I. Simple Harmonic Motion in one dimension.

(i) Introduction: The study of simple harmonic motion (SHM) has so far been useful for understanding a great many physical phenomena ranging from the motion of pendulum to the vibration of atoms or molecules in a crystal lattice. Since the days of Galileo's observation of pendulum motion, the subsequent determination of time period by Christian Huygens, the theory of oscillation has caught much attention through the ages both as a physical theory and as a genuine case of linear 2nd order differential equation. Our present discussion is essentially based on modelling through differential equations involving various cases of free, damped and forced vibrations.

(ii) Dynamical Consideration: Let's consider a particle of mass m moving along a st. line (x-axis for our present purpose). At any point of time t, x be the distance of the particle from a specified origin.

Definition 1: Let $x \doteq x(t)$ be a continuous function of time (t). It is called the trajectory.

Remark 1. We assume that the 2nd derivative of x exists and continuous.

2. The quantities $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$ are called velocity v and acceleration f respectively.

3. To determine $x \doteq x(t)$ we need to have a differential equation given by the following axiom.

Axiom 1: $\dfrac{d^2x}{dt^2} = F/m$ or $\ddot{x} = F/m$ (Newton's law)...[1]

where F is called the force acting on the mass.

For our present purpose we will construct the force term F on the r.h.s. of eq-[1] for a so-called spring-mass system (Diagram-1)

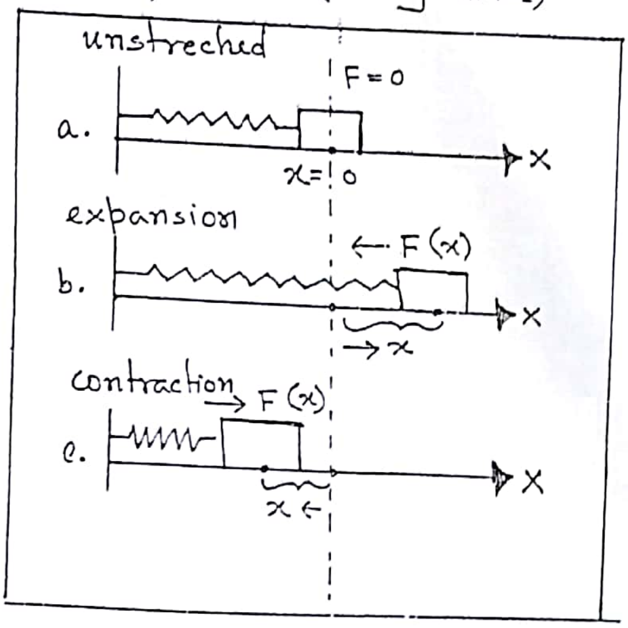

Diagram-1

• *Example-1:*
Considering the spring constant to be K, the force is a linear function of x acting opposite to the direction of increasing x (from the origin) i.e.;

$$F = -Kx$$

Hence for this system eq-[1] becomes

$$\ddot{x} = -\frac{Kx}{m}$$

$$\Rightarrow \quad \ddot{x} + \frac{K}{m}x = 0 \quad \text{---[2]}$$

As K and m are positive we can always write the ratio K/m to be equal to ω_0^2 ($\omega_0 \in \mathbb{R}$) leading to

$$\ddot{x} + \omega_0^2 x = 0 \quad \text{--- [3]}$$

• *Remark:* 1. Eq^n–[3] is linear because if $x_1(t)$ and $x_2(t)$ are two solutions their linear combination $c_1(x_1) + c_2(x_2)$, (c_1, c_2 being constants) is also a solution.

2. Eq^n–[3] is homogeneous in the sense that it contains no term independent of x and its derivative.

3. The point x=0 is called the point of equilibrium

In view of the above discussion we conclude the following definition.

• *Definition-2:* A motion is said to be an SHM about x=0, if the force F is given by the equation

$$\boxed{F = -Kx}$$

thus admitting a differential equation;

$$\boxed{\cancel{\ddot{x} = -\omega_0^2 x}}$$

$$\boxed{\ddot{x} + \omega_0^2 x = 0} \quad \cdots [3]$$

- General Solution of equation – [3] :

Let's choose $x(t) = \exp(\lambda t)$, $\lambda = $ const

Upon substitution in eq–[3]

$$(\lambda^2 + \omega_0^2) e^{\lambda t} = 0$$

If it holds for all t, $\lambda^2 + \omega_0^2 = 0 \Rightarrow \lambda = \pm i\omega_0$

where $i = \sqrt{-1}$. Two linearly independent solutions are possible for two values of λ. Hence, the general solution is given by

$$\boxed{x(t) = A\exp(i\omega_0 t) + B\exp(-i\omega_0 t)} \cdots\cdots [4]$$

where, A and B are arbitrary constants.

- Initial Value Problem and Specific solutions:

We can consider specific solutions by identifying the constants A and B corresponding to the physical situation involved at $t=0$. (Diagram – 2)

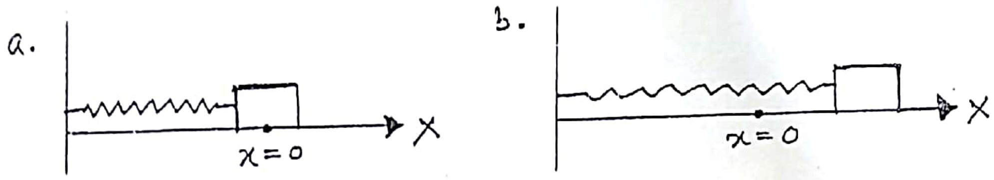

Diagram – 2

- Case – a. At $t = 0$; $x(t=0) = 0$, $v(t=0) = v_0$

From equation – [4],

$$\left.\begin{aligned} 0 &= A + B \\ v_0 &= i\omega_0(A - B) \end{aligned}\right\} \Rightarrow A = \frac{v_0}{2i\omega_0} = -B$$

So, $$x(t) = \frac{v_0}{2i\omega_0}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$$

$$\Rightarrow \boxed{x(t) = \frac{v_0}{\omega_0}\sin\omega_0 t} \cdots\cdots [5]$$

- Case – b. At $t = 0$; $x(t=0) = x_0$, $v(t=0) = 0$.

From equation – [4]

$$\left.\begin{aligned} x_0 &= A + B \\ 0 &= i\omega_0(A - B) \end{aligned}\right\} \Rightarrow A = B = \frac{x_0}{2}$$

$$\Rightarrow x(t) = \frac{x_0}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) = x_0\cos\omega_0 t.$$

$$\Rightarrow \boxed{x(t) = x_0 \cos \omega_0 t} \quad \text{....} \quad [6]^*$$

[N.B. Here we have used de-Moivre's identity

$$e^{\pm i\theta} = \cos\theta \pm i \sin\theta$$

Hence,
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$
$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})\ ]$$

- *Interpreting the trajectory:*

For our further discussion we can choose any one of the solutions [5] and [6]. Let's start with the later — $x(t) = x_0 \cos \omega_0 t$ giving

$$v = \frac{dx}{dt} = x_0(-\omega_0)\sin\omega_0 t = -\omega_0 x_0 \sin \omega_0 t$$

Writing $\max|v| = v_0$, we get $v_0 = \omega_0 x_0 = \omega_0 a$ where a is called the amplitude of oscillation. Thus we can write,

$$x(t) = a \cos \omega_0 t \quad \text{....} \quad [7]$$

- *Definition 3:* $x \doteq x(t)$ is called a periodic function with period T, if there exists a smallest number T such that

$$x(t+T) = x(t) \quad \text{....} \quad [8]$$

- *Remark:* 1. In view of equation -[8] we see that $x\left(t + \frac{2\pi}{\omega_0}\right) = x(t)$, giving $T = \frac{2\pi}{\omega_0}$. In our study of SHM, ω_0 is called the frequency and T, the time period.

