20. If $f(3) = a + \Delta f(1) + \Delta^2 f(1)$ then $a =$

(a) $f(0)$ (b) $f(1)$ (c) $f(2)$ (d) $f(3)$

[M.A.K.A.U.T., M(CS)-401, ...]

21. $\Delta\left(\frac{1}{x}\right)$ is equal to

(a) $\frac{1}{x(x+h)}$ (b) $\frac{1}{x+h}$ (c) $-\frac{h}{x+h}$ (d) ...

22. Which relations are true?

(a) $E = 1+\Delta,\ \Delta\nabla = \Delta - \nabla$

(b) $E = 1-\Delta,\ \Delta\nabla = \Delta + \nabla$

(c) $E = 1-\Delta,\ \Delta\nabla = \Delta - \nabla$

(d) $E = 1+\Delta,\ \Delta\nabla = \Delta + \nabla$ [M.A.K.A.U.T., M(CS)-301, ...]

23. The value of $(1+\Delta)(1-\nabla)$ is

(a) 0 (b) 1 (c) 2 (d) 3

[M.A.K.A.U.T., M(CS)-301, ...]

24. If $f(x) = \frac{1}{x}$, the divided difference $[a, b, c]$ is

(a) $\frac{1}{a+b+c}$ (b) $\frac{1}{abc}$ (c) $\frac{1}{a^2+b^2}$ (d) $a+b...$

[M.A.K.A.U.T., M(CS)-401, ...]

25. If the interval of differencing is unity and $f(x) = ax^2$ (... contant), which one of the following choices is wrong?

(a) $\Delta f(x) = a(2x+1)$ (b) $\Delta^2 f(x) = 2a$

(c) $\Delta^3 f(x) = 2$ (d) $\Delta^4 f(x) = 0$

[M.A.K.A.U.T., M(CS)-401, ...]

Answers

1.a	2.c	3.d	4.c	5.c	6.d	7.b	8.b	9.b	10...
11.b	12.b	13.c	14.b	15.c	16.c	17.a	18.d	19.c	20...
21.d	22.a	23.b	24.b	25.c					

3 INTERPOLATION

3.1 Introduction:

Let $f(x)$ be a function of x defined in the interval $I:(-\infty < x < \infty)$ in which it is assumed to be continuous and continuously differentiable for a sufficient number of time. Suppose the analytical formula for the function $y = f(x)$ is not known, but the values of $f(x)$ are known for $(n+1)$ distinct values of x, say $x_0, x_1, \ldots\ldots, x_n$, called *arguments of nodes* which are entered as $y_0 = f(x_0), y_1 = f(x_1), \ldots. y_n = f(x_n)$ and there is no other information available about the function. Our problem is to compute the value of $f(x)$, at least approximately, for a given arguments x in the vicinity of the above given values of the arguments. The process by which we can find the value of $f(x)$ for any other value of x in the interval $[x_0, x_1]$ is called *interpolation.* When x lies slightly outside the interval $[x_0, x_n]$, then the process is called *extrapolation.* [W.B.U.T., CS-312, 2006, 2008, 2009, M.A.K.A.U.T., 2013, 2015]

Since the analytical form i.e., explicit nature of $f(x)$ is not known, it is required to find a simpler function, say $p(x)$, such that

$$p(x_i) = f(x_i),\ i = 0, 1, 2, \ldots., n \qquad \ldots \quad (1)$$

This function $p(x)$ is known as interpolating function and in generals

$$f(x) \simeq p(x) \qquad \ldots \quad (2)$$

If $p(x)$ is a polynomial, then the process is called *polynomial interpolation* and $p(x)$ is called the *interpolating polynomial.* The justification of replacing a function by a polynomial rests on a theorem due to *Weierstrass* and is stated below without proof.

Theorem. Let $f(x)$ be a function defined and continuous on $a \le x \le b$. Then for $\varepsilon > 0$, there exist a polynomial $p(x)$ such that

$$|f(x) - p(x)| < \varepsilon,\ \ a \le x \le b$$

3.2. Error or remainder in polynomial interpolation

In virtue of (2), if we write

$$f(x) = p(x) + E(x) \quad \dots \quad (3)$$

then $E(x)$ is the error committed in replacing $f(x)$ by $p(x)$.

Using (1), we have

$$E(x_i) = 0, \quad i = 0, 1, 2, \dots n \quad \dots \quad (4)$$

By virtue of (4), let us assume $E(x) = k(x)\psi(x)$ ... (5)

where $\psi(x) = (x - x_0)(x - x_1)\dots(x - x_n)$ ... (6)

and $k(x)$ is to be determined such that (5) holds for any intermediate value of x, say $x = \alpha$, which is different from $x_i (i = 0, 1, 2, \dots, n)$

Hence $k(\alpha) = \dfrac{E(\alpha)}{\psi(\alpha)} = \dfrac{f(\alpha) - p(\alpha)}{\psi(\alpha)}$, by (3) ... (7)

Let us construct a function $F(x)$ such that

$$F(x) = f(x) - p(x) - k(\alpha)\psi(x) \quad \dots \quad (8)$$

Then $F(x_i) = 0$, $i = 0, 1,, 2, \dots n$, by (1) and (6)

Also $F(\alpha) = 0$, by (7) ... (9)

Hence $F(x)$ vanishes at $(n+2)$ number of points in the interval I. Then by repeated application of Rolle's theorem, we have

$$F^{n+1}(\xi) = 0 \text{ where } \xi \in I \quad \dots \quad (10)$$

Since $p(x)$ is a polynomial of degree not greater than n, so we must have

$$p^{n+1}(x) = 0 \quad \dots \quad (11)$$

Also, from (6), we have

$$\psi^{n+1}(x) = (n+1)! \quad \dots$$

$\therefore$ Hence (8) gives

$$F^{n+1}(x) = f^{n+1}(x) - 0 - (n+1)!\,k(\alpha)$$

or, $f^{n+1}(\xi) - (n+1)!\,k(\alpha) = 0$, by (10)

$$\therefore\ k(\alpha) = \frac{f^{n+1}(\xi)}{(n+1)!} \quad \dots \quad (13)$$

$\therefore$ From (7),

$$E(\alpha) = \frac{f^{n+1}(\xi)}{(n+1)!}\psi(\alpha)$$

Since α is an arbitrary value of x, so

$$E(x) = \frac{f^{n+1}(\xi)}{(n+1)!}\psi(x)$$

$$= \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\dots(x - x_n),\ \xi \in I \quad (14)$$

This expression gives the error in polynomial interpolation

3.3. Newton's forward interpolation formula.

[W.B.U.T., CS-312, 2002, 2006, 2013]

Let $y = f(x)$ be a real valued function of x defined in an interval $[a, b]$ and the $(n+1)$ entries $y_i = f(x_i)(i = 0, 1, 2, \dots, n)$ are known for the corresponding $(n+1)$ equispaced arguments $x_i (i = 0, 1, 2, \dots, n)$ such that $x_i = x_0 + ih\ (i = 0, 1, 2, \dots, n)$ with $x_0 = a, x_n = b$ and h is the space length. Let us now construct a polynomial function $p(x)$ of degree not greater than n such that

$$p(x_i) = f(x_i) = y_i \quad (i = 0, 1, 2, \dots, n) \quad \dots \quad (15)$$

Since $p(x)$ is a polynomial of degree $\le n$, so we assume $p(x)$ as

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1}) \quad \dots \quad (16)$$

where the coefficients $a_0, a_1, a_2, \dots, a_n$ are constants to be determined by (15).

