1.2 Probability Distribution And Expectation

1.2.1. Random Variable.

Definition. Let S be a given sample space. Then a real valued function X defined on S is called a random or a stochastic variable or sometimes a variate. Thus for every point s of a sample space S, we have a unique real value of X i.e $X(s)$

The range of the function X i.e. the set of all values assumed by X is called the ***Spectrum*** of the random variable.

Discrete Random Variable. A random variable X is said to be discrete if the spectrum of X is finite or countably infinite i.e an infinite sequence of distinct values.

Continuous Random Variable. A random variable X is said to be continuous if it can assume every value in an interval .

Event Described by Random Variable. The set of all sample points s for which $X(s) \in A$, a given set of real numbers, is an event and is denoted by $(X \in A)$. In particular, the event $(X = a)$ is the set of all sample points corresponding to which X takes the value a.

Let [a, b] be a given closed interval. Then the set of all sample points s for which $a \leq X(s) \leq b$ is an event and is denoted by $(a \leq X \leq b)$. Similarly the events $(a < X \leq b)$, $(a \leq X < b)$ and $(a < X < b)$ are defined. Also the event $(-\infty < X \leq x)$ is abbreviated as $(X \leq x)$ where x is a real number. Further the events $(-\infty < X < \infty)$ and $(-\infty < X < -\infty)$ denote respectively the certain event S and the impossible event ϕ.

Thus we see an event can be described by a random variable.

Illustration. **(i)** Let us consider the random experiment of tossing two (unbiased) coins. Then the sample space S contains 4 sample points.

i.e., $S = \{HH, HT, TH, TT\}$.

Let the random variable X be such that X (an outcome) = "the number of heads". Then X is a function over S defined by

$$X(HH)=2,\ X(HT)=X(TH)=1,\ X(TT)=0.$$

Thus the spectrum of X is $\{0,1,2\}$ which is a finite set. Hence X is a discrete random variable here. Here the event $(X=1)=\{TH,HT\}$ = 'one head', the event $(-1<X\leq 0)=\{TT\}$ = 'Two tails'

(ii) Let the random variable X denote the weights (in kg) of a group of individuals. Then X can assume every value in an interval say (30, 100), suppasing there is no individual having weight less than 30 and greater than 100. Hence X is a continuous random variable. Here the event $(42<X\leq 50)$ = the group of individuals whose weight lie between 42 and 50, including 50 ; the event $(X=70)$ = The group of individuals whose weight is 70 kg.

1.2.2. Probability Mass Function and Discrete Distribution

Let X be a discrete random variable which assumes the values $x_0, x_1, x_2, \cdots, x_n, \cdots$ Let $P(X=x_i)=f(x_i)=f_i$. So, the value of f_i depends on x_i i.e. i. This function f_i is called *Probability mass function* **(p.m.f)** of the random variable X. A particular value of f_i is called a probability mass.

The set of ordered pairs (x_i, f_i) is called the discrete probability distribution of the random variable X

Discrete distribution is presented in the following way :

X	:	x_0	x_1	x_2	$\cdots$	$\cdots$
f_i	:	f_0	f_1	f_2	$\cdots$	$\cdots$

Illustration. For the random experiment of tossing two coins as given in Illustration (i) of art 1.2.1 we see X assumes the values 0, 1 and 2. Moreover,

$$P(X=0)=\frac{1}{4},\ P(X=1)=\frac{1}{2},\ P(X=2)=\frac{1}{4}.$$

So, the distribution of the number of heads is given by

X	:	0	1	2
f_i	:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Fundamental Properties of pmf.

If

X	:	x_0	x_1	x_2	x_3	...	...
f_i	:	f_0	f_1	f_2	f_3	...	...

is a discrete distribution of X, then the pmf has following two properties :

(i) $f_i \geq 0$ (ii) $\sum_i f_i = 1$,

Proof : (i) $f_i = P(X=x_i)\geq 0$ since a probability is always ≥ 0

(ii) $\sum f_i = \sum P(X=x_i) = P(S) = 1$ as $S=\{x_0, x_1, x_2, \cdots \cdots\}$ = event space.

1.2.3. Distribution Function or Cumulative Distribution Function.

The distribution function (d.f) of a random variable X (discrete or continuous) is given by

$$F(x)=P(-\infty<X\leq x),\ -\infty<x<\infty$$

Thus, if $x_i \leq x < x_{i+1}$, then

$$F(x)=P(X=x_0)+P(X=x_1)+\cdots+P(X=x_i)=\sum_{\alpha=0}^{i} f_\alpha.$$

Illustration. In the discrete distribution

X	:	0	1	2
f_i	:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

, the d.f is

$$F(x)=0,\quad x<0$$
$$=\frac{1}{4},\quad 0\leq x<1$$
$$=\frac{1}{4}+\frac{1}{2},\quad 1\leq x<2$$
$$=\frac{1}{4}+\frac{1}{2}+\frac{1}{4},\quad 2\leq x.$$

Properties of Distribution Function.

(i) The distribution function $F(x)$ is a monotonic non-decreasing function.

(ii) $F(-\infty)=0$ and $F(\infty)=1$ and hence $0 \le F(x) \le 1$.

(iii) $F(x)$ is continuous on the right at all points and has a jump discontinuity on the left at $x=a$, the height of jump being equal to $P(X=a)$ i.e., $\lim_{x\to a+} F(x)=F(a)$ and

$$F(a)-\lim_{x\to a-} F(x)=P(x=a)$$

(iv) Suppose a and b are any real numbers such that $a<b$

Then $P(a<X\le b)=F(b)-F(a)$,

$P(a<X<b)=F(b)-F(a)-P(X=b)$

and $P(a\le X<b)=F(b)-F(a)-P(X=b)+P(X=a)$

Proof: Left as exercise.

Illustration. (i) Let X be a random variable denoting the number of points appearing in a toss of a die. The distribution of X is

X	:	1	2	3	4	5	6
f_i	:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Now, if $x<1$, $F(x)=P(X\le x)=0$

If $1\le x<2$, $\quad F(x)=P(X\le x)=f_1=\frac{1}{6}$

If $2\le x<3$, $\quad F(x)=P(X\le x)=f_1+f_2=\frac{1}{6}+\frac{1}{6}=\frac{2}{6}$

and so on.

Thus the distribution function $F(x)$ is given by :

$F(x)=0,\ -\infty<x<1$

$=\frac{1}{6},\ 1\le x<2$

$=\frac{2}{6},\ 2\le x<3$

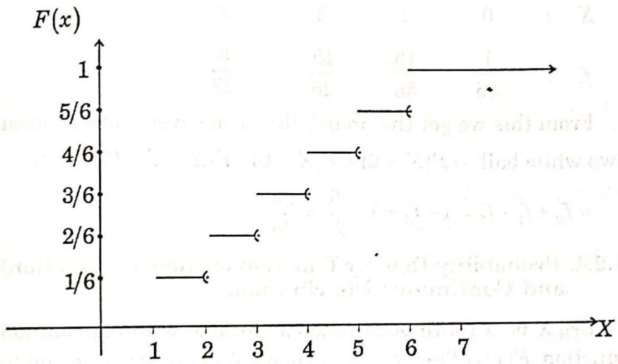
$=\frac{3}{6},\ 3\le x<4$

$=\frac{4}{6},\ 4\le x<5$

$=\frac{5}{6},\ 5\le x<6$

$=1,\ 6\le x<\infty$

The graph of the distribution $F(x)$ is as follow :



From the graph it is clear that $F(x)$ is a step function, non decreasing and is continuous on right at x = 1, 2, 3, ..., 6 and has a jump discontinuity on left at 1, 2, 3, ..., 6, the height of jump is $\frac{1}{6}$. Also, $F(-\infty)=0$, $F(\infty)=1$

(ii) Let three balls be drawn at random from a bag containing 5 white and 3 black balls ; X denotes the number of white balls drawn.

