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CSE-(A) 1751007

Morphisms, Rings and Fields

1. Show that the groups $(\mathbb{Z},+)$ and $(\mathbb{Q},+)$ are not isomorphic.

Solⁿ: $(\mathbb{Z},+)$ is a cyclic group. Isomorphic image of a cyclic group must be cyclic. But $(\mathbb{Q},+)$ is not a cyclic group.

$\therefore$ $(\mathbb{Z},+)$ and $(\mathbb{Q},+)$ are not isomorphic.

2. Determine whether the given map ϕ is a homomorphism.

(a) $\phi : \mathbb{R} \to \mathbb{Z}$ under addition given by $\phi(x) =$ the greatest integer $\leq x$.

(b) $\phi : \mathbb{R}^* \to \mathbb{R}^*$ under multiplication given by $\phi(x) = |x|$

(c) Let $M_n(\mathbb{R})$ be the additive group of all $n\times n$ matrices with real entries and $\phi(A) = \det(A)$, $A \in M_n(\mathbb{R})$

Solⁿ: (a) Let $x \in G$, $y \in G$. Then $x+y \in G$.

$\phi(x) = [x]$ $\quad \phi(y) = [y]$ $\quad \phi(x+y) = [x+y]$

We know

$[x+y] \neq [x] + [y]$

[eg: let $x = 2.5$ & $y = 1.5$
Then $[x+y] = 4$ $\quad [x] = 2$ $\quad [y] = 1$
$\therefore$ $[x]+[y] = 3 \neq [x+y]$]

$\therefore$ $\phi(x+y) \neq \phi(x) + \phi(y)$

Therefore ϕ is not a homomorphism.

(b) Let $x \in G$, $y \in G$. Then $xy \in G$.

$\phi(x) = |x|$ $\quad \phi(y) = |y|$ $\quad \phi(xy) = |xy|$

We know

$|xy| = |x||y|$

$\therefore$ $\phi(xy) = \phi(x)\phi(y)$

$\therefore$ ϕ is a homomorphism.

(c) Let $x \in G$, $y \in G$. Then $x+y \in G$.

where x & y are $n\times n$ matrices.

$\phi(x) = \det x$ $\quad \phi(y) = \det y$ $\quad \phi(x+y) = \det(x+y)$

But $\det|x+y| \neq \det|x| + \det|y|$

(eg: $x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $y = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\left[\det x = 1; \det y = 1 \atop \det|x+y| = 0\right]$

$\therefore \phi(x+y) \neq \phi(x) + \phi(y)$.

$\Rightarrow$ ϕ is not a homomorphism.

4) Prove that Z_8 is not a homomorphic image of Z_{15}.

Soln: We know that group G is said to be the homomorphic image of G' if $o(G)$ is a divisor of $o(G')$.

As we can see that 8 is not a divisor of 15.

So we can clearly say that Z_8 is not a homomorphic image of Z_{15}.

5) Prove that the cancellation law holds in a ring R iff R has no divisor of zero.

Soln: Let R be a ring in which cancellation law holds.

Let $a, b \in R$ and $a \cdot b = 0$ where $a \neq 0$

Then $a \cdot b = 0 = a \cdot 0$

$\because$ The cancellation law holds, $b = 0$.

This proves that a is not a left divisor of zero.

Let $a, b \in R$ and $a \cdot b = 0$ where $b \neq 0$

Then $a \cdot b = 0 = 0 \cdot b$

$\because$ The cancellation law holds, $a = 0$.

This proves that b is not the right divisor of zero.

Thus R has neither left or right divisor of zero.

Conversely, let R has no divisor of 0. Let $a, b, c \in R$ & $ab = ac$. $\because a \neq$

$a(b-c) = 0$, $\because$ R contains no divisor of 0 $b - c = 0 \Rightarrow b = c$.

$\therefore$ Left cancellation law holds in R.

Similar arguments prove that right cancellation law holds.
(proved)

6) Show that the ring of matrices $\left\{ \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ contains divisors of 0 and does not contain the unity.

Solⁿ: Let S be the ring and let $E = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$ in S be the unity.

Then $AE = EA = A$ $\forall$ A in S.

