6 PO SETS AND LATTICE

6.1 Partially Ordered Set (PO Set)

[W.B.U. Tech 2006]

Let S be a non empty set and $\preccurlyeq$ be a relation (as discussed in Chapter 1) in S. $\preccurlyeq$ is called a 'partially order' if the following three axioms are satisfied:

(i) For any a in S we have $a \preccurlyeq a$ (Reflexive)

(ii) If $a \preccurlyeq b$ and $b \preccurlyeq a$ then $a = b$ (Antisymmetric)

(iii) If $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$ (Transitive)

The set S with the partially order $\preccurlyeq$ is called a Partially Ordered Set or PO Set.

We write $(S, \preccurlyeq)$ is a PO set in short form.

Note. (1) *In the above definition we say $\preccurlyeq$ defines a partially ordering of S.*

(2) *As discussed in chapter 1 every two elements of S may not be related by $\preccurlyeq$. Because of this, the word 'partial' is used.*

'Precedes & Succeeds'

In a PO set $(S, \preccurlyeq)$ if for two elements a and b, $a \preccurlyeq b$ we say 'a precedes b' and 'b succeeds a.'

If $a \preccurlyeq b$ and $a \neq b$ we say 'a strictly precedes b' and 'b strictly succeeds a'. These are written as $a \prec b$.

Illustration.

(i) Let S be a set and $P(S)$ be its power set, i.e. $P(S)$ is collection of all subsets of S. Then the relation $\subseteq$ (subset) is a partially order in $P(S)$ because :

(a) $A \subseteq A$ for all A in $P(S)$

(b) $A \subseteq B$ and $B \subseteq A$ imply $A = B$

(c) $A \subseteq B$, $B \subseteq C$ imply $A \subseteq C$.

(ii) Let Z be set of all integers. Define ρ in such a way that $a \rho b$ hold if b can be expressed as $b = a^r$ for some positive integer r. (For example $2 \rho 8$). Now

(a) $\because a = a^1$ so $a \rho a$.

(b) $a \rho b$ and $b \rho a$ imply $b = a^{r_1}$ and $a = b^{r_2}$ (where r_1, r_2 are positive integers). This imply $b^{1/r_1} = b^{r_2}$. Since r_1, r_2 are positive integers so $r_1 = 1$, $r_2 = 1$ $\therefore a \rho b$ and $b \rho a \Rightarrow b = a$.

(c) $a\rho b, b\rho c \Rightarrow b=a^r, c=b^s$ (r, s are positive integers)
$\Rightarrow c=a^{rs} \Rightarrow a\rho c$ ($\because$ rs is a positive integer)

Thus ρ is a partially order and (Z,ρ) is a PO set.

Here 2 precedes 8 and 8 succeeds 2.

(iii) Let Z be set of all integers and '/' be a relation defined in such a way that a/b means b is divisible by a. (e.g. $5/60$). Then we see

(a) a/a for all a in Z.

(b) $3/-3$ and $-3/3$ but $3 \neq -3$, i.e. the relation '/' is not antisymmetric.

So $(Z,/)$ is not a PO set. [W.B.U.Tech 2006]

Comparable and Non-comparable elements in a PO set.

Two elements a and b in a PO set $(S, \preccurlyeq)$ are said to be comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

Two elements are non comparable if they are not comparable. We write this as $a \| b$.

Illustration : Let $S=\{1,2,3,4\}$ be the set . Then the two elements $A=\{2,4\}$ and $B=\{1,4\}$ are non-comparable in the PO set $(P(S),\subseteq)$.

Totally Ordered or Linearly Ordered Sets.

A PO set $(S, \preccurlyeq)$ is called totally ordered set if every pair of elements in S is comparable, i.e. for any two elements a,b in S either $a \preccurlyeq b$ or $b \preccurlyeq a$.

Illustration. (i) The PO set $(Z,\leq)$ is a totally ordered set because for any two integers a and b either $a \leq b$ or $b \leq a$.

(ii) The PO set $(P(S),\subseteq)$ is not totally ordered.

6.2 Hasse Diagram of PO set

Immediate predecessor & Immediate successor

Let $(S, \preccurlyeq)$ be a PO set and suppose $a, b \in S$. We say a is an immediate predecessor of b or b is an immediate successor of a if $a \prec b$ and there is no element c in S such that $a \prec c \prec b$, i.e. if a precedes b and S contains no element which lies between a and b.

This is written as $a \prec\!\prec b$. Sometimes we say b covers a.

Illustration.

(i) Let $S=\{1,2,3,4,5\}$ be a set. Consider the PO set $(S,\subseteq)$. We see the two elements $A=\{2,4\}$ and $B=\{2,4,5\}$ are such that A is a immediate predecessor of B, i.e. $A \subset\subset B$. On the other hand if $C = \{2, 3, 4, 5\}$ then A is a predecessor of C but not immediate because $A \subset B \subset C$.

(ii) In the totally ordered set $(Z, \leq)$ every element has immediate predecessor and immediate successor.

Hasse diagram of finite PO set

Let $(S, \preccurlyeq)$ be a finite PO set. We place the points on a plane which represent the elements of S. If an element y is an immediate successor of the element x then we place y higher than x and draw a line joining them. Thus a diagram is created whose vertices represents the elements of S and edge represents the immediate predecessor/ successor relationship. This diagram is known as Hasse diagram of the PO set.

Illustration :

Let $S = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be a set and '/' be a relation defined in S such that a/b mean b is divisible by a. e.g. 2/12 but 6/9 is not true etc. Clearly $(S, /)$ is a PO set. We show the Hasse diagram in the Fig. 6.1

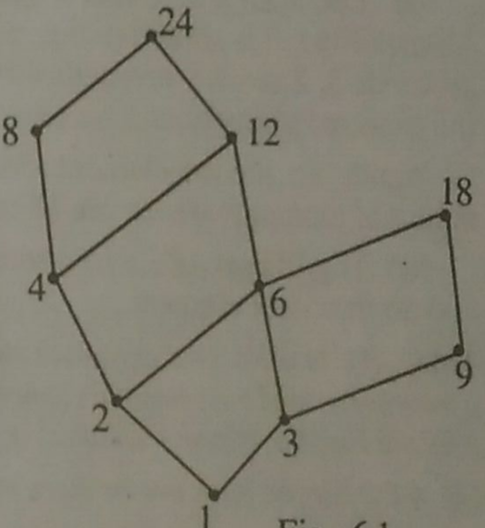


Fig. 6.1

Note that '6' is placed higher than '3' because 6 is an immediate successor of 3 and so on. The vertices '2' and '12' are not joined since 2/12 but 2//12 is not true.

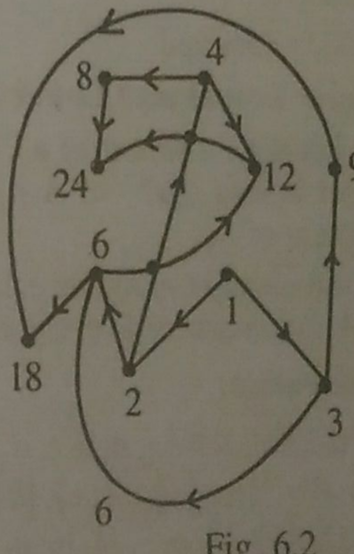


Fig.6.2

Note. (1) *Instead of placing the immediate successor at a higher position we could draw arrow from the immediate predecessor to the immediate successor.*

Thus we show an alternate Hasse diagram in Fig 6.2.

This is nothing but a Di-graph which would be discussed in Chapter 9.

(2) *There can be no circuits (directed) in this Di-graph.*

(3) The Di-graph of a PO set may not be connected.

(4) If $a \prec b$ then there must be a path (directed path) from the vertex a to b e.g. in Fig 6.2 we see there is a path $1 \to 2 \to 4 \to 8 \to 24$ because $1/24$.

(5) Any PO set gives a Hasse diagram and every Hasse diagram gives a PO set.

6.3. Bounds of a PO set

Minimal and Maximal elements in a PO set

Let S be a PO set. An element a in S is called a minimal element if no other element of S strictly preceeds a.

An element b in S is called a maximal element if no other element of S strictly succeeds b.

Illustration.

(i) Let $S = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be a PO set w.r.t the relation / (divisibility). It is shown in Fig. 6.1. From the figure it is seen that 1 strictly preceeds 3, 2 strictly preceeds 4 but no element strictly precceds 1. So 1 is the minimal element of S.

Again we see no element strictly succeeds 18 and 24. So S has two maximal elements which are 18 and 24.

(ii) The PO set $(Z, \leq)$ (where Z = set of all integers) has no minimal and no maximal element.

Note. (1) *In a Hasse diagram an element a is minimal element if no edge enters a from below and b is maximal element if no edge leaves b in the upward direction. [See the Fig. 6.1 of previous Illustration (i)]*

(2) A PO set can have two or more minimal element and maximal.

(3) A PO set may have no minimal/ maximal element.

Theorem *If a PO set is finite then it must have minimal and maximal elements.*

Proof: Omitted.

Upper Bound and Lower Bound.

Let A be a subset of a PO set $(S, \preccurlyeq)$. An element M in S is called an upper bound (u.b) of A if $x \preccurlyeq M$ for all x in A, i.e. M succeeds every element of A.

An element m in S is called a lower bound (l.b) of A if $m \preccurlyeq x$ for all x in A, i.e. m preceeds every element of A.

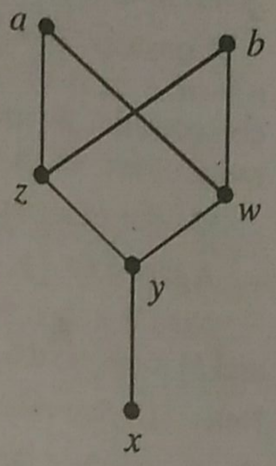
Illustration.

(i) Let $S = \{x, a, b, c, d, e\}$ be a set. Consider the PO set $(P(S), \subseteq)$. Let $A = \{\{b, c\}, \{a, c, e\}, \{b\}\}$ be a subset of $P(S)$. Here the element of $P(S)$, $M = \{a, b, c, e\}$ is an u.b of A because every set in the collection

A is $\subseteq M$. Note that the element $\{a,b,c,d,e\}$ is also an $u \cdot b$ of A. But the element $\{a,b,c,d\}$ is not an $u.b$ of A because of $\{a,c,e\} \not\subseteq \{a,b,c,d\}$.