2. ω_0 is a function of system's parameters (K and m) and is ~~related~~ independent of the amplitude a. (isochronicity condition). ω_0 is called natural frequency of oscillation.

3. The acceleration of the system $f(t) = -\omega_0^2 a \cos\omega_0 t$ and both velocity and acceleration & velocity are periodic function of t with same period.

* It is well known from the theory of 2nd order linear differential equation that given two initial conditions, the solution exists and it is unique. There is no solution other than eq^n [6] ~~is~~ possible for the initial condition: $x(0) = x_0$; $v(0) = 0$.

In our present discussion we frequent sinusoidal functions (sine/cosine). But what is that?

$\sin(?)$ $\cos(?)$

We see that if we replace $\omega_0 t$ by $\omega_0 t + \phi$ ($\phi \in \mathbb{R}$ and constant) all of our results remains intact, i.e.; $x(t) = a\cos(\omega t + \phi)$ is always a possible solution of eqn.[3]

We call (?) as phase of the said function, and ϕ is called the epoch. The following ~~the~~ diagram (Diagram-3) gives the phase relation among $x(t)$, $v(t)$ and $f(t)$

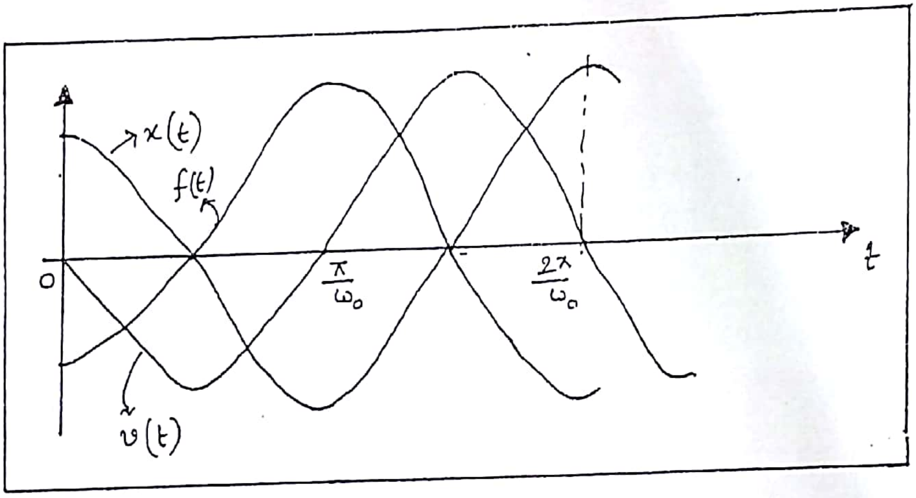

Diagram-3.

- *Energy Conservation*: That SHM is associated with ~~the~~ a conservative system is ensured by the following theorem.

- *Theorem 1*: An equation of motion of the form $\ddot{x} + \omega_0^2 x = 0$; $(\omega_0^2 = k/m)$ admits an energy conservation principle.

Proof: Given $\dfrac{d^2x}{dt^2} + \omega_0^2 x = 0$

$$\Rightarrow \frac{1}{2}m\frac{dx}{dt}\left(\frac{d^2x}{dt^2}\right) + \omega_0^2 \frac{1}{2}m\frac{dx}{dt}x = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2\right] + \frac{d}{dt}\left[\frac{1}{2}m\omega_0^2 x^2\right] = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2\right] = 0$$

$$\Rightarrow \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 = c \text{ (const)}$$

The constant c can be determined by imposing the condition that $v=0$ at $x=a$ giving

$$c = \frac{1}{2} m \omega_0^2 a^2$$

Hence $$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2 = \frac{1}{2} m \omega_0^2 a^2 \text{ (const)} \quad \text{---- [9]}$$

• **Remark:** 1. The 1st term on the l.h.s. is called kinetic energy and the 2nd term, the potential energy denoted by T and V respectively.

2. On the right hand side the single term depends only on the amplitude and all other system's parameters (m & ω_0)

3. In view of eqn. [9] one can write

$$v(x) = \pm \omega_0 \sqrt{a^2 - x^2} \quad \text{---- [10]}$$

4. It is often easier to establish the equation of motion for SHM by differentiating the total energy $E = T + V$ which is constant over time. The time variation of T (and V) is given by the following diagram (Diagram-4).

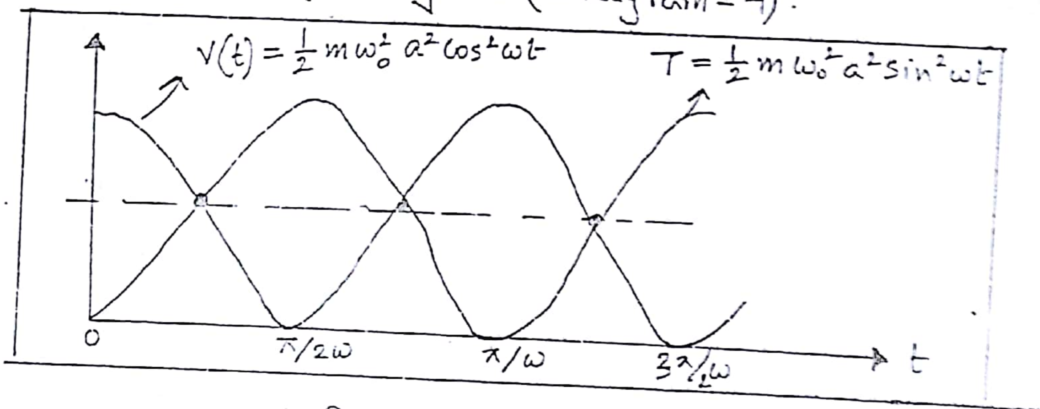

Diagram - 4.

• **Time Average of a physical quantity**: So many of our subsequent discussions will involve consideration of what is known as time average as defined below.

• **Definition-4**: Time average of a physical quantity $g \doteq g(t)$ ~~is~~ over the time period T is given by

$$\langle g(t) \rangle = \frac{1}{T}\int_0^T g(t)\,dt\,, \quad T = \frac{2\pi}{\omega_0} \quad \cdots [11]$$

• **Remark**: 1. The following results are very often used to calculate the time average of various quantities in our study of SHM ~~an~~ in particular and oscillation in general.

