8.1 CONSTRAINT (i.e., RESTRICTION) AND CONSTRAINT FORCE

Sometimes, it may be observed that the independent form of force may not be known though its effect is known for a system. Hence, in that case it is difficult to find the solution of the equation of motion.

For an example, in a simple pendulum, the independent form of the tension T (constraint force) that keeps its effective length constant, *is not known*. Though its value can be determined with the help of the known quantities of the system (e.g. $T = mg\cos\theta$, where m and θ are known for the system).

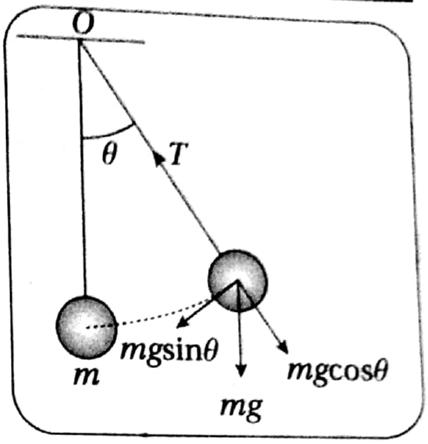

Fig. 1

For a rigid body, the distance between any two particles remains constant by the internal forces acting between its individual particles. The magnitudes of these internal forces are not known to us but its effect (*viz.* to keep the particle separation constant) is known. *These internal forces, which keep the particle separation constant are known as forces of constraint.*

Hence, ***the restriction on the movement of a body is called constraint*** **and** ***the forces which impart these restriction on the motion of the system of particles of the body are called constraint forces or forces of constraint.*** The corresponding equation which describes the motion of the system of particles of the body is known as **constraint equation.**

Special Note

The imposition of a constraint on a mechanical system simplifies the mathematical description.

8.1.1 Various Types of Constraint

Constraints can be divided depending on whether they are *time dependent* or *time independent, integrable or non-integrable, conservative or dissipative, algebraic equations or algebraic inequalities.*

(i) **Holonomic or integrable constraint :** If a constraint relation *is independent of velocity* and *can be expressed in the form of an algebraic equation* $f(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n, t) = 0$, then it is called *holonomic constraint*. This is also called as geometric constraint.

(ii) **Non-holonomic or non-integrable constraint :** If a constraint cannot be *expressed in an equation* of the form $f(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n, t) = 0$, then it is called non-*holonomic constraint. It depends on velocity.*

2. (i) **Scleronomic (or stationary) constraint :** If a constraint relation does not *explicitly depend on time,* then it is called scleronomic constraint.

   (ii) **Rheonomic (or time-dependent) constraint :** If a constraint relation depends *explicitly on time,* it is called rheonomic constraint.

3. (i) **Bilateral constraint :** If both the forward and backward motions are possible at any point on the constraint surface and the constraint equation *can be expressed only in form of equation,* this constraint is known as bilateral constraint.

   (ii) **Unilateral constraint :** If on the constraint surface *no forward motion* is possible, then it is called unilateral constraint. *Constraint relation is expressed in the form of inequality.*

4. (i) **Conservative constraint :** If the *total mechanical energy* of the system is *conserved* during constrained motion so that the work done by the constraint forces are zero, then this constraint is known as conservative constraint.

   (ii) **Dissipative constraint :** If the *total mechanical energy of a system is not conserved* in constrained motion and the forces of constraint do some work, then the constraint is called a dissipative constraint.

8.1.2 Some Examples of Constraints

1. **Rigid body:** In a rigid body, the distance between any two particles remains constant by the inter-atomic forces acting between them. The magnitudes of these forces are not exactly known though we know their effects which maintain a constant separation between any two particles of a rigid body under any circumstances.

For a rigid body, if $\vec{r}_i$ is the position vector of the ith particle and $\vec{r}_j$ is the position vector of the jth particle from a reference origin O, the distance between any two particles (i and j particles) is constant (for all i and j).

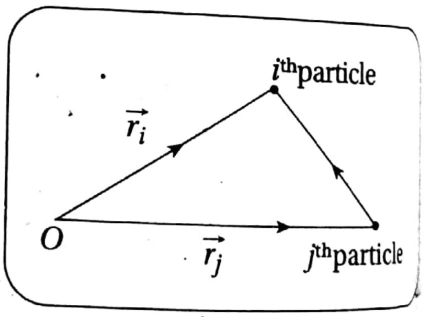

Fig. 2 ▷ Rigid body

$$\therefore \quad |\vec{r}_i - \vec{r}_j|^2 = \text{constant} = c^2 \text{ (say)}$$

or, $$|\vec{r}_i - \vec{r}_j|^2 - c^2 = 0 \qquad \cdots(8.1.2.1)$$

This is the constraint equation.

Here, the work done by the constraint force vanishes and constraint relation [equation (8.1.2.1)] is time independent.

So, the constraint will be *holonomic, scleronomic, bilateral* and *conservative.*

2. **Motion of a body on an inclined plane under gravity:** When a body P of mass m moves on a smooth inclined plane, the motion of the body is restricted on the surface of the inclined plane by the normal reaction force R. So, R is the force of constraint.

If (x, y) be the position coordinate of the body P at any instant on the inclined plane with angle of inclination θ with the horizontal, the **constraint equation** is in the form of

$$y - x\tan\theta = 0 \quad \left[\because \text{ slope, } \tan\theta = \frac{y}{x}\right] \qquad \cdots(8.1.2.2)$$

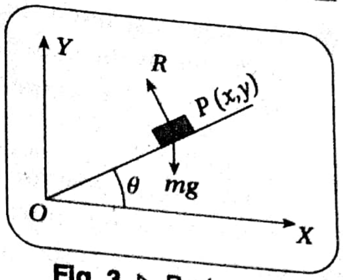

Fig. 3 ▷ Body on an inclined plane

Also it can be found that the work done by the constraint force vanishes.

So, the constraint is *holonomic, scleronomic, bilateral and conservative.*

- **Simple pendulum:** Let us consider a simple pendulum of mass m and effective length l. The weight mg of the bob at position B acts vertically downward. The tension T of the string maintains the effective length (l) of pendulum constant.

If (x, y) is position coordinate of the centre of the bob at any instant with respect to the origin 'O' at the point of suspension, then the **constraint equation** is

$$x^2 + y^2 = l^2 \qquad \cdots(8.1.2.3)$$

(as the effective length of the pendulum is constant)

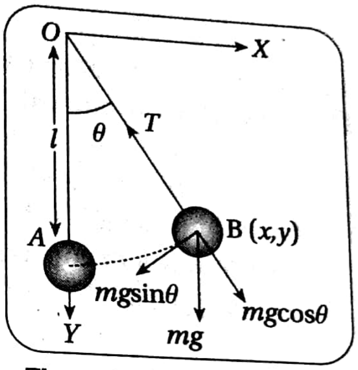

Fig. 4 ▷ Motion of simple pendulum

Therefore, the *constraint will be holonomic, scleronomic, conservative* (as the work done by the constraint force is zero) and *bilateral.*

- **Any deformable body:** For a deformable body, the shape of the body can be changed according to a certain prescribed function of time. So, for any two particles (i and jth particles with position vectors $\vec{r}_i$ and $\vec{r}_j$ respectively), the **constraint relation** can be expressed as,

$$|\vec{r}_i - \vec{r}_j| = f(t) \qquad \cdots(8.1.2.4)$$

Here, the constraint forces do work as the shape is changing with time and the constraint relation is a function of time.

So, the constraint will be *holonomic, rheonomic, dissipative* and *bilateral.*

- **A pendulum with variable length:** In this case, the length of the pendulum is changing with time. The position of the bob at a particular time t with respect to a fixed origin O can be written as

$$|r(t)|^2 = l^2(t) \qquad \cdots(8.1.2.5)$$

where $l(t)$, a function of time t, is the length of the pendulum and $r(t)$ is the position vector of the bob at time t.

Here, the work done by the constraint force T is not zero.

So, the constraint will be *holonomic, rheonomic, dissipative* (as the work done by the constraint force is not equal to zero) and *bilateral.*

- **Particle restricted to move on the surface of a sphere:** Here, the particle is restricted by the constraint force, so that it can only move on the surface of the sphere of radius a. The **equation of constraint** can be written as,

$$r = a \qquad \cdots(8.1.2.6)$$

where r is the position of the particle.

So, the constraint must be *holonomic, scleronomic, bilateral* and *conservative.*

- **Motion of gas molecules in a spherical container:** If a is the radius of the spherical container and r_i is the instantaneous position of the ith gas molecule with respect to the centre of the container, then the **constraint equation is,**

$$|\vec{r}_i| < a \qquad \cdots(8.1.2.7)$$

So, the constraint must be non-holonomic, scleronomic and dissipative.

- **Particle in a cubical box:** The particle is restricted by the constraints in such a way that it can move on the edge of the closed cubical box or inside the box. So, for the particle to be within the closed cubical box, the **equation of constraint** can be written as,

$$\begin{aligned} 0 < x < a \\ 0 < y < a \\ 0 < z < a \end{aligned} \qquad \cdots(8.1.2.8)$$

where a is the edge of the cube along X, Y or Z-axes. So, the constraint is *non-holonomic, unilateral* and *conservative.*

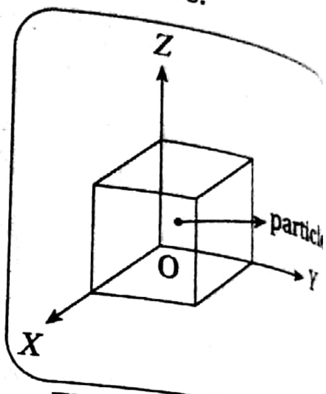

Fig. 5 ▷ Particle in a cubical box

- **A disc rolling down without slipping on a plane:** Here the disc is rolling without slipping. We assume, the surface in contact is rough and the starting point is A where the body makes first contact with the plane.

In that case, the **equation of constraint** is, $ds = a d\phi$, where a is the radius of the body and ϕ *is the angle of rotation of the disc about its own axis.*

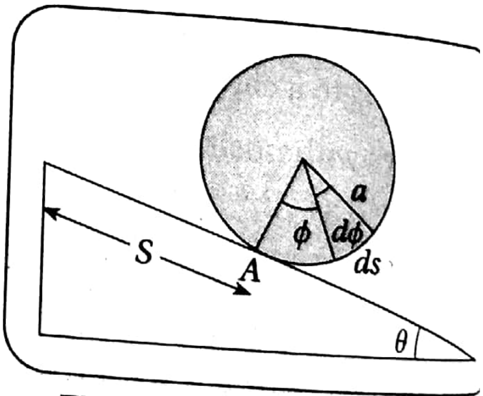

Fig. 6 ▷ Disc rolling down a plane without slipping

If θ is the angle of inclination of the plane,

$$v = \text{radius} \times \frac{d\phi}{dt} = a\frac{d\phi}{dt}$$

Again, $\frac{dx}{dt} = v\sin\theta$ or, $\frac{dx}{dt} = a\frac{d\phi}{dt}\sin\theta \quad \left[\because v = a\frac{d\phi}{dt}\right]$

or, $\quad dx = a\,d\phi\sin\theta \qquad \cdots(8.1.2.9)$

and $\quad \frac{dy}{dt} = -v\cos\theta = -a\frac{d\phi}{dt}\cos\theta \quad \left[\because v = a\frac{d\phi}{dt}\right]$

or, $\quad dy = -a\,d\phi\cos\theta \qquad \cdots(8.1.2.10)$

So, the equations of constraint are

$$dx - a\,d\phi\sin\theta = 0$$
$$dy + a\,d\phi\cos\theta = 0 \qquad \cdots(8.1.2.11)$$

Since, *equation* (8.1.2.11) *cannot be integrated*, it forms a *non-holonomic constraint.*

Similarly, it can be shown that the constraint is holonomic for a cylinder or for a spherical ball rolling down without slipping on a plane.

Special Note

Ideal constraint If the sum of work done by constraint forces on the particle of a dynamical system is zero, then the constraint is called the ideal constraint. In that case,

$\sum_i \vec{F}_i^{(c)} \cdot \delta\vec{r}_i = 0$, where $\vec{F}_i^{(c)}$ is the constraint force.

Example : A particle having circular motion or a sphere rolling down through a plane without slipping has ideal constraints.

8.2 DEGREES OF FREEDOM

The number of independent ways by which a dynamical system can move without violating any constraint imposed on it, is called the **degrees of freedom** of the system.

So, **the degrees of freedom can be defined as the minimum number of independent coordinates, which can specify the position of the system completely.**

A system of N free particles has $3N$ number of degrees of freedom (as 3 cartesian coordinates are required to specify the position of each particle).

But, if the distribution of the particles of the system be fixed to maintain the constrained motion of the system, then the number of degrees of freedom will be less than $3N$. So, due to constraints all the $3N$ number of coordinates are not all free or independent to each other. So, if **a system of N free particles has m number of constraints(i.e., m number of constraint equations), then the number of independent coordinates will be $(3N - m)$.**

Hence, $(3N - m)$ will be the degrees of freedom for a constrained system.

Examples :

1. When a particle is constrained to move along a straight line, it has only one degree of freedom. Because only one coordinate (x or y) is required to specify its position.
2. If a particle is allowed to move on a plane, then it has two degrees of freedom. Here two coordinates (x, y) are required to specify its position.
3. The degrees of freedom of a system of two particles separated by a fixed distance is $3 \times 2 - 1 = 5$.
4. For a rigid body, the number of degrees of freedom is 6 (3 are of translational and 3 of rotational).

   But a rigid body, constrained to move about a point has 3 degrees of freedom whereas a rigid body constrained to rotate about an axis has only one degree of freedom.

- For a particle constrained to move on the surface of a sphere, the degrees of freedom will be $= 3 \times 1 - 1 = 2$, as the number of constraint is 1.
- For a simple pendulum the degrees of freedom is $3 \times 1 - 2 = 1$, as the number of constraint is 2 (θ, the angle of deflection and l, the effective length).
- For a body on an inclined plane, the degrees of freedom will be $3 \times 1 - 1 = 2$

8.3 WORK DONE BY CONSTRAINT FORCE

Let us consider a rigid body having two point masses m_i and m_j with position vectors $\vec{r}_i$ and $\vec{r}_j$ from an arbitrary origin respectively.