The element of $P(S)$, $m=\{b\}$ is not a $l.b$ of A because $m \not\subseteq \{a,c,e\}$. Note that Φ (null set) is a l.b. of A.

(ii) Let $S=\{x,y,z,w,a,b\}$. The PO set S is given in terms of the shown Hasse-diagram



Consider $A=\{z,y,w\} \subset S$. Here x is a l.b of A and b is a $u \cdot b$ of A. Note that a is also a $u \cdot b$ of A.

Note that x is l.b of S. S has no $u \cdot b$. a and b are not $u \cdot b$ of S because $a \not\preccurlyeq b$ etc..

Note. *From the above illustration we see a set may have two or more u.b and l.b.*

A set may have no $u \cdot b$ or $l.b.$

Bounded Set.

If a set A has an $u \cdot b$ we say A is *bounded above* and if A has a l.b bound we say A is *bounded below.*

A set A is called bounded if A is bounded above and bounded below.

The set A considered in the above illustrations is bounded set.

Supremum (Least $u.b$) and Infimum (Greatest $l.b$)

Let A be a subset of a PO set $(S, \preccurlyeq)$.

An element M in S is said to be Supremum or Exact Upper Bound ($e.u.b$) or Least Upper Bound ($l.u.b$) of A if

(i) $x \preccurlyeq M$ for all x in A

(ii) $M \preccurlyeq y$ whenever y is an u.b of A. Supremum of A is denoted by $\text{Sup}(A)$.

An element m in S is said to be Infimum or Exact Lower Bound ($e.l.b$) or Greatest Lower Bound ($g.l.b$) of A if

(i) $m \preccurlyeq x$ for all x in A

(ii) $y \preccurlyeq m$ whenever y is an l.b of A

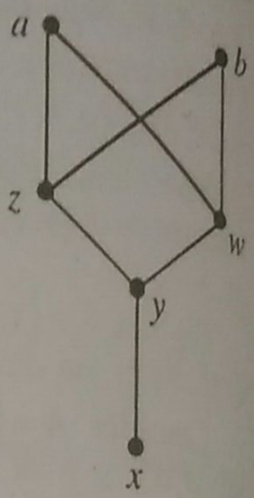
Infimum of A is denoted by $\text{Inf}(A)$.

Illustration.

(i) Let $S=\{x,a,b,c,d,e\}$ be a set. Consider the PO set $(P(S), \subseteq)$.

Let $A=\{\{b,c\},\{a,c,e\},\{b\}\}$ be a subset of $P(S)$.

Here Supremum of $A=\{a,b,c,e\}$ and Infimum of $A=\Phi$. Again Sup of $P(S)=S$, Inf of $P(S)=\Phi$.



(ii) Let $S = \{x, y, z, w, a, b\}$. The PO set S is given in terms of the shown Hass-diagram.

Consider $A = \{z, y, w\} \subset S$. Here we see b is an u.b of A, a is also an u.b of A but a and b are non comparable. So neither a nor b does not satisfy the requirement of supremum. So A has no supremum.

On the other hand Inf $(A) = y$.

Again Sup $\{y, a\} = a$ and Inf $\{y, a\} = y$.

Note that though S has maximul element (which are a and b) it has no u.b.

Note. **(1)** *Supremum / Infimum may not exist*

(2) *Supremum / Infimum, if exists, is unique.*

Illustrative Examples.

Ex. 1. *Let D_{40} be the set of all positive divisors of 40. Find whether D_{40} is a PO set w.r.t the relation ρ where $a \rho b$ means a divides b. Draw the Hasse diagram of the PO set (D_{40}, ρ). Find the Maximal, Minimal element of D_{40}. Find the Supremum and Infimum of the subset $A = \{4, 8, 10\}$. Find whether the set A is bounded. Is 5 a l.b of A ?*

$D_{40} = \{1, 2, 4, 5, 8, 10, 20, 40\}$.

Now a divides a, so $a \rho a$, so ρ is reflexive. Since D_{40} contains only positive integers so $a \rho b$ and $b \rho a$ imply $a = b$.

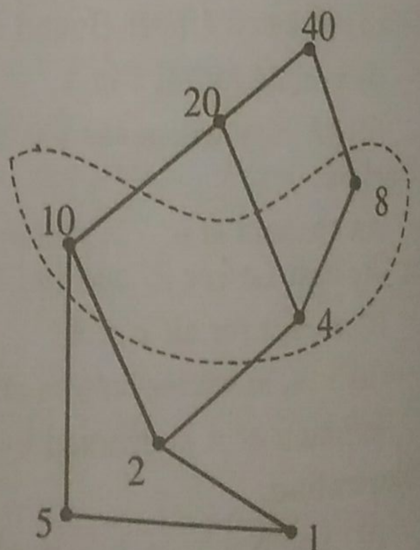
If a divides b and b divides c then a divides c. So $a \rho b$ and $b \rho c$ imply $a \rho c$ i.e. ρ is transitive. So (D_{40}, ρ) is a PO set.

The Hasse diagram of D_{40} is drawn in the given figure.

The minimal element is 1; the maximal element is 40.

$2 =$ H.C.F of 4, 10 and 8. So we see $2 \rho 4, 2 \rho 10, 2 \rho 8$. So 2 is a l.b. of A. 1 is another l.b of A and we also see $1 \rho 2$. So $2 =$ Infimum of A.

Again $40 =$ LCM of $4, 10, 8$. So we see $10 \rho 40$, $4 \rho 40$, $8 \rho 40$. So 40 is an u.b of A. It has no other u.b. So we conclude Sup $(A) = 40$.



Hasse diagram of D_{40}

Since A has a l.b and an u.b so A is bounded.

5 is not a *a l.b* of A since $5 \not\rho 8$.

Ex. 2. *Show that the set of all rational numbers with usual order '$\le$' is a PO set. Is it totally ordered ? Prove that no element in this set has an immediate successor or an immediate predecessor.*

Let Q = set of all rational numbers.

Obviously '$\le$' is reflexive, antisymmtric and transitive. So first part is obvious.

For any two rational numbers a and b either $a \le b$ or $b \le a$. So, $(Q, \le)$ is totally ordered.

Let a be an arbitrary element of Q. If possible, suppose b is an immediate successor of a. So $a \le b$ and $a \ne b$ i.e., $a < b$. Now $a < \frac{a+b}{2} < b$ and $\frac{a+b}{2} \in Q$. This contradicts the definition of 'immediate successor'. Thus there exists no immediate successor of a. Similarly a has no immediate predecessor.

Ex. 3. *Let (N, ρ) be a PO set where N is set of all natural numbers and 'ρ' stands for divisibility. (i) State which one of the following two subsets of N are linearly (totally) ordered.*

(i) $A = \{2, 8, 32, 4\}$, $B = \{3, 15, 20\}$

(ii) *Find the maximal and minimal element of A and B.* [W.B.U.Tech 05]

(i) *The set $A = \{2, 8, 32, 4\}$ is totally ordered because for any two elements a, b we have $a \rho b$ or $b \rho a$.*

$B = \{3, 15, 20\}$ is not linearly ordered because $3 \not\rho\, 20$ or $20 \not\rho\, 3$.

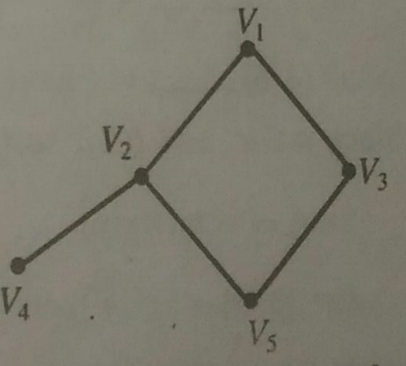
(ii) Maximal element of $A = 32$

Minimal element of $A = 2$

Maximal element of $B = 20$

Minimal element of $B = 3$.

Ex. 4. *Let $A = \{V_1, V_2, V_3, V_4, V_5\}$ This set is ordered by the following Hasse diagram*

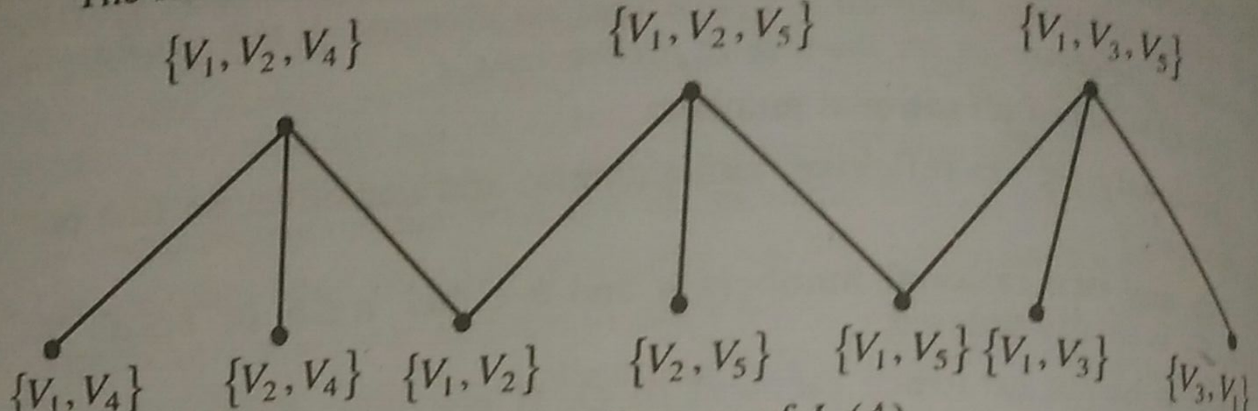


If $L(A)$ is the collection of totally ordered subset of A with 2 or more elements. Find $L(A)$. Draw the Hasse diagram of the PO set $(L(A), \subseteq)$. Find the minimal and maximal elements of $L(A)$.

