j \left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_1^2\right)\right\} - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right] \delta q_j$$

$$\text{or,} \quad \sum_j Q_j \delta q_j = \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] \delta q_j = 0,$$

where

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

is the *generalised force component* and

$$\sum_i \frac{1}{2} m_i v_i^2 = T$$

is the *total kinetic energy of the system*

$$\text{or,} \quad \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0. \tag{2.42}$$

The generalised displacements δq_j are independent due to linear independence of generalised coordinates q_j. Then the eqn. (2.42) holds only if the coefficients of δq_j vanish separately, i.e.,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j, \tag{2.43}$$

where $j = 1, 2, 3, \cdots, N$.

These equations are *second order* and their number is equal to the degrees of freedom of the system. Eqn. (2.43) is derived if the system does not involve constraints and is known as the general form of *Lagrange's equation*. Eqn. (2.43) also holds for systems involving holonomic constraints because such constraints may be used to minimise the degrees of freedom of the system and as such independent of generalised coordinates.

CASE 1. *When the system is wholly conservative.*

If the system under consideration be *conservative* then the forces will be derivable from a *scalar potential function* V. For the ith particle

$$\vec{F}_i = -\Delta V_i = -\frac{\partial V_i}{\partial r_i}. \tag{2.44}$$

$$\begin{aligned}
\therefore\ Q_k &= \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \\
&= -\sum_i \frac{\partial V_i}{\partial r_i} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \\
&= -\sum_i \frac{\partial V_i}{\partial q_k} = -\frac{\partial}{\partial q_k}\sum_i V_i = -\frac{\partial V}{\partial q_k}
\end{aligned} \tag{2.45}$$

where $V = \sum_i V_i =$ the *total potential energy of the system.*

Now from eqn. (2.43), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} \quad \left[\because\ Q_k = -\frac{\partial V}{\partial q_k}\right]$$

$$\text{or,} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0$$

$$\text{or,} \quad \frac{d}{dt}\left[\frac{\partial (T - V)}{\partial \dot{q}_j}\right] - \frac{\partial (T - V)}{\partial q_j} = 0$$

$$\because\ V \neq f(\dot{q}_j), \ \therefore\ \frac{\partial V}{\partial \dot{q}_j} = 0.$$

The *Lagrangian of the system* $L = T - V$.

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad (2.46)$$

$$\text{or, } \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = 0. \qquad (2.47)$$

Eqn. (2.47) is called the *Lagrange's equation* of motion of a system of particles.

CASE 2. *When the system is non-conservative.*

If the system is *not conservative*, then also the Lagrangian can exist if Q_k are derivable from a *velocity dependent or generalised potential function* $U(q_j, \dot{q}_j)$ in the form

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right). \qquad (2.48)$$

Using this value of Q_j in eqn. (2.46), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$\text{or, } \frac{d}{dt}\left[\frac{\partial (T-U)}{\partial \dot{q}_j}\right] - \frac{\partial}{\partial q_j}(T-U) = 0.$$

Putting $L = T - U$, where U *is the generalised potential*, then

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = 0. \text{ [Ref. eqn. (2.47)]}$$

This type of equations are used in the calculation of Lagrangian for the case of *electromagnetic forces on moving charges.*

CASE 3. *When the force is non-potential.*

If Q'_j denotes the *non-potential force*, then the Lagrange's equations of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q'_j \; (j = 1, 2, \cdots, f)$$

$$Q'_j = Q'_j(q_1, q_2, \cdots, q_f, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_f, t)$$

$$\text{and } L = T - V \text{ or, } L = T - U.$$

N.B. The Coriolis force $\vec{F} = 2m(\vec{v} \times \vec{\omega})$ and the Lorenz force $\vec{F} = q(\vec{v} \times \vec{B})$ are the examples of non-potential force and these forces cannot be derived from potential.

CASE 4. *When the force is dissipative.*

We known that the dissipative (e.g., frictional) force is proportional to the velocity of the particle.

