CHAPTER

2

Electrostatics

2.1 INTRODUCTION

Electrostatics is an important branch of physics which deals with electric charge at rest. The electric force between two electrons is the same as the electric force between two protons placed at the same distance. The amount of charge on an electron is the same as that on a proton. The charge on a proton is positive and that on an electron is negative. The net charge on the electron–proton system is zero. Stationary charges produce an electric field that is constant with time, hence the term electrostatics. Both electrostatics and magnetostatics can be explained by using vector calculus.

2.2 QUANTIZATION OF CHARGE

In 1911, Millikan successfully showed that charges in tiny oil drops are exact multiples of elementary charges. The magnitude of charge on a proton or an electron ($e = 1.6 \times 10^{-19}\,c$) is called elementary charge. Quantization of charge means that all observable charges are integral multiple of elementary charge $e(= 1.6 \times 10^{-19} c)$.

2.3 CONSERVATION OF CHARGE

The law of conservation of charge states that for an isolated system, the net charge always remains constant. In β-decay a neutron converts itself into a proton and creates an electron. The net charge remains zero before and after the decay. In nuclear fission, the total charge is always conserved.

$$_0n^1 + {}_{92}U^{235} \rightarrow {}_{56}Ba^{141} + {}_{36}K^{92} + 3{}_0n^1$$

Before collision total charge = + 92 e and total charge after collision = (56 + 36) e = 92 e. So total charge is conserved.

2.4 COULOMB'S LAW

Statement

The force between two small charged bodies separated by a distance in air is

(a) directly proportional to the magnitude of each charge

(b) inversely proportional to the square of the distance between them

(c) directed along the line joining the charges

The distance between charges must be large compared to their linear dimension.

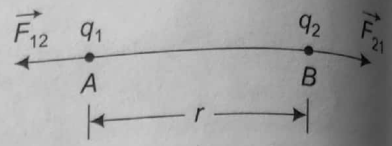

Fig. 2.1 Force between two charges.

In mathematical form, if q_1 and q_2 be two like charges and r is the distance between them [Fig. 2.1] then the force exerted on q_1 due to the charge q_2 is

$$\overrightarrow{F_{12}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{21} \qquad ...(2.1)$$

Here, $\hat{r}_{21}$ is unit vector pointing from q_2 to q_1 and ε_0 is the permittivity of free space. Experimentally, measured value of $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9\ \text{Nm}^2/\text{C}^2$

Similarly, the force exerted on q_2 due to the charge q_1 is

$$\overrightarrow{F_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{12} \qquad ...(2.2)$$

Here, $\hat{r}_{12}$ is a unit vector pointing from q_1 to q_2. So $\vec{F}_{12} = -\vec{F}_{21}$. For two unlike charges, $\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{12}$ and $\overrightarrow{F_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}_{21}$ are attractive

Relative permittivity (ε_r) The relative permittivity or dielectric constant of a medium is defined as the ratio of the force between two charges placed at a distance in vacuum (or air) to the force between the same charges placed at the same separation in that medium.

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}\left[F_{\text{vacuum}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \text{ and } F_{\text{medium}} = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2}\right] \qquad ...(2.3)$$

2.5 PRINCIPLE OF SUPERPOSITION

The principle states that when a number of charges are interacting, the total force on a given charge is the vector sum of the individual forces exerted by all other charges on the given charge.

If $q_1, q_2, q_3, q_4, ...$ are the charges situated at $A, B, C, D, ...$, as shown in Fig. 2.2.

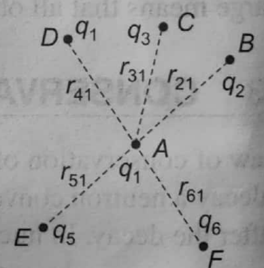

Fig. 2.2 Principle of superposition.

The total force on q_1 due to all other charges is

$$\vec{F}_1 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21} + \frac{q_1 q_3}{r_{31}^2}\hat{r}_{31} + ... + ...\right]$$

$$= \vec{F}_{12} + \vec{F}_{13} + ...$$

The total force on q_2 due to all other charges is

$$\vec{F}_2 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_2 q_1}{r_{12}^2}\hat{r}_{12} + \frac{q_2 q_3}{r_{32}^2}\hat{r}_{32} + ...\right]$$

If there is a test charge q_0, then total force on the test charge q_0 due to all other charges is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + ... = \sum_{i=1}^{N}\vec{F}_i$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_0 q_i}{r_i^2} \hat{r}_i \qquad ...(2.4)$$

2.6 ELECTRIC FIELD

The electric field due to a charge is the space around the charge in which any other charge is acted upon by an electrostatic force.

If we have many charges $q_1, q_2, ... q_n$ at distances $r_1\ r_2 ... r_n$ respectively from a test charge q_0 then from the principle of superposition, the total force on q_0 is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_0 q_i}{r_i^2} \hat{r}_i$$

$$= \frac{q_0}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{r}_i$$

The electric field intensity at the point is

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{r}_i \qquad ...(2.5)$$

Thus, the electric field intensity at a point in the electric field is the force on a unit test charge placed at the point concerned.

2.7 CONTINUOUS CHARGE DISTRIBUTIONS

On a uniform charge body, there are three types of distribution of charge:

(i) Line charge distribution If q is the total charge over a conducting wire of length l and infinitesimally small thickness, then charge per unit length λ (line charge density) is

$$\lambda = \frac{q}{l} \text{ coulomb/m}$$

For non-uniform distribution of charge

$$\lambda = \underset{\Delta l \to 0}{\text{Lt}} \frac{\Delta q}{\Delta l} = \frac{dq}{dl}$$

where Δl is a small element of length which carries a charge Δq.

So, total charge over the whole length

$$q = \int_l \lambda \, dl \qquad ...(2.6)$$

(ii) Surface charge distribution If q is the charge uniformly distributed over the conducting surface S, then surface density of charge (charge per unit area) σ is

$$\sigma = \frac{q}{S} \text{ coulomb/m}^2$$

If Δq be the charge contained by a small element ΔS then surface charge density,

$$\sigma = \underset{\Delta S \to 0}{\text{Lt}} \frac{\Delta q}{\Delta S} = \frac{dq}{dS}$$

So, total charge on the surface $q = \int_S \sigma \, dS$ ...(2.7)

(iii) Volume charge distribution If q is the charge uniformly distributed over the volume V then the volume density of charge (charge per unit volume)

$$\rho = \frac{q}{V} \text{ coulomb/m}^3$$

If charge distribution is not uniform then

$$\rho = \operatorname*{Lt}_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV}$$

where ΔV is a small volume element which carries a charge Δq.

So, total charge over the whole volume

$$q = \int_V \rho \, dV \qquad \text{...(2.8)}$$

2.8 ELECTRIC POTENTIAL

The concept of potential is based on energy consideration. The electric potential at a point in an electric field near a charged conductor is defined as the amount of work done in bringing a unit positive charge from infinity to that point against the electrostatic force. A positively charged body always tends to move from higher potential to lower potential.

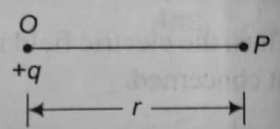

Fig. 2.3 **Potential at point *P* due to charge $+q$ at *O*.**

Potential at a point due to a point charge

The potential at P [Fig. 2.3] is given by

$$V = -\int_{\infty}^{r} \vec{E} \cdot \vec{dr}$$

$$= -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} dr = \frac{q}{4\pi\varepsilon_0 r} \qquad \text{...(2.9)}$$

Electric field Intensity as a gradient of potential

Let the electric field at a point $\vec{r}$ due to a charge distribution be $\vec{E}$ and electric potential at the same point be V. Suppose a test charge q_0 is displaced slightly from $\vec{r}$ to $\vec{r} + d\vec{r}$. Then the force on the test charge q_0 is $\vec{F} = q_0\vec{E}$ and work done for small displacement $\vec{dr}$ is $dw = \vec{F} \cdot d\vec{r} = q_0 \vec{E} \cdot d\vec{r}$

The change in potential energy $= -d\omega = -q_0\vec{E} \cdot d\vec{r}$

So the change in potential $dV = -\vec{E} \cdot d\vec{r}$

Again $\quad dV = \vec{\nabla} V \cdot d\vec{r} = -\vec{E} \cdot d\vec{r}$

or, $\quad \vec{E} = -\vec{\nabla} V \qquad \text{...(2.10)}$

Hence, the electric field at a point is equal to the negative gradient of the electrostatic potential at the point.

2.9 ELECTRIC POTENTIAL ENERGY

We define electric potential energy of a system of point charges as the work required to assemble this system of charges by bringing them from infinite distances.

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If two point charges q_1 and q_2 are separated by a distance r_{12} then potential energy of the system q_1 and q_2 is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

If another charge q_3 is at a distance r_{13} from q_1 and distance r_{23} from q_2 then potential energy of the system $(q_1 + q_2 + q_3)$ is

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

Generally, the potential energy for a system of n point charges is

$$U = \frac{1}{2} \times \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{q_i q_j}{r_{ij}} \qquad ...(2.11)$$

In summation, each term is counted twice, so half factor is included here to avoid double counting each term for calculation of potential energy.

2.10 ELECTRIC FLUX

We know that any area element dS is a vector $\overrightarrow{dS}$. If $\hat{n}$ is the unit normal to the area element then

$$\overrightarrow{dS} = \hat{n}\, dS$$

The total number of lines of force passing through a surface placed in an electric field is known as electric flux (φ_E).

Consider a surface of area S inside an electric field $\vec{E}$ [Fig. 2.4]. The surface S is divided into a number of elementary areas $\overrightarrow{dS}$ (known as area vector). The component of the electric field along the area vector $\overrightarrow{dS}$ is given by

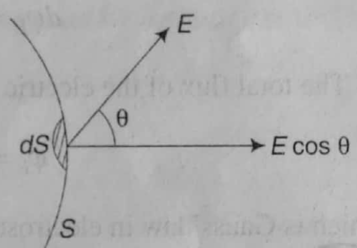


Fig. 2.4 Component of electric field $\vec{E}$ along $\overrightarrow{dS}$.

$$E_n = E\cos\theta$$

So, electric flux

$$d\varphi_E = E_n\, dS = E\cos\theta\, dS$$
$$= \vec{E}\cdot\overrightarrow{dS} = \vec{E}\cdot\hat{n}\, dS \qquad ...(2.12)$$

The total electric flux through S is

$$\varphi_E = \int_s \vec{E}\cdot\hat{n}\, dS \qquad ...(2.13)$$

2.11 SOLID ANGLE

In Fig. 2.5, the solid angle subtended by any surface dS at a point O, distance r away, is given by

$$d\omega = \frac{dS'}{r^2} = \frac{dS\cos\theta}{r^2}$$

where $dS' = dS\cos\theta$ is the projection of the surface dS.

For sphere $dS' = 4\pi r^2$ and $\theta = 0$, so solid angle subtended by the sphere at its center is

$$\omega = \int_s \frac{dS\cos\theta}{r^2} = 4\pi$$

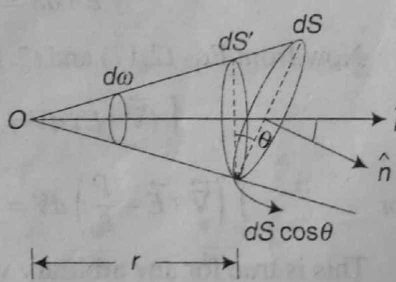


Fig. 2.5 Solid angle at point O.

2.12 GAUSS' LAW

Gauss' law states that the total electric flux through a closed surface in an electric field is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface, where ε_0 is the permittivity of free space.

Mathematically, $\oint_S \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$ when S encloses q

$= 0$ when S does not enclose q ...(2.14)

Proof of Gauss' law

We consider a spherically symmetric closed surface. Suppose a charge q is placed at the center of a sphere a radius r and $\vec{E}$ is the electric field intensity at a point R on the surface [Fig. 2.6].