(i). $\langle \cos(\omega_0 t + \phi) \rangle = \frac{1}{T}\int_0^T \cos(\omega_0 t + \phi)\,dt = 0$

(ii). $\langle \sin(\omega_0 t + \phi) \rangle = \frac{1}{T}\int_0^T \sin(\omega_0 t + \phi)\,dt = 0$

(iii). $\langle \cos^2(\omega_0 t + \phi) \rangle = \frac{1}{T}\int_0^T \cos^2(\omega_0 t + \phi)\,dt$

$$= \frac{1}{T}\,\frac{1}{2}\int_0^T [1 + \cos(2\omega_0 t + 2\phi)]\,dt.$$

$$= \frac{1}{2}$$

(iv). $\langle \sin^2(\omega_0 t + \phi) \rangle = \frac{1}{T}\int_0^T \sin^2(\omega_0 t + \phi)\,dt = \frac{1}{2}$

2. One can ~~very~~ verify the following table regarding time average. (Table-1)

Physical Quantity	Expression	Time Average
1. Position	$x(t) = a\cos(\omega_0 t + \phi)$	0
2. Velocity	$v(t) = -a\omega_0 \sin(\omega_0 t + \phi)$	0
3. Acceleration	$f(t) = -a\omega_0^2 \cos(\omega_0 t + \phi)$	0
4. Kinetic Energy	$T(t) = \frac{1}{2}m\omega_0^2 a^2 \sin^2(\omega_0 t + \phi)$	$\frac{1}{4}m\omega_0^2 a^2$
5. Potential Energy	$V(t) = \frac{1}{2}m\omega_0^2 \cos^2(\omega_0 t + \phi)$	$\frac{1}{4}m\omega_0^2 a^2$
6. Power	$P(t) = \frac{a^2\omega_0^3}{2}\sin 2(\omega_0 t + \phi)$	0

Table-1.

• (iii) Kinematical Consideration: Let's consider a point particle moving in X-Y plane. If the position vector of the particle is given by $\vec{r}$ then

$$\vec{r} = x\hat{i} + y\hat{j} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j},$$ where

$|\vec{r}| = a$. The projections of this motion along X and Y axis respectively are given by

$$\left.\begin{aligned} x &= \hat{i}\cdot\vec{r} = a\cos\theta \\ y &= \hat{j}\cdot\vec{r} = a\sin\theta \end{aligned}\right\} \cdots\cdots [12\ a, b]$$

• Definition 5: A motion (as given above) is said to be uniformly circular about the origin (0,0) if

$$\theta = \omega_0 t + \phi, \quad \omega_0 \text{ and } \phi \text{ are real consts.}$$

We call ω_0 and ϕ to be angular velocity and epoch respectively.

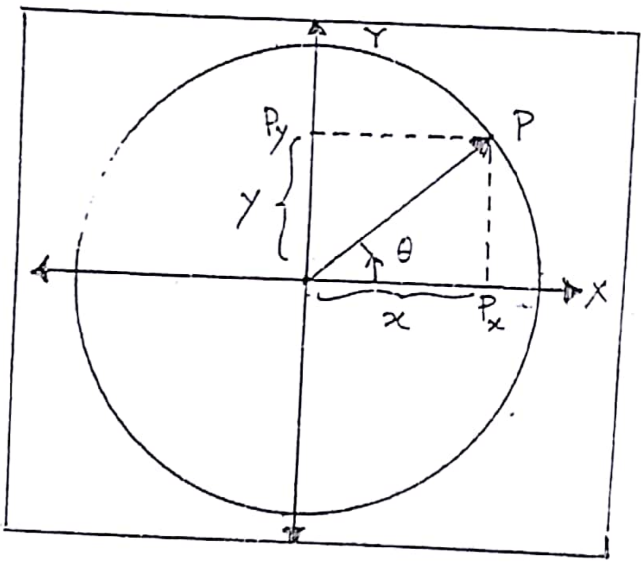

Diagram-5.

Taking θ to be the angle w.r.t +tive X-axis and θ = 0 at t = 0, we see φ = 0 and Diagram-5 results. Hence, in view of equation [12 a, b] we can write,

$$\left.\begin{aligned} x(t) &= a\cos\omega_0 t \\ y(t) &= a\sin\omega_0 t \end{aligned}\right\} [13\ a, b]$$

and the following kinematic definition of SHM —

• Definition 6: The projections of a uniform circular motion on either of the co-ordinate axes are called SHM.

• Remark: 1. In diagram-5 the projections of point P on the axes P_x & P_y respectively moves along the respective axis. Each of their motions is simple harmonic, with identical frequency ω_0 and a phase separation $\pi/2$.

In this section we shall consider some basic features of wave as a solution of classical wave equation with an objective to justify light as an electromagnetic wave.

Let us consider a collection of particles infinitesimally close to each other and located along the positive x-direction. Each point is vibrating simple-harmonically but not all at a time. If ψ represents the displacement of any particle along the vertical direction with respect to x-axis. This means if for the particle at the origin $(x=0)$

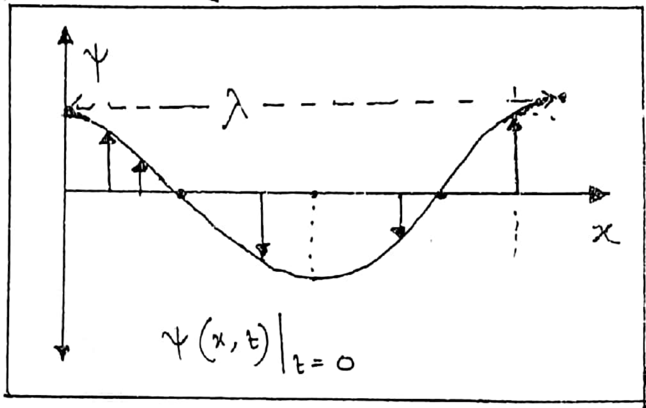

Diagram W-1.

$$\psi\Big|_{x=0} = a\cos\omega t$$

then that for a particle at $x = x (\neq 0)$

$$\psi\Big|_{x=x} = a\cos(\omega t - kx) \quad \cdots\cdots [1]$$

Diagram W-1 represents various ψ-values at $t=0$

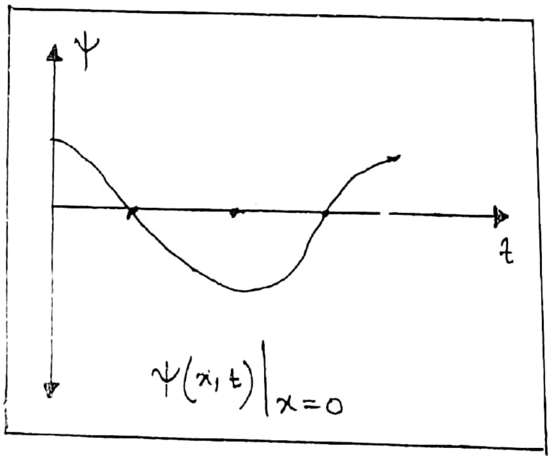

Diagram W-2

The quantity kx is the phase difference for a particle at $x=x$ as compared to that at $x=0$. Diagram W-1 therefore represents the wave ~~profile~~ profile at $t=0$. On the other hand diagram-W-2 represents the vibration of the particle at $x=0$ and does not represent a wave. Equation-[1] leads us to the following definition