So, $$\vec{F}_i^d = -\lambda_i \dot{\vec{r}}_i. \qquad (2.49)$$

where

$\vec{F}_i^d$ = the dissipative force,

r_i = the displacement

and λ_i = the constant of proportionality for the ith particle of the system.

Now *Rayleigh's dissipation function* is defined as,

$$R = \frac{1}{2}\sum_i \lambda_i \dot{r}_i^2. \tag{2.50}$$

$$\therefore \frac{\partial R}{\partial \dot{r}_i} = \lambda_i \dot{r}_i = -\vec{F}_i^d \text{ or, } \vec{F}_i^d = -\frac{\partial R}{\partial \dot{r}_i}. \tag{2.51}$$

The Lagrange's equation is then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \frac{\partial L}{\partial r_i} + \frac{\partial R}{\partial \dot{r}_i} = 0. \tag{2.52}$$

Again,

$$\begin{aligned} Q_k^d &= \sum_i \vec{F}_i^d \cdot \frac{\partial \vec{r}}{\partial q_k} = \sum_i -\lambda_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \\ &= -\sum_i \lambda_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} \quad \left[\because \frac{\partial \vec{r}}{\partial q_k} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k}\right] \\ &= -\sum_i \lambda_i \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\dot{r}_i^2\right) = \frac{\partial}{\partial \dot{q}_k}\sum_i -\frac{1}{2}\lambda_i \dot{r}_i^2 \\ \text{or, } Q_k^d &= \frac{\partial R}{\partial \dot{q}_k}. \end{aligned} \tag{2.53}$$

Thus the *Lagrange's equation of motion* for a system containing dissipative forces is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = 0. \tag{2.54}$$

2.8.1 Characteristics of Lagrange's equation

The Lagrange's equation has the following characteristics:

(i) For a system of particles, the Lagranges equations of motion have been described by the consideration of energy and not force.

(ii) The effect of constraint forces have been eliminated in the Lagrangian formulation

(iii) $L = T - V$ is a scalar quantity and the solutions of the Lagrange's equation of motion is easy. But Newton's equation of motion involves force having vectorial sense.

(iv) Since the Lagrange's equation of motion is second order equation, its solution contains two arbitrary constants and the form of the equation is independent of the coordinates used.

Finally, under certain conditions, Lagrange's equations of motion are basically improved form of Newton's equation of motion.

2.9 Lagrangian Formulation of Conservation Theorems

First we define *generalised momentum* and *cyclic coordinates* in view of Lagrangian of a system.

Generalised momentum

We know that for a conservative system force $\vec{F} = -\Delta V$, where the potential function V is a function of position coordinates only. For such a system,

Generalised momentum $= \frac{\partial L}{\partial \dot{q}_j}$

$$\begin{aligned}\frac{\partial L}{\partial \dot{q}_j} &= \frac{\partial (T - V)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j} \\ &= \frac{\partial T}{\partial \dot{q}_j} \left[\because \frac{\partial V}{\partial \dot{q}_j} = 0\right] \\ &= \frac{\partial}{\partial \dot{q}_j} \sum_j \frac{1}{2} m_j \dot{q}_j^2 = m_j \dot{q}_j = p_j. \end{aligned} \tag{2.55}$$

$\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j} \text{ is called the } \textit{generalised momentum}}$ *associated with the generalised coordinate* q_j.

Cyclic coordinates (or, ignorable coordinates)

The Lagrangian (L) of a physical system is in general a function of all the generalised coordinates (q_j) and time t, i.e.,

$$L = L(q_1, q_2, \cdots, q_j, \cdots, q_f, t).$$

If some of the generalised coordinates do not appear explicitly in the Lagrangian function, then those coordinates are called the cyclic or *ignorable coordinates.*

As the Lagrangian is not the function of *cyclic coordinates* q_j.

$$\therefore \frac{\partial L}{\partial q_j} = 0.$$

=> If any of the generalised co-ordinate is not present in the Lagrangian then that generalised co-ordinate is said or known as cyclic or ignorable.

