CHAPTER

4

Magnetostatics

4.1 INTRODUCTION

Moving charges, or current, are the sources of magnetic fields in the same way as static charges are the sources of electric fields. By using Biot–Savart law and Ampere's law we can calculate magnetic fields due to different current distribution. The magnetic field, like the electric field, is a vector field.

4.2 ELECTRIC CURRENT

Electric charge in motion produces electric current, and the current-carrying medium may be called a conductor. Electric current is simply a flow of charge. If a charge ΔQ crosses an area in time Δt, then average electric current

$$I_{av} = \frac{\Delta Q}{\Delta t}$$

The current at time t is

$$I = \underset{\Delta t \to 0}{\text{Lt}} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt} \text{ coulomb/s} \qquad ...(4.1)$$

If one coulomb of charge crosses an area in one second, the current is one ampere. The SI unit of current is ampere (A).

4.3 CURRENT DENSITY

A charged particle placed is an electric field $\vec{E}$ experiences a force $\vec{F}$. If the electric field $\vec{E}$ is constant, then the particle will have an average velocity and the average velocity of a charged particle is called drift velocity, $\vec{v}_d$.

Now we define a vector quantity known as *electric current density at a point*. To define current density, we consider a medium of uniform area of cross section S and volume charge density ρ. Then current I at a given point becomes

$$I = v_d \rho S$$

For uniformly distributed current, the magnitude of current density ...(4.2)

$$J = \frac{I}{S} = v_d \rho$$

But if current density is not uniform, then we define it as ...(4.3)

$$J = \underset{\Delta S \to 0}{\text{Lt}} \frac{\Delta I}{\Delta S}$$

Surface ΔS is normal to current direction. The total current through the entire surface S is

$$I = \int_S \vec{J} \cdot \vec{dS}$$

For many materials, it is found that the current density in the steady state is linearly proportional to the applied electric field intensity. Therefore ...(4.4)

$$\vec{J} \propto \vec{E} \quad \text{or,} \quad \vec{J} = \sigma \vec{E}\,*$$

The constant of proportionality is known as the conductivity of the medium at a given temperature. ...(4.5)

The drift velocity is directed along the direction of electric field and is related to by a constant called the mobility μ,

$$\vec{v}_d = \mu \vec{E}$$

Mobility (μ) is defined as the drift velocity per unit electric field. ...(4.6)

4.4 EQUATION OF CONTINUITY FOR CURRENT

Let us consider a volume V of the conductor enclosed by a surface S. If ρ is the volume charge density then the total charge (Q) within the volume is given by

$$Q = \int_V \rho\, dV$$

From conservation of charge (charge can neither be created nor destroyed), the amount of incoming flow of charge $\left(I = \oint_S \vec{J} \cdot \vec{dS}\right)$ must be equal to the rate of decrease of the total charge $\left(-\frac{dQ}{dt}\right)$ inside the volume.

i.e.

$$\oint_S \vec{J} \cdot \vec{dS} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho\, dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

By applying Gauss' divergence theorem

$$\int_V (\vec{\nabla} \cdot \vec{J})\, dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

Therefore,

$$\int_V \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}\right) dV = 0$$

[***Note**: From Eqs (4.2) and (4.6), we have $I = v_d \rho S = \mu \rho S E$

So, current density $J = \frac{I}{S} = \mu \rho E = \sigma E$ where $\sigma = \rho \mu$ is called electrical conductivity.

If $\rho = ne$ then $\sigma = ne\mu$ where n is the number of electrons per unit volume.]

For any arbitrary volume V, the integral must be zero

So, $$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \ldots(4.7)$$

This is known as the ***equation of continuity*** and represents the mathematical statement of local charge conservation.

If the region does not contain a source or sink of charge then $\frac{\partial \rho}{\partial t} = 0$ [for steady current] and Eq. (4.7) reduces to

$$\vec{\nabla} \cdot \vec{J} = 0 \quad \ldots(4.8)$$

Equation (4.7) represents the condition of steady current flow.

4.5 FORCE ON A MOVING CHARGE IN A STATIC MAGNETIC FIELD

If a charged particle moves across a magnetic field, it is accelerated at right angles to its direction of motion.

The particle experiences a force at right angles to its velocity, with a magnitude proportional to the component of velocity, charge and magnetic field.

So, we can write the infinitesimal magnetic force $\overrightarrow{dF}$ on an infinitesimal charge dq moving with a velocity $\vec{v}$ in a steady magnetic field as

$$d\vec{F} = dq\,(\vec{v} \times \vec{B}), \quad \ldots(4.9)$$

Since the electric force on an infinitesimal charge dq in an electric field is $dq\,\vec{E}$, so the total electromagnetic force on an infinitesimal charge is

$$d\vec{F} = dq\,(\vec{E} + \vec{v} \times \vec{B}) \quad \ldots(4.10)$$

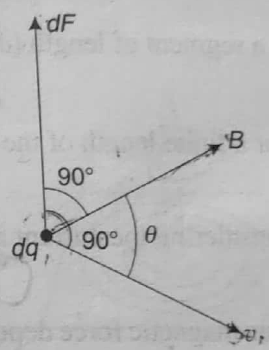


Fig. 4.1 Force on a moving charge in a constant magnetic field.

This is known as ***Lorentz force***.

Now for a single particle of charge e, the Lorentz force will be

$$\vec{F} = e\,(\vec{E} + \vec{v} \times \vec{B}) \quad \ldots(4.11)$$

In the absence of an electric field ($\vec{E}$), Lorentz force (magnetic force) for a single particle of charge e is

$$\vec{F} = e\,(\vec{v} \times \vec{B}) \quad \ldots(4.12)$$

The magnitude of the Lorentz force is

$$F = evB \sin\theta \quad \ldots(4.13)$$

where θ is the angle between $\vec{v}$ and $\vec{B}$ [Fig. 4.1].

No workforce If infinitesimal charge dq moves through a small amount dl then $\vec{dl} = \vec{v}dt$, the work done is [from Eq. (4.9)]

$$dW = d\vec{F} \cdot \vec{dl} = dq\,(\vec{v} \times \vec{B}) \cdot \vec{dl} = dq\,(\vec{v} \times \vec{B}) \cdot \vec{v}dt$$
$$= 0$$

Since, $\vec{v} \times \vec{B}$ is perpendicular to $\vec{v}$, so $(\vec{v} \times \vec{B}) \cdot \vec{v} = 0$

So, magnetic force does no work on a charged particle to move with a velocity v in a static magnetic field $\vec{B}$.

4.6 FORCE ON CURRENT ELEMENT PLACED IN A STATIC MAGNETIC FIELD

We know from Lorentz force [Eq. (4.9)] that

$$d\vec{F} = dq\,(\vec{v} \times \vec{B})$$

where $\overrightarrow{dF}$ is the infinitesimal force on an infinitesimal charge dq moving with a velocity v in a steady magnetic field B [Fig. 4.2]. Again, if dq is the amount of charge flow through the cross section of a conductor in time dt, then electric current

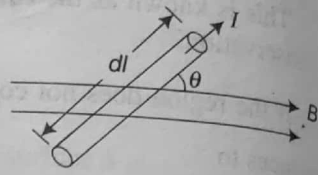


Fig. 4.2 Force on a current element which is in static magnetic field.

$$I = \frac{dq}{dt}$$

So,

$$d\vec{F} = I\,dt\,(\vec{v} \times \vec{B})$$
$$= I\,(dt\,\vec{v} \times \vec{B})$$

$v\,dt$, a segment of length (dl) gives the indication of the distance travelled by a particle in time dt, then

$$d\vec{F} = I\,(\vec{dl} \times \vec{B}) \quad \text{...(4.14)}$$

For a finite length of the conductor, the magnetic force

$$\vec{F} = I \int (\vec{dl} \times \vec{B}) \quad \text{...(4.15)}$$

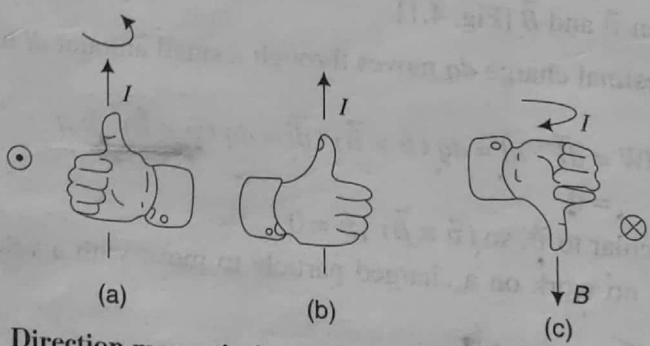
Considering the current as the vector along the length dl, the magnetic force per unit length

$$\vec{F} = \vec{I} \times \vec{B} \quad \text{...(4.16)}$$

Thus magnetic force depends only on the total current and applied magnetic field and is independent of the amount of charge carried by each particle. The direction of current is perpendicular to the plane containing $\vec{B}$ and $\vec{I}$.

4.7 BIOT–SAVART LAW

Steady currents produce magnetic fields which are constant in time. The right-hand thumb rule, Fig. 4.3(a, b) gives the direction of the magnetic field. According *to* **the thumb rule, if the current flows in the thumb's direction, right-handed fingers curl around in the direction of the magnetic field.** The symbol ⊙ gives the direction of the magnetic field perpendicular to the plane of the paper and ⊗ gives the direction of the magnetic field into the plane of the paper. If the fingers are curled along the current, the stretched thumb will point towards the magnetic field [Fig. 4.3(c)].



(a) (b) (c)

Fig. 4.3 **Direction magnetic field by using the right-hand thumb rule.**

The Biot–Savart law states that the magnetic field $\overrightarrow{dB}$ due to a current element $I\,\overrightarrow{dl}$ [Fig. 4.4(a, b)] is given by

$$\overrightarrow{dB} = \frac{\mu_0}{4\pi}\frac{I\overrightarrow{dl} \times \hat{r}}{r^2} \qquad \text{...(4.17)}$$

where $\hat{r}$ is the unit vector from the point of interest $I\,\overrightarrow{dl}$ towards the point of interest and r is the distance between the current element $I\,\overrightarrow{dl}$ and the point of observation.