The total work done on the i th particle due to a force $\vec{F}_{ij}$ by all the j th particles in rigid body will be,

$$W_i = \sum_j \vec{F}_{ij} \cdot \Delta\vec{r}_i \quad \cdots (8.3.1)$$

where $\vec{F}_{ii}$ is the self force (i.e., force by i on i) and is equal to zero.

So, the total work done (W) for all the particles in the rigid body becomes,

$$\sum_i \sum_j \vec{F}_{ij} \cdot \Delta\vec{r}_i = \sum_j \sum_i \vec{F}_{ji} \cdot \Delta\vec{r}_j = W \quad \cdots (8.3.2)$$

That is, $W = \frac{1}{2}\sum_i \sum_j (\vec{F}_{ij} \cdot \Delta\vec{r}_i + \vec{F}_{ji} \cdot \Delta\vec{r}_j)$

$= \frac{1}{2}\sum_i \sum_j \vec{F}_{ij} \cdot (\Delta\vec{r}_i - \Delta\vec{r}_j)$ [as from Newton's third law $\vec{F}_{ij} = -\vec{F}_{ji}$]

or,

$$W = \frac{1}{2}\sum_i \sum_j \vec{F}_{ij} \cdot \Delta(\vec{r}_i - \vec{r}_j) \quad \cdots (8.3.3)$$

Also, for a rigid body the **equation of constraint** is,

$$|\vec{r}_i - \vec{r}_j|^2 = \text{constant} \quad (\text{for all } i \text{ and } j) \quad \cdots (8.3.4)$$

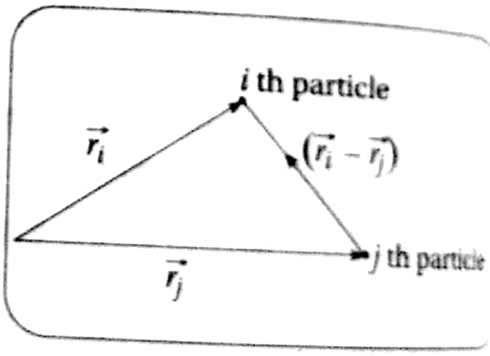

Fig. 7

Taking differentials we find that,

$$2(\vec{r}_i - \vec{r}_j) \cdot \Delta(\vec{r}_i - \vec{r}_j) = 0 \quad \cdots (8.3.5)$$

Again, the internal force $\vec{F}_{ij}$ between any two particles is directed along the line joining them i.e., $\vec{F}_{ij}$ is along $(\vec{r}_i - \vec{r}_j)$. So we may write $\vec{F}_{ij} = c_{ij}(\vec{r}_i - \vec{r}_j)$, where c_{ij} = real constant and symmetric in i and j. Substituting $\vec{F}_{ij}$ in equation (8.3.3), we have

$$W = \frac{1}{2}\sum_i \sum_j c_{ij}(\vec{r}_i - \vec{r}_j) \cdot \Delta(\vec{r}_i - \vec{r}_j) = 0 \quad \text{[using equation (8.3.5)]} \quad \cdots (8.3.6)$$

Hence, the work done by the constraint forces vanish.

Special case:

Simple pendulum with variable length: Let us consider a simple pendulum with variable length. So the length of the bob from the point of suspension changes from P_1 to P_2 (say). Here the displacement is not perpendicular to the force. So, the work done by constraint force does not vanish. But we know that it should be zero. Thus, *to make the work done by the constraint forces to be zero*, we consider an infinitesimally small displacement PQ_1 or PQ_2 with no passage of time. Now the tensional force is at right angle to this infinitesimal small displacement. Hence, the virtual work done on such a system for any arbitrary virtual displacement (PQ_1 or PQ_2) due to the constraint forces must be zero.

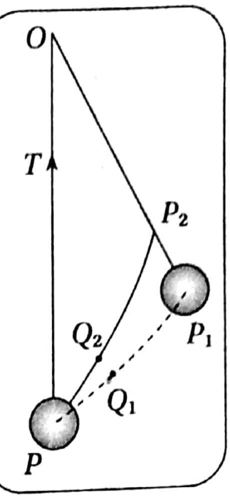

Fig. 8 ▷ Motion of a simple pendulum of variable length

8.4 VIRTUAL DISPLACEMENT

It is an infinitesimally small displacement that occur without any real change of time (i.e., instantaneously).

So, by definition, a virtual infinitesimal displacement is given by $\delta x_i = dx_i\big|_{dt = 0}$

8.4.1 Principle of Virtual Work

If a system is in equilibrium, the resultant force $\vec{F}_i$ acting on each particle must be zero i.e., $\vec{F}_i = 0$.

Hence, *the virtual work* done for a particle in equilibrium due to any arbitrary virtual displacement $\delta\vec{r}_i$ must be zero.

So, $$\boldsymbol{\vec{F}_i \cdot \delta\vec{r}_i = 0} \qquad \dots (8.4.1.1)$$

Hence, for all the particles of the system, the virtual work done is,

$$\boldsymbol{\sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0} \qquad \dots (8.4.1.2)$$

Now, splitting *the total force* $\vec{F}_i$ into *applied force* $\vec{F}_i^{(a)}$ and *the force of constraint* $\vec{F}_i^{(c)}$, equation (8.4.1.2) becomes

$$\sum_i (\vec{F}_i^{(a)} + \vec{F}_i^{(c)}) \cdot \delta\vec{r}_i = 0 \quad \text{or,} \quad \boldsymbol{\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i + \sum_i \vec{F}_i^{(c)} \cdot \delta\vec{r}_i = 0} \qquad \dots (8.4.1.3)$$

Since, the virtual work done by the constraint force is zero i.e., $\sum_i \vec{F}_i^{(c)} \cdot \delta\vec{r}_i = 0$, we get from equation (8.4.1.3)

$$\boldsymbol{\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0} \qquad \dots (8.4.1.4)$$

This is often called the **principle of virtual work done.**

It states that **the virtual work done by all the applied forces acting on a system in equilibrium is zero, provided that the total virtual work done by all the constraint forces is equal to zero.**

8.5 D'ALEMBERT'S PRINCIPLE[●]

For a dynamical system of particles, the equation of motion of one of the consisting particle such as ith particle of momentum $\vec{p}_i$ due to the applied force $\vec{F}_i$ can be written as

$$\vec{F}_i = \dot{\vec{p}}_i = m_i\ddot{\vec{r}}_i, \text{ where } \vec{r}_i = \text{position vector of } i\text{th particle.} \quad \cdots (8.5.1)$$

It implies that the total force acting on the ith particle must be equal to the effective force $(m_i\ddot{\vec{r}}_i)$ generated on the ith particle (i.e., the reverse force of inertia on it)

Now equation (8.5.1) can be written as,

$$(\vec{F}_i - m_i\ddot{\vec{r}}_i) = 0 \quad \cdots (8.5.2)$$

Equation (8.5.2) implies that *under the combined application of reverse effective force $(-m_i\ddot{\vec{r}}_i)$ and applied force $\vec{F}_i$, the system is in equilibrium.*

Now, we can apply the principle of virtual work done on the dynamical system.

The virtual work done for the system will be

$$\sum_i(\vec{F}_i - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i = 0, \text{ where } \delta\vec{r}_i \text{ is the virtual displacement} \quad \cdots (8.5.3)$$

The total force on the ith particle may be split up into two parts as

$$\vec{F}_i = \vec{F}_i^{(a)} + \vec{F}_i^{(c)} \quad \cdots (8.5.4)$$

where F_i^a = applied force on the ith particle, F_i^c = constraint force

∴ Equation (8.5.3) becomes

$$\sum_i(\vec{F}_i^{(a)} - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i + \sum_i\vec{F}_i^{(c)}\cdot\delta\vec{r}_i = 0 \quad \cdots (8.5.5)$$

As the virtual work done by the constraint force is zero, i.e., $\sum_i \vec{F}_i^{(c)}\cdot\delta\vec{r}_i = 0$, equation (8.5.5) reduces to

$$\sum_i(\vec{F}_i^{(a)} - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i = 0 \quad \cdots (8.5.6)$$

Equation (8.5.6) is known as **D'Alembert's principle.**

It states that **for the actual motion of any system of particles subjected to constraint motion, the sum of work done due to applied forces and due to the forces of inertia in any virtual displacement of the system for each given instant of time, is equal to zero.**

● Not included in W.B.U.T. syllabus.

8.5.1 Applications of D'Alembert's Principle

Simple pendulum:

Let us consider a simple pendulum of mass m and effective length l in X-Y-plane.

The **equation of constraint** for such a system is

$$x^2 + y^2 = l^2 \quad \cdots (8.5.1.1)$$

Differentiating equation (8.5.1.1) we get,

$$x\,dx + y\,dy = 0 \quad \cdots (8.5.1.2)$$

Using D'Alembert's principle, we can write the equation of motion as,

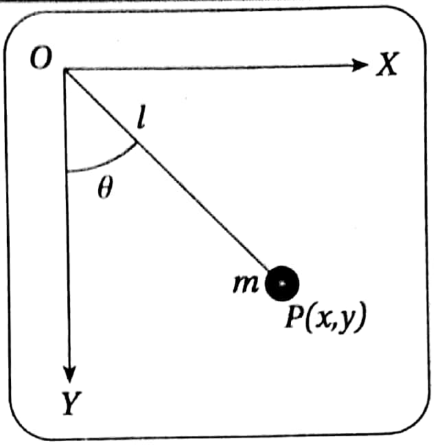

Fig. 9 ▷ Simple pendulum

$$\sum_i (m_i \ddot{\vec{r}}_i - \vec{F}_i^{(a)}) \cdot \delta\vec{r}_i = 0 \quad \cdots (8.5.1.3)$$

In terms of x, y coordinates, equation (8.5.1.3) can be written for this problem as,

$$(m\ddot{x} - F_x)dx + (m\ddot{y} - F_y)dy = 0 \quad \cdots (8.5.1.4)$$

where F_x and F_y are the components of the applied force along X and Y axes respectively.

Here, the applied forces do not include tension (the force of constraint).

Since, the only force (mg) acting on the bob is along Y-direction (i.e., downward).

$\therefore \quad F_x = 0$ and $F_y = -mg$

So, equation (6.8.1.4) reduces to

$$m\ddot{x}\,dx + (m\ddot{y} + mg)dy = 0 \quad \text{or,} \quad \ddot{x}\,dx + (\ddot{y} + g)dy = 0$$

or, $\ddot{x}\,dx + (\ddot{y} + g)\left(-\frac{x\,dx}{y}\right) = 0$ [Substituting $dy = -\frac{x\,dx}{y}$ from equation (8.5.1.2)]

or, $y\ddot{x}\,dx - x(\ddot{y} + g)\,dx = 0$ or, $y\ddot{x} - x(\ddot{y} + g) = 0 \quad \cdots (8.5.1.5)$

If the amplitude of oscillation is small enough, then $y = -l$, which implies that $\ddot{y} = 0$.

Hence, the equation (8.5.1.5) becomes, $-l\ddot{x} - xg = 0$.

or, $\ddot{x} + \frac{g}{l}x = 0$ or, $\ddot{x} + \omega^2 x = 0$, where $\omega^2 = \frac{g}{l}$ $\quad \cdots (8.5.1.6)$

and time period of oscillation, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$ $\quad \cdots (8.5.1.7)$

Atwood's machine: The Atwood's machine consists of two masses m_1 and m_2 suspended over a frictionless pulley of radius a and connected by an inextensible (i.e., not stretchable) string of length 'l'.

Here, the length of the string (l), connecting the two masses is constant.

The **equation of constraint** for such a system is,

$$x_1 + x_2 + \pi a = l \quad \cdots (8.5.1.8)$$

Differentiating equation (8.5.1.8) we get,

$$dx_1 + dx_2 = 0 \quad \cdots (8.5.1.9)$$

or, $\dot{x}_2 = -\dot{x}_1$ $\cdots$ (8.5.1.10)

D'Alembert's principle for such a system can be written as,

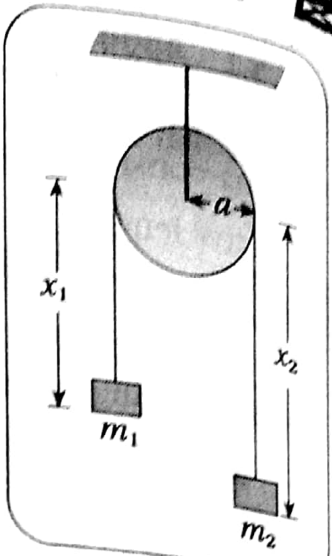

Fig. 10 ▷ Atwood's machine

$$\sum_i (m_i \ddot{\vec{r}}_i - \vec{F}_i^{(a)}) \cdot \delta \vec{r}_i = 0 \quad \cdots (8.5.1.11)$$

where $\vec{F}_i^{(a)}$ is the applied force on the system.

Rewriting equation (8.5.1.11) in terms of x_1 and x_2 for this problem we get,

$$(m_1\ddot{x}_1 - F_{x_1})dx_1 + (m_2\ddot{x}_2 - F_{x_2})dx_2 = 0 \quad \cdots (8.5.1.12)$$

where F_{x_1} and F_{x_2} are the components of the applied force and

$$F_{x_1} = m_1 g \quad \text{and} \quad F_{x_2} = m_2 g$$

Using the equations (8.5.1.9), (8.5.1.10) and substituting the values of the forces F_{x_1} and F_{x_2}, we get from equation (8.5.1.12),

$$(m_1\ddot{x}_1 - m_1 g)dx_1 + (-m_2\ddot{x}_1 - m_2 g)(-dx_1) = 0$$

$$[\because dx_1 + dx_2 = 0 \text{ and } \dot{x}_1 = -\dot{x}_2]$$

or, $[(m_1 + m_2)\ddot{x}_1 - (m_1 - m_2)g]\, dx_1 = 0$

as the variations dx_1 are arbitrary, the coefficient will be equal to zero.

$\therefore$ $(m_1 + m_2)\ddot{x}_1 = (m_1 - m_2)g$ or, $\ddot{x}_1 = \dfrac{m_1 - m_2}{m_1 + m_2} g$ $\cdots$ (8.5.1.13)

This is the equation of motion for the Atwood's machine.