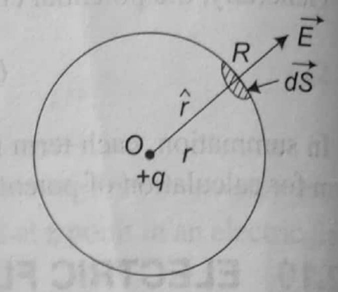


Fig. 2.6 **Proof of Gauss' law.**

From Coulomb's law,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

The flux of electric field through $\overrightarrow{dS}$ is

$$d\varphi_E = \vec{E} \cdot \overrightarrow{dS} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} \cdot \overrightarrow{dS} = \frac{q}{4\pi\varepsilon_0 r^2} dS$$

The total flux of the electric field due to the internal charge q through the closed surface

$$\varphi_E = \int d\varphi_E = \frac{q}{4\pi\varepsilon_0}\int_s \frac{dS}{r^2} = \frac{q}{4\pi\varepsilon_0}\frac{4\pi r^2}{r^2} = \frac{q}{\varepsilon_0}$$

which is Gauss' law in electrostatics.

2.12.1 Differential Form of Gauss' Law

The integral form of Gauss' law is

$$\oint_S \vec{E} \cdot \overrightarrow{dS} = \frac{q}{\varepsilon_0} \qquad ...(2.15)$$

If ρ be the volume charge density over a small volume element dV within the closed surface S, then

$$q = \int_V \rho \, dV \qquad ...(2.16)$$

Again from Gauss' divergence theorem

$$\oint_S \vec{E} \cdot \overrightarrow{dS} = \int_V (\vec{\nabla} \cdot \vec{E})\, dV \qquad ...(2.17)$$

Now from Eqs (2.17) and (2.16) into Eq. (2.15)

$$\int_V (\vec{\nabla} \cdot \vec{E})\, dV = \frac{1}{\varepsilon_0}\int_V \rho \, dV \qquad ...(2.18)$$

or $$\int_V \left(\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\varepsilon_0}\right) dV = 0$$

This is true for any arbitrary volume V.

$$\therefore \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad ...(2.19)$$

This is the differential form of Gauss' law.

In vacuum, electric displacement vector $\vec{D} = \varepsilon_0 \vec{E}$

So
$$\vec{\nabla} \cdot \vec{D} = \rho \quad ...(2.20)$$

This is also the differential form of Gauss' law in terms of electric displacement vector.

2.12.2 Coulomb's Law from Gauss' Law

Here we would like to deduce Coulomb's law from Gauss' law. Here, S is the spherical gaussian surface. In Fig. 2.7, $\vec{E}$ and $\vec{dS}$ on the gaussian surface are directed radially outward. So $\vec{E} \cdot \vec{dS} = E\,dS$.

Now from Gauss' law

$$\oint_s \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0}$$

or,
$$E \oint_s dS = E \times 4\pi r^2 = \frac{q}{\varepsilon_0} \quad [E \text{ is constant}]$$

or,
$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

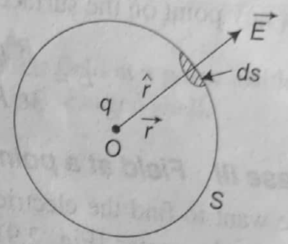


Fig. 2.7 Derivation of Coulomb's law from Gauss' law.

If another point charge q_0 be placed at the point at which $\vec{E}$ is calculated then the force on q_0 due to q is

$$\vec{F} = q_0 \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q_0 q}{r^2} \hat{r} \quad ...(2.21)$$

where $\hat{r}$ is the unit vector.

Equation (2.21) is Coulomb's law.

2.12.3 Application of Gauss' Law

For calculating electric field due to a charge distribution, Gauss' law provides the easiest way. Here, we consider some important applications of Gauss' law.

(i) Electric field due to uniformly charged sphere

Case I Field at a point outside the charged sphere

Let P be a point situated outside the charged sphere having charge q uniformly distributed throughout the volume of the sphere of radius R [Fig. 2.8]. In order to find the electric field intensity at P, a concentric sphere of radius r is drawn as gaussian surface, over which the electric field intensity is directed normal to every point of this surface. If $\vec{E}$ is the electric field intensity at the point P then the total flux through the gaussian surface is

$$\varphi_E = \oint_s \vec{E} \cdot \vec{dS} = E\,(4\pi r^2) = \frac{q}{\varepsilon_0}$$

or,
$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

For continuous charge distribution of density ρ within the sphere

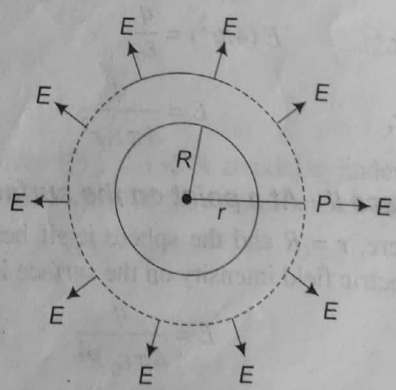


Fig. 2.8 Electric field outside of a charged sphere.

$$q = \frac{4}{3}\pi R^3 \rho$$

So,
$$E = \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi R^3 \rho}{r^2} = \frac{R^3\rho}{3\varepsilon_0 r^2} \qquad ...(2.22)$$

Case II Field at a point on the surface of the sphere

For any point on the surface of the sphere $r = R$, from Eq. (2.22), we have

$$E = \frac{R^3\rho}{3\varepsilon_0 R^2} = \frac{R\rho}{3\varepsilon_0} \qquad ...(2.23)$$

Case III Field at a point inside the charged sphere

We want to find the electric field at a point P inside the sphere at a distance r from the center [Fig. 2.9]. The total charge inside the gaussian surface of radius r

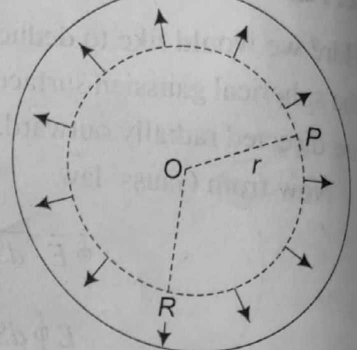


Fig. 2.9 Electric field inside a charged sphere.

$$q = \frac{4}{3}\pi r^3 \rho$$

The total electric flux over the gaussian surface is given by

$$\varphi_E = \oint_s \vec{E}\cdot d\vec{S} = E \times 4\pi r^2 = \frac{q}{\varepsilon_0} = \frac{4\pi r^3 \rho}{3\varepsilon_0}$$

or,
$$E = \frac{r\rho}{3\varepsilon_0} \qquad ...(2.24)$$

The variation of electric field intensity in different cases as discussed is shown in Fig. 2.10.

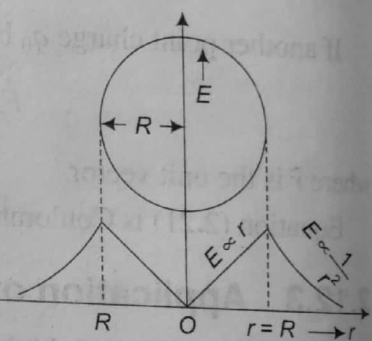


Fig. 2.10 Variation $\vec{E}$ with distance from the center of a charged sphere.

(ii) Electric field due to a charged spherical shell

Case I At a point outside the charged shell

Let P be a point outside the shell at a distance r [Fig. 2.11]. Here $r > R$

The total flux over the gaussian surface

$$\varphi_E = \oint_s \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

or,
$$E\,(4\pi r^2) = \frac{q}{\varepsilon_0}$$

or,
$$E = \frac{q}{4\pi\varepsilon_0 r^2} \qquad ...(2.25)$$

Case II At a point on the surface of the charged shell

Here, $r = R$ and the sphere itself behaves as gaussian surface. The electric field intensity on the surface is then

$$E = \frac{q}{4\pi\varepsilon_0 R^2}$$

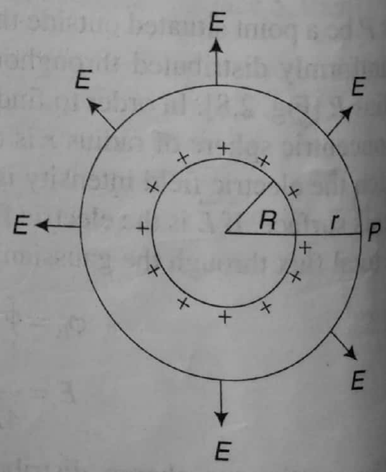


Fig. 2.11 Electric field due to a charged spherical shell.

Again for spherical shell the surface density of charge

$$\sigma = \frac{q}{4\pi R^2}$$

$$\therefore \quad E = \frac{\sigma}{\varepsilon_0} \qquad ...(2.26)$$

Case III At a point inside the shell

Here $r < R$, the total charge is situated on the surface of the shell of radius R and no charge is therefore enclosed by the gaussian surface, therefore,

$$\oint_s \vec{E} \cdot d\vec{S} = 0$$

$$\therefore \quad E = 0 \qquad ...(2.27)$$

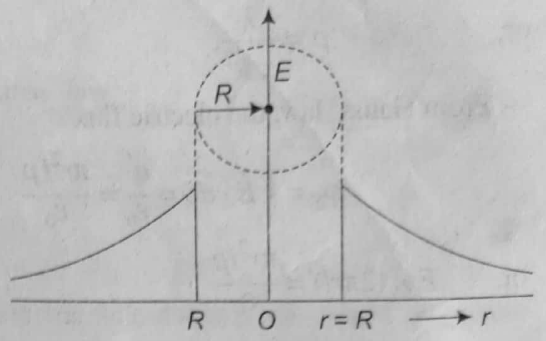


Fig. 2.12 Electric field at a point inside a spherical charged shell.

The variation of the electric field intensity with distance from the centre is shown in Fig. 2.12 with the help of Eqs (2.25), (2.26) and (2.27).

(iii) Electric field intensity due to long uniformly charged cylinder

Case I At a point outside of the cylinder

Let P_1 be the point at a distance r from the axis of the cylinder of radius R [Fig. 2.13]. Imagine a cylindrical gaussian surface of length l through P_1. The field will have a cylindrical symmetry and the total flux over the gaussian surface

$$\varphi_E = \oint_s \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0},$$

where λ is the line charge density.

$$\text{or,} \quad E \times 2\pi r l = \frac{\lambda l}{\varepsilon_0}$$

$$\therefore \quad E = \frac{\lambda}{2\pi r \varepsilon_0} \qquad ...(2.28)$$

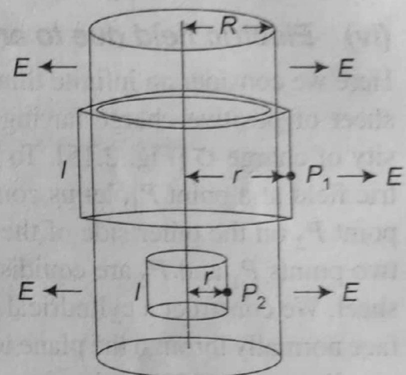


Fig. 2.13 Electric field at a point from the axis of a uniformly charged cylinder.

Case II At a point on the surface of the cylinder

Here $r = R$, the cylinder itself is the gaussian surface. The electric filed at the surface is obtained by replacing r by R in Eq. (2.28)

$$E = \frac{\lambda}{2\pi R \varepsilon_0} \qquad ...(2.29)$$

Case III At a point inside the cylinder

Let P_2 be the point at a distance r ($r < R$) from the axis of the cylinder [Fig. 2.13]. A coaxial cylinder of radius r and length l is constructed as gaussian surface such that the point P_2 lies on the curved surface of the cylinder.