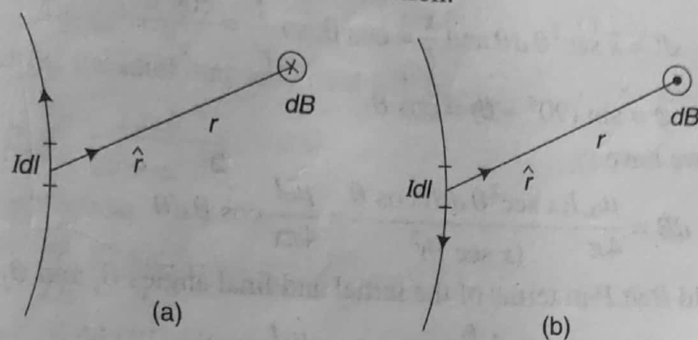


Fig. 4.4 Biot–Savart law for current element *Idl*.

The total field B due to the whole conductor can be obtained after taking the integration

$$\vec{B} = \frac{\mu_0}{4\pi}\int\frac{I\,\overrightarrow{dl}\times\hat{r}}{r^2} \qquad \text{...(4.18)}$$

or,

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\times\hat{r}}{r^2}\,dV \qquad \text{...(4.19)}$$

where $I\,\overrightarrow{dl} = \vec{J}\,dV$ 10^{-7}

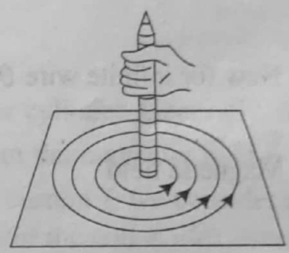


Fig. 4.5 Maxwell's cork-screw rule.

The constant μ_0 is called the permeability of free space and its value

$$\mu_0 = 4\pi \times 10^{-7}\ \text{N/A}^2.$$

Equation (4.18) is the integral form of the ***Biot–Savart law***.

The direction of magnetic field can also be obtained by **Maxwell's cork-screw** rule. Maxwell's cork-screw rule points that **if the direction of the current through a conductor is represented by the linear motion of the cork-screw motion then the direction of the magnetic field can be represented by the direction of rotation of the cork** [Fig. 4.5].

4.8 APPLICATIONS OF BIOT–SAVART LAW

(i) Magnetic field due to a long straight wire

In the diagram [Fig. 4.6] $(\overrightarrow{dl} \times \hat{r})$ points into (X) the paper. From Biot–Savart law, dB at P is

$$\overrightarrow{dB} = \frac{\mu_0}{4\pi}\frac{I\overrightarrow{dl}\times\hat{r}}{r^2}$$

where Idl is the small current element at a distance l from O. The magnitude of the magnetic field dB at the point P at a distance x from the wire due to the current element Idl is

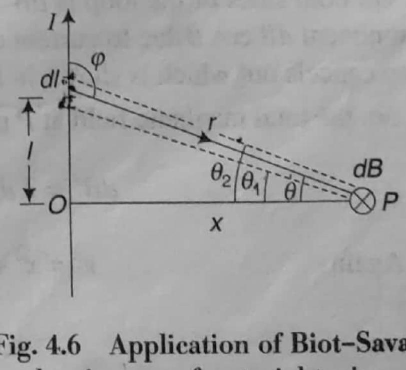


Fig. 4.6 Application of Biot–Savart law in case of a straight wire.

$$dB = \frac{\mu_0}{4\pi}\left|\frac{I\,\vec{dl}\times\hat{r}}{r^2}\right|$$

$$= \frac{\mu_0}{4\pi}\frac{I\,dl\sin\varphi}{r^2} \qquad \ldots(4.20)$$

Also,
$$l = x\tan\theta$$

$$dl = x\sec^2\theta\, d\theta \text{ and } \frac{x}{r} = \cos\theta, \text{ so } \frac{1}{r^2} = \frac{\cos^2\theta}{x^2}.$$

and
$$\sin\varphi = \sin(90° - \theta) = \cos\theta$$

Thus, from Eq. (4.20), we have

$$dB = \frac{\mu_0}{4\pi}\frac{I(x\sec^2\theta\, d\theta)\cos\theta}{(x\sec\theta)^2} = \frac{\mu_0 I}{4\pi x}\cos\theta\, d\theta$$

Now, total magnetic field B at P in terms of the initial and final angles θ_1 and θ_2 is

$$\therefore \quad B = \int dB = \frac{\mu_0 I}{4\pi x}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi x}(\sin\theta_2 - \sin\theta_1) \qquad \ldots(4.21)$$

Now for infinite wire $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$.

Magnetic field
$$B = \frac{\mu_0 I}{4\pi x}\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right)$$

$$= \frac{\mu_0}{4\pi}\left(\frac{2I}{x}\right) \qquad \ldots(4.22)$$

Equation (4.21) shows that magnetic field due to a straight infinite wire is inversely proportional to the distance from the wire.

(ii) Magnetic field at a point on the axis of a circular loop

Here, we consider the center of the loop to be at the origin and its axis is along the x direction [Fig. 4.7].

Now according to Biot–Savart law, the magnetic field at P due to the current element Idl of the loop is given by

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}$$

The total magnetic field at P is due to the current element $I\,dl$. On both sides of the loop is $dB' = 2\,dB\sin\theta$. Perpendicular component $dB\cos\theta$ due to current elements of both sides of the loop cancels out which is shown in Fig. 4.7.

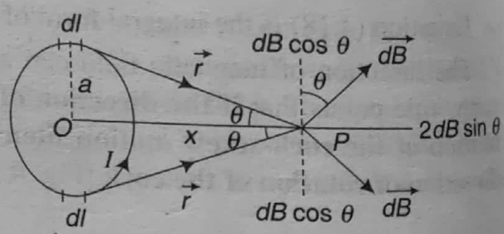


Fig. 4.7 Magnetic field on at a point on the axis of a circular current loop.

So, the total magnetic field at P due to current element Idl on both sides is

$$dB' = 2\,dB\sin\theta = 2\times\frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}\sin\theta$$

Again
$$r^2 = x^2 + a^2 \quad \text{and} \quad \sin\theta = \frac{a}{r} = \frac{a}{(x^2+a^2)^{1/2}}$$

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So, $$dB' = 2\frac{\mu_0}{4\pi}\frac{I\,dl\,a}{(x^2+a^2)^{3/2}}$$

Hence, total magnetic field at P is

$$B' = \int dB' = \frac{2\mu_0}{4\pi}\frac{Ia}{(x^2+a^2)^{3/2}}\int_0^{\pi a} dl$$

$$= \frac{\mu_0}{4\pi}\frac{2\pi I a^2}{(x^2+a^2)^{3/2}} \qquad \text{...(4.23)}$$

Now for n number of turns, the total magnetic field will be

$$B' = \frac{\mu_0}{4\pi}\frac{2\pi n I a^2}{(x^2+a^2)^{3/2}} \qquad \text{...(4.24)}$$

At the center of the loop $x = 0$ so,

$$B'_{\max} = \frac{\mu_0}{4\pi}\left(\frac{2\pi n I}{a}\right) \qquad \text{...(4.25)}$$

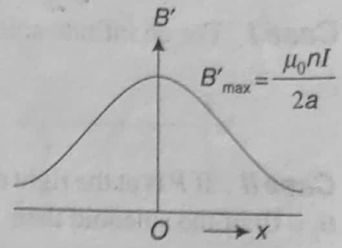

Fig. 4.8 Variation of magnetic field on the axis of a circular were.

The variation of magnetic field (B') with the axis of the coil is shown in Fig. 4.8. The graph shows that the magnetic field is maximum at the center of the coil.

(iii) Magnetic field along the axis of a solenoid

A solenoid is a wire wound closely in the form of a helix around a right circular cylinder. Generally, the length of the solenoid is large as compared to the transverse dimension. To find out the magnetic field $\vec{B}$ at an axial point P at a distance l from O of the solenoid of radius a and carrying a current I, we consider an elementary length dx at a distance x from O [Fig. 4.9]. The current in the section dx of the coil is $nIdx$, where n is the number of turns (N) per unit length, i.e., $\frac{N}{L}$.

The field at P due to the element dx is

$$dB = \frac{\mu_0 (ndx)\, Ia^2}{2[(l-x)^2+a^2]^{3/2}} \qquad \text{...[4.26]}$$

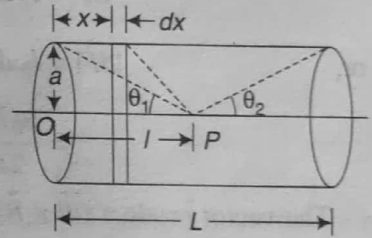

Fig. 4.9

The total magnetic field B at P due to the entire solenoid is

$$B = \int dB = \int_0^L \frac{\mu_0 n I a^2}{2}\frac{dx}{[(l-x)^2+a^2]^{3/2}}$$

$$= \frac{\mu_0 n I}{2}\left[\frac{x-l}{\sqrt{(l-x)^2+a^2}}\right]_0^L$$

$$= \frac{\mu_0 n I}{2}\left[\frac{l}{\sqrt{l^2+a^2}} + \frac{L-l}{\sqrt{(L-l)^2+a^2}}\right] \qquad \text{....(4.27)}$$

Again, from Fig. 4.9, $\cos\theta_1 = \frac{l}{\sqrt{l^2+a^2}}$ and $\cos\theta_2 = \frac{L-l}{\sqrt{(L-l)^2+a^2}}$

So from Eq. (4.27), total magnetic field

$$B = \frac{\mu_0 n I}{2}(\cos\theta_1 + \cos\theta_2) \qquad \text{...(4.28)}$$

Case I For an infinite solenoid $\theta_1 = \theta_2 = 0$; then $B = \mu_0 nI = \mu_0 \frac{NI}{L}$ ...(4.29)

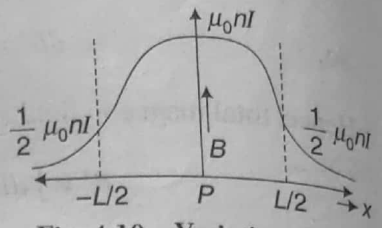

Fig. 4.10 Variation B along the axis of a solenoid.