Then X can assume the values 0, 1, 2, 3.

Here, $f_0=P(X=0)=$Probability of no white ball

$$={}^3C_3/{}^8C_3=\frac{1}{56}$$

$f_1=P(X=1)=$ Probability of one white ball

$$={}^5C_1\times{}^3C_2/{}^8C_3=\frac{15}{56}$$

$$f_2 = P(X=2) = {}^5C_2 \times {}^3C_1 / {}^8C_3 = \frac{15}{28}$$

$$f_3 = P(X=3) = {}^5C_3 / {}^8C_3 = \frac{5}{28}$$

Then the distribution of X is

X :	0	1	2	3
f_i :	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{15}{28}$	$\frac{5}{28}$

From this we get the probability of the event like 'at most two white ball' $= P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$$= f_0 + f_1 + f_2 = 1 - f_3 = 1 - \frac{5}{28} = \frac{23}{28}.$$

1.2.4. Probability Density Function (or density function) and Continuous Distribution.

Let X be a continuous random variable with distribution function $F(x) = P(-\infty < X \le x)$ Then a function $f(x)$ is said to be probability density function (*pdf*) of X if $f(x)$ is integrable on the interval $[a, x]$ for all a and if

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

This holds for every real x.

Fundamental Properties of pdf.

If $f(x)$ is a *pdf* of a random variable X then

(i) $f(x) \ge 0$ (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Proof : (i) From definition $F'(x) = f(x)$. Since $F(x)$ is increasing so $F'(x) \ge 0$

$\therefore f(x) \ge 0$

(ii) Since $F(\infty) = 1$ so, $\int_{-\infty}^{\infty} f(t)dt = F(\infty) = 1$.

Properties of pdf.

(i) As $F(x)$ is a continuous function, so we must have

$$P(X=a) = F(a) - \lim_{x \to a-} F(x) = 0.$$

(ii) For a continuous distribution,

$$P(a \le X \le b) = P(a < X < b) = P(a < X \le b)$$

$$= P(a \le X < b) = \int_a^b f(t)dt = F(b) - F(a),$$

where f is the probability density function.

(iii) If $f(x)$ is continuous, then from (definition of pdf), we must have $f(x) = F'(x)$

(iv) In differential notation we have,

$$P(x < X \le x + dx) = F(x+dx) - F(x) = dF(x) = F'(x)dx = f(x)\,dx$$

which is known as the prob. differential of X.

Density Curve. The curve given by $y = f(x)$ ($f(x)$ is pdf) is called the probability density curve which gives the graphical representation of the corresponding continuous distribution.

Illustration. Consider a function $f(x)$ ($f(x)$ is pdf) which is defined as $f(x) = \frac{2}{x^3}$, $1 \le x < \infty$

$= 0$, elsewhere.

As $f(x) \ge 0$ everywhere and $\int_{-\infty}^{\infty} f(x)dx$

$$= \int_1^{\infty} \frac{2}{x^3}dx = \underset{P\to\infty}{Lt} \int_1^{P} \frac{2}{x^3}dx = \underset{P\to\infty}{Lt}\left(1 - \frac{1}{P^2}\right) = 1 - 0 = 1,$$

so this $f(x)$ is a probability density function of some random variable.

Now $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} 0 \cdot dx = 0$ when $-\infty < x < 1$

and $F(x) = \int_{-\infty}^{x} f(x)dx = \int_1^{x} \frac{2}{x^3}dx$ when $1 \le x < \infty$

$$= 1 - \frac{1}{x^2}.$$

So the distribution function of the above *pdf* is

$$F(x) = 0, \quad -\infty < x < 1$$

$$= 1 - \frac{1}{x^2}, \quad 1 \le x < \infty.$$

1.2.5. Expectation or Mean of a Random Variable

Expectation of a Discrete Random Variable

Let X be a discrete random variable whose distribution is

X	:	x_0	x_1	x_2	$\cdots$	x_n	$\cdots$
f_i	:	f_0	f_1	f_2	$\cdots$	f_n	$\cdots$

Then the mean or expectation or expected value of X, denoted by $E(X)$ or $m(X)$ or simply m is defined as

$E(X) = x_0 f_0 + x_1 f_1 + x_2 f_2 + \cdots \cdots = \sum x_i f_i$, provided the series is absolutely convergent if the above sum is an infinite series.

Expectation for Continuous Random Variable.

For a continuous random variable X with probability density function $f(x)$, the mean or expectation of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx,$$

provided the infinite integral converges absolutely.

Similarly, the mean of a function $\psi(X)$ of the random variable X denoted by $E\{\psi(X)\}$ is defined as

$$E\{\psi(X)\} = \sum_i \psi(X_i) f_i, \quad \text{for a discrete distribution}$$

$$= \int_{-\infty}^{\infty} \psi(x) f(x) dx, \quad \text{for a continuous distribution.}$$

Illustration. (i) Suppose a die is rolled. Let X be the number of points on the die. Then its values are 1, 2, 3, 4, 5, 6.

$\therefore\ P(X = i) = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$.

So the distribution of X is

X	:	1	2	3	4	5	6
f_i	:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Therefore its expectation,

$$E(X) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + \cdots + 6\cdot\frac{1}{6} = \frac{7}{2} \qquad \text{[W.B.U.T. 2013]}$$

and $E(X^2+1) = (1^2+1)\cdot\frac{1}{6} + (2^2+1)\cdot\frac{1}{6} + (3^2+1)\cdot\frac{1}{6} + (4^2+1)\cdot\frac{1}{6} + (5^2+1)\cdot\frac{1}{6} + (6^2+1)\cdot\frac{1}{6} = \frac{97}{6}$

(ii) Let the p.d.f of a continuous random variable X is

$$f(x) = \frac{1}{2} \quad \text{in } -1 < x < 1$$

$$= 0, \quad \text{elsewhere.}$$

Then the mean or expectation of X is

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-1}^{1} x\,\frac{1}{2}\, dx = 0$$

and also, $E(2X^3) = \int_{-\infty}^{\infty} 2x^3 f(x) dx = \int_{-1}^{1} 2x^3\cdot\frac{1}{2} dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$

Properties of Expectation

(i) $E(a) = a$, where a is a constant.

(ii) $E(aX) = aE(X)$, a being a constant.

(iii) $E(X \pm Y) = E(X) \pm E(Y)$, X, Y are two r.v

(iv) $E(XY) = E(X)E(Y)$ if the two r.v X and Y are independent.

Proof: Left as an exercise.

Remark. The mean has an important physical significance. In fact this represents the centre of mass of the probability distribution.