Substituting $x = x_0, x_1, x_2, ..., x_n$ successively in (16) and using (15) we obtain

$$p(x_0) = a_0$$

i.e., $a_0 = y_0$,

$$p(x_1) = a_0 + a_1(x_1 - x_0)$$

i.e., $y_1 = y_0 + a_1.h$

$$\therefore\ a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{1!h}$$

$$p(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

or, $y_2 = y_0 + \frac{y_1 - y_0}{h} 2h + a_2.2h.h$

or, $a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2}$

Similarly, we get

$$a_3 = \frac{\Delta^3 y_0}{3!h^3}, ..., a_n = \frac{\Delta^n y_0}{n!h^n}$$

Substituting the above values of a_i's $(i = 0, 1, 2, ..., n)$ in (16) we obtain

$$p(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1)$$
$$+...+\frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)...(x - x_{n-1}) \quad ... \ (17)$$

On introducing the phase $s = \frac{x - x_0}{h}$ and noting that

$$x - x_r = (x_0 + sh) - (x_0 + rh) = (s - r)h, r = 0, 1, 2, ..., n-1 \quad (18)$$

we get

$$f(x) \simeq p(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0$$
$$+....+\frac{s(s-1)...(s-n+1)}{n!}\Delta^n y_0$$

The formula (17) or (19) is known as *Newton's forward interpolation formula.*

Newton's forward interpolation formula with the remainder or error term $E(x)$ can be written as

$$f(x) = p(x) + E(x)$$
$$= y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + ...+$$
$$+ \frac{s(s-1)(s-2)...(s-n+1)}{n!}\Delta^n y_0 + E(x), \quad ... \ (20)$$

where the remainder or error is given by

$$E(x) = (x - x_0)(x - x_1)...(x - x_n)\frac{f^{n+1}(\xi)}{(n+1)!}$$
$$= s(s-1)(s-2)...(s-n)\frac{h^{n+1}f^{n+1}(\xi)}{(n+1)!}, \quad ... \ (21)$$

$$\min\{x, x_0, x_1, ... x_n\} < \xi < \max\{x, x_0, x_1, x_2, ..., x_n\}$$

Note. (i) This formula is used only when the interpolating points are equally spaced.

(ii) The formula is used for interpolating the value of y near the beginning of the set of arguments and for extrapolating the values of y within a short distance backward to the left of y_0.

(iii) For better accuracy, x_0 should be chosen such that $s = \frac{x - x_0}{h}$ is as small as possible.

Example. From the following table, find $f(0.16)$ using Newton's forward interpolation formula :

x	:	0.1	0.2	0.3	0.4
$y = f(x)$	:	1.005	1.020	1.045	1.081

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	1.005			
		0.015		
0.2	1.020		0.010	
		0.025		0.001
0.3	1.045		0.011	
		0.036		
0.4	1.081			

To find $f(0.16)$, we put $x = 0.16$, $x_0 = 0.2$, $h = 0.1$

so that

$$s = \frac{x - x_0}{h} = \frac{0.16 - 0.2}{0.1} = -0.4$$

Then using (19), we get

$$f(0.16) \simeq y_0 + s\ \Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \dots$$

$$= 1.020 + (-0.4) \times 0.025 + \frac{(-0.4)(-0.4-1)}{2!} \times 0.011$$

$$= 1.01308$$

$\therefore\ f(0.16) \simeq 1.013$, correct upto three decimal places.

3.4. Newton's backward interpolation formula.

[W.B.U.T., M(CS)-401, 2013]

Let the values of the function $f(x)$ be given for the corresponding $(n+1)$ equispaced arguments $x_i (i = 0, 1, 2, \dots, n)$, the step length being h, such that $x_i = x_0 + ih\ (i = 0, 1, 2, \dots, n)$ and $y_i = f(x_i) (i = 0, 1, 2, \dots, n)$

Then $x_{n-i} - x_n = x_0 + (n-i)h - x_0 - nh$

$$= -ih \quad (i = 0, 1, 2, \dots, n).$$

Now we consider a polynomial $p(x)$ of degree $\le n$ which replaces $f(x)$ at the interpolating points $x_i (i = 0, 1, 2, \dots, n)$, i.e.,

$$p(x_i) = f(x_i) = y_i \quad (i = 0, 1, 2, \dots, n) \qquad \dots \quad (22)$$

Since $p(x)$ is a polynomial of degree $\le n$, we take

$$p(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \dots + a_n(x - x_n)(x - x_{n-1})\dots(x - x_1) \qquad \dots \quad (23)$$

where the coefficient $a_0; a_1, a_2, \dots, a_n$ are constants to be determined by (22)

Subtituting $x = x_n, x_{n-1}, x_{n-2}, \dots, x_0$ successively in (23) and using (22), we obtain

$$p(x_n) = a_0$$

i.e., $a_0 = y_n$,

$$p(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

i.e., $y_{n-1} = y_n + a_1(-h)$

$$\therefore a_1 = \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_n}{h},$$

$$p(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

i.e., $y_{n-2} = y_n + a_1(-2h) + a_2(-2n)(-h)$

leading to

$$a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2!\,h^2}$$

Proceeding in this way, we get

$$a_3 = \frac{\nabla^3 y_n}{3!\,h^3}, \dots, a_n = \frac{\nabla^n y_n}{n!\,h^n}$$

Substituting the above values of a_i's $(i = 0, 1, 2, \dots, n)$ in (23) we obtain

$$p(x) = y_n + \frac{\nabla y_n}{1!h}(x - x_n) + \frac{\nabla^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1}) + \dots + \frac{(x - x_n)(x - x_{n-1})\dots(x - x_1)}{n!\,h^n}\nabla^n y_n \qquad \dots \quad (24)$$

On introduction of the phase $s = \frac{x - x_n}{h}$ so that

$s + r = \frac{x - x_{n-r}}{h}$ $(r = 0, 1, 2, \dots, n)$ in (24) gives

$$f(x) \simeq p(x) = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \dots + \frac{s(s+1)(s+2)\dots(s+n-1)}{n!}\nabla^n y_n \qquad \dots \quad (25)$$

which is known as *Newton's backward interpolation formula.*

Newton's backward interpolation formula with remainder or error term $E(x)$ can be written as

$$f(x) = p(x) + E(x)$$

$$= y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \dots + \frac{s(s+1)(s+2)\dots(s+n-1)}{n!}\nabla^n y_n + E(x) \qquad \dots \quad (26)$$

where the remainder or error is given by

$$E(x) = (x - x_n)(x - x_{n-1})\ldots(x - x_1)(x - x_0)\frac{f^{n+1}(\xi)}{(n+1)!}$$

$$= s(s+1)\ldots(s+n)h^{n+1}\frac{f^{n+1}(\xi)}{(n+1)!}, \quad \ldots \quad (27)$$

$$\min\{x, x_0, x_1, \ldots, x_n\} < \xi < \max\{x, x_0, x_1, \ldots, x_n\}$$

Note. (i) The formula is used only when interpolating points are equally spaced.

(ii) The formula is used for interpolating the value of y near the end of the given set of arguments and for extrapolating the value of y within a short distance forward to the right of y_n.

(iii) For better accuracy x_n should be chosen such that $s = \frac{x - x_n}{h}$ is as small as possible.

Example. Find $f(2.28)$ from the following table :

x :	2.00	2.10	2.20	2.30
$y = f(x)$:	1.7314	1.7811	1.8219	1.8535

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2.00	1.7314			
		0.0497		
2.10	1.7811		-0.0089	
		0.0408		-0.0003
2.20	1.8219		-0.0092	
		0.0316		
2.30	1.8535			

To find $f(2.28)$, we put $x = 2.28$, $x_n = 2.30$, $h = 0.10$, so that

$$s = \frac{x - x_n}{h} = -0.2$$

Hence using (25) we obtain

$$f(2.28) \simeq y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \ldots$$

$$= 1.8535 + (-0.2) \times 0.0316 + \frac{(-0.2)(-0.2+1)}{2!} \times (-0.0092)$$

$$+ \frac{(-0.2)(-.02+1)(-0.2+2)}{3!} \times (-0.0003)$$

$$= 1.8464504$$

$\therefore$ $f(2.28) \simeq 1.8464$, correct upto four decimal places.

