# 1.3 SPECIAL TYPE OF DISTRIBUTION

### 1.3.1. Introduction :

A random variable is said to have a special type of distribution if its pmf / pdf gets a special form.

Seven such type of distributions are discussed in this chapter. Among these the two discrete distributions Binomial and Poisson are very well known and useful in everyfield of life. The continuous distribution 'Normal distribution' has also a wide range of application in sociology, market research and specially in industry. Definition of each of the distribution is first given and their properties and field of fitness are illustrated in this chapter.

### 1.3.2. Binomial Distribution.

A discrete random variable $X$ is said to have a binomial distribution with parameters $p(0<p<1)$ and $n$ (a positive integer) if its distribution is given by

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $X$ | : | 0 | 1 | 2 | ... | ... | $n$ |
| $f_i$ | : | $f_0$ | $f_1$ | $f_2$ | ... | ... | $f_n$ |

where the pmf $f_i = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$, $i = 0,1,2,\cdots,n$

For instant, one probability mass

$$f_1 = \binom{n}{1} p(1-p)^{n-1} = np(1-p)^{n-1} \text{ etc.}$$

**Note :** (1) The pmf $f_i$ satisfy the two fundamental properties $f_i \geq 0$ and $\sum_{i=0}^{n} f_i = 1$, which can be easily verified.

(2) When the random variable $X$ has a binomial distribution with parameters $n, p$ we write $X \sim b(n,p)$ and we say $X$ is a binomial variate.

(3) The significance of the parameters $n$ and $p$ would be given in subsequent theorem.



**Cases where Binomial Distribution fits.**

Let $A$ be an event of a random experiment $E$. We call "the event $A$" = 'success' and "the event Not $A$" = 'failure'. Let $p$ = probability of 'success' in a single trial of $E$. $E$ be repeated, independently, $n$ times. Let $X$ = number of succss in $n$ trials. Then $X$ may assume the values 0, 1, 2, ... ... $n$. For example, the event $(X=3)$ means " 3 success in $n$ trials". It can be shown that $P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$

Thus the distribution of $X$ becomes

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $X$ | : | 0 | 1 | 2 | 3 | ... | ... | $n$ |
| $f_i$ | : | $f_0$ | $f_1$ | $f_2$ | $f_3$ | ... | ... | $f_n$ |

where $f_i = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$ which is the pmf of Binomial distribution.

Thus 'No. of Success' in $n$ trials is a Binomial variate with parameter $p$ = probability of success in a single trial and $n$ = number of trial.

**Illustration.** The efficiency of a fighter-plane is such that the probability of a bomb hitting a target is $2/5$. The fighter is assigned to completely destroy a camp of enemy-side. The plane carries 6 bombs, i.e., 6 bombs can be aimed at the camp. Here throwing a bomb is the experiment. It can be repeated 6 times; ' A bomb hits the camp' = Success and $X$ = number of success in 6 trials. Then $X$ has the Binomial distribution,

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $X$ | : | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $f_i$ | : | $f_0$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ |

where $f_i = \binom{6}{i}\left(\frac{2}{5}\right)^i \left(1-\frac{2}{5}\right)^{6-i} = \binom{6}{i}\left(\frac{2}{5}\right)^i \left(\frac{3}{5}\right)^{6-i}$

If it is known that at least four direct hits are necessary to destroy the camp then the probability of complete destruction of the camp .

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$= P(X \geq 4) = P(X=4) + P(X=5) + P(X=6) = f_4 + f_5 + f_6$

$$= \binom{6}{4}\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^2 + \binom{6}{5}\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right) + \binom{6}{6}\left(\frac{2}{5}\right)^6 = \frac{112}{625}$$

**Theorem.** If $X$ has Binomial Distribution with parameter $n$ and $p$ then (i) its mean is $np$ (ii) its variance is $npq$ where $q = 1-p$. [W.B.U.Tech 2005]

*Proof.* Here $X : 0\ 1\ 2 \cdots n$

and its pmf is $f_i = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$

(i) Mean $= E(X) = \sum_{i=1}^{n} i f_i \quad = \sum_{i=1}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$

$$= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{r=0}^{n-1} \binom{n-1}{r} p^r (1-p)^{n-1-r}, \text{ replacing } i-1 \text{ by } r$$

$$= np(p+1-p)^{n-1} = np$$

(ii) Now

$$E\{X(X-1)\} = \sum_{i=0}^{n} i(i-1)\binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{n} i(i-1)\frac{n(n-1)}{i(i-1)}\binom{n-2}{i-2} p^i (1-p)^{n-i}$$

$$= n(n-1)p^2 \sum_{i=2}^{n} \binom{n-2}{i-2} p^{i-2}(1-p)^{n-i}$$

$$= n(n-1)p^2 \sum_{r=0}^{n-2} \binom{n-2}{r} p^r (1-p)^{n-2-r}, \text{ replacing } i-2 \text{ by } r$$

$$= n(n-1)p^2(p+1-p)^{n-2} = n(n-1)p^2.$$



$\therefore Var(X) = E\{X(X-1)\} - m(m-1)$

$= n(n-1)p^2 - np(np-1) = np(1-p) = npq$ where $q = 1-p$

$\therefore$ standard deviation $\sigma = \sqrt{npq}$.

