PROBABILITY (MATH 2202) (Mod 2 & 3).

Sample Space - In any random experiment, the sample space, where (Ω) is the set of all possible outcomes.

1) $\Omega = \{H, T\}$.
2) $\Omega = \{HH, HT, TH, TT\}$ } finite
3) of getting first head, when a toss is coined
$\{1, 2, 3, 4 \ldots\} \rightarrow$ countable infinite

4) There are two very important classification of a sample space.
→ We will call a sample space discrete, if it is finite or countably infinite.
→ We will call a sample space to be continuous, if it uncountably infinite.

We define an event in probability theory ($E \subseteq \Omega$) as a subset of sample space.
Events can be classified into three major categories:-
1 (1) Mutually exclusive events - Two events E & F will be
2 termed as mutually exclusive if $E \cap F = \phi$
3.

2) Events Equally likely events - Events will be called equally likely, if they have equal chance of happening

3) Mutually Independent events -

if $P(A_i \cap A_j) = P(A_i) \cdot P(A_j) \ \forall\ i \neq j, \ i,j = 1, 2 \ldots n$

$P(A_i \cap A_j \cap A_k) = P(A_i) \cdot P(A_j) \cdot P(A_k) \cdot \ \forall\ i \neq j \neq k$

Probability (**Classical definition**) – Suppose we have a sample space 'Ω' which has n equally likely points.

(Event) $E \rightarrow m$ points

$$P(E) = \frac{m}{n}$$

drawback:- (a) we use equally likely to define probability but earlier we don't know equally likely.

(b) set must be finite.

AXIOMATIC DEFINITION (BY:- A.N. KOLMOGOROV)

$\Omega \rightarrow$ Sample space and

$E \subseteq \Omega$ E is event then probability of 'E' written as $P(E)$ is a real number which obeys the following three axioms:-

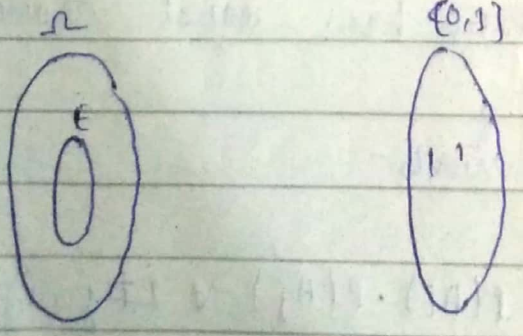
(i) $P(E) \in [0,1]$

(ii) $P(\Omega) = 1$.

(iii) If $E_1 \cap E_2 = \phi$.

$$P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

(Additive property of probability)



* Consider that 'A' is an event $A \subseteq \Omega$ then prove that

$$\boxed{P(A^c) = 1 - P(A)}$$

$E_1 = P(A)$, $E_2 = P(A^c)$

an $A \cap A^c = \phi$ $A \cup A^c = \Omega$

so, $P(A) + P(A^c) = P(\Omega)$

$P(A) + P(A^c) = 1.$

$P(A^c) = 1 - P(A)$

$$\boxed{P(\phi) = 0}$$

→ $P(A) = 0$ then A doesn't necessary that is null set.

→ Consider that 'A' and 'B' are two events such that $A \subseteq B$ prove that $P(A) \leq P(B)$

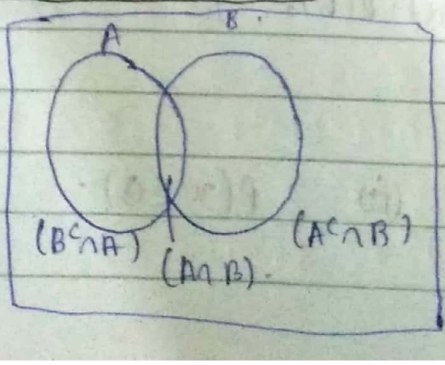
$B = A \cup (B - A)$

$= A \cup (A^c \cap B)$

$P(B) = P(A) + P(A^c \cap B) \geq 0$

so, $P(A) \leq P(B)$

$\boxed{P(A \cup B) \leq P(A) + P(B)}$ BOOLE'S Inequality.



as disjoint

$$(A \cup B) = (A^c \cap B) + (A \cap B) + (B^c \cap A)$$

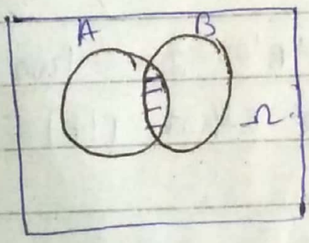
$$P(A \cup B) = P(A^c \cap B) + P(A \cap B) + P(B^c \cap A)$$
$$= P(A^c \cap B) + P(A)$$
$$= P(A) + P(B) - P(A \cap B)$$

Addition law.

Conditional Probability

Consider that there are two events A & B with the event 'A' already happend then the conditional probability of

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$



$$\boxed{P(A \cap B) = P(A) \cdot P(B/A)}$$ Multiplication law.

$$= P(B) \cdot P(A/B).$$

$\forall$ $P(B/A) = P(B)$

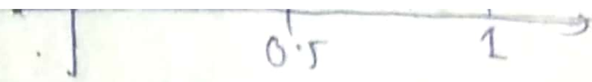
$P(A/B) = P(A)$

$P(A \cap B) = P(A) \cdot P(B).$

if ⇒ A & B are independent of each other.

Q→ $[0,1]$

① $P(x > 0.5)$ (ii) $P(x \in Q)$.



Baye's theorem

7R	4R	5R
3B	6B	5B.