Illustration. A number is chosen at random from the set $\{1, 2, \cdots 100\}$ and another number is chosen at random from the set $\{1, 2, 3, \cdots, 50\}$. We are to find the expected value of the product.

Let $X =$ first number and $Y =$ second number. So the distribution of X is

X	:	1	2	3	...	100
f_i	:	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	...	$\frac{1}{100}$

and the distribution of Y is

Y	:	1	2	3	...	50
f_i	:	$\frac{1}{50}$	$\frac{1}{50}$	$\frac{1}{50}$	...	$\frac{1}{50}$

So,

$$E(X) = \frac{1}{100}\cdot 1 + \frac{1}{100}\cdot 2 + \cdots + \frac{1}{100}\cdot 100 = \frac{1}{100}(1+2+3+\cdots+100)$$

$$= \frac{1}{100}\cdot\frac{100(100+1)}{2} = \frac{101}{2}$$

and $E(Y) = \frac{1}{50}(1+2+3+\cdots+50) = \frac{1}{50}\cdot\frac{50(50+1)}{2} = \frac{51}{2}$

The expected value of the product

$= E(XY) = E(X)E(Y)$ [$\because$ the two numbers are drawn independently]

$$= \frac{101}{2}\times\frac{51}{2} = \frac{5151}{4}.$$

1.2.6. Variance and S.D.

The variance of a r.v X, denoted by $Var(X)$ is defined as $Var(X) = E\left((X-m)^2\right)$, where $m = E(X)$

The positive square root of $Var(X)$ is called the *standard deviation* of X and is denoted by $\sigma(X)$ or σ_x or simply σ.

Thus $\sigma = +\sqrt{var(X)}$.

Remarks : (i) The variance describes how widely the probability masses are spread about the mean i.e it gives an inverse measure of concentration of the probability masses about the mean which is called the measure of dispersion.

(ii) As $Var(X) = 0$ only when $X - m = 0$ i.e, $X = m$, so in that case the whole mass is concentrated at the mean.

Theorem.

(i) $Var(X) = E(X^2) - m^2 = E(X^2) - \{E(X)\}^2$

(ii) $Var(aX+b) = a^2 Var(X)$

(iii) $Var(k) = 0$ where k is constant.

(iv) $Var(X) = E\{X(X-1)\} - m(m-1)$ where m is mean of X.

Proof: (i) $Var(X) = E\left\{(X-m)^2\right\} = E\left(X^2 - 2mX + m^2\right)$

$$= E(X^2) - E(2mX) + E(m^2)$$

$$= E(X^2) - 2mE(X) + m^2 = E(X^2) - 2m\cdot m + m^2 = E(X^2) - m^2$$

(ii) and (iii) are left as exercise.

(iv) $(X-m)^2 = X(X-1) - 2mX + X + m^2$

$$\therefore\ E\left\{(X-m)^2\right\} = E\{X(X-1)\} - 2mE(X) + E(X) + E(m^2)$$

$$= E\{X(X-1)\} - 2m\cdot m + m + m^2$$

$$= E\{X(X-1)\} - m(m-1).$$

Note : In fact the result (i) and (iv) of the above theorem are used to evaluate variance and standard deviation.

Illustration. Consider the following distribution of a random variable X :

$$f(x) = \frac{1}{2}x,\quad 0 \le x \le 2$$
$$= 0,\quad \text{elsewhere.}$$

Then the expectation of X is

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_0^2 x\cdot\frac{1}{2}x\, dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3}.$$

$$\text{Now, } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^2 x^2\cdot\frac{1}{2}x\, dx = \left[\frac{x^4}{8}\right]_0^2 = 2$$

$$\therefore\ Var(X) = E(X^2) - \{E(X)\}^2 = 2 - \frac{16}{9} = \frac{2}{9}.$$

So, s.d $= \sqrt{2}/3$.

1.2.7. Illustrative Examples.

Ex. 1. *Find the probability distribution (or probability function or p.m.f) of the number of heads when a fair coin is tossed repeatedly until the first tail appears.*

The sample space corresponding to the random experiment of tossing a fair coin is $S = \{T, HT, HHT, HHHT, \cdots\}$.

Let the random variable X denote "the number of heads in the experiment until the first tail appears".

Then the spectrum of X is $\{0, 1, 2, 3, \cdots\}$

Now, $P(X=0) = P(T) = \frac{1}{2}$

$P(X=1) = P(HT) = P(H)\cdot P(T)$

[$\because$ the trials are independent]

$= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^2}$

$P(X=2) = P(HHT) = P(H)P(H)P(T) = \frac{1}{2^3}$ and so on.

Hence the probability distribution of X is

X :	0	1	2	3	...	...
f_i :	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	...	...

Ex. 2. *A random variable X has the following probability mass function* [W.B.U.Tech 2007]

X :	0	1	2	3	4	5	6	7
$P(X=k)=f(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Determine the constant k

(ii) Evaluate $P(X<6), P(X\ge 6), P(3<X\le 6)$ and $P(3<X/X\le 6)$

(iii) Find the minimum value of x so that $P(X\le x) > \frac{1}{2}$

(iv) Obtain the distribution function $F(x)$. [W.B.U.Tech 2004]

(i) Since $f(x)$ is a p.m.f, $\sum_x f(x) = 1$

$\therefore \sum_{x=0}^{7} f(x) = 1 \Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow 10k^2 + 9k - 1 = 0 \Rightarrow (10k-1)(k+1) = 0$

$\Rightarrow k = -1, \frac{1}{10}$

$\therefore k = \frac{1}{10}$ [$\because f(x) \ge 0, \ \forall\ x = 0, 1, 2, \cdots 7$ and so $k \ne -1$]

(ii) $P(X<6) = 1 - P(X\ge 6) = 1 - \{P(X=6) + P(X=7)\}$

$= 1 - \left\{2\left(\frac{1}{10}\right)^2 + 7\cdot\left(\frac{1}{10}\right)^2 + \frac{1}{10}\right\} = \frac{81}{100}$

$\therefore P(X\ge 6) = 1 - P(X<6) = 1 - \frac{81}{100} = \frac{19}{100}$

$P(3<X\le 6) = P(X=4) + P(X=5) + P(X=6) = \frac{33}{100}$

$P(3<X/X\le 6) = \frac{P\{(3<X)\cap(X\le 6)\}}{P(X\le 6)} = \frac{P(3<X\le 6)}{P(X\le 6)} = \frac{33/100}{83/100} = \frac{33}{83}$

(iii) Now $P(X\le 1) = \frac{1}{10} < \frac{1}{2}$,

$P(X\le 2) = P(X=0) + P(X=1) + P(X=2) = \frac{1}{10} + 2\cdot\frac{1}{10} = \frac{3}{10} < \frac{1}{2}$

$P(X\le 3) = P(X=1) + P(X=2) + P(X=3)$

$= \frac{1}{10} + 2\cdot\frac{1}{10} + 2\cdot\frac{1}{10} = \frac{1}{2}$

$P(X\le 4) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= \frac{8}{10} = \frac{4}{5} > \frac{1}{2}$.

Thus the minimum value of x so that $P(X\le x) > \frac{1}{2}$ is 4.