$L(A) = \{\{V_1, V_2, V_4\}, \{V_1, V_2, V_5\}, \{V_1, V_3, V_5\}, \{V_1, V_2\}, \{V_1, V_4\}, \{V_1, V_3\}, \{V_1, V_5\}, \{V_2, V_4\}\ \{V_2, V_5\}, \{V_3, V_5\}\}$

(Note that $\{V_3, V_4\}$ is not totally ordered)

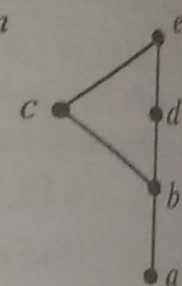
The Hasse diagram of $(L(A), \subseteq)$ is shown below :



Hasse diagram of $L(A)$

The set $\{V_1, V_4\}\{V_2, V_4\}\{V_1, V_2\}, \ldots$ are all minimal element ; the sets $\{V_1, V_2, V_4\}$, $\{V_1, V_2, V_5\}$, $\{V_1, V_3, V_5\}$ are maximal elements of $L(A)$.

Ex. 5. *Let* $A = \{a, b, c, d, e\}$ *Determine the relation represented by the given Hasse diagram .*



If R be the relation then R is a subset of $A \times A$ and $R = \{(a,b),(b,c),(b,d),(c,e),(d,e)\}$ i.e., aRb, bRc, bRd hold etc.

Ex. 6. *Prove that* $(N \times N, \preccurlyeq)$, *is a PO set where* $(a, b) \preccurlyeq (a', b')$ *if* $a \le a'$ *and* $b \le b'$ *; N is the set of all natural numbers.*

Insert the correct symbol $\prec$, $\succ$ *or* $\|$ *between each of the following two elements of* $N \times N$ *:*

(a) $(5, 5)$ ______ $(4, 8)$ (b) $(7, 9)$ ______ $(8, 2)$ (c) $(1, 3)$ ______ $(7, 2)$

(d) $(4, 6)$ ______ $(4, 2)$ (e) $(5, 7)$ ______ $(7, 1)$ (e) $(7, 0)$ ______ $(4, 1)$

Since $a \le a$ and $b \le b$ so $(a, b) \preccurlyeq (a, b)$

Thus $\preccurlyeq$ is reflexive.

Let $(a, b) \preccurlyeq (a', b')$ and $(a', b') \preccurlyeq (a, b)$.

Then $a \le a'$ and $a' \le a$; $b \le b'$ and $b' \le b$. This give $a = a', b = b'$ i.e. $(a, b) = (a', b')$. Thus $\preccurlyeq$ is anti symmetric.

Let $(a, b) \le (c, d)$ and $(c, d) \le (e, f)$.

Then $a \leq c \leq e$ and $b \leq d \leq f$. This give $a \leq e$, $b \leq f$ i.e. $(a,b) \leq (e,f)$. Thus $\preccurlyeq$ is transitive. Hence $(N \times N, \preccurlyeq)$ is a PO set.

The correct symbols are

(a) $\succ$ (b) $\|$ (c) $\|$ (d) $\succ$ (e) $\|$ (f) $\|$.

Ex. 7. *Prove that a finite subset of a PO set has at most one supremum.*

Let A be a subset of a PO set $(S, \preccurlyeq)$ and there are finite number of elements in A. Let, if possible, A has two supremums a and b. Since b is supremum so it is an u.b of A i.e. $b \preccurlyeq a$. Similarly $a \preccurlyeq b$. Since the relation $\preccurlyeq$ is antisymmetric so $a = b$. Thus there cannot be two distinct supremums of A.

6.4. Lattice.

Let L be a non-empty set ; $\wedge$ and $\vee$ are two binary operations defined on L. L is called a Lattice with respect to these two operations if the following axioms hold :

(1) For all a, b in L (i) $a \wedge b \in L$ (ii) $a \vee b \in L$

[called closure property]

(2) For all a, b in L (i) $a \wedge b = b \wedge a$ (ii) $a \vee b = b \vee a$

[called commutative property]

(3) For all a, b, c in L

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (ii) $(a \vee b) \vee c = a \vee (b \vee c)$

[called associative property]

(4) For all a, b in L (i) $a \wedge (a \vee b) = a$ (ii) $a \vee (a \wedge b) = a$

[called absorption law]

Illustrative Examples.

(i) *Let S be a set and $P(S)$ be its power set, i.e. set of all subsets of S. Prove that $P(S)$ is a lattice w.r.t the operations $\cap$ (intersection) and $\cup$ (union).*

Let us verify the axioms of Lattice.

(1) If $A, B \subset S$ then $A \cap B \subset S$ and $A \cup B \subset S$. So, $A, B \in P(S)$ $\Rightarrow A \cap B \in P(S)$ and $A \cup B \in P(S)$. Thus closure property is satisfied.

(2) It is well known that $A \cap B = B \cap A$ and $A \cup B = B \cup A$. Thus commutative property is satisfied.

(3) It is well known that $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$. Thus associative property hold in $P(S)$.

(4) (i) We shall show $A \cap (A \cup B) = A$. $x \in A \cap (A \cup B) \Leftrightarrow x \in A$ and $x \in A \cup B \Leftrightarrow x \in A$ and $x \in$ at least one of A and $B \Leftrightarrow x \in A$. So, $A \cap (A \cup B) = A$. (ii) Similarly it can be proved that $A \cup (A \cap B) = A$.

Thus absorption law hold.

So, $(P(S), \cap, \cup)$ is a lattice.

(ii) *Let N be the set of all positive integers.* $\wedge$ *and* $\vee$ *are defined as* $a \wedge b =$ *HCF of a and b ;* $a \vee b =$ *LCM of a and b. Prove that* $(N, \wedge, \vee)$ *is a lattice.*

Since HCF and LCM of two positive integers is positive integer so $a \wedge b \in N$ and $a \vee b \in N$. Obviously $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$.

Since $(a \wedge b) \wedge c =$ HCF of a, b and c $= a \wedge (b \wedge c)$ and $(a \vee b) \vee c$ $=$ LCM of a, b and c $= a \vee (b \vee c)$. So the associative property hold for both the two operations.

Since HCF of a and 'LCM of a, b' $= a$, so $a \wedge (a \vee b) = a$. Since, LCM of a and 'HCF of a, b, ' $= a$, so $a \vee (a \wedge b) = a$. So absorption law hold. So, $(N, \wedge, \vee)$ is a lattice.

Duality in a Lattice

In a lattice $(L, \wedge, \vee)$ the dual of any statement is defined to be the statement that is obtained by interchanging $\wedge$ and $\vee$. For example the dual of the statement "$(a \wedge b) \vee a = a \wedge (b \vee a)$" is the statement "$(a \vee b) \wedge a = a \vee (b \wedge a)$."

Principle of Duality in a Lattice.

If a Theorem hold in a Lattice $(L, \wedge, \vee)$ then another Theorem is obtained which is nothing but the dual of the former.

This follows from the fact that the dual of each axiom of a lattice also comes as an axiom.

Theorem 1 (Idempotent Law) In a lattice $(L, \wedge, \vee)$ for all a in L

(i) $a \wedge a = a$ (ii) $a \vee a = a$.

Proof: (i) $a \wedge a = a \wedge \{a \vee (a \wedge b)\}$ by Absorption Law

$= a \wedge (a \vee x)$ supposing $x = a \wedge b$

$= a$ by absorption law.

Though (ii) follows from (i) by principle of duality we give the independent proof.

(ii) $a \vee a = a \vee \{a \wedge (a \vee b)\}$ by Absorption Law

$= a \vee \{a \wedge x\}$ supposing $x = a \vee b$

$= a$ by absorption law.

Theorem 2. In a lattice $(L, \wedge, \vee)$ $a \wedge b = a$ if and only if $a \vee b = b$

Proof. $a \wedge b = a \Rightarrow (a \wedge b) \vee b = a \vee b$

$\Rightarrow b \vee (a \wedge b) = a \vee b$ by commutative property

$\Rightarrow b \vee (b \wedge a) = a \vee b$ by commutative property

$\Rightarrow b = a \vee b$ by Absorption law

$\Rightarrow a \vee b = b$

Conversely, $a \vee b = b \Rightarrow (a \vee b) \wedge a = b \wedge a$

$\Rightarrow a \wedge (a \vee b) = a \wedge b$ by commutative property

$\Rightarrow a = a \wedge b$ by absorption law

$\Rightarrow a \wedge b = a$.

Theorem 3 A PO set is a Lattice if the Supremum and Infimum of $\{a, b\}$ exist for every pair of elements a,b in the set.

Proof. Let $(L, \preccurlyeq)$ be a PO set. We define the two operations $\wedge$ and $\vee$ as $a \wedge b = \text{Inf}\{a, b\}$ and $a \vee b = \text{Sup}\{a, b\}$;

We shall show that L satisfies all the axioms of Lattice.

(1) Since, by hypothesis, $\text{Inf}\{a, b\}$ and $\text{Sub}\{a, b\}$ exist for every pair of elements a,b in L so for all a, b in L

(i) $a \wedge b \in L$ and (ii) $a \vee b \in L$. So closure property is satisfied.

(2) Since Inf $\{a, b\} = \text{Inf}\{b, a\}$ and $\text{Sup}\{a, b\} = \text{Sup}\{b, a\}$

so for all a, b in L (i) $a \wedge b = b \wedge a$ and (ii) $a \vee b = b \vee a$. So commutative property is satisfied.

(3) (i) We shall show for any three elements a, b, c in S $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. Let $(a \wedge b) \wedge c = \lambda$ and $a \wedge (b \wedge c) = \mu$. Since $\lambda = \text{Inf}\{a \wedge b, c\}$ so $\lambda \preccurlyeq a \wedge b$ and $\lambda \preccurlyeq c$. Again $a \wedge b \preccurlyeq a$ and b. So by transitivity $\lambda \preccurlyeq a$, $\lambda \preccurlyeq b$, $\lambda \preccurlyeq c$. Since $\lambda \preccurlyeq b$ and $\lambda \preccurlyeq c$ so λ is *l.b* of the set $\{b, c\}$. So $\lambda \preccurlyeq \text{Inf}\{b, c\}$ (By definition of Infimum).

i.e. $\lambda \preccurlyeq b \wedge c$. Thus $\lambda \preccurlyeq a$ and $\lambda \preccurlyeq b \wedge c$

So, λ is a *l.b* of the set $\{a, b \wedge c\}$.