$P(X) = P(Y) = P(Z) = 1/3$

$P(R) = P(X) \cdot P(R/X) + P(Y)\left(P\left(R/Y\right)\right) + P(Z) \cdot P(R/Z)$

$$P(X/R) = \frac{P(X \cap R)}{P(R)} = \frac{P(X) \cdot P(R/X)}{P(R)}.$$

$\cancel{P(X/R) = P(X) \cap R}$

→ if $A_1, A_2, A_3 \ldots A_n$ be a given set of n pairwise mutually exclusive & exhaustive events then for any event A, where $P(A) > 0$.

$$P(A) = P(A_1) \cdot P(A/A_1) + P(A_2) \cdot P(A/A_2) + \ldots P(A_n) P(A/A_n)$$

RMSG

Random Variables & Probability Distribution

Consider that we have a sample space denoted by ~~ct~~ Ω. Then a random variable usually denoted by X is a function defined on the sample space and its to the set of real nos.

$$X : \Omega \longrightarrow \mathbb{R}.$$

A random variable can be of two types:-

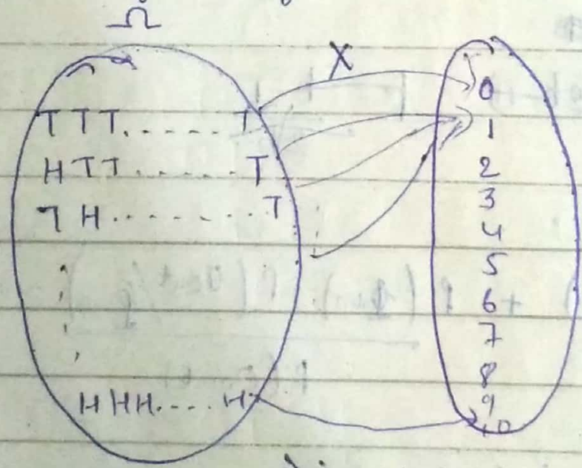
(1) Discrete

(2) Continuous

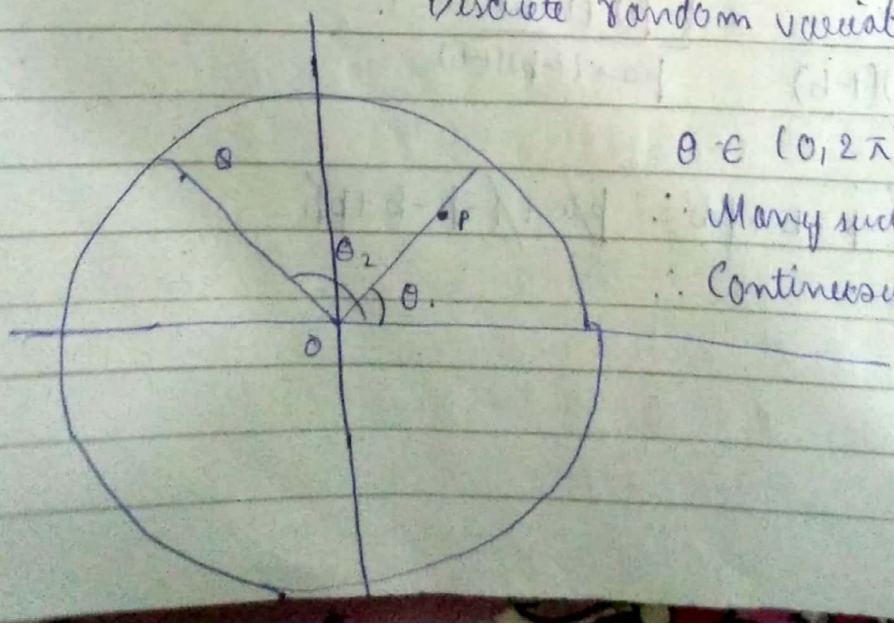
We will call a random variable to be discrete if Range(X) is a finite / countable set.

and continuous, if Range(X) = uncountable set.

eg→



Discrete random variable.



$\theta \in [0, 2\pi]$.

∴ Many such θ, ~~can~~

∴ Continuous random variable

or, drawing concentric circles, no. of circle $\in [0, \infty)$
but it is again uncountable & of infinite.

DISCRETE DISTRIBUTION

A discrete probability distribution is described by a probability mass function (PMF). It is defined as probability

$$P(X = x_i) = p_i$$

where x_i are points belonging to the range of X. and one very significant property of PMF is summation of $\sum_i p_i = 1$.

A discrete distribution can also be described in terms of a cumulative distribution function (cdf) which is defined as $F(x) = P(X \leq x)$.

Q-1. A coin is tossed 4 times in succession and the no. of heads is noted. Construct a pmf and a cdf for the random variable X = no. of heads.

→

$X = \{0, 1, 2, 3, 4\}$.

| H | H | T | T |

$^4C_2/16 = 6/16$.

$P(X=0) = 1/2^4 = 1/16$

$P(X=1) = 4/2^4 = 4/16$

$P(X=2) = 6/16$

$P(X=3) = {}^4C_3/16 = 4/16$

$P(X=4) = 1/16$.

$F(0) = P(X \leq 0) = 1/16.$

$F(1) = P(X \leq 1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$

$F(2) = \frac{5}{16} + \frac{6}{16} = \frac{11}{16}.$

$F(3) = \frac{11}{16} + \frac{4}{16} = \frac{15}{16}$

$F(4) = \frac{15}{16} + \frac{1}{16} = 1.$

Q.2. Consider that u have a biased coin where the probability of head is p. u are tossing the coin repeatedly till u get the first head. If we take the random variable X, which counts no. of your attempts, then find pmf & cdf of X.

$P(X=1) = p$

$P(X=2) = (1-p)p.$

$P(X=3) = (1-p)^2 p$

$P(X=4) = (1-p)^3 p$

$\vdots$

$P(X=i) = (1-p)^{i-1} p.$

$F(1) = P(X \leq 1) = p$

$F(2) = P(X \leq 1) = p + (1-p)p$

$F(3) = p + (1-p)p + (1-p)^2 p.$

$\vdots$

$F(i) = p\{1 + (1-p) + (1-p)^2 + \dots (1-p)^{i-1}\}$

$= p \cdot \frac{1}{1-(1-p)} = 1.$

Continuous Probability Distribution

In a continuous probability distribution, the probabilities are described by a function 'f' - probability density function (pdf). and it is defined as

$$P(a < X \leq b) = P(a \leq X < b) = P(a \leq x < b) = P(a \leq x \leq b) = \int_a^b f(x)\,dx$$

Properties:-

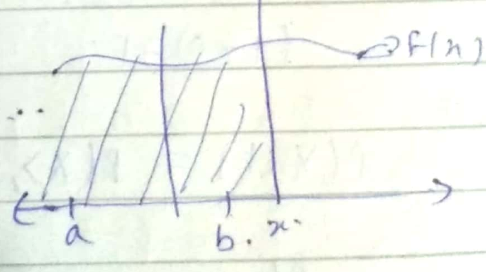
- The pdf has two very important properties:-

1. $f(x) \geq 0 \quad \forall x$
2. $\int_{-\infty}^{+\infty} f(x)\,dx = 1.$

The probabilities in a continuous probability distribution can also be described by a cumulative distribution function (cdf). It is denoted by F and

$$F(x) = P(-\infty \leq x \leq x).$$

$$= \int_{-\infty}^{x} f(x)\,dx.$$



$$P(a \leq x < b) = F(b) - F(a).$$

$$\boxed{F'(x) = f(x)}$$

$$\boxed{\begin{aligned} F(-\infty) &= 0. \\ F(\infty) &= 1. \end{aligned}}$$

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P.T. the cdf is a monotonically increasing function.

for any $x_1 < x_2$ &

$F(x_1) \leq F(x_2)$ - To proof

$F(x_2) = P(-\infty < x \leq x_2)$

$= P(-\infty < x \leq x_1) + P(x_1 \leq x < x_2)$.