- *Definition-1*: A plane harmonic wave propagating along the positive x-direction is defined by the function:-

$$\psi(x,t) = a\cos(\omega t - kx) \quad \cdots\cdots [2]$$

or

$$\psi(x,t) = a\sin(\omega t - kx) \quad \cdots\cdots [2]'$$

• **Remark** : 1. From equation-2 we see

$$\psi\left(x+\frac{2\pi}{k}, t\right) = a\cos\left[\omega t - k\left(x+\frac{2\pi}{k}\right)\right]$$
$$= a\cos(\omega t - kx) = \psi(x,t)$$

i.e.; ψ is a periodic function of x with $\frac{2\pi}{k}$ as period. We call $\lambda = \frac{2\pi}{k}$ as the wave-length. Physically speaking λ represents the minimum distance between two points with identical phase. (Diagram W-1)

2. Now as $\psi \doteq \psi(x,t) = a\cos(\omega t - kx)$

$$\frac{\partial\psi}{\partial x} = +ak\sin(\omega t - kx)$$
$$\frac{\partial^2\psi}{\partial x^2} = -ak^2\cos(\omega t - kx) \quad \cdots\cdots [3]$$

& $$\frac{\partial\psi}{\partial t} = -a\omega\sin(\omega t - kx)$$
$$\frac{\partial^2\psi}{\partial t^2} = -a\omega^2\cos(\omega t - kx) \quad \cdots\cdots [4]$$

From equation [3] and [4]

$$\frac{1}{k^2}\frac{\partial^2\psi}{\partial x^2} = \frac{1}{\omega^2}\frac{\partial^2\psi}{\partial t^2}$$

or $$\boxed{\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0} \quad \cdots\cdots [5]$$

where $c = \frac{\omega}{k}$ is a constant for obvious reason.

Equation-5 is known as classical wave equation.

3. For dimensional ground c has the dimension of velocity and we call it to be the wave velocity. We shall clarify its physical meaning later.

4. Equation-2 is not the only solution of equation-[5]. In fact equation-[5] has solutions with more general functional form as explained by the following theorem due to Hamilton :-

Taking up the 2nd one

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\psi = 0$$

$$\Rightarrow \quad \psi_x + \frac{1}{c}\psi_t = 0 \quad \cdots\cdots [9]$$

On the otherhand $\psi = \psi(x,t)$ gives.

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial t}dt = \psi_x\, dx + \psi_t\, dt \cdots [10]$$

Equation [9] gives $\left(0,\ 1,\ \frac{1}{c}\right)\cdot\begin{pmatrix}1\\ \psi_x\\ \psi_t\end{pmatrix} = 0 \quad \cdots [11]$

Equation [10] gives $\left(d\psi\ -dx,\ -dt\right)\cdot\begin{pmatrix}1\\ \psi_x\\ \psi_t\end{pmatrix} = 0 \quad [12]$

Comparing [11] and [12]

$$d\psi = 0$$
$$-dx = \beta$$
$$-dt = \beta/c$$

$$\Rightarrow \quad \frac{dx}{dt} = c \quad \text{or} \quad x - ct = \text{Const.}$$

Recalling, $c = \frac{\omega}{k}$; $x - \frac{\omega}{k}t = \text{Const}$ or $kx - \omega t = \text{Const.}$

We choose $\psi = f$ that maintains $x - ct = \text{Const}$ and implies a wave along the positive x direction. The same can be argued for $\psi = g$ ~~at ω~~ as well.

• *Remark*: 1. We call this quantity $kx - \omega t = \phi$ as the phase of the wave. So it is the phase of the wave which is propagating along the positive x-direction. This justifies $c = \frac{\omega}{k}$ as the <u>phase velocity</u> of the wave, which is nothing but the wave velocity as we've mentioned earlier.

2. For a ~~pl~~ plane harmonic wave moving along the x-direction ~~this f~~ and g function become sinusoidal.

Theorem-1: Let's write the classical wave equation (eq^n 5) In the following form:-

$$\psi_{xx} - \frac{1}{c^2}\psi_{tt} = 0 \quad \cdots\cdots [6]$$

where, $\psi_x = \partial\psi/\partial x$ and $\psi_t = \partial\psi/\partial t$. Then,

(i) The general solution can be given by

$$\psi(x,t) = f(x-ct) + g(x+ct)$$

(ii) While f is constant along the line $x-ct=$ const, g is constant along $x+ct=$ Const.

Proof :(i) We shall prove the case for $f \doteq f(x-ct)$ i.e., it satisfies equation-6

$$f_x = \frac{\partial}{\partial x} f(x-ct) = f'(x-ct)$$

$$\Rightarrow \quad f_{xx} = \frac{\partial^2}{\partial x^2} f(x-ct) = \frac{\partial}{\partial x} f'(x-ct) = f''(x-ct) \quad \cdots [7a]$$

and

$$f_t = \frac{\partial}{\partial t} f(x-ct) = -c f'(x-ct)$$

$$f_{tt} = \frac{\partial^2}{\partial t^2} f(x-ct) = -\frac{\partial}{\partial t}\left[c f'(x-ct)\right]$$

$$= \cancel{c^2 \frac{\partial}{\partial t}}\ c^2 f'(x-ct) \quad \cdots [7b]$$

Comparing [7a] and [7b]

$$f_{xx} - \frac{1}{c^2} f_{tt} = 0 \quad \cdots [8]$$

Hence, $f \doteq f(x-ct)$ is a solution of eq^n. [5] or [6]
The same can be verified for $g \doteq g(x+ct)$

(ii) The wave equation can be written as

$$\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\psi = 0$$

giving $\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)\psi = 0$, $\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\psi = 0$

3. The quantity k actually determines the direction of wave propagation and in 3-dimension it is replaced by a vector $\vec{k} = (k_x, k_y, k_z)$ known as wave vector. The expression for phase is now being modified as

$$\vec{k}\cdot\vec{r} - \omega t \quad , \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

The equation $\vec{k}\cdot\vec{r} - \omega t = \text{const}$ represents a plane that contains infinitely many points with identical phase. This plane moves as t varies leading to the following definition.

- *Definition - 2*: A plane (or a surface in general) that contains all the points which are in same phase is called a wave front.

- *Transreverse and Longitudinal waves, Light waves.*

So far we have discussed, the wave phenomena has been explained in terms of the vibrations of medium particles and the vibrations are always normal to the direction of propagation. Such types of waves are called transverse wave having a direct correspondence with the diagram W-1. Another kind of wave is called longitudinal wave where the vibrations of medium particles are along the direction of propagation. This type of wave is not of our present interest. In the following we shall discuss how transverse wave can be possible even in absence of medium, light wave as a case in point.

Recalling wave equation we replace ψ by a vector quantity
$$\vec{E}(x,t) = \vec{E}_0 \cos(kx - \omega t)$$
where $\vec{E}_0 = E_{0y}\hat{j} + E_{0z}\hat{k}$ is a vector in y-z plane, giving
$$\vec{E}(x,t) = E_{0y}\cos(kx-\omega t)\,\hat{j} + E_{0z}\cos(kx-\omega t)\,\hat{k}$$
E_{0y} and E_{0z} are constants.

