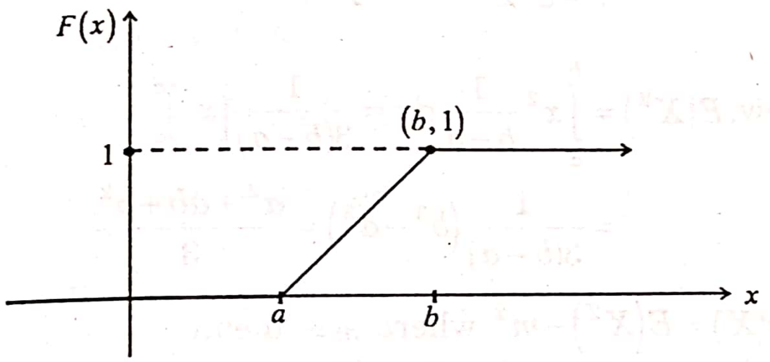
The distribution function $F(x)$ may be easily calculated by

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

and is given by

$$F(x) = 0, \qquad -\infty < x < a$$
$$= \frac{x-a}{b-a}, \quad a \le x < b$$
$$= 1, \qquad b \le x < \infty$$

The graph of the distribution function is given below.




**A Case where Uniform Distribution fits.**

Let [ $a$. $b$ ] be an interval. A point $P$ is taken at random in the interval [$a$, $b$]. Let $OP = x$.

Let $X$ be a random variable which assumes the values $x$. Then $X$ would have uniform distribution with pdf

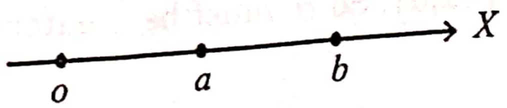
$$f(x) = \frac{1}{b-a}, \; a < x < b$$

$$= 0, \qquad \text{elsewhere}$$

**Theorem.** If a continuous random variable $X$ has uniform distribution with parameter $a$ and $b$ then

(i) the mean is $\frac{1}{2}(a+b)$

(ii) the variance is $\frac{(a-b)^2}{12}$



*Proof.* The p.d.f of $X$ is given by

$$f(x)=\frac{1}{b-a}, \quad a\le x\le b$$
$$=0, \quad \text{elsewhere}$$

(i) the mean $=E(X)=\int_{-\infty}^{\infty} xf(x)dx$

$$=\int_{-\infty}^{a} x\cdot 0dx+\int_{a}^{b} x\cdot\frac{1}{b-a}dx+\int_{b}^{\infty} x\cdot 0dx$$

$$=\frac{1}{b-a}\frac{1}{2}\left[x^2\right]_a^b=\frac{1}{2}(a+b)$$

(ii) Now, $E\left(X^2\right)=\int_a^b x^2\frac{1}{b-a}dx=\frac{1}{3(b-a)}\left[x^3\right]_a^b$

$$=\frac{1}{3(b-a)}\left(b^3-a^3\right)=\frac{a^2+ab+b^2}{3}$$

$\therefore$ $Var(X)=E\left(X^2\right)-m^2$ where $m=$ mean

$$=\frac{a^2+ab+b^2}{3}-\left(\frac{a+b}{2}\right)^2=\frac{(a-b)^2}{12}.$$

**Illustrative Examples.**

**Ex. 1.** *If $X$ is uniformly distributed in $[-\alpha,\alpha]$ with $\alpha>0$ then determine $\alpha$ such that $P(X>1)=\frac{1}{3}$.*

Here p.d.f of $X$ is $f(x)=\frac{1}{2\alpha}$ is $-\alpha<x<\alpha$

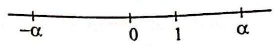
$=0$ elsewhere.

when $\alpha<1$,

$P(X>1)$ should be zero, as $X$ lies outside the given interval $[-\alpha,\alpha]$. So $\alpha$ must be greater than 1.



$\therefore$ $P(X>1)$




$$=\int_1^{\alpha} f(x)dx=\frac{\alpha-1}{2\alpha}$$

$\therefore$ $\frac{\alpha-1}{2\alpha}=\frac{1}{3}.$

$\therefore$ $\alpha=3.$

**Ex. 2.** *If $X$ is uniformly distributed in $-2\le x\le 2$, find $P\left(|X-1|\ge\frac{1}{2}\right)$.*

Here p.d.f of $X$ is

$$f(x)=\frac{1}{4} \text{ is } -2\le x\le 2$$
$$=0, \quad \text{elsewhere.}$$

$\therefore$ $P\left(|X-1|\ge\frac{1}{2}\right)$

$$=P\left(-2\le X\le\frac{1}{2}\right)+P\left(\frac{3}{2}\le X\le 2\right)$$

$$=\int_{-2}^{\frac{1}{2}} f(x)dx+\int_{\frac{3}{2}}^{2} f(x)dx$$

$$=\frac{\frac{1}{2}+2}{4}+\frac{2-\frac{3}{2}}{4}$$

$$=\frac{3}{4}.$$

**Ex. 3.** *If $X$ is uniformly distributed over* [1, 2] *find $U$ so that $P\left(X>U+\overline{X}\right)=\frac{1}{6}$.* [W.B.U.Tech 2006,2008 ]

Since $X$ is uniformly distributed over [1, 2] therefore its pdf,

$$f(x) = \frac{1}{2-1}, \quad 1 < x < 2$$
$$= 0, \quad \text{elsewhere}$$

i.e., $f(x) = 1, \quad 1 < x < 2$
$$= 0, \quad \text{elsewhere}$$

$$\therefore \ \bar{X} = E(X) = \int_{-\infty}^{\infty} x f(x)dx = \int_1^2 x \cdot 1\, dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{3}{2}$$

Now, $P(X > U + \bar{X}) = \frac{1}{6}$ or, $P\left(U + \frac{3}{2} < X < \infty\right) = \frac{1}{6}$

or, $\int_{U+\frac{3}{2}}^{2} dx = \frac{1}{6}$ or, $2 - U - \frac{3}{2} = \frac{1}{6}$ $\therefore U = \frac{1}{3}$

### 1.3.6. Exponential Distribution.

**Two parameters exponential distribution.** A random variable $X$ is said to have a two-parameter-exponential distribution if its probability density function is given by

$$f(x) = \frac{1}{b} e^{-\frac{x-a}{b}}, \quad x \geq a$$
$$= 0, \quad \text{elsewhere}$$

where $a, b (b > a)$ are two parameters of the distribution.