So, $\lambda \preccurlyeq \text{Inf}\{a, b \wedge c\}$ i.e. $\lambda \preccurlyeq a \wedge (b \wedge c)$ i.e. $\lambda \preccurlyeq u$.

Again since $\mu = \text{Inf}\ \{a, b \wedge c\}$ proceding exactly as above we get $\mu \preccurlyeq \text{Inf}\{a \wedge b, c\}$ i.e. $\mu \preccurlyeq \lambda$. Thus we get $\lambda \leq \mu$ and $\mu \leq \lambda$; since $\preccurlyeq$ is antisymmetric so $\lambda = \mu$ i.e., $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

(ii) Similarly we can prove that $(a \vee b) \vee c = a \vee (b \vee c)$ for all elements a, b, c in L. Thus associative property is satisfied.

(4) (i) We shall show for any two elements a, b in L, $a \wedge (a \vee b) = a$. Since $a \vee b = \text{Sup}\{a, b\}$ so $a \preccurlyeq a \vee b$; again $a \preccurlyeq a$ so a is $l.b$ of the set $\{a, a \vee b\}$.

So, by definition of Infimum, $a \preccurlyeq \text{Inf}\{a,\ a \vee b\}$

i.e., $a \preccurlyeq a \wedge (a \vee b)$

Again since $\text{Inf}\{a, a \vee b\} = a \wedge (a \vee b)$

so, $a \wedge (a \vee b) \preccurlyeq a$. Since $\preccurlyeq$ is antisymmetric so $a \wedge (a \vee b) = a$.

(ii) Similarly we can prove that $a \vee (a \wedge b) = a$

Thus absorption property is satisfied.

Hence $(L, \wedge, \vee)$ becomes a lattice.

Theorem 4. A Lattice is a PO set in which Supremum and Infimum of the set $\{a, b\}$ exist for every pair of elements in the set.

Proof. Let $(L, \wedge, \vee)$ be a lattice. We define the relation $\preccurlyeq$ as $a \preccurlyeq b$ if $a \wedge b = a$. We shall show L is a PO set w.r.t this relation.

Since $a \wedge a = a$ (by idempotent law) so $a \preccurlyeq a$. Thus the relation $\preccurlyeq$ is reflexive. Let $a \preccurlyeq b$ and $b \preccurlyeq a$ hold. So, $a \wedge b = a$ and $b \wedge a = b$ i.e., $a \wedge b = b$ (by commutative property)

Hence $a = b$. So $\preccurlyeq$ is antisymmetric.

Let $a \preccurlyeq b$ and $b \preccurlyeq c$. Then $a \wedge b = a$ and $b \wedge c = b$. So $a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a$. Hence $a \wedge c = a$ i.e., $a \preccurlyeq c$.

Thus the relation $\preccurlyeq$ is transitive. So, $(L, \preccurlyeq)$ is a PO set.

Now consider the set $\{a, b\}$ where a, b are arbitrary. We shall show $\text{Sup}\{a, b\} = a \vee b$ and $\text{Inf}\{a, b\} = a \wedge b$

Now, $a \wedge (a \vee b) = a$ by absorption law.

So, $a \preccurlyeq a \vee b$; $b \wedge (a \vee b) = b \wedge (b \vee a) = b$

So, $b \preccurlyeq a \vee b$. Let y be an $u.b$ of $\{a, b\}$

$\therefore a \preccurlyeq y$ and $b \preccurlyeq y$ $\quad \therefore a \wedge y = a$ and $b \wedge y = b$

$\therefore a \vee y = y$ and $b \vee y = y$ by Theorem 2

Now, $(a \vee b) \vee y = a \vee (b \vee y) = a \vee y = y$

$\therefore (a \vee b) \wedge y = a \vee b$ again by Theorem 2

So, $a \vee b \preccurlyeq y$. Hence, $a \vee b = \text{Sup}\{a, b\}$

Now, $(a \wedge b) \wedge a = a \wedge (b \wedge a) = a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b$.

So, $a \wedge b \preccurlyeq a$. Similarly $a \wedge b \preccurlyeq b$

Let x be a LB of $\{a, b\}$ $\therefore x \preccurlyeq a$ and $x \preccurlyeq b$

$\therefore x \wedge a = x$ and $x \wedge b = x$

Now, $x \wedge (a \wedge b) = (x \wedge a) \wedge b = x \wedge b = x$ $\quad \therefore x \preccurlyeq (a \wedge b)$

Thus $a \wedge b = \text{Inf}\{a, b\}$

Thus L is a PO set w.r.t the relation $\preccurlyeq$ and $\text{Sup}\{a, b\}$ and $\text{Inf}\{a, b\}$ exists for every pair of elements a, b in L.

Remark : *The above two Theorem 3 and 4 lead us to have an another equivalent definition of Lattice, which is given below .*

Alternative Definition of Lattice.

A partially ordered set, L in which $\text{Sup}\{a, b\}$ and $\text{Inf}\{a, b\}$ exist for all a, b in L, is called a lattice.

Illustrative Examples.

Show that $P(S)$, the power set is a lattice w.r.t the order relation $\subseteq$ (According to Alternate definition)

We already proved that $(P(S), \subseteq)$ is a PO set. Let $A, B \in P(S)$ be arbitrary. Then $A \subseteq A \cup B$, $B \subseteq A \cup B$

Let X be an u.b of the set $\{A, B\}$, i.e $A \subseteq X$, $B \subseteq X$.

Clearly then $A \cup B \subseteq X$ .So, $A \cup B = \text{Sup}\{A, B\}$.

Again $A \cap B \subseteq A$, $A \cap B \subseteq B$. Let X be a l.b of the set $\{A, B\}$, i.e., $X \subseteq A$, $X \subseteq B$. Clearly then $A \cap B \subseteq X$. So $A \cap B = \text{Inf}\{A, B\}$.

Thus $\text{Sup}\{A, B\}$ and $\text{Inf}\{A, B\}$ exist for all A, B in $P(S)$.

So, $P(S)$ is a lattice according to alternate definition.

Theorem 5. Every linearly ordered set is a lattice.

Proof. Let L be a linearly ordered set w.r.t the linear order relation $\preccurlyeq$. Let a, b be an arbitrary pair of L. Since $\preccurlyeq$ is linearly ordered so either $a \preccurlyeq b$ or $b \preccurlyeq a$. Let $a \preccurlyeq b$. Then we shall show that $\text{Sup}\{a, b\} = b$, $\text{Inf}\{a, b\} = a$. Now, $a \preccurlyeq b$ and $b \preccurlyeq b$. Let y be an $u.b$ of $\{a, b\}$. So, $a \preccurlyeq y$ and $b \preccurlyeq y$; this shows clearly $b \preccurlyeq y$. Thus $\text{Sup}\{a, b\} = b$.

Similarly it can be shown that $\text{Inf}\{a, b\} = a$.

Thus in L, $\text{Sup}\{a, b\}$ and $\text{Inf}\{a, b\}$ exist for all a, b in L. So L is a lattice (by the Alternate definition of Lattice)

Illustration.

(i) *Prove that $(N, \leq)$ is a lattice where N is set of all positive integers.*

Obviously $(N, \leq)$ is a PO set (in fact it is linearly ordered). Let a, b be an arbitrary pair of elements of L. Let $a \leq b$. Then $\text{Sup}\{a, b\} = b$ and $\text{Inf}\{a, b\} = a$ (Can be proved as the previous theorem). So in the PO set $(N, \leq)$ $\text{Sup}\{a, b\}$ and $\text{Inf}\{a, b\}$ exist for every pair of elements a, b. So $(N, \leq)$ is a lattice (by Alternate definition of Lattice)

(ii) *The set of divisors of m* (a positive integer) is a lattice under the relation of divisibility.

Let D_m be the set of all divisors of m, i.e. $D_m = \{n : n \text{ is a divisor of } m\}$. Let ρ be the relation such that $a \rho b$ means a divides b (e.g. $3 \rho 18$) Since a divides itself so, $a \rho a$ i.e. ρ is reflexive. Let $a \rho b$ and $b \rho a$. So a divides b and b divides a. This is possible only if $a = b$. Thus ρ is antisymmetric. Let $a \rho b$ and $b \rho c$. So, $b = pa$ and $c = qb \therefore c = pqa$ i.e., a divides c, i.e., $a \rho c$. So, ρ is transitive. This (D_m, ρ) is a PO set.

Let $a, b \in D_m$ be arbitrary. Let $h = H.C.F$ of a and b and $l = L.C.M$ of a and b. We shall prove that $h = \text{Inf}\{a, b\}$ and $l = \text{Sup}\{a, b\}$. Clearly $h \rho a, h \rho b$. Let y be a l.b of $\{a, b\}$. Then $y \rho a, y \rho b$. So y divides a and b. So y divides their *HCF*, h, i.e. $y \rho h$. Therefore $h = \text{Inf}\{a, b\}$. Similarly we can prove $l = \text{Sup}\{a, b\}$ (This is left to reader as Exercise)

Thus (D_m, ρ) is a PO set in which the supremum and the infimum of any two elements of D_m exists. So, by alternate definition, D_m is a lattice.

Sub Lattice.

Let $(L, \wedge, \vee)$ be a lattice and $M \subset L$. M is called a sublattice of L if M itself is a lattice w.r.t the same operations $\wedge$ and $\vee$.

Theorem 6. Let $(L, \wedge, \vee)$ be a lattice and $M \subset L$. M is sublattice of L if and only if for all a, b in M

(i) $a \wedge b \in M$ (ii) $a \vee b \in M$.

Proof. Obvious.

Illustrative Example.

Let $(N, \wedge, \vee)$ be a lattice where $a \wedge b$ means *HCF* of a and b; $a \vee b$ mean *LCM* of a and b. Then the set $M =$ The set of all divisors of 20 is a sub-lattice of N.