One can easily verify that $\vec{E}(x,t)$ satisfies classi[cal] wave equation admitting vector solutions. If we replace the material particles in diagram W-1 by some hypothetical points capable of producing time varying electric field (with phases depending on their respective location) in Y-Z plane, the following diagram can be possible as a snap-shot of $\vec{E}$ field at $t=0$

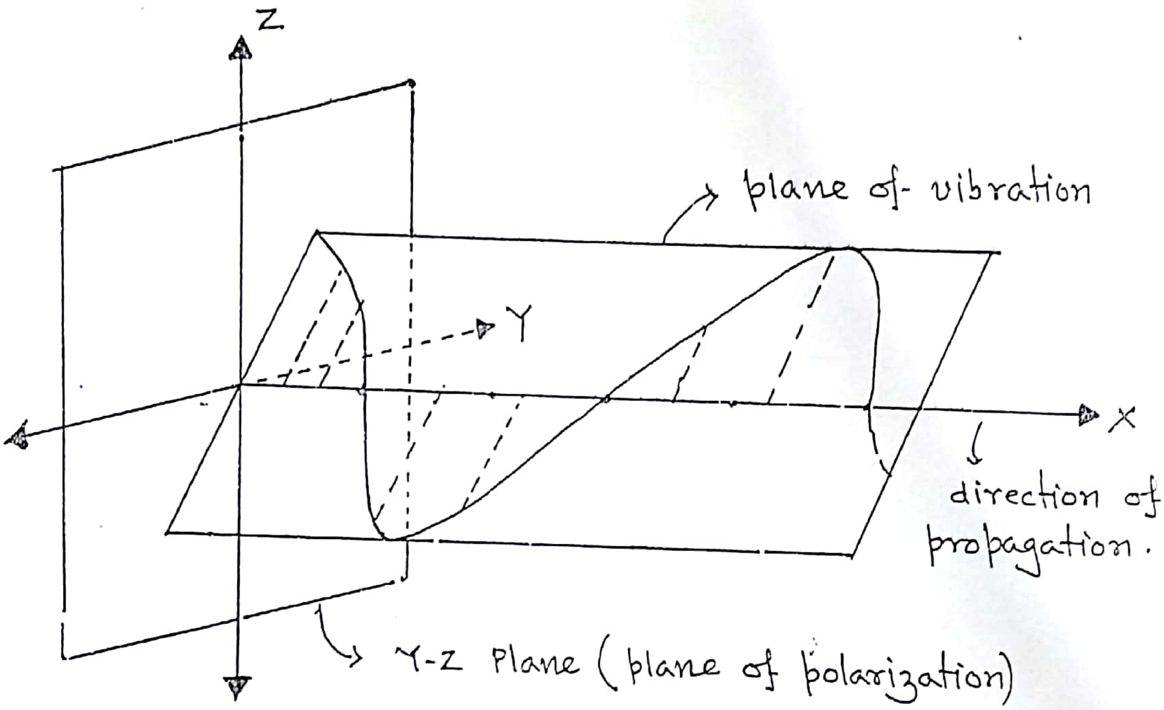

Diagram W-3.

• *Remark* : 1. When such a time varying ~~fie~~ electric field is known the corresponding magnetic field can easily be determined from Maxwell's theory. This magnetic field is time varying and directed normal to the electric field in YZ plane. In major part of our following discussion it is sufficient to consider only the electric field what we henceforth call light vector. The quantity E^2 implies the intensity of the field.

2. The plane normal to the propagation direction (Here YZ plane) is called plane of polarization. The plane determined by the vector $\vec{E}$ (light vector) and the propagation direction (here x-axis) is called plane of vibration. The wave understood in a such a manner is called a polarized wave.

Superposition and Interference.

The phenomenon of interference of waves is directly related to the superposition (linear) of two waves under certain restrictions. Before going into the details of superposition of two electromagnetic waves we'll consider the linear superposition of two simple harmonic oscillations with identical frequencies in view of the following theorem.

- *Theorem -1*: Let $x_1(t) = a_1 \cos(\omega t + \delta_1)$ and

$$x_2(t) = a_2 \cos(\omega t + \delta_2)$$

be two simple harmonic oscillations. Then,

(i) The linear superposition of these oscillations is again a simple harmonic motion with ~~identical~~ the same frequency.

(ii) The amplitude and phase of the ~~super~~ resultant motion are given by

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\delta_1 - \delta_2) \quad \cdots\cdots [1]$$

and
$$\delta = \tan^{-1}\left(\frac{a_1 \sin\delta_1 + a_2 \sin\delta_2}{a_1 \cos\delta_1 + a_2 \cos\delta_2}\right) \quad \cdots\cdots [2]$$

Proof: The trajectory of the superposed motion is given by

$$x(t) = x_1(t) + x_2(t)$$

$$\Rightarrow x(t) = a_1 \cos(\omega t + \delta_1) + a_2 \cos(\omega t + \delta_2)$$

$$\Rightarrow x(t) = a_1[\cos\omega t \cos\delta_1 - \sin\omega t \sin\delta_1] + a_2[\cos\omega t \cos\delta_2 - \sin\omega t \sin\delta_2]$$

$$\Rightarrow x(t) = (a_1 \cos\delta_1 + a_2 \cos\delta_2)\cos\omega t - (a_1 \sin\delta_1 + a_2 \sin\delta_2) \times \sin\omega t$$

$$\Rightarrow x(t) = A\cos\delta \cos\omega t - A\sin\delta \sin\omega t$$

[Identifying $A\cos\delta = a_1\cos\delta_1 + a_2\cos\delta_2$ --(a)

& $A\sin\delta = a_1\sin\delta_1 + a_2\sin\delta_2$] (b)

$$\Rightarrow x(t) = A\cos(\omega t + \delta)$$

Hence the motion is simple harmonic with the same frequency.

$(a)^2 + (b)^2 \Rightarrow \quad A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\delta_1 - \delta_2)$

$\cancel{(a)/(b)}$ $(b)/(a) \Rightarrow \quad \tan\delta = \dfrac{a_1 \sin\delta_1 + a_2 \sin\delta_2}{a_1 \cos\delta_1 + a_2 \cos\delta_2}$

• *Remark*: 1. The amplitude is independent of the sign of the phase difference $\delta_0 = \delta_1 - \delta_2$ of the superposing oscillations.

2. If the frequencies of the superposing waves are different, the ~~super~~ resultant oscillation is no longer simple harmonic giving rise to a number of frequency dependent behaviors which are not of our present interest.

3. The amplitude of the resultant oscillation has maximum and minimum value depending on ~~wt~~ $\delta_1 - \delta_2$ as ~~integral mu~~ even multiple ~~of~~ or odd multiple of π respectively.

• Superposition of two electromagnetic waves.

• *Definition-1*: By superposition of two electromagnetic waves given by the light vectors $\{\vec{E_1}, \vec{E_2}\}$ we mean the resultant vector ~~of~~

$$\vec{E} = \vec{E_1} + \vec{E_2} \quad \cdots \quad [3]$$

• *Remark*: 1. As $\vec{E_1}$ and $\vec{E_2}$ are solutions of classical wave equation (As being light vector) the resultant $\vec{E}$ is also a solution due to the linearity of the wave eqn.

2. If the intensities of the light vectors at any point are $I_1 \ (= E_1^2)$ and $I_2 \ (= E_2^2)$, the resultant intensity

$$I = (\vec{E_1} + \vec{E_2})^2 = (\vec{E_1} + \vec{E_2}) \cdot (\vec{E_1} + \vec{E_2}) = E_1^2 + E_2^2 + 2\vec{E_1} \cdot \vec{E_2}$$

As the light vectors are time varying we talk about the time average of resultant intensity given by

$$\langle I \rangle = \frac{1}{T} \int_0^T (E_1^2 + E_2^2 + 2\vec{E_1} \cdot \vec{E_2})\, dt. \quad [4]$$

The following theorem is immediate regarding the maxima and minima of $\langle I \rangle$.