**Note.** (1) Clearly $f(x) \geq 0$ for all $x$ and

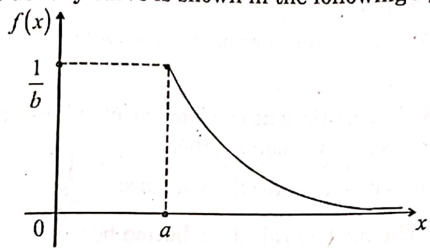
$$\int_{-\infty}^{\infty} f(x)dx = \int_a^{\infty} \frac{1}{b} e^{-\frac{x-a}{b}} dx = 1.$$

So the two fundamental properties of pdf are satisfied

(2) If $X$ has an exponential distribution with parameters $a$ and $b$ we write $X \sim E[a, b]$.



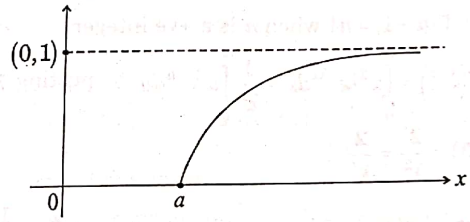
(3) The density curve is shown in the following figure




(4) The distribution function $F(x)$ is given by

$$F(x) = 0, \quad -\infty < x < a$$
$$= 1 - e^{-\frac{x-a}{b}}, \quad a \leq x < \infty$$

(5) The graph of the distribution function $F(x)$ is given in the following figure




**One Parameter Exponential Distribution**

We say that $X$ has one parameter exponential distribution if its pdf is $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$
$$= 0, \quad \text{elsewhere}$$

where $\lambda$ is the only parameter.

**Note. (1)** By the term $X$ has exponential distribution' we mean $X$ has one-parameter exponential distribution.

(2) Clearly the pdf $f(x)$ satisfies the two fundamental properties of pdf.

(3) If $X$ has exponential distribution with parameter $\lambda$ we write $X \sim E(0,\lambda)$.

(4) This distribution is obtained from the previous by putting $a=0,\ b=\frac{1}{\lambda}$.

**Theorem.** If a continuous random variable $X$ has exponential distribution with parameter $\lambda$ then

(i) the mean is $\frac{1}{\lambda}$ (ii) the variance is $\frac{1}{\lambda^2}$

*Proof.* The p.d.f. of this distribution be

$$f(x)=\lambda e^{-\lambda x}, \quad x \ge 0$$
$$=0, \quad \text{elsewhere}$$

$\therefore$ mean, $m=E(x)=\int_0^\infty x\lambda e^{-\lambda x}dx=\frac{1}{\lambda}\int_0^\infty ue^{-u}du$, by putting $\lambda x=u$

$=\frac{1}{\lambda}\Gamma(2)$ $\left[\because \Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx\right.$ from definition of Gamma function$]$

$=\frac{1}{\lambda}$ $[\because \Gamma(n+1)=n!$ when $n$ is a +ve integer$]$

Now, $E(X^2)=\int_0^\infty x^2\lambda e^{-\lambda x}dx=\frac{1}{\lambda^2}\int_0^\infty u^2e^{-u}du$ by putting $\lambda x=u$

$$=\frac{1}{\lambda^2}\Gamma(3)=\frac{2!}{\lambda^2}=\frac{2}{\lambda^2}$$

$\therefore Var(X)=E(X^2)-m^2$ [where $m$ is mean] $=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$.

## Illustrative Example.

**Ex. 1.** *Suppose that during rainy season, on a tropical island, the length of shower has an exponential distribution with average length of shower $\frac{1}{2}$ mins. What is the probability that a shower will last more than three minutes ? If a shower has already lasted for 2 minutes, what is the probability that it will last for at least one more minute ?*

Let $X$ = length of shower in minute.



By problem $X$ has exponential distribution with parameter $\lambda$ where $\frac{1}{\lambda}=\frac{1}{2}$, i.e., $\lambda=2$

$\therefore$ its p.d.f, $f(x)=2e^{-2x},\ x>0$

Now, Probability that a shower lasts more than three minutes

$$=P(X>3)=\int_3^\infty 2e^{-2x}dx=2\lim_{X\to\infty}\int_3^X e^{-2x}dx$$

$$=2\lim_{X\to\infty}\left[\frac{e^{-2x}}{-2}\right]_3^X=-\lim_{X\to\infty}\left\{\frac{1}{e^{2X}}-\frac{1}{e^6}\right\}=\frac{1}{e^6}$$

Probability that a shower lasts more than two minutes

$$=P(X>2)=\int_2^\infty 2e^{-2x}dx=\frac{1}{e^4}.$$

Now the required probability $=P(X\ge 3/X\ge 2)$

$$=\frac{P((X\ge 3)\cap(X\ge 2))}{P(X\ge 2)}=\frac{P(X\ge 3)}{P(X\ge 2)}=\frac{\frac{1}{e^6}}{\frac{1}{e^4}}=\frac{1}{e^2}.$$

### 1.3.7. Normal Distribution.

A continuous random variable $X$ is said to have a normal distribution if its probability density function is given by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty<x<\infty$$

where $\mu$ and $\sigma>0$ are the two parameter of the distribution.

**Note. (1)** In this case we say $X$ is a normal variate with parameters $\mu$ and $\sigma$ and we denote it by $X\sim N(\mu,\sigma)$

**(2)** Clearly $f(x)\ge 0$ for all $x$.

Moreover, $\int_{-\infty}^{\infty}f(x)dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}dz \text{, by putting } z=\frac{x-\mu}{\sigma\sqrt{2}}$$

$$=\frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-z^2}dz=\frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-u}u^{-\frac{1}{2}}du \text{, by putting } z^2=u$$

$$=\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right)=\frac{1}{\sqrt{\pi}}\cdot\sqrt{\pi}=1.$$

So the two necessary conditions for the probability density function are satisfied.