8.6 INTRODUCTION TO GENERALISED COORDINATES

Cartesian coordinates are usually used to solve the mechanical problems due to their utmost simplicity but it is not acceptable for all cases. For an example, consider the central force problem, where a particle is revolving around a fixed attracting centre (*viz, motion of earth around the sun*). Here it is easier to work with the spherical polar coordinates (r, θ, ϕ) rather than cartesian coordinates. These depend... to solve a particular mechanical problem...

Again for any system of N particles, the number of coordinates specifying the configuration of the system are somehow restricted by constraints. So, all the coordinates are not at all free and thus the number of degrees of freedom for constraints will be $(3N-m)$, where m is the number of constraint.

Therefore, we have to introduce a set of $(3N-m)$ number of independent coordinates known as generalised coordinates to specify the configuration of a dynamical system.

Therefore, **the generalised coordinates are defined as a set of independent coordinates that completely specify the configuration of a dynamical system.**

They are represented by n independent variables $q_1, q_2, \cdots, q_n$. Here, n *indicates the number of generalised coordinates which are equal to the number of degrees of freedom.*

8.6.1 Advantages of generalised Coordinates

1. It eliminates the dependence of coordinate system for describing the configuration of a dynamical system.
2. The generalised coordinates are independent to each other so that their individual variations (i.e., δq_j) can be considered.

Special Note

For holonomic system, the number of generalised coordinates is exactly equal to the number of degrees of freedom. But in non-holonomic system it is greater than the number of degrees of freedom.

8.7 GENERALISED VELOCITY

The time derivative of generalised coordinate is called generalised velocity.

If q_j be a generalised coordinate of a system, the generalised velocity of the system is defined as,

$$\dot{q}_j = \frac{dq_j}{dt} \quad \cdots (8.7.1)$$

where $j = 1, 2, \cdots, n$; n is the number of generalised coordinates.

It is to be noted that q_j need not have the dimension of length, so $\dot{q}_j$ also need not to have the dimension of velocity.

Now, for a constrained system of N particles, the position vector $\vec{r}_i$ depends on the generalised coordinates as,

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \cdots, q_n, t)$$

$$\therefore \quad d\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} dq_j + \frac{\partial \vec{r}_i}{\partial t} dt \quad \cdots (8.7.2)$$

$$\therefore \quad \vec{v}_i = \dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}, \text{ where } \dot{q}_j = \frac{dq_j}{dt} \quad \cdots (8.7.3)$$

8.8 CONCEPT OF GENERALISED FORCE

The position vector $\vec{r}_i$ of a holonomic system of N particles with m constraints can be written as,

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \cdots, q_n, t) \simeq \vec{r}_i(q_j, t)$$

where n is the number of degrees of freedom (i.e., number of generalised coordinates) of the system.

or, $$\delta\vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_1}\delta q_1 + \frac{\partial \vec{r}_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \vec{r}_i}{\partial t}\delta t$$

Now, for virtual displacement, $\delta\vec{r}_i = dr_i\big|_{dt=0}$

As virtual displacement is considered at a particular instant, virtual displacement,

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j \quad [\because \delta t = 0] \qquad \cdots (8.8.1)$$

Here, δq_j is called *generalised displacement* or virtual arbitrary displacement. For an example, if q_j is an angle coordinate, δq_j will be an angular displacement.

Hence, in terms of generalised coordinates, the virtual total work done due to applied force $\vec{F}_i$ is given by,

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_j\left(\sum_i F_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right)\delta q_j = \sum_j Q_j \delta q_j \qquad \cdots (8.8.2)$$

where Q_j is the jth component of the generalised force $\left(= \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right)$.

Since, generalised coordinates (q's) need not have the dimension of length, so generalised force (Q_j) does not necessarily have the dimension of force, but $\sum_j Q_j \delta q_j$ must have the dimension of work.

8.8.1 Generalised Force for Conservative System

For a conservative system where its total energy is conserved, the conservative force $\vec{F}_i$ can be expressed in terms of potential function $V\,[= V(q_j)]$ as,

$$\vec{F}_i = -\vec{\nabla}_i V \qquad \cdots (8.8.1.1)$$

We know that generalised force $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

or, $$Q_j = -\sum_i \frac{\partial V}{\partial \vec{r}_i}\cdot\frac{\partial \vec{r}_i}{\partial q_j} \qquad \cdots (8.8.1.2)$$

Now, for a conservative system scalar potential V is function of position only, so

$$Q_j = -\frac{\partial V}{\partial q_j} \qquad \cdots (8.8.1.3)$$

8.8.2 Generalised Potential

It is a velocity dependent potential and is called the generalised potential as it gives rise to generalised force.

For a conservative system, the force ($\vec{F} = -\vec{\nabla}V$) is derived from scalar potential V, which is a function of r only [i.e., $V = V(r)$].

$$V = V(r_1, r_2, r_3, \cdots, r_n) = V(q_1, q_2, q_3, \cdots, q_n) \qquad \cdots (8.8.2.1)$$

$$\therefore \quad dV = \sum_i \left[\frac{\partial V}{\partial x_i}dx_i + \frac{\partial V}{\partial y_i}dy_i + \frac{\partial V}{\partial z_i}dz_i\right] = \sum_i \vec{\nabla}_i V \cdot d\vec{r}_i \qquad \cdots (8.8.2.2)$$

$$\therefore \quad \frac{\partial V}{\partial q_j} = \sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} \qquad \cdots (8.8.2.3)$$

Again, we can write generalised force from equation (8.8.1.2) as,

$$Q_j = -\frac{\partial V}{\partial q_j}$$

But when *the system is not conservative*, potential depends on the generalised velocity $\dot{q}_j$ also. In that case the generalised force Q_j associated with $\dot{q}_j$ can be defined as,

$$Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \qquad \cdots (8.8.2.4)$$

where $U = U(q_j, \dot{q}_j)$ is called velocity dependent potential or generalised potential.

8.9 KINETIC ENERGY

The kinetic energy of the ith particle of mass m_i for a system is $\frac{1}{2}m_i\dot{r}_i^2$.

So, the kinetic energy of the system of N particles,

$$T = \sum_{i=1}^{N} \frac{1}{2}m_i\dot{r}_i^2 = \sum_{i=1}^{N} \frac{1}{2}m_i\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \qquad \cdots (8.9.1)$$

Now, the position vector $\vec{r}_i$ can be written as, $\vec{r}_i = \vec{r}_i(q_1, q_2, \cdots, q_n, t) = r_i(q_j, t)$,

where n = number of generalised coordinates

$$\therefore \quad \dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \qquad \cdots (8.9.2)$$

Substituting the value of $\dot{\vec{r}}_i$ in the expression of T, we get,

$$T = \sum_{i=1}^{N} \frac{1}{2}m_i\left(\sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\right)\cdot\left(\sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_k}\cdot \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)$$

$$= \sum_i \sum_j \sum_k \left\{ \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k + \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial q_k} \cdot \frac{\partial \vec{r}_i}{\partial t} \dot{q}_k + \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial t} \right\}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k + \sum_{i,j} \frac{1}{2}(2m_i) \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial t} \dot{q}_j + \sum_i \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial t} \right)^2$$

[$\because$ the 2nd and 3rd term of above equations are identical and so they are added]

or,
$$T = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} \dot{q}_j \dot{q}_k + \sum_{j=1}^{n} a_j \dot{q}_j + a \qquad \cdots (8.9.3)$$

where $a_{jk} = \sum_{i=1}^{N} \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$, $a_j = \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial t}$; $a = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial t} \right)^2$

This is the expression of kinetic energy of a system and it is a quadratic function of generalised velocities.

If $r_i = r_i(q_j)$ only, the transformation equation is independent of time i.e., $\frac{\partial \vec{r}_i}{\partial t} = 0$.

So, $a_j = a = 0$.

Hence, $T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k \qquad \cdots (8.9.4)$

8.10 GENERALISED MOMENTUM

The partial derivatives of the energy (E) of a system with respect to generalised velocities are called generalised momentum (p_j).

It is represented for a system of particles with n degrees of freedom as,

$p_k = (p_1, p_2, \cdots, p_n)$ where n is the number of generalised coordinates.

In general,

$$p_j = \frac{\partial E}{\partial \dot{q}_j} \quad \text{where } j = 1, 2, \cdots, n$$

$$= \frac{\partial (T+V)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

[$\because$ potential energy V depends only on q_j and independent of $\dot{q}_j$ $\therefore \frac{\partial V}{\partial \dot{q}_j} = 0$]

$$\therefore \quad p_j = \frac{\partial T}{\partial \dot{q}_j} \qquad \cdots (8.10)$$

LAGRANGIAN FORMULATION

It has been observed that, a particular coordinate system (e.g. cartesian, polar cylindrical) is incorporated to write the differential equation in Newtonian mechanics. But such a dependence of ... is n...

desirable. In the same way, it may not be possible to know about all the forces (constraints) acting on a system.

Lagrangian overcame all these difficulties of Newtonian mechanics by using generalised coordinates, which is independent from any particular coordinate system.

8.11.1 Derivation of Lagrange's Equation

The D'Alembert's principle can be written as,

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0 \qquad \cdots (8.11.1.1)$$

where $\vec{F}_i^{(a)}$ is the applied force on the ith particle of a system of N particles whose momentum is $\vec{p}_i$.

The above equation can be written as,

$$\sum_i \vec{F}_i^{(a)} \cdot \delta \vec{r}_i - \sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = 0 \qquad \cdots (8.11.1.2)$$

or,

$$\sum_j \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j - \sum_i \frac{d}{dt}(m_i \cdot \dot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0 \qquad \cdots (8.11.1.3)$$

$$\left[\because \text{ virtual displacement } \delta r_i = \sum_i \frac{\partial r_i}{\partial q_j} \delta q_j\right].$$

or,

$$\sum_j Q_j \delta q_j - \sum_i \frac{d}{dt}(m_i \dot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0 \qquad \cdots (8.11.1.4)$$

where Q_j is the jth component of the generalised force $\left(= \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right)$.

or

$$\sum_j Q_j \delta q_j - \sum_j \left(\sum_i \frac{d}{dt}(m_i \dot{\vec{r}}_i) \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) \delta q_j = 0 \qquad \cdots (8.11.1.5)$$

Now,

$$\sum_i \frac{d}{dt}(m_i \dot{\vec{r}}_i) \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - (m_i \dot{\vec{r}}_i) \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right] \qquad \cdots (8.11.1.6)$$

$$\left[\text{using } \ddot{\vec{A}} \cdot \vec{B} = \frac{d}{dt}(\dot{\vec{A}} \cdot \vec{B}) - \dot{\vec{A}} \cdot \frac{d}{dt}(\vec{B})\right]$$

Again,

$$\vec{r}_i = \vec{r}_i(q_j, t) \qquad \cdots (8.11.1.7)$$

$$\delta \vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_1} \delta q_1 + \frac{\partial \vec{r}_i}{\partial q_2} \delta q_2 + \cdots + \frac{\partial \vec{r}_i}{\partial t} \delta t \qquad \cdots (8.11.1.8)$$

or,

$$\dot{\vec{r}}_i = \frac{\partial \vec{r}_i}{\partial q_1} \dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2} \dot{q}_2 + \cdots + \frac{\partial \vec{r}_i}{\partial t} \qquad \cdots (8.11.1.9)$$

Differentiating both side with respect to $\dot{q}_j$ for virtual displacement,

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \qquad \cdots (8.11.1.10)$$

Hence, we get from equation (8.11.1.6)

$$\sum_i \frac{d}{dt}(m_i \dot{\vec{r}}_i) \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left\{ (m_i \dot{\vec{r}}_i) \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} \right\} - (m_i \dot{\vec{r}}_i) \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j} \right]$$

$$\left[\because \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} \text{ and from equation (8.11.1.10), we get } \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \right]$$

$$= \frac{d}{dt} \sum_i \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2} m_i \dot{r}_i^2\right) - \sum_i \frac{\partial}{\partial q_j}\left(\frac{1}{2} m_i \dot{r}_i^2\right)$$

or,

$$\sum_i \frac{d}{dt}(m_i \dot{\vec{r}}_i) \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \qquad \cdots (8.11.1.11)$$

where $T = \sum_i \frac{1}{2} m_i \dot{r}_i^2$ is the kinetic energy of the system.

Therefore, using equations (8.11.1.5) and (8.11.1.11) we get,

$$\sum_j \left[Q_j - \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} \right] \delta q_j = 0 \qquad \cdots (8.11.1.12)$$

Since δq_j's are independent, so, the coefficient of each of them must separately vanish. Hence, we get from equation (8.11.1.12),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad \cdots (8.11.1.13)$$

Equation (8.11.1.13) represents a set of j-equations, which are called Lagrange's equation of motion. Equation (8.11.1.13) is applicable to both conservative and non-conservative systems.

For a conservative system, $Q_j = -\frac{\partial V}{\partial q_j}$ [From equation (8.11.1.2)]

where V is a potential and it is a function of only generalised coordinates.

Now, equation (8.11.1.13) for a **conservative system** reduces to,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} \qquad \cdots (8.11.1.14)$$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}(T-V)\right\} - \frac{\partial}{\partial q_j}(T-V) = 0 \quad \left[\because \frac{\partial V}{\partial \dot{q}_j} = 0\right]$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{where } L = L(q_j, \dot{q}_j) = T - V \qquad \cdots (8.11.1.15)$$

L is called the **Lagrangian function** of the system.

In general, Lagrangian (L) is a function of generalised coordinate q_j, generalised velocity ($\dot{q}_j$) and time (t) i.e., $L = L(q_j, \dot{q}_j, t)$. *In case of a conservative system, it is only a function of generalised coordinate and generalised velocity.*

The equation (8.11.1.15) is a set of j-equations applicable to holonomic conservative system and is called **Lagrange's equation** or **Lagrangian equation of motion**.

8.11.2 Cyclic Coordinates

In general, Lagrangian of any dynamical system is a function of $q_j, \dot{q}_j$ and t i.e., $L = L(q_j, \dot{q}_j, t)$ for n number of generalised coordinates.