$F(x_1) = P(-\infty < x < x_1)$.

Thats help us to known the function is increasing.

26 (a) $f(x) = 6x(1-x) \quad 0 \leq x \leq 1$.

(1) $f(x) \geq 0 \quad \forall x \in [0,1]$.

(2) $\int_{-\infty}^{\infty} (6x - 6x^2)\,dx = 1$ or $\int_0^1 (6x - 6x^2)\,dx = 1$

$\left[\frac{6x^2}{2} - \frac{6x^3}{3}\right]_0^1$

$(1 - 0) = 1$.

(b) $P(X < b) = P(X \geq b)$.

$\Rightarrow \int_0^b (6x - 6x^2)\,dx = \int_b^1 (6x - 6x^2)\,dx$.

$6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^b = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_b^1$

$\frac{b^2}{2} - \frac{b^3}{3} = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{b^2}{2} - \frac{b^3}{3}\right) \Rightarrow 2\left(\frac{b^2}{2} - \frac{b^3}{3}\right) = \frac{1}{6}$

$$b^2\left(\frac{3-2b}{6}\right) = \frac{1}{6\times 2}$$

$$b^2(3-2b) = \frac{1}{2}.$$

$$6b^2 - 4b^3 = 1.$$

$$b^2(6-4b) = 1$$

$\cancel{b = 0 \text{ or } 6/4}$. $\quad 6b^2 - 4b^3 - 1 = 0.$

$$4b^3 - 6b^2 + 1 = 0.$$

$$\therefore b = 1/2.$$

Expectation & Variance

If a random variable X follows a discrete distribution, as described by the p.m.f. $P(X = x_i) = p_i$ then the expectation of X $(E(X) = \sum_i p_i x_i$.

If $g(X)$ is a function of random variable X, then

$$E(g(x)) = \sum_i p_i g(x_i)$$

If x follows a continuous distribution with the density function $f(x)$, then $E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx$.

If $g(x)$ is a function of x then, $E(g(x)) = \int_{-\infty}^{+\infty} g(x) \cdot f(x)\, dx$

For both discrete as well as continuous functions, the variance of X is given by $Var(X) = E(X^2) - \{E(X)\}^2$

standard deviation $= +\sqrt{\text{variance}}$

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$E(ax+b)$

$[E\{x - E(x)\}^2] = var(x)$. $\Rightarrow aE(x)+b$.

$\Rightarrow E(x^2 - 2(x)E(x) + E(x)^2)$. where $m = E(x)$

$E(x-m)^2 = E(x^2 + m^2 - 2mx)$

$\Rightarrow E(x^2) + m^2 - 2mE(x)$

$\Rightarrow E(x^2) + m^2 - 2m^2$

$\Rightarrow E(x^2) - m^2$

$\Rightarrow E(x^2) - \{E(x)\}^2$.

2.2

X	p	q	$(X-2)^2 \Rightarrow (x-m)^2$
0	1/16	$1/16 \times 1/16 = 1/256$	4
1	4/16	12/256	1
2	6/16	54/256	0
3	4/16	108/256	1
4	1/16	81/256	4

$E(x) = 0 \cdot 1/16 + 1 \cdot 4/16 + 2 \cdot 6/16 + 3 \cdot 4/16 + 4 \cdot 1/16$

$= \frac{4+12+12+4}{16} = 2$.

If coin is baised with $p(H) = 3/4$, $p(T) = 1/4$.

$E(x) = 0\ 12/256 + 2 \cdot 54/256 + 3 \cdot 108/256 + \frac{4 \times 81}{256}$

$\frac{12 + 108 + 324 + 324}{256} = 3$.

$Var(X) = E(X^2) - \{E(X)\}^2$

$= E[(X-m)^2]$, $m = E(X)$.

$= 4 \cdot 1/16 + 1 \cdot 4/16 + 0 + 4/16 \cdot 1 + 4 \cdot 1/16$

$= 1$.

2.6 X — DISCRETE / CONTINUOUS

(a).

To proof :- $E(aX+b) = aE(X) + b$.

(1) if X - DISCRETE.

$P(X = x_i) = p_i$

$E(g(X)) = \sum_i p_i g(x_i)$

$E(aX+b) = \sum_i p_i (ax_i + b)$

$= a \underbrace{\sum_i p_i x_i}_{E(X)} + b \underbrace{\sum_i p_i}_{1}$

$= a E(X) + b$.

(2) if X is CONTINUOUS.

$X \sim f(x)$.

$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx$.

$E(aX+b) = \int_{-\infty}^{+\infty} (ax+b) \cdot f(x) dx$.

$= a \int_{-\infty}^{+\infty} x f(x) dx + b \underbrace{\int_{-\infty}^{\infty} f(x) dx}_{1}$.

$= a E(X) + b$.

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To proof:-

(b) $Var(aX+b) = a^2 Var(X)$.

$E[(aX+b)^2] - E[(aX+b)]^2$

$\Rightarrow E[a^2x^2+b^2+2axb] - [a^2\{E(X)\}^2+2ab\,E(X) + b^2]$

$\Rightarrow a^2 E(X^2) + \cancel{2abE(X)} + \cancel{b^2} - a^2\{E(X)\}^2 - \cancel{2abE(X)} - \cancel{b^2}$

$\Rightarrow a^2 (E(X^2) - \{E(X)\}^2)$

$\Rightarrow a^2 (Var(X))$

23) $Var(X) = E(X^2) - \{E(X)\}^2 = E[(X-m)^2]$, $m = E(X)$.

X	p.	$(X-1)^2$.
0	p	
1	1-2p.	
2.	p	

~~E(X^2)~~

$E(X) = (1-2p) + 2p = 1$.