(3) The significance of the parameters $\mu$ and $\sigma$ are given in the next theorem.

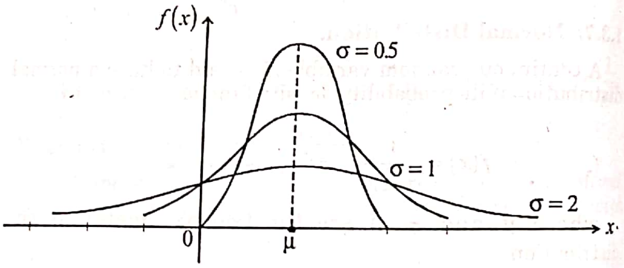
(4) The distribution function $F(x)$ is given by

$$F(x)=\int_{-\infty}^{x} f(u)du=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{(u-\mu)^2}{2\sigma^2}}du$$

**Normal Curve**

The graph of the pdf of a normal variate is called normal curve.

The curve is shown in the following figure for different values of $\sigma$.




The normal curve is bell shaped and symmetric about the ordinate $x=\mu$. For small values of $\sigma$, the curve has a small peak and as $\sigma$ increases, the normal curve tends to be flatter.

**Theorem 1.** If $X$ has normal distribution with parameter $\mu$ and $\sigma$ then (i) the mean of $X$ is $\mu$ (ii) the s.d of $X$ is $\sigma$.



*Proof.* Here p.d.f of $X$ is given by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty<x<\infty$$

(i) Mean $=E(X)=\dfrac{1}{\sigma\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} xe^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx+\frac{\mu}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} ue^{-u^2}du+\frac{\mu}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}du \text{, by putting } u=\frac{x-\mu}{\sigma\sqrt{2}}$$

$=0+\dfrac{\mu}{\sqrt{\pi}}2\cdot\displaystyle\int_{0}^{\infty} e^{-u^2}du$ [ $\because$ the integrand of the 1st integral is odd and that of the 2nd integral is even]

$$=\frac{\mu}{\sqrt{\pi}}\int_{0}^{\infty} z^{\frac{1}{2}-1}e^{-z}dz \text{, by putting } u^2=z$$

$$=\frac{\mu}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right)=\frac{\mu}{\sqrt{\pi}}\sqrt{\pi}=\mu \qquad \left[\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$$

(ii) Now, $Var(X)=E\{(X-\mu)^2\}$

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} z^2e^{-z^2}dz \text{ by putting } \frac{x-\mu}{\sigma\sqrt{2}}=z$$

$$=\frac{4\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty} z^2e^{-z^2}dz=\frac{2\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty} u^{\frac{1}{2}}e^{-u}du \text{ by putting } z^2=u$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)=\frac{2\sigma^2}{\sqrt{\pi}}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad [\because \Gamma(n+1)=n\Gamma(n)]$$

$$=\frac{\sigma^2}{\sqrt{\pi}}\cdot\sqrt{\pi}=\sigma^2$$

$\therefore$ the s.d of $X$ is $\sigma$.

### A Case where Normal Distribution Fits.

Let a thermal plant supplies electric power in a certain city. Let $X =$ amount of electric power (in watt) supplied by the plant in a day. Clearly $X$ varies from day to day. It can be assumed that $X$ is a continuous variate. It can be shown that $X$ has normal distribution with parameter $\mu =$ average power supply per day and $\sigma =$ standard deviation of all the values assumed by $X$.

### Standard Normal Distribution.

The normal distribution with mean 0 and standard deviation 1 is called standard normal distribution. The random variable having standard normal distribution is called standard normal variate or $X \sim N(0,1)$.

Thus the p.d.f of the standard normal distribution is given by

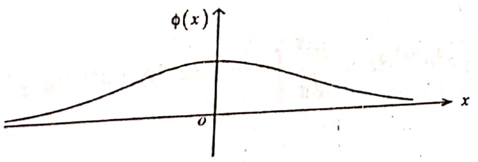
$$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.$$

and the corresponding distribution function is given by

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}du$$

### Standard Normal Curve.

The graph of the p.d.f of a standard normal distribution is called standard normal curve. This is shown in the following figure.

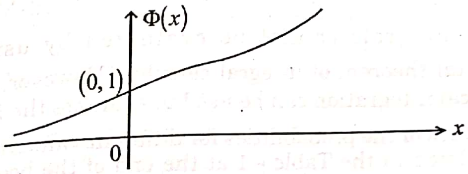





It is symmetric about $y$ axis, bell shaped. It has maximum value at $x=0$. Since $\int_{-\infty}^{\infty}\phi(x)dx=1$ so area under this curve is 1. $X$ axis is its asymptote.

### Standard Normal Distribution Curve.

The graph of the distribution function $\Phi(x)$ is shown in the following figure.




**Theorem 2. (An Important Result)**

If the continuous random variable $X$ has normal distribution with parameter $\mu$ and $\sigma$ then $Z=\frac{X-\mu}{\sigma}$ has standard normal distribution.

*Proof.* That $Z$ has normal distribution is shown in the next chapter. Yet it is observed that the mean of $Z$,

$$\bar{Z}=E(Z)=E\left(\frac{X-\mu}{\sigma}\right)=\frac{1}{\sigma}\{E(X)-E(\mu)\}$$

$$=\frac{1}{\sigma}\{\mu-\mu\}=0$$

$Var(Z)=Var\left(\frac{X-\mu}{\sigma}\right)=\frac{1}{\sigma^2}Var(X-\mu)$, from the property of variance

$=\frac{1}{\sigma^2}\{Var(X)-Var(\mu)\}$, from the property of variance

$$=\frac{1}{\sigma^2}\{\sigma^2-0\}=1.$$

**Tabulation of the Standard Normal Distribution.**

Let the random variable $X$ has standard normal distribution. Then its distribution function

$$\Phi(x) = P(-\infty < X \le x) = \int_{-\infty}^{x} \phi(z)dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz \therefore P(X \le a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{z^2}{2}} dz \text{ and } P(a \le X \le b) = \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{-\frac{z^2}{2}} dz$$

These integrals cannot be evaluated by using the fundamental theorem of integral calculus. However, method of numerical integration can be used to evaluate the integral.