(i) Conservation theorem of generalised momentum

We have the Lagrange's equation of motion for a conservative system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0.$$

If q_j is cyclic, then

$$\frac{\partial L}{\partial q_j} = 0.$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

$$\text{or, } \frac{\partial L}{\partial \dot{q}_j} = \text{constant or, } p_j = \text{constant} \tag{2.56}$$

Hence, *the conservation theorem of generalised momentum states that the generalised momentum corresponding to a cyclic or ignorable coordinate is conserved.*

(ii) Conservation theorem of energy

We consider a conservative system where the potential V is a function of position only and the constraints do not change with time.

$$L = L(q_j, \dot{q}_j),$$
$$\therefore \quad \frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}$$

Again,

$$\because \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad \therefore \quad \frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right).$$

So,

$$\frac{dL}{dt} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}\right]$$
$$= \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial T}{\partial \dot{q}_j}\right).$$

$\left[\because \text{ for a conservative system, } \frac{\partial V}{\partial \dot{q}_j} = 0, \ \therefore \ \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}.\right]$

Also $p_j = \frac{\partial T}{\partial \dot{q}_j}$.

$$\therefore \quad \frac{dL}{dt} = \sum_j \frac{d}{dt}(\dot{q}_j p_j) \text{ or, } \frac{dL}{dt} - \sum_j \frac{d}{dt}(\dot{q}_j p_j) = 0$$

$$\text{or, } \frac{d}{dt}\left(\sum_j \dot{q}_j p_j - L\right) = 0$$

$$\text{or, } \sum_j \dot{q}_j p_j - L = \text{constant } (J). \tag{2.57}$$

The constant J is called the *Jacobi's integral* for the system and it is one of the first integrals of equations of motion. It can be shown that J is the Hamiltonian (H) of the system

$$\therefore \ H = \sum_j \dot{q}_j p_j - L = \text{constant}. \tag{2.58}$$

If the system is *holonomic*, then the kinetic energy of the system is a *quadratic function of generalised velocities*, i.e.,

$$T = \sum_{jk} a_{jk}\dot{q}_j\dot{q}_k.$$

If f is a homogeneous function of order n of a set of variables q_j, $j = 1, 2, \cdots$, then according to *Euler's theorem*,

$$\sum_j q_j \frac{\partial f}{\partial \dot{q}_j} = nf.$$

Here $n = 2$ and $f = T$.

$$\therefore \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T \text{ or } \sum_j \dot{q}_j p_j = 2T.$$

Now from eqn. (2.58)

$$H = 2T - L = 2T - (T - V) \text{ or, } H = T + V. \tag{2.59}$$

Thus *if the Lagrangian function does not depend explicitly on time, then the total energy of a conservative system is constant.*

(iii) Conservation theorem of linear momentum

We consider a conservative system for which the potential energy is a function of position only and the kinetic energy is independent of position. Then,

$$\frac{\partial V}{\partial \dot{q}_j} = 0 \text{ and } \frac{\partial T}{\partial q_j} = 0.$$

For such a system, the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0.$$

But,

$$\frac{\partial L}{\partial q_j} = \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} = -\frac{\partial V}{\partial q_j} \left[\because \frac{\partial T}{\partial q_j} = 0\right]$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial V}{\partial q_j} = 0.$$

So,

$$\dot{p}_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = -\frac{\partial V}{\partial q_j} = Q_j. \tag{2.60}$$

Now,

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}.$$

If $\hat{n}$ be the unit vector along the translation of the system, then

$$\delta \vec{r}_i = \hat{n}\delta q_j \text{ and } \frac{\partial \vec{r}_i}{\partial q_j} = \hat{n}..$$

$$\therefore Q_j = \sum_i \vec{F}_i \cdot \hat{n} = \hat{n} \cdot \vec{F}. \tag{2.61}$$

This shows, *the component of total force* along the direction of motion ($\hat{n}$).

