7 Numerical Solution of Ordinary Differential Equation

7.1 Introduction:

Differential equation are involved in many problems in Engineering and Science. In this chapter, we discuss various numerical methods for solving ordinary differential equations. Our aim is to study the solution of the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y_0 = y(x_0) \qquad \dots \qquad (1)$$

in which $f(x, y)$ is a continuous function of x and y in some domain D of the xy-plane and $(x_{0,} \; y_0)$ is a given point in D. The condition $y_0 = f(x_0)$ is known as the initial condition. Sufficient conditions for the existence and uniqueness of the solution of the equation (1) are the well known *Lipschitz conditions* given by

(i) $f(x, y)$ is defined and continuous in D, the region containing (x_0, y_0)

(ii) there exists a constant L such that

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2| \qquad \dots \qquad (2)$$

for all $(x, y_1), (x, y_2) \in D$.

We now proceed to consider numerical techniques for solving (1) at a sequence of points $x_i = x_0 + ih$, called the *mesh points*, h being the step length. Let y_i be the approximation to the exact solution $y(x_i)$ of (1). A continuous approximation to y is then obtained by interpolating the data points (x_i, y_i).

7.2. Euler's method.

We shall now describe a method, known as Euler's method, which gives the solution in the form of a set of tabulated values. In single step method, we determine a function $\phi(x, y; h)$ of x, y and h (the step length) depending on $f(x, y)$ and its derivatives such that

$$y(x+h) = y(x) + h\phi(x, y; h) + 0(h^{p+1}), \qquad \dots \qquad (3)$$

where p is a positive integer, called the order of the method.

A general single-step method of order p can be obtained by expanding $y(x+h)$ by Taylor's theorem as follows:

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \ldots + \frac{h^p}{p!}y^{(p)}(x)$$
$$+\frac{h^{p+1}}{(p+1)!}y^{(p+1)}(x+\theta h),\ 0<\theta<1 \qquad \ldots (4)$$

When $p = 1$, we have, from (4)

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x+\theta h),\ 0<\theta<1 \qquad \ldots (5)$$

so that $\phi(x, y; h) = y'(x) = f(x, y)$ $\qquad \ldots (6)$

Let $y_n = y(x_n)$ and $y_{n+1} = y(x_{n+1}) = y(x_n + h), (n = 1, 2, \ldots)$

Then neglecting the last term in (5) and putting $x = x_n$, we have

$$y_{n+1} = y_n + hy'(x_n)$$
$$= y_n + h\,f(x_n, y_n), \text{ by } (6)$$

Thus we get

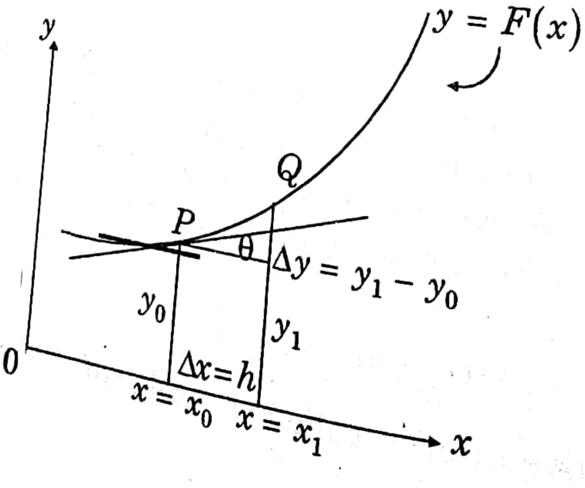
$$y_{n+1} = y_n + h\,f(x_n, y_n),\ n = 0, 1, 2, \ldots \qquad \ldots (7)$$

This is the general *recursion formula for Euler's method* which is a single step method of order 1.

The truncation error of Euler's method is given by

$$\frac{h^2}{2}\cdot y''(x_n+\theta h),\ 0<\theta<1 \text{ and } n = 1, 2, \ldots$$

The geometrical illustration of Euler's method is given below



Let $y = F(x)$ be the solution of (1) and its graph be as shown in the adjoining figure. Since a very small portion of a smooth curve can be thought of as a line segment, so we can write

$$\frac{\Delta y}{\Delta x} = \tan\theta$$

$\therefore$ From the adjacent figure, we have

$$\Delta y = \left(\frac{dy}{dx}\right)_p .\Delta x \text{ and } y_1 = y_0 + \Delta y$$

$$\therefore y_1 = y_0 + \left(\frac{dy}{dx}\right)_p .\Delta x$$

or, $y_1 = y_0 + hf(x_0, y_0)$

which is the approximate value of y for $x = x_1$. On the same lines, the approximate value of y for $x = x_2$ is given by

$$y_2 = y_1 + hf(x_1, y_1)$$

Thus, in general we have

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Note. A great disadvantage of the method lies in the fact that if h is not small enough then the method yields erroneous result; on the otherhand, if h is taken too small enough then the method becomes very slow.

Example.1. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x=0$, find y for $x = 0.1$ by Euler's method, correct upto 4 decimal places, taking step length $h = 0.02$.

[*W.B.U.T., CS-312, 2007*]

Solution. Here $f(x, y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$ and $h = 0.02$

$\therefore$ From (7), we get

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$= 1 + 0.02\left(\frac{1-0}{1+0}\right)$$

$\therefore y(0.02) = 1.02$

$$y(0.04) = y_2 = y_1 + h\,f(x_1, y_1)$$

$$= 1.02 + 0.02\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right)$$

$$= 1.039231$$

Similarly $y(0.06) = 1.039231 + 0.02\,\dfrac{1.039231 - 0.04}{1.03923 + 0.04}$

$$= 1.057748$$

$$y(0.08) = 1.057748 + 0.02 \times 0.892641$$

$$= 1.075601$$

$$y(0.10) = 1.075601 + 0.02 \times 0.861544 = 1.092832$$

$\therefore y(0.1) = 1.0928$, correct upto 4 decimal places.

7.3. Modified Euler's Method.

To remove the drawback to some extent, we shall discuss modified Euler's method starting with the initial value y_0 an approximate value for y_1 is computed from the Euler's method as

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) \quad \dots \quad (8)$$

Then to get the second approximation for y_1 we replace $f(x_0, y_0)$ in (8) by the average value of $f(x_0, y_0)$ and $f\left(x_1, y_1^{(0)}\right)$.

Thus the second approximation for y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

Similarly, third approximation for y_1 is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

Thus, in general

$$y_n^{(k)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f\left(x_n, y_n^{(k-1)}\right)\right], \quad k = 1, 2, 3, \dots$$

is used to approximate y_n $\quad \dots \quad (9)$

Example.2. Given $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x^2}$, $y(1)=1$. Evaluate $y(1.2)$ by modified Euler's method correct upto 4 decimal places.