Let $A = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$. Then $AE = A$ implies

$$\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4ax & 0 \\ 0 & 4by \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

$$\Rightarrow 4ax = 2a \text{ \& } 4by = 2b$$

$\therefore$ $x = 1/2$ & $y = 1/2$ & $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ not possible as $x, y \in \mathbb{Z}$.

$\therefore$ S does not contain unity.

$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$ are non zero elements of S &

$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ shows that S contains divisors of 0.

7) Examine the ring of matrices $\left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ contains divisors of zero.

Solⁿ: Let S be the ring and $A = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$ be a non zero element of S. Then $a, b \in \mathbb{R}$ and $(a, b) \neq (0, 0)$.

Let $\underbrace{AB = 0}_{(1)}$, where $B = \begin{pmatrix} p & q \\ 2q & p \end{pmatrix}$.

This (1) gives $ap + 2bq = 0$

$bp + aq = 0$

This is a homogeneous system of equations giving non zero solutions for p, q if $\begin{vmatrix} a & 2b \\ b & a \end{vmatrix} = 0$ i.e. if $\underbrace{a^2 - 2b^2 = 0}_{(2)}$,

Now from (2) we can get non zero values of a & b eg: $a = \sqrt{2}$ & $b = 1$. In this case a non zero matrix B exists such that $AB = 0$

$\therefore S$ contains divisors of zero.

eg: $\begin{pmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{pmatrix}\begin{pmatrix} -\sqrt{2} & 1 \\ 2 & -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

8) Prove that the ring $Z[x]$, the ring of all polynomials with integer coefficients is an integral domain.

Solⁿ:- The ring $Z[x]$ is a commutative ring with unity, the constant polynomial 1 being the identity element.

The zero element in the ring is the constant polynomial 0.

Let $f(x), g(x)$ be non zero polynomials in $Z[x]$ of degree m, n respectively

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m$ & $g(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$ where $a_m \neq 0$ & $b_n \neq 0$.

Then $f(x) g(x)$ is a polynomial in $Z[x]$ with a non zero polynomial term $a_m b_n x^{m+n}$ since $a_m b_n \neq 0$ and therefore $f(x) g(x)$ is a non zero polynomial in $Z[x]$. Thus $Z[x]$ has no divisors of zero and therefore it is an integral domain.

9) Prove that the ring of matrices $\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in R \right\}$ is a field.

Solⁿ:- Let $S = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in R \right\}$, $(S, +, \cdot)$ is a ring with unity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ being the unity.

Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $B = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \in S$. Then

$$A \cdot B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} p & q \\ -q & p \end{pmatrix} = \begin{pmatrix} ap - bq & aq + bp \\ -bp - aq & -bq + ap \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} p & q \\ -q & p \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} ap - bq & aq + bp \\ -bp - aq & -bq + ap \end{pmatrix}$$

Therefore $AB = BA \quad \forall A, B \in S$.

Hence $(S, +, \cdot)$ is a commutative ring with unity.

Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ be a non zero element of S.

Then $(a,b) \neq (0,0)$ & $\det A = a^2 + b^2 \neq 0$.

Hence A^{-1} exists & $A^{-1} = \frac{1}{a^2+b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in S$

$\therefore$ Every non zero element of a ring is a unit.

Hence $(S,+,\cdot)$ is a field.

10) Prove that the ring of matrices $\left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a,b \in \mathbb{Q} \right\}$ is a field.

Sol^n: Let $S = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a,b \in \mathbb{Q} \right\}$. $(S,+,\cdot)$ is a ring with unity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ being the unity.

Let $A = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$, $B = \begin{pmatrix} p & q \\ 2q & p \end{pmatrix} \in S$. Then

$$A \cdot B = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \begin{pmatrix} p & q \\ 2q & p \end{pmatrix} = \begin{pmatrix} ap + 2bq & aq + bp \\ 2bp + 2aq & 2bq + ap \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} p & q \\ 2q & p \end{pmatrix} \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} = \begin{pmatrix} ap + 2bq & aq + bp \\ 2bp + 2aq & 2bq + ap \end{pmatrix}$$

$\therefore$ $AB = BA$ $\forall A, B \in S$.

$\Rightarrow$ $(S,+,\cdot)$ is a commutative ring with unity.