The kinetic energy of the system is

$$T = \frac{1}{2}\sum_i m_i \dot{r}_i^2.$$

Now the component of *generalised momentum*

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}.$$

$$\left[\because T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2}, \therefore \frac{\partial T}{\partial \dot{q}_j} = \frac{1}{2}\sum_i m_i \frac{1}{2}\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right]$$

$$\text{or, } p_j = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} = \sum_i m_i \vec{v}_i \cdot \hat{n} = \hat{n} \cdot \sum \vec{p}_i. \tag{2.62}$$

Eqn. (2.62) shows that *the component of total linear momentum is along the direction* ($\hat{n}$) *of translation of the system.*

Hence we can say that the eqn. (2.60), i.e., $\dot{p}_j = Q_j$ *is the equation of motion for total linear momentum of the system.* If $Q_j = 0$, then $\dot{p}_j = 0$ and p_j = constant.

Hence, *if a given component of the total applied force vanishes, the corresponding component of the linear momentum is conserved.*

(iv) Conservation theorem of angular momentum

The theorem states that, *if the rotation coordinates q_j is cyclic, i.e., if the component of applied torque along the axis of rotation vanishes, then the component of total angular momentum along the axis of rotation is conserved.*

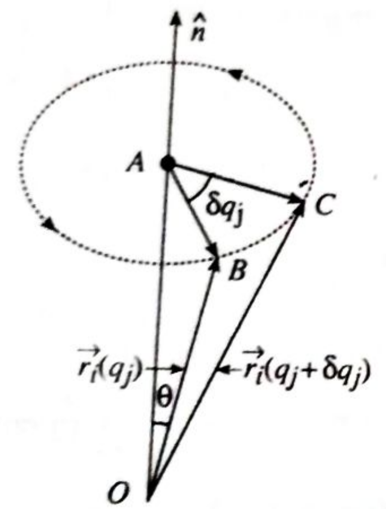


FIG. 2.3 : Conservation of angular momentum.

If V does not depend on $\dot{q}_j$ and T does not depend on q_j, then for a conservative system,

$$\frac{\partial V}{\partial \dot{q}_j} = 0 \text{ and } \frac{\partial T}{\partial q_j} = 0.$$

The Lagrangian equation for such a system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$\text{or, } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\text{or, } \dot{p}_j = Q_j \text{ [cf. eqn. (2.60)].} \tag{2.63}$$

If now, we can show that with q_j, the generalised force Q_j is the component of total applied torque about the axis of rotation and the generalised momentum p_j is the component of total angular momentum along the same axis then eqn. (2.63) will represent the *equation of motion for the total angular momentum.*

The generalised force

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}.$$

The magnitude of change in $\vec{r}_i(q_j)$ due to a change in rotation coordinate q_j is given by

$$|\delta \vec{r}_i| = |\overrightarrow{AB}|\delta q_j = |\vec{r}_i| \sin\theta \delta q_j \text{ or, } \left|\frac{\partial \vec{r}_i}{\partial q_j}\right| = r_i \sin\theta. \tag{2.64}$$

If $\hat{n}$ be the unit vector along the axis of rotation, then

$$\frac{\partial \vec{r}_i}{\partial q_j} = \hat{n} \times \vec{r}_i. \tag{2.65}$$

Now the generalised force

$$\begin{aligned} Q_j &= \sum_i \vec{F}_i \cdot \hat{n} \times \vec{r}_i = \sum_i \hat{n} \cdot (\vec{r}_i \times \vec{F}_i) \\ &= \sum_i \hat{n} \cdot N_i = \hat{n} \cdot \vec{N}, \end{aligned} \tag{2.66}$$

where $\vec{N}$ is the *total torque*. *Thus Q_j is the component of the total torque along the axis of rotation.*