3.5. Lagrange's interpolation formula.

Let $y = f(x)$ be a function of x, continuous and $(n+1)$ times continuously differentiable in $[a, b]$. Let us divide the interval $[a, b]$ by $(n+1)$ points $a = x_0, x_1, \ldots, x_n = b$ which are not necessarily equispaced and the corresponding entries are $y_i = f(x_i)$ $(i = 0, 1, 2, \ldots, n)$. We now wish to find a polynomial $L_n(x)$ in x of degree n such that

$$L_n(x_i) = f(x_i) = y_i (i = 0, 1, 2, \ldots, n) \quad \ldots \quad (28)$$

Since $L_n(x)$ is a polynomial of degree n, so we may take $L_n(x)$ as

$$L_n(x) = a_0(x - x_1)(x - x_2)\ldots(x - x_n) + a_1(x - x_0)(x - x_2)\ldots(x - x_n) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \quad \ldots \quad (29)$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants to be determined by (28).

Putting $x = x_0$ in (29) and using (28), we get

$$L_n(x_0) = a_0(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)$$

$$\therefore \; a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

Now putting $x = x_1$ in (29) and using (28), we get

$$L_n(x_1) = a_1(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)$$

$$\therefore a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}$$

Proceeding in the same way, we have

$$\therefore a_n = \frac{y_n}{(x_n - x_0)(x_n - x_2)\ldots(x_n - x_{n-1})}$$

Substituting the above values of a_i's $(i = 0, 1, 2, \ldots, n)$ in (29), we obtain

$$L_n(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}y_1 + \ldots + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}y_n$$

$$= \sum_{i=0}^{n} l_i(x) y_i \quad \ldots \quad (30)$$

where

$$l_i(x)=\frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} \quad (31)$$

is called the *Lagrangian function*.

Now let us set

$$p_{n+1}(x)=(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_i)(x-x_{i+1})\dots(x-x_n)$$

so that

$$p'_{n+1}(x_i)=(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)$$

Thus we may write (31) in the form

$$l_i(x)=\frac{p_{n+1}(x)}{(x-x_i)p'_{n+1}(x_i)}$$

and therefore, from (30), we have

$$f(x)\simeq L_n(x)=\sum_{i=0}^{n}\frac{p_{n+1}(x)}{(x-x_i)p'_{n+1}(x_i)}y_i \quad \dots (32)$$

which is called *Lagrange's interpolation formula*.

The remainder or error in Lagrange's interpolation formula is given by

$$E(x)=f(x)-L_n(x)$$

$$=\frac{p_{n+1}(x)f^{n+1}(\xi)}{(n+1)!}, \quad \dots (33)$$

$$\min\{x, x_0, x_1, \dots, x_n\}<\xi<\max\{x, x_0, x_1, \dots, x_n\}$$

Note. (1) Some advantage of Lagrange's interpolation are given below :

(i) The formula is applicable to both equispaced and unequispaced interpolating points.

(ii) There is no restriction on the order of the interpolating points $x_0, x_1, x_2, \dots, x_n$.

(iii) The value of x corresponding to which the value of $y=f(x)$ is to be determined may lie anywhere of the tabulated values i.e., x may lie near the begining, end or middle of the tabulated values.

Note. (2) Some disadvantage of Lagrange's interpolation are given below :

(i) For increase of the degree of the interpolating polynomial by adding new interpolating point, the whole calculation would be made afresh.

(ii) The calculations provide no check whether the functional values used are taken correctly or not.

Example. Find the polynomial of degree ≤ 3 passing through the points $(-1, 1)$, $(0, 1)$, $(1, 1)$ and $(2, -3)$.

Solution. Using Lagrange's interpolation formula, we have

$$L_n(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_1-x_2)(x_0-x_3)}\cdot y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\cdot y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\cdot y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\cdot y_3$$

$$=\frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)}.1+\frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}.1$$

$$+\frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)}.1+\frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}(-3)$$

$$=\tfrac{1}{3}\left(-2x^3+2x+3\right).$$

Hence the required polynomial is

$$\tfrac{1}{3}\left(-2x^3+2x+3\right)$$

3.6. Newton's divided difference interpolation.

The Lagrange's interpolation formula has the disadvantage that whenever a new data is added to an existing set, then the interpolating polynomial has to be completely recomputed. In this section, we describe Newton's general interpolation formula based on divided difference to overcome the above diadvantage.

Let $y=f(x)$ be a real valued function defined in $[a, b]$ and known at $(n+1)$ distinct arguments $x_0, x_1, x_2, \dots, x_n$ not in order in any way. We seek a polynomial $p(x)$ of degree not greater than n such that

$$y_i=f(x_i)=p(x_i),\ i=0, 1, 2, \dots, n \quad \dots (34)$$

and $$f(x)=p(x)+R_{n+1}(x), \quad \dots (35)$$

$R_{n+1}(x)$ being the remainder or error in interpolation of $f(x)$.

From the definition of divided difference, we have

$$f[x, x_0] = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f[x, x_0, x_1] = \frac{f[x, x_0] - f[x_0, x_1]}{x - x_1}$$

$$f[x, x_0, x_1, x_2] = \frac{f[x, x_0, x_1] - f[x_0, x_1, x_2]}{x - x_2}$$

... ...

$$f[x, x_0, x_1, x_2, ..., x_n] = \frac{f[x, x_0, x_1, ..., x_{n-1}] - f[x_0, x_1, x_2 \cdots x_n]}{x - x_n}$$

Multiplying the above $(n+1)$ relations successively by

$(x-x_0), (x-x_0)(x-x_1), (x-x_0)(x-x_1)(x-x_2), ...(x-x_0)(x-x_1)...(x-x_n)$

and then adding we get the following identity which holds for all values of x except possibly at $x = x_i$ $(i = 0, 1, 2, ... n)$:

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2] +$$
$$...+(x - x_0)(x - x_1)...(x - x_{n-1})\, f[x_0, x_1, x_2, ..., x_{n-1}]$$
$$+(x - x_0)(x - x_1)...(x - x_n)\, f[x, x_0, x_1, ..., x_n]$$
$$= p(x) + R_{n+1}(x) \quad ... \quad (36)$$

where

$$p(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] +$$
$$...+(x - x_0)(x - x_1)(x - x_2)....(x - x_{n-1})f[x_0, x_1, x_2, ..., x_n]$$

and

$$R_{n+1}(x) = (x - x_0)(x - x_1)....(x - x_n)\, f[x, x_0, x_1, ..., x_n]$$

It can be easily verify that

$$f(x_i) = p(x_i) \ \forall i, \ i = 0, 1, 2, ..., n$$

Also, clearly

$$R_{n+1}(x_i) = 0, \text{ for } i = 0, 1, 2, ..., n$$

Thus $p(x)$ is the required interpolating polynomial

i.e.,

$$f(x) \simeq p(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1) \times$$
$$f[x_0, x_1, x_2] + ... + (x - x_0)(x - x_1)...(x - x_{n-1}) \times$$
$$f[x_0, x_1, x_2, ..., x_n] \quad ... \quad (37)$$

This formula is known as Newton's divided difference interpolation formula with remainder or error as

$$R_{n+1}(x) = (x - x_0)(x - x_1)...(x - x_n)f[x, x_0, x_1, ..., x_n] ... \quad (38)$$

Example. Apply Newton's divided difference formula to find the polynomial of lowest possible degree which satisfies the conditions $f(-1) = 21$, $f(1) = 15$, $f(2) = 12$, $f(3) = 3$

Solution. Let us first construct the following divided difference table :

x	$f(x)$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
–1	–21			
		18		
1	15		–7	
		–3		1
2	12		–3	
		–9		
3	3			

Using the above table, we have from Newton's divided difference formula,

$$f(x) \simeq -21 + (x+1) \times 18 + (x+1)(x-1)(-7) + (x+1)(x-1)(x-2) \times 1$$
$$= x^3 - 9x^2 + 17x + 6 \cdot$$

ILLUSTRATIVE EXAMPLES

Ex.1. Given the following table of function $F(x) = \frac{1}{x}$, find $\frac{1}{2.72}$ using the suitable interpolation formula. Find an estimate of the error

x	:	2.7	2.8	2.9
$F(x)$	:	0.3704	0.3571	0.3448

[W.B.U.T., CS-312, 2008]

Solution. The difference table is

x	$F(x)$	$\Delta F(x)$	$\Delta^2 F(x)$
2.7	0.3704		
		-0.0133	
2.8	0.3571		0.0010
		-0.0123	
2.9	0.3448		

To find $F(2.72)$, we use Newton's forward difference interpolation formula

$$F(x) \simeq y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \dots \quad \dots \quad (1)$$

Here $x_0 = 2.7$, $h = 0.1$

$$\therefore s = \frac{x - x_0}{h} = \frac{2.72 - 2.7}{0.1} = 0.2$$

$\therefore$ From (1), we get

$$F(2.72) \simeq 0.3704 + 0.2 \times (-0.0133) + \frac{0.2(0.2-1)}{2!} \times 0.0010$$

$$= 0.36766$$

Thus $\frac{1}{2.72} \simeq 0.36766$.