~~E(X^2)~~ $E(X^2) = 1^2 \times (1-2p) + 2^2 \times p$

$= 1-2p+4p$

$= 1+2p$.

$Var(X) = \cancel{1}+2p - \cancel{1} = 2p$.

$max(Var(X)) = 2 \times 0.5$ | at $p = 0.5$

$= 1$

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2.5 $f(x) = 1 \quad 1 \le x \le 2.$
$f(x) = 0. \quad$ else
$X \to$ Radius

$$E(\pi X^2) = \int_1^2 \pi x^2 dx \Rightarrow \frac{\pi x^3}{3}\Big|_1^2$$

$$\Rightarrow \pi\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7\pi}{3}.$$

$$var(\pi X^2) = E(\pi X^2)^2 - \{E(\pi X^2)\}^2.$$

$$E((\pi X^2)^2) = E(\pi^2 X^4)$$
$$= \pi^2 E(X^4).$$
$$\pi^2 \int_1^2 x^4 dx \Rightarrow \pi^2 \frac{x^5}{5}\Big|_1^2$$
$$\Rightarrow \pi^2\left(\frac{32-1}{5}\right) = \frac{31}{5}\pi^2.$$

$$Var(\quad) = \left(\frac{31}{5} - \frac{49}{9}\right)\pi^2$$
$$= \frac{34}{45}\pi^2.$$

$-\frac{ax^2}{2} + 3ax.$
$3ax - \frac{ax^2}{2}\Big|_2^3$
$\left(9a - \frac{9a}{2}\right)(6a - 2a)$
$\frac{9a}{2} - 4a$
$\frac{a}{2}.$

2.7 (a) $\int_{-\infty}^{+\infty} f(x)dx = 1$

$$\Rightarrow \int_0^3 f(x)dx = 1$$

$$\Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (-ax+3a)dx = 1.$$

$\frac{ax^2}{2}\Big|_0^1$

$$\frac{a}{2} + a + \frac{a}{2} = 1 \Rightarrow \frac{3a}{2} + \frac{4a}{3} = 1$$
$$\Rightarrow \frac{9a + 8a}{6} = 1 \quad a = \frac{6}{17}$$
$$\Rightarrow a = 1/2.$$

Special Discrete Probability Distributions

BINOMIAL DISTRIBUTION

Consider a random experiment, where the following three conditions are satisfied.

(1) No. of trials is finite and they are independent of each other.

(2) each trial has exactly 2 outcomes. (success / failure)

(3) the probability of success in each trial is constant.

$$\boxed{P(X=i) = {}^{n}C_i \, p^i (1-p)^{n-i}}$$

where i is no. of success
$n \to$ no. of trials

$X \sim B(n, p)$.

POISSON DIST.

(1) $n \to \infty$

(2) $p \to 0$

(3) $\lambda = np$ (moderate no.).

$$\boxed{P(X=i) = \frac{e^{-\lambda} \lambda^i}{i!}}$$

$i = 0, 1, 2, \ldots$

(3)

(3.1) $n = 3$

$i = 1$, $p = 0.1$

$P(X \leq 1) = {}^3C_1 (0.1) \cdot (0.9$

$= {}^3C_0 (0.1)^0 (0.9)^3 + {}^3C_1 (0.1)^1 (0.9)^2$

$= 0.972$

(3.2) $A \to \frac{3}{5}$ $B \to 2/5$, $n = 5$, $X \to$ no. of wins by A.

$$P(X \geq 3) = \sum_{i=3}^{5} {}^5C_i (3/5)^i (2/5)^{5-i}$$

(3.4) 4-engine $\to$ $n = 4$,

2-engine $\to$ $n = 2$,

$X =$ no. of operative engines.

for 4-engine $P(X \geq 2) = 1 - P(X < 2)$

$= 1 - [P(X=0) + P(X=1)]$

$S_1 = 1 - [{}^4C_0 (p)^0 (1-p)^4 + {}^4C_1 (p)^1 (1-p)^3]$.

for 2-engine

$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X=0)$

$S_2 = 1 - {}^2C_0 \, p^0 (1-p)^2$.

$\therefore S_1 > S_2$.

$\Rightarrow \cancel{1} - [{}^4C_0 (1-p)^4 + {}^4C_1 \, p(1-p)^3] > \cancel{1} - {}^2C_0 (1-p)^2$

3.5

$n = 52$

$p = 0.95$

X = No of passengers turning up.

$P(X \leq 50) = 1 - P(X=51) - P(X=52)$

$n = 52$

$p = 0.05$

Y = No. of passengers not turning up.

$P(Y \geq 2) = 1 - P(Y=0) - P(Y=1)$.

3.6 $n = 100$

$p = 0.05$

X = No. of pins damaged.

$$P(X \geq 10) = \sum_{i=10}^{100} {}^{100}C_i \cdot (0.05)^i (0.95)^{100-i} = 1 - P(X \leq 10)$$

3.7. $\lambda = 2.5 \times 4 = 10$.

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - \frac{e^{-10} 10^0}{0!} - \frac{e^{-10} 10^1}{1!}$$