**Illustration.** An unbiased die is tossed four times. Let 'multiple of three' be success; otherwise it is failure. Here $p$ = probability of sucess in a single trial $= \frac{2}{6} = \frac{1}{3}$. Let $X$ = number of 'multiple of three' appeared among these four trial. Then, as we discussed before, $X$ has Binomial distribution with parameter $n = 4$ and $p = \frac{1}{3}$. The expected number of 'multiple of 3' = mean of $X = 4 \times \frac{1}{3} = \frac{4}{3}$. The standard deviation of

$$X = \sqrt{4 \times \frac{1}{3}\left(1 - \frac{1}{3}\right)} = \sqrt{4 \times \frac{1}{3} \times \frac{2}{3}} = \frac{2\sqrt{2}}{3}.$$

**Illustrative Examples**

**Ex. 1.** *The mean and s.d of a binomial distribution are respectively 4 and* $\sqrt{\frac{8}{3}}$. *Find the values of n and p. Hence evaluate* $P(X=0)$. [W.B.U.Tech 2006]

We know the mean and s.d of a binomial variate are respectively $np$ and $\sqrt{np(1-p)}$.

$\therefore np = 4$ or, $np(1-p) = \frac{8}{3}$

$\therefore 4(1-p) = \frac{8}{3} \Rightarrow p = \frac{1}{3}$

$\therefore n = 4 \times 3 = 12$

$$P(X=0) = f_0 = {}^{12}C_0\left(\frac{1}{3}\right)^0\left(1-\frac{1}{3}\right)^{12-0} = \left(\frac{2}{3}\right)^{12}.$$

**Ex. 2.** *Comment on the statement "a binomial variate has mean 4 and s.d 3".*

Here, $np = 4$ and $\sqrt{np(1-p)} = 3$ i.e. $np(1-p) = 9$

$\therefore\ 4(1-p)=9 \quad \therefore\ 1-p=\frac{9}{4} \ \therefore\ p=-\frac{5}{4}$ which is not possible since $0<p<1$. So the statement is false.

**Ex. 3.** *If the mean of a binomial distribution is 3 and the variance is $\frac{3}{2}$, find the probability of obtaining at most 3 success.* [W.B.U.Tech 2007]

Let $X$ be the r.v corresponding to the number of success. Then the *pmf* of $X$ is

$$f_i = P(X=i) = {}^nC_i p^i (1-p)^{n-i},\ i = 0,1,2,\cdots,n$$

$\therefore$ mean $= np = 3$, Variance $= np(1-p) = \frac{3}{2}$

or, $\frac{np(1-p)}{np} = \frac{3}{2}\times\frac{1}{3} \quad \therefore\ p=\frac{1}{2}$ and so $n=6$

Now probability of at most 3 success $= P(X\le 3)$

$= P(X=0)+P(X=1)+P(X=2)+P(X=3)$

$$= {}^6C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^6 + {}^6C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^5 + {}^6C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 + {}^6C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 = \frac{21}{32}$$

**Ex. 4.** *The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the prob. that (i) exactly two will be defecive (ii) none will be defective (iii) at least two will be defective*

Let 'defective pen' $=$ success

$\therefore$ Prob of success in a single trial $p=\frac{1}{10}$.

The experiment is repeated 12 times.

Let the random variable $X$ = number of defective pens. Then $X$ is a binomial variate where the parameters $n=12$ and $p=\frac{1}{10}$.

$\therefore$ Its pmf is $f_i = P(X=i) = \binom{12}{i}\left(\frac{1}{10}\right)^i\left(1-\frac{1}{10}\right)^{12-i}$



$\therefore$ (i) The required probability $= P(X=2) = \binom{12}{2}\left(\frac{1}{10}\right)^2\left(1-\frac{1}{10}\right)^{12-2}$

$$= \frac{66\times 9^{10}}{10^{12}} = 0.2301.$$

(ii) required probality $P(X=0) = \binom{12}{0}\left(\frac{1}{10}\right)^0\left(1-\frac{1}{10}\right)^{12-0} = \left(\frac{9}{10}\right)^{12}$

$= 0.2833$.

(iii) $P(X=1) = \binom{12}{1}\left(\frac{1}{10}\right)^1\left(1-\frac{1}{10}\right)^{12-1} = 12\times\frac{9^{11}}{10^{12}} = .3755$.

$\therefore\ P(X\ge 2) = 1-P(X<2)$

$= 1-[P(X=0)+P(X=1)] = 0.3412$.

**Ex. 5.** *A defective die is thrown ten times independently. The probability that an even number will appear 5 times is twice the probability that an even number will appear 4 times. What is the probability that odd face appear in each of the ten throws.*

Let "even face" = "success" and $X$ = number of even face among 10 trials.