So the error is

$$\frac{1}{2.72} - 0.36766 \simeq -1.3 \times 10^{-5}$$

Ex.2. Find the polynomial of the least degree which attains the prescribed values of the given points :

x	:	0	1	2	3
y	:	3	6	11	18

Hence find y for $x = 1.1$

[M.A.K.A.U.T., M(CS)-401, 2014]

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		
3	18			

Here $x_0 = 0$, $h = 1$ so that $s = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$

$\therefore$ From, Newton's forward difference interpolation formula,

$$y \simeq y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \dots,$$

we have

$$y \simeq 3 + x \times 3 + \frac{x(x-1)}{2!} \times 2 + 0$$

$$= x^2 + 2x + 3$$

So the required polynomial is

$$y = x^2 + 2x + 3$$

$$\therefore\ y(1.1) = (1.1)^2 + 2 \times 1.1 + 3 = 6.41$$

Ex.3. What is the lowest degree polynomial which takes the following value?

x	:	0	1	2	3	4	5
$f(x)$	:	1	4	9	16	25	36

[W.B.U.T., CS-312, 2007]

Solution. The forward difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$
0	1		
		3	
1	4		2
		5	
2	9		2
		7	
3	16		2
		9	
4	25		2
		11	
5	36		

Choose $x_0 = 0$

$\therefore\ s = \dfrac{x - x_0}{h} = \dfrac{x-0}{1} = x$

$\therefore$ From Newton's forward difference interpolation formula

$$f(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2}\Delta^2 y_0 + \ldots$$

we have

$$f(x) = 1 + x \times 3 + \frac{x(x-1)}{2} \times 2$$

$$= 1 + 3x + x^2 - x$$

$$= 1 + 2x + x^2$$

Ex.4. Compute the value of $f(3.5)$ and $f(7.5)$ using Newtan's interpolation from the following table:

x	:	3	4	5	6	7	8
$f(x)$	:	27	64	125	216	343	512

[*W.B.U.T.*, *CS*-312, 2008]

Solution. First we construct the difference table as given below:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
3	27			
		37		
4	64		24	
		61		6
5	125		30	
		91		6
6	216		36	
		127		6
7	343		42	
		168		
8	512			

To compute $f(3.5)$, we choose $x_0 = 3$

Here $h = 1$

$\therefore\ s = \dfrac{x - x_0}{h} = \dfrac{3.5 - 3}{1} = 0.5$

$\therefore$ By Newton's forward interpolation formula

$$f(3.5) = 27 + 0.5 \times 37 + \frac{0.5(0.5-1)}{2} \times 24 + \frac{0.5(0.5-1)(0.5-2)}{6} \times 6$$

$$= 42.875$$

To compute $f(7.5)$, we choose $x_n = 7$

$\therefore\ s = \dfrac{x - x_n}{h} = 0.5$

$\therefore$ By Newton's backward difference interpolation formula

$$f(x) = y_n + s\nabla y_n + \frac{s(s+1)}{2}\nabla^2 y_n + \ldots,$$

we get

$$f(7.5) = 343 + 0.5 \times 127 + \frac{0.5(0.5+1)}{2} \times 42 + \frac{0.5(0.5+1)(0.5+2)}{6} \times 6$$

$$= 424.125.$$

Ex.5. If $y(10) = 35.3$, $y(15) = 32.4$, $y(20) = 29.2$, $y(25) = 26.1$, $y(30) = 23.2$, and $y(35) = 20.5$, find $y(12)$ using Newton's forward interpolation formula. [*W.B.U.T.*, *M(CS)*-301, 2009]

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	35.3					
		−2.9				
15	32.4		−0.3			
		−3.2		0.4		
20	29.2		0.1		−0.3	
		−3.1		0.1		0.2
25	26.1		0.2		−0.1	
		−2.9		0.0		
30	23.2		0.2			
		−2.7				
35	20.5					

To find $y(12)$, we choose $x_0 = 10$

Here $n = 12$, $h = 5$

$\therefore\ s = \dfrac{x - x_0}{h} = \dfrac{12-10}{5} = 0.4$

$\therefore$ From Newton's forward difference interpolation formula,

$$y(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \dots,$$

we have

$$y(12) = 35.3 + 0.4(-29) + \frac{0.4(0.4-1)}{2!}\times(-0.3) + \frac{0.4(0.4-1)(0.4-2)}{3!}\times(0.4) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}\times(-0.3) = 34.21408$$

Ex.6. Values of x (in degree) and $\sin x$ are given in the following table :

x (in degree):	15	20	25	30	35	40
$y = f(x)$:	0.2588190	0.3420201	0.4226183	0.5	0.5735764	0.6427876

Determine the value of $\sin 38^0$ by Newton's backward difference interpolation formula. [*W.B.U.T., CS-312, 2010*]

Solution. The difference table :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
15	0.2588190				
		0.0832011			
20	0.3420201		−0.0026029		
		0.0805982		−0.0006136	
					−0.0000248
25	0.4226183		−0.0032165		
				−0.0005888	
		0.0773817			
30	0.5		−0.0038053		−0.0000289
		0.0735764		−0.0005599	
35	0.5735764		−0.0043652		
		0.0692112			
40	0.6427876				

To find $\sin 38^0$, we choose $x_n = 40$

Here $h = 5, x = 38$

$\therefore\ s = \dfrac{x - x_n}{h} = -0.4$

So the Newton's backward difference formula

$$f(x) \simeq y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \dots$$

gives

$$f(38) \simeq 0.6427876 - 0.4\times 0.0692112 + \frac{(-0.4)(-0.4+1)}{2!}(-0.0043652) + \frac{(-0.4)(-0.4+1)(-0.4+2)}{3!}(-0.005599) + \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{4!}(-0.0000289)$$

$$= 0.615662777$$

$\therefore\ \sin 38 \simeq 0.615663$, correct upto six decimal places.

Ex.7. Using approximate formula find $f(0.23)$ and $f(0.29)$ from the following table [*M.A.K.A.U.T., 2013, 2015, 2016*]

x :	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$:	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

Solution. First we construct the difference table as given below:

x	y	Δy	$\Delta^2 y$
0.20	1.6596		
		0.0102	
0.22	1.6698		0.0001
		0.0106	
0.24	1.6804		0.0002
		0.0108	
0.26	1.6912		0.0004
		0.0112	
0.28	1.7024		0.0003
		0.0115	
0.30	1.7139		

Here we apply Newton's backward difference interpolation formula for finding $f(0.29)$,

For that we take $x_n = 0.30$ as $x = 0.29$

$\therefore\ s = \dfrac{x - x_n}{h} = \dfrac{0.29 - 0.30}{0.02} = -0.5$

Then using Newton's backward formula

$$f(x) = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \dots,$$

we get

$$f(0.29) \simeq 1.7139 + (-0.5)\times 0.0115 + \frac{(-0.5)(-0.5+1)}{2!}\times 0.0003$$

$$= 1.70777$$

$$= 1.7078, \text{ correct upto four decimal places.}$$

To find $f(0.23)$, we choose $x = 0.23,\ x_0 = 0.22$

$$\therefore\ h = \frac{x - x_0}{h} = 0.5$$

$\therefore$ By Newton's forward interpolation formula

$$f(x) = y_0 + s\Delta y_0 + s\frac{(s-1)}{2!}\Delta^2 y_0 + \dots$$

we get

$$f(0.23) = 1.6698 + 0.5\times 0.0106 + \frac{0.5(0.5-1)}{2!}\times 0.0001$$

$$= 1.675087$$

$$= 1.6751, \text{ correct upto four decimal places.}$$

Ex.8. The function $y = \sin x$ is tabulated as given below :

x	:	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\sin x$	:	0	0.70711	1.0

Find the value of $\sin\frac{\pi}{3}$ using Lagrange's interpolation formula correct upto 5 places of decimal. [*W.B.U.T., C.S-312, 2004*]