3.8 $\lambda = \frac{n}{10}$ $n = ?$

$$P(X=7) = \frac{e^{-\lambda} \cdot \lambda^7}{7!} > \frac{9}{10}$$

$$\frac{e^{-n/10} \cdot (n/10)^7}{7!} > \frac{9}{10}$$

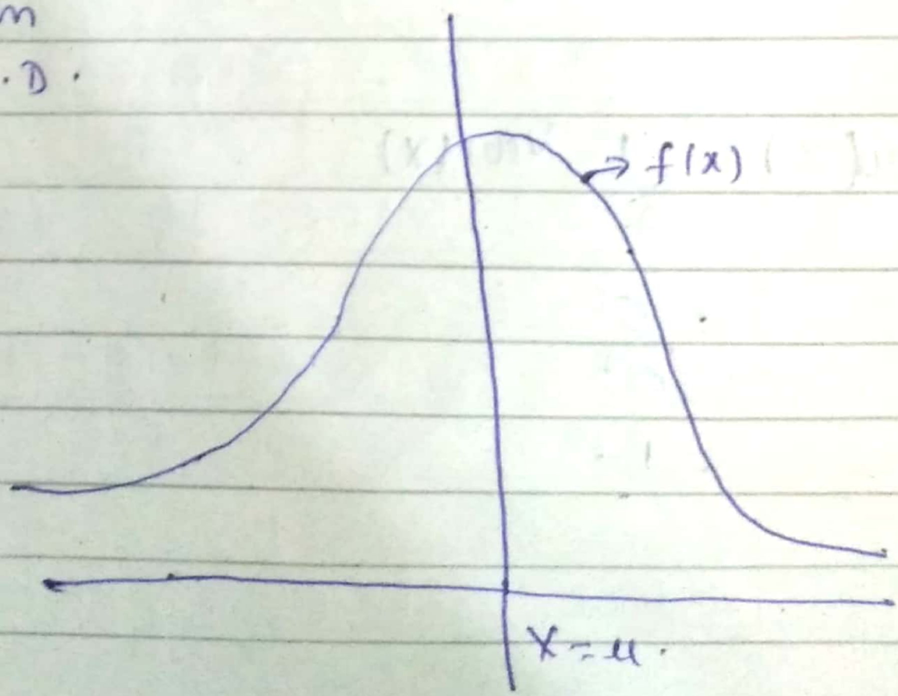
NORMAL DISTRIBUTION

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < +\infty .$$

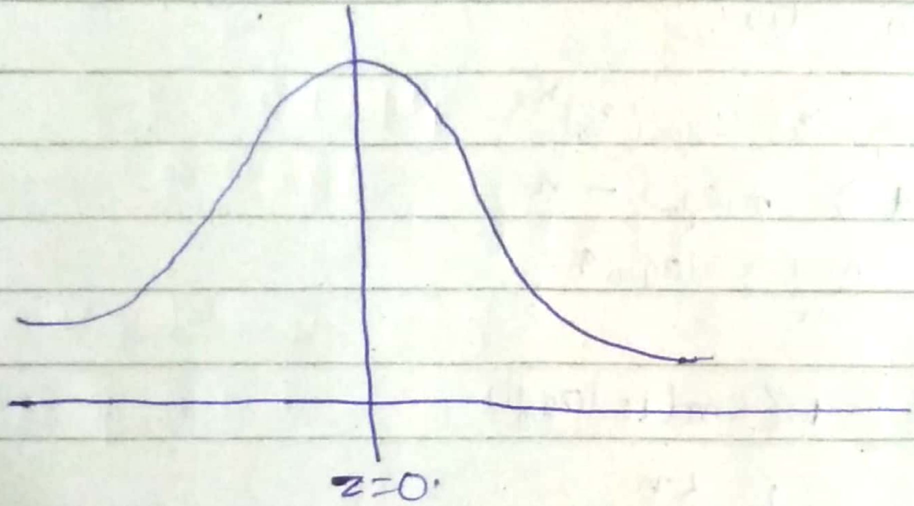
~~X~~ $X \sim N(\mu, \sigma)$

$\mu \to$ Mean

$\sigma \to$ S.D.



In most cases in a normal distribution, we need to transform, the general normal random variable X, $(X \sim N(\mu, \sigma))$ to a standard normal random variable $Z \sim N(0,1)$



The transformation which serves this purpose is given by $Z = \frac{X-\mu}{\sigma}$

<u>proof:-</u>

$$Z = \frac{X}{6} + -\left(-\frac{\mu}{6}\right).$$

$$E(X) = \frac{1}{6}E(x) + \left(-\frac{\mu}{6}\right) \qquad \because E(aX+b) = aE(X)+b$$

$$= \frac{\sigma}{u} - \frac{\sigma}{u} = 0 \qquad (\because E(X) = 6.$$

$$Var(Z) = \frac{1}{6^2} Var(X) \qquad \because Var(aX+b) = a^2 Var(X).$$

$$= \frac{6^2}{6^2}$$

$$= 1.$$

$X \sim N(\mu, \sigma)$

$z = \frac{x-\mu}{\sigma}$

$P(a < X < b)$

$\Rightarrow P\left(\frac{a-\mu}{\sigma} < X < \frac{b-\mu}{\sigma}\right)$

$= \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$

$\downarrow$

C.D.F of Z.

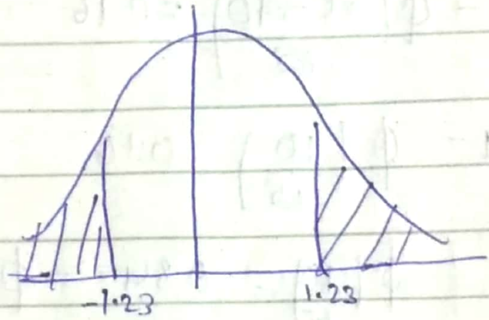
eg-1 ~~①~~ $\phi(1.23)$

$P(-\infty < Z \leq 1.23)$ $= 1.2 + 0.03$

$= 0.8907$

② $\phi(-1.23)$

$\phi(-1.23) = 1 - \phi(1.23)$.



(4.1). $\mu = 4.35$, $\sigma = 0.59$.

1(a) $P(4 < X < 5)$

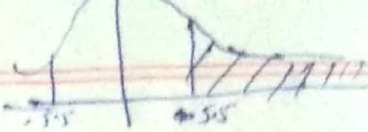
$= \cancel{\phi(5) - \phi(4)}.\ \phi\left(\frac{5-4.35}{0.59}\right) - \phi\left(\frac{4-4.35}{0.59}\right)$

$\phi(1.10) - \phi(-0.59)$.

$= 0.8643 - (1 - 0.7224)$

$= -1 + 1.5867$

$= 0.5867$.



$P(X \geq 5.5)$

$P(5.5 \leq X < \infty)$.

$= 1 - \phi\left(\frac{5.5 - 4.35}{0.59}\right)$

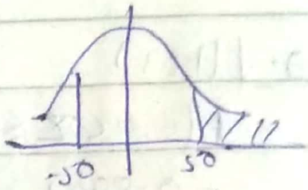
$= 1 - \phi(1.95)$

$= 1 - 0.9744$

$= 0.0256$

(4.2) $\mu = 40$ secs., $\sigma = 2$

$P(X > 50) = 0.16$.