[*W.B.U.T., CS-312, 2003, 2004, 2006, M(CS)-301, 2015, M(CS)-401, 2013, 2015*]

Solution. Here $f(x,y)=\frac{1}{x^2}-\frac{y}{x}$, $x_0=1$, $y_0=1$

Let $h=0.1$ so that $x_1=1+0.1=1.1$

$\therefore y_1^{(0)}=y_0+hf(x_0,y_0)=1+0.1\times(1-1)=1$

$\therefore$ From (9), we get

$$y_1^{(1)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f\left(x_1,y_1^{(0)}\right)\right]$$

$$=1+\frac{0.1}{2}\left[(1-1)+\left\{\frac{1}{(1.1)^2}-\frac{1}{1.1}\right\}\right]$$

$$=1+0.05(-0.08264)$$

$$=0.99587$$

$$\therefore y_1^{(2)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f\left(x_1,y_1^{(1)}\right)\right]$$

$$=1+\frac{0.1}{2}\left[(1-1)+\left(\frac{1}{(1.1)^2}-\frac{0.99587}{1.1}\right)\right]$$

$$=1+0.05(-0.078888)$$

$$=0.99606$$

Similarly $$y_1^{(3)}=1+0.05\left[(1-1)+\left(\frac{1}{(1.1)^2}-\frac{0.99606}{1.1}\right)\right]$$

$$=1+0.05(-0.079063)$$

$$=0.99607$$

Hence $y_1 = y(1.1) \simeq 0.9961$

$\therefore x_1 = 1.1, y_1 = 0.9961$

$$\therefore f(x_1, y_1) = \frac{1}{(1.1)^2} - \frac{0.9961}{1.1} = -0.079$$

$$\therefore y_2^{(0)} = y_1 + hf(x_1, y_1)$$

$$= 0.9961 + 0.1 \times (-0.079)$$

$$= 0.98819$$

$\therefore$ From (9), we have

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$$

$$= 0.9961 + 0.05\left[-0.079 + \frac{1}{(1.2)^2} - \frac{0.98819}{1.2}\right]$$

$$= 0.98569$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$$

$$= 0.9961 + 0.05\left[-0.079 + \frac{1}{(1.2)^2} - \frac{0.98569}{1.2}\right]$$

$$= 0.98580$$

Similarly,

$$y_2^{(3)} = 0.9961 + 0.05\left[-0.079 + \frac{1}{(1.2)^2} - \frac{0.98580}{1.2}\right]$$

$$= 0.985797$$

Thus $y_2 \simeq 0.9858$, correct upto four decimal places

$\therefore y(1.2) \simeq 0.9858$

7.4. Runge-Kutta method.

This method is one of the most widely used methods to obtain greater accuracy and most suitable in case when computation of higher order derivatives is complicated. In single step method, it follows from (3) that

$$y(x+h) = y(x) + h\phi(x, y; h) + 0\left(h^{p+1}\right) \qquad \ldots \qquad (10)$$

When $p = 2$, we get from (8)

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x+\theta h), 0 < \theta < 1 \ldots \qquad (11)$$

so that $\phi(x, y; h) = y'(x) + \frac{h}{2}y''(x)$

$$= f + \frac{h}{2}(f_x + f\, fy) \qquad \ldots \qquad (12)$$

In an 2-stage Runge-Kutta method, we set

$$k_1 = hf(x, y)$$

$$k_2 = hf(x + \alpha h, y + \beta k_1)$$

$$k = w_1 k_1 + w_2 k_2 \qquad \ldots \qquad (13)$$

The constants α, β, w_1 and w_2 are determined so that (12) agree with Taylor's series of order as high as possible.

$\therefore$ From (10), we get

$$y(x+h) = y(x) + k + 0\left(h^3\right) \qquad \ldots \qquad (14)$$

From (12) and (13), it follows that

$$f + \frac{1}{2}h\left(f_x + ff_y\right) = w_1 f(x, y) + w_2 f(x + \alpha h, y + \beta k_1)$$

$$= \omega_1 f + w_2\left(f + \alpha h f_x + \beta k_1 f_y\right) + 0\left(h^2\right)$$

$$= (\omega_1 + \omega_2)f + h\left(\omega_2 \alpha f_x + \omega_2 \beta f f_y\right) + 0\left(h^2\right) \quad [\because k_1 = hf]$$

which is true for all values of the constant α, β, ω_1 and ω_2 and therefore, for arbitrary f. Thus we have

$$w_1 + w_2 = 1$$

$$w_1\alpha = w_2\beta = 1/2 \qquad \ldots \qquad (15)$$

Any set of values of the constants α, β, w_1, w_2 satisfying (15) gives a one-parameter family of solutions and each of these corresponds to a 2-stage Runge-Kutta method of order 2.

A possible solution of (15) is

$$w_1 = w_2 = \frac{1}{2}, \alpha = \beta = 1$$

Thus $k_1 = hf(x, y)$

$$k_2 = hf(x + h, y + k_1)$$

$$k = \frac{1}{2}(k_1 + k_2) = \frac{h}{2}[f(x, y) + f(x + h, y + hf(x, y))]$$

$$y(x + h) = y(x) + k + 0(h^3)$$

Hence the iterative formula is

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))],$$

$$n = 0, 1, 2, \ldots \qquad \ldots (16)$$

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This is know as *Runge-Kutta method of order 2* with truncation error of order h^3

In the similar manner the *Runge-Kutta method of order 4* can be written as

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_n, y_n)$

$$\left.\begin{aligned} k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\ k_4 &= hf(x_n + h, y_n + k_3), \end{aligned}\right\} \qquad \ldots (17)$$

$n = 0, 1, 2, \ldots$

Here the truncatior error is of order h^5

Note. (i) The advantage of this method is that the method is stable and self starting. It is easy to change the step size h for higher order accuracy.

(ii) For this method several evaluations of the first derivative are require and so the method is time consuming. Most disadvantage of this method is that neither the truncation errors nor the estimates of them are obtained in the computation procedure.