- *Theorem-2*: Given two light vectors $\{\vec{E_1}, \vec{E_2}\}$ with same frequencies i.e.;

$$\vec{E_1} = \vec{E}_{10} \cos(\vec{K_1}\cdot\vec{r} - \omega t + \delta_1)$$

and
$$\vec{E_2} = \vec{E}_{20} \cos(\vec{K_2}\cdot\vec{r} - \omega t + \delta_2)$$

Let's denote the phase difference at any point with position vector $\vec{r_0}$ by

$$\delta_0 = \vec{K_1}\cdot\vec{r_0} - \vec{K_2}\cdot\vec{r_0} + \delta_1 - \delta_2 .$$

Then the intensity of superposition at the point $\vec{r_0}$ is

a. Maximum for $\delta = 0, \pm 2\pi, \pm 4\pi \cdots\cdots$

b. Minimum for $\delta = \pm\pi, \pm 3\pi, \cdots\cdots$

c. Uniform when $\vec{E_1}$ and $\vec{E_2}$ are polarized along mutually perpendicular directions.

Proof: The average intensity for the light vector $\vec{E_1}$ is

$$\langle I_1\rangle = \frac{1}{T}\int_0^T E_{10}^2 \cos^2(\vec{K_1}\cdot\vec{r_0} - \omega t + \delta_1)\, dt.$$

$$= \frac{E_{10}^2}{2T}\int_0^T [1 + \cos\{2(\vec{K_1}\cdot\vec{r_0} - \omega t + \delta_1)\}]\, dt$$

$$= \frac{E_{10}^2}{2}$$

Similarly $\langle I_2\rangle = \frac{E_{20}^2}{2}$

Now, $\langle I\rangle = \langle E^2\rangle = \langle E_1^2\rangle + \langle E_2^2\rangle + 2\langle \vec{E_1}\cdot\vec{E_2}\rangle$.

$$\langle \vec{E_1}\cdot\vec{E_2}\rangle = \frac{1}{T}\int_0^T \vec{E}_{10}\cdot\vec{E}_{20} \cos(\vec{K_1}\cdot\vec{r_0} - \omega t + \delta_1) \cos(\vec{K_2}\cdot\vec{r_0} - \omega t + \delta_2)\, dt$$

$$= \frac{1}{2T}\vec{E}_{10}\cdot\vec{E}_{20}\int_0^T [\cos\{(\vec{K_1}+\vec{K_2})\cdot\vec{r_0} - 2\omega t + \delta_1 + \delta_2\} + \cos\{(\vec{K_1}-\vec{K_2})\cdot\vec{r_0} + \delta_1 - \delta_2\}]\, dt$$

$$= \frac{\vec{E}_{10}\cdot\vec{E}_{20}}{2}\cos\delta$$

Hence $\langle I \rangle = \frac{E_{10}^2}{2} + \frac{E_{20}^2}{2} + 2 \cdot \frac{\vec{E}_{10} \cdot \vec{E}_2}{2} \cos\delta.$

Now if $\vec{E}_{10}$ and $\vec{E}_{20}$ are parallel

$$\langle I_1 \rangle = \frac{E_{10}^2}{2} + \frac{E_{20}^2}{2} + 2 \frac{E_{10}}{\sqrt{2}} \cdot \frac{E_{20}}{\sqrt{2}} \cos\delta$$

$$\Rightarrow \langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2\sqrt{\langle I_1 \rangle \langle I_2 \rangle} \cos\delta \quad \text{- - - [5]}$$

Then a. $\langle I \rangle = \langle I \rangle_{max} = \langle I_1 \rangle + \langle I_2 \rangle + 2\sqrt{\langle I_1 \rangle \langle I_2 \rangle}$

i.e., $\cos\delta = 1 \Rightarrow \delta = 0, \pm 2\pi, \pm 4\pi \ldots\ldots$

b. $\langle I \rangle = \langle I \rangle_{min} = \langle I_1 \rangle + \langle I_2 \rangle - 2\sqrt{\langle I_1 \rangle \langle I_2 \rangle}$

i.e.; $\cos\delta = -1 \Rightarrow \delta = \pm\pi, \pm 3\pi \ldots\ldots$

c. If $\vec{E}_{10} \perp \vec{E}_{20}$ $\vec{E}_1 \cdot \vec{E}_2 = 0$. This happens when the light vectors are polarized along the mutually perpendicular directions giving

$$\langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle$$

resulting to uniform intensity irrespective of δ and (and hence $\vec{r}_0$).

- *Definition-2*: The term $2\langle \vec{E}_1 \cdot \vec{E}_2 \rangle$ is called the interference term which is

a. positive for $\delta = 0, \pm 2\pi, \pm 4\pi$ implying constructive interference ($I = I_{max}$) (Diagram-1)

b. negative for $\delta = \pm\pi, \pm 3\pi$ implying destructive interference. ($I = I_{min}$) and the whole phenomenon is known as interference. (Diagram-2)

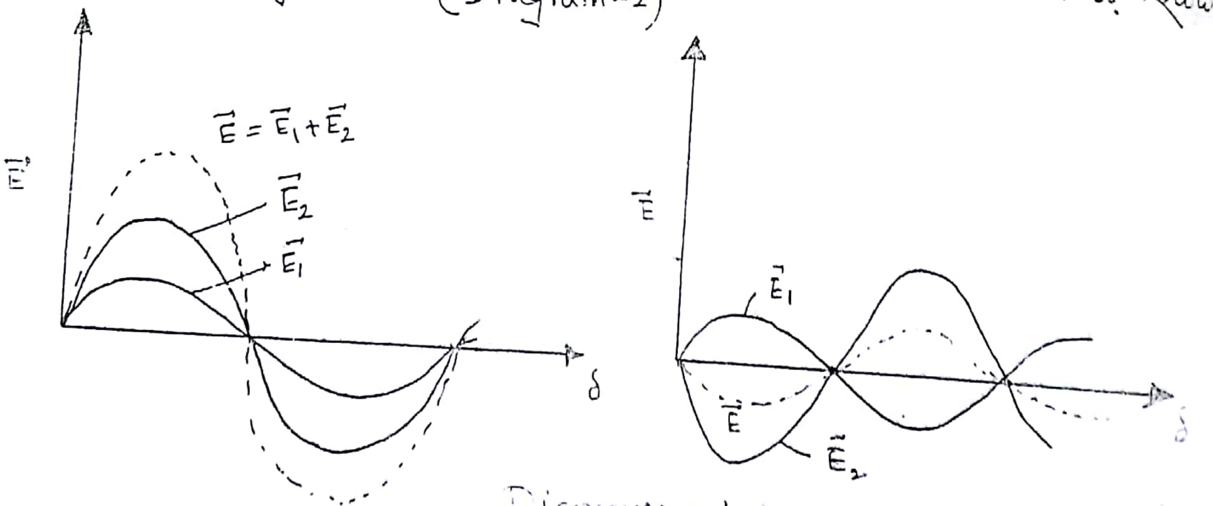

Diagram – 1, 2

• *Remark* : 1. When $\langle I_1 \rangle = \langle I_2 \rangle = \langle I_0 \rangle$, the resultant intensity

$$\langle I \rangle = 2\langle I_0 \rangle (1+\cos\delta) \quad (\text{from eq}^n\text{ [5]})$$

$$\Rightarrow \langle I \rangle = 4\langle I_0 \rangle \cos^2 \delta/2. \quad (\text{Diagram-[3]})$$