Solution. Using Lagrange's interpolation formula, we obtain

$$L_n(x) \simeq \frac{\left(x-\frac{\pi}{4}\right)\left(x-\frac{\pi}{2}\right)}{\left(0-\frac{\pi}{4}\right)\left(0-\frac{\pi}{2}\right)}\times 0 + \frac{(x-0)\left(x-\frac{\pi}{2}\right)}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}\times 0\cdot 70711$$

$$+ \frac{(x-0)\left(x-\frac{\pi}{4}\right)}{\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}\times 1$$

$$= -x\left(x-\frac{\pi}{2}\right)\cdot\frac{16\times 0.70711}{\pi^2} + \frac{16x}{\pi^2}\left(x-\frac{\pi}{4}\right)$$

$$\therefore\ \sin x \simeq \frac{8x}{\pi^2}(-0.41422x + 0.45711\pi)$$

$$\therefore\ \sin\frac{\pi}{3} \simeq \frac{8.\pi}{3\pi^2}\left(-0.41422\frac{\pi}{3} + 0.45711\pi\right)$$

$$= 0.850764$$

$$\simeq 0.85076, \text{ correct upto 5 places of decimal.}$$

Ex.9. Construct Lagrange's interpolation polynomial by using the following data :

x	:	40	45	50	55
$y = f(x)$	:	15.22	13.99	12.62	11.13

[*W.B.U.T., CS-312,2007*]

Solution. Using Lagrange's interpolation formula, we have

$$L_n(x) = \frac{(x-45)(x-50)(x-55)}{(40-45)(40-50)(40-55)}\times 15.22$$

$$+ \frac{(x-40)(x-50)(x-55)}{(45-40)(45-50)(45-55)}\times 13.99$$

$$+ \frac{(x-40)(x-45)(x-55)}{(50-40)(50-45)(50-55)}\times 12.62$$

$$+ \frac{(x-40)(x-45)(x-50)}{(55-40)(55-45)(55-50)}\times 11.13$$

$$= 2.7\times 10^{-5}x^3 - 6.4\times 10^{-3}x^2 + 15.3\times 10^{-2}x + 17.62$$

Ex.10. Find the polynomial $f(x)$ and hence calculate $f(5.5)$ for the given data :

x	:	0	2	3	5	7
$f(x)$	:	1	47	97	251	477

[*W.B.U.T., CS-312, 2006, 2008, M(CS)-401, 2016, M(CS)-301, 2015*]

Solution. Applying Lagrange's interpolation formula, we have

$$L_n(x)=\frac{(x-2)(x-3)(x-5)(x-7)}{(0-2)(0-3)(0-5)(0-7)}\times 1$$
$$+\frac{(x-0)(x-3)(x-5)(x-7)}{(2-0)(2-3)(2-5)(2-7)}\times 47$$
$$+\frac{(x-0)(x-2)(x-5)(x-7)}{(3-0)(3-2)(3-5)(3-7)}\times 97$$
$$+\frac{(x-0)(x-2)(x-3)(x-7)}{(5-0)(5-2)(5-3)(5-7)}\times 251$$
$$+\frac{(x-0)(x-2)(x-3)(x-5)}{(7-0)(7-2)(7-3)(7-5)}\times 477$$
$$=\frac{(x-2)(x-3)(x-5)(x-7)}{210}+\frac{x(x-3)(x-5)(x-7)}{-30}$$
$$+\frac{x(x-2)(x-5)(x-7)}{24}\times 97+\frac{x(x-2)(x-3)(x-7)}{-60}\times 251$$
$$+\frac{x(x-2)(x-3)(x-5)}{280}\times 477$$
$$\therefore\ f(x)\simeq 9x^2+5x+1$$
$$\therefore\ f(5.5)\simeq 9(5.5)^2+5\times 5.5+1=300.75$$

Ex.11. Use Lagrange's interpolation formula to find the value of $f(x)$ for $x=0$, given

x	:	-1	-2	2	4
$f(x)$	:	-1	-9	11	69

[W.B.U.T.,MCS-301, 2007]

Solution. Applying Lagrange's interpolation formula, we have

$$f(x)=\frac{(x+2)(x-2)(x-4)}{(-1+2)(-1-2)(-1-4)}\times(-1)+\frac{(x+1)(x-2)(x-4)}{(-2+1)(-2-2)(-2-4)}\times(-9)$$
$$+\frac{(x+1)(x+2)(x-4)}{(2+1)(2+2)(2-4)}\times 11+\frac{(x+1)(x+2)(x-2)}{(4+1)(4+2)(4-2)}\times 69$$

$$\therefore\ f(0)=\frac{(0+2)(0-2)(0-4)}{1\times(-3)\times(-5)}\times(-1)+\frac{(0+1)(0-2)(0-4)}{(-1)\times(-4)\times(-6)}\times(-9)$$
$$+\frac{(0+1)(0+2)(0-4)}{3\times 4\times(-2)}\times 11+\frac{(0+1)(0+2)(0-2)}{5\times 6\times 2}\times 69$$
$$=\frac{16}{15}+2+\frac{11}{3}-\frac{23}{5}=1$$

Ex.12. Find Logrange's interpolation polynomial passing through the set of points

x	:	0	1	2
y	:	4	3	6

Use it to find y at $x=1.5$, $\frac{dy}{dx}$ at $x=0.5$ and evaluate $\int_0^3 y\,dx$.

[W.B.U.T.,MCS-301, 2008]

Solution. Using Lagrange's interpolation formula, we have

$$y=f(x)=\frac{(x-1)(x-2)}{(0-1)(0-2)}\times 4+\frac{(x-0)(x-2)}{(1-0)(1-2)}\times 3+\frac{(x-0)(x-1)}{(2-0)(2-1)}\times 6$$
$$=2(x^2-3x+2)-3(x^2-2x)+3(x^2-x)$$
$$=2x^2-3x+4$$
$$\therefore\ y(15)=2(15)^2-3\times 15+4=4$$

Now $\frac{dy}{dx}=4x-3$

$$\therefore\ \left(\frac{dy}{dx}\right)_{x=0.5}=4\times 0.5-3=-1$$

Also $\int_0^3 y\,dx=\int_0^3\left(2x^2-3x+4\right)dx$

$$=\left[\frac{2x^3}{3}-\frac{3x^2}{2}+4x\right]_0^3=16.5.$$

Ex.13. Apply Lagnange's interpolation formula to find $f(x)$, if $f(1)=2,\ f(2)=4,\ f(3)=8,\ f(4)=16$ and $f(7)=128$.