$\therefore X$ has Binomial distribution with parameter $n=10$ and $p$. So the pmf is given by $P(X=i) = f_i = {}^{10}C_i p^i (1-p)^{10-i}$.

By problem, $P(X=5) = 2P(X=4)$ or, $f_5 = 2\times f_4$

or, ${}^{10}C_5 \times p^5(1-p)^{10-5} = 2\times{}^{10}C_4 p^4(1-p)^{10-4}$

or, ${}^{10}C_5 p = 2\times{}^{10}C_4(1-p)$

or, $252p = 2\times 210(1-p)$ or, $3p = 5(1-p)$ or $p=\frac{5}{8}$

Required probability

$$= P(X=0) = f_0 = {}^{10}C_0 p^0 (1-p)^{10-0}$$

$$= \left(1-\frac{5}{8}\right)^{10} = \left(\frac{3}{8}\right)^{10}.$$

**Ex. 6.** *The overall percentage of failures in a certain examination is 40. What is the probability that out of a group of 6 candidates at least 4 passed the examination?*

Now the overall percentage of success in a certain exam is 60.

Consider the experiment that one student is drawn and seen whether he is passed. Probability that he is a passed student is $p = \frac{60}{100} = 0.6$. The experiment be repeated 6 times. Let $X$ = number of passed-student in 6 such trials. So $X$ is a Binomial variate with parameter $n=6$, $p=0.6$.

So the distribution of $X$ is

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $X$ | : | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $P(X=i)=f_i$ | : | $f_0$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$, |

where $f_i = {}^6C_i p^i (1-p)^{6-i} = {}^6C_i (0.6)^i (1-0.6)^{6-i} = {}^6C_i (0.6)^i (0.4)^{6-i}$

Required probability $= P(X \geq 4) = f_4 + f_5 + f_6$

$$= {}^6C_4 (0.6)^4 (0.4)^{6-4} + {}^6C_5 (0.6)^5 (0.4)^{6-5} + {}^6C_6 (0.6)^6 (0.4)^{6-6} = 0.54432.$$

**Ex. 7.** *A family has 6 children. Find the probability that (i) 3 boys and 3 girls (ii) fewer boys than girls.*

Probability of any particular child being a boy is $\frac{1}{2}$.

Let 'boy' = success. The experiment of noticeing whether the child is boy or girl is repeated 6 times. $X$ = No. of boy



i.e. No. of success

$\therefore$ $X \sim b(n, p)$ where $n=6$, $p=\frac{1}{2}$.

So the pmf of $X$ is

$$f_i = P(X=i) = {}^6C_i \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{6-i} = {}^6C_i \left(\frac{1}{2}\right)^6.$$

(i) Now Probability of " 3 boys and 3 girls"

$$= P(X=3) = {}^6C_3 \left(\frac{1}{2}\right)^6 = 5/16$$

(ii) Probability of "fewer boys than girls'

$$= P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= {}^6C_0 \left(\frac{1}{2}\right)^6 + {}^6C_1 \left(\frac{1}{2}\right)^6 + {}^6C_2 \left(\frac{1}{2}\right)^6 = \frac{11}{32}.$$

**Ex. 8.** *A die is tossed thrice. A success is "getting 1 or 6" on a toss. Find the mean and variance of the number of success.*

Let $X$ denote the number of successs. Clearly $X$ can take the values 0, 1, 2 or 3

and

$X$ follows binomial distribution with $n=3$

$p =$ Probability of success $= \frac{2}{6} = \frac{1}{3}$

and $q = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore$ Mean $= E(X) = np = 3 \times \frac{1}{3} = 1$

and

variance $= npq = 3 \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{3}$

**Ex. 9.** *Find the probability distribution of the number of boys in a family with 3 children, assuming equal probabilities for boys and girls. Graph the distributions. Also find the distribution function $F(x)$ for the random variable $X$.*

Let $E$ be the experiment of picking a child in the family.

The event 'boy' = success.

We have $p$ = probability of boy

$= \frac{1}{2}$ by hypothesis. $E$ is repeated 3 times.

Let $X$ denote the number of boys.

Then $X$ can assume the values 0, 1, 2, 3. $X$ has the Binomial distribution.