Let $A = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$ be a non zero element of S. Then $(a,b) \neq (0,0)$

$\det A = a^2 - 2b^2 \neq 0$ as $(a,b) \neq (0,0)$ and a, b are rational.

Hence A^{-1} exists and $A^{-1} = \frac{1}{a^2-2b^2} \begin{pmatrix} a & -b \\ -2b & a \end{pmatrix} \in S$.

$\therefore$ Each non zero element of S is a unit.

$\therefore$ $(S,+,\cdot)$ is a field.

11) Examine the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a,b \in R\right\}$ is a field.

Soln: Let $S = \left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a,b \in R\right\}$. S is a commutative ring with unity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Let $A = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$ be a non zero element of S. Then $(a,b) \neq (0,0)$. A^{-1} exists only if $\det A \neq 0$.

Now $\det A = a^2 - 2b^2$. There exist non zero real no:s for which $\det A = 0$ eg: $a = \sqrt{2}$ $b = 1$.

$\therefore$ Non zero matrix $\begin{pmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{pmatrix}$ has no inverse.

$\therefore$ $(S,+,\cdot)$ is not a field.

12) Prove that the set S of matrices $\left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a,b \in Z\right\}$ is a subring of $M_2(Z)$.

Soln: Now when $a,b,c = 0$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$ $\therefore S \neq \phi$

Let $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ $B = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}$

$$A - B = \begin{pmatrix} a-p & 0 \\ b-q & c-r \end{pmatrix} \in S$$

$$A \cdot S = \begin{pmatrix} ap & 0 \\ bp+cq & cr \end{pmatrix} \in S.$$

$\therefore$ S is a subring of $M_2(Z)$.

13) Prove that $(Z_n,+,\cdot)$ is an integral domain iff n is prime.

Soln: Let n be a composite number.

$n = pq$, $1 < p < n$, $1 < q < n$

$\therefore$ $\bar{n} = \bar{p}\cdot\bar{q} = \bar{0}$ $\Rightarrow$ Existence of divisors of 0.

$\therefore$ It is a contradiction

$\therefore$ n is prime.

Let n be a prime no:.

$(Z_n, +, \cdot) \to$ commutative ring with unity.

Let $\bar{m} \in Z_n$ gcd $(m,n) = 1$.

$\Rightarrow mu + nv = 1$

$\Rightarrow n \mid mu - 1$

$\overline{mu-1} = \bar{0} \Rightarrow \bar{m}\bar{u} = \bar{1}$

If $u > n$ $u = nq + r$

$\Rightarrow \bar{u} = \bar{r}$

$\therefore \bar{m} \cdot \bar{r} = \bar{1} = \bar{r} \cdot \bar{m}$

$\therefore$ Every element of Z_n is an unit.

14) Prove that the ring $Z[i] = \{a + ib : a, b \in Z\}$, the ring of Gaussian integers, is an integral domain.

Sol^n: The ring of Gaussian integers is a commutative ring with unity. 1 being the unit $(1 + 0i)$

Let $(a + bi)(c + di) = 0$

Let $a + bi \neq 0$ Then $(a, b) \neq (0, 0)$

$(a + bi)(c + di) = 0 \Rightarrow$ ~~$a+bi=0$ & $c+d$~~

with ~~$a+bi \neq 0 \Rightarrow c+di$~~

gives $ac - bd = 0$, $ad + bc = 0$

This is a homogeneous system of eq^{n}s in a,b having non zero solutions. Therefore the coefficient of det of the system, $c^2 + d^2$ is 0. This gives $c = 0$ & $d = 0$

Thus $(a + bi)(c + di)$ with $a + bi \neq 0 \Rightarrow c + di = 0$.

$\therefore$ It contains no divisor of 0. Hence it is an integral domain.

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15) Prove that Z_{11}, the ring of all integers modulo 11, is a field. State any theorem that you use. Find the multiplicative inverses of the non zero elements of Z_{11}.

Sol^n. As we know that $(Z_n, +, \cdot)$ is an integral domain ~~when~~ iff n is prime.

Here we can see that $n = 11$ is prime.

$\therefore (Z_{11}, +, \cdot)$ is an integral domain.

Also we know that any finite integral domain is a field.

$\therefore (Z_{11}, +, \cdot)$ is a field.

We know m is a multiplicative inverse if $(m, 11) = 1$.