[W.B.U.T., M(CS)-401,2006]

Solution. Apply Lagnange's interpoletion formula, we have

$$f(x)=\frac{(x-2)(x-3)(x-4)(x-7)}{(1-2)(1-3)(1-4)(1-7)}\times 2$$

$$+\frac{(x-1)(x-3)(x-4)(x-7)}{(2-1)(2-3)(2-4)(2-7)}\times 4$$

$$+\frac{(x-1)(x-2)(x-4)(x-7)}{(3-1)(3-2)(3-4)(3-7)}\times 8$$

$$+\frac{(x-1)(x-2)(x-3)(x-7)}{(4-1)(4-2)(4-3)(4-7)}\times 16$$

$$+\frac{(x-1)(x-2)(x-3)(x-4)}{(7-1)(7-2)(7-3)(7-4)}\times 128$$

$$=\frac{1}{18}\left(x^4-16x^3+89x^2-206x+168\right)$$

$$-\frac{2}{5}\left(x^4-15x^3+31x^2-101x+84\right)$$

$$+\left(x^4-14x^3+63x^2-106x+56\right)$$

$$-\frac{8}{9}\left(x^4-13x^3+53x^2-83x+42\right)$$

$$+\frac{16}{45}\left(x^4-10x^3+35x^2-50x+24\right)$$

$$=0.1222x^4-0.8889x^3+20.8778x^2-21.0444x+2.9333$$

Ex.14. Use Newton's divided difference formula to find $f(5)$ from the following data :

x :	0	2	3	4	7	8
$f(x)$:	4	26	58	112	466	668

[W.B.U.T. CS-312, 2009, M.A.K.A.U.T, M(CS)-401, 2016, M(CS)-301, 2014]

Solution. The divided difference table is given below :

x	$f(x)$	1st div.	2nd div.	3rd div.	4th div.
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		21		
		202			
8	668				

Using Newton's divided difference interpolation formula

$$f(x)\simeq f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+\ldots,$$

we get

$$f(5)\simeq 4+(5-0)\times 11+(5-0)(5-2)\times 7+(5-0)(5-2)(5-3)\times 1$$

$$=194.$$

Ex.15. Find the equation of the cubic curve which passes through the points $(4,-43),(7,83),(9,327)$ and $(12,1053)$ Hence find $f(10)$

Solution. Here we use Newton's divided difference formula,

$$f(x)\simeq f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+\ldots. \quad \ldots (1)$$

The divided difference table is

x	$f(x)$	$f[x_0,x_1]$	$f[x_0,x_1,x_2]$	$f[x_0,x_1,x_2,x_3]$
4	−43			
		42		
7	83		16	
		122		1
9	327		24	
		242		
12	1053			

From (1), we get

$$f(x) \simeq -13+(x-4)\times 42+(x-4)(x-7)\times 16+(x-4)(x-7)(x-9)\times 1$$

$$= x^3-4x^2-7x-15$$

$$\therefore f(10)=10^3-4\times 10^2-7\times 10-15$$

$$=515$$

Ex.16. Using Newton's forward formula compute y_{12} given that $y_{10}=600,\ y_{20}=512,\ y_{30}=439,\ y_{40}=346,\ y_{50}=243$

Solution. The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	600				
		−88			
20	512		15		
		−73		−35	
30	439		−20		45
		−93		10	
40	346		−10		
		−103			
50	243				

To find y_{12}, we choose $x_0=10$, so that

$$s=\frac{x-x_0}{h}=\frac{12-10}{10}=0.2$$

Then Newton's forward difference interpolation formula

$$y \simeq y_0+s\Delta y_0+\frac{s(s-1)}{2!}\Delta^2 y_0+\ldots$$

gives

$$y_{12}\simeq 600+0.2\times(-88)+\frac{0.2(0.2-1)}{2!}\times 15+\frac{0.2(0.2-1)(0.2-2)}{3!}\times(-35)$$

$$+\frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!}\times 45$$

$$=578.008$$

$$\therefore\ y_{12}\simeq 578$$

Ex. 17. Find the fourth degree curve $y=f(x)$ passing through the points (2, 3), (4, 43), (7, 778) and (8, 1515) using Newton's divided difference formula.

[W.B.U.T. MCS-401, 2006]

Solution. The divided difference table is

x	$f(x)$	1st order div. diff.	2nd order div. diff.	3rd order div. diff.
2	3			
		20		
4	43		45	
		245		13
7	778		123	
		737		
8	1515			

Using Newton's divided difference interpolation formula

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+\ldots$$

we get

$$f(x)=3+(x-2)\times 20+(x-2)(x-4)\times 45+(x-2)(x-4)(x-7)\times 13$$

$$=3+(20-40)+45\left(x^2-6x+8\right)+19\left(x^3-13x^2+80x-56\right)$$

$$=13x^3-124x^2+400x-405$$

Ex.18. Using Newton divide difference formula find $y(3.4)$:

x :	2.5	2.8	3.0	3.1	3.6
y :	12.1825	16.4446	20.0855	22.1980	36.5982

[W.B.U.T. CS-312, 2007]

Solution. The divided difference table is

x	y	1st order div. diff	2nd order div. diff.	3rd order div. diff.	4th order div. diff
2.5	12.1825				
		14.207			
2.8	16.4446		7.995		
		18.2045		2.9	
3.0	20.0855		9.735		0.8378
		21.125		3.8216	
3.1	22.1980		12.7923		
		28.8004			
3.6	36.5982				

Using Newton's divided difference interpolation formula

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x, x_2) + \dots$$

we get,

$$f(3.4) = 12.1825 + (3.4 - 2.5) \times 14.207 + (3.4 - 2.5)(3.4 - 2.8) \times 7.995$$
$$+ (3.4 - 2.5)(3.4 - 2.8)(3.4 - 3.0) \times 2.9$$
$$+ (3.4 - 2.5)(3.4 - 2.8)(3.4 - 3.0)(3.4 - 3.1) \times 0.8378$$
$$= 29.96679$$

$\therefore\ y(3.4) = 29.9668$, correct upto four decimal places.

Ex.19. Calculate $f(1.135)$ using suitable formula:

x :	1.140	1.145	1.150	1.155	1.160	1.165
y :	0.131030	0.13541	0.13976	0.14410	0.14842	0.15272

[W.B.U.T. CS-312, 2007]

Solution. The difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1.140	0.13103	0.00438	−0.00003	0.00002
1.145	0.13541	0.00435	−0.00001	−0.00001
1.150	0.13976	0.00434	−0.00002	0
1.155	0.14410	0.00432	−0.00002	
1.160	0.14842	0.00430		
1.165	0.15272			

To find $f(1.135)$, we choose $x_0 = 1.140$.

Here $h = 0.005$, $x = 1.135$

$$\therefore\ s = \frac{x - x_0}{x} = -1$$

$\therefore$ Using Newton's forward differene interpolation formula

$$f(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \dots$$

we get

$$f(1.135) = 0.13103 + (-1) \times 0.00438 + \frac{(-1)(-1-1)}{2} \times (-0.00003) + \frac{(-1)(-1-1)(-1-2)}{6} \times 0.00002$$
$$= 0.1266$$

Exercise

I. SHORT ANSWER QUESTIONS

1. Fit a polynomial of degree three which takes the following values

x	:	3	4	5	6
y	:	6	24	60	120

Hence find $y(1)$.

2. Find $f(0.3)$ where $f(x) = 5^x$, taking 0 and 1 as interpolating points by the methods of interpolation.

3. Using Newton's forward interpolation formula find the polynomial of degree 3 passing through the points (–1,1), (0,1), (1, 1) and (2, –3).

4. Find the forward interpolation polynomial for the function $f(x)$ where $f(0) = -1, f(1) = 1, f(2) = 1$ and $f(3) = -2$.

5. Find Newton's forward interpolation polynomial of the function $f(x)$ when $f(0) = 1, f(1) = 2, f(2) = 1$ and $f(3) = 10$

6. Using appropriate interpolation formula, find the value of $f(5)$ from the following data :

x	:	3	4	6	8
$f(x)$	:	4.5	13.2	43.7	56.4

7. Find $f(1.02)$ having given

x	:	1.00	1.10	1.20	1.30
$f(x)$	:	0.8415	0.8912	0.9320	0.9636

8. Evaluate $f(1)$ from the following values of x and $f(x)$:

x	:	0	2	4	6
$f(x)$	:	2	6	10	15

9. Using appropriate interpolation formula, find the value of the function $f(x)$ when $x = 7$ from the following data.

x	:2	4	6	8
$f(x)$	: 15	28	56	89

10. Find Newton's backward difference interpolating polynomial against the tabulated values :

x	:	3	4	5	6
y	:	6	24	60	120

11. Find the value of y when $x = 19$; given

x	:	0	1	20
y	:	0	1	2

12. Compute $f(21)$ using the following datas :

x	:	0	5	10	20
$f(x)$	:	1.0	1.6	3.8	15.4

13. Use Lagrange's interpolation formula to find the value of $f(x)$ for $x = 0$, given the following table :

x	:	–1	–2	2	4
$f(x)$	:	–1	–9	11	69

[W.B.U.T., CS-312, 2007]

14. $f(x)$ is a function defined on [0, 1] having values 0, –1 and 0 at $x = 0, \frac{1}{2}$ and 1. Find the two degree polynomial $\phi(x) \simeq f(x)$ such that $\phi(0) = f(0)$, $\phi\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right)$ and $\phi(1) = f(1)$

15. Find the Lagrangian polynomial for the following tabulated value :

x	:	0	1	3
y	:	0	3	1

16. Find Lagrange's interpolation polynomial for the function $f(x)$ when $f(0) = 4, f(1) = 3, f(2) = 6$

17. Find Lagrange's interpolation polynomial for the function $f(x) = \sin x\pi$ when $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{1}{2}$. Also compute the value of $\sin\pi/3$.