Example.3. Use Runge-Kutta method of order 2 to calculate $y(0.2)$ for the equation

$$\frac{dy}{dx} = x + y^2, y(0) = 1$$

Solution. Here $f(x, y) = x + y^2, x_0 = 0, y_0 = 1$

We take $h = 0.1$. Then

$k_1 = hf(x_0, y_0) = 0.1 \times (0 + 1) = 0.1$

$k_2 = hf(x_0 + h, y_0 + k_1) = 0.1 f(0.1, 1.1)$

$= 0.1 \times \left\{0.1 + (1.1)^2\right\} = 1.31$

$\therefore$ From iterative formula (16) of Runge-Kutta method of order 2 we get

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 1 + \frac{1}{2}(0.1 + 1.31)$$

$$= 1.1155$$

$\therefore y(0.1) \simeq 1.1155$

Thus $x_1 = 0.1, y_1 = 1.1155$

$\therefore k_1 = hf(x_1, y_1) = 0.1 \times \left\{0.1 + (1.1155)^2\right\}$

$\simeq 0.1344$

$k_2 = hf(x_1 + h, y_1 + k_1)$

$= 0.1 \times f(0.2, 1.2499)$

$\simeq 0.1762$

$\therefore$ From (16), we get

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

$$= 1.1155 + \frac{1}{2}(0.1344 + 0.1762)$$

$$= 1.2708$$

Hence $y(0.2) \simeq 1.2708$

Example.4. Find $y(1.1)$ using Runge-Kutta method of fourth order, given that

$$\frac{dy}{dx} = y^2 + xy, \; y(1) = 1$$

[W.B.U.T., CS-312, 2005
M.A.K.A.U.T., MCS-401, 2014]

Solution. Here $f(x, y) = y^2 + xy, \;\; x_0 = 1, y_0 = 1$

Taking $h = 0.1$, we have

$$k_1 = hf(x_0, y_0) = 0.1(1^2 + 1 \times 1) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, \, y_0 + \frac{1}{2}k_1\right)$$

$$= 0.1f(1.05, 1.1)$$

$$= 0.1\left\{(1.1)^2 + 1.05 \times 1.1\right\}$$

$$= 0.2365$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, \, y_0 + \frac{1}{2}k_2\right)$$

$$= 0.1f(1.05, 1.11825)$$

$$= 0.1\left\{(1.11825)^2 + 1.11825 \times 1.05\right\}$$

$$= 0.2425$$

$$k_4 = hf(x_0 + h, \, y_0 + k_3)$$

$$= 0.1f(1.1, 1.2425)$$

$$= 0.1\left\{(1.2425)^2 + 1.1 \times 1.2425\right\}$$

$$= 0.2910556$$

$\therefore$ From iterative formula (17) of Runge-Kutta method of order 4, we get

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.2 + 2 \times 0.2365 + 2 \times 0.2425 + 0.2910556)$$

$\simeq 1.2415$, correct upto four decimal places.

$\therefore y(1.1) \simeq 1.2415$

7.5. Predictor-Corrector methods.

In order to solve the differential equation (1), by this method, we first obtain the approximate value of $y_{n+1} = y(x_{n+1})$ by predictor formula and then improve this value by means of a corrector formula. It may be noted that the corrector formula is more accurate than the predictor one although it requires a companion of predictor formula and knowledge of the initial set of values $y_0, y_1, \cdots y_n$

The simplest formula of this type is Euler's formula and the modified Euler's one is given by

$$y_{n+1}^{(p)} = y_n + hf(x_n, y_n) \qquad \cdots \quad (18)$$

$$y_{n+1}^{(c)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y_{n+1}^{(p)}\right)\right] \qquad \cdots \quad (19)$$

The first is an open formula which can be used for predicting y_{n+1} and this value can be used to compute $f(x_{n+1}, y_{n+1})$ to get a corrector formula which can be used in an iterative manner.

Ex.5. Solve the equation $\frac{dy}{dx} = x + y$ with initial condition $y(0) = 1$, taking step length 0.1 to find $y(0.2)$ by predictor-corrector method. *[M.A.K.A.U.T., MCS-401, 2014]*

Solution. Here $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$\therefore y_1^p = y_0 + h\ f(x_0, y_0)$$

$$= 1 + 0.1\ f(0, 1) = 1 + 0.1(0 + 1) = 1.1$$

$$y_1^{c(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(p)}\right)\right]$$

$$= 1 + \frac{0.1}{2}[f(0, 1) + f(0.1, 1.1)]$$

$$= 1 + 0.05\,[(0+1) + (0.1+1.1)] = 1.11$$

$$y_1^{c(2)} = 1 + \frac{0.1}{2}[f(0.1) + f(0.1, 1.11)]$$

$$= 1 + 0.05\,[1 + (0.1 + 1.11)] = 1.1105$$

$$y_1^{c(3)} = 1 + \frac{0.1}{2}[1 + (0.1 + 1.1105)]$$

$$= 1.110525$$

$$\therefore\; y_1 = y(0.1) \simeq 1.1105$$

$$\therefore\; y_2^p = y_1 + h\, f(x_1, y_1)$$

$$= 1.1105 + 0.1\, f(0.1, 1.1105)$$

$$= 1.1105 + 0.1(0.1 + 1.1105) = 1.23155$$

$$y_2^{c(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_1^p\right)\right]$$

$$= 1.1105 + \frac{0.1}{2}[f(0.1, 1.1105) + f(0.2, 1.23155)]$$

$$= 1.1105 + 0.05\,[(0.1 + 1.1105) + (0.2 + 1.23155)]$$

$$= 1.1105 + 0.05\,(1.2105 + 1.43155)$$

$$= 1.242602$$

$$y_2^{c(2)} = 1.1105 + \frac{0.1}{2}[f(0.1, 1.1105) + f(0.2, 1.242602)]$$

$$= 1.1105 + 0.05(1.2105 + 1.442602)$$

$$= 1.243155$$

$y_2^{c(3)} = 1.1105 + 0.05$

$= 1.243183$

$\therefore\; y_2 = y(0.2) = 1.2432,$

Let us now proceed to d
are of much practical use

I. Adams-Bashforth m

We consider the diffe

$\frac{dy}{dx} = f$

which when integrated

$y_{n+1} = y_n +$

To evaluate the inte
can replace $f(x, y)$ b
$f(x, y(x))$ at the $(k+1)$
the Newton's backward

$P_k(x) = \sum_{j=0}^{k}\Big(\,s + j -$
j

where $s = \frac{x - x_n}{h}$

Then, in virtue of (2

$y_{n+1} = y_n + h\sum_{j=0}^{k}\alpha$

where $\alpha_j = \int_0^1 \Big(\,s + j - 1$
j

The formula (22) i
Bashforth formula and

A few values of α_j

$\alpha_0 = 1, \alpha_1 = \frac{1}{2},$

Solving the system we get

$$y_1 = 0.2943, y_2 = 0.5701, y_3 = 0.8108$$

i.e., $y(0.25) = 0.2943, y(0.5) = 0.5701, y(0.75) = 0.8108$

ILLUSTRATIVE EXAMPLES

Ex.1. Find the solution of the differential equation

$$\frac{dy}{dx} = x^2 - y, \; y(0) = 1$$

for $x = 0.3$ taking $h = 0.1$ and using Euler's method. Compare the result with the exact solution.