(iv) The distribution function $F(x)$ is given below

$$F(x)=\begin{cases} 0, & -\infty < x < 1 \\ \frac{1}{10} & 1 \le x < 2 \\ \frac{1}{10}+2\cdot\frac{1}{10}=\frac{3}{10} & 2 \le x < 3 \\ \frac{3}{10}+2\cdot\frac{1}{10}=\frac{1}{2} & 3 \le x < 4 \\ \frac{1}{2}+3\cdot\frac{1}{10}=\frac{4}{5} & 4 \le x < 5 \\ \frac{4}{5}+\left(\frac{1}{10}\right)^2=\frac{81}{100} & 5 \le x < 6 \\ \frac{81}{100}+2\cdot\left(\frac{1}{10}\right)^2=\frac{83}{100} & 6 \le x < 7 \\ \frac{83}{100}+7\left(\frac{1}{10}\right)^2+\frac{1}{10}=1 & 7 \le x < \infty \end{cases}$$

Ex. 3. *Let* $F(x)=0, \quad -\infty < x < 0$

$$=\frac{1}{5}, \quad 0 \le x < 1$$
$$=\frac{3}{5}, \quad 1 \le x < 3$$
$$=1, \quad 3 \le x < \infty$$

Show that $F(x)$ *is a possible distribution function. Determine the spectrum and the probability mass of the distribution.* [W.B.U.Tech 2003]

Clearly $F(x)$ is monotonic non-decreasing function and is continuous on the right at all points.

Also, $F(-\infty)=0$ and $F(\infty)=1$. Hence $F(x)$ is a possible distribution function.

Again $F(x)$ is step function and step points are 0, 1, 3. So the spectrum is {0, 1, 3 }.

Now, $P(X=0)=F(0)-\lim_{x\to 0-} F(x)=\frac{1}{5}-0=\frac{1}{5}$

$P(X=1)=F(1)-\lim_{x\to 1-} F(x)=\frac{3}{5}-\frac{1}{5}=\frac{2}{5}$

$P(X=3)=F(3)-\lim_{x\to 3-} F(x)=1-\frac{3}{5}=\frac{2}{5}$.

So the required prob. mass of the distribution is

X	:	0	1	3
f_i	:	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

Ex. 4. *The distribution function* $F(x)$ *of a variate X is defined as follows*

$$F(x)=A, \quad -\infty < x < -1$$
$$=B, \quad -1 \le x < 0$$
$$=C, \quad 0 \le x < 2$$
$$=D, \quad 2 \le x < \infty$$

where A,B,C, D are constants. Determine the values of A, B, C, D, given that $P(X=0)=\frac{1}{6}$ *and* $P(X>1)=\frac{2}{3}$. [W.B.U.Tech 2004]

We have $F(-\infty)=0$

$\therefore \lim_{x\to-\infty} F(x)=0$ i.e., $\lim_{x\to-\infty}(A)=0 \quad \therefore A=0$

Again $F(\infty)=1 \quad \therefore \lim_{x\to\infty} F(x)=1$

i.e., $\lim_{x\to\infty}(D)=1 \quad \therefore D=1$

Now, $\frac{1}{6}=P(X=0)=F(0)-\lim_{x\to 0-} F(x)$

$$[\because P(X=a)=F(a)-\lim_{x\to a-} F(x)]$$

$$=C-\lim_{x\to 0-} F(x) \;=C-B$$

$$\therefore C-B=\frac{1}{6} \quad \dots (1)$$

Again $P(-\infty < X < \infty) = P(-\infty < X \le 1) + P(1 < X < \infty)$

$\therefore\ 1 = P(-\infty < X \le 1) + P(X > 1) \quad = P(-\infty < X \le 1) + \frac{2}{3}$

$\therefore P(-\infty < X \le 1) = 1 - \frac{2}{3}$

i.e., $F(1) = \frac{1}{3}$ $\quad [\because F(x) = P(-\infty < X \le x)]$

$\therefore\ C = \frac{1}{3}$

$\therefore$ From (1) $B = C - \frac{1}{6} = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

$\therefore A = 0,\ B = \frac{1}{6},\ C = \frac{1}{3},\ D = 1$

Ex. 5. *The probability density function of a random variable X is* $f(x) = k\ (x-1)(2-x)$ *for* $1 \le x \le 2$. *Determine (i) the value of the constant* k *; (ii) the distribution function* $F(x)$ *(iii)* $P\left(\frac{5}{4} \le X \le \frac{3}{2}\right)$. [*W.B.U.Tech 2007*]

(i) Since $f(x)$ is a p.d.f,

we have, $\int_{-\infty}^{\infty} f(x)dx = 1$

i.e., $k\int_{1}^{2}(x-1)(2-x)dx = 1$ i.e., $k\int_{1}^{2}(3x - x^2 - 2)dx = 1$

i.e., $k\left[\frac{3x^2}{2} - \frac{x^3}{3} - 2x\right]_1^2 = 1$

i.e., $\frac{k}{6} = 1 \quad \therefore k = 6$.

$\therefore$ The complete p.d.f of X is therefore

$f(x) = 6(x-1)(2-x)$, for $1 \le x \le 2$

$= 0$, elsewhere.

(ii) The distribution function $F(x)$ is defined by

$F(x) = \int_{-\infty}^{x} f(x)\,dx \quad \therefore\ F(x) = 0$, for $x < 1$

Also $F(x) = \int_{-\infty}^{x} f(x)\,dx + \int_{1}^{x} f(x)dx$, for $1 \le x \le 2$

$$= \int_{-\infty}^{x} 0 \cdot dx + \int_{1}^{x} 6(x-1)(2-x)\,dx$$

$$= 6\left[\frac{3x^2}{2} - \frac{x^3}{3} - 2x\right]_1^x = 5 - 12x + 9x^2 - 2x^3.$$

Again $F(x) = \int_{-\infty}^{1} 0 \cdot dx + \int_{1}^{2} 6(x-1)(2-x)\,dx + \int_{2}^{x} 0 \cdot dx$, for $x > 2$

$$= 0 + 6\left[\frac{3x^2}{2} - \frac{x^3}{3} - 2x\right]_1^2 + 0 = 1.$$

Therefore the distribution function is given by :

$F(x) = 0$, for $x < 1$

$= 5 - 12x + 9x^2 - 2x^3$, for $1 \le x \le 2$

$= 1$, for $x > 2$

(iii) Using the properties of distribution function

$$P\left(\frac{5}{4} \le X \le \frac{3}{2}\right) = F\left(\frac{3}{2}\right) - F\left(\frac{5}{4}\right)$$

$$= \left\{5 - 12\cdot\frac{3}{2} + 9\cdot\left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right)^3\right\} - \left\{5 - 12\cdot\frac{5}{4} + 9\cdot\left(\frac{5}{4}\right)^2 - 2\cdot\left(\frac{5}{4}\right)^3\right\}$$

$$= \frac{1}{2} - \frac{5}{32} = \frac{11}{32}$$

Alternative by (Using p.d.f) :

$$P\left(\frac{5}{4}\le X\le\frac{3}{2}\right)=\int_{5/4}^{3/2}f(x)\,dx=\int_{5/4}^{3/2}6(x-1)(2-x)dx=\frac{11}{32}.$$

Ex. 6. *A random variable X has the density function*

$$f(x)=\frac{a}{x^2+1},\quad -\infty<x<\infty.$$

Find (i) a (ii) the probability that X^2 lies between $\frac{1}{3}$ and 1 (iii) the distribution function of X.