$1 - \phi\left(\frac{50 - 40}{\sigma}\right) = 0.16$

$1 - \phi\left(\frac{10}{\sigma}\right) = 0.16$

$\phi\left(\frac{10}{\sigma}\right) = 0.84 = \phi(0.99)$

$\sigma = \frac{10}{0.99} = 10.10$

(4.3) $\mu = 36{,}000$ ₹, $n = 5000$ firms

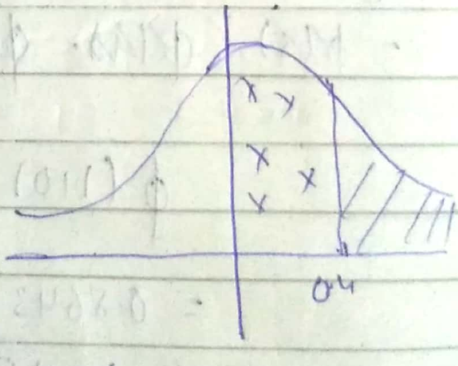
(a) $\sigma = 10{,}000$

$P(X \geq 40{,}000)$

$= P(Z \geq 0.4)$

$= 0.5 - 0.1554$

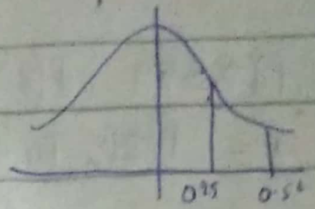
$= 0.3446$.



$= 0.3446 \times 5000 = 344.6 \times 5$

$= 1723.0$

(b) $P(38500 \leq X \leq 4100)$



$-\phi(.25) + \phi(.50)$

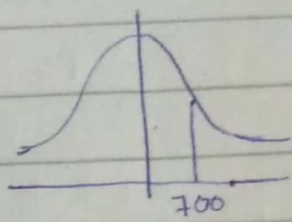
$= 0.1915 - 0.0987$

$= 0.0928$

$= 9.28\%$

(4.4) $n = 2000$ $u = 1000$ hrs $\sigma = 200$ hrs.

(i) ~~no.~~ $P(X \leq 700)$.



$t- \phi\, P\left(Z < \frac{700-1000}{200}\right)$

$= P(Z < -1.5)$

$= \phi(-1.5)$

$= 1 - \phi(1.5)$

$= 0.0668$.

(ii) no. of lamps $= 0.0668 \times 2000 = 133.6$

$= \underline{133}$

(iii) $P(X \leq t) = 0.1$

$P\left(Z \leq \frac{t-1000}{200}\right) = 0.1$

$\phi\left(\frac{t-1000}{200}\right) = 0.1$

$\phi\left(-\frac{t+1000}{200}\right) = \cancel{0.5398}\ 0.9$

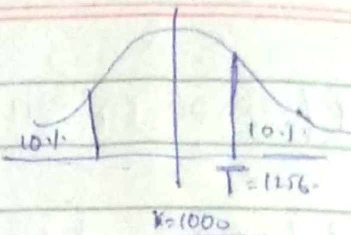
~~$t - 1000 = 107.96$~~

~~$t = 1107.96$ hrs.~~

$\frac{1000 - t}{200} = 1.28$

$t = 744$.

(iii) $P(X \le 90$ $\quad P(X \ge T) = 0.1$



$$1 - P\left(Z \le \frac{T-1000}{200}\right) = 0.1$$

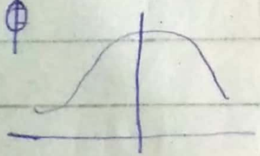
$$\phi\left(\frac{T-1000}{200}\right) = 0.9$$

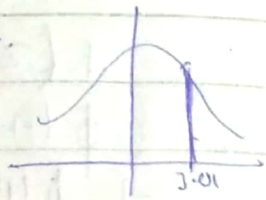
$$\phi\left(\frac{-1000+T}{200}\right) = \phi\, 0.9.$$

$$\frac{T-1000}{200} = 1.28.$$

$$T = \underline{1256}.$$

(4.5) $P(X \le 2.99) = 0.05 \qquad P(X \ge 3.01) = 0.05$





$$\phi\left(\frac{2.99-\mu}{\sigma}\right) = 0.05 \qquad 1 - \phi\left(\frac{3.01-\mu}{\sigma}\right) = 0.05$$

$$\phi\left(\frac{\mu-2.99}{\sigma}\right) = \phi(0.95) \qquad \phi\left(\frac{3.01-\mu}{\sigma}\right) = 0.95$$

$$\frac{\mu-2.99}{\sigma} = 1.64 \qquad \frac{3.01-\mu}{\sigma} = 1.64$$

$\mu = 1.64\sigma + 2.99 \qquad \mu = 3.01 - 1.64\sigma$

$1.64\sigma = \mu - 2.99 \qquad 1.64\sigma = 3.01 - \mu$

$$2\mu = 3.01 + 2.99$$
$$= 6.00$$
$$\mu = 3.0$$
$$\sigma = \frac{3-2.99}{1.64} = 0.00609.$$

or
$$\mu = \frac{2.99+3.01}{2}$$
$$= 3.00$$

UNIFORM DISTRIBUTION

The uniform distribution is a continuous probability distribution, which describes the behaviour of a random variable X, which is equally distributed over some closed interval [a, b] and the density function is given by $f(x) = \frac{1}{b-a}$, $a \le x \le b$

$= 0$ otherwise

EXPONENTIAL DISTRIBUTION

The exponential dist. is again another continuous distribution, whose density function is given by:-

$f(x) = \alpha e^{-\alpha x}$, for $x \geq 0$, α is some real parameter.

$= 0$, otherwise

Mean and Variance for uniform distribution

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, d(x).$$

$$= \int_a^b \frac{x}{(b-a)} dx \Rightarrow \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}.$$

$$\frac{(b-a)^2}{2(b-a)} = \frac{b-a}{2}.$$

$$Var(X) = E(X^2) - \{E(X)\}^2.$$

$$E(X^2) = \int_a^b \frac{x^2}{(b-a)} dx = \frac{x^3}{3(b-a)}\Big|_a^b = \frac{b^3-a^3}{3.(b-a)}$$

$$Var(x) = \frac{(b^3-a^3)}{3(b-a)} - \frac{(b+a)^2}{4}$$

$$= \frac{(b^2+ab+a^2)}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(a-b)^2}{12} = \frac{(b-a)^2}{12}$$

Exponential dist.