18. Find the parabola passing through the points (0, 1), (1, 3) and (3, 55) using Newton's divide interpolation formula.

19. Using Newton's divide interpolation formula, find $p(4)$ when $p(1) = 10$ $p(2) = 15$ and $p(5) = 42$.

20. Given

x	:	1	2	5
$f(x)$	:	10	15	42

Find $f(4)$

Answers

1. $x^3 - 3x^2 + 2x, 0$ 2. 2.2 3. $1 - \frac{2}{3}x(x^2 - 1)$

4. $-\frac{1}{6}(x^3 + 3x^2 - 16x + 6)$ 5. $2x^3 - 7x^2 + 6x + 1$

6. 26.7 7. 0.8521 8. 4.0625 9. 72.5

10. $x^3 - 3x^2 + 2x$ 12. 17.23 14. $\phi(x) = 4x^2 - 4x$

15. $-\frac{4}{3}x^2 + \frac{13}{3}x$ 16. $2x^2 - 3x + 4$ 17. $-3x^2 + \frac{7}{2}x$, 0.8333

18. $y = 8x^2 - 6x + 1$ 19. 31 20. 31

II. Long Answer Questions

1. If $y(10) = 35.3, y(15) = 32.4, y(20) = 29.2, y(25) = 26.1$, $y(30) = 23.2$ and $y(35) = 20.5$, find $y(12)$ using Newton's forward interpolation formula.

[W.B.U.T., CS-312, 2010]

2. Find $f(2.5)$ using Newton's forward difference formula for the given data:

x	:	1	2	3	4	5	6
$y = f(x)$	:	0	1	8	27	64	125

3. A function $y = f(x)$ is given by the following table. F $f(0.2)$ by a suitable formula.

x :	0	1	2	3	4	5	6
$y = f(x)$:	176	185	194	203	212	220	229

4. Find the value of $\sqrt{5}$ correct upto four significant figu from the following table:

x :	1.9	2.1	2.3	2.5	2.7
$\sqrt{x}$:	1.3784	1.4491	1.5166	1.5811	1.6432

5. Calculate $f(1.135)$ using suitable formula

x :	1.140	1.145	1.150	1.155	1.160	1.165
$f(x)$:	0.13103	0.13541	0.13976	0.14410	1.14842	0.1527

[W.B.U.T., CS-312, 2007]

6. Compute $y(0.5)$ using the following table:

x:	0	1	2	3	4	5
y:	5.2	8.0	10.4	12.4	14.0	15.2

7. Determine the polynomial of degree 3 from the following table

x:	0	1	2	3	4	5
y:	-3	-5	-11	-15	-11	-7

8. Find the equation of the cubic curve that passes through the points $(0, -5)$, $(1, -10)$, $(2, -9)$, $(3, 4)$ and $(4, 35)$.

[W.B.U.T., CS-312, 2004]

9. Compute the values of $f(3.5)$ and $f(7.5)$ using Newton's interpolation from the following table:

x :	3	4	5	6	7	8
$f(x)$:	27	64	125	216	343	512

[W.B.U.T., CS-312, 2008]

10. The values of $y = \sin x$ are given below for different values of x. Find the values of y for (i) $x = 32^0$, (ii) $x = 52^0$.

x :	30^0	35^0	40^0	45^0	50^0	55^0
$y = \sin x$:	0.5000	0.5735	0.6428	0.7071	0.7660	0.8192

11. Using appropriate formula find $f(0.29)$ from the following table:

x :	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$:	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

12. The population of a city for five census are given below:

year :	1941	1951	1961	1971	1981	1991
population : (in lacs)	46.52	66.23	81.01	93.70	101.58	120.92

Using suitable formula estimate the population of the city for the year 1985

13. Apply Lagrange's interpolation formula to find $f(x)$, if $f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(4) = 16$ and $f(7) = 128$.

[W.B.U.T., CS-312, 2006, 2010]

14. Use Lagrange's interpolation formula to fit a polynomial to the following data. Hence find $y(4)$

x :	-1	0	2	3
y :	-8	3	1	2

15. Find the fourth degree curve $y = f(x)$ passing through the points $(2, 3)$, $(4, 43)$, $(7, 778)$ and $(8, 1515)$, using Newton's divided difference formula. [W.B.U.T., CS-312, 2006]

16. Using Newton's divided difference formula to find $y(3.4)$

x :	2.5	2.8	3.0	3.1	3.6
y :	12.1825	16.4446	20.0855	22.1980	36.5982

[W.B.U.T., CS-312, 2007]

17. Using divided difference interpolation formula, compute $f(27)$ from the following data:

x :	14	17	31	35
$f(x)$:	68.7	64.0	44.0	39.1

18. Find $f(8)$ using Newton's divided difference formula given that

x:	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

19. Use Newton's divided difference formula to approximate $f(0.05)$ from the following table

x:	0.0	0.2	0.4	0.6	0.8
$f(x)$:	1.0000	1.22140	1.49182	1.82212	2.22554

[W.B.U.T., CS-312, 2007]

20. Find the values of (i) $\log_{10}(11.1)$ and $\log_{10}(17.8)$ from the following table

x:	11	12	13	14	15	16	17
$\log_{10} x$:	1.0414	1.0792	1.1139	1.1461	1.1761	1.2041	1.2304

Answers

1. 8.8345 2. 3.4 3. 177.67 4. 1.414 5. 6.65

8. x^3-5x^2+2x-3 10. 0.5299, 0.7888 11. 1.708

12. 107.03 14. $\frac{1}{6}(7x^3-31x^2+28x+18)$, 13.66

15. $13x^3-124x^2+400x-405$ 17. 49.3 18. 448

20. 1.0453, 1.2504

III. Multiple Choice Questions

1. In Newton's forward interpolation, the interval should be
   (a) equally spaced (b) not equally spaced
   (c) may be equally spaced (d) both (a) and (b)
   [W.B.U.T., CS-312, 2008, MCS-401,2014]

2. Newton's forward interpolation formula is used to interpolate
   (a) near end (b) near central position
   (c) near beginning (d) none of these

3. The coefficient of Newton's forward difference interpolation formula are
   (a) $\frac{s(s-1)\dots(s-n+1)}{n!}$ (b) $\frac{s(s+1)\dots(s+n-1)}{n!}$
   (c) $\frac{s(s-1)\dots(s-n+1)}{(n-1)!}$ (d) none of these
   [where $s=\frac{x-x_0}{h}$]

4. In Newton's forward difference interpolation, the value of $s=\frac{x-x_0}{h}$ lies between
   (a) 1 and 2 (b) −1 and 1 (c) 0 and ∞ (d) 0 and 1
   [M.A.K.A.U.T. M(CS)-301, 2015, M(CS)-401, 2016]

5. Newton's backward interpolation formula is used to interpolate
   (a) near end (b) near central position
   (c) near the beginning (d) none of these

6. The restriction on the interpolating points for Newton's forward and backward formulae is
   (a) should not be so large
   (b) should be in arithmetic progression
   (c) should be in geometric progression
   (d) should be in positive