(i) As $f(x)$ is a p.d.f, so we must have $\int_{-\infty}^{\infty}f(x)\,dx=1$

i.e., $\int_{-\infty}^{\infty}\frac{a}{1+x^2}\,dx=1$

i.e., $a\left[\tan^{-1}x\right]_{-\infty}^{\infty}=1$ i.e., $a\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=1 \quad \therefore\ a=\frac{1}{\pi}$

(ii) As $\frac{1}{3}\le X^2\le 1\Rightarrow\left(-1\le X\le-\frac{1}{\sqrt{3}}\right)\cup\left(\frac{1}{\sqrt{3}}\le X\le 1\right)$

$$P\left(\frac{1}{3}\le X^2\le 1\right)=P\left(-1\le X\le-\frac{1}{\sqrt{3}}\right)+P\left(\frac{1}{\sqrt{3}}\le X\le 1\right)$$

$$=\frac{1}{\pi}\int_{-1}^{-\frac{1}{\sqrt{3}}}\frac{dx}{1+x^2}+\frac{1}{\pi}\int_{\frac{1}{\sqrt{3}}}^{1}\frac{dx}{1+x^2}$$

$$=\frac{1}{\pi}\left[\tan^{-1}x\right]_{-1}^{-\frac{1}{\sqrt{3}}}+\frac{1}{\pi}\left[\tan^{-1}x\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$=\frac{1}{\pi}\left(-\frac{\pi}{6}+\frac{\pi}{4}\right)+\frac{1}{\pi}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\frac{1}{6}.$$

(iii) The distribution function of X is given by

$$F(x)=\frac{1}{\pi}\int_{-\infty}^{x}\frac{dx}{1+x^2}=\frac{1}{\pi}\left[\tan^{-1}x\right]_{-\infty}^{x}=\frac{1}{\pi}\left(\tan^{-1}x+\frac{\pi}{2}\right).$$

Ex. 7. *A random variable X is exponentially distributed with p.d.f*

$$f(x)=\frac{1}{40}e^{-\frac{x}{40}},\ x>0$$

$$=0,\qquad x\le 0$$

Find (i) $P(X\le 20)$ (ii) $P(32\le X\le 48)$ (iii) $P(X\ge 25)$

(i) $P(X\le 20)=\int_0^{20}\frac{1}{40}e^{-\frac{x}{40}}dx=\left[-e^{-\frac{x}{40}}\right]_0^{20}=1-e^{-\frac{1}{2}}$

(ii) $P(32\le X\le 48)=\int_{32}^{48}\frac{1}{40}e^{-\frac{x}{40}}dx=\left[-e^{-\frac{x}{40}}\right]_{32}^{48}=e^{-\frac{4}{5}}-e^{-\frac{6}{5}}$

(iii) $P(X\ge 25)=1-P(X<25)=1-\int_0^{25}\frac{1}{40}e^{-\frac{x}{40}}dx$

$$=1-\left[-e^{-\frac{x}{40}}\right]_0^{25}=1-\left(1-e^{-\frac{5}{8}}\right)=e^{-\frac{5}{8}}.$$

Ex. 8. *A special un-biased die with $n+1$ faces is rolled. Its faces are marked by the number $0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n},\frac{n}{n}$. If X denotes the number shown then find (i) the expectation of X (ii) standard deviation of X (iii) $E\left(X-\frac{1}{2}\right)^3$.*

Note that probability of each face is $\frac{1}{n+1}$. So, the distribution of X is

X	:	0	$\frac{1}{n}$	$\frac{2}{n}$	....	$\frac{n-1}{n}$	$\frac{n}{n}$
f_i	:	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$	....	$\frac{1}{n+1}$	$\frac{1}{n+1}$

Obviously X assumes values $x_i=\frac{i}{n}$.

(i) $E(X) = 0 \times \frac{1}{n+1} + \frac{1}{n} \times \frac{1}{n+1} + \frac{2}{n} \times \frac{1}{n+1} + \cdots + \frac{n-1}{n} \times \frac{1}{n+1} + \frac{n}{n} \times \frac{1}{n+1}$

$$= \sum_{i=0}^{n} \frac{i}{n} \cdot \frac{1}{n+1} = \frac{1}{n(n+1)} \sum_{i=0}^{n} i = \frac{1}{n(n+1)} \cdot \frac{n(n+1)}{2} = \frac{1}{2}$$

(ii) $E\left(X^2\right) = \sum_{i=0}^{n} x_i^2 f_i = \sum_{i=0}^{n} \left(\frac{i}{n}\right)^2 \times \frac{1}{n+1} = \frac{1}{n^2(n+1)} \sum_{i=0}^{n} i^2$

$$= \frac{1}{n^2(n+1)} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{2n+1}{6n}$$

$\therefore Var(X) = E\left(X^2\right) - \{E(X)\}^2 = \frac{2n+1}{6n} - \frac{1}{4} = \frac{n+2}{12n}$

$\therefore \sigma_X = \sqrt{\frac{n+2}{12n}}$

(iii) $E\left(X - \frac{1}{2}\right)^3 = E\left(X^3\right) - \frac{3}{2} E\left(X^2\right) + \frac{3}{4} E(X) - \frac{1}{8}$

Now, $E\left(X^3\right) = \sum_{i=0}^{n} x_i^3 f_i = \sum_{i=0}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n+1} = \frac{1}{n^3(n+1)} \sum_{i=0}^{n} i^3$

$$= \frac{1}{n^3(n+1)} \left(1^3 + 2^3 + \cdots + n^3\right) = \frac{1}{n^3(n+1)} \left\{\frac{n(n+1)}{2}\right\}^2 = \frac{n+1}{4n}$$

Hence $E\left(X - \frac{1}{2}\right)^3 = 0$.

Ex. 9. *If t is a positive real number and n is a discrete r.v assuming the values $0, 1, 2, \ldots, \infty$ with p.m.f $P(X = i) = e^{-t}\left(1 - e^{-t}\right)^{i-1}$. Find the mean and $E(3X+2)$.*

The p.m.f. $f_i = e^{-t}\left(1 - e^{-t}\right)^{i-1}$

$\therefore$ the mean, $E(X) = \sum_{i=0}^{\infty} i\, f_i = \sum_{i=0}^{\infty} i e^{-t}\left(1 - e^{-t}\right)^{i-1}$

$$= e^{-t} \sum_{i=0}^{\infty} i\left(1 - e^{-t}\right)^{i-1} = e^{-t} \sum_{i=0}^{\infty} i z^{i-1} \quad [\text{ put } z = 1 - e^{-t}]$$

$$= e^{-t}\left(1 + 2z + 3z^2 + \cdots up\ to\ \infty\right) = e^{-t}(1-z)^{-2} \quad [\because z < 1]$$

$$= e^{-t}\left\{1 - \left(1 - e^{-t}\right)\right\}^{-2} = e^{t}$$

Now $E(3X+2) = 3E(X) + 2 = 3e^t + 2$

Ex. 10. *If a person gains or loses an amount equal to the number appearing when a balanced die is rolled once according to whether the number is even or odd, how much money can he expect from the game in the long run ?*

Let $X =$ amount of gain. Then X may assume the values $-1, 2, -3, 4, -5$ and 6.