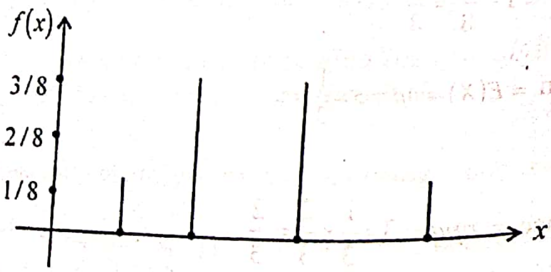
| | | | | |
|---|---|---|---|---|
| $X$ : | 0 | 1 | 2 | 3 |
| $f_i$ : | $f_0$ | $f_1$ | $f_2$ | $f_3$ |

$$\text{where } f_i = \binom{3}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{3-i} = \binom{3}{i}\left(\frac{1}{2}\right)^3 = \binom{3}{i}\frac{1}{8}$$

Therefore, the probability distribution of boys is

| | | | | |
|---|---|---|---|---|
| $X$ : | 0 | 1 | 2 | 3 |
| $f_i$ : | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

The graph of the distribution is given below :






Using the above table we obtain the distribution function $F(x)$ as

$$F(x) = 0, \quad -\infty < x < 0$$
$$= \frac{1}{8}, \quad 0 \le x < 1$$
$$= \frac{1}{8} + \frac{3}{8}, \quad 1 \le x < 2$$
$$= \frac{1}{2} + \frac{3}{8}, \quad 2 \le x < 3$$
$$= \frac{7}{8} + \frac{1}{8}, \quad 3 \le x < \infty$$

i.e., $F(x) = 0, \quad -\infty < x < 0$

$$= \frac{1}{8}, \quad 0 \le x < 1$$
$$= \frac{1}{2}, \quad 1 \le x < 2$$
$$= \frac{7}{8}, \quad 2 \le x < 3$$
$$= 1, \quad 3 \le x < \infty.$$

**Ex. 10.** *Suppose that half the population of a town are consumers of rice. 100 investigators are appointed to find out its truth. Each investigator interviews 10 individuals. How many investigator do you expect to report that three or less of the people interviewed are consumers of rice ?*

Consider the experiment "One investigator is drawn and seen whether he reports that three or less of the people interviewed are consumers of rice" ≡ whether it is success. Let probability of success = $p$

Now the investigator draws one individual and sees whether he is consumer of rice. Let $q$ be the probability of this event = $\frac{1}{2}$. The investigator repeats this experiment 10 times.

Let $Y$ = Number of such individual among ten. So $Y$ has binomial distribution with parameter $n = 10$, $q = \frac{1}{2}$

Therefore $p = P(Y \le 3)$

$$= P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3)$$

$$= {}^{10}C_0\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^{10-0} + {}^{10}C_1\left(\frac{1}{2}\right)^1\left(1-\frac{1}{2}\right)^{10-1}$$

$$+ {}^{10}C_2\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{10-2} + {}^{10}C_3\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{10-3}$$

$$= \left(\frac{1}{2}\right)^{10} + 10\times\frac{1}{2}\left(\frac{1}{2}\right)^9 + 45\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^8 + 120\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)$$

$$= \left(\frac{1}{2}\right)^{10}\{1+10+45+120\} = 176\times\left(\frac{1}{2}\right)^{10}.$$

Now, $X$ has bonomial distribution with parameter $n=100$, $p=176\times\left(\frac{1}{2}\right)^{10}$. Therefore required expectation $= E(X)$

$$= 100\times 176\times\left(\frac{1}{2}\right)^{10} = \frac{17600}{2^{10}} = \frac{17600}{1024} \simeq 17$$

$\therefore$ 17 investigators are expected to report so.

**Ex. 11.** *If X be a binomially distributed with* $E(X) = 2$ *and* $\text{var}(X) = \frac{4}{3}$, *find the distribution of X.*

We have $E(X) = 2, \text{var}(X) = \frac{4}{3}$.

$\therefore\ np = 2$ ...(1)

$np(1-p) = \frac{4}{3}$ ...(2)

Solving (1), (2), we get

$p = \frac{1}{3}, n = 6,\ q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}.$



$$\therefore\ f_0 = P(X=0) = {}^6C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^6 = \frac{64}{729}$$

$$f_1 = P(X=1) = {}^6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^5 = \frac{64}{249}$$

$$f_2 = P(X=2) = {}^6C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 = \frac{80}{243}.$$

Similarly $f_3 = P(X=3) = \frac{160}{729}, f_4 = P(X=4) = \frac{20}{243}$

$$f_5 = P(X=5) = \frac{4}{243}, f_6 = P(X=6) = \frac{1}{729}.$$

Thus the required distribution of $X$ is

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $X$ : | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $f_i$ : | $\frac{64}{729}$ | $\frac{64}{243}$ | $\frac{80}{243}$ | $\frac{160}{729}$ | $\frac{20}{243}$ | $\frac{4}{729}$ | $\frac{1}{729}$. |

**Ex. 12.** *The probability of a man hitting a target is* $\frac{1}{3}$

*(a) If he fires 5 times, what is the probability of his hitting the target at least twice?*

*(b) How many times must he fire so that the probability of his hitting the target at least once is more than 90%.*

Here $p$ = probability of hitting $= \frac{1}{3}$.

$\therefore\ q$ = probability of no hit $= \frac{2}{3}$.

(a) Let $X$ be the number of hits. Here $n = 5$

$$\therefore\ P(X \ge 2) = 1 - P(X < 2)$$

$$= 1 - P(X=0) - P(X=1)$$

$$= 1 - {}^5C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^5 - {}^5C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^4$$

$$= \frac{131}{243}.$$

(b) Let $n$ be the smallest number of fires so that the probability of hitting the target of least once is more than 90%.