The values of the probabilities for different values of $a$ have been tabulated in the Table - 1 at the end of the book. From this we can find $\Phi(x)$ for different values of $x$.

Since $P(a \le x \le b) = \Phi(b) - \Phi(a)$, we have if $X$ has distribution $N(0,1)$ then from the tabulated value of $\Phi$ we can evaluate the probabilities associated with $X$.

**An Illustrative Examples**

Let a thermal plant supplies electric power in a certain city. Let $X =$ amount of electric power (in watt) supplied by the plant in a day. Clearly $X$ varies from day to day.

It can be assumed that this $X$ has normal distribution with parameter $\mu =$ average power supply per day and $\sigma =$ standard deviation of all the values assumed by $X$. Now if it is known that these $m = 300$ and $\sigma = 10$ then by the previous theorem $Z = \dfrac{X-300}{10}$ is a standard normal variate.

Suppose we are asked to find the number of days on which power supplied will lie between 280 to 310 MW.

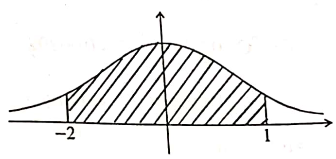
Then we find $P(280 < X < 310)$

$$= P\left(\frac{280-300}{10} < \frac{X-300}{10} < \frac{310-300}{10}\right) = P(-2 < Z < 1)$$



$= \int_{-2}^{1} \phi(z)dz$ [where $\phi$ is the pdf of standard normal distribution]

$=$ area enclosed by standard normal curve, $X$ axis, the ordinates $z = -2$ and $z = 1$ (shown by shade in the figure)




$= 0.4772 + 0.3413$ (obtained from the tabulated value)

$= 0.8185$. So the probability of having power supply between 280 and 310 MW in a day is 0.8185. Thus 81.85% day will receive a power supply between 280 and 310 MW.

**1.3.8. Binomial Approximation to Normal Distribution.**

**Theorem.** Let the random variable $X$ follows Binomial distribution with pmf

$$f_i = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, 2, \cdots, n$$

where $n$ and $p$ are parameters. If $n \to \infty$ and $p$ is not very small then the distribution of the r.v $Z = \dfrac{X - np}{\sqrt{np(1-p)}}$ approches to the standard normal distribution.

*Proof.* Beyond the scope.

**Note.** **(1)** In light of above theorem we understand $X$ is approximately a normal variate with mean $np$ and s.d. $\sqrt{npq}$

**(2)** The variable $Z = \dfrac{X - np}{\sqrt{np(1-p)}}$ is called standardized Binomial variate.

**Illustration.** Let an unbiased coin is tossed 12 times. Let $X =$ number of heads appeared. Then $X$ has binomial

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distribution with parameter $n=12$ and $p=\frac{1}{2}=0.5$. So th... probability masses are

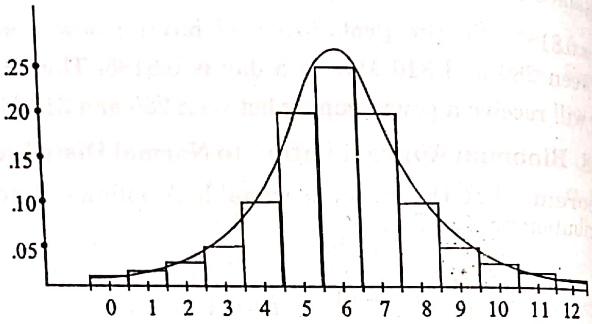
$$f_0={}^{12}C_0(0.5)^0(0.5)^{12-0}=0.000244$$

$$f_1={}^{12}C_1(0.5)^1(0.5)^{12-1}=0.00292$$

$$f_2={}^{12}C_2(0.5)^2(0.5)^{12-2}=0.01611$$

and so on.

In the following diagram the heights of the rectangles ar... these.



We think $n$ as large. So $X$ is approximately normal variate with parameter $\mu=np=12\times 5=6$ and

$$\sigma=\sqrt{np(1-p)}=\sqrt{12\times 5\times 5}=1.73$$

i.e., $Z=\dfrac{X-6}{1.73}$ is approximately a standard normal variate

In the above figure the curve is $(6,1.73)$ normal curve. This approximately fits the height of the rectangle.

(i) Now, suppose we seek $P(4\le X\le 7)$. But if we find this from the normal distribution then we must find $P(3.5\le X\le 7.5)$ as indicated in the above diagram. Now whe...

$X=3.5,\ Z=\dfrac{3.5-6}{1.73}=-1.46$, when $X=7.5, Z=\dfrac{7.5-6}{1.73}=0.87$



Then $P(4\le X\le 7)\approx P(-1.46\le X\le 0.87)$

$$=\int_{-1.46}^{0.87}\phi(t)dt$$

$$=\int_{-1.46}^{0}\phi(t)dt+\int_{0}^{0.87}\phi(t)dt$$

$$=0.4279+0.3078$$

(obtained from the statistical table-I)

$$=0.7357.$$

(ii) Now, suppose we seek $P(X=8)$.

We calculate this with the help of approximate normal distribution then we find $P(7.5\le X\le 8.5)=P(0.87\le Z\le 1.44)$ which would be obtained from statistical table.