$\therefore m = \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}$

16) Let R be a ring. The centre of R is the set $\{x \in R : ax = xa\}$ for all $a \in R$. Prove that centre of a ring is subring.

Sol^n. Let $Z(R) = \{x \in R : ax = xa\}$

Aim : to show,
$a(x-y) = (x-y)a$
$a(xy) = (xy)a$.

Let $x, y \in Z(R)$

$\therefore ax = xa$ & $ay = ya$.

$a(xy) = (ax)y = (xa)y = x(ay) = (xy)a$ — (1)

Now $a(x-y) = ax - ay = xa - ya$
$= (x-y)a$. — (2)

From (1) & (2) we see that centre of a ring is subring.

17) In a ring $(Z_n, +, \cdot)$, $[m]$ is a unit if and only if the $\gcd(m, n) = 1$.

Sol^n. Let $\bar{m}$ be the unit of $(Z_n, +, \cdot)$. Then $\exists\ \bar{u} \in Z_n$ s.t

$\bar{m} \cdot \bar{u} = 1 \Rightarrow \bar{m} \cdot \bar{u} - 1 = \bar{0}$

$\Rightarrow \overline{mu - 1} = \bar{0}$

$\Rightarrow n \mid mu - 1$

$\Rightarrow mu - 1 = nv$

$\Rightarrow mu - nv = 1$

$\Rightarrow \gcd(m,n) = 1.$

Conversely let $\gcd(m,n) = 1$

$\Rightarrow mu + nv = 1$

$\Rightarrow n \mid mu - 1 \Rightarrow \overline{mu} = \overline{1}$

If $u > n$ by division algorithm,

$u = nq + r$

Now $\overline{m}.\overline{u} = 1 \Rightarrow \overline{m}.\overline{r} = 1 = \overline{r}.\overline{m}$

Hence $\overline{m}$ is a unit in Z_n.

18) In a ring R with unity $(xy)^2 = x^2y^2 \ \forall x, y \in R$, then show that R is commutative.

Soln: Since the unity of ring $1 \in R$, so for all $x, y \in R$

$[x(y+1)]^2 = x^2(y+1)^2$, by given condition.

$\Rightarrow [x(y+1)][x(y+1)] = x^2(y+1)(y+1)$

$\Rightarrow (xy+x)(xy+x) = x^2[y(y+1) + 1(y+1)]$

$\Rightarrow (xy)^2 + xyx + xxy + x^2 = x^2[y^2 + y + y + 1]$

$\Rightarrow (xy)^2 + xyx + xxy + x^2 = x^2y^2 + x^2y + x^2y + x^2$

$\Rightarrow x^2y^2 + xyx + x^2y + x^2 = x^2y^2 + x^2y + x^2y + x^2$

$\Rightarrow xyx + x^2y = x^2y + x^2y$

$\Rightarrow xyx = x^2y$

Replacing x by $x+1$ we get

$(x+1)y(x+1) = (x+1)^2y.$

$\Rightarrow (xy+y)(x+1) = (x+1)(x+1)y.$

$\Rightarrow (xy+y)x + (xy+y) = (x^2+x+x+1)y.$

$\Rightarrow x^2y + yx + xy + y = x^2y + xy + xy + y$

$\Rightarrow yx + xy = xy + xy$

$\Rightarrow yx = xy$ (proved)

1a) In a field F, prove that $a^2 = b^2$ implies either $a = b$ or $a = -b$ $\forall a, b \in F$.

Soln: Now $(a-b)(a+b) = (a-b)\cdot a + (a-b)\cdot b$

By right Distributive law

$= a\cdot a - b\cdot a + a\cdot b - b\cdot b$

$= a^2 - a\cdot b + a\cdot b - b^2$ $[\because a\cdot b = b\cdot a \; \forall a, b \in F]$

$= a^2 - b^2 = a^2 - a^2$ $[\because a^2 = b^2]$

$= 0$.

Either $a - b = 0$ or $a + b = 0$

i.e. either $a = b$ or $a = -b$.

2a) Find all the solutions of the equation $x^2 + x - 6 = 0$ in the ring Z_{14} by factoring the quadratic polynomial.