Due to some reasons, when some of the generalised coordinates (say q_k) do not occur explicitly in the expression of Lagrangian, then those coordinates are called ignorable or cyclic coordinates. In that case, any change of these coordinates can not affect the Lagrangian.

Lagrangian equation of motion from the equation (8.11.1.15) for a cyclic coordinate (q_k) is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \quad \left(\because \frac{\partial L}{\partial q_k} = 0\right) \qquad \cdots (8.11.2.1)$$

On integration we have,

$$\frac{\partial L}{\partial \dot{q}_k} = p_k = \text{constant} \qquad \cdots (8.11.2.2)$$

Thus, whenever a coordinate q_k does not appear explicitly in L the corresponding conjugate momentum (may be linear or angular) is constant of motion.

- For an example, $\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}(T-V) = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i}$

Since, $V = V(x, y, z) \quad \therefore \frac{\partial V}{\partial \dot{x}_i} = 0$

$$\therefore \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\sum_i \frac{1}{2} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) = m_i \dot{x}_i = p_{i_x}$$

i.e., the x component of the linear momentum associated with the ith particle and its value is constant for that particle.

Hence, *the generalised momentum conjugate to a cyclic coordinate is conserved*. The pair (q_k, p_k) is known as *canonical or conjugate variables*.

Examples :

- For the motion of a projectile with position coordinate (x, y, z) under the constant gravitational field, the Lagrangian L can be written as,

$$L = \frac{1}{2}mv^2 - mgz$$

where z is the vertical component of projectile motion of mass m.

or $$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \quad \cdots (8.11.23)$$

Since L is *not depending on x and y* (although they are present in corresponding velocities), so *x and y are called cyclic coordinates*.

- Similarly, the motion of a free particle is also an example of cyclic coordinates. Here, we can show all the 3 coordinates x, y, z are cyclic coordinate.

8.11.3 Advantages of Lagrangian Formulation over Newtonian Mechanics

Derivation or formulation of Lagrange's equation of motion for a system is not a new theory, it is an alternative and equivalent formulation of Newtonian equation of motion. Both of them are of second order differential equation of motion of the system. When they are solved, we get the nature of motion of the system. But still there are so many advantages of Lagrangian formulation, which makes it different from Newtonian mechanics. These advantages are given below.

- In Lagrangian equation of motion, we use generalised coordinates which do not depend on any particular coordinate system. On the other hand, we have to use a particular coordinate system depending on the symmetry of the dynamical system in Newtonian mechanics.
- Lagrange's equation of motion is invariant in form with respect to any coordinate system but the form of Newtonian equation of motion is not invariant.
- Lagrange's equation of motion deals with scalar quantities (i.e., kinetic energy T and potential energy V). But in Newtonian mechanics vector quantities like force, momentum appear. This vector quantities make the equation of motion more difficult to solve.
- In Newtonian mechanics, we deal with many forces some of which (when particularly constraints are introduced) may not be known. But under these situation kinetic and potential energies required in Lagrangian approach, may be calculated. So Lagrangian formulation is more useful and convenient.

These above points may also be considered as the differences between Newtonian mechanics and Lagrangian mechanics.

8.11.4 Applications of Lagrangian Formulation

- **Motion of free particle :** When a particle is not acted upon by any external force, the particle is called free particle. So, the potential energy function V is zero here. Therefore, the Lagrangian L for a free particle will be,

$$L = T - V$$

or, $$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \cdots (8.11.4.1)$$

where (x, y, z) is the set of cartesian coordinates of the free particle of mass m. **Here all the coordinates are cyclic as Lagrangian L is not explicitly depending on them though they are present in corresponding velocities.**

So the Lagrangian equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \quad \text{or, } m\ddot{x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \quad \text{or, } m\ddot{y} = 0 \quad \cdots (8.11.4.2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{\partial L}{\partial z} \quad \text{or, } m\ddot{z} = 0$$

Simple harmonic oscillation: If the free vibration of any vibrating system is simple harmonic, this vibrating system is called **harmonic oscillator.**

Examples: An atom at the lattice sites of crystals may be considered as a harmonic oscillator.

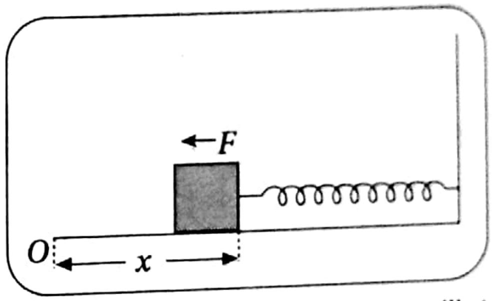

Fig. 11 ▷ A simple harmonic oscillator

Let us consider a vibrating body which is acted upon by a force directed towards or away from a fixed point O (equilibrium position). Let x be the displacement of the body at any instant and the magnitude of the force (F) varies linearly with displacement (x) from the fixed point. i.e., $F \propto -x$.

Here, force $F = -kx$, k is the spring or force constant.

So, the kinetic energy (T) and potential energy (V) of a simple harmonic oscillator of mass m are given by,

$$T = \frac{1}{2}m\dot{x}^2 \quad \text{and} \quad V = -\int F\,dx = -\int_{\infty}^{x} -kx\,dx = \frac{1}{2}kx^2$$

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad \cdots (8.11.4.3)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad \frac{\partial L}{\partial x} = -kx$$

Now, the Lagrange's equation of motion for a simple harmonic oscillator will be,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{or,} \quad \frac{d}{dt}(m\dot{x}) + kx = 0$$

or, $$m\ddot{x} + kx = 0 \quad \text{or,} \quad \ddot{x} + \frac{k}{m}x = 0$$

or, $\ddot{x}+\omega^2 x = 0$, where $\omega = \sqrt{\frac{k}{m}}$ ... (8.11.4.4)

So, the time period of oscillation, $T = \frac{2\pi}{\omega}$ or, $T = 2\pi\sqrt{\frac{m}{k}}$ ... (8.11.4.5)

Special Note

A harmonic oscillator is imagined to consist of a point mass m attached to a spring of force constant k. Here, the spring is being used only as representation of the field forces on the system due to its environment.

- **Simple pendulum:** Let us consider a simple pendulum of mass m and effective length l suspended from O. The angle θ between the rest position and deflected position of the bob is considered as generalised coordinate.

The kinetic energy of the simple pendulum $T = \frac{1}{2}ml^2\dot{\theta}^2$

and the potential energy $V = mg(OA - OC) = mg(l - l\cos\theta)$

$$= mgl(1-\cos\theta)$$

So, the **Lagrangian,** $L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$ ... (8.11.4.6)

$$\therefore \quad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad \text{and} \quad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

So, the Lagrangian equation of motion will become,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{or,} \quad ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\text{or,} \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

If the amplitude of oscillation is small enough then $\sin\theta$ can be approximated as, $\sin\theta \simeq \theta$.

So, $\ddot{\theta} + \frac{g}{l}\theta = 0$

or, $\ddot{\theta} + \omega^2\theta = 0$ $\left[\text{where, } \omega = \sqrt{\frac{g}{l}}\right]$ ... (8.11.4.7)

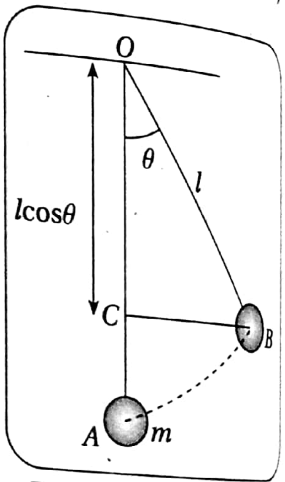

Fig. 12 ▷ Motion of an simple pendulum

and the time period of oscillation $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$ ... (8.11.4.8)

- **Central force field problem:** Any force which is directed towards a fixed centre is called a central force and is defined as,

$$f(r) = -\frac{k}{r^2}$$

where k is the force constant, r is the radius of the orbit.

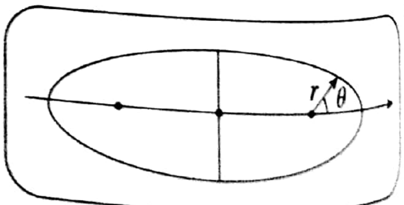

Fig. 13 ▷ Central force on a particle

Central force depends on the radial distance only.

The kinetic energy of a particle for such a system in terms of its generalised coordinate (r, θ) can be written as,

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ and the potential energy

$$V = -\int f(r)\,dr = -\int_{\infty}^{r} -\frac{k}{r^2}dr \quad \left[\because f(r) = -\frac{dV}{dr} \quad \therefore V = -\int f(r)dr\right]$$

$$= -\left[\frac{k}{r}\right]_{\infty}^{r} = -\frac{k}{r} + 0 = -\frac{k}{r}$$

So, the Lagrangian, $\boxed{\mathbf{L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}}}$ ... (8.11.4.9)

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{k}{r^2}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = 0$$

Now, the Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \text{or,} \quad \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{k}{r^2} = 0$$

or, $\boxed{\mathbf{m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} = 0}}$ ... (8.11.4.10)

and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$ or, $\frac{d}{dt}(mr^2\dot{\theta}) = 0$

or, $mr^2\dot{\theta} = \text{constant} = l$ or, $\boxed{\mathbf{\dot{\theta} = \frac{l}{mr^2}}}$ ... (8.11.4.11)

Substituting $\dot{\theta}$ from equation (8.11.4.11) into equation (8.11.4.10) we get,

$$m\ddot{r} = mr\frac{l^2}{m^2r^4} + f(r) \quad \text{since } f(r) = -\frac{k}{r^2}$$

or, $\boxed{\mathbf{m\ddot{r} = \frac{l^2}{mr^3} + f(r)}}$ ... (8.11.4.12)

Let $r = \frac{1}{u}$,

$$\frac{dr}{dt}\left(=\dot{r}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta}$$

$$\frac{d^2r}{dt^2}\left(=\ddot{r}\right) = -\frac{1}{u^2}\dot{\theta}\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -\frac{1}{u^2}\dot{\theta}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

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or, $\ddot{r} = -\frac{1}{u^2}\dot{\theta}^2\frac{d^2u}{d\theta^2}$ ··· (8.11.4.13)

Substituting $\ddot{r}$ from equation (8.11.4.13) into equation (8.11.4.12) we get,

$$m\left(-\frac{1}{u^2}\dot{\theta}^2\frac{d^2u}{d\theta^2}\right) = \frac{l^2}{mr^3} + f(r) \quad \text{or,} \quad -m\frac{l^2}{m^2r^4u^2}\frac{d^2u}{d\theta^2} = \frac{l^2}{mr^3} + f(r)$$

$$\text{or,} \quad -\frac{l^2}{mr^2}\frac{d^2u}{d\theta^2} = \frac{l^2}{mr^3} + f(r) \quad \text{or,} \quad \frac{d^2u}{d\theta^2} = -\frac{l^2}{mr^3} \times \frac{mr^2}{l^2} - \frac{mr^2}{l^2}f(r)$$

$$\text{or,} \quad \frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2}f\left(\frac{1}{u}\right) \qquad \cdots (8.11.4.14)$$

This is the ***equation of motion for central force field***. Example of such motion is planetary motion.

5 Compound pendulum: If a rigid body is capable of oscillating in a vertical plane (X-Y) above a fixed horizontal axis, it is called a compound pendulum.

In the figure, O is the point through which axis of rotation passes and B is the centre of mass. Here θ is the angle by which the body is deflected and is considered as generalised coordinate.

The kinetic energy of the compound pendulum of moment of inertia I about the axis of rotation can be written as,

$$T = \frac{1}{2}I\dot{\theta}^2$$

And the potential energy about the reference level (i.e., horizontal plane) passing through O will be

$$V = -mgl\cos\theta$$

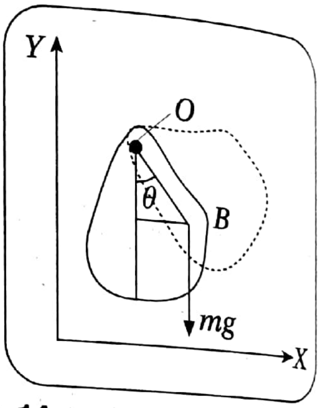

Fig. 14 ▷ Compound pendulum

So, the Lagrangian,

$$L = T - V = \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta \qquad \cdots (8.11.4.15)$$

$$\therefore \quad \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \quad \text{and} \quad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

Now, the Lagrange's equation for a compound pendulum will become,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{or,} \quad I\ddot{\theta} + mgl\sin\theta = 0 \quad \text{or,} \quad \ddot{\theta} + \frac{mgl\sin\theta}{I} = 0$$

If the amplitude of oscillation is small enough, then $\sin\theta \simeq \theta$.

Hence, $\ddot{\theta} + \frac{mgl}{I}\theta = 0$ or, $\ddot{\theta} + \omega^2\theta = 0$ $\left[\because \omega = \sqrt{\frac{mgl}{I}}\right]$ ··· (8.11.4.16)

So, the time period of oscillation for a compound pendulum,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}} \qquad \cdots (8.11.4.17)$$

6 Atwood's machine: Let two masses m_1 and m_2 are suspended over a frictionless pulley of radius a by an inextensible string of length l.

$\therefore$ The equation of constraint is $x_1 + x_2 + \pi a = l$

Differentiating the above equation we get

$$\dot{x}_1 = -\dot{x}_2$$

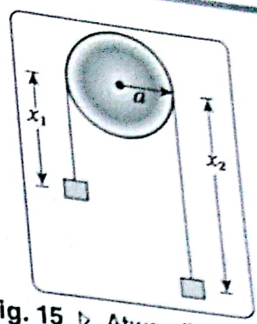

Fig. 15 ▷ Atwood's machine

$\therefore$ The kinetic energy of the system,

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 \quad [\text{As } \dot{x}_1 = -\dot{x}_2]$$

And the potential energy of the system

$$V = -m_1gx_1 - m_2gx_2 = -m_1gx_1 - m_2g(l - \pi a - x_1) \; [\because x_1 + x_2 + \pi a = l]$$

$$= -m_1gx_1 + m_2gx_1 - m_2g(l - \pi a)$$

So, the Lagrangian

$$L = T - V$$

or, $$L = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + m_1gx_1 - m_2gx_1 + m_2g(l - \pi a)$$

or $$\boldsymbol{L = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + (m_1 - m_2)gx_1 + V_0} \quad \cdots (8.11.4.18)$$

where $V_0 = m_2g(l - \pi a)$

So, the Lagrangian equation (in terms of x_1) will be,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \quad \text{or,} \quad \frac{d}{dt}[(m_1 + m_2)\dot{x}_1] - (m_1 - m_2)g = 0$$

$$\text{or,} \quad (m_1 + m_2)\ddot{x}_1 = (m_1 - m_2)g$$

or, $$\boldsymbol{\ddot{x}_1 = \frac{m_1 - m_2}{(m_1 + m_2)}g} \quad \cdots (8.11.4.19)$$

which is the acceleration of the system.