Solution. Here $f(x, y) = x^2 - y, x_0 = 0, y_0 = 1, h = 0.1$ so that $x_i = x_o + ih (i = 0, 1, 2, \ldots)$ gives

$$x_1 = 0.1, x_2 = 0.2 \text{ etc}$$

Thus the recursion formula

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, 2, \ldots$$

yields

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1(0 - 1) = 0.9$$

$$y_2 = y_1 + hf(x_1, y_1) = 0.9 + 0.1\{(0.1)^2 - 0.9\} = 0.811$$

$$y_3 = y_2 + hf(x_2, y_2) = 0.811 + 0.1\{(0.2)^2 - 0.811\} = 0.7339$$

Hence $y(0.3) \simeq 0.7339$

The given equation can be written as

$$\frac{dy}{dx} + y = x^2$$

which is a linear equation in y.

$$\therefore I.F. = e^{\int 1.dx} = e^x$$

Multiplying both sides of the equation by e^x and then integrating we get

$$ye^x = \int x^2 e^x dx + c$$

$$= x^2 e^x - 2xe^x + 2e^x + c$$

$$\therefore y = x^2 - 2x + 2 + ce^{-x}$$

Also given $y(0) = 1$

$\therefore 1 = 0 - 2 \times 0 + 2 + c$

$\therefore c = -1$

$\therefore y = x^2 - 2x + 2 - e^{-x}$

$\therefore y(0.3) = (0.3)^2 - 2 \times 0.3 + 2 - e^{-0.3} = 0.7492$

Hence the error is $0.7492 - 0.7339 = 0.0153$

Ex.2. Using Euler's method, find an approximate value of y at $x = 0.5$ given that

$$\frac{dy}{dx} = x + y, \; y(0) = 1$$

Solution. We have $f(x, y) = x + y, x_0 = 0, y_0 = 1$

Taking $h = 0.1$, we have from recursion formula of Euler's method,

$$y_{n+1} = y_n + hf(x_n, y_n)$$
$$= y_n + 0.1(x_n + y_n), n = 0, 1, 2, \ldots$$

$$\therefore y_1 = y(0.1) = 1 + 0.1(0 + 1) = 1.10$$

$$y_2 = y(0.2) = y_1 + 0.1(x_1 + y_1)$$
$$= 1.10 + 0.1(0.1 + 1.10)$$
$$= 1.22$$

$$y_3 = y(0.3) = y_2 + 0.1(x_2 + y_2)$$
$$= 1.22 + 0.1(0.2 + 1.22)$$
$$= 1.36$$

$$y_4 = y(0.4) = y_3 + 0.1(x_3 + y_3)$$
$$= 1.36 + 0.1(0.3 + 1.36)$$
$$= 1.53$$

$$y_5 = y(0.5) = y_4 + 0.1(x_4 + y_4)$$
$$= 1.53 + 0.1(0.4 + 1.53)$$
$$= 1.72$$

Thus $y(0.5) = 1.72$

Ex.3. Solve the equation

$$5x\frac{dy}{dx} + y^2 - 2 = 0;\ y(4) = 1$$

for $y(4.1)$, taking $h = 0.1$ and using modified Euler's method.

Solution. The given equation can be written as

$$\frac{dy}{dx} = \frac{2 - y^2}{5x}$$

$$\therefore f(x, y) = \frac{2 - y^2}{5x}$$

Here $x_0 = 4,\ y_0 = 1,\ h = 0.1$

So the recursion formula of modified Euler's method gives

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.1 f(4, 1)$$

$$= 1 + 0.1 \times 0.05 \quad \left[\because f(4,1) = \frac{2-1}{5 \times 4} = 0.05\right]$$

$$= 1.005$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$= 1 + \frac{0.1}{2}\left[f(4,1) + f(4.1, 1.005)\right]$$

$$= 1 + 0.05\left[0.05 + \frac{2 - (1.005)^2}{5 \times 4.1}\right]$$

$$= 1.0049$$

Similarly, $$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 1 + 0.05\left[f(4,1) + f(4.1, 1.0049)\right]$$

$$= 1 + 0.05\left[0.05 + \frac{2 - (1.0049)^2}{5 \times 4.1}\right]$$

$$= 1.0049$$

$\therefore y(4.1) \simeq 1.005$, correct upto three decimal places

Ex.4. Find $y(0.10)$ and

differential equation $\frac{dy}{dx} =$

decimal places, taking st

Solution. Here $f(x, y) =$

$\therefore x_1 = 0.05,\ x_2 = 0.1$

$\therefore$ The recursion for

$y_{n+1} = y_n +$

gives

$y_1 = y_0 + h\, f(x_0,$

$= 0 + 0.05\ f(0$

$y_2 = y_1 + h\ f(x_1,$

$= 0 + 0.05\ f($

$= 0.05\{(0.05$

$= 1.25 \times 10^{-4}$

$y_3 = y_2 + h\ f(x$

$= 0.000125$

$= 6.25 \times 10^{-}$

Ex.5. Solve by usin

equation for $x = 1$ b

$\frac{dy}{dx} = xy,\ y = 1$

Solution. Here $f($

$\therefore$ By Euler's

$y_{n+1} =$

Ex.4. Find $y(0.10)$ and $y(0.15)$ by Euler's method, from the differential equation $\frac{dy}{dx} = x^2 + y^2$, with $y(0) = 0$, correct to four decimal places, taking step length $h = 0.05$.

[*W.B.U.T., MCS-301, 2007*]

Solution. Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.05$

$\therefore x_1 = 0.05$, $x_2 = 0.10$, $x_3 = 0.15$

$\therefore$ The recursion formula

$$y_{n+1} = y_n + h\ f(x_n, y_n),\ x = 0,1,2.....$$

gives

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$
$$= 0 + 0.05\ f(0, 0) = 0 + 0.05 \times 0 = 0$$
$$y_2 = y_1 + h\ f(x_1, y_1)$$
$$= 0 + 0.05\ f(0.05, 0)$$
$$= 0.05\left\{(0.05)^2 + 0^2\right\}$$
$$= 1.25 \times 10^{-4}$$
$$y_3 = y_2 + h\ f(x_2, y_2)$$
$$= 0.000125 + 0.05\left\{(0.1)^2 + (0.000125)^2\right\}$$
$$= 6.25 \times 10^{-4}$$

Ex.5. Solve by using Euler's method the following differential equation for $x = 1$ by taking $h = 0.2$:

$$\frac{dy}{dx} = xy,\ y = 1 \text{ when } x = 0 \quad [\textit{W.B.U.T., MCS-301, 2008}]$$

Solution. Here $f(x, y) = xy$, $x_0 = 0$, $y_0 = 1$ and $h = 0.2$

$\therefore$ By Euler's iterative formula

$$y_{n+1} = y_n + h\ f(x_n, y_n)\ \ n = 0,1,2,....$$

we get

$$y_1 = y(0.2) = y_0 + h\,f(x_0, y_0)$$
$$= 1 + 0.2(0 \times 1) = 1$$
$$y_2 = y(0.4) = y_1 + h\,f(x_1, y_1)$$
$$= 1 + 0.2(0.2 \times 1) = 1.04$$
$$y_3 = y(0.6) = y_2 + h\,f(x_2, y_2)$$
$$= 1.04 + 0.2(0.4 \times 1.04) = 1.1232$$
$$y_4 = y(0.8) = 1.1232 + 0.2(0.6 \times 1.1232$$
$$= 1.25798$$
$$y_5 = y(1.0) = 1.25798 + 0.2(0.8 \times 1.25798)$$
$$= 1.45926$$