**Illustrative Examples.**

**Ex. 1.** *If $X$ is normally distributed with zero mean and unit variance find the expectation of $X^2$.* [W.B.U.Tech, 2002, 2007]

By problem $E(X)=0$, $Var(X)=1$ or, $E(X^2)-\{E(X)\}^2=1$

or, $E(X^2)-0^2=1$ $\therefore E(X^2)=1$

**Ex. 2.** *If $X$ is normally distributed with mean 3 and s.d. 2, find c such that $P(X>c)=2\ P(X\le c)$.*

*Given* $\int_{-\infty}^{.43}\phi(t)dt=\frac{1}{3}$. [W.B.U.Tech 2007]

Let $Z$ be the standard normal variate. Then $Z=\dfrac{X-3}{2}$

$$\therefore P(X>c)=P\left(\frac{X-3}{2}>\frac{c-3}{2}\right)=P\left(Z>\frac{c-3}{2}\right)$$

$$=1-P\left(Z\le\frac{c-3}{2}\right)=1-\int_{-\infty}^{\frac{c-3}{2}}\phi(t)dt$$

Also $P(X \le c) = P\left(\frac{X-3}{2} \le \frac{c-3}{2}\right) = P\left(Z \le \frac{c-3}{2}\right) = \int_{-\infty}^{\frac{c-3}{2}} \phi(t)dt$

As $P(X > c) = 2P(X \le c)$, we have

$$1-\int_{-\infty}^{\frac{c-3}{2}} \phi(t)dt = 2\times \int_{-\infty}^{\frac{c-3}{2}} \phi(t)dt \quad \text{or,} \quad \int_{-\infty}^{\frac{c-3}{2}} \phi(t)dt = \frac{1}{3}$$

or, $\frac{c-3}{2} = .43$ (from the given data)

$\therefore\ c = 3.86$.

**Ex. 3.** *The length of bolts produced by a machine is normally distributed with mean 4 and and s.d. 0.5. A bolt is defective if its length does not lie in the interval* (3.8, 4.3). *Find the percentage of defective bolts produced by the machine.*

$$\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.6} e^{-\frac{t^2}{2}}dt = 0.7257,\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.4} e^{-\frac{t^2}{2}}dt = 0.6554\right]$$ [*W.B.U.Tech 2004*]

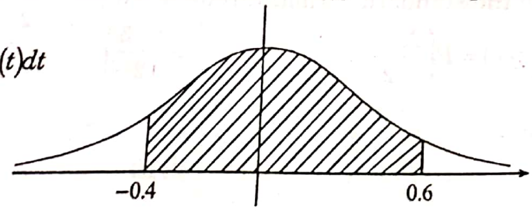
Let $X =$ length of bolt. By problem $X$ has normal distribution with mean $\mu = 4$ and s.d $\sigma = 0.5$. First we shall find the probability $P(3.8 < X < 4.3)$.

Now, $Z = \frac{X-4}{0.5}$ has standard normal distribution. Now when $X = 3.8$, $Z = \frac{3.8-4}{0.5} = -0.4$; when $X = 4.3$, $Z = \frac{4.3-4}{0.5} = 0.6$

$\therefore\ P(3.8 < X < 4.3) = P(-0.4 < Z < 0.6)$

= Area under standard normal curve enclosed between the two ordinate $Z = -.04$ and $Z = 0.6$ (shaded part in figure)

$= \int_{-0.4}^{0.6} \phi(t)dt$




$= \int_{-\infty}^{0.6} \phi(t)dt - \int_{-\infty}^{-0.4} \phi(t)dt = 0.7257 - \int_{0.4}^{\infty} \phi(t)dt$ [$\because$ $\phi$ curve is symmetric]

$= 0.7257 - \left(1 - \int_{-\infty}^{0.4} \phi(t)dt\right) = 0.7257 - (1 - 0.6554) = 0.3811$

$\therefore$ Probability that the length of the bolt lies between 3.8 and 4.3 is $0.3811$.

$\therefore$ Probability that the length of the bolt does not lie between 3.8 and $4.3 = 1 - 0.3811 = 0.6189$.

$\therefore$ Probability that the bolt is defective $= 0.6189$

$\therefore$ Percentage of defective bolts produced $= 0.6189 \times 100 = 61.89 \simeq 62$.

**Ex. 4.** *If the weekly wage of 10,000 workers in a factory follows normal distribution with mean and s.d Rs. 70 and Rs. 5 respectively, find the expected number of workers whose weekly wages are (i) between Rs. 66 and Rs. 72 (ii) less than Rs. 66 and (iii) more than Rs. 72* [*W.B.U.Tech 2006*]

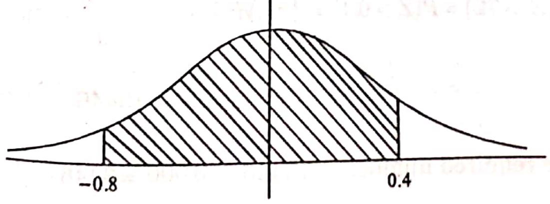
[*Given that* $\frac{1}{\sqrt{2\pi}}\int_0^z e^{-t^2/2}dt = 0.1554$ *and* 0.2881 *according as* $z = 0.4$ *and* $z = 0.8$]

Let $X =$ wage of a worker. $X$ has normal distribution with $\mu = 70,\ \sigma = 5$.

$\therefore\ Z = \frac{X-70}{5}$ has standard normal distribution.

(i) When $X = 66$, $Z = \frac{66-70}{5} = -0.8$; when $X = 72$, $Z = \frac{72-70}{5} = 0.4$

$\therefore\ P(66 < X < 72) = P(-0.8 < Z < 0.4)$



= area under st. normal curve enclosed between the two ordinates $Z = -0.8$ and $Z = 0.4$ (shown by shade in the figure)

$$= \int_{-0.8}^{0.4} \phi(t)dt = \int_{-0.8}^{0} \phi(t)dt + \int_{0}^{0.4} \phi(t)dt$$

$$= \int_{0}^{0.8} \phi(t)dt + \int_{0}^{0.4} \phi(t)dt \text{, since } \phi \text{ curve is symmetric about } Y \text{ axis.}$$

$$= 0.2881 + 0.1554 = 0.4435$$

$\therefore$ Probability that the wage of a worker lies between Rs 66 and Rs. 72 is $0.4435$.