Now, $P(X = -1) =$ Prob. of 'face 1' $= \frac{1}{6}$

$P(X = 2) =$ Prob. of 'face 2' $= \frac{1}{6}$

$P(X = -3) =$ Prob. of 'face 3' $= \frac{1}{6}$

and so on.

Therefore the distribution of X is

X	:	-1	2	-3	4	-5	6
f_i	:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$\therefore$ The amount of expected money $= E(X)$

$$= -1 \times \frac{1}{6} + 2 \times \frac{1}{6} + (-3) \times \frac{1}{6} + 4 \times \frac{1}{6} + (-5) \times \frac{1}{6} + 6 \times \frac{1}{6}$$

$$= \frac{1}{6}(-1 + 2 - 3 + 4 - 5 + 6) = \frac{1}{6} \times 3 = \frac{1}{2}.$$

Ex. 11. *If a person gets Rs. $(2x+5)$, where x denotes the number appearing when a balanced die is rolled once, then how much money can he expect in the long run per game ?*

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By problem $x =$ number appearing on the die.

$\therefore$ the distribution of x is

x	:	1	2	3	4	5	6
f_i	:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Now, $E(x) = 1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}+6\cdot\frac{1}{6}=\frac{21}{6}=\frac{7}{2}$

Expected amount of money $= E(2x+5) = E(2x)+E(5)$

$$= 2E(x)+5 = 2\times\frac{7}{2}+5 = 12.$$

Ex. 12. *If the random variable X has p.d.f* $f(x)=\frac{1}{4}, -2\le x\le 2$, *find (i)* $P(X<1)$ *(ii)* $P\left(|X-1|\ge\frac{1}{2}\right)$

The p.d.f of X is given by

$$f(x)=\frac{1}{4} \quad \text{in } -2\le x\le 2$$
$$=0 \quad \text{elsewhere}$$

(i) $P(X<1) = \int_{-\infty}^{1} f(x)dx = \int_{-\infty}^{-2} f(x)dx + \int_{-2}^{1} f(x)dx$

$$= \int_{-\infty}^{-2} 0\,dx + \int_{-2}^{1}\frac{1}{4}\,dx = \frac{1}{4}(1+2)=\frac{3}{4}$$

(ii) Now $|X-1| > \frac{1}{2} \Rightarrow \left(X-1\ge\frac{1}{2}\right)\cup\left(X-1\le-\frac{1}{2}\right)$

$$\Rightarrow\left(\frac{3}{2}\le X<\infty\right)\cup\left(-\infty<X\le\frac{1}{2}\right)$$

$\therefore P\left(|X-1|>\frac{1}{2}\right) = P\left(\frac{3}{2}\le X<\infty\right)+P\left(-\infty<X\le\frac{1}{2}\right)$

$$= \int_{\frac{3}{2}}^{\infty} f(x)dx + \int_{-\infty}^{\frac{1}{2}} f(x)dx$$

$$= \int_{\frac{3}{2}}^{2}\frac{1}{4}dx+0+0+\int_{-2}^{\frac{1}{2}}\frac{1}{4}dx$$

$$= \frac{1}{4}\left(2-\frac{3}{2}\right)+\frac{1}{4}\left(\frac{1}{2}+2\right)=\frac{3}{4}.$$

Ex. 13. *Consider the p.d.f* $f(x)=ae^{-b|x|}$ *where x is a random variable whose allowable range of value are from* $x=-\infty$ *to* ∞. *Find (i) cumulative distribution function (ii) the relation between a and b (iii)* $P(1\le X\le 2)$.

(i) The cumulative distribution function is given by

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} ae^{bx}dx, \text{ for } -\infty<x<0$$

$$= \frac{a}{b}\left[e^{bx}\right]_{-\infty}^{x} = \frac{a}{b}e^{bx}$$

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx \text{ for } 0\le x<\infty$$

$$= \int_{-\infty}^{0} ae^{bx}dx + \int_{0}^{x} ae^{-bx}dx = \frac{a}{b}\left[e^{bx}\right]_{-\infty}^{0}+\frac{a}{b}\left[-e^{-bx}\right]_{0}^{x}$$

$$= \frac{a}{b}-\frac{a}{b}e^{-bx}+\frac{a}{b}=\frac{a}{b}\left(2-e^{-bx}\right)$$

Thus $F(x) = \frac{a}{b}e^{bx}, \quad x<0$

$$= \frac{a}{b}\left(2-e^{-bx}\right), \; x\ge 0$$

(ii) Since $F(\infty)=1$, so, $\frac{a}{b}(2-0)=1$

i.e., $\frac{2a}{b}=1 \;\therefore\; b=2a$

(iii) Now $P(1\le X\le 2) = F(2)-F(1)$

$$= \frac{a}{b}\left(2-e^{-2b}-2+e^{-b}\right) = \frac{e^{-b}}{2}\left(1-e^{-b}\right).$$

Ex.14. *Show that a function which is $|x|$ in $(-1,1)$ and zero elsewhere is a possible p.d.f and find the corresponding distribution function.* [W.B.U.T. 2013, 2006]

Let us denote the given function by $f(x)$. Then

$$f(x) = |x| \text{ in } -1 < x < 1$$
$$= 0, \quad \text{elsewhere}$$

i.e.,
$$f(x) = -x, \quad -1 < x \le 0$$
$$= x, \quad 0 < x < 1$$
$$= 0, \quad \text{elsewhere}$$

Clearly we see that $f(x) \ge 0$ everywhere.

Also, $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$

$$= 0 + \int_{-1}^{0} (-x)dx + \int_{0}^{1} x\,dx + 0 = \frac{1}{2} + \frac{1}{2} = 1.$$

Hence $f(x)$ is a possible prob. density function.

Now the distribution function $F(x)$ is given by

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} 0 \cdot dx = 0, \text{ for } -\infty < x \le -1$$

$$F(x) = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{x} f(x)dx, \text{ for } -1 < x \le 0$$

$$= \int_{-\infty}^{-1} 0 \cdot dx + \int_{-1}^{x} (-x)dx = \frac{1}{2}\left(1 - x^2\right)$$

$$F(x) = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx + \int_{0}^{x} f(x)dx, \text{ for } 0 < x \le 1$$

$$= 0 + \int_{-1}^{0} (-x)dx + \int_{0}^{x} x\,dx = \frac{1}{2}\left(1 + x^2\right), \text{ for } 0 < x \le 1.$$

$$F(x) = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)dx + \int_{1}^{x} f(x), \text{ for } 1 < x < \infty$$

$$= 0 + \int_{-1}^{0} (-x)dx + \int_{0}^{1} x\,dx + 0 = \frac{1}{2} + \frac{1}{2} = 1.$$

So the distribution function is given by

$$F(x) = 0, \quad -\infty < x \le -1$$
$$= \frac{1}{2}\left(1 - x^2\right), \quad -1 < x \le 0$$
$$= \frac{1}{2}\left(1 + x^2\right), \quad 0 < x \le 1$$
$$= 1, \quad 1 < x < \infty$$

Ex. 15. *X is a continuous random variable having p.d.f*

$$f(x) = \frac{4x}{5}, \quad 0 < x \le 1$$
$$= \frac{2}{5}(3 - x), \quad 1 < x \le 2$$
$$= 0, \quad \textit{elsewhere.}$$

Find the mean value of X.