7 A two-particle system: Consider a two-particle system of masses m_1 and m_2 whose position vectors are $\vec{r}_1$ and $\vec{r}_2$ respectively with respect to (some reference) origin O. Let $\vec{R}$ be the position vector of the centre of mass C.

We define a vector $\vec{r}$ as $\vec{r} = \vec{r}_1 - \vec{r}_2$.

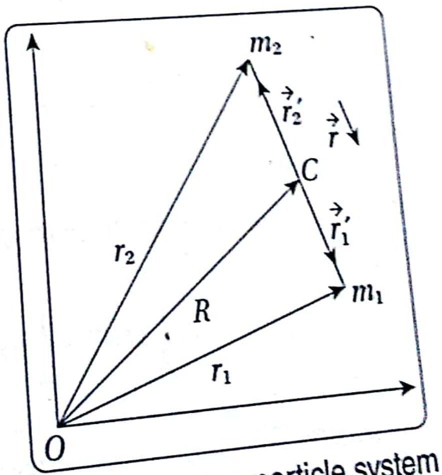

Fig. 16 ▷ A two-particle system

Let $\vec{r}_1{}'$ and $\vec{r}_2{}'$ be the position vectors of m_1 and m_2 with respect to centre of mass C.

$\therefore$ Lagrangian for the system will be $L = T - V$

where T = kinetic energy

$$= \frac{1}{2}(m_1 + m_2)\dot{R}^2 + T'$$

[T' = kinetic energy of the system about the centre of mass, $\vec{R}$ = position vector of centre of mass from the origin O]

$$T' = \frac{1}{2}m_1\dot{r}_1'^2 + \frac{1}{2}m_2\dot{r}_2'^2$$

Again $\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{r}_1' - \vec{r}_2'$

Now, we have

$$\sum_i m_i \cdot \vec{r}_i' = 0 \quad \text{or,} \quad m_1\vec{r}_1' + m_2\vec{r}_2' = 0 \quad \text{or,} \quad \vec{r}_2' = -\frac{m_1}{m_2}\vec{r}_1'$$

Substituting $\vec{r}_2'$ in $\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{r}_1' - \vec{r}_2'$,

$$\vec{r} = \vec{r}_1' + \frac{m_1}{m_2}\vec{r}_1' = \frac{m_1 + m_2}{m_2}\vec{r}_1' \quad \text{or,} \quad \vec{r}_1' = \frac{m_2\vec{r}}{m_1 + m_2}$$

Similarly, $\vec{r}_2' = -\dfrac{m_1\vec{r}}{m_1 + m_2}$

Now, $$T = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + T'$$

or, $$T = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}m_1\dot{r}_1'^2 + \frac{1}{2}m_2\dot{r}_2'^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}m_1\frac{m_2^2}{(m_1 + m_2)^2}\dot{r}^2 + \frac{1}{2}m_2\frac{m_1^2}{(m_1 + m_2)^2}\dot{r}^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}\dot{r}^2\left[\frac{m_1m_2^2 + m_2m_1^2}{(m_1 + m_2)^2}\right]$$

$$= \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}\cdot\frac{m_1m_2(m_1 + m_2)}{(m_1 + m_2)^2}\dot{r}^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}\dot{r}^2$$

$$= \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2$$

where $M = m_1 + m_2$ = total mass of the system and $\mu = \dfrac{m_1m_2}{m_1 + m_2}$ = reduced mass of the system.

$\therefore$ Lagrangian for the system $L = T - V = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2 - V(r)$ ... (8.11.4.20)

$\therefore$ The Lagrangian equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \text{or,} \quad \frac{d}{dt}\left(\mu\dot{\vec{r}}\right) + \frac{\partial V(r)}{\partial r} = 0$$

$$\text{or,} \quad \mu\ddot{\vec{r}} + \frac{\partial V(r)}{\partial r} = 0 \qquad \cdots (8.11.4.21)$$

$$\text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) - \frac{\partial L}{\partial R} = 0 \quad \text{or,} \quad \frac{d}{dt}\left(M\dot{\vec{R}}\right) = 0 \quad \text{or,} \quad M\ddot{\vec{R}} = 0 \qquad \cdots (8.11.4.22)$$

$$\text{or,} \quad \dot{\vec{R}} = \text{constant} \qquad \cdots (8.11.4.23)$$

Electrical circuit: For an electrical circuit, containing an inductance, capacitance and resistance, the Lagrangian is given by,

$$L_E = T_M - V_E \qquad \cdots (8.11.4.24)$$

where T_M is *the magnetic energy of electrical circuit* (analogous to kinetic energy of mechanical system) and V_E is *the electrical energy of the electrical circuit* (analogous to the potential energy of the mechanical system).

The Lagrange's equation of motion can be written as,

$$\frac{d}{dt}\left(\frac{\partial L_E}{\partial \dot{q}_j}\right) - \frac{\partial L_E}{\partial q_j} = Q_j \qquad \cdots (8.11.4.25)$$

where Q_j is the generalised force due to friction.

If the system is free from any friction or dissipative forces, $Q_j = 0$ and the corresponding Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L_E}{\partial \dot{q}_j}\right) - \frac{\partial L_E}{\partial q_j} = 0 \qquad \cdots (8.11.4.26)$$

But when *current* i *flows through a resistance* R, *the dissipative force* $Q = Ri$.

PROBLEM 1 **Find the Lagrangian and Lagrange's equation of motion for the electrical circuit containing an inductance L and capacitance C.**

Solution Let the current flowing through an electrical circuit be i. The capacitor is now charged by q amount of charge.

$$\therefore \; T_M = \text{magnetic energy} = \frac{1}{2}Li^2 = \frac{1}{2}L\dot{q}^2 \quad \left[\because i = \frac{dq}{dt}\right]$$

$$V_E = \text{electrical energy} = \frac{q^2}{2C}$$

So, Lagrangian of this electrical circuit,

$$L_E = T_M - V_E = \frac{1}{2}L\dot{q}^2 - \frac{q^2}{2C}$$

If q be the generalised coordinate, the Lagrange's equation of motion in absence of dissipative frictional force is given by,

$$\frac{d}{dt}\left(\frac{\partial L_E}{\partial \dot{q}}\right) - \frac{\partial L_E}{\partial q} = 0 \qquad \left[\text{From eqn. (1) we get, } \frac{\partial L_E}{\partial \dot{q}} = L\dot{q} \text{ and } \frac{\partial L_E}{\partial q} = -\frac{q}{C}\right]$$

or, $$L\ddot{q} + \frac{q}{C} = 0.$$

PROBLEM 2 **Find Lagrange's equation of motion for the following circuit.**

Solution Let q be the charge flowing through the circuit,

$$T_M = \frac{1}{2}Li^2 = \frac{1}{2}L\dot{q}^2$$

$$V_E = \frac{q^2}{2C} - qE$$

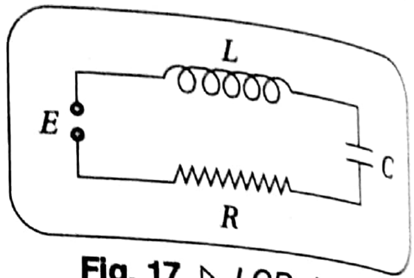

Fig. 17 ▷ LCR circuit

Lagrangian for this electrical circuit,

$$L_E = T_M - V_E = \frac{1}{2}L\dot{q}^2 - \left(\frac{q^2}{2C} - qE\right)$$

Now, $$\frac{\partial L_E}{\partial \dot{q}} = L\dot{q} \quad \text{and} \quad \frac{\partial L_E}{\partial q} = -\left(\frac{q}{C} - E\right)$$

So, the Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L_E}{\partial \dot{q}}\right) - \frac{\partial L_E}{\partial q} = -Ri$$ as the dissipative frictional force due to resistance R is $Q = -Ri$

or, $$L\ddot{q} + \frac{q}{C} - E = -Ri$$

or, $$L\ddot{q} + \frac{q}{C} = E - Ri.$$

PROBLEM 3 **If $L = \frac{1}{2}m\dot{x}^2 - \beta x\dot{x} - \frac{kx^2}{2}$, find the Lagrangian equation of motion.**

Solution $L = \frac{1}{2}m\dot{x}^2 - \beta x\dot{x} - \frac{kx^2}{2}$; $\frac{\partial L}{\partial \dot{x}} = m\dot{x} - \beta x$; $\frac{\partial L}{\partial x} = -kx - \beta\dot{x}$

The Lagrange's equation of motion becomes,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{or,} \quad \frac{d}{dt}(m\dot{x} - \beta x) + kx + \beta\dot{x} = 0 \quad \text{or,} \quad m\ddot{x} - \beta\dot{x} = -\beta\dot{x} - kx$$

or, $$m\ddot{x} + kx = 0$$

This is the required equation of motion.

...BLEM If a proton is 1800 times heavier than an electron, then show that the centre of mass of hydrogen atom is located practically at proton. Find the reduced mass of this system and write the Lagrangian of the reduced mass. [W.B.U.T. 2007]

...tion Let the mass of an electron is m_e.

the mass of a proton $m_p = 1800\, m_e$.

...the distances of the electron and proton from the centre of mass are r_1 and r_2 ...spectively, the coordinate of center of mass of hydrogen atom (system)

$$\vec{R} = \frac{m_e\vec{r}_1 + m_p\vec{r}_2}{m_e + m_p} = \frac{m_e\vec{r}_1 + 1800\, m_e\vec{r}_2}{m_e + 1800\, m_e}$$

$m_p \gg m_e$ we can write $R \simeq \dfrac{1800 m_e\vec{r}_2}{1801 m_e}$ or, $R \simeq \vec{r}_2$

...the centre of mass of hydrogen atom is located practically at proton.

...ow, μ = reduced mass of the system

$$= \frac{m_e m_p}{m_e + m_p} = \frac{m_e \times 1800\, m_e}{1801\, m_e} = \frac{1800}{1801} m_e$$

...ow Lagrangian of the system can be written as

$$L = \frac{1}{2} M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2 - V(r)$$

...here $M = 1800\, m_e + m_e = 1801\, m_e$ = Total mass of the system

$\mu = \dfrac{1800}{1801} m_e$ = reduced mass of the system

$V(r)$ = potential energy of the system.

...12 HAMILTONIAN FORMULATION

Lagrange's equation of motion for a holonomic system is a second order differential equation. Therefore, to know the nature of the motion of the system we have to solve second order equation which sometimes may be a difficult job.

Hamilton derived a set of first order differential equation of motion in the form of generalised coordinate q_j and momenta $p_j \left(= \dfrac{\partial L}{\partial \dot{q}_j}\right)$ to know the nature of motion of a dynamical system. Since, **it is a first order differential equation, we can solve it very easily. In this way we can make a difference between Lagrangian and Hamiltonian equation.**

Hamiltonian : In general, Lagrangian $L = L(q_j, \dot{q}_j, t)$. If L is not an explicit function of time then, $L = L(q_j, \dot{q}_j)$

Now, $dL = \sum_j \frac{\partial L}{\partial q_j} dq_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j$

$\therefore \quad \frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}(\dot{q}_j)$

or, $\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}(\dot{q}_j)$ $\left[\because \text{from Lagrange's equation } \frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]$

or, $\frac{dL}{dt} = \sum_j \frac{d}{dt}(p_j)\,\dot{q}_j + \sum_j p_j \frac{d}{dt}(\dot{q}_j)$ $\left[\because \text{from equation (8.11.2.2), we have } p_j = \frac{\partial L}{\partial \dot{q}_j}\right]$

or, $\frac{dL}{dt} = \sum_j \frac{d}{dt}(p_j\,\dot{q}_j)$ or, $\frac{d}{dt}\left(\sum_j p_j\,\dot{q}_j - L\right) = 0$

or,
$$\sum_j p_j\,\dot{q}_j - L = \text{constant} \qquad \cdots(8.12.1)$$

The function $\sum_j p_j\,\dot{q}_j - L$ is a conserved quantity. It is known as the **Hamiltonian, H** of the system. So, $H = \sum_j p_j\,\dot{q}_j - L$. (Of course the name Hamiltonian will remain whether H is conserved or not.)