$\therefore\ y(1.0) = 1.4593$; correct upto four decimal places

Ex.6. Use Runge-Kutta method of order two to find $y(0.2)$ and $y(0.4)$ given that

$$y\frac{dy}{dx} = y^2 - x,\ y(0) = 2,\ \text{taking } h = 0.2$$

Solution. The given equation can be written as

$$\frac{dy}{dx} = \frac{y^2 - x}{y}$$

$\therefore$ Here $f(x, y) = \dfrac{y^2 - x}{y}$, $x_0 = 0, y_0 = 2, h = 0.2$

$\therefore$ By Runge-Kutta method of order 2, we have

$$y(x_0 + h) = y_0 + k$$

where $k = \dfrac{1}{2}(k_1 + k_2)$,

$$k_1 = hf(x_0, y_0) = 0.2 \times \frac{2^2 - 0}{2} = 0.4$$

$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$
$$= 0.2\ f(0.2, 2.4)$$
$$= 0.2 \times \frac{(2.4)^2 - 0.2}{2.4}$$
$$= 0.46333$$

Thus $y(0 + 0.2) = 2 + \frac{1}{2}(0$

$= 2.43166$

$\therefore\ y(0.2) \simeq 2.43166$

To compute $y(0.4)$ we

$\therefore\ y(x_1 + h) = y(x_1) + k$

where $k = \dfrac{1}{2}(k_1 + k_2)$,

$k_1 = hf(x_1, y_1) = 0.2f($

$k_2 = hf(x_1 + h,\ y_1 + k_1$

$= 0.2f(0.4, 0.42533$

$= -0.10302$

$\therefore\ y(0.2 + 0.2) = 2.431$

$\therefore\ y(0.4) = 2.37698$

Hence $y(0.2) = 2.432$ places.

Ex.7. Use the fourth when $x = 0.2$ given th

Solution. Here $f(x,$

We take $h = 0.2$

$\therefore\ k_1 = h\,f(x_0, y_0)$

$k_2 = hf(x_0 + h,\ y_0 + k_1)$

$= 0.2\ f(0.2,\ 2.4)$

$= 0.2 \times \dfrac{(2.4)^2 - 0.2}{2.4}$

$= 0.46333$

Thus $y(0+0.2) = 2 + \frac{1}{2}(0.4 + 0.46333)$

$= 2.43166$

$\therefore y(0.2) \simeq 2.43166$

To compute $y(0.4)$ we have $x_1 = 0.2, y_1 = 2.43166$

$\therefore y(x_1 + h) = y(x_1) + k$

where $k = \frac{1}{2}(k_1 + k_2)$,

$k_1 = hf(x_1, y_1) = 0.2f(0.2, 0.432) = -0.00633$

$k_2 = hf(x_1 + h,\ y_1 + k_1)$

$= 0.2f(0.4, 0.42533)$

$= -0.10302$

$\therefore y(0.2 + 0.2) = 2.43166 + \frac{1}{2}(-0.00633 - 0.10302)$

$\therefore y(0.4) = 2.37698$

Hence $y(0.2) = 2.432$, $y(0.4) = 2.377$ correct upto three decimal places.

Ex.7. Use the fourth order RK-method to find the value of y when $x = 0.2$ given that $y = 0$ when $x = 0$ and $\dfrac{dy}{dx} = 1 + y^2$.

[W.B.U.T., MCS-301, 2010]

Solution. Here $f(x, y) = 1 + y^2$, $x_0 = 0$, $y_0 = 0$

We take $h = 0.2$

$\therefore k_1 = h\ f(x_0, y_0) = 0.2\ f(0, 0) = 0.2$

$\therefore k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.2\ f(0.1, 0.1) = 0.202$

$\therefore k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.2\ f(0.1, 0.202) = 0.2020$

$\therefore k_4 = h\, f(x_0 + h, y_0 + k_3)$

$= 0.2\ f(0.2, 0.20204) = 0.2082$

$\therefore y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0 + \frac{1}{6}(0.2 + 2\times 0.202 + 2\times 0.202 + 0.2082)$

$= 0.2027$

$\therefore y(0.2) = 0.2024$

Ex.8. Comute $y(0.2)$ from $\frac{dy}{dx} = x + y,\ y(0) = 1$ taking step $h = 0.1$ by 4th order RK method. [*W.B.U.T., MCS-301, 2007*]

Solution. Here $f(x, y) = x + y,\ x_0 = 0,\ y_0 = 1,\ h = 0.1$

$\therefore y(0.1) = y(0) + k$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0) = 0.1 f(0, 1) = 0.1 \times (0 + 1) = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 f(0.05, 1.05) = 0.1(0.05 + 1.05) = 0.11$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 f(0.05, 1.055)$

$= 0.1(0.05 + 1.055) = 0.1105$

$k_4 = hf(x_0 + h, y_0 + k_3) =$

$=$

$\therefore y(0.1) = y_0 + k$

$= 1 + \frac{1}{6}(0.1 + 2\times$

$= 1.0901667$

$\therefore y(0.1) \simeq 1.0902$

To compute $y(0.2)$, we

$\therefore k_1 = hf(x_1, y_1) = 0.1f$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k}{2}\right.$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right.$

$k_4 = hf(x_1 + h, y_1 + k$

$\therefore y(0.2) = 1.0902 + \frac{1}{6}(0.1$

$= 1.220545$

$\therefore y(0.2) \simeq 1.2205$, co

Ex.9. Find the values of Kutta method of fourth

$\frac{dy}{dx} = xy + y^2$,

Solution. Here $f(x, y$

Taking $h = 0.1$, we

$y(0.1) =$

where $k = \frac{1}{6}(k_1 + 2k_2$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1105) = 0.1(0.1 + 1.1105)$
$= 0.12105$

$\therefore y(0.1) = y_0 + k$
$= 1 + \frac{1}{6}(0.1 + 2 \times 0.11 + 2 \times 0.1105 + 0.12105)$
$= 1.0901667$

$\therefore y(0.1) \simeq 1.0902$

To compute $y(0.2)$, we have $x_1 = 0.1, y_1 = 1.0902, h = 0.1$

$\therefore k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.0902) = 0.11902$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f(0.15, 1.149708) = 0.12997$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f(0.15, 1.155185) = 0.13052$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1f(0.2, 1.220718) = 0.142072$

$\therefore y(0.2) = 1.0902 + \frac{1}{6}(0.11902 + 2 \times 0.12997 + 2 \times 0.13052 + 0.142072)$
$= 1.220545$

$\therefore y(0.2) \simeq 1.2205$, correct upto four decimal places.