The total charge enclosed by the gaussian surface

$$q' = \pi r^2 l \rho$$

[where, ρ is the volume charge density]

$$\text{Again} (\pi R^2 l)\rho = \lambda l$$

or, $$\rho = \frac{\lambda}{\pi R^2},$$

From Gauss' law, the electric flux

$$\varphi_E = \oint_S \vec{E} \cdot \vec{dS} = \frac{q'}{\varepsilon_0} = \frac{\pi r^2 l \rho}{\varepsilon_0}$$

or, $$E \times (2\pi r l) = \frac{\pi r^2 l \rho}{\varepsilon_0}$$

or, $$E = \frac{r\rho}{2\varepsilon_0} = \frac{r}{2\varepsilon_0}\left(\frac{\lambda}{\pi R^2}\right) = \frac{\lambda r}{2\pi R^2 \varepsilon_0} \quad \text{...(2.30)}$$

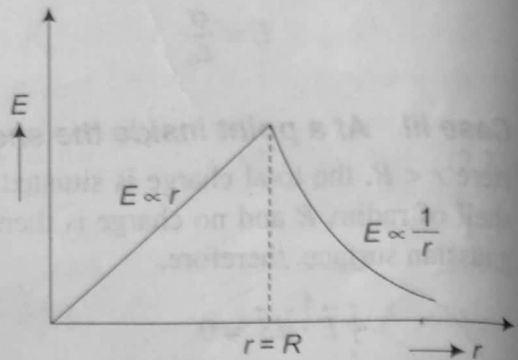


Fig. 2.14 Variation of $\vec{E}$ with distance r from the axis of a charged cylinder.

The variation of electric field intensity with distance r from the axis of a charged cylinder is shown in Fig. 2.14.

[Note: For a hollow charged cylinder, the charge inside the cylinder is zero and so the electric flux inside the cylinder, $\varphi_E = 0$ which gives $E = 0$.]

(iv) Electric field due to an infinite plane charge sheet

Here we consider an infinite thin plane charge sheet of positive charge having surface density of charge σ [Fig. 2.15]. To find the electric field at a point P_1, let us consider another point P_2 on the other side of the sheet so that two points P_1 and P_2 are equidistant from the sheet. We construct a cylindrical gaussian surface normally through the plane which extends equally on two sides of the plane.

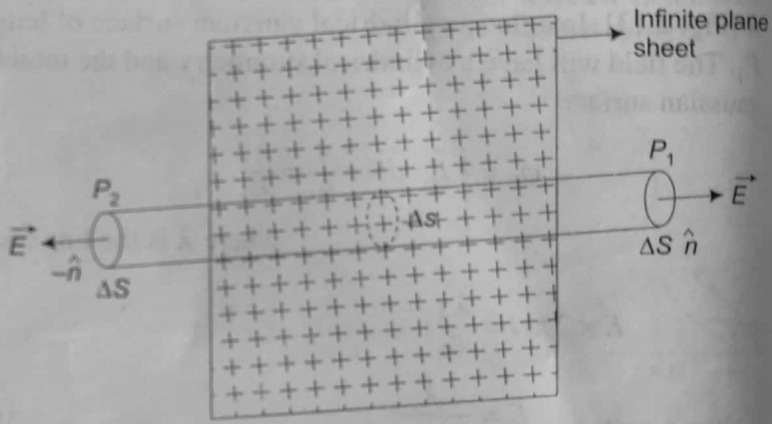


Fig. 2.15 Electric field due to an infinite charged sheet.

The flux of the electric field crossing through the gaussian surface

$$\varphi_E = E\,\Delta s + E\,\Delta S$$

$= 2E\,\Delta S$ [where ΔS is the area of cross section of each end face]

$$= \frac{\sigma\,\Delta S}{\varepsilon_0}$$

or, $$E = \frac{\sigma}{2\varepsilon_0} \quad \text{...(2.31)}$$

We see that electric field is uniform and does not depend on the distance from the charge sheet.

(v) Electric field near a charged conducting surface

Here we consider a plane conducting sheet. All the charges of the conductor lie on the surface, so the electric field inside the conductor is zero. We have to find the electric field at a point P which is near but outside the conductor. To find the electric field, we construct a gaussian surface as follows [Fig. 2.16]. The total flux through the gaussian surface is

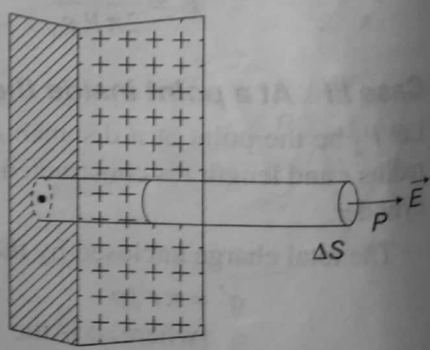


Fig. 2.16 Electric field at a point near a charged conducting surface.

$$\varphi_E = E\,\Delta S$$

and charge enclosed inside the closed surface is $\sigma\,\Delta S$. So from Gauss' law

$$E\,\Delta S = \frac{\sigma\,\Delta S}{\varepsilon_0}$$

or, $$E = \frac{\sigma}{\varepsilon_0} \quad \ldots(2.32)$$

The electric field near a plane charged conductor is twice the electric field due to a non-conducting plane charge sheet. This is also known as *Coulomb's theorem*. The theorem states that the electric field at any point very close to the surface of a charged conductor is equal to charge density of the surface divided by free space permittivity.

2.13 POISSON'S AND LAPLACE'S EQUATIONS

The differential form of Gauss' law is

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{[From Eq. (2.19)]}$$

Again, the electric field ($\vec{E}$) at any point is equal to the negative gradient of the potential V

i.e., $$E = -\vec{\nabla} V \quad \text{[From Eq. (2.10)]}$$

Now combining these two equations, we have

$$\vec{\nabla}\cdot(-\vec{\nabla}V) = -\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

or, $$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \quad \ldots(2.33)$$

This is known as Poisson's equation.

Now in a charge-free region ($\rho = 0$), the Poisson's equation becomes

$$\nabla^2 V = 0 \quad \ldots(2.34)$$

This equation is known as Laplace's equation and is valid only in the charge-free region.

The laplacian operator ∇^2 is a scalar operator and its form in three coordinate systems are:

(a) Cartesian system (x, y, z): $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

(b) Cylindrical coordinate system (ρ, ϕ, z):

$$\nabla^2 \equiv \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

(c) Spherical polar coordinate system (r, θ, φ)

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

2.13.1 Application of Laplace's Equation (Effective 1D Problem)

(i) Potential between the plates of a parallel-plate capacitor

Let us consider a parallel-plate condenser (capacitor) having two plates, one at $z = 0$ and other at $z = d$ [Fig. 2.17]. The potential at the upper plate is V_A and potential at the lower plate is zero. So, the potential exists only along the z direction.

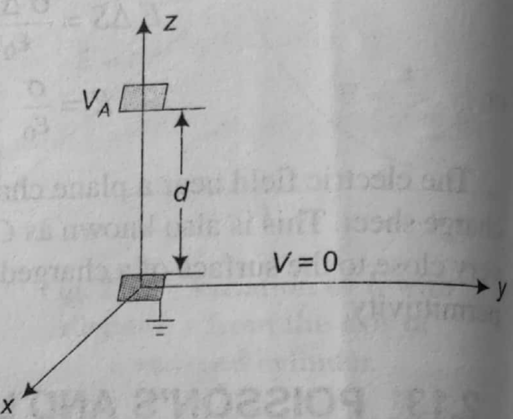


Fig. 2.17 Potential difference between the plates of a parallel-plate capacitor.

The Laplace's equation in the cartesian system is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Here $$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = 0$$

Since the potential exists only along the z direction

So, $$\frac{\partial^2 V}{\partial z^2} = 0 \quad \text{or} \quad \frac{\partial V}{\partial z} = C \text{ (constant)}$$

or, $$V = Cz + D \quad (D \text{ is another constant}) \qquad \ldots(2.35)$$

Now applying boundary conditions, i.e.,

$$z = 0,\ V = 0$$

and $$z = d,\ V = V_A$$

The first condition gives $D = 0$

From the second condition at $z = d$, $V = V_A$

$$V_A = Cd$$

or $$C = \frac{V_A}{d}$$

So, from Eq. (2.35), $$V = \frac{V_A z}{d} \qquad \ldots(2.36)$$

Equation (2.36) is the solution of Laplace's equation, which gives the potential between the plates.

(ii) Potential of a coaxial cylindrical capacitor

Here, we consider a cylindrical capacitor of inner and outer radii a and b $(b > a)$ as shown in Fig. 2.18. The potential of the inner cylinder is V_A and the potential of the outer cylinder is zero.

Since the variation of potential exists only along the radial direction.

then $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \left[\because \frac{\partial^2 V}{\partial \varphi^2} = \frac{\partial^2 V}{\partial z^2} = 0\right] \qquad \ldots(2.37)$$

Integrating Eq. (2.37) twice with respect to r, the potential at an arbitrary distance r is

$$V = C \ln r + D \qquad \ldots(2.38)$$

[where C and D are constants of integration]

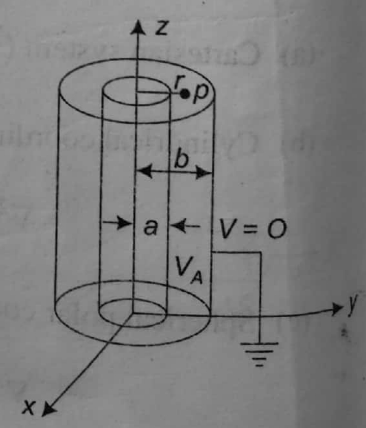


Fig. 2.18 Potential of a cylindrical capacitor.

Now applying boundary conditions, i.e.,

$$r = b,\ V = 0$$

and $$r = a,\ V = V_A$$

we have, from Eq. (2.38), $D = -C \ln b$ and $V_A = C \ln \frac{a}{b}$

so, $$C = \frac{V_A}{\ln \frac{a}{b}} \quad \text{and} \quad D = -V_A \frac{\ln b}{\ln \frac{a}{b}}$$

Now from Eq. (2.38), we have

$$V = \frac{V_A}{\ln \frac{a}{b}} \ln r - V_A \frac{\ln b}{\ln \frac{a}{b}}$$

$$= V_A \frac{\ln \frac{r}{b}}{\ln \frac{a}{b}} \qquad \text{....(2.39)}$$

Equation (2.39) is the solution of Laplace's equation in cylindrical coordinate, which gives the potential inside a cylindrical capacitor.

(iii) Potential of a concentric spherical capacitor

Let us consider a spherical capacitor of inner and outer radii a and b ($b > a$) as shown in Fig. 2.19. The potential of the inner sphere is V_A and the outer sphere is zero.

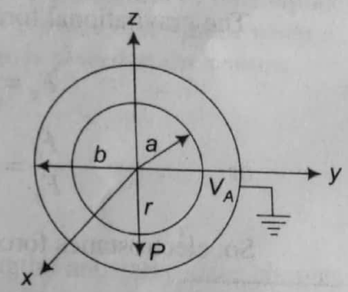


Fig. 2.19 Potential of concentric spherical capacitor.

Since the variation of potential exists only along the radial direction then from Laplace's equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \quad \left[\text{Here, } \frac{\partial V}{\partial \theta} = \frac{\partial^2 V}{\partial \varphi} = 0\right] \qquad \text{...(2.40)}$$

Integrating Eq. (2.40) twice with respect to r, the potential at P will be

$$V = -\frac{C}{r} + D \qquad \text{...(2.41)}$$

[where C and D are integrating constants]

Now applying boundary conditions, i.e.,

$$r = b,\ V = 0$$

and $$r = a,\ V = V_A$$

we have from Eq. (2.41) $D = \frac{C}{b}$

So, $$V = -\frac{C}{r} + \frac{C}{b} = C\left(\frac{1}{b} - \frac{1}{r}\right) \qquad \text{...(2.42)}$$

and from second boundary condition ($r = a$, $V = V_A$), Eq. (2.41) gives $V_A = C\left(\frac{1}{b} - \frac{1}{a}\right)$

or, $$C = \frac{V_A}{\left(\frac{1}{b} - \frac{1}{a}\right)}$$

Now putting the values of c in Eq. (2.42)
we have

$$V = \frac{V_A}{\left(\frac{1}{b} - \frac{1}{a}\right)} \left(\frac{1}{b} - \frac{1}{r}\right) \quad \ldots(2.43)$$

Equation (2.43) is the solution of Laplace's equation in spherical polar coordinate, which gives the potential inside a spherical capacitor.