Therefore, we can say, **when L is not an explicit function of time, Hamiltonian is a constant of motion.**

8.12.1 Formulation of Hamilton's Canonical Equations of Motion

In general, with giving equal footings to position coordinate q_j and momentum coordinate p_j, we can write the Hamiltonian as $H = H(p_j, q_j, t)$

We know, $H = \sum_j p_j\,\dot{q}_j - L.$

$\therefore$
$$dH = \sum_j \dot{q}_j\,dp_j + \sum_j p_j\,d\dot{q}_j - dL \qquad \cdots(8.12.1.1)$$

Again $L = L(q_j, \dot{q}_j, t)$

$\therefore$
$$dL = \sum_j \frac{\partial L}{\partial q_j} dq_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt \qquad \cdots(8.12.1.2)$$

Substituting dL from equation (8.12.1.2) into equation (8.12.1.1) we get,

$$dH = \sum_j \dot{q}_j\,dp_j + \sum_j p_j\,d\dot{q}_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt \qquad \cdots(8.12.1.3)$$

$p_j = \dfrac{\partial L}{\partial \dot{q}_j}$ and from Lagrange's equation we have,

Again,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \quad \text{or,} \quad \dot{p}_j = \frac{\partial L}{\partial q_j}$$

Substituting $\dfrac{\partial L}{\partial \dot{q}_j} = p_j$ and $\dfrac{\partial L}{\partial q_j} = \dot{p}_j$ into equation (8.12.1.3) we get,

$$dH = \sum_j \dot{q}_j\, dp_j + \sum_j p_j\, d\dot{q}_j - \sum_j \dot{p}_j\, dq_j - \sum_j p_j\, d\dot{q}_j - \frac{\partial L}{\partial t}dt$$

$$\boldsymbol{dH = \sum_j \dot{q}_j\, dp_j - \sum_j \dot{p}_j\, dq_j - \frac{\partial L}{\partial t}dt} \qquad \cdots(8.12.1.4)$$

or,

$$H = H(q_j, p_j, t)$$

Again,

$$\boldsymbol{dH = \sum_j \frac{\partial H}{\partial q_j}dq_j + \sum_j \frac{\partial H}{\partial p_j}dp_j + \frac{\partial H}{\partial t}dt} \qquad \cdots(8.12.1.5)$$

$\therefore$ Equating equations (8.12.1.4) and (8.12.1.5) we get,

$$\sum_j \dot{q}_j\, dp_j - \sum_j \dot{p}_j\, dq_j - \frac{\partial L}{\partial t}dt = \sum_j \frac{\partial H}{\partial q_j}dq_j + \sum_j \frac{\partial H}{\partial p_j}dp_j + \frac{\partial H}{\partial t}dt$$

Comparing the coefficients from both sides of the above equation we get,

$$-\dot{p}_j = \frac{\partial H}{\partial q_j} \quad \text{or,} \quad \boldsymbol{\dot{p}_j = -\frac{\partial H}{\partial q_j}} \qquad \cdots(8.12.1.6)$$

$$\boldsymbol{\dot{q}_j = \frac{\partial H}{\partial p_j}} \quad \cdots(8.12.1.7) \qquad \text{and} \qquad \boldsymbol{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}} \qquad \cdots(8.12.1.8)$$

Equations (8.12.1.6) to (8.12.1.8) are known as **Hamiltons equations of motion**. The integration of these two equations give $2n$ integration constants which can be find in terms of initial conditions.

8.12.2 Hamiltonian for a Conservative System

Hamiltonian H for a system can be written as $H = H(p_j, q_j, t) = \sum_j p_j \dot{q}_j - L$

$\because$ For a conservative system potential energy $V = V(q_j)$ only.

and, generalised momentum $p_j = \dfrac{\partial L}{\partial \dot{q}_j} = \dfrac{\partial (T-V)}{\partial \dot{q}_j} = \dfrac{\partial T}{\partial \dot{q}_j} \quad \left[\because \dfrac{\partial V}{\partial q_j} = 0\right]$

or, $p_j = \dfrac{\partial}{\partial \dot{q}_j}\left[\sum_{j,k} a_{jk}\, \dot{q}_j\, \dot{q}_k\right]$, where $T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k$ [from equation (8.12.4)]

$$= 2\sum_{j,k} a_{jk}\dot{q}_k, \text{ where } \boldsymbol{a_{jk} = \sum_i \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k}} \qquad \cdots(8.12.2.1)$$

Special Note

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right) \cdot \left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k\right)$$

Now $\vec{r}_i = \vec{r}_i(q_j)$. Since for virtual displacement $\delta t = 0$; $\dot{\vec{r}}_i = \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j$

$$\therefore \quad T = \sum_{j,k}\left(\sum_i \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial q_k}\right)\dot{q}_j \dot{q}_k = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k$$

Hence, the **Hamiltonian H for a conservative system**

$$H = \sum_j p_j \dot{q}_j - L = \sum_{j,k} 2a_{jk}\dot{q}_k\dot{q}_j - L = 2\sum_{j,k} a_{jk}\dot{q}_k\dot{q}_j - L$$

$$= 2T - L = 2T - (T - V)$$

or, $$\boldsymbol{H = T + V = E} \qquad \cdots(8.12.2.2)$$

where E = total energy $= T + V$.

Hence, *only for a conservative system, the Hamiltonion indicates the total energy of the system.*

Again, from equation (6.15.1.9), we can write for $L \neq L(t)$,

$$\frac{dH}{dt} = 0 \text{ i.e., } H = \text{constant and } H \textit{ is not a function of time.}$$

So, *the Hamiltonian is a constant of motion and hence conserved.*

8.12.3 General Notes on Hamiltonian Function

1. It is defined as a function of generalised coordinate q_j, generalised momenta p_j and time t i.e., $H = H(q_j, p_j, t)$
2. It can be defined from $H = \sum_j p_j \dot{q}_j - L(q_j, \dot{q}_j, t)$, where $p_j = \frac{\partial L}{\partial \dot{q}_j}$.
3. Hamiltonian equations of motion can be derived using Lagrangian function L. These are $\frac{\partial H}{\partial q_j} = -\dot{p}_j$, $\frac{\partial H}{\partial p_j} = \dot{q}_j$ and $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

8.12.4 General Significance of the Hamiltonian

1. If the Lagrangian L *of a system does not depend on time explicitly, the Hamiltonian* H *is a constant of motion* i.e., *conserved.*
2. *For a conservative system* $[V = V(q_j)$ and $V \neq V(\dot{q}_j)]$; the *Hamiltonian* $H = T + V = E$ = *total energy of the system.* In that case, the constraints are independent on time.

- If the constraints are time dependent or the transformation equations $r_j = r_j(q_j, t)$ contain time explicitly, $H \neq E$. But still the total energy is conserved.

8.12.5 Applications of Hamiltonian Formulation

- **Simple harmonic oscillator:** The kinetic energy and potential energy of a simple harmonic oscillator of spring constant k and mass m can be written (like example 2 of article 8.11.4)

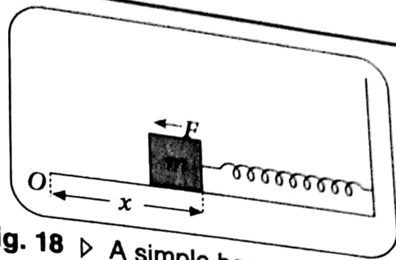

Fig. 18 ▷ A simple harmonic oscillator

$$T = \frac{1}{2}m\dot{x}^2 \text{ and } V = -\int_{\infty}^{x} -kx\,dx = \frac{1}{2}kx^2$$

So, the Lagrangian

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 = \frac{p_x^2}{2m} - \frac{1}{2}kx^2 \quad [\because p_x = m\dot{x}] \qquad \cdots(8.12.5.1)$$

Hence, the generalised momentum

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{or,} \quad \dot{x} = \frac{p_x}{m}$$

So, Hamiltonian $H = \sum_j p_j \dot{q}_j - L = p_x \dot{x} - L = p_x \frac{p_x}{m} - \left(\frac{p_x^2}{2m} - \frac{1}{2}kx^2\right)$

$$\text{or,} \quad H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 = \text{total energy} \qquad \cdots(8.12.5.2)$$

The Hamiltonian equations of motion are,

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad \text{and} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -kx$$

So, $p_x = m\dot{x}$

Now $\dot{p}_x = -kx$ or, $\frac{d}{dt}(m\dot{x}) + kx = 0 \quad [\because p_x = m\dot{x}]$

or, $m\ddot{x} + kx = 0$ or, $\ddot{x} + \frac{k}{m}x = 0$

$$\text{or,} \quad \ddot{x} + \omega^2 x = 0, \text{ where, } \omega = \sqrt{\frac{k}{m}} \qquad \cdots(8.12.5.3)$$

This equation is Hamiltonian equation of motion.

Hence, the time period of oscillation

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad \cdots(8.12.5.4)$$

The solution is similar to that obtained in Lagrangian formulation.

- **Simple pendulum:** The kinetic and potential energies of a simple pendulum of mass m and effective length l are given by (as discussed in example 3 of article 8.11.4)

$$T = \frac{1}{2}ml^2\dot{\theta}^2 \quad \text{and} \quad V = mgl(1-\cos\theta)$$

Hence, the Lagrangian $L = T - V$

or, $$\boldsymbol{L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)} \quad \cdots (8.12.5.5)$$

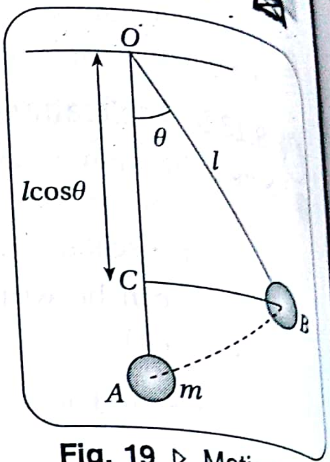

Fig. 19 ▷ Motion of a simple pendulum

So, the generalised momentum p_θ is given by

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad \text{or,} \quad \dot{\theta} = \frac{p_\theta}{ml^2}$$

The Hamiltonian $H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta} - L$

$$= \frac{p_\theta^2}{ml^2} - \left[\frac{ml^2p_\theta^2}{2m^2l^4} - mgl(1-\cos\theta)\right]$$

$\left[\text{substituting the value } \dot{\theta} = \frac{p_\theta}{ml^2}\right]$

or, $$\boldsymbol{H = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)} \quad \cdots (8.12.5.6)$$

So, the Hamilton's equations of motion are,

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta$$

So, $p_\theta = ml^2\dot{\theta}$.

Now $\dot{p}_\theta = -mgl\sin\theta$ or, $\frac{d}{dt}(p_\theta) = -mgl\sin\theta$

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0 \quad [\because p_\theta = ml^2\dot{\theta}]$$

or, $$ml^2\ddot{\theta} + mgl\sin\theta = 0 \quad \text{or,} \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

If the amplitude of oscillation is small enough, then $\sin\theta \simeq \theta$.

$$\therefore \quad \ddot{\theta} + \frac{g}{l}\theta = 0 \quad \text{or,} \quad \boldsymbol{\ddot{\theta} + \omega^2\theta = 0} \quad \left[\because \omega = \sqrt{\frac{g}{l}}\right] \quad \cdots (8.12.5.7)$$

This equation is Hamiltonian equation of motion.

The solution is similar to that obtained in Lagrangian formulation.

$\therefore$ the time period of oscillation $$\boldsymbol{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}} \quad \cdots (8.12.5.8)$$

❸ **Central force field problem:** The kinetic and potential energies of a particle of mass m under central force can be written as,

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad \text{and} \quad V = V(r)$$

where r and θ are the polar co-ordinates of the particle P (as discussed in example 4 of Article 8.11.4).

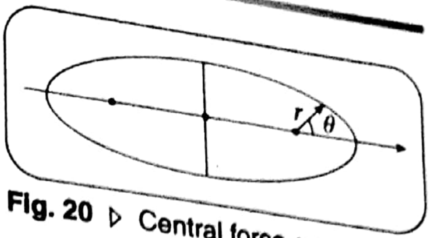

Fig. 20 ▷ Central force on a particle

Hence, the Lagrangian

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int_{\infty}^{r} -\frac{k}{r^2}dr$$

or,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r} \quad \cdots(8.12.5.9)$$

So, the generalised momenta in terms of r and θ can be written as,

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{and} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

The Hamiltonian $H = \sum_j p_j \dot{q}_j - L = p_r\dot{r} + p_\theta\dot{\theta} - \left[\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}\right]$

$$= p_r \cdot \frac{p_r}{m} + p_\theta \cdot \frac{p_\theta}{mr^2} - \left[\frac{1}{2}m\left(\frac{p_r^2}{m^2} + r^2 \cdot \frac{p_\theta^2}{m^2r^4}\right) + \frac{k}{r}\right]$$

or,

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r} \quad \cdots(8.12.5.10)$$

So, the Hamilton's equations of motion are,

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \quad \text{and} \quad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

or, $p_\theta = \text{constant} = l$ (say)

This implies that the angular momentum for a central force field is conserved.

Again $\dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2}$ or, $\frac{d}{dt}(m\dot{r}) = \frac{l^2}{mr^3} - \frac{k}{r^2}$ (since $p_\theta = l$)

or,

$$m\ddot{r} = \frac{l^2}{mr^3} + f(r) \quad \left(\text{since } f(r) = -\frac{k}{r^2}\right) \quad \cdots(8.12.5.11)$$

Again it can be showed that

$$\ddot{r} = -\frac{l^2u^2}{m^2}\frac{d^2u}{d\theta^2}\text{•}, \text{ where } r = \frac{1}{u} \quad \cdots(8.12.5.12)$$

• The same equation was derived in the Lagrangian formulation of the same problem [equation (8.11.4.13)].

Using equations (8.12.5.11) and (8.12.5.12) we get,

$$m\left(-\frac{l^2u^2}{m^2}\frac{d^2u}{d\theta^2}\right) = \frac{l^2}{mr^3}+f(r) \quad \text{or,} \quad -\frac{l^2u^2}{m}\cdot\frac{d^2u}{d\theta^2} = \frac{l^2}{mr^3}+f(r)$$

or, $$\boldsymbol{\frac{d^2u}{d\theta^2}+u = -\frac{m}{l^2u^2}f\left(\frac{1}{u}\right)} \quad \cdots(8.12.5.13)$$

This is the equation of motion for central force problem. It is also similar to that obtained in Lagrangian formulation.

4. **Compound pendulum:** As discussed in example 5 of Article 8.11.4, kinetic energy of the compound pendulum is

$T = \frac{1}{2}I\dot{\theta}^2$, where I is the moment of inertia about the axis of rotation.

The potential energy with reference to a reference level passing through O can be written as $V = -mgl\cos\theta$.