Case II If P is at the right end ($\theta_1 = 0, \theta_2 = 90°$) or at the left end ($\theta_1 = 90°$, $\theta_2 = 0$) of the solenoid then

$$B = \frac{\mu_0 nI}{2} = \frac{\mu_0 N}{2L} I \qquad ...(4.30)$$

The variation of the magnetic field along the axis of the solenoid is shown in Fig. 4.10. Figure 4.10 shows that for a long solenoid, the magnetic field is maximum at center (P) and just half at the ends of the solenoid.

4.9 FORCE BETWEEN TWO STRAIGHT PARALLEL WIRES

Let C_1 and C_2 be two long parallel wires carrying currents I_1 and I_2 respectively in the same direction [Fig. 4.11(a)]. The separation between the wires is d. The magnetic field at dl, a small element of the wire C_2 due to the current I_1 in C_1 is

$$B = \frac{\mu_0}{4\pi}\left(\frac{2I_1}{d}\right) \qquad ...(4.31)$$

The direction of B is perpendicular to C_2. The magnetic force at the element dl due to B is

$$\vec{dF} = I_2 \vec{dl} \times \vec{B}$$

or,

$$|\vec{dF}| = I_2\, dl \frac{\mu_0}{4\pi}\left(\frac{2I_1}{d}\right)$$

$$= \frac{\mu_0 I_1 I_2}{2\pi d} dl \qquad ...(4.32)$$

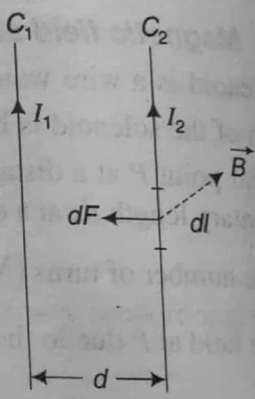

Fig. 4.11 (a) Magnetic force between the parallel wires carrying current.

The vector product ($\vec{dl} \times \vec{B}$) has a direction towards the wire C_1. So, the direction of the force $\vec{dF}$ is towards the wire C_1. The force per unit length of the wire C_2 due to the wire C_1 is

$$F = \frac{\mu_0 I_1 I_2}{2\pi d} \qquad ...(4.33)$$

If the parallel wires carrying currents are in opposite directions, the force will be repulsive in nature.

4.10 MAGNETIC FORCE BETWEEN TWO FINITE ELEMENTS OF CURRENT

From Fig. 4.11(b), the magnetic field due to current I_1 of the conductor A on dl_2 at a distance r is

$$dB = \frac{\mu_0}{4\pi} \frac{I_1 \vec{dl_1} \times \hat{r}}{r^2}$$

where $I_1 \vec{dl}_1$ is the current element of the conductor A.

Now, the force on current I_2, due to current I_1 is

$$d\vec{F} = I_2 \vec{dl}_2 \times d\vec{B} = \frac{\mu_0}{4\pi} I_1 I_2 \frac{\vec{dl}_2 \times (\vec{dl}_1 \times \hat{r})}{r^2} \quad \ldots(4.34)$$

where $I_2 \vec{dl}_2$ is the current elements of the conductor B and $\hat{r}$ is the unit vector in the direction of r.

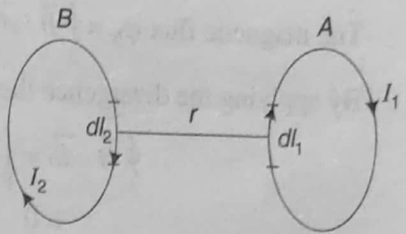

Fig. 4.11 (b) Magnetic field between two finite current-carrying elements.

4.11 DIVERGENCE OF MAGNETIC FIELD

We know from Biot–Savart law, the magnetic field at P [Fig. 4.12] is

$$\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \hat{r}}{r^2} dV \quad \ldots(4.35)$$

where $dV\,(dx'dy'dz')$ is the volume element or source element of current. The position vectors of the source and field points are suppose $\vec{r}'\ (\hat{i}x' + \hat{j}y' + \hat{k}z')$ and $\vec{r}\ (\hat{i}x + \hat{j}y + \hat{k}z)$. Again current density $\vec{J}$ is the function of (x', y', z') and magnetic field $\vec{B}$ is the function of (x, y, z).

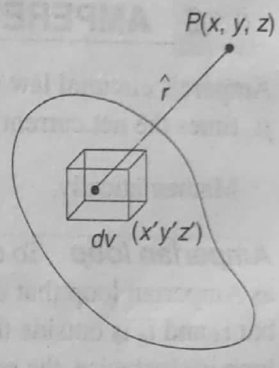

Fig. 4.12 Divergence of magnetic field.

Now taking divergence of equation

$$\vec{\nabla} \cdot \vec{B}(r) = \frac{\mu_0}{4\pi} \int_V \vec{\nabla} \cdot \left(\vec{J}(r') \times \frac{\hat{r}}{r^2} \right) dV$$

Now using vector identity

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \vec{\nabla} \times \vec{A} - \vec{A} \cdot \vec{\nabla} \times \vec{B}$$

we have

$$\vec{\nabla} \cdot \vec{B}(r) = \frac{\mu_0}{4\pi} \int_V \left[\frac{\hat{r}}{r^2} \cdot (\vec{\nabla} \times J(r') - \vec{J}(r') \cdot \vec{\nabla} \times \frac{\hat{r}}{r^2} \right] dV$$

But $\vec{\nabla} \times \vec{J}(r') = 0$ because $\vec{\nabla}$ operator derivatives with respect to the field point $(\vec{r})$ while $\vec{J}$ is the function of the source point$(\vec{r}')$ only.

Again $\vec{\nabla} \times \frac{\hat{r}}{r^2} = 0$ from vector calculus.

Hence, $$\vec{\nabla} \cdot \vec{B} = 0 \quad \ldots(4.36)$$

Thus, the magnetic field is solenoidal.

Physical significance Since magnetic lines of force are continuous, the magnetic flux entering any region of volume is equal to the magnetic flux leaving the volume. Hence the net flux over the volume is equal to zero. Divergence of magnetic field B is defined as the flux of B through the surface enclosing per unit volume. Since, net flux per unit volume is zero, so mathematically

$$\vec{\nabla} \cdot \vec{B} = 0$$

which is known as the differential form of Gauss' law in magnetostatics. Comparing it with Gauss' law in electrostatics $\left(\vec{\nabla} \cdot E = \frac{\rho}{\varepsilon_0} \right)$, we may conclude that monopole does not exist.

The magnetic flux $\varphi_B = \oint_S \vec{B} \cdot \vec{dS}$

By applying the divergence theorem,

$$\oint_S \vec{B} \cdot \vec{dS} = \int_V (\vec{\nabla} \cdot \vec{B})\, dV$$
$$= 0 \qquad \ldots(4.36a)$$

Equation (4.36a) states that there are no magnetic flux sources, and magnetic flux lines always close upon themselves. So, there is no source or sink of magnetic flux (law of conservation of magnetic flux), i.e., magnetic monopole does not exist.

Equations (4.36) and (4.36a) are differential and integral forms of Gauss' law in magnetostatics.

4.12 AMPERE'S CIRCUITAL LAW

Ampere's circuital law states that the line integral of the magnetic field $\vec{B}$ around any closed path is equal to μ_0 times the net current enclosed by the path.

Mathematically, $$\int_c \vec{B} \cdot \vec{dl} = \mu_0 I \qquad \ldots(4.37)$$

Amperian loop To explain net current we consider a loop known as Amperian loop that encloses four wires of currents i_1, i_2, i_3 and i_4 but i_5 and i_6 is outside the loop [Fig. 4.13]. Since the direction of the loop is clockwise, the positive side of the plane (lower plane) is away from the viewer, i.e., into the plane of the paper. So i_1 and i_3 are positive and i_2 and i_4 are negative. Hence, total current is $i_1 + i_3 - (i_2 + i_4)$. Any current outside the loop is not included in writing the right-hand side of Eq. (4.37).

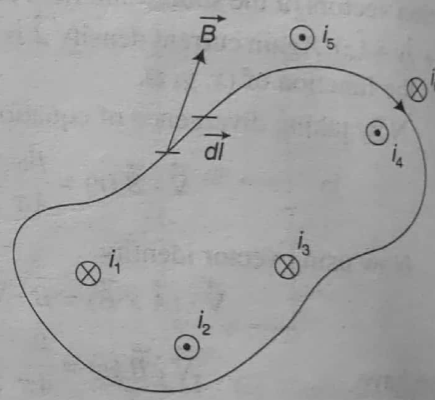

Fig. 4.13 **Amperian loop.**

Ampere's law is valid for a closed path of any shape. If the path does not include the current then

$$\int_c \vec{B} \cdot \vec{dl} = 0$$

To find magnetic field, there must be two conditions: (i) At each point on the closed path, $\vec{B}$ is either tangential or normal to the path. (ii) If $\vec{B}$ is tangential then at all points of the path, $\vec{B}$ must have the same value.

Ampere's law plays the same role in magnetostatics as Gauss' law plays in electrostatics and is very helpful in determining the magnetic field around a conductor for symmetrical distribution.

4.12.1 Differential Form of Ampere's Law

The total current (I) enclosed by a path enclosing a surfaces S is

$$I = \iint_S \vec{J} \cdot \vec{dS}$$

where $\vec{J}$ is the current density in an element $\vec{dS}$ of the surface bounded by the closed path.

Now, Ampere's law in terms of current density $\vec{J}$ is

$$\int_c \vec{B} \cdot \vec{dl} = \mu_0 \iint_S \vec{J} \cdot \vec{dS} \qquad \ldots(4.38)$$

which is the integral form of the Ampere's circuital law.