Worked Out Problems

Example 2.1 Compare the electrostatics force and gravitational force between a proton and electron in a hydrogen atom. Given $e = 1.6 \times 10^{-19}$ C, $m_e = 9.1 \times 10^{-31}$ kg, $m_p = 1.7 \times 10^{-27}$ kg and $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}

Sol. The electrostatic force between a proton and electron is

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = 9 \times 10^9 \times \frac{(1.6 \times 10^{-19})^2}{r^2} \quad \left[\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \text{ Nm}^2/\text{C}^2\right]$$

The gravitational force between a proton and electron

$$F_g = G\frac{m_p m_e}{r^2} = 6.67 \times 10^{-11} \frac{1.7 \times 10^{-27} \times 9.1 \times 10^{-31}}{r^2}$$

or
$$\frac{F_e}{F_g} = \frac{9 \times 10^9}{6.67 \times 10^{-11}} \times \frac{(1.6 \times 10^{-19})^2}{9.1 \times 1.7 \times 10^{-58}} = 2.2 \times 10^{34}$$

So, electrostatics force between electron and proton is much greater than gravitational force.

Example 2.2 Two particles P and Q having charges 8.0×10^{-6} C and -2.0×10^{-6} C respectively are held fixed with a separation of 20 cm. Where should a third charged particle be placed so that it does not experience a net electric force?

Sol. Since the particles are charged with opposite sign, the point R where net electric force is zero, can't be between P and R.

Suppose $QR = x$ metre and charge on R is q.

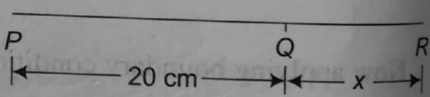


Fig. 2.1W

The force at R due to P is $= \dfrac{8.0 \times 10^{-6} \times q}{4\pi\varepsilon_0 (x + 0.2)^2}$

The force at R due to Q is $= \dfrac{2.0 \times 10^{-6} \times q}{4\pi\varepsilon_0 x^2}$

They are oppositely directed and the resultant is zero.

So, $$\frac{8 \times 10^{-6}}{(0.2 + x)^2} = \frac{2 \times 10^{-6}}{x^2}$$

or, $$0.2 + x = 2x$$

or, $$x = 0.2 \text{ m}$$

Example 2.3 Four equal charges $+q$ are placed at the corner of a square. Find the point charge at the center of the square so that the system will remain in equilibrium.

Sol. The charge Q at the center (O) must be negative [Fig. 2.2W]. If the net force on a charge at D is zero, then by symmetry it follows that the net force experienced by charges at other points will also be zero.

The resultant force (F_R) at D due to all other charges at different corner will be

$$F_R = F_B + F_C \cos 45° + F_A \cos 45°$$

$$= \frac{1}{4\pi\varepsilon_o}\left[\frac{q^2}{(DO)^2} + \frac{q^2}{a^2}\frac{1}{\sqrt{2}} + \frac{q^2}{a^2\sqrt{2}}\right]$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1}{(\sqrt{2}a)^2} + \frac{2}{\sqrt{2}a^2}\right]$$

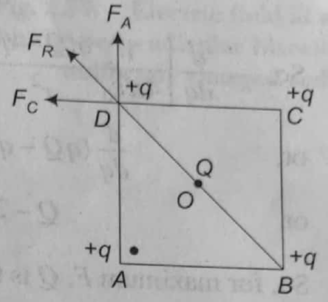


Fig. 2.2W **Equilibrium of four equal charges at corners of a square when a charge is placed at the center.**

That force must be equal to $\dfrac{Qq}{4\pi\varepsilon_0\left(\frac{1}{\sqrt{2}}a\right)^2}$

So $$\frac{Qq \times 2}{4\pi\varepsilon_0 a^2} = \frac{q^2}{4\pi\varepsilon_0 a^2}\left(\frac{1}{2} + \frac{2}{\sqrt{2}}\right)$$

or, $$Q = q\frac{(1+2\sqrt{2})}{2}$$

So, the charge at the center will be $-\dfrac{q(1+2\sqrt{2})}{2}$.

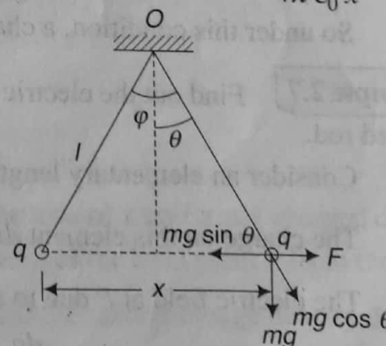
Example 2.4 Two similar balls of mass m are hung from silk threads of length l and carry same charges. Prove that for a small angle φ, the separation of the charges, will be

$$x = \left(\frac{q^2 l}{2\pi\varepsilon_o mg}\right)^{\frac{1}{3}}$$

Sol. In Fig. 2.3W, each ball of charge $+q$ are suspended from O by silk threads. Here $\theta = \dfrac{\varphi}{2}$

The restoring force $= mg \sin\theta$ and electrostatic repulsive force between the balls is $= \dfrac{qq}{4\pi\varepsilon_0 x^2}$

In equilibrium, $mg\sin\theta = \dfrac{q^2}{4\pi\varepsilon_0 x^2}$ where x is the separation of the balls.



From the figure for small θ, $\sin\theta = \dfrac{x/2}{l} = \dfrac{x}{2l}$

$\therefore$ $$mg\frac{x}{2l} = \frac{q^2}{4\pi\varepsilon_0 x^2}$$

or, $$x^3 = \frac{q^2 l}{2\pi\varepsilon_0 mg} \quad \text{or,} \quad x = \left(\frac{q^2 l}{2\pi\varepsilon_0 mg}\right)^{\frac{1}{3}}$$

Fig. 2.3W **Separation of two like and equal charges suspended from a point.**

Hence proved.

Example 2.5 An amount of charge Q is divided into two particles. Find the charge on each particle so that the effective force between them will be maximum.

Sol. Suppose the charge on one particle be q, then charge on the other is $(Q-q)$. If they are separated by a distance r then force

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q(Q-q)}{r^2}$$

For maximum F, $\frac{dF}{dq} = 0$

So, $$\frac{d}{dq}\left[\frac{1}{4\pi\varepsilon_0}\frac{q(Q-q)}{r^2}\right] = 0$$

or, $$\frac{d}{dq}(qQ - q^2) = 0$$

or, $$Q - 2q = 0 \quad \text{or,} \quad q = \frac{Q}{2}$$

So, for maximum F, Q is to be equally divided in the particles.

Example 2.6 Find out electric field intensity at any point on the axis of the uniformly charged rod.

Sol. In Fig. 2.4W, let L is the length of the rod AB uniformly charged (q). If λ be the linear charge density then $\lambda = \frac{q}{L}$ and charge on an elementary length dx is $dq = \lambda\, dx$. The electric field at P due to dx is

$$d\vec{E} = \frac{dq}{4\pi\varepsilon_0 x^2}\hat{x} \text{ along } \overrightarrow{BP}$$

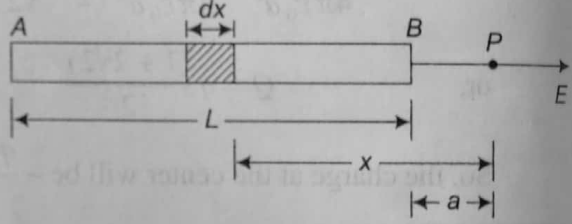


Fig. 2.4W Electric field at a point P on the axis of a uniformly charged rod AB.

For the entire charged rod, electric field acts in the same direction and total field at P is

$$E = \int_a^{a+L} \frac{\lambda}{4\pi\varepsilon_0}\frac{dx}{x^2} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{a+L}\right] = \frac{\lambda L}{4\pi\varepsilon_0\, a\,(a+L)}$$

If $a >> L$, $$E = \frac{1}{4\pi\varepsilon_o}\frac{q}{a^2}.$$

So under this condition, a charged rod behaves as a point charge.

Example 2.7 Find out the electric field intensity at any point on the perpendicular bisector of a uniformly charged rod.

Sol. Consider an elementary length dx at a distance x from the center of the rod of length L [Fig. 2.5W].

The charge on this element dx is $dq = \frac{q}{L}dx = \lambda\, dx$, where $\lambda = \frac{q}{L}$ the linear charge density.

The electric field at P due to this element is

$$dE = \frac{dq}{4\pi\varepsilon_0 (AP)^2} = \frac{dq}{4\pi\varepsilon_0 (a^2 + x^2)}$$

Again, the component of dE along OP is $dE \cos\theta$.

The resultant field at P due to the whole charged rod is

$$E = \int dE \cos\theta = \frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{L/2} \frac{q\,a\,dx}{L(a^2+x^2)^{3/2}}$$

$$\left[\because \cos\theta = \frac{a}{\sqrt{x^2+a^2}}\right]$$

$$= \frac{qa}{4\pi\varepsilon_0 L}\int_{-L/2}^{L/2}\frac{dx}{(a^2+x^2)^{3/2}}$$

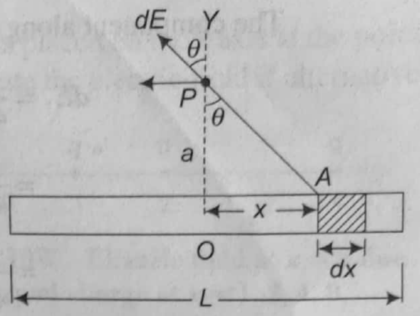


Fig. 2.5W Electric field at a point on the perpendicular bisector of a uniformly charged rod.

We have $x = a\tan\theta$ or $dx = a\sec^2\theta\, d\theta$

or, $$E = \frac{qa}{4\pi\varepsilon_0 L}\int \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta} = \frac{qa^2}{4\pi\varepsilon_0 a^3 L}\int \cos\theta\, d\theta$$

$$= \frac{q}{4\pi\varepsilon_0 La}\sin\theta = \frac{q}{4\pi\varepsilon_0 La}\left[\frac{x}{(a^2+x^2)^{1/2}}\right]_{-L/2}^{L/2}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{2q}{a\sqrt{L^2+4a^2}} = \frac{q}{2\pi\varepsilon_0 a\sqrt{L^2+4a^2}}$$

Example 2.8 Find out the electric field intensity at a point on the axis of a uniformly charged ring.

Sol. Consider an elementary length dl of the ring (Fig. 2.6W). The charge of the ring $dq = \lambda\, dl$. The field at P due to dq of dl is

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} \text{ along } AP$$

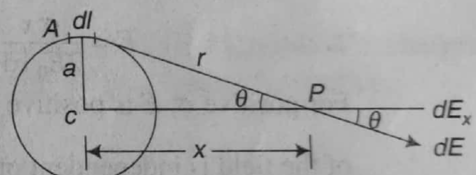


Fig. 2.6W Electric field intensity at a point on the axis of a charged ring.