$\therefore$ By the condition

$$P(X \geq 1) > \frac{90}{100}$$

or, $1 - P(X = 0) > 0.9$

or, $1 - \left(\frac{2}{3}\right)^n > 0.9$

or, $\left(\frac{2}{3}\right)^n < 0.1$

or, $n \log\frac{2}{3} < \log 0.1$

or, $n > \dfrac{\log 0.1}{\log\frac{2}{3}}$

$n > 5{\cdot}679$

$\therefore\ n = 6$.

Thus he must fire 6 times.

### 1.3.3. Poisson Distribution.

A discrete random variable $X$ is said to have a poisson distribution with parameter $\mu(> 0)$ if its distribution is given by

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $X$ | : | 0 | 1 | 2 | 3 | ... | ... |
| $f_i$ | : | $f_0$ | $f_1$ | $f_2$ | $f_3$ | ... | ... |

where the pmf, $f_i = P(X = i) = \dfrac{e^{-\mu}\mu^i}{i!}$

e.g one probability mass, $f_2 = \dfrac{e^{-\mu}\mu^2}{2!} = \dfrac{e^{-\mu}\mu^2}{2}$ etc.

[W.B.U.T. 2013]

**Note :** (1) Since the parameter $\mu > 0$, $f_i = \dfrac{e^{-\mu}\mu^i}{i!} \geq 0$.



Again $\displaystyle\sum_{i=0}^{\infty} f_i = \sum_{i=0}^{\infty} \frac{e^{-\mu}\mu^i}{i!} = e^{-\mu}\sum_{\mu=0}^{\infty}\frac{\mu^i}{i!} = e^{-\mu}e^{\mu} = 1$

Thus the pmf $f_i$ satisfies the two fundamental properties of pmf.

(2) When the random variable $X$ has a poisson distribution with parameter $\mu(> 0)$, we write $X \sim P(\mu)$ and we say $X$ is a poisson variate.

(3) Actually poisson distribution is a limiting case of Binomial distribution when $n$ is very large and $p$ is very small so that $\mu = np$ is of finite magnitude.

(4) The significance of the parameter $\mu$ is given in next theorem.

**Cases where Poisson Distribution fits.**

Let us consider a sequence of changes. If the random variable $X(t)$ denotes the number of changes during the interval $(0, t)$, then $X(t)$ assumes the values 0, 1, 2, 3, ... ... .

It can be shown that $P(X(t) = i) = f_i = e^{-\lambda t}\dfrac{(\lambda t)^i}{i!}$, $i = 0, 1, 2, \cdots \cdots$

where $\lambda$ = number of changes per unit time.

Thus the distribution of $X(t)$ becomes

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $X(t)$ | : | 0 | 1 | 2 | 3 | ... | ... |
| $f_i$ | : | $f_0$ | $f_1$ | $f_2$ | $f_3$ | ... | ... |

where $f_i = e^{-\lambda t}\dfrac{(\lambda t)^i}{i!}$ which is the pmf of poisson distribution.

Thus 'changes in an interval' is a poisson variate with parameter $\mu = \lambda t$ = average changes in the interval.

**Note.** (1) The interval $(0, t)$ may not be a time-interval. See the following illustration

(2) We use the notation $X(t)$ in place of $X$ because it depends on $t$.

**Illustration.** (i) Let a huge metal sheet be produced by a machine in a factory. Defects are noticed in the sheet. The machine is such that the average number of defects per unit area is 3. A piece of '10 unit area of the sheet is purchased by a company. Let $X$ = Number of defects in this piece. Then $X$ may assume values 0, 1, 2, 3 ... up to $\infty$. (Here 'change' means 'defect' and the interval $(0,t)$ stands for '10 unit area of the sheet' or $(0,10)$. Then $X$ (or, $X(10)$) has the poisson distribution

| | | | | | | |
|---|---|---|---|---|---|---|
| $X$ | : | 0 | 1 | 2 | 3 | ... |
| $f_i$ | : | $f_0$ | $f_1$ | $f_2$ | $f_3$ | ... |

where $f_i = e^{-\mu}\dfrac{\mu^i}{i!}$ with the parameter $\mu = 3\times 10 = 30 =$ average number of defects per 10 unit area of the sheet.

For example, $f_0 = P(X=0) = e^{-30}\dfrac{30^0}{0!} = e^{-30}$

i.e., probability no defects in the piece $= e^{-30}$

In other words $100e^{-30}\%$ such pieces will be free from defects.

(ii) The number of deaths in a state in one year is a poisson variate

(iii) The number of radio active atoms decaying in time $t$ follows the poisson distribution with parameter $\mu = t\lambda$, $\lambda$ = average number of decayed radioactive atoms per unit time.