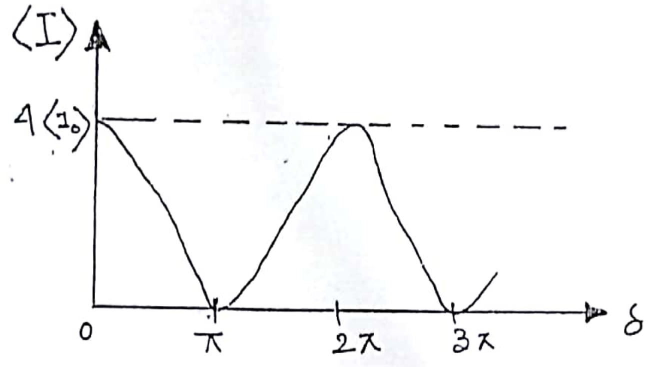

Diagram - 3.

2. The most important fact regarding the interference phenomenon is that the intensity at any particular point is no-longer the sum total of individual intensities of the superposing light vectors. Due to the presence of the interference term involving the phase difference the resultant intensity varies from point to point. Apperently this implies a violation of energy conservation principle if a particular point or a region of space is concerned. Now as the phase difference (that depends on the position vector of the said point) takes up values in a periodic interval $[0, 2\pi]$ for which the contribution to energy involves the average of $\cos\delta$ i.e.;

$$\int_0^{2\pi} \cos\delta \, d\delta = 0$$

Then in view of eq^n [5] the total energy is globally conserved.

3. Througout ~~at~~ our calculation we assume the phase difference δ between two interfering waves to fixed i.e., the wave reaching a particular point maintains a fixed phase difference δ independent of time and for a pair of strictly monochromatic waves it is independent of frequency. The constancy of δ is majorly attributed to the sources called coherent sources.

- Experimental Realization

In order to have a sustained interference pattern we therefore require two co-herent sources. Initially, it was not possible to produce two independent coherent sources ~~unit~~ until the discovery of modern day laser. But from a single source it is possible to produce coherent sources in the following ways.

A. Division of wave front

B. Division of amplitude.

- A. Interference by Division of Wavefront.

The idea to produce two coherent sources from a given wave front is motivated from what is known as Huygen's principle.

- Huygens principle: Let Σ_0 be the wave front at any instant t_0 and Σ' be the same at some letter instant $t' (> t_0)$, then,

(i) Σ' is possibly determined from the knowledge of Σ_0

(ii) Σ_0 serves as the source of secondary wavelets and Σ' becomes the envelope of all these wavelets.

(iii) The secondary wavelets have same velocity and frequency of the primary wave-let.

Diagram - 4.

Remark: 1. In Huygens principle it is assumed that the secondary wave has no effect except at the point where it ~~touce~~ touches the envelope.

2. It is also assumed that only the forward

moving envelope is to be considered — an assumption very often valid in one and three dimension, and not so obvious in two dimension.

3. Huygens principle and its subsequent extension can successfully explain the phenomena of reflection, refraction, interference, diffraction of wave in general. In the present context we exploit it to describe the so called double slit interference pattern as done by Young.

- Young's Double-slit experiment.

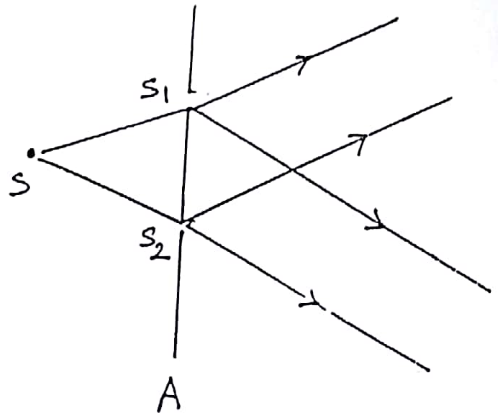

Diagram - 5.

- Experiment

The rays coming from a point source S fall on two pinholes at S_1 and S_2 situated close to each-other. S_1 and S_2 act as secondary monochromatic point sources which are in phase and beams from them are superposed in the region beyond A. An interference pattern is formed on a screen quite afar from the sources.

- Explanation.

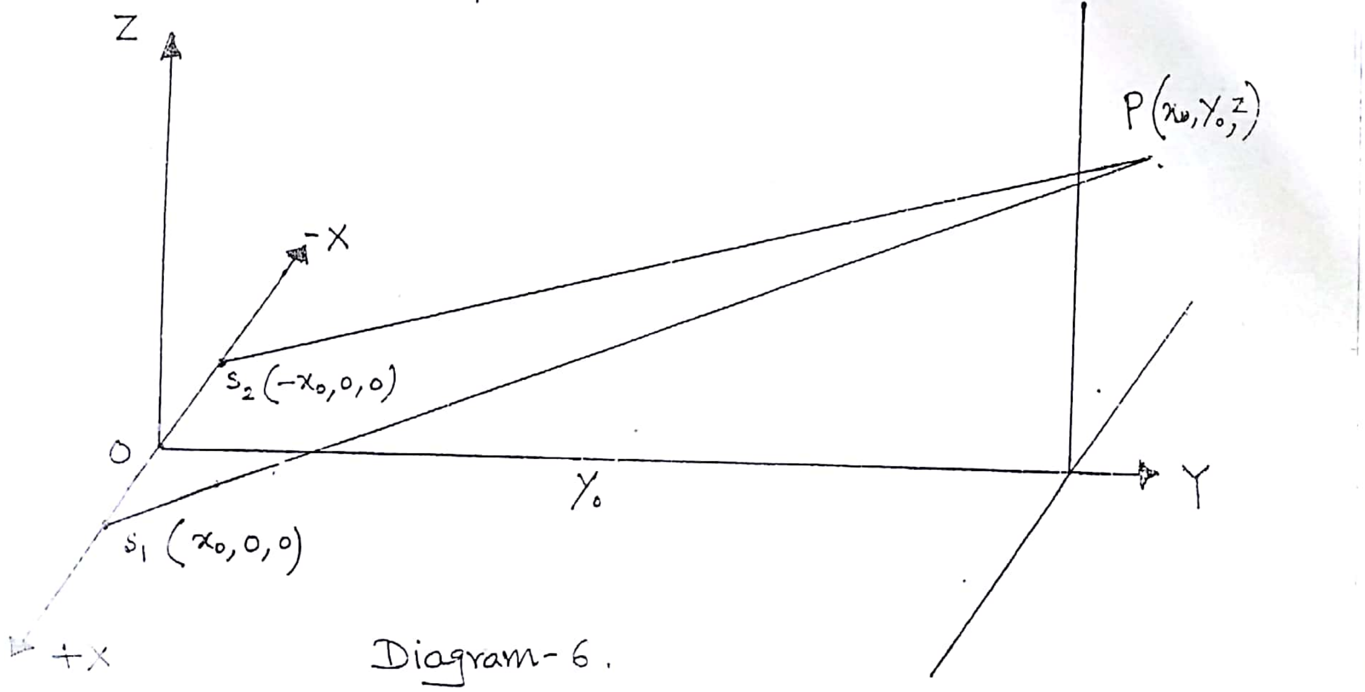

Diagram-6.