The component of dE along the x axis is $dE_x = dE\cos\theta$

$$\therefore \quad dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dl}{r^2}\cos\theta$$

Total field intensity at P due to the whole ring

$$E = \frac{\lambda}{4\pi\varepsilon_0}\int\frac{dl}{r^2}\cos\theta = \frac{\lambda}{4\pi\varepsilon_0 r^2}\int\frac{x}{r}\,dl$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{x}{r^3}\times 2\pi a$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi a x}{(a^2+x^2)^{3/2}} = \frac{qx}{4\pi\varepsilon_0(a^2+x^2)^{3/2}} \text{ along the } x \text{ axis}$$

When $x >> a$, $\quad E = \dfrac{q}{4\pi\varepsilon_0 x^2}$

So, at large distance the ring behaves like a point charge.

Example 2.9 Find out the electric field intensity at any point on the axis of a uniformly charged disc.

Sol. In Fig. 2.7W, Let R be the radius of the disc and x be the distance of the field point P from the center O. We consider a concentric ring within radii r and $r + dr$. If σ is the surface charge density then total charge on a surface element dS is $\sigma\, dS$. The field at P due to ds is given by $d\vec{E} = \dfrac{\sigma\, dS}{4\pi\varepsilon_0 a^2}\hat{a}$, here $a^2 = x^2 + r^2$.

The component along the axis of the disc

$$dE_1 = \frac{1}{4\pi\varepsilon_o}\frac{\sigma\, dS}{(r^2+x^2)}\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\sigma\, dS}{(r^2+x^2)}\frac{x}{(x^2+r^2)^{1/2}}$$

$$= \frac{\sigma\, dS\, x}{4\pi\varepsilon_0 (r^2+x^2)^{3/2}}$$

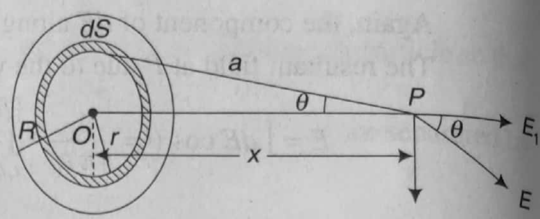


Fig. 2.7W Electric field at a point P on the axis of a charged disc.

Again $ds = 2\pi r\, dr$

or, $$dE_1 = \frac{\sigma \times 2\pi r\, dr\, x}{4\pi\varepsilon_0 (r^2+x^2)^{3/2}}$$

Total intensity at P is $E = \int dE_1 = \frac{2\pi x\sigma}{4\pi\varepsilon_0}\int_0^R \frac{r\, dr}{(r^2+x^2)^{3/2}}$

$$E = \frac{x\sigma}{2\varepsilon_0}\left[\frac{1}{x} - \frac{1}{(R^2+x^2)^{1/2}}\right]$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{(R^2+x^2)^{1/2}}\right]$$

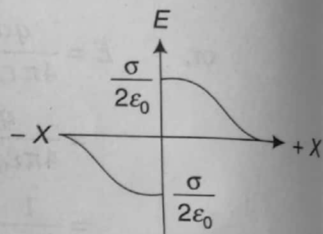


Fig. 2.8W Variation of field intensity with distance from the center of a charged disc.

For infinite charge sheet $R \to \alpha$

$\therefore$ $$E = \frac{\sigma x}{2E_0\, |x|}$$

For positive σ, E is positive when x is positive and it is negative when x is negative. The magnitude of the field is independent of the distance x and is given by $\frac{\sigma}{2\varepsilon_0}$. The variation of field intensity with distance from the center of a uniformly charged disc is shown in Fig. 2.8W.

Example 2.10 Five equal charges of 40 nC each are placed at five vertices of a regular hexagon of 6 cm side. The sixth vertex is free. Determine the electric field at the center of the hexagon due to the distribution.

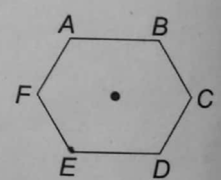


Fig. 2.9W Electric field at center due to five equal charges placed at five corners of a regular hexagon.

Sol. The field at the center due to the charges located at two opposite vertices is zero. Since there is no charge at F (free vertex) [Fig. 2.9W] so the resultant field will be due to charge q located at C. The field is directed from the center to the vacant corner and its magnitude is $\frac{q}{4\pi\varepsilon_0 a^2}$, where a is the distance of the center from each of the vertices. So, the field is $\frac{9\times10^9\times40\times10^{-9}}{(6\times10^{-2})^2}$ N/C or, 10^5 N/C

Example 2.11 Infinite number of positive charges, each of magnitude q is placed on the x axis at the point $x = 1, 2, 4, 8, \dots$ What will be intensity of electric field at $x = 0$? Also calculate the electric field if alternative charges are of opposite signs.

Sol. The resultant intensity at $x = 0$ [Fig. 2.10W] is

$$E = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{1^2}+\frac{q}{2^2}+\frac{q}{4^2}+\frac{q}{8^2}+\dots\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\dots\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\left(1-\frac{1}{4}\right)}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{4}{3}\text{ units}$$

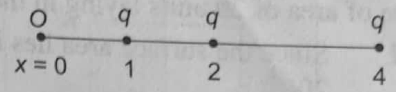


Fig. 2.10W Electric field at $x = 0$ due to equal charge at $x = 1, 2, 4, 8, \dots$

If alternate charges are of opposite signs then electric intensity at $x = 0$ is

$$E_1 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{1^2}-\frac{q}{2^2}+\frac{q}{4^2}-\frac{q}{8^2}+\dots\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\left(1+\frac{1}{4}\right)} = \frac{q}{4\pi\varepsilon_0}\frac{4}{5}\text{ units}$$

Example 2.12 Three charges q_1 q_2 and q_3 are at the vertice of an equilateral triangle of 1 m side. The charges are $q_1 = -2\,\mu\text{C}$, $q_2 = 6\,\mu\text{C}$ and $q_3 = 4.5\,\mu\text{C}$. Find the total potential energy of this charge distribution.

Sol. The total potential energy, $U = \dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{q_1 q_2}{r_{12}}+\dfrac{q_2 q_3}{r_{23}}+\dfrac{q_1 q_3}{r_{13}}\right]$

$$= 9\times 10^9 \times \frac{1}{a}(q_1 q_2 + q_2 q_3 + q_1 q_3) \quad [r_{12} = r_{23} = r_{31} = a \text{ (Say)}]$$

$$= 9\times 10^9 \times [-2\times 6 + 6\times 4.5 - 2\times 4.5]\times 10^{-12}$$

$$= 0.054 \text{ Joule}$$

Example 2.13 Three charges q, $2q$, and $4q$ are placed along a straight line of 6 cm length. Where should the charges be placed so that potential energy of the system is minimum. Find out the distance of the charges.

Sol. Let $2q$ be placed between the other two charges [Fig. 2.11W]. Suppose, the distance between q and $2q$ is x m. so, the distance between $2q$ and $4q$ is $(0.06 - x)$ m.

The potential energy of the whole system,

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q\times 2q}{x}+\frac{2q\times 4q}{0.06-x}+\frac{q\times 4q}{0.06}\right]$$

For minimum value of U $\quad \dfrac{dU}{dx} = 0$

or, $$-\frac{1}{x^2}+\frac{4}{(0.06-x)^2} = 0$$

or, $$x = 0.02 \text{ m}$$

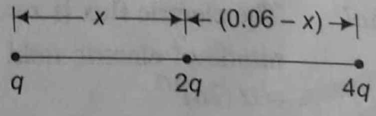


Fig. 2.11W Minimum PE due to charges q, $2q$, $4q$ placed on a line of 6 cm length.

So, the distance between q and $2q$ is 0.02 m and the distance between $2q$ and $4q$ is 0.04 m.

Example 2.14 If the electric field is given by $\vec{E} = 6\hat{i} + 3\hat{j} + 4\hat{k}$, calculate the electric flux through a surface of area of 20 units laying in the yz plane.

Sol. Since the surface area lies in the yz plane, the area vector $\vec{S}$ is directed along the x direction. So $\vec{S} = 20\hat{i}$.

Here $\vec{E} = 6\hat{i} + 3\hat{j} + 4\hat{k}$

∴ electric flux through the surface is $\varphi_E = \vec{E} \cdot \vec{S}$

$$= (6\hat{i} + 3\hat{j} + 4\hat{k}) \cdot 20\hat{i}$$

$$= 120 \text{ units}$$

Example 2.15 Find out electric field intensity at regions I, II, III due to two infinite plane parallel sheets of charge [Fig. 2.12W].

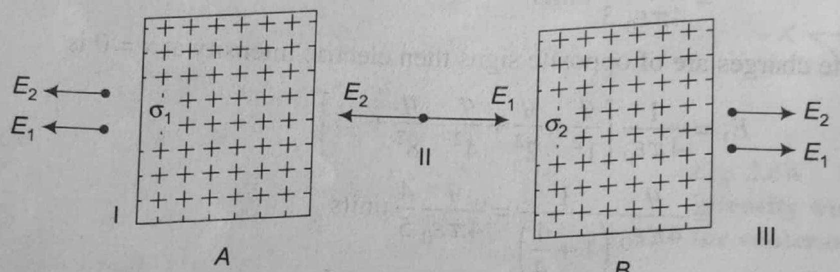


Fig. 2.12W Electric field at three points I, II and III due to two infinite plate charge sheets.

Sol. Let A and B be two infinite plane parallel charge sheets and σ_1, σ_2 be uniform surface densities of charge on A and B respectively. Here $\sigma_1 > \sigma_2$ (say).

In region I, the net electric field

$$E_{\text{I}} = E_1 + E_2 = \left(\frac{-\sigma_1}{2\varepsilon_0}\right) + \left(\frac{-\sigma_2}{2\varepsilon_0}\right) = -\frac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)$$

In region II, $$E_{\text{II}} = E_1 - E_2 = \frac{\sigma_1}{2\varepsilon_0} - \frac{\sigma_2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}(\sigma_1 - \sigma_2)$$

In region III, $$E_{\text{III}} = E_1 + E_2 = \frac{\sigma_1}{2\varepsilon_0} + \frac{\sigma_2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2)$$

Now if $\sigma_1 = \sigma$ and $\sigma_2 = -\sigma$ then $E_{\text{I}} = 0$ $E_{\text{III}} = 0$, and $E_{\text{II}} = \dfrac{\sigma}{\varepsilon_0}$

Example 2.16 The electric field components in Fig. 2.13W are $E_x = \alpha x^{1/2}$, $E_y = E_z = 0$ in which $\alpha = 800$ NC^{-1} m$^{-1/2}$. Calculate electric flux through the cube and the charge within the cube. Assume that $a = 0.1$ m.

Sol. The electric flux is zero for each face of the cube except the two faces $ABCD$ and $EFGH$. The magnitude of electric field at the face $ABCD$, $E_1 = \alpha x^{1/2} = \alpha a^{1/2}$ and at the face $EFGH$, $E_2 = \alpha x^{1/2} = \alpha (2a)^{1/2}$.

So, flux $\varphi_1 = \vec{E}.\vec{dS} = E_1 S \cos 180°$

$= \alpha a^{1/2} \times a^2 (-1) = -a^{5/2}\alpha$

and flux $\varphi_2 = \vec{E}.\vec{dS} = E_2 S \cos 0°$

$= \alpha (2a)^{1/2} a^2 = 2^{1/2} a^{5/2} \alpha$

So, net flux $\varphi = (E_2 - E_1) = 2^{1/2} a^{5/2} \alpha - a^{5/2} \alpha$

$= a^{5/2} \alpha (2^{1/2} - 1)$

Now, putting the value $a = 0.1$ m and

$\alpha = 800 \text{ NC}^{-1} \text{ m}^{-1/2}$,

we have net flux = 1.05 $\text{Nm}^2 \text{ C}^{-1}$ and

charge $q = \varepsilon_0 \varphi$

So, $q = 8.85\ 10^{-12} \times 1.05$

$= 9.3 \times 10^{-12}$ C

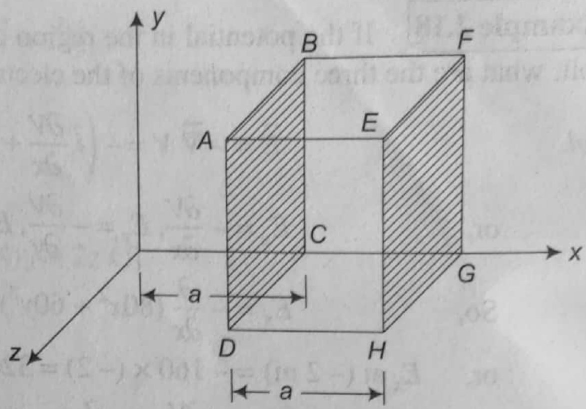


Fig. 2.13W Electric flux through a cube and charge inside it.