Clearly $M \subset N$. Since the *HCF* of two divisors of 20 is also a divisor so $a \wedge b \in M$ for all a, b, in M. Since the *LCM* of two divisors of 20 is also a divisor so $a \vee b \in M$ $\forall a, b$ in M. Thus M is a sub-lattice of N.

6.5. Distributive Lattice

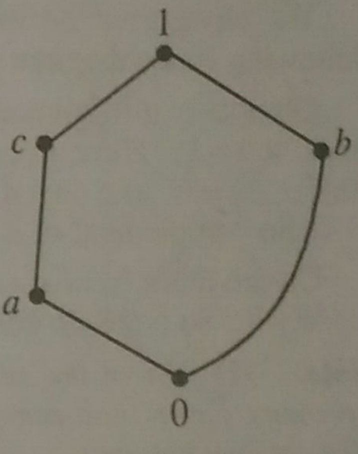
A lattice $(L, \wedge, \vee)$ is said to be a Distributive Lattice if the following two properties (known as distributive property) hold :

For all a, b, c in L,

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Illustrative Example.

(i) Let S be a set and $P(S)$ be its power set, i.e. the set of all subset of S. We have shown $(P(S), \cap, \cup)$ is a lattice. We know further $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ hold. So this lattice is a Distributive lattice.



(ii) The PO set $S = \{0, 1, a, b, c\}$ is a lattice which is shown by the following Hasse diagram:

(See Alternate Definition of Lattice). Here we know the two binary relation $\wedge$ and $\vee$ are defined as $a \wedge b = \text{Inf}\{a, b\}$ and $a \vee b = \text{Sup}\{a, b\}$.

(See proof of the Theorem 4)

Now, from the given Hasse diagram, $b \wedge c = \text{Inf}\{b, c\} = 0$ and $a \vee (b \wedge c) = a \vee 0 = \text{Sup}\{a, 0\} = a$ but $a \vee b = \text{Sup}\{a, b\} = 1, a \vee c = \text{Sup}\{a, c\} = c$.

and $(a \vee b) \wedge (a \vee c) = 1 \wedge c = \text{Inf}\{1, c\} = c$.

This shows $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$.

So this lattice S is not a distributive lattice.

6.6. Bounded Lattice and Complemented Lattice

Bounded Lattice.

Lattice is a PO set. If this PO set is a bounded set then the lattice is called a Bounded Lattice.

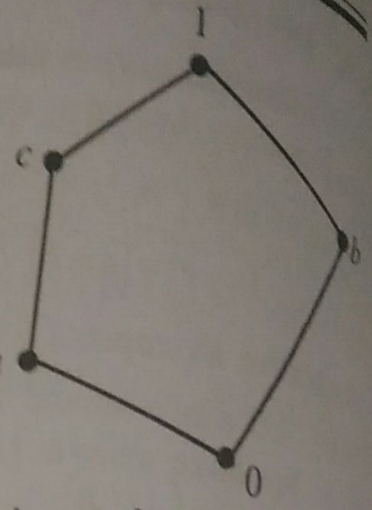
Illustrative Example

(i) Let S be a set. We know the PO set $(P(S), \subseteq)$ is a Lattice. Here we see for every element $A \in P(S)$, $\Phi \subseteq A \subseteq S$

$\therefore$ $l.b$ and $u.b$ of $P(S)$ are Φ and S respectively. $\therefore$ $P(S)$ is a bounded lattice.

(ii) The PO set $S = \{0, 1, a, b, c\}$ shown by the Hasse diagram is a lattice. Its $l.b$ is 0 and $u.b$ is 1. So it is a bounded lattice.

(iii) The PO set $(N, /)$ is a lattice. It has no $u.b$. So it is not a bounded lattice.

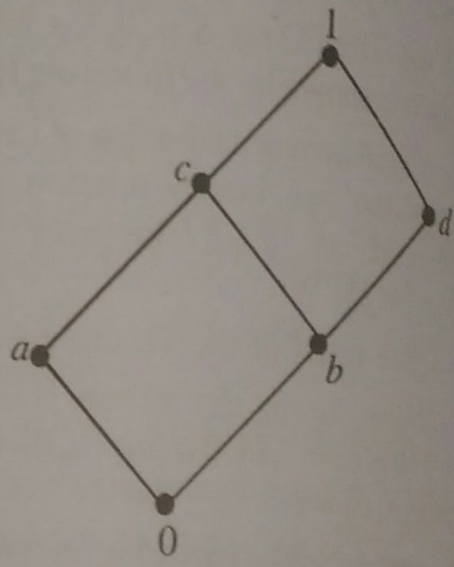


Complements.

Let $(L, \wedge, \vee)$ be a bounded lattice where O be its $l.b$ and I be its $u.b$. Let a be an element of L. Another element a' in L is said to be complement of a if $a \wedge a' = O$ and $a \vee a' = I$.

Illustrative Examples.

We have seen before the set $S = \{0, a, b, c, d, 1\}$ shown by the following Hasse diagram is a lattice.



Obviously it is bounded with $l.b = 0$ and $u.b = 1$. Here we see $a \wedge d = \text{Inf}\{a, d\} = 0$ and $a \vee d = \text{Sup}\{a, d\} = 1$. So complement of a is d.

On the other hand $c \wedge d = b \neq 0$ and $c \vee d = 1$. So c has no complement.

Note. (1) *From the above example it becomes clear that complement of an element may not exist.*

(2) *Complement of an element is not unique. (Find an example)*

Theorem If the complement of an element exists in a bounded distributive lattice then the complement is unique.

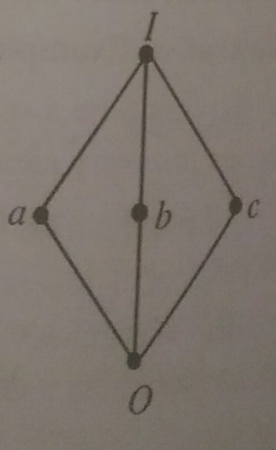
Proof. Beyond the scope of the text.

Complemented Lattice

A lattice is called a complemented lattice if it is bounded and every element of it has a complement.

Illustrative Example.

(i) The lattice $S = \{0, a, b, c, I\}$ shown by the Hasse diagram is bounded with $u.b$ I and $l.b$ 0.



Here we see, $a \vee b = I$, $b \vee c = I$; $a \wedge b = O$, $b \wedge c = O$. So complement of a, b, c exist. So this lattice is a complemented lattice.

Note that the complement of a is not unique. Infact b and c both are complements of a.

(ii) $(P(S), \subseteq)$ is a complemented lattice because for every element A in $P(S)$ its complement $A' = S - A \in P(S)$.

Illustrative Examples

Ex. 1. *Write the dual of the statement* $(a \wedge b) \vee a = a \wedge (b \vee a)$.

The dual is obtained by replacing $\wedge$ and $\vee$ by $\vee$ and $\wedge$ respectively. So the required dual is $(a \vee b) \wedge a = a \vee (b \wedge a)$.

Ex. 2. *Find which of the following three PO set are lattices :*

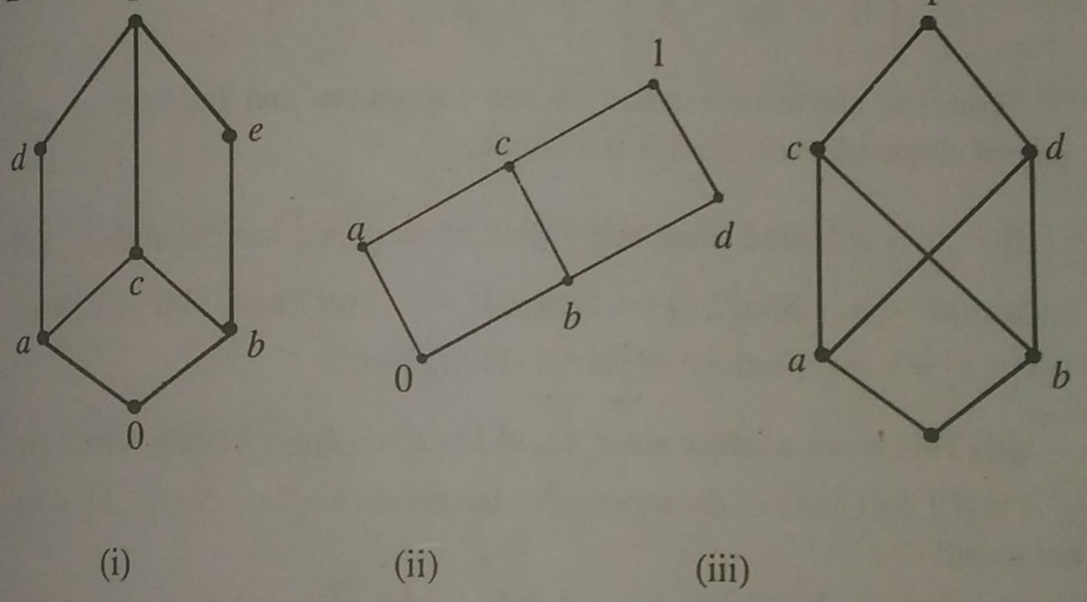


Table showing the supremum of a pair of elements:

(i) Sup	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	a	c	c	d	1	1
b	b	c	b	c	1	e	1
c	c	c	c	c	1	1	1
d	d	d	1	1	d	1	1
e	e	1	e	1	1	e	1
1	1	1	1	1	1	1	1

Table showing the Infimum of a pair of elements:

Inf	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	0	a	a	0	a
b	0	0	b	b	0	b	b
c	0	a	b	c	a	b	c
d	0	a	0	a	d	0	d
e	0	0	b	b	0	e	e
1	0	a	b	c	d	e	1

From the above two tables we see supremum and Infimum of any pair of elements exist. So this is a lattice.

(ii) It is a lattice since every pair of elements has Supremum and Infimum e.g. $\text{Sup}\{0, c\} = 1, \text{Sup}\{b, 1\} = 1$ etc. and $\text{Inf}\{a, b\} = 0$, $\text{Inf}\{c, b\} = b$ etc. [find for all pair in tabular form].

(iii) This is not a lattice since $\{a, b\}$ has three upper bounds which are c, d, and 1 and none of them precedes the other two, i.e., $\text{Sup}\{a, b\}$ does not exist.