Now applying Stoke's law

$$\int_c \vec{B} \cdot \vec{dl} = \iint_S (\vec{\nabla} \times \vec{B}) \cdot \vec{dS} \qquad \text{...(4.39)}$$

Now from Eqs. (4.38) and (4.39), we get

$$\iint_S (\vec{\nabla} \times \vec{B}) \cdot \vec{dS} = \mu_0 \iint_S \vec{J} \cdot \vec{dS}$$

or, $$\iint_S [\vec{\nabla} \times \vec{B} - \mu_0 \vec{J}] \cdot \vec{dS} = 0$$

Since the surface element $\vec{dS}$ is arbitrary, so

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \text{...(4.40)}$$

which is the differential form of Ampere's law.

In electrostatics $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \vec{\nabla} \times \vec{E} = 0$

In magnetostatics $\vec{\nabla} \cdot \vec{B} = 0, \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

4.13 APPLICATIONS OF AMPERE'S CIRCUITAL LAW

Ampere's circuital law is applicable for line currents, sheet currents or volume currents.

4.13.1 Long Straight Cylindrical Wire

Let us consider an infinitely long conducting wire of radius R, carrying current I as shown in Fig. 4.14. Suppose the current distribution is uniform throughout the cross section of the wire. Now applying Eq. (4.37) to an amperian loop at A_1 [Fig. 4.14] of radius r is

$$\oint \vec{B}.\vec{dl} = \mu_0 I_1 \qquad \text{...(4.41)}$$

where $$I_1 = \frac{I}{\pi R^2} \times \pi r^2 = I \frac{r^2}{R^2}$$

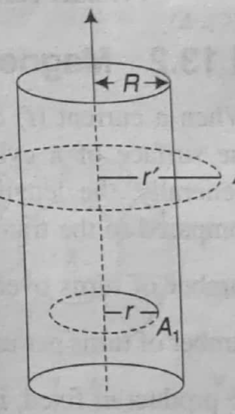

Fig. 4.14 Magnetic field due to a long straight wire of radius R.

Now, from Eq. (4.40)

$$\oint \vec{B} \cdot \vec{dl} = I \frac{r^2}{R^2} \mu_0$$

or, $$B \times 2\pi r = \mu_0 \frac{I r^2}{R^2}$$

so that $$B = \mu_0 \frac{Ir}{2\pi R^2} \qquad \text{...(4.42)}$$

within the wire. Now, outside the wire, applying Eq. (4.37) to an amperian loop at A_2 [Fig. 4.14] of radius $r' > R$ is

$$\int_c \vec{B} \cdot \vec{dl} = \mu_0 I$$

or, $$B \times 2\pi r' = \mu_0 I$$

or, $$B = \frac{\mu_0 I}{2\pi r'} \qquad \text{...(4.43)}$$

At the surface of the wire, $r' = R$, $B = \frac{\mu_0 I}{2\pi R}$ ...(4.44)

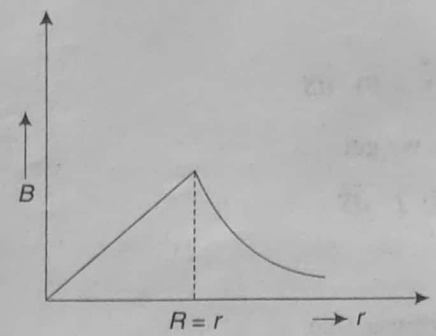

Fig. 4.15 Variation of magnetic field with distance of a current carrying cylindrical wire.

The variation of the magnetic field with distance from the axis of the cylinder is shown in Fig. 4.15.

Note:

(i) For long straight conducting wire of infinite length carrying current I, the magnetic field at any point at a perpendicular distance r from the wire will be $B = \frac{\mu_0 I}{2\pi r}$.

(ii) For a hollow cylinder, since the current I exists only on the surface of the cylinder and inside the cylinder current is zero, so magnetic field inside the cylinder will be zero.

4.13.2 Magnetic Field Inside a Long Solenoid

When a current (I) carrying wire is wound tightly on the surface of a cylindrical tube, we get a solenoid. Generally, the length (L) of the solenoid is large as compared to the transverse dimension. If N is the total number of turns over a length L, we get $\frac{N}{L} = n$ as the number of turns per unit length of the solenoid. Keeping the product nI fixed, if we make n very large and corresponding I very small, then we obtain a surface current of value nI over the curved surface of the cylinder. It turns out that the magnetic field inside a closely wound solenoid is almost uniform over its cross section and can be taken to be negligible outside the volume of the solenoid. Ampere's law thus can easily applied to find out the value of $\vec{B}$ inside the solenoid.

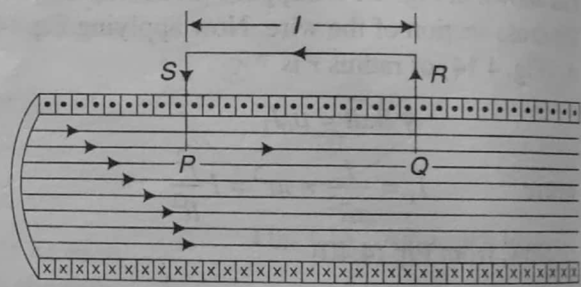

Fig. 4.16 Magnetic field inside a solenoid.

In Fig. 4.16, we draw a rectangle $PQRS$ of length l. The line PQ is parallel to the solenoid axis and hence parallel to the field $\vec{B}$ inside the solenoid. Thus,

$$\int_P^Q \vec{B} \cdot \vec{dl} = Bl$$

Along QR, RS and SP, $\vec{B}.\vec{dl}$ is zero everywhere as $\vec{B}$ is either zero (outside the solenoid) or perpendicular to $\vec{dl}$ (inside the solenoid).

Thus, from $PQRSP$,

$$\oint \vec{B} \cdot \vec{dl} = Bl \quad ...(4.45)$$

If n be the number of turns per unit length, then total of nl turns cross the rectangle $PQRS$. Each turn carries a current I. Hence net current passes through the area $PQRS$ is nlI.

Now, from Eq. (4.45) we have from Ampere's law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \, nlI$$

or,

$$Bl = \mu_0 \, nlI$$

or,

$$B = \mu_0 \, nI \quad ...(4.46)$$

The above equations gives the magnetic field inside a long closely wound solenoid. The relation does not depend on the diameter or the length of the solenoid and magnetic field B is constant over the solenoid cross section.

4.13.3 Magnetic Field Due to Toroid

An endless solenoid in the form of circular shape is called toroid. The magnetic field in such a toroid can be obtained by using Ampere's law.

Let P be a point on the concentric circular path at which magnetic field $\vec{B}$ is to be calculated. By symmetry, the field will have equal magnitude at all points of this circle [Fig. 4.17]. Let the distance of P from the center be r. The field B is tangential at every point of the circle. Hence,

$$\oint \vec{B} \cdot \vec{dl} = B \int dl = B \times 2\pi r$$
$$= \mu_0 NI \quad ...(4.47)$$

where N is the total number of turns and the current crossing the area bounded by the circle is NI.

So,

$$B = \frac{\mu_0 NI}{2\pi r} \quad ...(4.48)$$

Thus B is inversely proportional to r. If the cross section of the toroid is very small, the variation in r can be neglected and $\frac{N}{2\pi r}$ can be written as n, the number of turns per unit length. So

$$B = \mu_0 nI \quad ...(4.49)$$

The field at an external point (P') of the toroid, from Ampere's law

$$\oint B \cdot dl = 0 \text{ or, } B = 0 \quad ...(4.50)$$

Thus the field outside the toroid is zero.

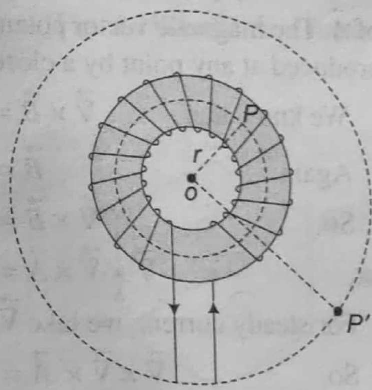

Fig. 4.17 Magnetic field due to a toroid.

4.14 MAGNETIC POTENTIALS

4.14.1 Magnetic Scalar Potential

In electrostatics, scalar potential V plays an important role to find electric field intensity $\vec{E}$. Since $\vec{\nabla} \times \vec{E} = 0$, $\vec{E}$ can be expressed as the gradient of a scalar quantity V. The two powerful relations are:

$$\vec{E} = -\vec{\nabla} V \text{ and} \quad ...(4.51)$$

$$\nabla^2 V = 0 \quad ...(4.52)$$

If in some region of space, current density $J = 0$, then from Ampere's circuital law in magnetostatics, $\vec{\nabla} \times \vec{B} = 0$. We may therefore express $\vec{B}$ as a gradient of scalar quantity V_m

i.e.
$$\vec{B} = -\vec{\nabla} V_m \quad \ldots(4.53)$$

where V_m is called the magnetic scalar potential.

Since $\vec{\nabla} \cdot \vec{B} = 0$ so $\vec{\nabla} \cdot (-\vec{\nabla} V_m) = 0$ or, $\nabla^2 V_m = 0$ $\quad \ldots(4.54)$

We see that V_m satisfies *Laplace's equation* in homogeneous magnetic materials, it is not defined in any region where the current density exists.

The magnetic scalar potential may be defined as a scalar whose negative gradient at any point gives the magnetic induction at that point due to a close loop of carrying current.