Ex.9. Find the values of $y(0.1)$, $y(0.2)$ and $y(0.3)$ using Runge-Kutta method of fourth order, given that

$$\frac{dy}{dx} = xy + y^2, \; y(0) = 1$$ [W.B.U.T., MCS-301, 2009]

Solution. Here $f(x, y) = xy + y^2$, $x_0 = 0$, $y_0 = 1$

Taking $h = 0.1$, we have

$$y(0.1) = y(0) + k$$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = hf(x_0, y_0) = 0.1(0 \times 1 + 1^2) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.05, 1.05)$$

$$= 0.1\{0.05 \times 1.05 + (1.05)^2\} \quad = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05, 1.05775)$$

$$= 0.1\{0.05 \times 1.05775 + (1.05775)^2\} \quad = 0.1172$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1172)$$

$$= 0.1\{0.1 \times 1.1172 + (1.1172)^2\}$$

$$= 0.13598$$

$$\therefore y(0.1) = 1 + \frac{1}{0}(0.1 + 2 \times 0.1155 + 2 \times 0.1172 + 0.135598)$$

$$= 0.1168$$

Similarly we can find out

$y(0.2) = 1.2689$

$y(0.3) = 1.4856$

Ex.10. Solve the equation $\frac{dy}{dx} = \frac{1}{x+y}$, $y(0) = 1$, for $y(0.1)$ and $y(0.2)$ using Runge-Kutta method of the fourth order.

[M.A.K.A.U.T., M(CS)-401, 2006, 2013]

Solution. Here $f(x, y) = \frac{1}{x+y}$, $x_0 = 0$, $y_0 = 1$

Taking $h = 0.1$, we have

$$y(x_0) = y(x_0 + h) = y_0 + k$$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = h\, f(x_0, y_0) = 0.1 \times \frac{1}{0+1} = 0.1$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) =$$

$$= 0.1 \times \frac{1}{0.5 + 1.05}$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1 \times \frac{1}{0.05 + 1.0}$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) =$$

$$= 0.08394$$

$$\therefore k = \frac{1}{6}(0.1 + 2 \times 0.0909$$

$$= 0.09139$$

$$\therefore y(0.1) = 1 + 0.09139 =$$

To find $y(0.2)$, we h

$x_1 = 0.1$, $y_1 = 1.0914$

$$\therefore k_1 = h\, f(x_1, y_1) = 0.1$$

$$k_2 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k}{}\right.$$

$$= 0.07792$$

$$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 +\right.$$

$$= 0.07810$$

$$k_4 = h\, f(x_1 + h, y_1 + k$$

$$= 0.07302$$

$$\therefore y_2 = y(x_1 + h) = y_1$$

$$= 1.16957$$

$$\therefore y(0.2) \approx 1.1696$$

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$$k_2 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.1 \times f(0.05, 1.05)$$

$$= 0.1 \times \frac{1}{0.5 + 1.05} = 0.09091$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0.1 \times f(0.05, 1.04545)$$

$$= 0.1 \times \frac{1}{0.05 + 1.04545} = 0.09129$$

$$k_4 = h\, f(x_0 + h,\ y_0 + k_3) = 0.1 \times f(0.1, 1.09129)$$

$$= 0.08394$$

$$\therefore\ k = \frac{1}{6}(0.1 + 2 \times 0.09091 + 2 \times 0.09129 + 0.08394)$$

$$= 0.09139$$

$$\therefore\ y(0.1) = 1 + 0.09139 = 1.0914$$

To find $y(0.2)$, we have

$$x_1 = 0.1,\ y_1 = 1.0914$$

$$\therefore\ k_1 = h\, f(x_1, y_1) = 0.1 \times f(0.1, 1.0914) = 0.08393$$

$$k_2 = h\, f\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right) = 0.1 \times f(0.15, 1.3337)$$

$$= 0.07792$$

$$k_3 = h\, f\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right) = 0.1 \times f(0.15, 1.13036)$$

$$= 0.07810$$

$$k_4 = h\, f(x_1 + h,\ y_1 + k_3) = 0.1 \times f(0.2, 1.16950)$$

$$= 0.07302$$

$$\therefore\ y_2 = y(x_1 + h) = y_1 + \frac{1}{6}(k_1 + k_2 + 2k_3 + k_4)$$

$$= 1.16957$$

$$\therefore\ y(0.2) \approx 1.1696$$

Ex.11. Using Runge-Kutta method of order 4 obtain the solution of $\frac{dy}{dx} = 2x + y^2$, $y(0) = 1$ and $h = 0.1$ at $x = 0.2$.
[*M.A.K.A.U.T., M(CS)-401, 2015*]

Solution. Here $f(x, y) = 2x + y^2$, $x_0 = 0$, $y_0 = 1$

Given $h = 0.1$

$\therefore k_1 = h\, f(x_0, y_0) = 0.1(0 + 1^2) = 0.1$

$k_2 = h\, f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

$= 0.1\, f(0.05, 1.05)$

$= 0.1\,\{2\times 0.05 + (1.05)^2\} = 0.12025$

$k_3 = h\, f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1\, f(0.05, 1.06012)$

$= 0.1\,\{2\times 0.05 + (1.060128)^2\} = 0.12239$

$k_4 = h\, f(x_0 + h, y_0 + k_3) = 0.1\, f(0.1, 1.12239)$

$= 0.1\,\{2\times 0.1 + (1.2239)^2\} = 0.14598$

$\therefore y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0.1 + 2\times 0.12025 + 2\times 0.12239 + 0.14598)$

$= 1.12188$

$\therefore y(0.1) = \mathbf{1.12188}$

To compute $y(0.2)$, we have

$x_1 = 0.1$, $y_1 = 1.12188$, $h = 0.1$

$\therefore k_1 = h\, f(x_1, y_1) = 0.1\, f(0.1, 1.12188)$

$= 0.1\,\{(2\times 0.1 + (1.12188)^2\}$

$= 0.14586$

$\therefore k_1 = h\, f(x_1, y_1) = 0.1\, f(0.$

$= 0.1\{2\times 0.1 + (1.1218$

$= 0.14586$

$k_2 = h\, f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right.$

$= 0.17276$

$k_3 = h\, f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k\right.$

$= 0.1\, f(0.15, 1.20826)$

$= 0.17599$

$k_4 = h\, f(x_1 + h, y_1 + k_3)$

$= 0.1\, f(0.2, 1.29787)$

$= 0.208446$

$\therefore y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 +$

$= 1.12188 + \frac{1}{6}(0.1$

$= 1.2971877$

$\therefore y(0.2) = 1.2972$ cor

Ex.12. Find the value o fourth order with $h = 0$

$$\frac{dy}{dx} = \sqrt{x^2 +}$$

Solution. Here $f(x, y)$

$\therefore k_1 = hf(x_0, y_0) = ($

$\therefore k_1 = h\, f(x_1, y_1) = 0.1\, f(0.1, 1.12188)$

$= 0.1\{2\times 0.1 + (1.12188)^2\}$

$= 0.14586$

$k_2 = h\, f\left(x_1 + \frac{1}{2}h,\, y_1 + \frac{1}{2}k_1\right) = 0.1\, f(0.15, 1.19481)$