**Theorem.** If $X$ is a poisson variate with parameter $\mu$ then

(i) Mean of $X$ is $\mu$

(ii) Variance of $X$ is $\mu$ [*W.B.U.T. 2006,2007,2012,2013*]

*Proof.* The values assumed by $X$ are 0, 1, 2, ..., ... ... with probability $P(X=i) = f_i = \dfrac{e^{-\mu}\mu^i}{i!}$.

$$\text{(i) Mean} = E(X) = \sum_{i=0}^{\infty} i\,f_i = \sum_{i=0}^{\infty} i\,\frac{e^{-\mu}\mu^i}{i!}$$

$$= e^{-\mu}\sum_{i=0}^{\infty}\frac{\mu^i}{(i-1)!} = e^{-\mu}\left(\frac{\mu}{0!}+\frac{\mu^2}{1!}+\frac{\mu^3}{2!}+\frac{\mu^4}{3!}+\cdots \text{ up to } \infty\right)$$

$$= \mu e^{-\mu}\left(1+\mu+\frac{\mu^2}{2!}+\frac{\mu^3}{3!}+\cdots \text{ up to } \infty\right) = \mu e^{-\mu}\cdot e^{\mu} = \mu$$



(ii) Now, $E\{X(X-1)\} = \sum_{i=0}^{\infty} i(i-1)e^{-\mu}\dfrac{\mu^i}{i!} = e^{-\mu}\mu^2\sum_{i=2}^{\infty}\dfrac{\mu^{i-2}}{(i-2)!}$

$$= e^{-\mu}\mu^2\left(1+\frac{\mu}{1!}+\frac{\mu^2}{2!}+\frac{\mu^3}{3!}+\cdots \cdots \text{ up to } \infty\right)$$

$$= e^{-\mu}\mu^2 e^{\mu} = \mu^2$$

So, by an earlier result, (See the Theorem of Art 2.6) the variance,

$Var(X) = E\{X(X-1)\} - m(m-1)$, where $m$ = mean

$= \mu^2 - \mu(\mu-1) = \mu$

**Note.** In light of the above theorem we considered the parameter $\mu = \lambda t$ in our previous illustration.

### 1.3.4. Binomial Approximation to Poisson Distribution.

The range of applications of Poisson Distribution becomes more wider as it is used as an approximation for a Binomial distribution. In case of a Binomial distribution when $n$ becomes large, $p$ is small enough so that $np$ is a moderate fixed value, the Binomial variate becomes approximately equal to a Poisson variate. This is given by the following theorem.

**Theorem.** Let the random variable $X$ follows Binomial distribution with *pmf*, $f_i = \binom{n}{i}p^i(1-p)^{n-i}$, $i = 0,1,2,\cdots,n$ where $n$ and $p$ are parameters. If $n\to\infty$ and $p\to 0$ such that $np = \mu$, a fixed quantity then

$\lim\limits_{n\to\infty}\binom{n}{i}p^i(1-p)^{n-i} = \dfrac{e^{-\mu}\mu^i}{i!}$ for a fixed $i$, *i.e.* $f_i$ of Binomial distribution $\approx f_i$ of Poisson distribution.

*Proof.* Beyond the scope of the book.

**Illustration.** Let a box contains 200 fuses. Experience tells that 2% of such fuses are defective. Let us consider the experiment of drawing a fuse and testing whether this is defective or not. Let $X$ = number of defective fuse. $\therefore$ $X$ has Binomial distribution with parameter

$n = 200$, $p = \dfrac{2}{100} = .02$. The pmf of $X$ is

$$f_i = \binom{n}{i} p^i (1-p)^{n-i} = \binom{200}{i} (.02)^i (1-.02)^{200-i}.$$

Here we see $n$ is so large and $p$ is small so we can write

$$\binom{200}{i}(.02)^i(1-.02)^{200-i} \simeq \frac{e^{-200\times.02}}{i!} = \frac{e^{-4}}{i!}$$

Now $P(X \le 3) = f_0 + f_1 + f_2 + f_3 \simeq \frac{e^{-4}}{0!} + \frac{e^{-4}}{1!} + \frac{e^{-4}}{2!} + \frac{e^{-4}}{3!}$.

**Illustrative Examples.**

**Ex. 1.** *For a poisson variate if* $P(X=2) = P(X=1)$, *find* $P(X = 1 \text{ or } 0)$. *Find also mean of X.*

Let $m$ be the parameter of the poisson variate.

$$\therefore P(X=i) = f_i = \frac{e^{-m} m^i}{i!}.$$

Now, $f_2 = f_1$ , or, $\frac{e^{-m}\cdot m^2}{2!} = \frac{e^{-m}\cdot m^1}{1!}$

$\therefore m = 2$

$$\therefore P(X = 1 \text{ or } 0) = f_1 + f_0 = e^{-m}(1+m) = e^{-2}(1+2) = 3e^{-2}$$

Mean of $X = m = 2$

**Ex. 2.** *A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a poisson distribution with average number of demand per day 1.5. Calculate the proportion of days on which neither car is used and the proportion of days on which some demand is refused* $(e^{-1.5} = 0.2231)$.