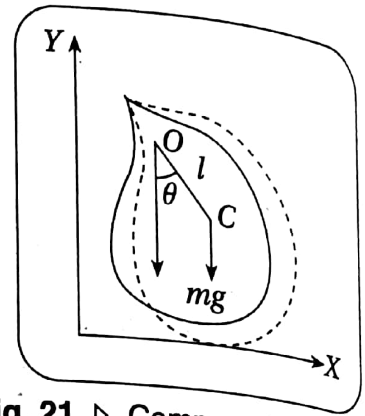

Fig. 21 ▷ Compound pendulum

So, the Lagrangian $\boldsymbol{L = T - V = \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta}$ $\cdots(8.12.5.14)$

The generalised momenta corresponding to θ can be written as,

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = I\dot{\theta} \quad \text{or, } \dot{\theta} = \frac{p_\theta}{I}$$

Hence, the Hamiltonian

$$H = \sum_j p_j\dot{q}_j - L = p_\theta\dot{\theta} - L = p_\theta\cdot\frac{p_\theta}{I} - \frac{1}{2}I\cdot\frac{p_\theta^2}{I^2} - mgl\cos\theta$$

$$= \frac{p_\theta^2}{2I} - mgl\cos\theta$$

or, $$\boldsymbol{H = \frac{p_\theta^2}{2I} + (-mgl\cos\theta) = \text{total energy}} \quad \cdots(8.12.5.15)$$

So, the Hamilton's equations of motion can be written as,

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I} \quad \text{or,} \quad p_\theta = I\dot{\theta} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial\theta} = -mgl\sin\theta$$

Now $\dot{p}_\theta = -mgl\sin\theta$

$$\frac{d}{dt}(p_\theta) = -mgl\sin\theta \quad \text{or,} \quad \frac{d}{dt}(I\dot{\theta}) + mgl\sin\theta = 0 \quad [\because p_\theta = I\dot{\theta}]$$

or, $$I\ddot{\theta} + mgl\sin\theta = 0 \quad \text{or,} \quad \ddot{\theta} + \frac{mgl}{I}\sin\theta = 0$$

If the amplitude of the oscillation is very small, then $\sin\theta \simeq \theta$.

$$\therefore \quad \ddot{\theta} + \frac{mgl}{I}\theta = 0 \quad \text{or,} \quad \ddot{\theta} + \omega^2\theta = 0 \quad \left[\because \omega = \sqrt{\frac{mgl}{I}}\right]$$

This is the Hamiltonian equation of motion. ...(8.12.5.16)

The time period of oscillation $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}$ ...(8.12.5.17)

The above solution is similar to that obtained from Lagrangian.

Here, we have solved the same types of problems by using Lagrangian and Hamiltonian formulation. But *it has been seen that the solution in each case is same by these two methods*. Therefore, **the Hamiltonian approach is nothing but a new approach to solve the dynamical problem with the sake of simplicity in calculation.**

5 Atwood's machine: Let two masses m_1 and m_2 are suspended over a frictionless pulley of radius a by an inextensible string of length l.

$\therefore$ The equation of constraint is

$$x_1 + x_2 + \pi a = l$$

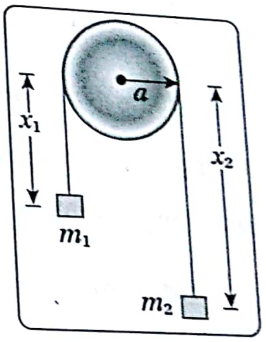

Fig. 22 ▷ Atwood's machine

Differentiating the above equation we get

$$\dot{x}_1 = -\dot{x}_2$$

$\therefore$ The kinetic energy of the system,

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 \quad [\text{As } \dot{x}_1 = -\dot{x}_2]$$

And the potential energy of the system

$$V = -m_1gx_1 - m_2gx_2$$

$$= -m_1gx_1 - m_2g(l - \pi a - x_1) \quad [\because x_1 + x_2 + \pi a = l]$$

$$= -m_1gx_1 + m_2gx_1 - m_2g(l - \pi a)$$

So, the Lagrangian

$$L = T - V$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + m_1gx_1 - m_2gx_1 + m_2g(l - \pi a)$$

or, $$L = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + (m_1 - m_2)gx_1 + V_0$$ ...(8.12.5.18)

where $V_0 = m_2g(l - \pi a)$

The generalised momentum corresponding to x_1 will be

$$p_{x_1} = \frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1$$

$\therefore$ The Hamiltonian

$$H = \sum_j p_j \dot{q}_j - L = p_{x_1}\dot{x}_1 - L$$

$$= (m_1 + m_2)\dot{x}_1^2 - \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 - (m_1 - m_2)gx_1 - V_0$$

$$\therefore \quad H = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 - (m_1 - m_2)gx_1 - V_0$$

or, $$H = \frac{p_{x_1}^2}{2(m_1 + m_2)} - (m_1 - m_2)gx_1 - V_0 \quad \left[\because \dot{x}_1 = \frac{p_{x_1}}{(m_1 + m_2)}\right] \qquad \cdots (8.12.5.19)$$

$\therefore$ The Hamiltonian equations are

$$\dot{p}_{x_1} = -\frac{\partial H}{\partial x_1} = (m_1 - m_2)g \quad \text{and} \quad \dot{x}_1 = \frac{\partial H}{\partial p_{x_1}} = \frac{p_{x_1}}{(m_1 + m_2)}$$

or, $p_{x_1} = (m_1 + m_2)\dot{x}_1$ or, $\dot{p}_{x_1} = (m_1 + m_2)\ddot{x}_1$

Comparing the above two equations we get,

or, $(m_1 - m_2)g = (m_1 + m_2)\ddot{x}_1 \quad [\because \dot{p}_{x_1} = (m_1 - m_2)g]$

$$(m_1 + m_2)\ddot{x}_1 = (m_1 - m_2)g$$

or, $$\ddot{x}_1 = \frac{m_1 - m_2}{(m_1 + m_2)}g \qquad \cdots (8.12.5.20)$$

This is the Hamiltonian equation of motion and it is the same as obtained by usin[g] Lagrangian formulation.

PROBLEM 1 **Derive the Lagrangian and Hamilton's equations of motion for a particle falling freely under the influence of gravity.**

[W.B.U.T. 2005]

Solution Initially, let the body of mass m be at a height h from ground. Then the body be released and after covering a distance x from the top it comes to P.

So, the potential energy $V = mg(h - x)$ and its kinetic energy $T = \frac{1}{2}m\dot{x}^2$.

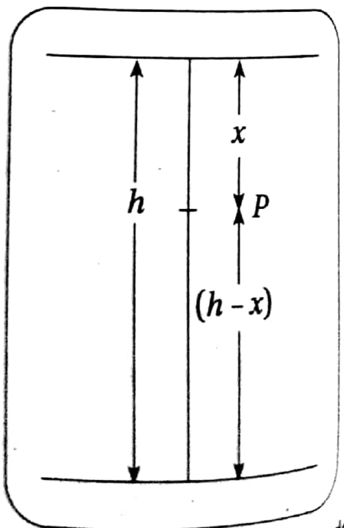

Fig. 23 ▸ Particle falling under

Hence, the Lagrangian

$$L = T - V = \frac{1}{2}m\dot{x}^2 - mg(h - x)$$

$$\therefore \quad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{or,} \quad \dot{x} = \frac{p_x}{m}$$

Hamiltonian $H = p_x\dot{x} - L = \frac{p_x^2}{m} - \left[\frac{1}{2}m\frac{p_x^2}{m^2} - mg(h-x)\right]$

$$= \frac{p_x^2}{2m} + mg(h-x)$$

= total energy

So, the Hamilton's equations of motion are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = mg \quad \text{and} \quad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

or, $p_x = m\dot{x}$ or, $\dot{p}_x = m\ddot{x}$ or, $mg = m\ddot{x}$ [substituting $\dot{p}_x = mg$]

or, $\ddot{x} = g$ or, $\ddot{x} - g = 0$

This is the required equation of motion.

PROBLEM 2

Show that, if a given coordinate is cyclic in the Lagrangian, it will also be cyclic in Hamiltonian.

Solution If q_j is a cyclic coordinate, $\frac{\partial L}{\partial q_j} = 0$

or, $\dot{p}_j = 0$ $\left[\because \frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \text{ and } \frac{\partial L}{\partial \dot{q}_j} = p_j\right]$ or, p_j = constant

Again, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$ or, $\frac{\partial H}{\partial q_j} = 0$

So, H is not depending on the generalised coordinate q_j. Therefore, the *co-ordinate q_j will also be cyclic in Hamiltonian.*

PROBLEM 3

If Lagrangian of a system is given by $L = \frac{1}{2}\dot{x}^2 + \dot{x} - \frac{x^2}{2}$, find the Hamiltonian and equation of motion.

Solution Hamiltonian $H = \sum_j p_j\dot{q}_j - L = p_x\dot{x} - L$ ··· (1)

Since $p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} + 1$ or, $\dot{x} = p_x - 1$

Substituting $\dot{x}$ in equation (1) we get,

$$H = p_x(p_x - 1) - \left(\frac{1}{2}\dot{x}^2 + \dot{x} - \frac{x^2}{2}\right) = \frac{p_x^2}{2} - p_x + \frac{x^2}{2} + \frac{1}{2} \quad \cdots (2)$$

The Hamillton's equations of motion are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -x \quad \cdots (3)$$

and $\dot{x} = \frac{\partial H}{\partial p_x} = (p_x - 1)$ or, $\ddot{x} = \dot{p}_x$ ... (4)

Using (3) and (4) we get, $\ddot{x} + x = 0$

This is the required equation of motion.

PROBLEM 4 **For a bead sliding on a wire in the form of a cycloid the equations are $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, $0 \leq \theta \leq 2\pi$. Derive the Lagrangian equation and Hamiltonian for the system.**

Solution If $A(x, y)$ be the position of the bead of mass m describing by the equations

$$x = a(\theta - \sin\theta) \qquad y = a(1 + \cos\theta)$$

then the kinetic energy of the system,

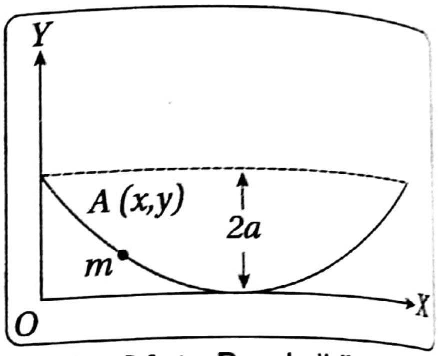

Fig. 24 ▷ Bead sliding on the given wire

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m[a^2(\dot{\theta} - \cos\theta\,\dot{\theta})^2 + a^2(-\sin\theta\,\dot{\theta})^2]$$

$$= \frac{1}{2}ma^2[\dot{\theta}^2 + \cos^2\theta\,\dot{\theta}^2 - 2\dot{\theta}^2\cos\theta + \dot{\theta}^2\sin^2\theta]$$

$$= \frac{1}{2}ma^2[\dot{\theta}^2 + \dot{\theta}^2 - 2\dot{\theta}^2\cos\theta]$$

$$= ma^2\dot{\theta}^2(1 - \cos\theta)$$

and the potential energy with reference to X axis will be,

$$V = mgy = mga(1 + \cos\theta)$$

∴ The Lagrangian $L = T - V = ma^2\dot{\theta}^2(1 - \cos\theta) - mga(1 + \cos\theta)$

∴ The Lagrangian equation will be,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or, $\frac{d}{dt}[ma^2 2\dot{\theta}(1 - \cos\theta)] - (ma^2\dot{\theta}^2\sin\theta + mga\sin\theta) = 0$

or, $2ma^2\ddot{\theta}(1 - \cos\theta) + 2ma^2\dot{\theta}^2\sin\theta - ma^2\dot{\theta}^2\sin\theta - mga\sin\theta = 0$

or, $2ma^2\ddot{\theta}(1 - \cos\theta) + ma^2\dot{\theta}^2\sin\theta - mga\sin\theta = 0$

or, $\ddot{\theta}(1 - \cos\theta) + \frac{ma^2}{2ma^2}\dot{\theta}^2\sin\theta - \frac{mga}{2ma^2}\sin\theta = 0$ [Dividing throughout by $2ma^2$]

or, $\ddot{\theta}(1 - \cos\theta) + \frac{1}{2}\dot{\theta}^2\sin\theta - \frac{g}{2a}\sin\theta = 0$

The generalised momentum corresponding to $\dot{\theta}$ will be,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = 2ma^2\dot{\theta}(1 - \cos\theta)$$

The Hamiltonian

$$\therefore \quad H = p_\theta \dot{\theta} - L = 2ma^2\dot{\theta}^2(1-\cos\theta) - ma^2\dot{\theta}^2(1-\cos\theta) + mga(1+\cos\theta)$$

$$= 2ma^2\dot{\theta}^2(1-\cos\theta) + mga(1+\cos\theta)$$

$$= 2ma^2(1-\cos\theta)\cdot\frac{p_\theta^2}{4m^2a^4(1-\cos\theta)^2} + mga(1+\cos\theta).$$

$$= \frac{p_\theta}{2ma^2(1-\cos\theta)} + mga(1+\cos\theta).$$

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. **A constraint is said to be holonomic when**
   Ⓐ they are non-integrable and in equational form
   Ⓑ they are integrable but in inequational form
   Ⓒ they are integrable and in equational form **Ans. Ⓒ**

2. **Rheonomic constraint**
   Ⓐ depends on time
   Ⓑ does not depend on time
   Ⓒ depends on velocity **Ans. Ⓐ**

3. **If the total mechanical energy of a system is conserved during its constrained motion then the constraint is known as**
   Ⓐ dissipative constraint
   Ⓑ conservative constraint
   Ⓒ bilateral constraint **Ans. Ⓑ**

4. **Constraint means**
   Ⓐ restriction on the shape of the system
   Ⓑ restriction on the movement of the system
   Ⓒ none of the above **Ans. Ⓑ**

5. **A rigid body, given by the constraint equation $|\vec{r}_i - \vec{r}_j|^2$ = constant, always constitute a**
   Ⓐ holonomic, conservative and rheonomic constraint
   Ⓑ non-holonomic, conservative and scleronomic constraint
   Ⓒ holonomic, scleronomic and conservative constraint **Ans. Ⓒ**

6. **In a deformable body constraint is**
   Ⓐ holonomic, rheonomic, dissipative and bilateral
   Ⓑ non-holonomic, rheonomic, conservative and unilateral
   Ⓒ holonomic, scleronomic, dissipative and bilateral **Ans. Ⓐ**

7. **A particle in a cubical box is under**

Ⓐ holonomic constraint

Ⓑ non-holonomic constraint

Ⓒ rheonomic constraint

Ans. Ⓑ

8. **When a disc rolls down on an inclined plane without slipping, it constitutes**

Ⓐ holonomic constraint

Ⓑ conservative constraint

Ⓒ non-holonomic constraint

Ans. Ⓒ

9. **Degrees of freedom of a constraint system of N particles with K constraints is**

Ⓐ $3N-K$ Ⓑ $3N$ Ⓒ $3N+K$

Ans. Ⓐ

10. **The number of degrees of freedom for two particles separated by a fixed distance is**

Ⓐ 2 Ⓑ 3 Ⓒ 5

Ans. Ⓒ

11. **For a rigid body the total number of degrees of freedom is**

Ⓐ 2 Ⓑ 3 Ⓒ 6

Ans. Ⓒ

12. **The number of degrees of freedom of a rigid body constrained to move about a fixed point is**

Ⓐ 3 Ⓑ 2 Ⓒ 6

Ans. Ⓐ

13. **For a simple pendulum of constant length, the degrees of freedom will be**

Ⓐ 1 Ⓑ 2 Ⓒ 5

Ans. Ⓐ

14. **A particle constrained to move on the surface of a sphere has**

Ⓐ 3 Ⓑ 2 Ⓒ 1

degrees of freedom.