$= 0.17276$

$k_3 = h\, f\left(x_1 + \frac{1}{2}h,\, y_1 + \frac{1}{2}k_2\right)$

$= 0.1\, f(0.15, 1.20826)$

$= 0.17599$

$k_4 = h\, f(x_1 + h,\, y_1 + k_3)$

$= 0.1\, f(0.2, 1.29787)$

$= 0.208446$

$\therefore y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.12188 + \frac{1}{6}(0.14586 + 2\times 0.17278$

$+ 2\times 0.17599 + 0.208446)$

$= 1.2971877$

$\therefore y(0.2) = 1.2972$ correct up to four decimal places

Ex.12. Find the value of $y(0.4)$ using Runge-Kutta method of fourth order with $h = 0.2$, given that

$$\frac{dy}{dx} = \sqrt{x^2 + y},\; y(0) = 0.8$$

Solution. Here $f(x, y) = \sqrt{x^2 + y}$, $x_0 = 0$, $y_0 = 0.8$, $h = 0.2$

$\therefore k_1 = hf(x_0, y_0) = 0.2 f(0, 0.8) = 0.2\sqrt{0^2 + 0.8} = 0.17889$

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$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 0.88944)$$

$$= 0.2\sqrt{(0.1)^2 + 0.88944} = 0.18968$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1, 0.89484)$$

$$= 0.2\sqrt{(0.1)^2 + 0.89484}$$

$$= 0.19025$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 0.99025)$$

$$= 0.2\sqrt{(0.2)^2 + 0.99025}$$

$$= 0.20300$$

$$\therefore y_1 = y(x_0 + h)$$

$$= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.8 + \frac{1}{6}(0.17889 + 12 \times 0.18968 + 2 \times 0.19025 + 0.20300)$$

$$= 0.99029$$

$$\therefore y(0.2) = 0.99029$$

To compute $y(0.4)$, we have $x_1 = 0.2, y_1 = 0.99029, h = 0.2$

$$\therefore k_1 = hf(x_1, y_1) = 0.2f(0.2, 0.99029) = 0.20301$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f(0.3, 1.09180) = 0.21742$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2f(0.3, 1.09901) = 0.21808$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.20838) = 0.23396$$

$$\therefore y_2 = y(x_1 + h)$$

$$= y_1 + \frac{1}{6}(k_2 + 2k_2 + 2k_3$$

$$= 0.99029 + \frac{1}{6}(0.20301$$

$$= 1.20832$$

$\therefore y(0.4) \simeq 1.2083$, corr

Ex.13. Solve initial valu

$$10\frac{dy}{dx} = x^2 +$$

for $x = 0.1, 0.2$ by using R
find the solution correct

Solution. Here $f(x, y)$

Let $h = 0.1$

$\therefore$ By fourth order R

$$y(x_0 + h) = y_0 + k$$

where $k = \frac{1}{6}(k_1 + 2k_2 +$

$$k_1 = hf(x_0, y_0) = 0.1($$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0\right.$$

$$= 0.1f(0.05, 1.005)$$

$$= 0.1 \times \frac{(0.05)^2 + (}{10}$$

$\therefore y_2 = y(x_1 + h)$

$= y_1 + \frac{1}{6}(k_2 + 2k_2 + 2k_3 + k_4)$

$= 0.99029 + \frac{1}{6}(0.20301 + 2 \times 0.21742 + 2 \times 0.21808 + 0.23396)$

$= 1.20832$

$\therefore y(0.4) \simeq 1.2083$, correct upto four decimal places.

Ex.13. Solve initial value problem

$$10\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1$$

for $x = 0.1, 0.2$ by using Runge-Kutta fourth order method and find the solution correct upto 4 places of decimal.

[*W.B.U.T., CS-312, 2004*]

Solution. Here $f(x, y) = \dfrac{x^2 + y^2}{10}$, $x_0 = 0$, $y_0 = 1$

Let $h = 0.1$

$\therefore$ By fourth order Runge-Kutta method,

$$y(x_0 + h) = y_0 + k$$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

$$k_1 = hf(x_0, y_0) = 0.1\left(\frac{0+1}{10}\right) = 0.01$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1f(0.05, 1.005)$$

$$= 0.1 \times \frac{(0.05)^2 + (1.005)^2}{10} = 0.010125$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05, 1.0050625)$$

$$= 0.1 \times \frac{(0.05)^2 + (1.0050625)^2}{10}$$

$$= 0.000025$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.000025)$$

$$= 0.1 \times \frac{(0.1)^2 + (1.000025)^2}{10} = 0.0101$$

$$\therefore k = \frac{1}{6}(0.01 + 2 \times 0.010125 + 2 \times 0.000025 + 0.0101)$$

$$= 0.00673$$

$$\therefore y(x_0 + h) = y(0.1) = 1 + 0.00673 \simeq 1.0067$$

For $y(0.2)$, we have $x_1 = 0.1, y_1 = 1.0067$

$$\therefore k_1 = hf(x_1, y_1) = 0.1 \times \frac{(0.1)^2 + (1.0067)^2}{10} = 0.010235$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f(0.15, 1.0118175)$$

$$= 0.1 \times \frac{(0.15)^2 + (1.0118175)^2}{10}$$

$$= 0.01046$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f(0.15, 1.01193)$$

$$= 0.1 \times \frac{(0.15)^2 + (1.01193)^2}{10}$$

$$= 0.010465$$

$k_4 = hf(x_1 + h, y_1 + k_3$

$= 0.1f(0.2, 1.017165$

$= 0.1 \times \frac{(0.2)^2 + (1.0}{10}$

$= 0.010746$

$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2$

$= \frac{1}{6}(0.010235 +$

$= 0.01047$

$\therefore y(x_1 + h) = y(0.2)$

$\simeq 1.0172$

Ex.14. Compute $y(0.8$ method from $\frac{dy}{dx}$

given $y(0.2) = 0.2027,$

Solution. Here $x_3 = 0.6$ and $y_0 = 0,$

Hence $f_0 = f(x_0,$

$f_3 = 1.4680$

Now fourth order

$y_4^{(p)} = y_3 + \frac{h}{24}(55$

and Adam's Moul

$y_4^{(c)} = y_3 + \frac{h}{24}(9f$

$\therefore$ From (1),

$y_4^{(p)} = 0.6842 +$

$= 1.0235$

50625)

).0101)