*[W.B.U.Tech,2003, 2006,2007]*

Let $X$ be the random variable denoting the number of demands for a car on any day. Then $X$ is poisson distributed with parameter $\mu = 1.5$. So its pmf $P(X=i) = f_i = \frac{e^{-\mu}\mu^i}{i!}$ where $\mu = 1.5$.



$\therefore$ Proportion of days on which neither car is used
= Prob. of there being no demand for the car

$$= P(X=0) = \frac{\mu^0 e^{-\mu}}{0!} = e^{-1.5} = 0.2231.$$

Proportion of days on which some demand is refused
= Prob. for the number of demands to be more than two

$$= P(X>2) = 1 - P(X \le 2)$$

$$= 1 - \{P(X=0) + P(X=1) + P(X=2)\}$$

$$= 1 - \left\{e^{-\mu} + \frac{\mu e^{-\mu}}{1!} + \frac{\mu^2 e^{-\mu}}{2!}\right\} = 1 - e^{-1.5}\left(1 + 1.5 + \frac{(1.5)^2}{2}\right) = 0.19126$$

**Ex. 3.** *A radio active source emits on the average 2.5 particles per second. Calculate the prob. that 2 or more particles will be emitted in an interval of 4 seconds.*

Here $\lambda$ = number of changes (which is particle emitted) per unit time on an average $= 2.5$.

Let $X$ be the random variable denoting the number of particles emitted in the given interval. Then $X$ is poisson distributed with parameter $\mu$ = average number of particle in 4 seconds $= 2.5 \times 4 = 10$.

So the p.m.f, $f_i = P(X=i) = \mu^i e^{-\mu}/i! = 10^i e^{-10}/i!$

So the required Prob. $= P(X \ge 2) = 1 - P(X < 2)$

$$= 1 - \{P(X=0) + P(X=1)\}$$

$$= 1 - \{e^{-10} + 10e^{-10}\} = 1 - 11e^{-10}.$$

**Ex. 4.** *In a certain factory turning razor blades, there is a small chance, 1/500 for any blade to be defective. The blades are in packets of 10. Use poisson distribution to calculate the approximate number of packets containing (i) no defective (ii) one defective (iii) two defective blades respectively in one consignment of 10, 000 packets.* (*Given* $e^{-.02} = .9802$).

*[W.B.U.Tech 2004 ]*

On an average there are 1 defective blade per 500 blades.

So the average number of defective blades in a packet of 10 is $10\times\frac{1}{500}=\frac{1}{50}=0.02$.

Let $X$ = number of defective blades in a packet. $X$ follows poisson distribution with parameter $\mu=0.02$. So the pmf is

$$P(X=i)=f_i=\frac{e^{-\mu}\mu^i}{i!}.$$

(i) Now probability that one packet contains no defective blade

$$=P(X=0)=f_0=\frac{e^{-\mu}\mu^0}{0!}=e^{-\mu}=e^{-0.02}=0.9802.$$

$\therefore$ Number of packets in the consignment containing no defective blades $=0.9802\times10{,}000=9802$

(ii) Probability that one packet contains one defective blade

$$=P(X=1)=f_1=\frac{e^{-\mu}\mu}{1!}=e^{-0.02}\times0.02$$

$$=0.9802\times0.02=0.019604$$

$\therefore$ Number of blades in the consignment

$$=0.019604\times10{,}000=196.04\simeq196$$

(iii) $P(X=2)=\frac{e^{-\mu}\mu^2}{2!}=e^{-.02}\times\frac{(.02)^2}{2}$

$$=0.9802\times.0002=0.00019604$$

$\therefore$ Required number $=0.00019604\times10{,}000=1.9604\simeq2$.

**Ex. 5.** *If a random variable has a poisson distribution such that $P(1)=P(2)$, find (i) mean of the distribution (ii) standard derivation (iii) $P(X=4)$* [W.B.U.Tech 2007]

Let $X$ be a poisson variate. Then the p.m.f of $X$ is

$$f_i=P(X=i)=e^{-\mu}\frac{\mu^i}{i!},\quad i=0,1,2,\cdots$$

As $P(1)=P(2)$, so $e^{-\mu}\mu=e^{-\mu}\frac{\mu^2}{2!}$ $\quad\therefore\ \mu=2$

(i) So the mean of the distribution is 2

(ii) Now $Var(X)=\mu=2$ $\quad\therefore$ standard derivation $=\sqrt{2}$.

(iii) $P(X=4)=e^{-2}\frac{2^4}{4!}=\frac{2}{3}e^{2}$.