Moment generating function :-

$$\phi(t) = E(e^{tx}).$$

$$\phi'(t) = E[x e^{tx}]$$

$$\phi'(0) = E(x) = \mu.$$

$$\phi''(t) = E[x^2 e^{tx}].$$

$$\phi''(0) = E(x^2)$$

$$Var(x) = \phi''(0) - [\phi'(0)]^2.$$

$$\therefore E[e^{tx}] = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx.$$

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$$= \frac{\alpha}{(t-\alpha)} e^{(t-\alpha)x} \Big|_{-\infty}^{\infty}$$

$$\frac{\alpha}{t-\alpha} \left(0 - 1 \right)$$

$$= \frac{\alpha}{\alpha - t} \qquad [\alpha > t].$$

$\phi'(t) \rightarrow \phi'(0)$

$$= \frac{(\alpha-t)\times 0 - \alpha(-1)}{(\alpha-t)^2} = \frac{\alpha}{(\alpha-t)^2}.$$

$$\phi'(0) = \frac{1}{\alpha} = E(X) = \mu$$

$$\phi''(t) = \frac{-\alpha \cdot 2(\alpha-t)(-1)}{(\alpha-t)^4} = \frac{2\alpha}{(\alpha-t)^3}.$$

$$\phi''(0) = \frac{2}{\alpha^2}.$$

$$Var(X) = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}.$$

Q→ If a random variable X follows α dist with parameter α, then find $P[X>t] \rightarrow Re(t)$

→ (Realibility of f^n).

$\alpha \rightarrow$ failure rate of a machine

$X \sim EXP(\alpha) = f(x) = \alpha e^{-\alpha x}, \; x \geq 0$

$X = \cancel{\alpha}$.

$$P[X>t] = \int_t^\infty (\alpha e^{-\alpha x})\, dx.$$

$$= 1 - \int_0^t (\alpha e^{-\alpha x})\, dx$$

$$= 1 - \left(\frac{\alpha e^{-\alpha x}}{-\alpha} \right)_0^t \Rightarrow 1 - \left(-e^{-\alpha x} \right)_0^t = 1 - \left(-e^{-\alpha t} + 1 \right)$$

$$= e^{-\alpha t}.$$

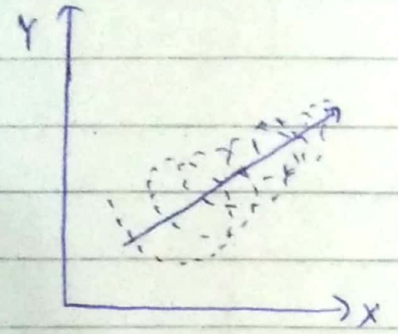
CORRELATION COEFFICIENT

$X \quad x_1 \;\; x_2, \; x_3 \; \ldots \; x_n$

$Y \quad y_1 \;\; y_1 \;\; y_2, \; \ldots \; y_n.$

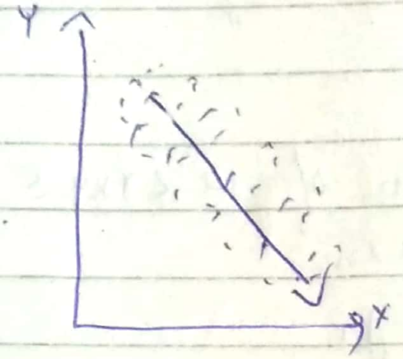
bivariant univariant

(x_1, y_1) $(x_1, x_2, x_3, \ldots)$



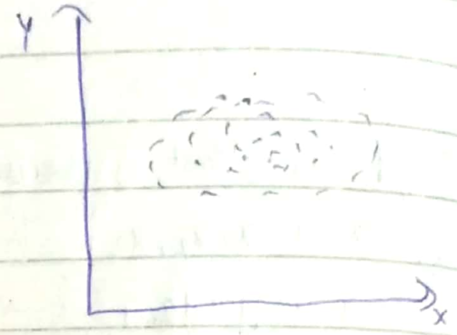
+vely correlated

$Cov(X,Y) > 0, \; r_{xy} > 0$



-vely correlated

$Cov(X,Y) < 0, \; r_{xy} < 0$



no correlation

$Cov(X,Y) = 0, \; r_{xy} = 0$

$$r_{xy} = \frac{Cov(x,y)}{\sigma_x \sigma_y}.$$

$$Cov(x,y) = E(XY) - E(X)E(Y).$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} x_i y_i\right) - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)$$

$$\sigma_x^2 = E(X^2) - \{E(X)\}^2 = \left\{\frac{1}{n}\sum_{i=1}^{n} x_i^2\right\} - \left\{\frac{1}{n}\sum_{i=1}^{n} x_i\right\}^2$$

Similarly σ_y^2.

*. When X and Y are independent random variables, then $cov(X,Y)=0$, i.e. $E(XY) = E(X)E(Y)$, but converse is not true.

eg →

X	-3	-2	-1	0	1	2	3
Y	9	4	1	0	1	4	9

$\rightarrow Y = x^2$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$E(XY) = \frac{1}{7}(-27 - 8 - 1 + 1 + 8 + 27) = 0$

∴ Converse is not always true.

→ Prove that value of $-1 \leq r_{xy} \leq 1$.

Proof: → $X \rightarrow x_1, x_2, x_3, \dots x_n$

$Y \rightarrow y_1, y_2, \dots y_n$

$u_i = \frac{x_i - \bar{x}}{\sigma x}$, $v_i = \frac{y_i - \bar{y}}{\sigma y}$.

$(u_i + v_i)^2 \geq 0$.

$\sum_{i=1}^{n} (u_i + v_i)^2 \geq 0$.

$\sum_{i=1}^{n} u_i^2 + 2\sum_{i=1}^{n} u_i v_i + \sum_{i=1}^{n} v_i^2 \geq 0$.