Example 2.17 Using Gauss' law in integral form, obtain the electric field due to the following charge distribution in spherical coordinates, [WBUT 2012]

$$\rho(r, \theta, \varphi) = \rho_0 \left(1 - \frac{r^2}{a^2}\right) \quad 0 < r < a$$

$$= 0 \quad a < r < \infty$$

Sol. Consider region $0 < r < a$, in spherical coordinate volume element $dv = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

Now, from Gauss' law $\int_S \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0}$

or, $$E \times 4\pi r^2 = \frac{1}{\varepsilon_0} \int_{r=0}^{r} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \rho_0 \left(1 - \frac{r^2}{a^2}\right) r^2 \sin\theta\, d\theta\, dr\, d\varphi$$

$$E \times 4\pi r^2 = \frac{\rho_0}{\varepsilon_0}\left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right) 4\pi$$

or, $$E = \frac{\rho_0 (5a^2 r^3 - 3r^5)}{15 a^2 r^2 \varepsilon_0} \quad \text{for } 0 < r < a$$

For region $a < r < \infty$, apply Gauss' law

$$\oint_S \vec{E} \cdot \vec{dS} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \rho_0 \left(1 - \frac{r^2}{a^2}\right) r^2 \sin\theta\, dr\, d\theta\, d\varphi \quad \text{[Charge enclosed up to } r = a\text{]}$$

or, $$E \times 4\pi r^2 = \rho_0 \left(\frac{a^3}{3} - \frac{a^5}{5a^2}\right) 4\pi$$

or, $$E = \frac{2\rho_0 a^3}{15 \varepsilon_0 r^2} \quad a < r < \infty$$

and at $r = a$ $$E = \frac{2}{15} \frac{\rho_0 a}{\varepsilon_0}$$

Example 2.18 If the potential in the region of space near the point (– 2 m, 4 m, 6 m) is $V = 80x^2 + 60y^2$ volt, what are the three components of the electric field at that point?

Sol.
$$\vec{E} = -\vec{\nabla} V = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

or,
$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z}$$

So,
$$E_x = -\frac{\partial}{\partial x}(80x^2 + 60y^2) = -160x$$

or, E_x at (– 2 m) = – 160 × (– 2) = 320 Vm^{-1}

$$E_y = -\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}(80x^2 + 60y^2) = -120\,y$$

E_y at (4 m) = – 120 × 4 = – 480 Vm^{-1}

$$E_z = -\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(80x^2 + 60y^2) = 0$$

Example 2.19 If $V\left(= \frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right)$ is the potential, at *a* distance *r*, due to *a* point charge *q*, then determine the electric field due to point charge *q*, at *a* distance *r*.

Sol.
$$\vec{E} = -\vec{\nabla}V = -\hat{r}\frac{\partial}{\partial r}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right)$$
$$= -\hat{r}\frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r^2}\right) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}.$$

Example 2.20 Three point charges *q*, 2*q* and 8*q* are placed on a 9 cm long straight line Fig. 2.14 W. Determine the position where the charges should be placed such that the potential energy of this system is minimum.

Sol. Let *q* charge be placed at a distance *x* from the charge 2*q*.
Now potential energy,

$$U = \frac{2q^2}{x} + \frac{8q^2}{9 - x}$$

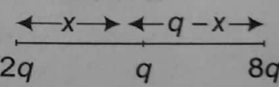


Fig. 2.14 W Placing of three charges *q*, 2*q*, 8*q* on a line of 9 cm length to make PE minimum.

For minimum *U*,
$$\frac{dU}{dx} = 0 = -\frac{2q^2}{x^2} + \frac{8q^2}{(9 - x)^2}$$

or, $(9 - x)^2 = 4x^2$

or, $9 - x = \pm 2x$

or, $x = 3, -9$ cm

But $x = -9$ is not possible so $x = 3$ cm.

Example 2.21 Show that the potential function $V = V_0(x^2 - 2y^2 + z^2)$ satisfies Laplace's equation, where V_0 is a constant. [WBUT 2004]

Sol. Here $V = V_0(x^2 - 2y^2 + z^2)$

or, $$\vec{\nabla}V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)[V_0(x^2 - 2y^2 + z^2)]$$

$$= V_0(2x\,\hat{i} - 4y\hat{j} + 2z\,\hat{k})$$

Again $$\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla}V$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot[V_0(2x\hat{i} - 4y\hat{j} + 2z\,\hat{k})]$$

$$= V_0(2 - 4 + 2) = 0$$

or, $$\nabla^2 V = 0$$

So, the potential function V satisfies Laplace's equation

Example 2.22 A very long cylindrical object carries charge distribution proportional to the distance from the axis (r). If the cylinder is of radius a, then find the electric field both at $r > a$ and $r < a$ by the application of Gauss' law in electrostatics. [WBUT 2007]

Sol. Let A_1 and A_2 be the points [Fig. 2.15W] at a distance r such that (*i*) $r < a$ (ii) $r > a$

(i) ***Inside*** ($r < a$)

Here we consider a coaxial cylinder of radius $r < a$ and length l.

Total flux through the cylindrical surface

$$\oint_S \vec{E}\cdot\vec{dS} = \frac{q}{\varepsilon_0}$$ (q is the total charge enclosed within the cylinder).

Here, if $\rho(r)$ be the charge density then $\rho(r) = \lambda r$ where λ is constant.

Then total charge $$q = \int_0^r 2\pi r l\, dr\, \rho$$

$$= \int 2\pi r l\, dr\, \lambda r = 2\pi l\lambda\int_0^r r^2\, dr$$

$$= \frac{2}{3}\pi l\lambda r^3$$

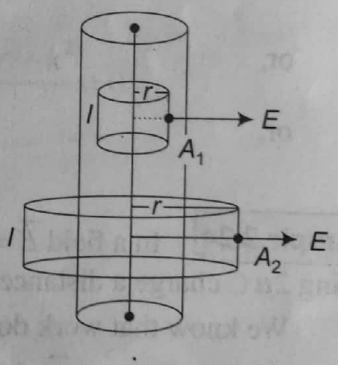


Fig. 2.15W Electric field at points where $r < a$ and $r > a$ due to long cylindrical charged body.

Now, from Gauss' law $$\oint_S \vec{E}\cdot\vec{dS} = \frac{2\pi l\lambda r^3}{3\varepsilon_0}$$

or, $$E\times 2\pi r l = \frac{2\pi l\lambda r^3}{3\varepsilon_0}$$

or, $$E = \frac{\lambda r^2}{3\varepsilon_0}$$

(ii) ***Outside*** ($r > a$)

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

But $$q = \int_0^a 2\pi r l\, dr\, \rho = 2\pi l\int_0^a r\, dr\, \lambda r = 2\pi\lambda l\int_0^a r^2\, dr$$

Now $$\oint_S \vec{E}\cdot\vec{dS} = \frac{2\pi\lambda l}{\varepsilon_0}\int_a^0 r^2\, dr$$

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or,

$$E \times 2\pi r l = \frac{2}{3}\frac{\pi a^3}{\varepsilon_0}\lambda l$$

or,

$$E = \frac{1}{3}\frac{a^3 \lambda}{\varepsilon_0 r}$$

Example 2.23 The potential field at any point in free space is given by $V = 5x^2y + 3yz^2 + 6xz$ volt, where x, y, z are in meters. Calculate the volume charge density at point (2, 5, 3) m.

Sol. From Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Here

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

So,

$$\frac{\partial^2 V}{\partial^2 x^2} = \frac{\partial^2}{\partial x^2}(5x^2y + 3yz^2 + 6xz) = \frac{\partial}{\partial x}(10xy + 6z) = 10y$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2}{\partial y^2}(5x^2y + 3yz^2 + 6xz) = \frac{\partial}{\partial y}(5x^2 + 3z^2) = 0$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2}{\partial z^2}(5x^2y + 3yz^2 + 6xz) = \frac{\partial}{\partial z}(6yz + 6x) = 6y$$

or, $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 10y + 6y = 16y$$

Now at point (2, 5, 3), $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 80$$

or,

$$\nabla^2 V = 80 = -\frac{\rho}{\varepsilon_0}$$

or,

$$\rho = -80\,\varepsilon_0$$

$$= -80 \times 8.854 \times 10^{-12}\ \text{C/m}^3$$

Example 2.24 In a field $\vec{E} = -50y\,\hat{i} - 50x\,\hat{j} + 30\,\hat{k}$ V/m, calculate the differential amount of work done in moving $2\mu C$ charge a distance $5\,\mu$m from P_1 (1, 2, 3,) to P_2 (2, 4, 1).

Sol. We know that work done $dw = -q\vec{E}.\vec{dl}$

Here $$\vec{dl} = 5\hat{e}_{P_1P_2}$$

$$\hat{e}_{P_1P_2} = \frac{(2-1)\,\hat{i} + (4-2)\,\hat{j} + (1-3)\,\hat{k}}{\sqrt{1^2 + 2^2 + (-2)^2}} = \frac{1}{3}(\hat{i} + 2\hat{j} - 2\hat{k})$$

or, $$\vec{dl} = \frac{5}{3}(\hat{i} + 2\hat{j} - 2\hat{k})\ \mu\text{m}.$$

or, $$dw = -q\vec{E}\cdot\vec{dl} = -2 \times 10^{-6}(-50y\,\hat{i} - 50x\hat{j} + 30\,\hat{k}).\frac{5}{3}(\hat{i} + 2\hat{j} - 2\hat{k}) \times 10^{-6}$$

At initial point (1, 2, 3)

$$dw = -2 \times 10^{-6}(-100\,\hat{i} - 50\,\hat{j} + 30\,\hat{k}).\,\frac{5}{3} \times 10^{-6}\,(\hat{i} + 2\hat{j} - 2\hat{k})$$

$$= -\frac{10}{3} \times 10^{-12}\,(-100 - 100 - 60)$$

$$= -\frac{10}{3} \times 260 \times 10^{-12} \text{ Joule} = 8.66 \times 10^{-10} \text{ Joule}$$

Example 2.25 Show that $V = \frac{1}{r}$ satisfies Laplace's equation.

Sol. The position vector $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

Magnitude of $\vec{r}$, $|\vec{r}| = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2}$

$$\therefore \quad \nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2 + y^2 + z^2)^{-1/2}$$

Now, $$\frac{\partial^2}{\partial x^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{\partial}{\partial x}\{-x(x^2 + y^2 + z^2)^{-3/2}\}$$

$$= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

Similarly, $$\frac{\partial^2}{\partial y^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

and $$\frac{\partial^2}{\partial z^2}(x^2 + y^2 + z^2)^{-1/2} = \frac{2z^2 - y^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\therefore \quad \nabla^2\left(\frac{1}{r}\right) = \frac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 - 2z^2 - y^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}} = 0$$

So, $V = \frac{1}{r}$ satisfies Laplace's equation.

Example 2.26 The potential in a medium is given by $\varphi(r) = \dfrac{qe^{-r/\lambda}}{4\pi\varepsilon_0 r}$

(i) Obtain the corresponding electric field.