$\therefore$ the number of worker whose wage lie between Rs. 66 and Rs. 72 $= 0.4435 \times 10{,}000 = 4435$.

(ii) When $X = 66$, $Z = -0.8$

So, $P(X < 66) = P(Z < -0.8)$

= area under the st. normal curve enclosed on left side of the ordinate $Z = -0.8$

$$= \int_{-\infty}^{-0.8} \phi(t)dt = \int_{0.8}^{\infty} \phi(t)dt \text{ (by symmetry)}$$

$$= 0.5 - \int_{0}^{0.8} \phi(t)dt = 0.5 - 0.2881 = 0.2119$$

$\therefore$ the expected number of workers $= 0.2119 \times 10{,}000 = 2119$

(iii) When $X = 72$, $Z = 0.4$

$$\therefore P(X > 72) = P(Z > 0.4) = \int_{0.4}^{\infty} \phi(t)dt$$

$$= 0.5 - \int_{0}^{0.4} \phi(t)dt = 0.5 - 0.1554 = 0.3446$$

$\therefore$ the required number $= 0.3446 \times 10{,}000 = 3446$.

**Ex. 5.** *The mean of a normal distribution is 50 and 5% of the values are greater than 60. Find the standard deviation of the distribution (Given that the area under standard normal curve between $z = 0$ and $z = 1.64$ is $0.45$)*

Let $X$ be the normal variate. Let its s.d. be $\sigma$ $\therefore$ $Z = \dfrac{X - 50}{\sigma}$ is standard normal variate. By problem

$$P(X > 60) = \frac{5}{100} = 0.05.$$

When $X = 60$, $Z = \dfrac{60 - 50}{\sigma} = \dfrac{10}{\sigma}$

$\therefore$ from above, $P\left(Z > \dfrac{10}{\sigma}\right) = 0.05$ ... (1)

From the supplied data we have $P(0 < Z < 1.64) = .45$

or, $P(Z > 1.64) = 0.5 - 0.45 = 0.05$ ... (2)

Comparing (1) and (2) we get $\dfrac{10}{\sigma} = 1.64$

or, $\sigma = \dfrac{10}{1.64} = 6.097 \approx 6.1$ $\therefore$ the s.d is 6.1.

**Ex. 6.** *In a normal distribution, 31% of the items are under 45 and 8% are above 64. Find the mean and standard deviation. [Given*

$$P(0 < Z < 1.405) = 0.42,\ P(-0.496 < Z < 0) = 0.19$$

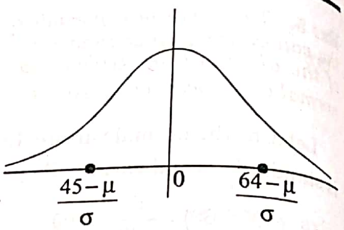
[ W.B.U.Tech 2003 ]

Let $X$ be the normal variate with mean $\mu$ and s.d $\sigma$. So, $Z = \dfrac{X - \mu}{\sigma}$ is standard normal variate. Now, by problem, $P(X < 45) = \dfrac{31}{100}$ and $P(X > 64) = \dfrac{8}{100}$. When $X = 45, Z = \dfrac{45 - \mu}{\sigma}$; when $X = 64$, $Z = \dfrac{64 - \mu}{\sigma}$. So from above $P\left(Z < \dfrac{45 - \mu}{\sigma}\right) = 0.31$ and $P\left(Z > \dfrac{64 - \mu}{\sigma}\right) = 0.08$.

Since $0.31 < 0.5$ so

$\frac{45-\mu}{\sigma}$ is on negative side.

From above




$$P\left(\frac{45-\mu}{\sigma} < z < 0\right) = 0.5 - 0.31$$

$= 0.19$

Comparing this with the second given data we have

$$\frac{45-\mu}{\sigma} = -1.496$$

or, $45-\mu = -1.496\,\sigma$ or, $\mu - 0.496\,\sigma = 45$ ... (1)

Since $0.08 < 0.5$ so $\frac{64-\mu}{\sigma}$ is lies on +ve side.

$$\therefore\ P\left(0 < z < \frac{64-\mu}{\sigma}\right) = 0.5 - 0.08 = 0.42$$

Comparing this with the first given data we get

$$\frac{64-\mu}{\sigma} = 1.405$$

or, $64-\mu = 1.405\,\sigma$

or, $\mu + 1.405\sigma = 64$ ... (2)

Solving (1) and (2) $\sigma = 9.995$ and $\mu = 49.958$.

$\therefore$ Mean $= 49.958$ and s.d. $= 9.995$.

**Ex. 7.** *If X is normally distributed with mean 12 and s.d. 4, find the probability of (i)* $X \geq 20$ *(ii)* $0 \leq X \leq 12$ *and (iii) also find a such that* $P(X > a) = 0.24$. [Use table]

Let $Z$ be the standard normal variate. Then

$$Z = \frac{X-\mu}{\sigma} = \frac{X-12}{4} \qquad \therefore\ X = 4Z + 12$$

$\therefore$ (i) $P(X \geq 20) = P(4Z+12 \geq 20) = P(Z \geq 2) = 1 - P(Z < 2) = 1 - \Phi(2)$

$= 1 - 0.9772 = 0.0228$.