$u_i^2 = \frac{1}{\sigma x^2} (x_i - \bar{x})^2$

$\sum_{i=1}^{n} u_i^2 = \frac{1}{\sigma x^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 = n$

$\sum_{i=1}^{n} v_i = \frac{1}{\sigma y^2} \sum_{i=1}^{n} (y_i - \bar{y})^2 = n$.

$$\sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} \frac{1}{\sigma_x \sigma_y} (x_i - \bar{x})(y_i - \bar{y})$$

$$= \frac{1}{\sigma_x \sigma_y} \sum_{i=1}^{n} (x_i y_i) - (\bar{x} y_i) - (x_i \bar{y}) + (\bar{x}\bar{y})$$

$$= \frac{1}{\sigma_x \sigma_y} \left[\sum x_i y_i - \bar{x} \sum y_i - \bar{y} \sum x_i + \bar{x}\bar{y} \right].$$

$$= \frac{1}{\sigma_x \sigma_y} \left[\sum x_i y_i - n\bar{x}\bar{y} - \cancel{n\bar{x}}\bar{y} + \cancel{\bar{x}}\bar{y}n \right]$$

$$= \frac{1}{\sigma_x \sigma_y} \left[\sum x_i y_i - n\bar{x}\bar{y} \right].$$

$$= n\,\mathrm{cov}(X, Y)$$

$$= n\, r_{xy}.$$

$$\sum_{i=1}^{n} (u_i + v_i)^2 \geq 0$$

$$\Rightarrow \left(\sum u_i^2 + 2 \sum u_i v_i + \sum v_i^2 \right) \geq 0.$$

$$n + 2 n r_{xy} + n \geq 0.$$

$$n r_{xy} \geq -n$$

$$r_{xy} \geq -1$$

$$(u_i - v_i)^2 \geq 0$$

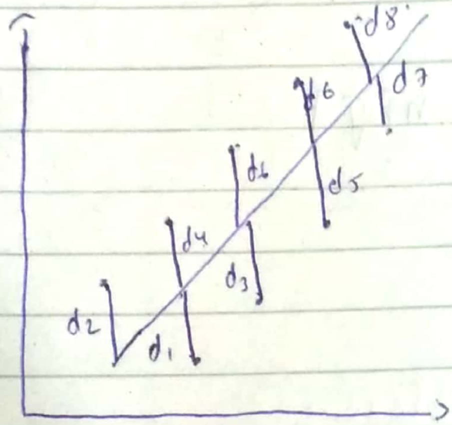
$$n - 2n r_{xy} + n \geq 0$$

$$2n \geq 2n r_{xy}$$

$$1 \geq r_{xy}$$

$$\therefore \boxed{1 \geq r_{xy} \geq -1}$$

Method of least square



$$S = d_1^2 + d_2^2 + \ldots\ldots d_8^2 \rightarrow$$

Since we take dist parallel to y, it is called regression line for y.

(1) Regression line of y on x

$$y - \bar{y} = b_{yx}(x - \bar{x}) \qquad b_{yx} = \frac{\sigma_y}{\sigma_x} r_{xy}$$

(2) Regression line of x on y

$$x - \bar{x} = b_{xy}(y - \bar{y}) \qquad \underset{\text{reg. coefficient}}{b_{xy}} = \frac{\sigma_x}{\sigma_y} r_{xy} \rightarrow$$

$y - \bar{y} = b_{yx} (x - \bar{x})$.

Properties of regression line

(1) The gradient of the regression line of y on x is b_{yx} & of $x_{,y} = \frac{1}{b_{xy}}$.

(2) $b_{xy} * b_{yx} = r^2_{xy} < 1$.

(3) The sign of two reg. coeff. b_{xy} & b_{yx} are same as the sign of r_{xy}.

(4) Intersecting point of the two reg. line is the $\bar{x}, \bar{y}$.

(5) $\frac{b_{yx}}{b_{xy}} = \frac{\sigma_y^2}{\sigma_x^2}$

(6) The reg. line of y on x is used to predict the value of y when x is known & the reg. line of x on y is used to predict the value of x when y is known.

Q → If the eqns of reg. line obtained in a co-orelation analysis are:-

$$x + 4y + 5 = 0 \quad \text{—} (1)$$
$$4x + 9y + 3 = 0 \cdot (2)$$

identify which eqn is of y on x & which one is of x on y. Find the correlation coeff. b/w x & y.
Find $\bar{x}$, $\bar{y}$. If $\sigma_x = \sqrt{3}$, $\sigma_y = ?$
Predict the value of x when $y = 1$.

Soln:- Let us assume eq (1) is reg. line of x on y.
(2) is reg. line of y on x.

$$\frac{1}{b_{xy}} = \frac{-1}{4} \qquad b_{yx} = \frac{-4}{9}.$$

$$b_{xy} = -4.$$

$$b_{xy} \cdot b_{yx} = \frac{16}{9}.$$

$$r_{xy} = \pm\frac{4}{3} \quad \times \quad (\text{since } r_{xy} \not\leq 1).$$

∴ (1) is reg. line of y on x
(2) " " " " x on y.

$$\therefore \frac{1}{b_{xy}} = \frac{-4}{9} \qquad b_{yx} = \frac{-1}{4}.$$

$$b_{xy} = \frac{-9}{4}.$$

$$b_{xy} \cdot b_{yx} = \frac{9}{16} \Rightarrow r_{xy} = \pm\frac{3}{4}. \Rightarrow \text{but cannot be +ve}$$

since b_{xy} b_{yx} are -ve, r_{xy} = -ve.

$4(-4y-5)+9y+3=0$.

$-7y-17=0$.

$\bar{y} = \frac{-17}{7}$. $\bar{x} = -5-4\left(\frac{-17}{7}\right) = -5+\frac{68}{7}$

$= \frac{33}{7}$.

$\left(\frac{\sigma_{xy}}{\sigma_{yx}}\right)^2 = \left(\frac{b_{yx}}{b_{xy}}\right)$.

$\frac{-1}{4} \times \left(\frac{-4}{9}\right) = \frac{3}{\sigma_{yx}^2}$ => $\sigma_{yx}^2 = \frac{3 \times 1\cancel{2}}{\cancel{9}3}$.

$\sigma_{yx} = \frac{\cancel{4}1}{\sqrt{3}}$.

We have to choose x on y.

$4x+9y+3$.

$4x = -3-9$

$x = \frac{-\cancel{1}2}{4} = -3$.

Q-> Find the corr. coeff. r_{xy} from the following set of data.

x	1	5	3	2	1	1	7	3
y	6	1	0	0	1	2	1	5

(2) Find also the correlation coeff. of y on x and x on y.

x	y	x^2	y^2	xy
1	6	1	36	6
5	1	25	1	5
3	0	9	0	0
2	0	4	0	0
1	1	1	1	1
1	2	1	4	2
7	1	49	1	7
3	5	9	25	15
$\Sigma x = 23$.	$\Sigma y = 16$	$\Sigma y^2 = 68$		$\Sigma xy = 36$

$$Cov(x,y) = \frac{1}{n} \sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}.$$

<u>Note</u>:- If $U = ax+b$, $V = cy+d$ then $r_{UV} = \frac{ac}{|a||c|} r_{xy}$.