(ii) Find the charge density that may produce the potential mentioned above. [WBUT 2008]

Sol. Here $\varphi(r)$ is purely a function of r. So, electric field

(i) $$\vec{E} = -\vec{\nabla}\varphi = -\hat{r}\frac{\partial\varphi}{\partial r} = -\hat{r}\frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\frac{e^{-r/\lambda}}{r}\right)$$

$$= -\hat{r}\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{\lambda r}e^{-r/\lambda} - \frac{1}{r^2}e^{-r/\lambda}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\hat{r}\left(\frac{1}{\lambda r} + \frac{1}{r^2}\right)e^{-r/\lambda}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{r}}{\lambda r^2} + \frac{\vec{r}}{r^3}\right)e^{-r/\lambda} \qquad \left[\text{where } \hat{r} = \frac{\vec{r}}{r}\right]$$

(ii) We know that $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$

or,

$$\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E} = \varepsilon_0 \vec{\nabla}\cdot\left[\frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{r}}{\lambda r^2}+\frac{\vec{r}}{r^3}\right)e^{-r/\lambda}\right]$$

$$= \frac{q}{4\pi}\left[\vec{\nabla}\cdot\left(\frac{\vec{r}}{\lambda r^2}\right)e^{-r/\lambda} + \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right)e^{-r/\lambda}\right]$$

By applying $\vec{\nabla}\cdot(\varphi\vec{A}) = \vec{\nabla}\varphi\cdot\vec{A} + \varphi\vec{\nabla}\cdot\vec{A}$

We have

$$\rho = \frac{q}{4\pi}\left[\frac{1}{\lambda}\vec{\nabla}\left(e^{-\frac{r}{\lambda}}\right)\cdot\frac{\vec{r}}{r^2} + \frac{1}{\lambda}e^{-r/\lambda}\left(\vec{\nabla}\cdot\frac{\vec{r}}{r^2}\right) + \vec{\nabla}e^{-r/\lambda}\cdot\frac{\vec{r}}{r^3} + e^{-r/\lambda}\left(\vec{\nabla}\cdot\frac{\vec{r}}{r^3}\right)\right]$$

Again $\vec{\nabla}(e^{-r/\lambda}) = -\dfrac{1}{\lambda}e^{-\frac{r}{\lambda}}\hat{r}$

$$\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 0 \quad \text{and} \quad \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}$$

So,

$$\rho = \frac{q}{4\pi}\left[-\frac{1}{\lambda^2}e^{-\frac{r}{\lambda}}\frac{1}{r} + \frac{1}{\lambda}e^{-\frac{r}{\lambda}}\frac{1}{r^2} - \frac{1}{\lambda}e^{-\frac{r}{\lambda}}\frac{1}{r^2}\right]$$

$$= -\frac{q}{4\pi\lambda^2 r}e^{-r/\lambda}.$$

Example 2.27 Show that $\vec{F} = (2xy + z^3)\,\hat{i} + x^2\hat{j} + 3xz^2\,\hat{k}$ is a conservative field. Find also the scalar potential. [WBUT 2003]

Sol. We know that for conservative force field $\vec{\nabla}\times\vec{F} = 0$

Here $\vec{F} = (2xy + z^3)\,\hat{i} + x^2\,\hat{j} + 3xz^2\,\hat{k}$

$$\therefore \quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy + z^3) & x^2 & 3xz^2 \end{vmatrix} = (0-0)\,\hat{i} + (3z^2 - 3z^2)\,\hat{j} + (2x - 2x)\,\hat{k} = 0$$

So, $\vec{F}$ is a conservative field.

Again, we know that for a conservative field $\vec{F} = \vec{\nabla}V$, where V is the scalar potential.

$$\therefore \quad \vec{F}\cdot\vec{dr} = \vec{\nabla}V.\,\vec{dr} = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = dV$$

$\therefore$ here $\vec{F}\cdot\vec{dr} = (2xy + z^3)dx + x^2dy + 3xz^2\,dz$

So,

$$dV = \vec{F}.\,\vec{dr} = (2xy + z^2)\,dx + x^2dy + 3xz^2\,dz$$

$$= 2xy\,dx + x^2dy + z^3\,dx + 3xz^2\,dz$$

$$= d\,(x^2y) + d\,(xz^3)$$

$$= d\,(x^2y + xz^3)$$

$$\therefore \quad V = \int dV = \int d\,(xy^2 + xz^3) = x^2y + xz^3 + \text{constant.}$$

Example 2.28 Find the electric field due to the following electric potential.

$$V = \frac{\sin\theta\cos\varphi}{r^2}$$

Sol. We know $\vec{E} = -\vec{\nabla} V$. Here, V is the function of r, θ, φ. So in spherical polar coordinates

$$\vec{E} = -\vec{\nabla} V = -\left[\frac{\partial V}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{e}_\varphi\right]$$

$$\frac{\partial V}{\partial r} = -\frac{2\sin\theta\cos\varphi}{r^3},\ \frac{\partial V}{\partial \theta} = \frac{\cos\theta\cos\varphi}{r^2},\ \frac{\partial V}{\partial \varphi} = -\frac{\sin\theta\sin\varphi}{r^2}$$

So, $$\vec{E} = \frac{2\sin\theta\cos\varphi}{r^3}\hat{e}_r - \frac{1}{r^3}\cos\theta\cos\varphi\,\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\sin\theta\sin\varphi}{r^2}\hat{e}_\varphi$$

$$= \frac{1}{r^3}[2\sin\theta\cos\varphi\,\hat{e}_r - \cos\theta\cos\varphi\,\hat{e}_\theta + \sin\theta\,\hat{e}_\varphi] \text{ units}$$

Example 2.29 Is it possible for the electric potential in a charge-free space to be given by (a) $V = x^2 + y^2 - y^2$ (b) $x^2 + y^2 - 2z^2$. If not, find the charge density.

Sol. (a) $$V = x^2 + y^2 - z^2$$

or, $$\vec{\nabla} V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z^2)$$

$$= 2x\,\hat{i} + 2y\,\hat{j} + -2z\,\hat{k}$$

and $$\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla} V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(2x\,\hat{i} + 2y\,\hat{j} - 2z\,\hat{k})$$

$$= 2 + 2 - 2 = 2.$$

Now by using Poisson's equation, $\nabla^2 V = \frac{\rho}{\varepsilon_0}$

$\therefore$ $$\rho = -2\,\varepsilon_0$$

The space is not charge-free.

(b) $$V = x^2 + y^2 - 2z^2$$

or, $$\vec{\nabla} V = 2x\,\hat{i} + 2y\,\hat{j} - 4z\,\hat{k}$$

$$\nabla^2 V = 2 + 2 - 4 = 0$$

By Poisson's equation, $\nabla^2 V = -\frac{\rho}{\varepsilon_0} = 0$ or, $\rho = 0$

The space is a charge-free region.

Example 2.30 Region between the two coaxial cones is shown in Fig. 2.16W. A potential V_a exists at θ_1 and $V = 0$ at θ_2. The cone vertices are insulated at $r = 0$. Solve Laplace's equation to get potential at a cone at any angle θ.

Sol. The potential is constant with respect to r and φ. So, in spherical polar coordinates, Laplace's equation takes from

$$\frac{1}{r^2}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

Now after integration with respect to θ

$$\sin\theta\frac{\partial V}{\partial\theta}=\text{constant}=C_1\ (\text{say})$$

Again integrating
$$V=C_1\ln\left(\tan\frac{\theta}{2}\right)+C_2\ (\text{constant})$$

Here, boundary constants are $\theta=\theta_1,\ V=V_a$
$$\theta=\theta_2,\ V=0$$

Now using boundary conditions

$$V_a=C_1\ln\left(\tan\frac{\theta_1}{2}\right)+C_2$$

$$0=C_1\ln\left(\tan\frac{\theta_2}{2}\right)+C_2$$

After simplification
$$V=V_a\frac{\ln\left(\tan\frac{\theta}{2}\right)-\ln\left(\tan\frac{\theta_2}{2}\right)}{\ln\left(\tan\frac{\theta_1}{2}\right)-\ln\left(\tan\frac{\theta_2}{2}\right)}$$

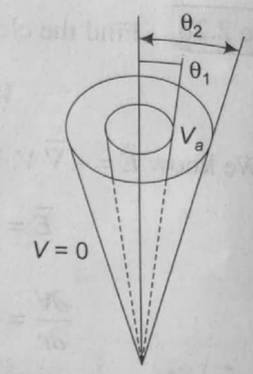

Fig. 2.16W **Potential at a cone of angle θ due to two other equipotential cones at angles θ_1 and θ_2.**

Review Exercises

Part 1: Multiple Choice Questions

1. Which of the following statements is not correct regarding electrostatic field vector E? [WBUT 2008]

(a) $\oint_C \vec{E}\cdot\vec{dr}=0$, where c is a simple closed curve.

(b) $\int_a^b \vec{E}\cdot\vec{dr}$ is independent of the path for given end points a and b.

(c) $\vec{E}=\vec{\nabla}\times\vec{A}$ for some vector potential $\vec{A}$.

(d) $\vec{E}=\vec{\nabla}\varphi$, for some scalar field φ.

2. Flux of the electric field for a point charge (q) at origin through a spherical surface centered at the origin is [WBUT 2006]

(a) $\frac{2q}{\varepsilon_0}$ (b) $\frac{q}{\varepsilon_0}$ (c) $\frac{q}{4\pi\varepsilon_0}$ (d) zero

3. Two concentric spheres of radii r_1 and r_2 carry charges q_1 and q_2 respectively. If the surface charge density (σ) is the same for both spheres, the electric potential at the common center will be

(a) $\frac{\sigma}{\varepsilon_0}\frac{r_1}{r_2}$ (b) $\frac{\sigma}{\varepsilon_0}\frac{r_2}{r_1}$ (c) $\frac{\sigma}{\varepsilon_0}(r_1-r_2)$ (d) $\frac{\sigma}{\varepsilon_0}(r_1+r_2)$

4. Six charges, each equal to $+q$, are placed at the corners of a regular hexagon of side a. The electric potential at the point where the diagonals of the hexagon intersect will be given by
   (a) zero (b) $\frac{1}{4\pi\varepsilon_0}\frac{q}{a}$ (c) $\frac{1}{4\pi\varepsilon_0}\frac{6q}{a}$ (d) $\frac{1}{4\pi\varepsilon_0}\frac{\sqrt{3}q}{2a}$
5. In free space Poisson's equation is [WBUT 2005]
   (a) $\nabla^2 V = 0$ (b) $\nabla^2 V = \frac{\varepsilon_0}{\rho}$ (c) $\nabla^2 V = \alpha$ (d) None of these
6. The electric flux through each of the faces of a cube of 1 m side if a charge q coulomb is placed at its centre is [WBUT 2007]
   (a) $\frac{q}{4\varepsilon_0}$ (b) $4\varepsilon_0 q$ (c) $\frac{q}{6\varepsilon_0}$ (d) $\frac{\varepsilon_0}{6q}$
7. Let (r, θ, φ) represent the spherical polar coordinates of a point in a region where the electrostatics potential V is given by $V = K\varphi^2$. Then the charge density in that region [WBUT 2007]
   (a) is also a function of φ only (b) is constant in that region
   (c) is a function of all the coordinates (r, θ, φ) (d) is a function of (r, θ) only
8. The electrostatic potential energy of a system of two charges q_1 and q_2 separated by a distance r is
   (a) $\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$ (b) $\frac{\varepsilon_0}{4\pi}\frac{q_1 q_2}{r}$ (c) $\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$ (d) $\frac{1}{4\pi\varepsilon_0}\frac{q_1^2 q_2}{r^2}$
9. The magnitude of electric field $\vec{E}$ in the annular region of a charged cylindrical capacitor
   (a) same anywhere (b) varies as $\frac{1}{r}$ (c) varies as $\frac{1}{r^2}$ (d) None of these
10. Electric field and potential inside a hollow charged conducting sphere are respectively
    (a) $0, 4\pi\varepsilon_0 \frac{q}{r}$ (b) $\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}, 0$ (c) $0, \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ (d) $\frac{q}{4\pi\varepsilon_0 r^2}, \frac{q}{4\pi\varepsilon_o r}$
11. For a closed surface which does not include any charge, the Gauss's law will be
    (a) $\oint_S \vec{E}.\vec{ds} = 0$ (b) $\oint_S \vec{E}.\vec{ds} = \frac{q}{\varepsilon_o}$ (c) $\oint_S \vec{E}.\vec{ds} = \frac{q}{4\pi\varepsilon_0}$ (d) None of these
12. Electrostatic field is
    (a) conservative (b) non-conservative (c) rotational (d) None of these
13. Laplace's equation for an electrostatic field is
    (a) $\nabla^2 V = \frac{\rho}{\varepsilon_0}$ (b) $\nabla^2 V = 0$ (c) $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ (d) $\vec{\nabla} V = \frac{\rho}{\varepsilon_0}\hat{r}$
14. Electric field intensity at any point distant r from a plane charged conducting sheet varies as
    (a) r^{-1} (b) r^0 (c) r^{-2} (d) r
15. In a region of space, if the electrostatic potential is constant, then the electric field at that region is
    (a) zero (b) infinite (c) constant (d) None of these