Again

$$\begin{aligned} p_j &= \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \\ &= \sum_i m_i \vec{v}_i \cdot \hat{n} \times \vec{r}_i = \sum_i \hat{n} \cdot \vec{r}_i \times m_i \vec{v}_i \\ \text{or, } p_j &= \hat{n} \cdot \sum_i \vec{L}_i = \hat{n} \cdot \vec{L}. \end{aligned} \tag{2.67}$$

$$\vec{L} = \sum_i \vec{L}_i = \text{total angular momentum.}$$

Eqn. (2.67) shows that, *p_j is the component of the total angular momentum along the axis of rotation.*

So, $\dot{p}_j = Q_j$ represents the equation of motion of the system.

If q_j is cyclic, then

$$\frac{\partial L}{\partial q_j} = 0 \text{ or, } \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} = 0 - \frac{\partial V}{\partial q_j} \text{ or, } \frac{\partial L}{\partial q_j} = -\frac{\partial V}{\partial q_j}.$$

$$\therefore \quad \hat{n} \cdot \vec{N} = Q_j = -\frac{\partial V}{\partial q_j} = \frac{\partial L}{\partial q_j} = 0.$$

$$\therefore \quad \dot{p}_j = Q_j = -\frac{\partial V}{\partial \dot{q}_j} = 0$$

$$\text{or,} \quad p_j = \text{constant or, } p_j = \hat{n} \cdot \vec{L} = \text{constant.} \tag{2.68}$$

Eqn. (2.68) represents the *conservation law of angular momentum for a conservative system.*

2.9.1 Advantage of Lagrangian approach over Newtonian approach

(1) In Lagrangian formulation we have only to deal with two scalar functions T and V which greatly simplifies the problem, But in Newtonian formulation we have to work with many forces and accelerations which are vector quantities.

(2) Lagrangian formulation, unlike Newtonian formulation, is not restriced to Cartesian (or its equivalent) coordinates as well as the forces of constraints.

(3) T and V are scalar functions and are invariant under coordinate transformations. In

Newtonian approach force, momentum, torque, etc. vector quantities are involved which introduce difficulty in solving problems.

(4) When constraints are involved, it may not be possible to know the forces but KE (T) and PE (V) may be given. Under such situation, Lagrangian formulation is far more convenient and useful.

(5) Newtonian approach is valid only in *inertial frame*. But Lagrangian approach is valid in both *inertial and non-inertial frame of reference*; only one must calculate T in an inertial frame.

2.10 The Hamiltonian Formulation

The Lagrangian formulation is described in terms of generalised coordinate (q_j), generalised momentum ($\dot{q}_j$) and time which are independent variables. Here the generalised velocity ($\dot{q}_j$) being the time derivative of $q_j, \dot{q}_j$ are treated ultimately as *dependent variables.* In Hamiltonian formulation this dependence is removed by introducing a new independent variable p_j, where p_j is the *generalised momentum.* This means that the basis has been changed from $(q_j, \dot{q}_j, t)$ to (q_j, p_j, t). Like the Lagrangian $L = L(q_j, \dot{q}_j, t)$ a new function H is defined as

$$H = H(q_j, p_j, t)$$

where H is called the *Hamiltonian function.*

2.10.1 The Hamiltonian Function (H)

We know that the Lagrangian of a system is

$$L = L(q_j, \dot{q}_j, t).$$

$$\therefore \quad \frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t}. \tag{2.69}$$

The Lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \text{ or, } \frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = p_j. \tag{2.70}$$

Using eqn. (2.70) in eqn. (2.69), we get

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t} = \sum_j \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right] + \frac{\partial L}{\partial t}$$

$$\text{or, } \frac{d}{dt}\left[L - \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right] = \frac{\partial L}{\partial t} \tag{2.71}$$

$$\text{or, } \frac{d}{dt}\left[L - \sum_j p_j\dot{q}_j\right] = \frac{\partial L}{\partial t} \text{ or, } \frac{d}{dt}\left[\sum_j p_j\dot{q}_j - L\right] = -\frac{\partial L}{\partial t}. \tag{2.72}$$