Mean of $X = E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$

$$= \int_{-\infty}^{0} x \times 0\, dx + \int_{0}^{1} x \cdot \frac{4x}{5} dx + \int_{1}^{2} x \frac{2}{5}(3 - x)dx + \int_{2}^{\infty} x \cdot 0\, dx$$

$$= \frac{4}{5}\int_{0}^{1} x^2 dx + \frac{2}{5}\int_{1}^{2} x(3 - x)dx = \frac{17}{15}$$

Ex. 16. *Find the mean, variance of a continuous random variable having p.d.f.*

$$f(x) = 1 - |1 - x|, \quad 0 < x < 2$$
$$= 0, \quad \textit{elsewhere}$$

Find also $E\left(X^3\right)$

The mean

$$= E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_{-\infty}^{0} x \cdot 0 dx + \int_{0}^{2} x\{1 - |1 - x|\}dx + \int_{2}^{\infty} x \cdot 0\, dx$$

$$= \int_0^2 x\{1-|1-x|\}dx = \int_0^1 x\{1-(1-x)\}dx + \int_1^2 x\{1+(1-x)\}dx$$

$$= \int_0^1 x^2 dx + \int_1^2 (2x - x^2)dx = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^2 x^2\{1-|1-x|\}dx$$

$$= \int_0^1 x^2\{1-(1-x)\}dx + \int_1^2 x^2\{1+(1-x)\}dx$$

$$= \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3)dx = \frac{7}{6}$$

So, $Var(X) = E(X^2) - \{E(X)\}^2 = \frac{7}{6} - (1)^2 = \frac{1}{6}$

$$E(X^3) = \int_{-\infty}^{\infty} x^3 f(x)dx = \int_{-\infty}^{0} x^3 \cdot 0\, dx + \int_0^2 x^3\{1-|1-x|\}dx$$

$$+ \int_2^{\infty} x^3 \cdot 0 dx = \int_0^2 x^3\{1-|1-x|\}dx$$

$$= \int_0^1 x^3\{1-(1-x)\}dx + \int_1^2 x^3\{1+(1-x)\}dx = \frac{3}{2}$$

Ex. 17. *A variable X has the density function*

$$f(x) = \frac{x}{2}, \quad 0 \le x \le 1$$

$$= \frac{1}{2}, \quad 1 < x \le 2$$

$$= \frac{1}{2}(3-x), \quad 2 < x \le 3$$

Find the r… …ı and variance of X. [W.B.U.Tech 2002]

Mean $= E(X) = \int_0^3 x f(x)dx = \int_0^1 x \cdot \frac{x}{2}dx + \int_1^2 x \cdot \frac{1}{2}dx + \int_2^3 x \cdot \frac{1}{2}(3-x)dx$

$$= \frac{1}{6}(1-0) + \frac{1}{4}(4-1) + \frac{1}{2}\left(\frac{27}{2} - 9 - 6 + \frac{8}{3}\right) = \frac{3}{2}$$

$$E(X^2) = \int_0^3 x^2 f(x)dx = \int_0^1 x^2 \cdot \frac{x}{2}dx + \int_1^2 x^2 \cdot \frac{1}{2}dx + \int_2^3 x^2 \frac{1}{2}(3-x)dx$$

$$= \frac{8}{3}$$

$$\therefore \quad Var(X) = E(X^2) - \{E(X)\}^2 = \frac{8}{3} - \left(\frac{3}{2}\right)^2 = \frac{5}{12}$$

Ex. 18. *The demand for a new product of a company is assumed to be a random variable with p.d.f*

$$f(x) = \frac{x}{\lambda^2} e^{-\frac{x^2}{2\lambda^2}}, \quad x \ge 0$$

$$= 0, \quad x < 0$$

Find the mean, variance of this random variable and also find the probability that it will exceed λ.

Mean $= E(X) = \int_0^{\infty} x \cdot \frac{x}{\lambda^2} e^{-\frac{x^2}{2\lambda^2}} dx$

$$= \lambda\sqrt{2}\int_0^{\infty} u^{\frac{3}{2}-1} e^{-u} du, \text{ by putting } \frac{x^2}{2\lambda^2} = u$$

$$= \lambda\sqrt{2}\,\Gamma\left(\frac{3}{2}\right) = \lambda\sqrt{2}\frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \lambda\sqrt{2} \cdot \frac{1}{2}\sqrt{\pi} = \lambda\sqrt{\frac{\pi}{2}}$$

(using the property $\Gamma(n+1) = n\Gamma(n)$)

Now, $E(X^2) = \int_0^{\infty} x^2 \cdot \frac{x}{\lambda^2} e^{-\frac{x^2}{2\lambda^2}} dx$

$$= 2\lambda^2 \int_0^{\infty} u^{2-1} e^{-u} du, \text{ by putting } \frac{x^2}{2\lambda^2} = u$$

$$= 2\lambda^2 \Gamma(2) = 2\lambda^2 \qquad [\because \Gamma(2) = 1!\ ]$$

$\because\ Var(X) = E(X^2) - \{E(X)\}^2 = 2\cdot\lambda^2 - \lambda^2\frac{\pi}{2} = (4-\pi)\frac{\lambda^2}{2}$

Now, $P(X > \lambda) = 1 - P(X \le \lambda) = 1 - \int_0^{\lambda} \frac{x}{\lambda^2} e^{-\frac{x^2}{2\lambda^2}} dx$

$= 1 - \int_0^{\frac{1}{2}} e^{-u} du$, by putting $\frac{x^2}{2\lambda^2} = u$

$= 1 - \left(1 - e^{-\frac{1}{2}}\right) = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$.

Ex. 19. *A random variable X has probability density*

$f(x) = e^{-x}, \quad x \ge 0$

$= 0, \quad$ *otherwise*

Find (a) $E(X)$ *(b)* $E(X^2)$ *(c)* $E\{(X-1)^2\}$ *(d)* $E\left(e^{\frac{2X}{3}}\right)$.

(a) $E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x e^{-x} dx = \Gamma(2) = 1$

$\left[\because \text{the Gamma Function } \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx\right]$

(b) $E(X^2) = \int_0^{\infty} x^2 e^{-x} dx = \Gamma(3) = 2! = 2$

(c) $E\{(X-1)^2\} = \int_0^{\infty} (x-1)^2 e^{-x} dx$

$= \int_0^{\infty} x^2 e^{-x} dx - 2\int_0^{\infty} x e^{-x} dx + \int_0^{\infty} e^{-x} = \Gamma(3) - 2\ \Gamma(2) + \Gamma(1)$

$= 2! - 2\cdot 1 + 1 = 1$

(d) $E\left(e^{\frac{2X}{3}}\right) = \int_0^{\infty} e^{\frac{2x}{3}} \cdot e^{-x} dx = \int_0^{\infty} e^{-\frac{x}{3}} dx$

$= \underset{U\to\infty}{Lt} \int_0^{U} e^{-\frac{x}{3}} dx = \underset{U\to\infty}{Lt}\left(-3e^{-U/3} + 3\right) = 0 + 3 = 3.$

Ex. 20. *The distribution function of a random variable X is*

$F(X) = cx^3, \quad 0 \le x < 3$

$= 1, \quad x \ge 3$

$= 0, \quad x < 0$

If $P(X = 3) = 0$, find (i) the constant c (ii) the density function (iii) $P(X > 1)$ (iv) $P(1 < X \le 2)$ (v) $P(3X + 2 < 8)$

(i) The density function $f(x)$ is given by $f(x) = F'(x)$

$\therefore\ f(x) = 3cx^2, \quad 0 \le x < 3$

$= 0, \quad$ elsewhere

Also we have $\int_{-\infty}^{\infty} f(x) dx = 1 \quad \therefore \quad \int_0^3 3cx^2 dx = 1$

i.e., $27c = 1 \quad \therefore\ c = \frac{1}{27}$.