Soln: By factorizing we get $(x+3)(x-2)$

For x, y in Z_{14} we have $xy = 0$ if xy is a multiple of 14.

$\therefore$ either x is a multiple of 7 or y is a multiple of 7.

The only 2 multiples of 7 in Z_{14} are either 7 & 0.

So $x + 3 = 0$ or $x - 2 = 0$

$x = -3$ $\qquad x = 2$

or $x = 11$.

& $x + 3 = 7$ or $x - 2 = 7$

$\Rightarrow x = 4$ or $x = 9$.

$\therefore$ Solutions of x are $2, 4, 9, 11$.

21) Find all solutions of the equation $x^3 - 2x^2 - 3x = 0$ in ring Z_{12}.

Soln. Factorizing we get,

$$x(x+1)(x-3) = 0$$

In Z_{12}, the product of two non zero elements may be 0.

We have to find all elements of x where $x \in Z_{12}$ & $0 \leq x \leq 11$

Thus zero occur in Z_{12} when $x = 0, 3, 5, 8, 9, 11$ in

$$x(x+1)(x-3) = 0.$$

$\therefore$ Solutions of the eqn. $x^3 - 2x^2 - 3x = 0$ in Z_{12}

is $0, 3, 5, 8, 9, 11$.

22) An element of a ring R is idempotent if $a^2 = a$.

a) Show that the set of all idempotent elements of a commutative ring is closed under multiplication.

Let $a \in R$ then $a + a \in R$.

$$(a+a)^2 = a+a$$

$\Rightarrow (a+a)(a+a) = a+a$

$\Rightarrow (a^2 + a^2) + (a^2 + a^2) = a + a$

$\Rightarrow a + a + a + a = a + a$ [As $a^2 = a$]

$\Rightarrow a + a = 0$ [Applying cancellation law].

also $b + b = 0$ $\Rightarrow a = b$.

Now we have to see whether $a+b$ is an idempotent element or not.

$(a+b)^2 = (a+b)(a+b)$

$= a(a+b) + b(a+b)$

$= a^2 + ab + ba + b^2$

$= a^2 + a^2 + a^2 + b^2$ [as $a = b$]

$= a + a + a + b$ [as $a^2 = a$ $b^2 = b$]

$= a + b$ [as $a + a = 0$]

(proved)

b) Find all idempotents in the ring $Z_6 \times Z_{12}$

The idempotents of this ring are the ordered pair (a,b) such that a is an idempotent of Z_6 and b is an idempotent of Z_{12}.

Idempotent of $Z_6 = \{0, 1, 3, 4\}$

Idempotent of $Z_{12} = \{0, 1, 4, 9\}$

$\therefore$ Idempotent of $Z_6 \times Z_{12} = \{0,1,3,4\} \times \{0,1,4,9\}$

23) Let R be a commutative ring with unity of characteristic 3. Compute and simplify $(a+b)^6$ $\forall a,b \in R$.

Soln: Now,

$$(a+b)^6 = a^6 + {}^6C_1 a^5 b + {}^6C_2 a^4 b^2 + {}^6C_3 a^3 b^3 + {}^6C_4 a^2 b^4 + {}^6C_5 ab^5 + b^6$$

$$= a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6ab^5 + b^6$$

Given R is of characteristic 3 then $6a^5 b + 15a^4 b^2 + 15a^2 b^4$ are 0 as they have a factor of 3.

$\therefore (a+b)^6 = a^6 + 20a^3 b^3 + b^6$.

24) Prove that intersection of 2 subring is a subring.

Let R be a ring & S & T be 2 subrings of R.

Soln: $S \cap T$ is an non empty set as $0 \in S \cap T$.

Case 1: Let $S \cap T = \{0\}$. Then $S \cap T$ is a subring of R.

Case 2: Let $S \cap T \neq \{0\}$ & let $p \in S \cap T$ $q \in S \cap T$.

Then $p \in S$, $q \in S$, $p \in T$, $q \in T$.

Since S is a subring of R then $p-q \in S$ & $p \cdot q \in S$

Since T is a subring of R then $p-q \in T$ & $p \cdot q \in T$

Thus $p \in S \cap T, q \in S \cap T \Rightarrow p-q \in S \cap T$

$p \cdot q \in S \cap T$

$\therefore S \cap T$ is a subring of R.