We now introduce a new function such that

$$H = \sum_j p_j \dot{q}_j - L(q_j, \dot{q}_j, t), \tag{2.73}$$

where H is called the *Hamiltonian function* of the independent variables q_j, p_j, t. Hence,

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}. \tag{2.74}$$

2.10.2 Hamilton's canonical equations of motion

We know that the Hamiltonian is a function of q_j, p_j, t, i.e., $H = H(q_j, p_j, t)$

$$\therefore \; dH = \sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt. \tag{2.75}$$

Again, since $H = \sum_j p_j \dot{q}_j - L$,

$$\therefore \; dH = \sum_j \dot{q}_j dp_j + \sum_j p_j d\dot{q}_j - dL. \tag{2.76}$$

Again, $\because \; L = L(q_j, \dot{q}_j, t)$,

$$\begin{aligned}\therefore \; dL &= \sum_j \frac{\partial L}{\partial q_j} dq_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt \\ &= \sum_j \left[\frac{\partial L}{\partial q_j} dq_j + p_j d\dot{q}_j\right] + \frac{\partial L}{\partial t} dt.\end{aligned} \tag{2.77}$$

Using this value of dL in eqn. (2.76), we get

$$\begin{aligned} dH &= \sum_j \dot{q}_j dp_j + \sum_j p_j d\dot{q}_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j p_j d\dot{q}_j - \frac{\partial L}{\partial t} dt \\ dH &= \sum_j \dot{q}_j dp_j - \sum_j \dot{p}_j dq_j - \frac{\partial L}{\partial t} dt \quad \left[\because \; \dot{p}_j = \frac{\partial L}{\partial q_j}\right]. \end{aligned} \tag{2.78}$$

Comparing the coefficients of dp_j, dq_j and dt in eqns. (2.75) and (2.78), we get

$p_j = \frac{\partial L}{\partial \dot{q}_j}$

V.V.I.

$$\left.\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} \end{aligned}\right\} \tag{2.79}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \tag{2.80}$$

$\dot{p}_j = -\frac{\partial H}{\partial q_j}$

$\dot{q}_j = +\frac{\partial H}{\partial p_j}$

$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$

The eqns. (2.79) are called *Hamilton's canonical equations of motion.* These are $2N$ first order equations of motion for a system containing N particles in absence of holonomic constraints in place of $3N$ Lagrangian equations of the 2nd order. Integrating $2N$ first order (partial) differential equations we get $6N$ constants which can be found out from the knowledge of the initial conditions of the problem.

2.10.3 Physical significance of H

Though both L and H have the dimension of energy but H is not equal to the total energy in all situations. This is due to the following facts:

(i) *For conservative system the potential energy is coordinate dependent and not velocity dependent.*

(ii) *The coordinate transformation equations are independent of time, so that* $\sum_j p_j\dot{q}_j = 2T$.

If the above conditions are fulfilled, then only H can represent the total energy E.

If the coordinates transformation equations involve time explicitly, L may still be independent of time. Then

$$H = \sum_j p_j\dot{q}_j - L(q_j, \dot{q}_j)$$

is a constant of motion but not equal to total energy.