(ii) $P(0 \leq X \leq 12) = P(0 \leq 4Z + 12 \leq 12) = P(-3 \leq Z \leq 0)$

$= P(0 \leq Z \leq 3)$ (due to symmetry)

$= \Phi(3) - \Phi(0) = 0.9986 - 0.5 = 0.4986$ (from table)

(iii) $P(X > a) = 0.24$ or, $P(4Z + 12 > a) = 0.24$

or, $P\left(Z > \frac{a-12}{4}\right) = 0.24$ or, $1 - P\left(Z \leq \frac{a-12}{4}\right) = 0.24$

or, $1 - \Phi\left(\frac{a-12}{4}\right) = 0.24$, where $\Phi$ is c.d.f of st. normal variate

or, $\Phi\left(\frac{a-12}{4}\right) = 0.76$

or, $\frac{a-12}{4} = 0.71$, from table-I

$\therefore\ a = 14.84$

**Ex. 8.** *If a random variable X follows a normal distribution such that* $P(9.6 \leq X \leq 13.8) = 0.7008$, *and* $P(X \geq 9.6) = 0.8159$ *where* $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.9} e^{-t^2/2}dt = 0.8159$, $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1.2} e^{-t^2/2}dt = 0.8849$, *find mean and variance of X.*

Let $E(X) = \mu$, $Var(X) = \sigma^2$. Now, if $\Phi$ is c.d.f of st normal variate

$$\Phi(x) = P\left(\frac{X-\mu}{\sigma} \leq x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du$$

$\therefore\ \Phi(0.9) = .8159$, $\Phi(1.2) = 0.8849$

$\therefore\ \Phi(-0.9) = 1 - \Phi(0.9) = 0.1841$

Now, $P\left(-0.9 \le \frac{X-\mu}{\sigma} \le 1.2\right) = \Phi(1.2) - \Phi(-0.9)$

$$= 0.8849 - 0.1841 = 0.7008$$

$$= P(9.6 \le X \le 13.8)$$

$$= P\left(\frac{9.6-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{13.8-\mu}{\sigma}\right)$$

$\therefore \quad \frac{13.8-\mu}{\sigma} = 1.2, \quad \frac{9.6-\mu}{\sigma} = -0.9.$

Solving we get $\mu = 11.4, \ \sigma = 2$

$\therefore$ mean $= 11.4$, $\text{var}(X) = 4$.

**Ex. 9.** *A fair cain is tassed 400 times. Using normal approximation to binomial distribution find the probability of obtaining (i) exactly 200 heads (ii) between 190 and 210 heads, both inclusive. Given that the area under standard normal curve between Z=0 and Z=0.05 is 0.0199 and between Z=0 and Z=1.05 is 0.3531.* [ W.B.U.Tech 2007]

Let the random variable $X$ denotes the number of heads in 400 tosses. Then clearly $X$ has a binomial distribution with parameter $n = 400, p = \frac{1}{2}$. Since $n$ is large we suppose $X$ is an approximately normal variate with parameter $\mu = np = 400 \times \frac{1}{2} = 200$

and $\sigma = \sqrt{np(1-p)} = \sqrt{400 \times \frac{1}{2} \times \frac{1}{2}} = 10$

i.e. $Z = \frac{X-200}{10}$ is approximate standard normal variate.

(i) Now, $X = 199{\cdot}5 \Rightarrow X = \frac{199{\cdot}5-200}{10} = -0{\cdot}05$ and $X = 200{\cdot}5 \Rightarrow Z = 0{\cdot}05$.

Then using the normal approximation, we have the required probability, $P(X=200) \approx P(199{\cdot}5 \le X \le 200{\cdot}5) = P(-0{\cdot}05 \le Z \le 0{\cdot}05)$

$$= 2P(0 \le Z \le 0{\cdot}05) = 2 \times 0{\cdot}0199 = 0{\cdot}0398$$



(ii) Now, $X = 189.5 \Rightarrow Z = -1.05$ and $X = 210{\cdot}5 \Rightarrow Z = 1.05$.

Then using the normal approximation, we have the required probability,

$P(190 \le X \le 210)$

$\approx P(189.5 \le X \le 210.5)$ (in terms of normal approximation)

$= P(-1.05 \le Z \le 1.05) = 2P(0 \le Z \le 1.05) = 2 \times 0.3531 = 0.7062$

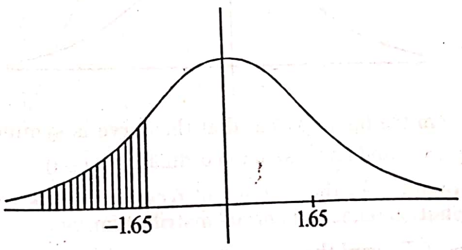
**Ex. 10.** *Among 10,000 random digits, find the probability that the digit 3 appears at most 950 times.*

Let $X$ denotes the number of times the digit 3 appears. Then $X$ is a $b(n, p)$ variate where $n = 10{,}000, \ p = \frac{1}{10}$

$\therefore \ np = 1000, \ np(1-p) = 10000 \times \frac{1}{10} \times \frac{9}{10} = 900$

$\because n$ is large we find $P(X \le 950)$ by approximating with normal distribution. $X$ is approximately $\left(np, \sqrt{np(1-p)}\right) = (1000, 30)$ normal variate i.e., $Z = \frac{X-1000}{30}$ is approximately standard normal variate. Since discrete variate is approximated by continuous variate so we find $P(X \le 950.5)$.

Now $X = 950.5 \Rightarrow Z = -1.65$



$\therefore$ the required probability $\simeq P(X \le 950.5) = P(Z \le -1.65)$

$= 0.5 - P(-1.65 \le Z < 0) = 0.5 - P(0 \le Z \le 1.65)$

$= 0.5 - \int_0^{1.65} \phi(t)dt = 0.5 - 0.4505$ (From statistical table)

$= 0.0495$.

### 1.3.9. $t$ - Distribution or Student's Distribution.

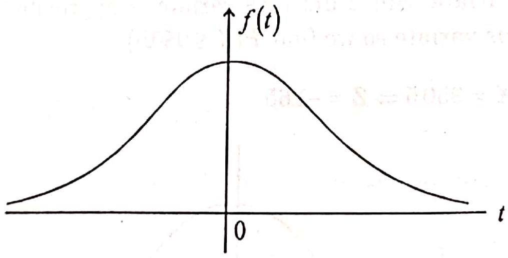
The t-distribution is a continuous distribution whose probability density function is given by

$$f(t) = \frac{1}{\sqrt{n}B\left(\frac{1}{2},\frac{n}{2}\right)}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}, \quad -\infty < t < \infty$$

where $n$ is a positive integer called the number of degres of freedom of the distribution and is the only parameter of the distribution. $B\left(\frac{1}{2},\frac{n}{2}\right)$ is the welknown Beta-Function.