The magnetic scalar potential is useful in describing the magnetic field around a current source

4.14.2 Magnetic Vector Potential

Gauss' law in magnetostatics state that always $\vec{\nabla} \cdot \vec{B} = 0$. Again we know that divergence of any curl = 0, i.e., $\vec{\nabla} . \vec{\nabla} \times \vec{A} = 0$. Since the curl of $\vec{B}$ is not necessarily zero (only if $\vec{J} = 0$, $\vec{\nabla} \times \vec{B} = 0$), so $\vec{B}$ can't be the gradient of a scalar potential in general but as the curl of a vector field, in the form

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \ldots(4.55)$$

The vector function $\vec{A}$ which satisfies Eq. (4.54) is known as *vector potential.* The vector potential $\vec{A}$ is as important in magnetostatics as the scalar potential function V in electrostatics. The vector potential $\vec{A}$ does not have any physical significance. It can help to determine $\vec{B}$ at a given point, since $\vec{B}$ is the space derivative of $\vec{A}$. The magnetic vector potential may be defined as a vector, the curl of which gives the magnetic induction produced at any point by a closed-loop carrying current.

We know that $\quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ $\quad \ldots(4.56)$

Again $\quad \vec{B} = \vec{\nabla} \times \vec{A}$

So, $\quad \vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0 \vec{J}$ $\quad \ldots(4.57)$

but, $\quad \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} . \vec{A}) - \nabla^2 \vec{A}$

For steady current, we take $\vec{\nabla} . \vec{A} = 0$

So, $\quad \vec{\nabla} \times \vec{\nabla} \times \vec{A} = -\nabla^2 \vec{A}$

Now from Eq. (4.57) for dc current only, $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ $\quad \ldots(4.58)$

Equation (4.58) is the same as Poisson's equation in electrostatics,

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \quad \ldots(4.59)$$

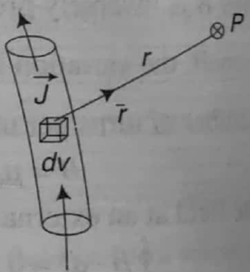

Fig. 4.18 **Magnetic vector potential at a distance r from a current element.**

where V is the electrostatics potential and satisfies

$$V = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho\, dV}{r} \quad \ldots(4.60)$$

Similarly for Eq. (4.58) we have the general solution

$$\vec{A} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}}{r}\, dV \quad \ldots(4.61)$$

The magnetic vector potential is useful for studying radiation in transmission lines, wave guides, antennas.

Here r is the distance from the current element to the point at which the magnetic vector potential is being calculated [Fig. 4.18]. Thus the field $\vec{B}$ produced by a current can be calculated by first determining $\vec{A}$ using Eq. (4.61) and substituting this in Eq. (4.55).

Worked Out Problems

Example 4.1 How many electrons pass through a wire in 1 minute if the current passing through the wire is 200 mA?

Sol. We know $I = \frac{q}{t} = \frac{ne}{t}$

or, $n = \frac{It}{e} = \frac{200 \times 10^{-3} \times 60}{1.6 \times 10^{-19}} = 7.5 \times 10^{19}$.

Example 4.2 What is the drift velocity of electrons in a Cu conductor having a cross-sectional area of 5×10^{-6} m^2 if the current is 10 A? Assume that there are 8×10^{28} electrons/m^3.

Sol. Here, area of cross section $A = 5 \times 10^{-6}$ m^2

Current $I = 10$ A

Number density of free electrons, $n = 8 \times 10^{28}$ electrons/m^3.

We know $v_d = \frac{I}{neA} = \frac{10}{8 \times 10^{28} \times 1.6 \times 10^{-19} \times 5 \times 10^{-6}} = 1.56 \times 10^{-4}$ ms^{-1}.

Example 4.3 Calculate the magnetic field at the center of a regular hexagon [Fig. 4.1W] of side a meter and carrying a current I A.

Sol. The magnetic field at O due to part AB of the hexagon is

$$B' = \frac{\mu_0}{4\pi}\frac{I}{r}(\sin\theta_1 + \sin\theta_2)$$

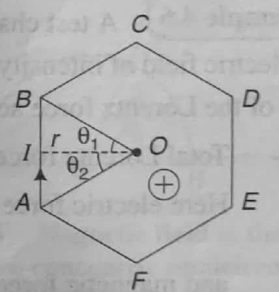

Fig. 4.1W Magnetic field at the center of a hexagon.

Here $r = \frac{\sqrt{3}}{2}a$ and $\theta_1 = \theta_2 = 30°$, $\frac{\mu_0}{4\pi} = 10^{-7}$

So
$$B' = 10^{-7}\frac{I}{\frac{\sqrt{3}}{2}a}(\sin 30° + \sin 30°)$$

$$= 10^{-7} \times \frac{2I}{\sqrt{3}a}\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$= 10^{-7} \times \frac{2I}{\sqrt{3}a}$$

So, total magnetic field at O of the hexagon is

$$B = 6B' = 6 \times 10^{-7} \times \frac{2I}{\sqrt{3}a} = \frac{4\sqrt{3}}{a} \times 10^{-7} \text{ Tesla}$$

Example 4.4 A circular segment QR of a wire $PQRS$ [Fig. 4.2W] of 0.1 m radius subtends an angle of 60° at its center. A current of 6 amperes is flowing through it. Find the magnitude and direction of the magnetic field at the center of the segment.

Sol. Magnetic field due to current in a circular segment making an angle θ at the center is

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}$$

Here, $dl = r\,d\theta$ So, $dB = \dfrac{\mu_0}{4\pi}\dfrac{Ird\theta}{r} = \dfrac{\mu_0}{4\pi}\dfrac{Id\theta}{r}$

So
$$B = \frac{\mu_0}{4\pi}\frac{I\theta}{r}$$
$$= 10^{-7} \times 6 \times \frac{\pi}{3} \times \frac{1}{0.1} \text{ Tesla}$$
$$= 6.28 \times 10^{-6} \text{ Tesla}$$
$$= 6.28\ \mu \text{ Tesla}$$

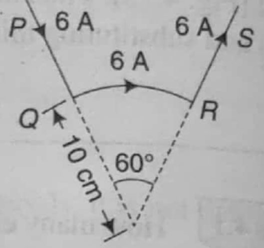

Fig. 4.2W Magnetic field at the center of a circular element of a current-carrying wire.

Example 4.5 A magnetic field $4 \times 10^{-3}\,\hat{k}$ tesla exerts a force of $(4\hat{i} + 3\hat{j}) \times 10^{-10}$ N or a particle having change of 1×10^{-9} C and moving in the xy plane. Calculate the velocity of the particle.

Sol. Lorentz force $\vec{F} = q(\vec{v} \times \vec{B})$

Here $(4\hat{i} + 3\hat{j}) \times 10^{-10} = 1 \times 10^{-9}\,[(v_x\hat{i} + v_y\hat{j})] \times 4 \times 10^{-3}\,\hat{k}$

or, $(4\hat{i} + 3\hat{j}) \times 10^{-10} = 4 \times 10^{-12}\,[(v_x(-\hat{j}) + v_y(\hat{i})]$

or,
$$v_x = -\frac{3 \times 10^{-10}}{4 \times 10^{-12}} = -\frac{3}{4} \times 10^2 = -75$$
$$v_y = \frac{4 \times 10^{-10}}{4 \times 10^{-12}} = 100$$

So,
$$v = -75\,\hat{i} + 100\,\hat{j}\ \text{ms}^{-1}$$

Example 4.6 A test charge having a charge of 0.4 C is moving with a velocity of $4\hat{i} - \hat{j} + 2\hat{k}$ m/s through an electric field of intensity $10\,\hat{i} + 10\,\hat{k}$ and a magnetic field $2\hat{i} - 6\hat{j} - 6\hat{k}$. Determine the magnitude and direction of the Lorentz force acting on the test charge. [WBUT 2007]

Sol. Total Lorentz force $= q\vec{E} + q(\vec{v} \times \vec{B})$

Here electric force $= q\vec{E} = 0.4\,(10\,\hat{i} + 10\,\hat{k})$

and magnetic force $= q(\vec{v} \times \vec{B}) = 0.4 \begin{vmatrix} i & j & k \\ 4 & -1 & 2 \\ 2 & -6 & -6 \end{vmatrix}$

$$= 0.4\,[(6 + 12)\,\hat{i} + (4 + 24)\,\hat{j} + (-24 + 2)\,\hat{k}]$$
$$= 0.4\,(18\,\hat{i} + 28\,\hat{j} - 22\,\hat{k})$$

Now total Lorentz force,
$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$$
$$= 0.4\,(10\,\hat{i} + 10\,\hat{k}) + 0.4\,(18\,\hat{i} + 28\,\hat{j} - 22\,\hat{k})$$
$$= 0.4\,(28\,\hat{i} + 28\,\hat{j} - 12\,\hat{k})$$

The magnitude of the force $= 0.4\sqrt{(28)^2 + (28)^2 + (-12)^2} = 16.6$ N

Suppose, the total force makes an θ with the x axis, then

$$\vec{F}.\hat{i} = 0.4\,(28\,\hat{i} + 28\,\hat{j} - 12\,\hat{k}).\,\hat{i} = 0.4 \times 28$$

or, $(0.4)\sqrt{(28)^2+(28)^2+(-12)^2}\cos\theta = 0.4\times 28$

or, $$\cos\theta = \frac{28}{\sqrt{(28)^2+(28)^2+(-12)^2}}$$

or, $$\cos\theta = \frac{7}{\sqrt{107}}$$

$\therefore$ $$\theta = \cos^{-1}\left(\frac{7}{\sqrt{107}}\right) = 47.41°$$

Example 4.7 A straight wire carrying a current of 10 A is bent into a semicircular arc of π cm radius as shown in Fig. 4.3W. What is the magnetic field and direction of the magnetic field at center O of the arc?

Sol. The magnetic field at center O due to each straight portion of the wire is O. The magnetic field at center O is only due to half the circular loop. The magnitude of magnetic field

$$B = \frac{\mu_0}{4\pi}\left(\frac{\pi I}{a}\right) = \frac{\mu_0 I}{4a} = \frac{4\pi\times 10^{-7}\times 10}{4\times\pi\times 10^{-2}} = 10^{-4}\text{ T}$$

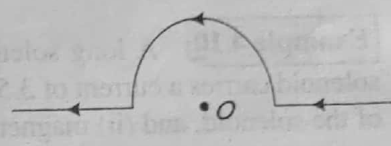

Fig. 4.3W **Magnetic field at the center of a semicircular wire carrying current.**

The current in the loop is anticlockwise and the direction, of the field is perpendicular to the paper.