7. The coefficient of Newton's backward difference interpolation formula are
   (a) $\frac{u(u-1)\dots(u-n+1)}{n!}$ (b) $\frac{u(u+1)\dots(u+n-1)}{n!}$
   (c) $\frac{u(u-1)\dots(u-n+1)}{(n-1)!}$ (d) none of these
   [where $u=\frac{x-x_0}{h}$]

8. In Newton's backward difference interpolation formula, the value of $s=\frac{x-x_n}{h}$ should lie between
   (a) 0 and 1 (b) 0 and ∞
   (c) greater than 1 (d) no restriction

4 Numerical Integration

4.1 Introduction:

In this chapter we derive and analyse numerical methods to evaluate definite integrals of the form

$$I = \int_a^b f(x)dx$$

for any finite interval $[a, b]$ by replacing the function $f(x)$ with a suitable polynomial $p(x)$ such that $\int_a^b p(x)dx$ is taken to be an approximation of the integral I. The approximation of I is usually known as *numerical integration or quadrature.*

[*W.B.U.T., CS-312, 2006*]

Let $y = f(x)$ be a real valued function defined in $[a, b]$ such that the values of $f(x)$ are known for $x = x_i$ $(i = 0, 1, 2, \dots n)$ whose all x_i lies in $[a, b]$ and $y_i = f(x_i)$, $(i = 0, 1, 2, \dots n)$. Also let $p(x)$ be the interpolating polynomial of degree at most n such that

$$p(x_i) = f(x_i) = y_i \qquad \dots \quad (1)$$

Thus $\quad p(x) \simeq f(x)$ and so

$$I = \int_a^b f(x)dx \simeq \int_a^b p(x)dx \qquad \dots \quad (2)$$

Then the expression

$$E(x) = \int_a^b f(x)dx - \int_a^b p(x)dx \qquad \dots \quad (3)$$

is known as the error of integration.

4.2. The Important Concepts.

(a) Degree of precision.

A quadrature formula is said to have a degree of precision m (m being a positive integer) if it is exact i.e. the error is zero for an arbitrary polynomial of degree $m \le n$ but there exists a so that

$$I = \int_{x_0}^{x_n} f(x)dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{2n^3 - 3n^2}{12}\Delta^2 y_0 + \frac{n^4 - 4n^3 + 4n^2}{24}\Delta^3 y_0 + \ldots\right] \quad \ldots (5)$$

The formula (5) is known as general integration formula when the interval of integration is divided into n equal sub-intervals.

We can derive some integration formulae from (5) as particular cases by putting $n = 1, 2, 3, \ldots$

4.4. Trapezoidal rule.

Putting $n = 1$ in (5), we obtain

$$\int_{x_0}^{x_1} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{1}{2}h(y_0 + y_1) \quad \ldots (6)$$

This is called Trapezoidal rule for numerical integration for two points.

Similarly we have

$$\int_{x_1}^{x_2} f(x)dx \simeq \frac{1}{2}h(y_1 + y_2)$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\int_{x_{n-1}}^{x_n} f(x)dx \simeq \frac{1}{2}h(y_{n-1} + y_n)$$

Adding the above integrals, we obtain

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})\right] \quad \ldots (7)$$

which is known as composite Trapezoidal rule.

Error in Trapezoidal rule

The error committed in (6) is given by

$$E = \int_{x_0}^{x_1} f(x)dx - \frac{1}{2}h(y_0 + y_1)$$

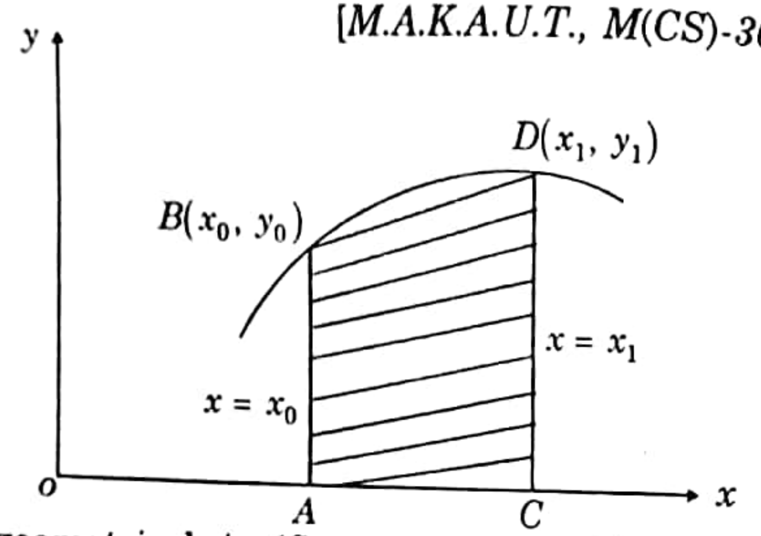
$$= -\frac{1}{12}h^3 f''(x_0) \quad \cdots \quad (8)$$

The total error committed in composite Trapezoidal rule (7) is

$$E_T = -\frac{1}{12}h^3 n f''(\xi), x_0 < \xi < x_n \quad \cdots \quad (9)$$

Geometrical Significance of Trapezoidal rule.

[*M.A.K.A.U.T., M(CS)-301, 2014*]



The geometrical significance of Trapezoidal rule li...

(ii) Since the error involves fourth order derivatives of $f(x)$, the Simpson's one third rule yields an exact value of the integral if $f(x)$ is a polynomial of degree less than or equal to three. Hence the degree of precision of this formula is three.

4.6. Weddle's rule

Putting $n = 6$ in (5), we obtain

$$\int_{x_0}^{x_6} f(x)dx = 6h\left[y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{41}{20}\Delta^4 y_0 + \frac{11}{20}\Delta^5 y_0 + \frac{41}{840}\Delta^6 y_0\right]$$

$$= h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{3}{10}\Delta^6 y_0\right]$$

$$- \frac{h}{140}\Delta^6 y_0$$

If we now choose h in such a way that the sixth order difference are very small, then we may neglect the small term $\frac{h}{140}\Delta^6 y_0$.

Thus we have

$$\int_{x_0}^{x_6} f(x)dx \cong \frac{3h}{10}\left[20y_0 + 60(y_1 - y_0) + 90(y_2 - 2y_1 + y_0)\right.$$
$$+ 80(y_3 - 3y_2 + 3y_1 - y_0) + 41(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)$$
$$+ 11(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)$$
$$\left. + (y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0)\right]$$

so that

$$\int_{x_0}^{x_6} f(x)dx \cong \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right] \quad \ldots \quad (14)$$

This is called Weddle's rule for numerical integration.

Note. **(i)** The Weddle's rule is applicable only when the number of sub-intervals is multiple of six.

(ii) Since, the error involves sixth order derivatives of $f(x)$, the Weddle's rule yields an exact value of the integral if $f(x)$ is a polynomial of degree less than or equal to five. Hence the degree of precision of this formula is five.

Example.1. Find the approximate value of $I=\int\frac{dx}{1+x}$ when the interval is $(0,1)$ and $h=\frac{1}{2}$. Use Trapezaidal rule. [W.B.U.T.,MCS-301, 2009]

Solution. Here $f(x)=\frac{1}{1+x}$, $h=\frac{1}{2}=0.5$

$$\therefore\ n=\frac{1-0}{0.5}=2$$

The different values of x and $f(x)$ are given below :

x	:	0	0.5	1
$f(x)$	:	1	0.6667	0.5

$\therefore$ By Trapezoidal rule, we get

$$\int_0^1\frac{dx}{1+x}=\frac{h}{2}(y_0+2y_1+y_2)$$

$$=\frac{0.5}{2}[1+2\times0.6667+0.5]$$

$$=0.70835$$

Example.2. Calculate the area of the function $f(x)=\sin x$ with limits $(0, 90^\circ)$ by simpson's $\frac{1}{3}rd$ rule using 11 ordinates.

[W.B.U.T., MCS-301, 2008]

Solution. Here $f(x)=\sin x$, $n=11-1=10$

$$\therefore\ h=\frac{\frac{\pi}{2}-0}{10}=\frac{\pi}{20}$$