0235

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1f(0.2, 1.017165)$$

$$= 0.1 \times \frac{(0.2)^2 + (1.017165)^2}{10}$$

$$= 0.010746$$

$$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.010235 + 2 \times 0.01046 + 2 \times 0.010465 + 0.010746)$$

$$= 0.01047$$

$$\therefore y(x_1 + h) = y(0.2) = 1.0067 + 0.01047$$

$$\simeq 1.0172$$

Ex.14. Compute $y(0.8)$ by Adams-Moulton predictor-corrector method from

$$\frac{dy}{dx} = 1 + y^2, \; y(0) = 0$$

given $y(0.2) = 0.2027$, $y(0.4) = 0.4228$, $y(0.6) = 0.6842$

Solution. Here $f(x, y) = 1 + y^2$, $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$ and $y_0 = 0$, $y_1 = 0.2027$, $y_2 = 0.4228$, $y_3 = 0.6842$

Hence $f_0 = f(x_0, y_0) = 1$, $f_1 = f(x_1, y_1) = 1.0411$, $f_2 = 1.1788$, $f_3 = 1.4680$

Now fourth order Adam's Bashforth formula is

$$y_4^{(p)} = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0) \quad \ldots \quad (1)$$

and Adam's Moulton formula is

$$y_4^{(c)} = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1) \quad \ldots \quad (2)$$

$\therefore$ From (1),

$$y_4^{(p)} = 0.6842 + \frac{0.2}{24}(80.3960 - 69.5468 + 38.5203 - 9)$$

$$= 1.0235$$

$\therefore f_4 = f\left(x_4, y_4^{(p)}\right) = 2\cdot0475$

$\therefore$ From (2),

$y_4^{c(1)} = 0.6842 + \frac{0.2}{24}(18.4275 + 27.8945 - 5.8938 + 1.0411)$

$= 1.0298$

$\therefore f_4 = f\left(x_4, y_4^{c(1)}\right) = 2\cdot0604$

$\therefore$ From (2),

$y_4^{c(2)} = 0.6842 + \frac{0.2}{24}(18.5440 + 27.8945 - 5.8938 + 1.0411)$

$= 1.0308$

$\therefore \tilde{f}_4 = f\left(x_4, y_4^{c(2)}\right) = 2.0624$

$\therefore$ From (2), $y_4^{c(3)} = 1.0309$

$\therefore \tilde{f}_4 = f\left(x_4, y_4^{c(3)}\right) = 2.0628$

$\therefore$ From (2), $y_4^{c(4)} = 1.0309$

Hence $y_4^{c(3)} = y_4^{c(4)} = 1.0309$, correct upto four decimal places

$\therefore y(0.8) \simeq 1.0309$

Ex.15. Apply Milne's method to find the solutions of the differential equation

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

at $x = 0.08$,

given $y(0) = 1, y(0.02) = 1.02, y(0.04) = 1.0392\ y(0.06) = 1.0577$

Solution. Here $x_0 = 0, x_1 = 0.02, x_2 = 0.04, x_3 = 0.06$

and $y_0 = 1, y_1 = 1.02, y_2 = 1.0392,\ y_3 = 1.0577$

Also, $f(x, y) = \frac{y - x}{y + x}$

$\therefore f_1 = f(x_1, y_1) = \dfrac{1.02 - 0.02}{1.02 + 0.02} = 0.9615$

Similarly, $f_2 = f(x_2, y_2) = 0.9259$

$f_3 = f(x_3, y_3) = 0.8926$

Now Milne's predictor formula of order 4 is

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \quad \dots \quad (1)$$

and the corrector formula of order 4 is

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \quad \dots \quad (2)$$

$\therefore$ From (1),

$$y_4^{(p)} = 1 + \frac{4 \times 0.02}{3}(2 \times 0.9615 - 0.9259 + 2 \times 0.8926)$$

$$\simeq 1.0742$$

$\therefore \tilde{f}_4 = f\left(x_4, y_4^{(p)}\right) = 0.8614$

$\therefore$ From (2),

$$y_4^{c(1)} = 1.0392 + \frac{0.02}{3}(0.9259 + 4 \times 0.8926 + 0.8614)$$

$$\simeq 1.0749$$

$\therefore \tilde{f}_4 = f\left(x_4, y_4^{c(1)}\right) = 0.8615$

$\therefore$ From (2),

$$y_4^{c(2)} = 1.0749$$

$\therefore y_4^{c(1)} = y_4^{c(2)} = 1.0749$, correct upto four decimal places.

$\therefore y_4 = 1.0749$ i.e. $y(0.08) = 1.0749$

Ex.16. Using the method of finite difference find the solution of the boundary value problem

$$x^2 y'' + xy' = 1;\ y(1) = 0,\ y(1.4) = 0.0566$$

Solution. The finite difference form of the given equation is

$$x_i^2\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}+x_i\frac{y_{i+1}-y_{i-1}}{2h}=1$$

i.e. $\left(2x_i^2-hx_i\right)y_{i-1}-4x_i^2y_i+\left(2x_i^2+hx_i\right)y_{i+1}=2h^2$

$$i=1,2,\dots n-1$$

with the boundary conditions $y_0=0, y_n=0.0566$

Taking $h=0.1$ i.e. $n=4$, the above system becomes

$$2.31y_0-4.84y_1+2.53y_2=0.02$$

$$2.76y_1-5.76y_2+3y_3=0.02$$

$$3.25y_2-6.76y_3+3.51y_4=0.02$$

where $y_0=0, y_n=0.0566$

Solving the system we get

$$y_1=0.0046, y_2=0.0167, y_3=0.0345$$

Hence $y(1.1)=0.0046$, $y(1.2)=0.0167$, $y(1.3)=0.0345$

Ex. 17. Using finite difference method, solve the following BVP.

$$\frac{d^2y}{dx^2}+y+1=0$$

with $y(0)=0, y(1)=0$ [M.A.K.A.U.T., MCS-401, 2014,2016]

Solution. The finite difference form of the given equation is

$$\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}+y_i+1=0$$

i.e. $y_{i-1}+\left(h^2-2\right)y_i+y_{i+1}=-h^2$, $i=1,2,\dots n-1$

with the boundary conditions at $x_0=0$ and at $x_n=1$ i.e. $y_0=0, y_n=0$.

Taking $h=0.25$ i.e. $n=4$, the above system becomes

$$y_0-1.9375y_1+y_2=-0.0625$$

$$y_1-1.9375y_2+y_3=-0.0625$$

$$y_2-1.9375y_3+y_4=-0.0625$$

where $y_0=0, y_4=0$

Solving the system

$y_1=0.10468, y_2$

Hence $y(0.25)=0$

and $y(0.75)=0.1046$

I. SHO

1. Describe Euler's

equation

Interpret the e

2. Evaluate $y(0.1$

3. Solve $\frac{dy}{dx}=x$

by Euler's meth

4. Find $y(0.10$

differential equ

$\frac{dy}{dx}$

correct to four

5. Solve by u

equation for