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16. If the flux of the electric field through *a* closed surface is zero,
    (a) the charge inside the surface must be zero
    (b) the electric field must be zero everywhere on the surface
    (c) the charge in the vicinity of the surface must be zero
    (d) None of these

[Ans. 1 (c), 2 (b), 3 (d), 4 (c), 5 (a), 6 (c), 7 (d), 8 (c), 9 (b), 10 (c), 11 (a), 12 (a), 13 (b), 14 (b), 15 (a), 16 (a)]

Short Questions with Answers

1. What is an electric line of force? What is its importance?

Ans. An electric line of force is an imaginary straight or curved path along which *a* positive test charge is supposed to move. The lines of force originate from a single positive charge and converge at an isolated negative charge.

The relative closeness of electric line of force in *a* certain region provides an estimate of the electric field strength in that region.

2. Sketch the electric lines of force due to point charges (i) $q > 0$ (ii) $q < 0$.

Ans.

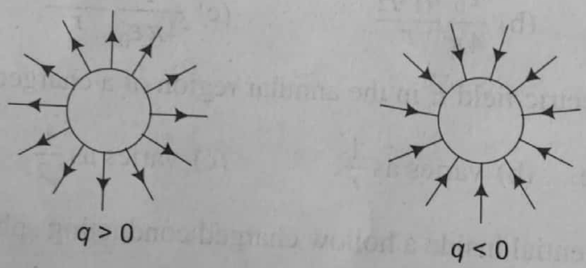

Fig. 2.17W

3. Electrostatic force are much stronger than gravitational force. Give an example.

Ans. A charged glass rod can lift a piece of paper. This shows that the electrostatic force of attraction between the glass rod and paper is much stronger than the gravitational force of attraction between them.

4. State the principle of superposition of electric forces.

Ans. See Section 2.5.

5. State the importance of Gauss' law.

Ans. By applying Gauss' law one can calculate in a simple manner the field intensity due to many different symmetrical configurations of charge. Gauss' law is also important to gain information about the properties of conductors.

6. Obtain Coulomb's theorem from lines of force concept.

Ans. See Section 2.12.3(v).

7. Why is electrostatic field called conservative field?

Ans. A field is conservative when the work done is independent of the path followed and depends only on the initial and final position. For *a* close path work done is zero. In electric field, work done to bring a charge from one point to another point depends on initial and final points. So, the electric field is conservative.

8. Show that electric field is always perpendicular to the equipotential surface.

Ans. In Fig. 2.18W, S is an equipotential surface. A and B are two very close points on the surface. Let electric field $\vec{E}$ make an angle θ with the equipotential surface. The work done for moving a charge q from A to B along the surface is

$$W = qE \cos\theta \times AB$$

Again work done $\quad W = q(V_A - V_B)$

So, $\quad qE\cos\theta \times AB = q(V_A - V_B)$

But $\quad V_A = V_B$ (equipotential surface)

$\therefore \quad qE\cos\theta\,(AB) = 0$

$\therefore \quad \cos\theta = 0 \quad$ or, $\quad \theta = 90°$

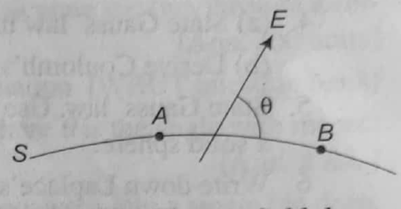

Fig. 2.18W Electric field due to equipotential surface.

9. If Coulomb's law involved $\frac{1}{r^3}$ dependence instead of $\frac{1}{r^2}$, would Gauss' law be still true?

Ans. No, Gauss' law would not hold good.

10. Can electric potential at any point in space be zero while intensity of electric field at that point is not zero?

Ans. Yes, at a point midway between two equal and opposite charges, electric potential is zero but electric field is not zero.

11. No two equipotential surfaces intersect each other. Why?

Ans. We know that two electric lines of force can't intersect, therefore two equipotential surfaces also can't intersect. This is because the electric lines of force and the equipotential surface are mutually perpendicular.

12. Define positive and negative electric flux.

Ans. The electric flux linked with a•surface is said to be positive if the electric field vector appears to be leaving the surface.

The electric flux linked with a surface is said to be negative if the electric field vector appears to be entering the surface.

13. Show that Coulomb's law can be derived from Gauss' law.

Ans. See Section 2.12.2.

14. The electric potential is constant in *a* region. What can you say about the electric field there?

Ans. We know that $\quad E = -\dfrac{dV}{dr}$

If V is constant then $\quad \dfrac{dV}{dr} = 0 \quad \therefore E = 0$

So, electric field is zero.

15. Is it possible for a metal sphere of 1 cm radius to hold a charge of one coulomb?

Ans. $V = 9 \times 10^9 \times \dfrac{1}{10^{-2}} = 9 \times 10^{11}$ volt

This is so high a potential that there will be an electrical breakdown of air. On account of ionization of air, the charge on the sphere will leave away.

Part 2: Descriptive Questions

1. What do you mean by conservation of charge? Explain.
   Find out the relation between electric field intensity and potential. What is equipotential surface? [WBUT 2002]
2. State and explain Gauss' law in electrostatics. Obtain its differential form. [WBUT 2007]
3. Derive Coulomb's law from Gauss' law in electrostatics.
4. (a) State Gauss' law in electrostatics and hence obtain Poisson's equation.
   (b) Derive Coulomb's law from Gauss' law. [WBUT 2008]
5. State Gauss' law. Use Gauss' law to find electric field intensity outside, inside and on the surface of a solid sphere.
6. Write down Laplace's equation in cylindrical coordinates and find the solution.
7. If in the region of space electric field is always in the x direction then prove that the potential is independent of y and z coordinates. If the field is constant there is no free charge in that region. [WBUT 2007]
8. (a) State and Prove Gauss' law in electrostatics.
   (b) Using Gauss' law, obtain an expression for the electric field around a charged hollow cylinder. [WBUT 2004]
9. Show that the potential $V = V_0 (x^2 - 2y^2 + z^2)$ satisfies Laplace's function where V_0 is a constant [WBUT 2004]
10. Write down Laplace's equation in spherical coordinate system and hence find the solution.
11. (a) State Gauss' law of electrostatics.
    (b) Use this law to calculate the electric field between two infinite extent parallel-plate capacitors carrying charge density σ and mutual separation d. Draw the necessary diagram. [WBUT 2006]
12. State Gauss' theorem in electrostatics. Using this theorem, derive an expression for the electric field intensity due to an infinite plane sheet of charge density σ coulomb/m^2.
13. Using Gauss' law, determine the electric field intensity due to a long thin wire of uniform linear charge density.
14. Derive Poisson's and Laplace's equations from fundamentals.

Part 3: Numerical Problems

1. Two point charges Q and q are placed at distance x and $\frac{x}{2}$ respectively from a third charge $4q$. All the three charges are on the same straight line. Calculate Q in terms of q such that the net force on q is zero. **[Ans.** $Q = 4q$**]**
2. Charge is distributed along the x axis from $x = 0$ to $x = L = 50.0$ cm in such a way that its linear charge density is given by $\lambda = ax^2$ where $a = 18.0\ \mu$ cm^{-3}. Calculate the total charge in the region $0 \le x \le l$.
   [Ans. 0.75 μc**]** $\left[\textbf{Hints: } q = \int_0^L \lambda\, dx\right]$
3. Consider a uniform electric field $\vec{E} = 3 \times 10^3\, \hat{i}$ NC^{-1}. What is the flux of this field through a square of 10 cm on a side whose plane is parallel to the yz plane? **[Ans.** $\varphi_E = 30$ Nm^{2}C^{-1}**]**

4. A point charge of 2.0 μ C is at the centre of a cubic gaussian surface, 9.0 cm on edge. What is the net electric flux through the surface? [Ans. 2.26×10^5 Nm^2 C^{-1}]

5. The electric potential $V(x)$ in a region along the x axis varies with distance x (in meter) according to the relation $V(x) = 4x^2$. Calculate the force experienced by 1 μ C charge placed at point $x = 1$ m. [Ans. $F = 8 \times 10^{-6}$ N]

6. In a region of space, the electric field is given by $\vec{E} = 8\hat{i} + 4\hat{j} + 3\hat{k}$. Calculate the flux through a surface of 1000 units area in the xy plane. [Ans. 300 units]

7. Show that the potential function $V = x^2 + z - y^2$ satisfies Laplace's equation [WBUT question bank]

8. A circular wire of radius R has a linear charge density $\lambda = \lambda_0 \cos^2\theta$, where θ is the angle with respect to a fixed radius. Calculate total charge. [Ans. $\pi R\lambda_0$]

9. n charged spherical water drops, each having a radius r and charge q, coalesce into a single big drop. What is the potential of the big spherical drop? $\left[\textbf{Ans. } \dfrac{1}{4\pi\varepsilon_0}\dfrac{n^{2/3}q}{r}\right]$

10. An infinite line charge produces a field of 9×10^4 N C^{-1} at a distance of 2 cm. Calculate the linear charge density. [Ans. 10^{-7} cm^{-1}]

11. Determine the charge distribution at $r \neq 0$ which gives a spherically symmetrical potential $V(r) = \dfrac{e^{-\lambda r}}{r}$ where λ is a constant.

12. The volume charge density of a spherical body of radius a centered at the origin is given by

$$\rho(r, \theta, \varphi) = \frac{\rho_0}{r} \quad \text{where } \rho_0 \text{ is constant.}$$

Calculate the total charge in the sphere. [Ans. $\varphi = 2\pi\rho_0 a^2$]

13. Is is possible for the electric potential in a charge-free region to be given by (i) $V = x^2 + y^2 - z^2$? (ii) $V = x^2 + y^2 + z^2$? If not find the charge density. [Ans. (i) $-4\varepsilon_0$ (ii) $-6\varepsilon_0$] [WBUT Question Bank]

14. Two concentric spheres of radii a and b are kept in potential V_a and V_b. If the intervening space is vacuum then write the appropriate differential equation that the electrostatics potential satisfies. Solve this equation to find out the potential in any point between the spheres and also for a point outside the sphere. Calculate the total charge on the outer sphere. [WBUT 2007]

15. A uniformly charged conducting sphere of 2.4 m diameter has a surface charge density of 80.0 μC/m^2. (i) Find the charge on the sphere. (ii) What is the total electric flux leaving the surface of the sphere? [Ans. (i) 1.45 mC (ii) 1.6×10^8 Nm^2 C^{-1}]