Ex. 3. *Let* $(L, \preccurlyeq)$ *be a lattice. Let* $a, b \in L$. *Let* $[a, b] = \{x : x \in L \text{ and } a \preccurlyeq x \preccurlyeq b\}$. *Prove that* $[a, b]$ *is a sub-lattice of L.*

Let $x, y \in [a, b]$. Then $x, y \in L$. So, $x \vee y$ and $y \wedge x \in L$ as L is lattice Now a is a $l.b$ of x and y. Then $a \preccurlyeq x \preccurlyeq x \vee y \preccurlyeq b$. So, $x \vee y \in [a, b]$ and similarly $a \preccurlyeq x \wedge y \preccurlyeq x \preccurlyeq b$. So $x \wedge y \in [a, b]$. Hence $[a, b]$ is a lattice.

Ex. 4. *Prove that intersection of two sub lattices is a sub-lattice.*

Let S and T be two sub lattices of a lattice $(L, \wedge, \vee)$. Let $a, b \in S \cap T$

$\therefore a, b \in S \quad \therefore a \wedge b \in S$ and $a \vee b \in S$ as S is a sub-lattice.

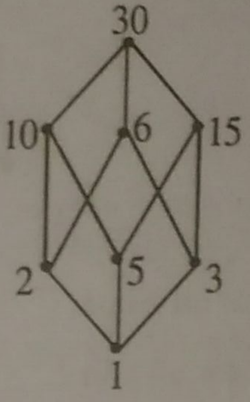
Similarly $a, b \in T$ and $\therefore a \wedge b \in T$ and $a \vee b \in T$ as T is a sub-lattice.

Hence $a \wedge b \in S \cap T$ and $a \vee b \in S \cap T$. Since $S \cap T \subset L$ so $S \cap T$ is a sub lattice.

Ex. 5. *Give an example to show that union of two sub-lattices may not be a sub-lattice.*

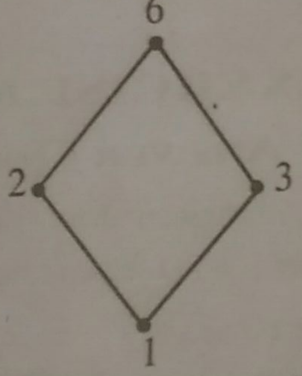
Let $L=$ set of all factors of 12 (including 1 and 12). Then $(L,\wedge,\vee)$ is a lattice where $\wedge$ and $\vee$ stand for HCF and LCM. Obviously $S=\{1,3\}$ and $T=\{1,2\}$ are sub-lattices of L. Now $S\cup T=\{1,2,3\}$ is not a sub lattice since $2,3\in S\cup T$ but $2\vee 3=6\notin S\cup T$.

Ex. 6. *Find the sub-lattices of the lattice*



Consider the subset $M=\{1,2,3,6\}$; this PO set is shown

by the Hasse diagram



We see Supremum and Infimum of any pair of elements exist. So this is a sub lattice. Similarly the set $\{1,3,5,15\}$ is also a sub-lattice.

Ex. 7. *In a distributive lattice* $(L,\wedge,\vee)$, $a\wedge b=a\wedge c$ *and* $a\vee b=a\vee c\Rightarrow b=c$.

Since we are given $a\wedge b=a\wedge c$ and $a\vee b=a\vee c$ so by using commutative property we have $b\wedge a=c\wedge a$ and $b\vee a=c\vee a$

Now $b=b\wedge(b\vee a)$ by absorption law

$=b\wedge(c\vee a)$ by hypothesis

$=(b\wedge c)\vee(b\wedge a)$ by distributive law

$=(b\wedge c)\vee(c\wedge a)$ by hypothesis

$=(c\wedge b)\vee(c\wedge a)$ by commutative law

$=c\wedge(b\vee a)$ by distributive law

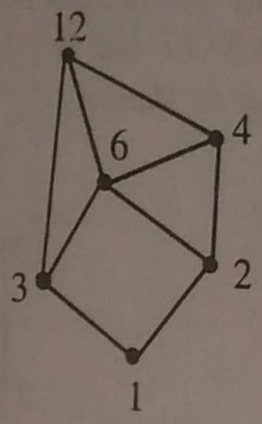
$=c\wedge(c\vee a)=c$ by absoption law

Thus $b=c$.

Ex. 8. *Consider the lattice* $L=\{1,2,3,4,6,12\}$ *ordered by divisibility* $(/)$. *Find the tower and upper bound of L. Is L a complemented Lattice ?*

[W.B.U.Tech 2006, 2008]

The Hasse diagram of the lattice is



We see $1/x$ for all x in L.

We see $x/12$ for all x in L. $\therefore$ 1 is LB and 12 is UB of L. Complement of 1, 2, 3, 4, 5, 6, 12 are respectively 12, 6, 4, 3, 2, 1 all of which $\in L$. $\therefore L$ is complemented.

EXERCISE 6

I. Short Answer Questions

1. Define partially ordered set. Illustrate with an example.
2. Prove that the set of integers is a PO set w.r.t the relation $\geq$.
3. Which of the following are PO sets ?

   (i) $(Z, >)$ (ii) $(Z, \neq)$, (iii) $(Z, =)$ where Z is set of all integers.
4. Write down all possible partial order relation of the set $\{a, b\}$.
5. Prove that the set $Y=\{1,2,3,4,6,9\}$

   forms a PO set w.r.t the 'divide' relation.

   Draw the Hasse diagram for each. Find the maximal and minimal element; *u.b.*, *l.b.* Supremum and Infimum of each of the two sets.
6. Draw the Hasse diagram for the PO set $(A, /)$ where / stand for divisibility and (i) $A=\{2,3,6,12,24,36\}$

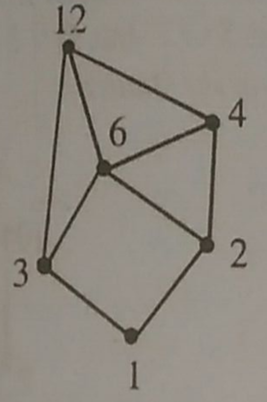
   [W.B.U.Tech 2006]

   (ii) $A=\{3,6,12,24,48\}$

Ex. 8. *Consider the lattice* $L = \{1, 2, 3, 4, 6, 12\}$ *ordered by divisibility* $(/)$. *Find the tower and upper bound of L. Is L a complemented Lattice ?*

[*W.B.U.Tech 2006, 2008*]

The Hasse diagram of the lattice is



We see $1/x$ for all x in L.

We see $x/12$ for all x in L. $\therefore$ 1 is LB and 12 is UB of L. Complement of 1, 2, 3, 4, 5, 6, 12 are respectively 12, 6, 4, 3, 2, 1 all of which $\in L$.
$\therefore L$ is complemented.

EXERCISE 6

I. Short Answer Questions

1. Define partially ordered set. Illustrate with an example.
2. Prove that the set of integers is a PO set w.r.t the relation $\geq$.
3. Which of the following are PO sets ?

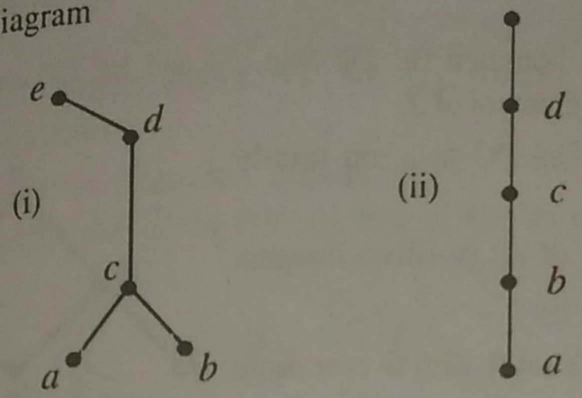
   (i) $(Z, >)$ (ii) $(Z, \neq)$, (iii) $(Z, =)$ where Z is set of all integers.
4. Write down all possible partial order relation of the set $\{a, b\}$.
5. Prove that the set $Y = \{1, 2, 3, 4, 6, 9\}$

   forms a PO set w.r.t the 'divide' relation.

   Draw the Hasse diagram for each. Find the maximal and minimal element; *u.b.*, *l.b.* Supremum and Infimum of each of the two sets.
6. Draw the Hasse diagram for the PO set $(A, /)$ where / stand for divisibility and (i) $A = \{2, 3, 6, 12, 24, 36\}$ [*W.B.U.Tech 2006*]

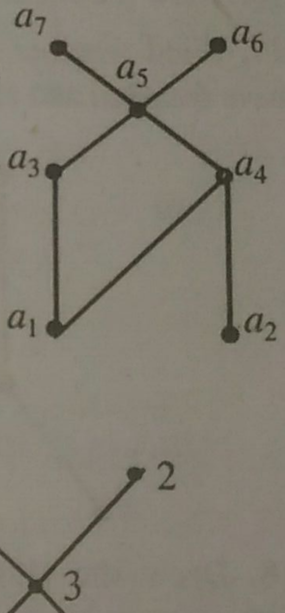
   (ii) $A = \{3, 6, 12, 24, 48\}$

7. Let $X=\{a,b,c,d,e\}$. Determine the relation represented by the following Hasse diagram



8. Define Maximal element, Upper bound and supremum (or least upper bound) of a PO set. Illustrate with an example to show they are not same.

9. Let $S=\{a_1,a_2,a_3,a_4,a_5,a_6,a_7\}$ be ordered as the given Hasse diagram :

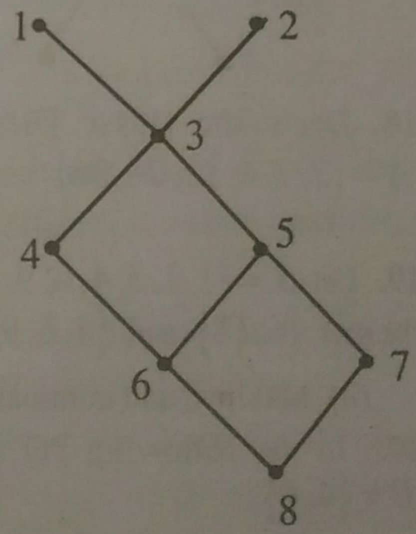


Find the upper and lower bounds of X. Find Sup (X) and Inf (X) if exists when (i) $X=\{a_3,a_4,a_5\}$,

(ii) $X=\{a_1,a_3,a_4\}$,

(iii) $X=\{a_2,a_3,a_5\}$.