(ii) The required density function is given by

$f(x) = \frac{x^2}{9}, \quad 0 \le x < 3$

$= 0, \quad$ elsewhere

(iii) $P(X > 1) = \int_1^{\infty} f(x) dx = \frac{1}{9}\int_1^3 x^2 dx = \frac{26}{27}$

(iv) $P(1 < X \le 2) = \int_1^2 f(x) dx = \frac{1}{9}\int_1^2 x^2 dx = \frac{2}{27}$

(v) $P(3X + 2 < 8) = P(3X < 6) = P(X < 2)$

$= \int_{-\infty}^{2} f(x) dx = \int_{-\infty}^{0} 0 \cdot dx + \int_0^2 \frac{x^2}{9} dx = \frac{8}{27}$.

Ex. 21. *Verify that the following is a distribution function*

$F(x) = 0, \quad x < -a$

$= \frac{1}{2}\left(\frac{x}{a} + 1\right), \quad -a \le x \le a$

$= 1, \quad x > a$

In order that $F(x)$ may be a distribution function we should have

(i) $F(-\infty)=0$ (ii) $F(\infty)=1$

(iii) $\frac{d\,F(x)}{dx}=f(x)\ge 0,\ -a\le x\le a$ (iv) $\int_{-\infty}^{\infty} f(x)\,dx=1.$

As $F(x)=0$ for $x<-a$ and $F(x)=1$ for $x>a$, so the conditions (i) and (ii) are satisfied.

Now $\frac{d\,F(x)}{dx}=\frac{1}{2a}\ge 0$ for $-a\le x\le a$

Again $\int_{-\infty}^{\infty} f(x)\,dx=\int_{-\infty}^{-a} 0\cdot dx+\int_{-a}^{a}\frac{1}{2a}dx+\int_{a}^{\infty} 0\cdot dx=\left[\frac{x}{2a}\right]_{-a}^{a}=1$

Hence $F(x)$ is a distribution function.

Ex. 22. *A continuous cumulative distribution function $F(x)$ is defined as follow:*

$$F(x)=0,\quad x\le 1$$
$$=\frac{1}{16}(x-1)^4,\quad 1<x\le 3$$
$$=1,\quad x>3$$

Find the probability density function $f(x)$. Also find the mean of X.

The p.d.f $f(x)$ is given by $F'(x)=f(x)$

i.e., $f(x)=\frac{1}{4}(x-1)^3,\ 1\le x\le 3$

$=0,$ elsewhere

$\therefore$ mean of $X=\int_1^3 x\cdot\frac{1}{4}(x-1)^3\,dx=\frac{1}{4}\int_1^3\left(x^4-3x^3+3x^2-x\right)dx$

$$=\frac{1}{4}\left[\frac{x^5}{5}-\frac{3x^4}{4}+x^3-\frac{x^2}{2}\right]_1^3=2\cdot 65\cdot$$

Ex. 23. *Find the value of the constant k such that*

$$f(x)=kx(1-x),\ 0<x\le 1$$
$$=0\quad,\text{ else where.}$$

is a possible density function and compute $P\left(X>\frac{1}{2}\right)$. Also find $E(X)$.

Since $f(x)$ is a p.d.f, so

$$\int_{-\infty}^{\infty} f(x)dx=1$$

i.e., $k\int_0^1 x(1-x)dx=1$

i.e., $k\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=1$

i.e., $k\left(\frac{1}{2}-\frac{1}{3}\right)=1$

$\therefore\ k=6.$

$\therefore\ f(x)=6x(1-x),\ 0<x\le 1$

$=0,$ else where.

$\therefore\ P\left(X>\frac{1}{2}\right)$

$=1-P\left(X\le\frac{1}{2}\right)$

$=1-\int_0^{\frac{1}{2}} f(x)dx=1-6\int_0^{\frac{1}{2}} x(1-x)dx$

$=1-6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{\frac{1}{2}}=1-\frac{1}{2}=\frac{1}{2}.$

Now, $E(x)=6\int_0^1 x^2(1-x)dx$

$$=6\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1$$

$$=6\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{1}{2}.$$

EXERCISES

[I] SHORT ANSWER QUESTIONS

1. Find mean and variance of the following distribution:

x_i	:	-1	0	1	2	3
f_i	:	0.3	0.1	0.1	0.3	0.2

[Hints : mean $=(-1)\times 0.3+0\times 0.1+1\times 0.1+2\times 0.3+3\times 0.2=1$

Var $=(-1-1)^2\times 0.3+(0-1)^2\times 0.1+(1-1)^2\times 0.1+(2-1)^2\times 0.3$
$+(3-1)^2\times 0.2=2.4$]

2. Find mean and S.D of the following distribution :

$$f(x)=\frac{1}{4}e^{-\frac{x}{4}},x>0$$
$$=0 \quad \text{, else where.}$$

[Hints : mean $=\frac{1}{4}\int_0^\infty xe^{-\frac{x}{4}}dx=4$ $\quad \sigma=\frac{1}{4}\int_0^\infty (x-4)^2 e^{-\frac{x}{4}}dx=80$]

3. Let $f(x)=\begin{cases} x & 0<x\le 1 \\ 2-x & 1\le x\le 2 \\ 0 & \text{elsewhere} \end{cases}$

Is the following function is a p.d.f. of random variable X ?

[Hints : Yes, as $f(x)\ge 0$ and $\int f(x)dx=1$]

4. Is the funtion $f(x)$ defined as

$$f(x)=\begin{cases} 0 & ,x<2 \\ \frac{1}{18}(3+2x),2\le x\le 4 \\ 0 & ,x>4 \end{cases}$$

a density function ?

5. A random variable X has the p.d.f

$f(x)=Axe^{-\lambda^2x^2},x>0$

Find A

[Hints : Use the result $\int_{-\infty}^{\infty} f(x)dx=1$].

6. For what values of λ will the function

$f(x)=\lambda x,x=1,2,3,\cdots n$

be a probability mass function of a random variable.

[Hints : Use the result

$\sum_i f_i=1$ gives

$\lambda(1+2+\cdots+n)=1\Rightarrow\lambda\cdot\frac{n(n+1)}{2}=1$].

7. If the random variable X has mean 'm' and S.D σ show that

$$E\left(\frac{X-m}{\sigma}\right)^2=1,E\left(\frac{X-m}{\sigma}\right)=0.$$

8. Find the probability distribution of X, the number of 'sixes' in two tosses of a die.

9. Two cards are drawn successively without replacement from a well-shuffled deck of 52 cards. Find the probability distribution of the number of aces.

10. Five balls are drawn from a box containing 4 black and 6 white balls. Find the probability distribution of the number of black balls drawn without replacement.