We have the Hamiltonian function

$$\begin{aligned}
H &= H(q_1, q_2, \cdots, q_j \cdots, p_1, p_2, \cdots, p_j, \cdots, t) \\
\therefore \quad \frac{dH}{dt} &= \sum_j \frac{\partial H}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial H}{\partial p_j}\dot{p}_j + \frac{\partial H}{\partial t} \\
&= -\sum_j \dot{p}_j\dot{q}_j + \sum_j \dot{q}_j\dot{p}_j + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.
\end{aligned}$$

But

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \quad \therefore \quad \frac{dH}{dt} = -\frac{\partial L}{\partial t}. \tag{2.81}$$

If $L \neq L(t)$, then $\frac{\partial L}{\partial t} = 0$ and so

$$\frac{dH}{dt} = 0 \text{ or, } H = \text{constant}. \tag{2.82}$$

Thus, *if the Lagrangian is not an explicit function of time, then the Hamiltonian is a constant of motion.* Now *for a conservative system*, V does not depend on $\dot{q}_j$, i.e., $\frac{\partial V}{\partial \dot{q}_j} = 0$.

Also,

$$\begin{aligned}
H &= \sum_j p_j\dot{q}_j - L = \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L \quad \left[\because \; p_j = \frac{\partial L}{\partial \dot{q}_j}\right] \\
&= \sum_j \dot{q}_j\left[\frac{\partial}{\partial \dot{q}_j}(T - V)\right] - L = \sum_j \dot{q}_j\frac{\partial T}{\partial \dot{q}_j} - L \quad \left[\because \; \frac{\partial V}{\partial \dot{q}_j} = 0\right] \\
&= \sum_j \dot{q}_j\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m\dot{q}_j^2\right) - L \quad \left[\because \; T = \frac{1}{2}m\dot{q}_j^2\right] \\
&= \sum_j m\dot{q}_j^2 - L = 2T - L \\
&= 2T - (T - V) = T + V = E, \text{ the total energy.}
\end{aligned} \tag{2.83}$$

Hence *if the coordinate transformation is independent of time for a conservative system then the Hamiltonian represents the total energy of the system.*

2.10.4 If a given coordinate is cyclic in Lagrangian, it will also be cyclic in Hamiltonian

$\frac{\partial L}{\partial q_j} = 0$

If q_j is cyclic in L, then

$$\dot{p}_j = \frac{\partial L}{\partial q_j} = 0; \text{ so } p_j \text{ is constant of time.}$$

$\dot{p}_j = 0$

$\dot{p}_j = -\frac{\partial H}{\partial q_j}$

Again, if p_j is constant, then

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = 0.$$

This shows that q_j does not appear in H, i.e., q_j is also cyclic in H. A cyclic coordinate reduces the number of variables by two in Hamiltonian formulation.

2.10.5 Characteristics of Hamiltonian function

(i) The Hamiltonian function is defined as

$$H(p_j, q_j, t) = \sum_j \dot{q}_j p_j - L(q_j, \dot{q}_j, t).$$

Thus H is a function of p_j, q_j and t. But Lagrangian is a function of $q_j, \dot{q}_j, t$.

(ii) If the KE T of the system is a homogeneous quadratic function of velocities and PE V is independent of velocity and time, then only the Hamiltonian (H) is equal to the total energy of the system, i.e., $H = T + V$.

(iii) If the constraints are time dependent (*rheonomic*) or the transformation equations

$$r_i = r_i(q_i, t)$$

contains time explicitly, then $H \neq E$ but still E is constant of time.

2.10.6 Advantage of Hamiltonian formulation over Lagrangian formulation

(i) The two variables q_j and $\dot{q}_j$ are not given equal status in Lagrangian formulation since $\dot{q}_j$ is simply the time derivative of q_j. But in Hamiltonian formulation both coordinate (q_j) and momentum (p_j) are given equal status. This provides a frequent freedom of choosing coordinates and momenta.

(ii) The fact that the numerical value of H is equal to the total energy of the conservative system is very important in the cases of energy changes in atoms and molecules.

(iii) The 'equality status' of p_j and q_j in Hamiltonian formulation provides a convenient basis for the development of *quantum mechanics* and *statistical mechanics*.

(iv) In quantising a dynamical system, the knowledge of the Hamiltonian is extremely important. In setting up Schrödinger equation in *wave mechanics* we replace generalised momenta by corresponding differential operators.