Example 4.8 The wire loop $ABCDA$ formed by joining two semicircular wires of radii R_1 and R_2 carries a current I as shown in Fig. 4.4W. Find out the magnetic field at center O.

Sol. The magnetic field due to a semicircular loop of radius R_1 is

$$B_1 = \frac{\mu_o}{4\pi}\left(\frac{\pi I}{R_1}\right)$$

[Here, direction of current is anticlockwise]

and direction of the field is normal to the plane of the loop, directed upward. For a bigger loop, direction of the current is clockwise. The value of the magnetic field due to semi-circular loop of radius R_2 is

$$B_2 = \frac{\mu_0}{4\pi}\left(\frac{\pi I}{R_2}\right)$$

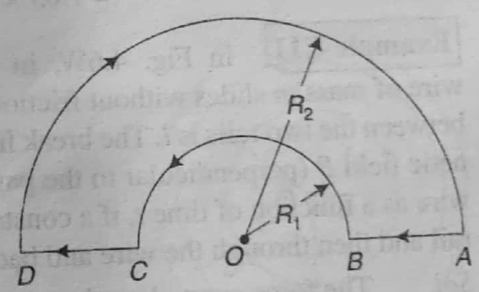

Fig. 4.4W **Magnetic field at the center of two concentric semicircular wires carrying current.**

Here direction of the magnetic field is into the plane of the paper. So, net magnetic field

$$B = B_1 - B_2 = \frac{\mu_0}{4\pi}\pi I\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$= \frac{\mu_0 I}{4}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

and direction is perpendicular to the plane of the paper, hence directed upward.

Example 4.9 A current-carrying straight wire cannot move but a current-carying square loop adjacent to it can move under the influence of a magnetic force. Show that the square loop in Fig. 4.5W will move towards the wire.

Sol. In Fig. 4.5W, we see that the force acting on arms AB and DC are equal and opposite. But the force on arm AD is given by

$$F_1 = \frac{\mu_0}{4\pi}\left(\frac{2Ii}{a}\right)$$

which is directed towards the wire. The force on arm BC is given by

$$F_2 = \frac{\mu_0}{4\pi}\left(\frac{2Ii}{b}\right)$$

which is directed away from the wire. Here, $b > a$ hence $F_1 > F_2$. So the loop will move towards the wire.

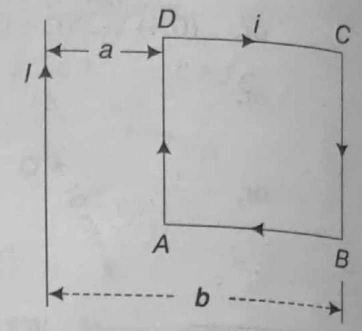

Fig. 4.5W Current-carrying square loop moves under the influence of magnetic force due to the current-carrying wire.

Example 4.10 A long solenoid of 40 cm length has 300 turns. If the solenoid carries a current of 3.5 A, calculate (i) magnetic field at the center of the solenoid, and (ii) magnetic field of the axis at one end of the solenoid.

Sol. (i) The magnetic field at the center of the solenoid is

$$B = \mu_0\, n\, I = \mu_0 \frac{N}{l} I = 4\pi \times 10^{-7} \times \frac{300}{0.4} \times 3.5 = 3.3 \times 10^{-3} \text{ Tesla}$$

(ii) The magnetic field at one end of the solenoid

$$B = \frac{1}{2}\mu_0\, n\, I = \frac{1}{2}\mu_0 \frac{N}{l} I = \frac{1}{2} \times \frac{300}{0.4} \times 3.5$$

$$= 1.65 \times 10^{-3} \text{ Tesla.}$$

Example 4.11 In Fig. 4.6W, in between two rails, a metal wire of mass m slides without friction. The distance of separation between the two rails is l. The break lies in a vertical uniform magnetic field B (perpendicular to the paper). Find the velocity of the wire as a function of time t, if a constant current I flows along one rail and then through the wire and back down the other rail.

Sol. The force exerted on the wire of length l is

$$F = B\,I\,l \sin 90° = B\,I\,l$$

The direction of the force will be to the left according to Fleming's left-hand rule.

The acceleration of the wire

$$a = \frac{F}{m} = \frac{B\,I\,l}{m}$$

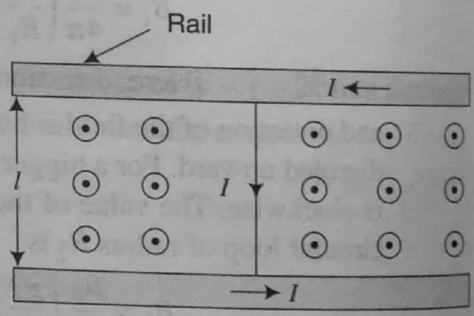

Fig. 4.6W A metal wire slides without friction between two rails carrying current in an opposite direction.

Let initial velocity of wire $u = 0$, then velocity at any time t is $v = at = \frac{B\,I\,l}{m}\,t$.

Example 4.12 Two straight wires, each 2 m long, are parallel to one another and are separated by a distance of 2 cm. If each carries a current of 8 A, calculate the force experienced by either of the wires.

Sol. The force per unit length experienced by each wire carrying currents I_1, I_2 separated by a distance d is given by

$$F = \frac{\mu_0 I_1 I_2}{2\pi a}$$

Here $I_1 = I_2 = 8$ A $d = 2$ cm $= 0.02$ m

so, $$F = \frac{4\pi \times 10^{-7} \times 8 \times 8}{2\pi \times 0.02} = 64 \times 10^{-5} \text{ N/m}$$

The total force on either of the wires is

$$F = 2 \times 64 \times 10^{-5} \text{ N} = 128 \times 10^{-5} \text{ N}$$

Example 4.13 In the Bohr model of a hydrogen atom, an electron is revolving in a circular path of 0.4 Å radius with a speed of 10^6 m/s. What is the value of magnetic field at the center of the orbit?

Sol. Here $r = 0.4$ Å $= 0.4 \times 10^{-10}$ m, $v = 10^6$ m/s.

We know that time period $T = \frac{2\pi r}{v} = \frac{2\pi \times 0.4 \times 10^{-10}}{10^6} = 8\pi \times 10^{-17}$ s

Again, current $I = \frac{e}{T} = \frac{1.6 \times 10^{-19}}{8\pi \times 10^{-17}} = \frac{2}{\pi} \times 10^{-3}$ A

The magnetic field at the center of the orbit

$$B = \frac{\mu_o}{4\pi}\left(\frac{2\pi I}{r}\right) = 10^{-7} \times \frac{2 \times \pi}{0.4 \times 10^{-10}} \times \frac{2}{\pi} = 10 \text{ Tesla.}$$

Example 4.14 The volume current density distribution in cylindrical coordinates is

$$J(r, \varphi, z) = 0 \qquad 0 < r < a$$

$$= J_0\left(\frac{r}{a}\right)\hat{e}_z \qquad a < r < b$$

$$= 0 \qquad b < r < \infty$$

Find the magnetic field in various regions [Fig. 4.7W].

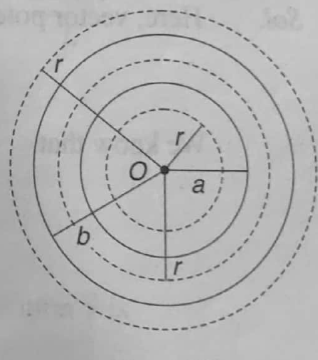

Fig. 4.7W

Sol. From Ampere's circuital law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \int_S J \cdot d\vec{S}$$

For region $0 < r < a$, $J = 0$

So, $\oint \vec{B} \cdot \vec{dl} = 0$ or, $B = 0$

For region $a < r < b$ $\quad J(r, \varphi, z) = J_0\left(\frac{r}{a}\right)\hat{e}_z$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 \int_{\varphi=0}^{2\pi} \int_a^r J_0\left(\frac{r}{a}\right) \cdot r\, dr\, d\varphi$$

or, $$B \times 2\pi r = \mu_0 \frac{J_0}{a}\left[\frac{r^3}{3}\right]_a^r [\varphi]_0^{2\pi} = \mu_0 \frac{2\pi}{3a} J_0 (r^3 - a^3)$$

or, $$B = \frac{\mu_0 J_0}{3ar}(r^3 - a^3)$$

Now at $r = b$, $$B = \frac{\mu_0 J_0}{3ab}(b^3 - a^3)$$

For region $b < r < \alpha$, $J = 0$

we have $\oint \vec{B} \cdot \vec{dl} = \mu_0 \int\limits_{\varphi=0}^{2\pi} \int\limits_{r=a}^{b} \frac{J_0 r}{a} r\, dr\, d\varphi$

or, $$B = \frac{\mu_0 J_0}{3ar}(b^3 - a^3)$$

Example 4.15 Show that $\oint\limits_S \vec{B} \cdot \vec{dS} = 0$ where $\vec{B}$ is the magnetic field and S is a closed surface. [WBUT 2006]

Sol. We have by applying divergence theorem

$$\oint\limits_S \vec{B} \cdot \vec{dS} = \int\limits_V (\vec{\nabla} \cdot \vec{B})\, dV$$

But $\vec{\nabla} \cdot \vec{B} = 0$ [$\because \vec{B}$ is solenoidal field]

$$\therefore \quad \oint\limits_S \vec{B} \cdot \vec{dS} = \int\limits_V (\vec{\nabla} \cdot \vec{B})\, dV = 0$$

$$\therefore \quad \oint\limits_S \vec{B} \cdot \vec{dS} = 0$$

which shows that the lines of induction are continuous, meaning, it has no sources or sinks.