**t-Probability Density Curve or *t-Curve*.**

The prob. density curve is shown in the following fig. for a fixed $n$, say for $n = 4$.




From the fig. it is clear that the curve is symmetric about the $y$-axes and has maximum ordinate at $t = 0$.

**Theorem.** As the degrees of freedom tends to $\infty$ the $t$-distribution tends to a normal distribution.

*Proof.* Beyond the scope of the text.



**Note.** Because of having the above result $t$ distribution has an important role in theory of estimation, test of hypothesis. These are discussed in the later chapters.

### 1.3.10. $\chi^2$-Distribution (Chi-Square Distribution).

$\chi^2$ distribution is a continuous distribution whose probability density function is given by

$$f(\chi^2) = \frac{e^{-\frac{\chi^2}{2}}\left(\frac{1}{2}\chi^2\right)^{\frac{n}{2}-1}}{2\Gamma\left(\frac{1}{2}n\right)}, \quad \chi^2 > 0$$

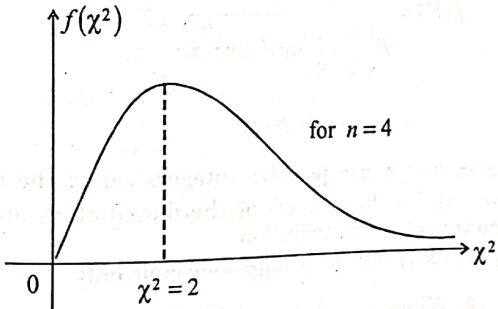
$$= 0, \quad \text{elsewhere}$$

where $n$ is a positive integer called the number of degrees of freedom (d.o.f) of the distribution and is the only parameter of the distribution. $\Gamma\left(\frac{1}{2}n\right)$ is a Gamma-Function

**Note.** (1) $\chi^2$ stands for a single notation-it only tells us that it is always non-negative. But never think it is $\chi \cdot \chi$. (2) Here $X$ is a chi-square variate with parameter $n$. We denote it by $X \sim \chi^2(n)$

**The $\chi^2$ Prob. Density Curve or, $\chi^2$-Curve.**

It is shown in the following figure for fixed $n$ (say $n = 4$)




From the figure it is clear that $f(\chi^2)$ is maximum at $\chi^2 = 2$. It is positively skew. Starting from 0 it extends to $\infty$ on the right.

**Theorem 1.** If $X$ and $Y$ are two independent random variables having $\chi^2$ distribution with degree of freedom $n_1$ and $n_2$ then their sum $X+Y$ has $\chi^2$ distribution with d.o.f $n_1+n_2$.

*Proof.* Beyond the scope.

**Theorem 2.** If a $\chi^2$ variate's d.o.f $n \to \infty$ then the variate $\sqrt{2\chi^2}-\sqrt{2n-1}$ tends to a normal variate.

*Proof.* Beyond the scope of the book.

**Theorem 3.** If $Z_1, Z_2, \cdots, Z_n$ are $n$ independent standard normal variates then $Z_1^2+Z_2^2+\cdots+Z_n^2$ has $\chi^2$ distribution with d.o.f $n$.

*Proof.* Beyond the scope.

**Note.** Because of the above important results $\chi^2$ distribution has a significant role in theory of estimation and test of hypothesis which are discussed in the later chpaters.

### 1.3.11. F-Distribution (Snedecesor's Distribution).

$F$ distribution is a continuous distribution whose probability density function is given by

$$f(F)=\frac{m^{\frac{m}{2}}n^{\frac{n}{2}}F^{\frac{m}{2}-1}}{B\left(\frac{m}{2},\frac{n}{2}\right)(mF+n)^{\frac{m+n}{2}}}, F>0$$

$$=0, \quad \text{elsewhere}$$

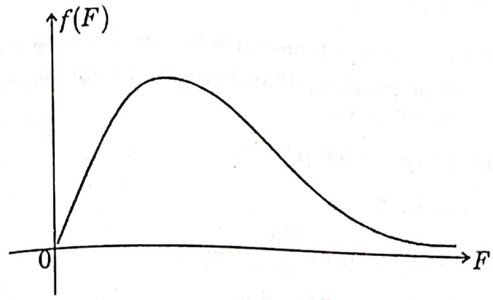
where $m$ and $n$ are positive integers called the number of degrees of freedom (d.o.f) of the distribution and are the parameters of the distribution.

**Note : (1)** $F$ stands for a single variable only.

**(2)** When a random variable $F$ has a $F$ distribution with parameter $m$ and $n$ we write $F \sim F(m,n)$ and we say is a $F$ variate with $(m,n)$ degrees of freedom.

**The F Prob. Density curve or F-curve**

The curve of the pdf of F variate is highly positively skew.




**Theorem 1.** If $\chi_1^2$ and $\chi_2^2$ are two independent $\chi^2$ variates with $m$ an $n$ degrees of freedom then $F=\frac{n\chi_1^2}{m\chi_2^2}$ is $F$ variate with $m$ and $n$ degrees of freedom.

*Proof.* Beyond the scope of the book.

**Theorem 2.** If $F$ is a $F$ variate with $(m,n)$ d.o.f then $\frac{1}{F}$ is a $F$ variate with $(n,m)$ d.o.f.

*Proof.* Beyond the scope of the book.

## EXERCISES

**[I] SHORT ANSWER QUESTIONS**

1. If a conference room cannot be reserved for more than 4 hours, find the probability that a given conference lasts more than three hours.

   [Hints : Here $f(x)=\frac{1}{4}$ $P(X\geq 3)=\int_3^4 f(x)dx=\frac{1}{4}$]

2. If $X$ is uniformly distributed in $[-\alpha,\alpha]$ with $\alpha>0$. Then find $\alpha$ such that $P\left(X<\frac{1}{2}\right)=0.7$.