10. Let $S=\{1,2,3,4,5,6,7,8\}$ be a PO set defined by the given Hasse diagram.



Find the $u.b$, $l.\ b$, supremum and infimum of the set (i) $B=\{1,2,5\}$,

(ii) $C=\{3,6,7\}$, (iii) $D=\{1,2,4,7\}$.

[Hint of (i) : There is no element succeeding both 1 and 2. So B has no $u.b$. The $l.b$ are 6, 7, 8; Sup (B) does not exist and Inf $(B)=5$]

11. Let $X=\{x: x\in Q \text{ and } 8<x^3<15\}$ where Q is set of rational numbers. Find whether the PO set $(X,\leq)$ is bounded above or below. Do $\text{Sup}(X)$ and $\text{Inf}(X)$ exists?

[Hint : e.g. 1 is a l.b and 100 is u.b ; For any rational number x, $x>\sqrt[3]{15}$ there exists a rational y s.t $x>y>\sqrt[3]{15}$. So Sup (X) does not exists. Inf (X) $=2$]

12. Define Minimal element, Lower Bound and Infimum (or greatest lower bound) of a PO set. Illustrate with an example to show they may not be same.

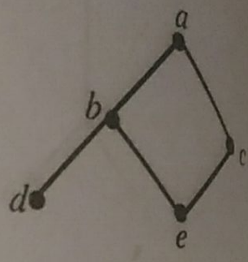
13. Let $X = \{x : x \in Q \text{ and } 5 < x^3 < 27\}$ where Q is set of all rational numbers.

(a) Is the PO set $(X, \leq)$ bounded (b) Do Sup (X) and Inf (X) exists. (c) Find the maximal and minimal element of X.

14. Find whether the following PO sets are totally ordered :

(i) $(N, /)$ where N is set of all positive integers and a / b means b is divisible by a.

(ii) The power set $P(A)$ of a set A with two or more elements, respect to $\subseteq$.

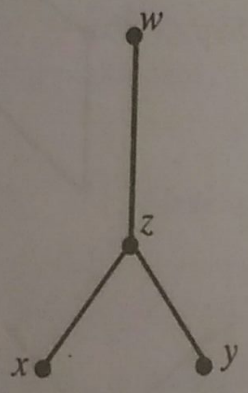


15. Draw the Hasse diagram of the PO set $\{P(S), \subseteq\}$ where $S = \{x, y, z\}$

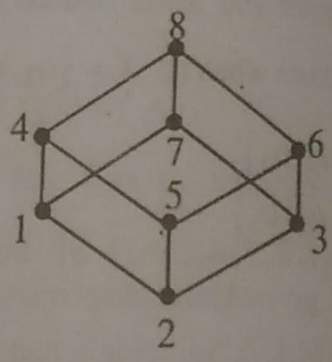
16. Draw the Hasse diagram of the PO set $\{A, /\}$ where $A = \{81, 27, 9, 3, 1\}$.

17. Find whether the PO sets represented by the following Hasse diagram have maximal and minimal element, $u.b$, $l.b$, Supremum and Infimum :

(i)



(ii)



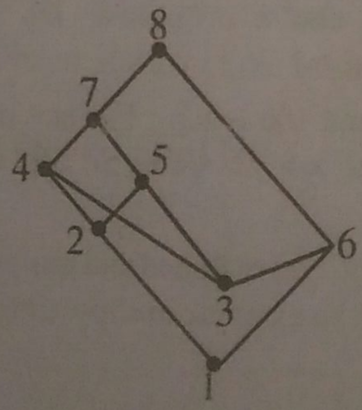
18. Draw the Hasse Diagram for the divisibility relation on the set $A = \{2, 3, 6, 12, 24, 36\}$ and find the maximal and minimal elements

[W.B.U.Tech (MCA) 2003]

19. Let $S = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$. Find the (i) Infimum and Supremum of the sets $\{6, 18\}$ and $\{4, 6, 9\}$ in the PO set $(S, /)$ [W.B.U.Tech (MCA) 2003]

(ii) Maximal and minimal element of S

20. In the following PO set find $u.b$ and Supremum for $A = \{2, 3\}$ and $B = \{4, 6\}$:

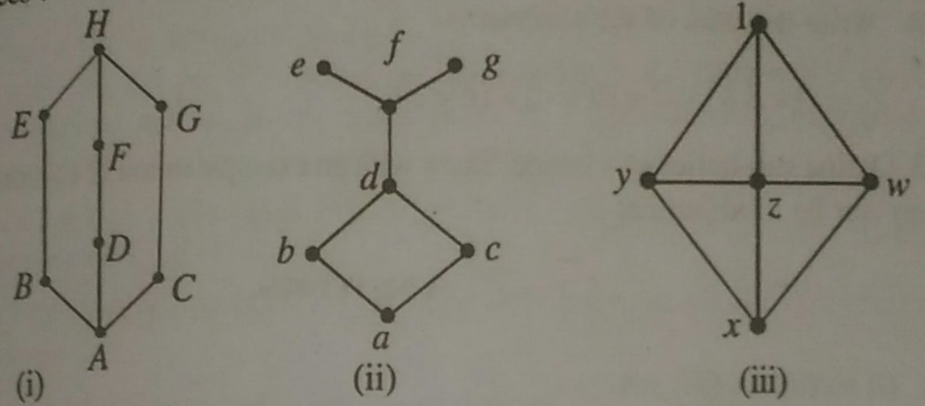


21. Find all the $u.b$ and $l.b$, Supremum and Infimum, Maximal and Minimal of the PO set $(X, \leq)$ where $X = \{x : x \text{ is real and } 1 < x < 2\}$

22. In the following PO set $(P(A), \subseteq)$ where $A = \{0, 1, 2\}$ find two non-comparable elements.

23. Define Lattice. Construct an example

24. Find which one of the PO sets represented by the following Hasse diagram are lattices :



[Hint : (i) lattice because Sup and Inf exist which are H and A

(ii) a lattice since sup and inf exist for all pair.

(iii) a lattice.]

25. In a lattice $(L, \wedge, \vee)$ prove that $a \vee a = a$ for all a in L. what is its dual.

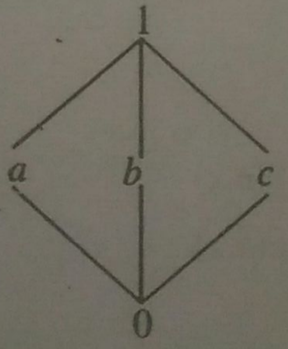
26. Draw Hase diagram of all lattice with five elements.

27. In a lattice $(L, \cap, \cup)$ prove that $a \cap b = a$ if and only if $a \cup b = b$.

28. Prove that the set $\{1, 2, 3, 4, 6, 12\}$ is a bounded lattice w.r.t the operation $\wedge$ and $\vee$ when $a \vee b = LCM$ of a and b, $a \wedge b = HCF$ of a and b. Find the complements of 3. Are they unique ?

29. Prove that the set $\{1, 3, 5, 15\}$ is a bounded, complemented lattice w.r.t the relation ρ where $a \rho b$ *means* a divides b. Draw its Hasse diagram.

30. Prove that the PO set $\{0, 1, a, b, c\}$ shown by the given Hasse diagram is a non-distributive lattice



31. If $S = \{a, b, c, d\}$ then find whether

(i) $\{\phi, \{a\}, \{a, c\}, \{c\}, \{a, b, c\}\}$

(ii) $\{\phi, \{a\}, \{b\}, \{c\}, \{b, c\}\}$

are sub-lattice of the lattice $(S, \subseteq)$

32. Write the dual of the statements

$$(a \wedge b) \vee c = (b \vee c) \wedge (c \vee a)$$

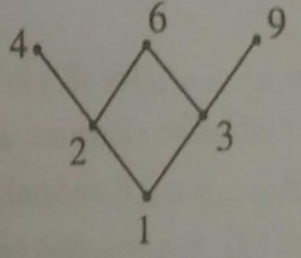
33. Define sub-lattice of a lattice. Show with an example union of two sub-lattice may not be a sublattice.

ANSWERS

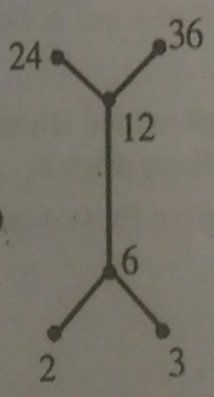
3. (i) no (ii) no, (iii) yes

4. aRa, bRb ; aRa, bRb, aRb ; aRa, bRb, bRa

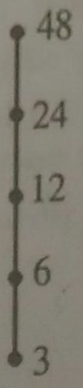
5. Maximal elements are 4, 6, 9, minimal elements are 1 ; $l.b$ is 1, no $u.b$, no Sup, Inf is 1



6. (i)

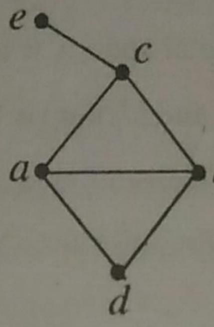


(ii)



7. (i) $(a, c), (b, c), (c, d), (d, e)$

(ii) $(a, b), (b, c), (c, d), (d, e)$

8. The PO set $\{a,b,c,d,e\}$ 

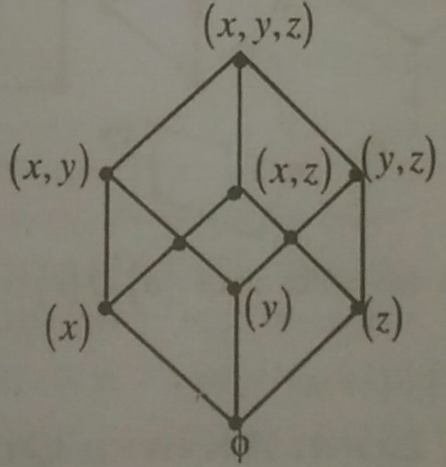
defined by the Hasse diagrams. Let us take the set $S=\{a,b,d\}$ Here a,b are maximal element of S, c and e are upper bound of S, c is supremum of S.