Example 4.16 If the vector potential $\vec{A} = (x^2 + y^2 - z^2)\,\hat{j}$ at position (x, y, z), find the magnetic field at (1, 1, 1). [WBUT 2007]

Sol. Here, vector potential $\vec{A} = (x^2 + y^2 - z^2)\,\hat{j}$

We know that $$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (x^2 + y^2 - z^2) & 0 \end{vmatrix}$$

$$= \hat{i}\left[-\frac{\partial}{\partial z}(x^2 + y^2 - z^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2 + y^2 - z^2)\right] = 2z\,\hat{i} + 2x\,\hat{k}$$

At (1, 1, 1) $\quad \vec{B} = 2\hat{i} + 2\hat{k}$

Example 4.17 If the vector potential $\vec{A} = \frac{1}{2}(\vec{a} \times \vec{r})$, where $\vec{a}$ is a constant vector, find the associated magnetic field.

Sol. Let $\quad \vec{a} = \hat{i}\, a_1 + \hat{j}\, a_2 + \hat{k}\, a_3$

So $\quad \vec{a} \times \vec{r} = \hat{i}\,(z\, a_2 - y\, a_3) + \hat{j}\,(x\, a_3 - z\, a_1) + \hat{k}\,(y\, a_1 - x\, a_2)$

We know that $$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\, a_2 - y\, a_3 & x\, a_3 - z\, a_1 & y\, a_1 - x\, a_2 \end{vmatrix}$$

$$= \frac{\hat{i}}{2}\left[\frac{\partial}{\partial y}(y\,a_1 - x\,a_2) - \frac{\partial}{\partial z}(x\,a_3 - z\,a_1)\right] + \frac{\hat{j}}{2}\left[\frac{\partial}{\partial z}(z\,a_2 - y\,a_3) - \frac{\partial}{\partial x}(y\,a_1 - x\,a_2)\right]$$

$$+ \frac{\hat{k}}{2}\left[\frac{\partial}{\partial x}(x\,a_3 - z\,a_1) - \frac{\partial}{\partial y}(z\,a_2 - y\,a_3)\right]$$

$$= a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k} = \vec{a}$$

Example 4.18 Two circular coils having identical turns and radii in the ratio 1:3 are joined in series. Find the ratio of the magnetic fields at the center of the coils.

Sol. Here, $B = \dfrac{\mu_0 N I}{2R}$. Since the coils are connected in series, therefore I is constant. N is also given to be constant.

So, $B \propto \dfrac{1}{R}$ $\quad \therefore \dfrac{B_1}{B_2} = \dfrac{3}{1}$ or, $B_1 : B_2 = 3{:}1$

Review Exercises

Part 1: Multiple Choice Questions

1. A wire of length L carrying a current I is bent into a circle. The magnitude of the magnetic field at the center of the circle is

   (a) $\dfrac{\pi \mu_0 I}{L}$ (b) $\dfrac{\mu_0 I}{2\pi L}$ (c) $\dfrac{\mu_0 I}{2L}$ (d) $\dfrac{2\pi \mu_0 I}{L}$

2. In the region around a moving charge, there is
   (a) electric field only
   (b) magnetic field only
   (c) neither electric field nor magnetic field
   (d) electric as well as magnetic field

3. The magnetic field at the origin due to a current element $i\,\vec{dl}$ placed at a position $\vec{r}$ is

   (a) $\dfrac{\mu_0}{4\pi}\dfrac{i\,\vec{dl}\times\vec{r}}{r^2}$ (b) $\dfrac{\mu_0}{4\pi}\dfrac{i\,\vec{dl}\times\vec{r}}{r^3}$ (c) $\dfrac{\mu_0}{4\pi}\dfrac{i\,\vec{dl}\times\hat{r}}{r^3}$ (d) zero

4. A current-carrying straight wire is kept along axis of a circular loop carrying current. The straight wire
   (a) will exert an inward force on the circular loop
   (b) will exert an outward force on the circular loop
   (c) will not exert any force on the circular loop
   (d) None of these

5. A moving charge produces
   (a) electric field only (b) magnetic field only
   (c) Both of them (d) None of these

6. Which of the following statements is not characteristic of a static magnetic field? [WBUT 2006]
   (a) It is solenoid. (b) It is conservative.
   (c) Magnetic flux lines are always closed.
   (d) It has no sink or source.

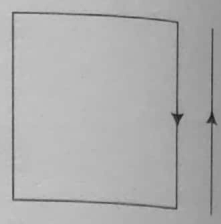

Fig. 4.8W Moving of current-carrying square loop.

7. A current-carrying straight wire cannot move, but a current-carrying square loop adjacent to it can move under the influence of a magnetic force [Fig. 4.8W]. [WBUT 2008]
   The square loop will
   (a) remain stationary (b) move towards the wire
   (c) move away from the wire (d) None of these
8. The direction of magnetic induction due to a straight infinitely long current carrying wire is [WBUT 2008]
   (a) perpendicular to the wire (b) parallel to the wire
   (c) at an inclination of 30° to the wire (d) None of these
9. The equation of continuity in a steady charge distribution is
   (a) $\vec{\nabla} \cdot \vec{J} = 0$ (b) $\vec{\nabla} \times \vec{J} = 0$ (c) $\vec{\nabla} \cdot \vec{J} = \rho$ (d) $\vec{\nabla} \cdot \vec{J} = \frac{\rho}{\varepsilon_0}$
10. The work done by the Lorentz force $\vec{F}$ on a charged particle is
    (a) $\vec{F} \cdot d\vec{r}$ (b) zero (c) $\frac{q}{\varepsilon_0}$ (d) qF
11. $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$.
    (a) integral form of law of conservation of charge
    (b) differential form of law of conservation of charge
    (c) Poisson's equation
    (d) None of these

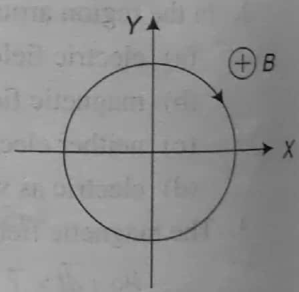

Fig. 4.9W A current-carrying loop in a magnetic field.

12. A conduction loop carrying a current I is placed in a uniform magnetic field pointing into the plane of the paper as shown in Fig. 4.9W. The loop will have a tendency to [WBUT 2007]
    (a) contract
    (b) expand
    (c) move towards positive the x axis
    (d) move towards negative the x axis

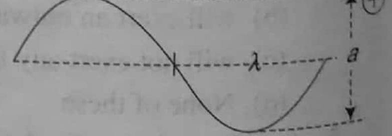

Fig. 4.10W Current-carrying sinusoidal wire in a magnetic field.

13. A copper wire is bent in the form of a sine wave of wavelength λ and peak-to-peak value as shown in Fig. 4.10W. A magnetic field of flux density B tesla acts perpendicular to the plane of the figure in the entire region. If the wire carries a steady current I ampere, the magnetic force on the wire is [WBUT 2007]
    (a) $I\sqrt{(a^2 + \lambda^2)}\, B$ (b) IaB
    (c) $I(a + \lambda)B$ (d) $I\lambda B$

14. If the current density $\vec{J} = k\,\hat{r}$ where $\hat{r}$ is a unit vector along $x\,\hat{i} + y\,\hat{j}$, the current through the surface $x^2 + y^2 = a^2$, bounded by $z = 0$ and $z = h$ is [WBUT 2007]

(a) $\pi a^2 hk$ (b) zero (c) $2\pi ahk$ (d) $\frac{a^3 k}{\pi h}$

15. If $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{B}$ and $\vec{A}$ are any vectors then [WBUT 2005]

(a) $\vec{\nabla} \cdot \vec{B} = 0$ (b) $\vec{\nabla} \cdot \vec{B} = +1$ (c) $\vec{\nabla} \cdot \vec{B} = -1$ (d) None of these

16. Magnetic field due to an infinitely long straight conductor carrying current I is

(a) $\frac{\mu_0}{4\pi}\left(\frac{2\pi I}{a}\right)$ (b) $\frac{\mu_0}{4\pi}\left(\frac{2I}{a}\right)$ (c) $\frac{1}{4\pi\,\mu_0}\frac{I}{a}$ (d) zero

17. Two thin, long parallel wires, separated by a distance 'd' carry a current of I A, in the same direction. They will

(a) attract each other with a force of $\frac{\mu_0 I^2}{2\pi d^2}$ (b) repel each other with a force of $\frac{\mu_0 I^2}{2\pi d^2}$

(c) attract each other with a force of $\frac{\mu_0 I^2}{2\pi d}$ (d) None of these

18. A long wire carries a steady current. It is bent into a circle of one turn and the magnetic field at the center of the coil is B. It is then bent into a circular loop of n turns. The magnetic field at the center of the coil will be

(a) $n^2 B$ (b) $n^3 B$ (c) $n B$ (d) $2\,n\,B$

19. A 1.5 m long solenoid 0.4 cm in diameter possesses 10 turns per cm length. A current of 5 A flows through it. The magnetic field at the axis inside the solenoid is

(a) $4\pi \times 10^{-4}$ Tesla (b) $2\pi \times 10^{-3}$ Tesla

(c) $2\pi \times 10^{-6}$ Tesla (d) None of these

20. A long straight wire along the z axis carries a current I in the negative z direction. The magnetic vector field $\vec{B}$ at a point having coordinates (x, y) in the $z = 0$ plane is

(a) $\frac{\mu_0 I}{4\pi}\left(\frac{x\,\hat{i} - y\,\hat{j}}{x^2 + y^2}\right)$ (b) $\frac{\mu_0 I}{2\pi}\left(\frac{y\,\hat{i} - x\,\hat{j}}{x^2 + y^2}\right)$

(c) $\frac{\mu_0 I}{2\pi}\left(\frac{y\,\hat{i} + x\,\hat{j}}{2x^2 + y^2}\right)$ (d) None of these

[**Ans.** 1 (a), 2 (d), 3 (b), 4 (c), 5 (c), 6 (b), 7 (c), 8 (a), 9 (a), 10 (b), 11 (b), 12 (b), 13 (d), 14 (b), 15 (a), 16 (b), 17 (c), 18 (a), 19 (b), 20 (b)]

Short Questions with Answers

1. **The net charge on a current-carrying conductor is zero. Then, why does it experience a force in a magnetic field?**

Ans. In a conductor, positive ions are stationary. So, they do not experience any force. But the free electrons drift towards the positive end of the conductor with some drift velocity and experience a magnetic field.