By definition the phase difference at a point P is given by

$$\delta = \vec{K_1}\cdot\vec{r} - \vec{K_2}\cdot\vec{r} = \left(\vec{K_1} - \vec{K_2}\right)\cdot\vec{r}$$

where $\vec{r}$ is the position vector of the point P, $\vec{K_1}$ the wave vector along the direction $\vec{S_1P}$ and $\vec{K_2}$, the wave vector along the direction $\vec{S_2P}$.

Hence, $\vec{K_1} = \frac{2\pi}{\lambda}\left(\text{unit vector along } \vec{S_1P}\right)$

$$= \frac{2\pi}{\lambda}\,\frac{(x-x_0)\hat{i} + y_0\hat{j} + z\hat{k}}{\left[(x-x_0)^2 + y_0^2 + z^2\right]^{1/2}}$$

Similarly $\vec{K_2} = \frac{2\pi}{\lambda}\left(\text{unit vector along } \vec{S_2P}\right)$

$$= \frac{2\pi}{\lambda}\,\frac{(x+x_0)\hat{i} + y_0\hat{j} + z\hat{k}}{\left[(x+x_0)^2 + y_0^2 + z^2\right]^{1/2}}$$

Therefore,

$$\delta = \frac{2\pi}{\lambda}\left[\frac{(x-x_0)\hat{i} + y_0\hat{j} + z\hat{k}}{\left\{(x-x_0)^2 + y_0^2 + z^2\right\}^{1/2}} - \frac{(x+x_0)\hat{i} + y_0\hat{j} + z\hat{k}}{\left\{(x+x_0)^2 + y_0^2 + z^2\right\}^{1/2}}\right]\cdot\left(x\hat{i} + y_0\hat{j} + z\hat{k}\right)$$

$$= \frac{2\pi}{\lambda}\left[\frac{x(x-x_0) + y_0^2 + z^2}{\left\{(x-x_0)^2 + y_0^2 + z^2\right\}^{1/2}} - \frac{x(x+x_0) + y_0^2 + z^2}{\left\{(x+x_0)^2 + y_0^2 + z^2\right\}^{1/2}}\right]$$

$$\simeq \frac{2\pi}{\lambda y_0}\left(-2xx_0\right) = -\frac{4xx_0}{\lambda y_0}\pi \quad (\text{for } y_0 \text{ being very large})$$

Choosing only the magnitude

The location of m-th bright fringe x_m satisfyies.

$$\frac{4x_m x_0}{\lambda y_0}\pi = 2m\pi \Rightarrow x_m = \frac{m\lambda y_0}{2x_0},$$

Similarly, the location of the m-th drak fringe

$$x_m = \frac{(2m+1)\lambda y_0}{4x_0},$$

The fringe width is defined as the spacing between two consecutive dark ~~of~~ or bright fringe and calculated a.s

for bright fringe $x_{m+1} - x_m = \frac{\lambda y_0}{2x_0}\left[m+1-m\right] = \frac{\lambda y_0}{2x_0}$

& for dark fringe $x_{m+1} - x_m = \frac{\lambda y_0}{\ }$

The socalled fringe-width = spacing between two consecutive dark (or bright) fringe —

For bright fringe $x_{m+1} - x_m = \frac{y_0 \lambda_0}{2x_0}(m+1-m) = \frac{y_0\lambda_0}{2x_0}$

Dark fringe $x_{m+1} - x_m = \frac{y_0 \lambda_0}{4x_0}\left[2(m+1)+1-(2m+1)\right]$

$$= \frac{y_0\lambda_0}{2x_0}$$

- Remark : 1. The fringes are formed on the x-z plane (i.e. plane of the screen), they are parallel to z-axis. The fringe pattern is given below.

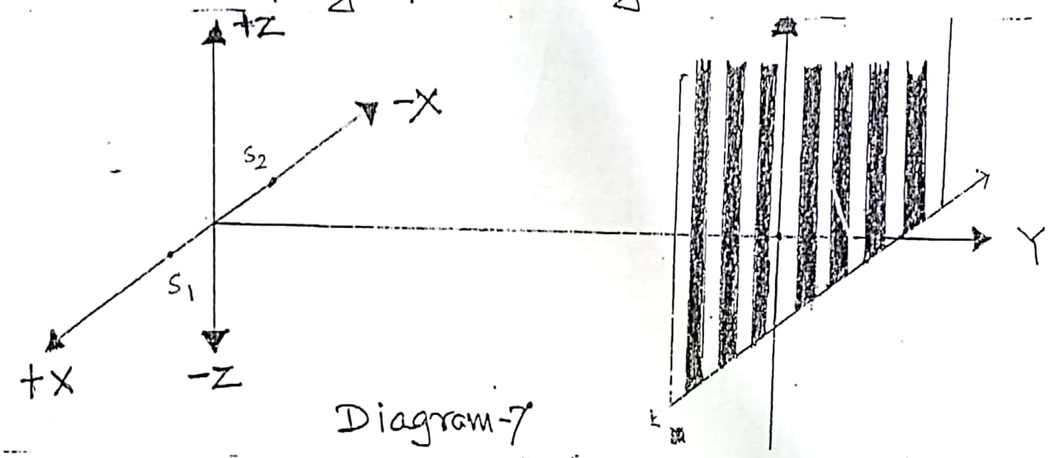

Diagram-7

2. For the central fringe $m=0$ and $x_m = x_0 = 0$. This can happen when it is a bright fringe.

- Displacement of fringe

Placing a transparent sheet of refractive index μ can cause a shift in the position of fringes due to the following consideration.

Let the thickness of the sheet is d and it is placed somewhere between s_1 and P_0

The time required for light to reach P from s_1

$$t = \frac{s_1P - d}{c} + \frac{d}{v} = \frac{1}{c}\left[s_1P - d + \mu d\right]$$

$$= \frac{1}{c}\left[s_1P + (\mu - 1)d\right]$$

The new path difference between the rays becomes

$$\Delta' = s_2P - \left[s_1P + (\mu-1)d\right] = (s_2P - s_1P) - (\mu-1)d$$

For bright fringe $\Delta' = 2m\left(\frac{\lambda}{2}\right)$

If the new location of the fringe is x'_m then

$$2m\left(\frac{\lambda}{2}\right) = \frac{2x'_m x_0}{y_0} - (\mu-1)d$$

$$\Rightarrow \quad x'_m = \left[m\lambda + (\mu-1)d\right]\frac{y_0}{2x_0}$$

$$\Rightarrow \quad x'_m = x_m + (\mu-1)\frac{dy_0}{2x_0}$$

$$\Rightarrow \quad x'_m - x_m = (\mu-1)\frac{dy_0}{2x_0}$$

The quantity $\Delta x_m = x'_m - x_m = (\mu-1)\frac{dy_0}{2x_0}$ is the amount of fringe shift. One can also start with the condition of dark fringe and find the same result. The amount of fringe shift is independent of the order of the fringe.

Remark : 1. This method can be used to determine the thickness and refractive index of the sheet.