Ans. Ⓑ

15. **Virtual displacement occurs with**

Ⓐ finite change in time

Ⓑ infinite change in time

Ⓒ no real change in time

Ans. Ⓒ

16. **Principle of virtual work states that**

Ⓐ the virtual work done only by the constraint force is zero

Ⓑ the virtual work done by the applied force is zero provided that the virtual work done by the constraint force is zero

Ⓒ none of the above

Ans. Ⓑ

17. **For a holonomic system the number of generalised coordinates is**

Ⓐ greater than the number of degrees of freedom

Ⓑ less than the number of degrees of freedom

Ⓒ equal to the number of degrees of freedom

Ans. Ⓒ

18. **$Q_j dq_j$, [where Q_j is the generalised force and dq_j is the virtual displacement], must have the dimension of**

Ⓐ momentum

Ⓑ energy Ⓒ work

Ans. Ⓒ

19. For a conservative system, kinetic energy (T) and potential energy (V) can be represented in terms of generalised coordinate and generalised velocity as

Ⓐ $T = T(\dot{q}_j)$, $V = V(\dot{q}_j)$

Ⓑ $T = T(q_j)$, $V = V(q_j)$

Ⓒ $T = T(\dot{q}_j)$, $V = V(q_j)$

Ans. Ⓒ

20. generalised momentum p_j can be represented as

Ⓐ $p_j = \frac{\partial T}{\partial \dot{q}_j}$ Ⓑ $\dot{p}_j = \frac{\partial T}{\partial q_j}$ Ⓒ $p_j = \frac{\partial T}{\partial q_j}$

Ans. Ⓐ

21. Lagrangian L can be written in terms of kinetic energy (T) and potential energy (V) as

Ⓐ $L = T - V$ Ⓑ $L = V - T$ Ⓒ $L = T + V$

Ans. Ⓐ

22. For a projectile with position coordinate (x, y, z), the number of cyclic coordinate is

Ⓐ 3 Ⓑ 2 Ⓒ 1

Ans. Ⓑ

23. The Lagrangian equation of motion for a simple pendulum is

Ⓐ $\ddot{\theta} - \omega^2\theta = 0$ Ⓑ $\ddot{\theta} + \omega^2\dot{\theta} = 0$ Ⓒ $\ddot{\theta} + \omega^2\theta = 0$

Ans. Ⓒ

24. Hamiltonian for a conservative system can be written as

Ⓐ $H = \sum_j p_j\dot{q}_j + L$ Ⓑ $H = \sum_j p_j\dot{q}_j - L$ Ⓒ $H = \sum_j p_j\dot{q}_j$

Ans. Ⓑ

25. For a conservative system, $H = \sum_j p_j\dot{q}_j - L$ is

Ⓐ a conserved quantity

Ⓑ not conserved Ⓒ none of the above

Ans. Ⓐ

26. If a coordinate is cyclic in the Lagrangian then the

Ⓐ corresponding momentum is also cyclic

Ⓑ corresponding momentum is constant

Ⓒ none of the above

Ans. Ⓑ

27. If a coordinate is cyclic in the Lagrangian then the

Ⓐ corresponding momentum is also cyclic

Ⓑ corresponding velocity is constant

Ⓒ corresponding momentum is also cyclic in the Hamiltonian

Ans. Ⓒ

28. Degrees of freedom of an atom in a hydrogen molecule are [W.B.U.T. 2005]

Ⓐ 1 Ⓑ 2 Ⓒ 5

Ans. Ⓒ

29. A system consists of 3 point masses. If the mutual distances between the masses do not vary with time then the degrees of freedom of such a system are [W.B.U.T. 2006]

Ⓐ 0 Ⓑ 6 Ⓒ 9

Ans. Ⓑ

30. $\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0$ [where $\vec{F}_i^{(a)}$ is the applied force] represents

Ⓐ D'Alembert's principle
Ⓑ principle of virtual work
Ⓒ Hamilton's principle

Ans. Ⓑ

31. If a system has f degrees of freedom, then the number of Hamilton's equations for the system is

Ⓐ 2 Ⓑ f Ⓒ $2f$

Ans. Ⓒ

32. The Hamilton's equations of motion are of

Ⓐ first order second degree
Ⓑ first order first degree
Ⓒ second order second degree

Ans. Ⓑ

33. If a system has f degrees of freedom, then the number of Lagrange's equations for the system is

Ⓐ 3 Ⓑ f Ⓒ $2f$

Ans. Ⓑ

34. Hamiltonian is defined as

Ⓐ $H = T - V$ Ⓑ $H = T + V$ Ⓒ $H = 2T + L$

Ans. Ⓑ

35. For a conservative system, Hamiltonian indicates

Ⓐ T (Kinetic energy) Ⓑ V (Potential energy) Ⓒ $E\ (= T + V)$

Ans. Ⓒ

36. In absence of external force, the linear momentum of a particle is

Ⓐ conserved Ⓑ not conserved Ⓒ none

Ans. Ⓐ

37. In absence of external torque, the angular momentum of a particle with respect to arbitrary origin is

Ⓐ conserved Ⓑ not conserved Ⓒ none

Ans. Ⓐ

38. Rigid body has constraints classified in which of the following group? [W.B.U.T. 2007]

Ⓐ rheonomic and holonomic
Ⓑ rheonomic and non-holonomic
Ⓒ scleronomic and holonomic

Ans. Ⓒ

SHORT ANSWER TYPE QUESTIONS

1. What is meant by 'ignorable or cyclic coordinate'? Explain with an example.
[W.B.U.T 2006] *[See article 8.11.2]*

2. Obtain the differential equation of motion of a simple pendulum from its Lagrangian representation.
[W.B.U.T. 2006] *[See article 8.11.4 example 3]*

3. Write down the Lagrangian for a particle moving freely in one dimension.
[W.B.U.T. 2005] *[See article 8.11.4 example 1]*

4. Define Hamiltonian of a system. When is it equal to the total energy of the system?
[W.B.U.T. 2005]

5. A particle is falling vertically under the influence of gravity. Obtain the Hamiltonian and the equations of motion corresponding to that Hamiltonian. [W.B.U.T. 2005]

6. Define constraints. How do they effect the motion of a mechanical system?
[See article 8.1]

7. Define generalised coordinates. Explain the statement 'generalised forces do not necessarily have the dimension of force.' [See article 8.6 and 8.8]
8. Applying D'Alembert's principle, obtain the acceleration of the masses under gravity in Atwood's machine. [See article 8.5.1 example 2]
9. Using D'Alembert's principle, obtain the equation of motion of a simple pendulum. [See article 8.5.1 example 1]
10. When a body is moving on an inclined plane under the action of gravity, find its constraint equation indicating the nature of constraint acting on the system. [See article 8.1.2 example 2]
11. Write down the Lagrangian equation and mention the corresponding terms appearing in the equation. [See article 8.11.1]

    For the motion of a projectile under the constant gravitational force find the Lagrangian and hence the Lagrangian equation.
12. Define Hamiltonian. Find the equation of motion and time period of a simple harmonic oscillator using Hamiltonian equations. [See article 8.12 and 8.12.5 example 1]
13. Obtain the equation of motion of a simple pendulum and its time period by using Hamiltonian method. [W.B.U.T. 2006] [See article 8.12.5 example 2]
14. When does the Hamiltonian of the system represent the total energy of the system? [W.B.U.T. 2006] [See article 8.12.2]

    or, Prove that for a conservative system, Hamiltonian represents the total energy of the system.
15. **[a]** What is the form of Lagrange's equation of motion for a non-conservative force? [W.B.U.T. 2007] [See equation 8.11.1.13]

    [b] Show that if a coordinate is cyclic in Lagrangian it will be cyclic in Hamiltonian also. [W.B.U.T. 2007] [See article 8.12.5 problem 2]

    [c] Give example of system with non-holonomic and rheonomic (one each) constraints. [W.B.U.T. 2007] [See article 8.1.2]

LONG ANSWER TYPE QUESTIONS

1. **[a]** What are generalised coordinates? What are the advantages of using them? Obtain the expression for generalised forces. [See article 8.6 and 8.8]

   [b] Write down the Lagrangian of a simple pendulum with explaining each term. Hence, obtain the equation of motion. [W.B.U.T. 2004] [See article 8.11.4 example 3]
2. **[a]** Define Hamiltonian of a system. [See article 8.12]

   [b] When is it equal to the total energy of the system? Explain it. [See article 8.12.2]

   [c] A particle is falling vertically under the influence of gravity. Obtain the Hamiltonian and the equation of motion corresponding to that Hamiltonian. [W.B.U.T. 2005]
3. **[a]** What are the limitations of Newtonian mechanics? [See article 8.2]

   [b] What are constraints? Give specific example to explain the forces of constraint. [See article 8.1]

   [c] Define holonomic and non-holonomic constraints with example. [See article 8.1.1]

[See article 8.2]

4. [a] What is degrees of freedom? Explain with an example.
   [b] Write down the equation of constraint in each of the following cases :
   (i) a particle constrained to move on the surface of a sphere.
   (ii) a simple pendulum with a fixed support.
   Specify the nature of constraint and calculate the degrees of freedom in each case. [W.B.U.T. 2005]
   [See article 8.3]

5. [a] Prove that the work done by a constraint force is zero.
   [b] Obtain the expression for (i) generalised force and (ii) generalised potential. [See article 8.2]
   [See article 8.4]

6. [a] What is virtual displacement? [See article 8.4.1]
   [b] Explain the principle of virtual work and define it.
   [c] Deduce the D'Alembert's principle from the principle of virtual work.
   [See article 8.5]
   [See article 8.5]

7. [a] What is D'Alembert's principle?
   [b] Explain the use of this principle.
   [c] Obtain the equation of motion of a simple pendulum using D'Alembert's principle. [See article 8.5.1 example 1]

8. [a] Define generalised momentum for a conservative system. [See article 8.10]
   [b] Show that if a given coordinate is cyclic in the Lagrangian then it is also cyclic in the Hamiltonian.
   [c] Show that for a conservative system the generalised force can be represented in terms of potential as $Q_j = -\dfrac{\partial V}{\partial q_j}$. [See article 8.8.1]

9. [a] What are the advantages of Lagrangian formulation over Newtonian approach in classical mechanics? [See article 8.11.3]
   [b] What is Lagrangian (or Lagrangian function)?
   [c] Using Lagrangian equation of motion, find the motion and time period of a simple harmonic oscillator. [See article 8.11.4 example 2]

10. [a] Write down the Lagrangian equation of motion for a conservative system and mention each term.
    [b] Show that if the Lagrangian does not depend on time, then the Hamiltonian is constant of motion.

Hint : From Hamiltonian principle $\dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t}$

If $L \neq L(t)$ then $\dfrac{\partial L}{\partial t} = 0$, which implies $\dfrac{\partial H}{\partial t} = 0$ i.e., it is constant.

11. A Lagrangian is given by $L = \frac{1}{2}\alpha\dot{q}^2 - \frac{1}{2}\beta q^2$, where α and β are constants.
    [a] Obtain the Lagrangian equation of motion.
    [b] Find the Hamiltonian of the system.

Hint: Lagrangian equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

$$\text{or, } \frac{d}{dt}(\alpha\dot{q}) - (-\beta q) = 0 \quad \text{or, } \quad \alpha\ddot{q} + \beta q = 0$$

$$\text{or, } \ddot{q} + \frac{\beta}{\alpha}q = 0 \quad \text{or, } \quad \ddot{q} + \omega^2 q = 0 \quad \left[\because \omega = \sqrt{\frac{\beta}{\alpha}}\right]$$

$$H = p_k\dot{q}_k - L; \left[p_k = \frac{\partial L}{\partial \dot{q}_k} = \alpha\dot{q}\right]$$

$$= \alpha\dot{q}^2 - \left(\frac{1}{2}\alpha\dot{q}^2 - \frac{1}{2}\beta q^2\right) = \frac{1}{2}\alpha\dot{q}^2 + \frac{1}{2}\beta q^2.$$

12. **[a]** Define Hamiltonian function or Hamiltonian and explain its general significance. *[See article 8.12 and 8.12.4]*

[b] Derive the Hamiltonian and equation of motion for simple harmonic oscillation.

13. **[a]** Write down the Hamiltonian equations of motion (canonical) and mention each term. *[See article 8.12.1]*

[b] Deduce the Hamiltonian function for a system when the Lagrangian is not an explicit function of time. *[See article 8.12]*

[c] Prove that for a conservative system, its Hamiltonian indicates the total energy of the system. *[See article 8.12.2]*

[d] Prove that for a conservative system, the total energy is conserved.

or,

Find the Hamiltonian for a conservative system.

14. Find out the degrees of freedom of a rigid body constrained to move along the equator and remain on the surface of the earth. [W.B.U.T. 2007]

Hint: $r = a$ is the equation of constraint, degrees of freedom $= 3 \times 1 - 1 = 2$

15. Find out Hamilton's equations of motion for a system comprising masses m_1 and m_2 connected by a massless string of length a through a frictionless pulley such that $m_1 > m_2$. [W.B.U.T. 2007]