2. Why does a solenoid contract when current is passed through it?

Ans. When current is passed through a solenoid, the currents in the different turns of the solenoid flow in the same direction. Again we know that when currents in two parallel conductors flow in the same direction, the conductors attract each other. So, the solenoid contracts.

3. Define current density.

Ans. Current density is defined as the current through an infinitesimal area at any point inside a conductor, the area held perpendicular to the direction of flowing positive charge.

4. What is Lorentz force? Show that Lorentz force does not work on a charged particle.

Ans. See Section 4.5.

5. State Ampere's law both in integral and differential form.

Ans. See Section 4.12.

6. Compare between Lorentz electric force and Lorentz magnetic force.

Ans. Lorentz electric force $F_e = qE$, direction along the field does not depend on velocity and work is done. Lorentz magnetic force $F_m = Bq\, v \sin\theta$, direction perpendicular to plane containing B and v and depends on velocity of the charge, no workforce.

7. Define magnetic scalar potential and magnetic vector potential.

Ans. See Section 4.14.

8. A proton moving through a magnetic field region experiences maximum force. When does this occur?

Ans. When the proton moves perpendicular to the magnetic field, $\theta = 90°$, $\vec{v} \times \vec{B}$ will be maximum. So $\vec{F}$ is maximum.

9. Write the one condition under which an electric charge does not experience a force in a magnetic field.

Ans. Either the electric charge is at rest or it is moving parallel to the direction of the magnetic field.

10. Define an ampere in terms of the force between current-carrying conductors.

Ans. One ampere is that current which if passed in each of the two parallel conductors of infinite length and 1 m apart in vacuum, causes each conductor to experience a force of 2×10^{-7} Nm^{-1} length of conductor.

11. Apply Ampere's law qualitatively to the three parts as shown in Fig. 4.11W.

Ans. For paths I and III, $\oint B.\, \vec{dl} = \mu_0 I$

For path II, the net current is zero, i.e., for path II, $\oint \vec{B} \cdot \vec{dl} = 0$

12. A charge 4C is moving with a velocity $\vec{v} = (2\hat{j} + 3\hat{k})$ in a magnetic field $B = (2\hat{j} + 3\hat{k})$ Wbm^{-2}. Find the force acting on the charge.

Ans. Here, $\vec{v}$ and $\vec{B}$ are parallel vectors.

So $\vec{v} \times \vec{B} = 0 \quad \therefore \vec{F} = q(\vec{v} \times \vec{B}) = 0$

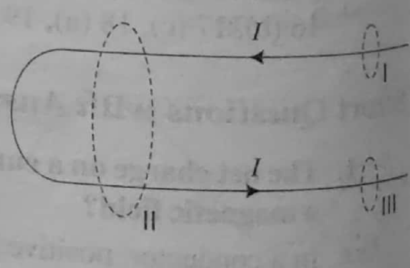

Fig. 4.11W

Part 2: Descriptive Questions

1. State Biot–Savart law. Using Biot–Savart law, calculate the field at the center of a circular current-carrying coil. [WBUT 2002]
2. (a) State Biot–Savart law.
   (b) Using Biot–Savart law, obtain an expression for the magnetic flux intensity at the center of a long current-carrying solenoid
   (c) Show that the field at the end of such a solenoid is half of that at the center. [WBUT 2003]
3. (a) State Biot–Savart law in magnetostatics. Find the magnetic field of an infinitely long straight wire at a transverse distance of d from the expression of $\vec{B}$ found in Biot–Savart law.
   (b) Express Biot–Savart law in terms of current density and hence show that the magnetic field is solenoidal.
   (c) Express Ampere's circuital law in terms of vector potential. (You may use $\vec{\nabla}\cdot\vec{A}=0$, where $\vec{A}$ is the vector potential.) [WBUT 2008]
4. Find the magnetic induction $\vec{B}$ at a point on the axis of an infinitely long solenoid carrying a current I, number of turns per unit length being n. [WBUT 2007]
5. Find the magnetic field of a circular loop carrying field due to a long solenoid at a point. [WBUT 2007]
6. (a) State Ampere's circuital law.
   (b) By applying Ampere's circuital law, find out magnetic field due to a long solenoid at a point (i) inside the solenoid, and (ii) outside the solenoid.
7. What is Lorentz force? Show that Lorentz force does not work on a charged particle.
8. What do you mean by magnetic vector potential? Why is it called so? [WBUT 2002]
9. Show that $\oint \vec{B}\cdot\vec{dS}=0$, when $\vec{B}$ is the magnetic field and S is a closed surface. State the theorem that you have used. [WBUT 2006]
10. Show that the equation of continuity is given by $\vec{\nabla}\cdot\vec{J}+\dfrac{\partial\rho}{\partial t}=0$, where $\vec{J}$ and ρ have their usual meaning. [WBUT 2005]
11. (a) Give an example of an electrical circuit carrying a non-steady current when Ampere's circuital law is not applicable.
    (b) Write the expression of the magnetic field due to a current-carrying conductor. Draw a diagram necessary to explain the symbols. Show that this field is solenoidal. [WBUT 2006]
12. Starting from the definition of current density, derive the equation of continuity in current electricity. [WBUT 2007]
13. State Ampere's law in magnetostatics in integral form and from that deduce its differential form. [WBUT 2007]
14. Write down the condition of steady-state current. Show that Ampere's law implies that the current is in the steady state. [WBUT 2007]
15. Prove that the magnetic field inside a toroid having n numbers of turns per unit length and carrying a current I is $\mu_0 nI$.
16. Find the force per unit length of a current-carrying conductor placed in a uniform magnetic field. Hence find the force between the straight conductors carrying currents.

Part 3: Numerical Problems

1. Calculate the drift speed of the electrons when 1 A of current exists in a copper wire of 2 mm^2 cross section. The number of free electrons in 1 cm^3 of copper is 8.5×10^{22}. [**Ans.** 36×10^{-3}]

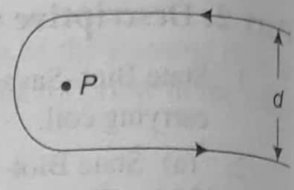

Fig. 4.12W

2. Find the magnetic field at the point P in Fig. 4.12W. The curve portion is a semicircle and the straight wires are long. $\left[\textbf{Ans. } \frac{\mu_o}{2d}\left(\frac{2}{\pi}+\frac{1}{d}\right)\right]$

3. Consider a coaxial cable which consists of an inner wire of radius 'a' surrounded by an outer shell of inner and outer radii 'b' and 'c' respectively. The inner wire carries an equal current in opposite direction. Find the magnetic field at a distance 'r' from the axis, where

   (a) $r < a$ (b) $a < r < b$ (c) $b < r < c$ (d) $r > c$.

   Assume that the current density is uniform in the inner wire and also uniform in the outer shell.

   [WBUT (Question Bank)] $\left[\textbf{Ans. } \text{(a) } B = \frac{\mu_0 Ir}{2\pi a^2} \text{ (b) } \frac{\mu_0 I}{2\pi r} \text{ (c) } \frac{\mu_0 Ir}{2\pi(c^2 - b^2)} \text{ (d) } B = 0\right]$

4. Find the magnetic field B due to a semicircular wire of 10.00 cm radius carrying a current of 5.0 A at its center of curvature. [**Ans.** 1.6×10^{-4} Tesla]

5. Two parallel wires carry equal currents of 10 A along the same direction and are separated by a distance of 2.0 cm. Find the magnetic field at a point which is 2.0 cm away from each of these wires. [**Ans.** 1.7×10^{-9} Tesla]

6. An infinite wire carrying a current I is bent in the form of a parabola. Find the magnetic field at the focus of the parabola. $\left[\textbf{Ans. } B = \frac{\mu_0 I}{4a}\right]$

7. If the magnetic scalar potential $\varphi_m = x^2 + y^2 - z^2$ at any point (x, y, z) in current free space, then find the magnetic field at the point (1, 2, 2). [**Ans.** $-2\hat{i} - 4\hat{j} + 4\hat{k}$] [**Hints:** Let $\vec{B} = -\vec{\nabla}\varphi_m$]

8. If the vector potential $\vec{A} = (2z + 5)\hat{i} + (3x - 2)\hat{j} + (4x - 1)\hat{k}$, find the magnetic field. [**Ans.** $B = -2\hat{i} + 3\hat{k}$]

9. A solenoid has 4 layers of 1200 turns each. Its length and mean radius are 3 m and 0.25 m respectively. Find the magnetic field at the center if a current of 2.5 A flows through it. [**Ans.** $B = 5.02 \times 10^{-3}$ Tesla]

10. Figure 4.13W shows a current-carrying system of straight wire and loop. Determine the magnetic field at the center O of the loop. Given R is the radius of the loop and I is the current flowing in the system. $\left[\textbf{Ans. } B = \frac{\mu_0 I}{2R}\left(1 - \frac{1}{\pi}\right)\right]$

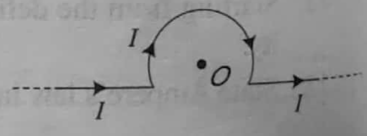

Fig. 4.13W

11. A proton enters a magnetic field of 4 Tesla with a velocity of 2.5×10^6 ms^{-1} at an angle of 30° with the direction of the field. Find the magnitude of the force acting on the proton. [**Ans.** $F = 8 \times 10^{-13}$ N]

12. A particle of charge q moves with a velocity $v = a\hat{i}$ in a magnetic field $B = b\hat{j} + c\hat{k}$ where a, b and c are constants. Find the magnitude of the force experienced by the particle. [**Ans.** $qa(b^2 + c^2)^{1/2}$]