**Ex. 6.** *Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2 percent of such fuses are defective.*

Let $X$ denote the number of defective fuses in the box. Then clearly $X$ has a binomial distribution with parameters

$$n=200,\ p=\frac{2}{100}=\frac{1}{50}$$

$$\therefore\ \mu=np=200\times\frac{1}{50}=4$$

Using an approximation, by the poisson distribution, we have

$$P(X\le5)=\sum_{i=0}^{5}\frac{e^{-4}4^i}{i!}=e^{-4}\left(1+\frac{4}{1!}+\frac{4^2}{2!}+\cdots+\frac{4^5}{5!}\right)=0.785$$

**Ex. 7.** *Six coins are tossed 6400 times. Using the poisson distribution find the approximate probability of getting six heads 8 times.*

Let $X$ denote the number of six heads in the toss of six coin. Then $X$ is a binomial variate with parameter $n=6400,\ p=\left(\frac{1}{2}\right)^6=\frac{1}{64}$. Here $n$ is so large and $p$ is small, but $\mu=np=6400\times\frac{1}{64}=100$

So using the poisson approximation, we have,

$$P(X=8)={}^nC_8p^8(1-p)^{n-8}\simeq e^{-100}\frac{(100)^8}{8!}.$$

**Ex. 8.** *2% of the items made by a machine are defective. Find the probability that 3 or more items are defective in a sample of 100 items. (Given $e^{-1}=0.368$, $e^{-2}=0.135$, $e^{-3}=0.498$)*

Consider the experiment – one item is drawn and found whether it is defective (success). Let this experiment be repeated 100 times. $X$ = number of defective items. Then $X$ follows Binomial distribution with parameter $n=100$, $p=2\%=\frac{2}{100}=.02$. $\therefore$ the pmf, $f_i={}^nC_ip^i(1-p)^{n-i}$.

Now since $n$ is large and $p$ is small and $np = 100 \times .02 = 2$ so Binomial pmf is approximately equal to Poisson pmf with parameter $np$.

$$\therefore f_i \simeq \frac{e^{-2} \cdot 2^i}{i!}$$

Required probability $= P(X \geq 3) = 1 - P(X < 3)$

$$= 1 - \{f_0 + f_1 + f_2\} = 1 - e^{-2}\left(1 + 2 + \frac{2^2}{2!}\right) = 1 - 0.135 \times 5 = 0.325.$$

### 1.3.5. Uniform or Rectangular Distribution.

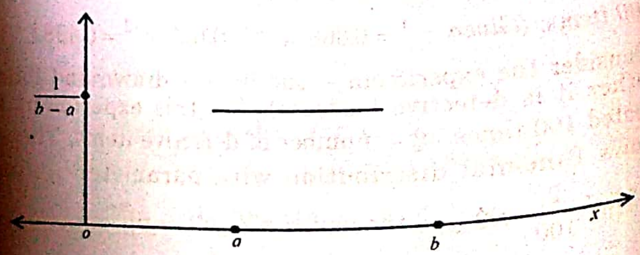
A random variable $X$ is said to have a uniform distribution on the interval $[a, b], -\infty < a < b < \infty$, if its probability density function is given by

$$f(x) = \frac{1}{b-a}, \ a \leq x \leq b$$

$= 0$, otherwise

where $a$, $b$ are two parameters of the distribution. Clearly $f(x) \geq 0$, for $a < x < b$ and $\int\limits_{-\infty}^{\infty} f(x)dx = \int\limits_a^b f(x)dx = \int\limits_a^b \frac{dx}{b-a} = 1$. So the fundamental properties of the density function is satisfied. If $X$ has a uniform distribution on $[a, b]$, then we write $X \sim U[a, b]$.

The density curve is shown in the following fig.




The distribution function $F(x)$ may be easily calculated by

$$F(x) = \int\limits_{-\infty}^{x} f(x)dx$$

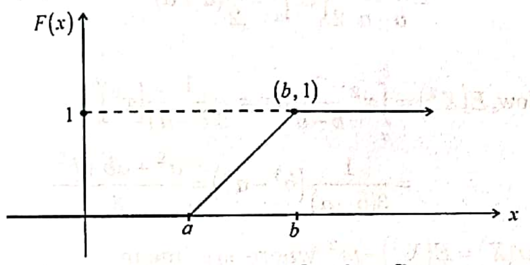
and is given by

$F(x) = 0, \quad -\infty < x < a$

$= \dfrac{x-a}{b-a}, \quad a \leq x < b$

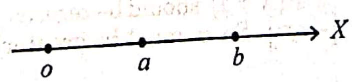
$= 1, \quad b \leq x < \infty$

The graph of the distribution function is given below.




**A Case where Uniform Distribution fits.**

Let $[a. b]$ be an interval. A point $P$ is taken at random in the interval $[a, b]$. Let $OP = x$.




Let $X$ be a random variable which assumes the values $x$. Then $X$ would have uniform distribution with pdf

$$f(x) = \frac{1}{b-a}, \quad a < x < b$$

$= 0$, elsewhere

**Theorem.** If a continuous random variable $X$ has uniform distribution with parameter $a$ and $b$ then

(i) the mean is $\frac{1}{2}(a+b)$ [W.B.U.Tech 2014]

(ii) the variance is $\frac{(a-b)^2}{12}$