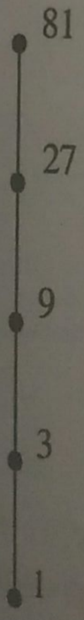
9. (i) u.b are a_5, a_6, a_7; (ii) u.b. are a_5, a_7, a_6; l.b. is a_1 ; Sup (X) = a_5, Inf (X) = a_1 (iii) u.b are a_5, a_7, a_6; there is no l.b ; Sup $(X) = a_5$, Inf (X) = does not exist.

10.(ii) $u.b$ are 1, 2, 3 ; $l.b$ are 8, Sup (C)=3, Inf(C) =8, (iii) no $u.b$; $l.b$ is 8; Sup (D) does not exists, Inf (D) = 8

13. (a) Bounded (b) Sup (X) = 3 ; Inf (X) = does not exist (c) No min element, Max element $= 3$.

14. (i) Not totally ordered (ii) Not totally ordered.

15. 

16. 

17. (i) W is max and $u.b$, x, y are minimal and $l.b$, w is sup, no Inf exists

(ii) 8 is maximal and $u.b$, 2 is minimal and $l.b$. Sup is 8, Inf is 2.

18. max elements are 24, 36, minimal elements are 2,3.

19. (i) 6 is Infimum, Sup is 18 ; 1 is Infimum, Sup is 36. (ii) Maximal element of S are 36, minimal element of S is 1.

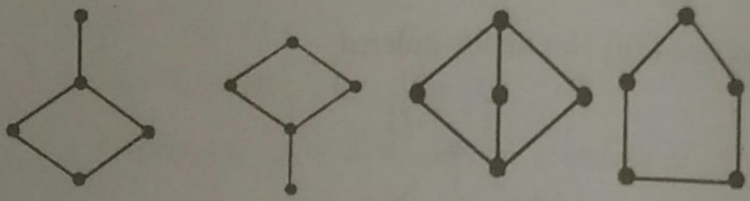
20. $u.b$ of A are 4, 5, 7, 8, Sup A does not exist ; $u.b$ of B are 8

Sup $B = 8$.

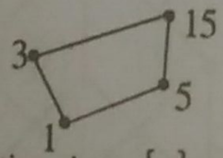
21. If $x \geq 2, x$ is $u.b$, If $x \leq 1, x$ is $l.b$. $1 =$ Infimum, $2 =$ Supremum of X. X has no maximal or minimal element.

22. $\{0\}$ and $\{1\}$

25. $a \wedge a = a$

26. 

28. 4

29. 

31. (i) sub-lattice (ii) not sub-lattice since $\{a\} \cup \{c\} = \{a, c\} \notin$

32. $(a \vee b) \wedge c = (b \wedge c) \vee (c \wedge a)$

II. Long Answer Questions

1. Find which of the following are partial order set

   (i) (Z, ρ) when $a \rho b$ means $|a| \leq |b|$

   (ii) (Z, ρ) when $a \rho b$ means $|a - b| \leq 1$

   (iii) (Z, ρ) when $a \rho b$ means $a - b \leq 0$

   (iv) (Z, ρ) when $a \rho b$ means $a + b$ is an even integer.

2. Let $A = \{a, b, c, d\}$ and consider the relation $R = \{(a,a), (a,b), (a,c), (b,b), (c,b), (c,c), (d,b), (d,c), (d,d)\}$. Show that R is a PO relation. Draw its Hasse diagram.

3. Prove that the sets

$X = \{2, 3, 5, 30, 60, 120, 180, 360\}$

forms a PO set w.r.t the 'divide' relation.

Draw the Hasse diagram for each. Find the maximal and minimal element ; *u.b.*, *l.b.* Supremum and Infimum of each of the two sets.

4. Let $X = \{24, 18, 12, 9, 8, 6, 4, 3, 2, 1\}$ be ordered by the relation 'x divides y'. Find the Hasse diagram.

5. (a) Draw the Hasse diagram for the P.O.Set $(A, /)$ where / stand divisibility

(i) $A =$ set of all factors of 30 (including 1 and 30)

(ii) $A =$ set of all factors of 17

(b) Let $A = \{2,3,5,30,60,120,180,360\}$. Prove that $(A, /)$ is a PO set. Is it well ordered ? Find the

(i) Successors of 30 (ii) Immediate successor of 120

(iii) Predecessors of 180 (iv) Immediate predecessor of 5

6. In the *B*. Tech course we say $A < B$ if the paper A is must for studying paper *B*. In the *B*. Tech course there are eight papers on Mathematics. The paper codes and their prerequisities are given below :

Paper	: M101	M201	M250	M251	M340	M341	M450	M500
Prerequisities	: None	M101	M101	M250	M201	M340	M201,M250	M450, M251

Construct a PO set regarding this problem and draw a Hasse diagram

7. Find whether the following set is a PO set w.r.t the relation mentioned :

$(N, /)$ where $N =$ set of all positive integers and a / b means a divides b.

8. Let D_{36} be the set of all positive divisors of 36. Show that D_{36} is a PO set w.r.t the relation 'divisor'. Draw the Hasse diagram of this PO set. Find (i) the Supremum and Infimum of the set $\{4, 9\}$ (ii) the maximal and minimal element of D_{36}. (iii) the maximal element of $\{12, 6, 18\}$.

9. Let $X = \{1, 2, 3, 4, 5, 6\}$. Prove that X is a PO set w.r.t the relation / (divisibility). Draw the Hasse diagram of $(X, /)$.

10. Consider the PO set $(S, /)$ where $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find the maximal, minimal *u.b*, *l.b*, Supremum and Infimum of (i) S (ii) $A = \{2, 7\}$ (iii) $B = \{1, 2, 3\}$ (iv) $C = \{1, 2, 4\}$ (v) State with reasons whether 2 is a *l.b* of

A and 3 is an $u.b$ of B.

11. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ be an ordered set as shown by the given Hasse diagram:

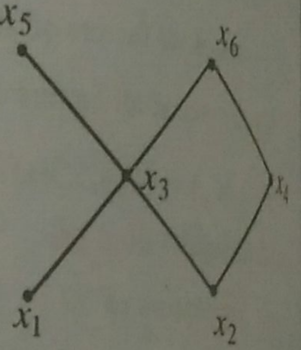
Find (i) all minimal and maximal elements of A

(ii) all $l.b$ and $u.b$ of A

(iii) Supremum and Infimum of A

(iv) Three linearly ordered subsets of A, each of which contains at least three elements.

12. Let $A = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be ordered as shown in the following Hasse diagram :



Find (i) all $u.b$ and $l.b$ of A

(ii) Supremum and Infimum of A

(iii) All linearly ordered subsets with three or more elements.

13. Consider the PO set $(S, \subseteq)$ where

$$S = \{\{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$$

Find (i) the maximal elements (ii) the minimal elements

(iii) all the $u.b$ of $\{\{2\},\{4\}\}$ and the supremum, if exists

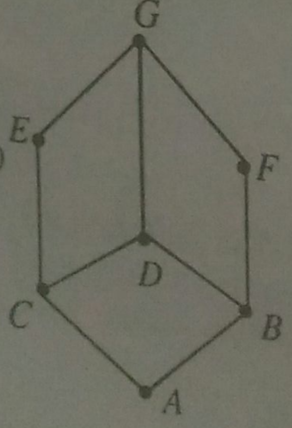
(iv) all the $l.b$ of $\{1,3,4\}$ and the Infimum of $\{1,3,4\}$.

14. Prove that a finite subset of a PO set has at most one Infimum.

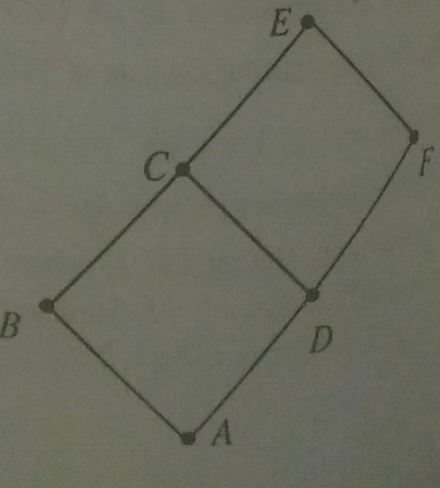
15. Let $L = \{a_1, a_2, a_3, \cdots, a_n\}$ be a lattice with respect to the PO relation $\preccurlyeq$. Let $a_1 \preccurlyeq a_2 \preccurlyeq a_3 \preccurlyeq \ldots \preccurlyeq a_n$, i.e., L is a *chain*. Prove that L is a distributive lattice.

16. Prove that the PO sets shown by the following Hassee Diagram are lattices. Are they distributive ? Bounded ? Complemented ? Give reasons.

(i)



(ii)

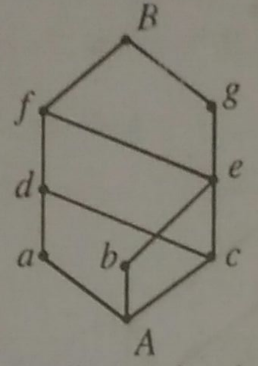


17. Prove that the set $\{\phi, \{a\}, \{a, c\}, \{c\}, \{a, b, c\}\}$ is a lattice w.r.t the relations $\cap$ and $\cup$. Is it complemented ? [W.B.U.Tech 2007]

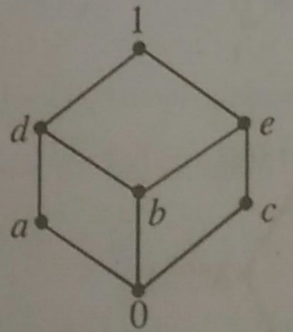
18. Prove that $([0, 1], \leq)$ is a lattice. What are the operation $\wedge$ and $\vee$ here ?

19. Prove that the set of all divisors of an integer m is a lattice w.r.t the operation $\wedge$, $\vee$ where $\wedge$ and $\vee$ stand for HCF and LCM respectively.

20. Find all the sublattices with five elements of the following lattice



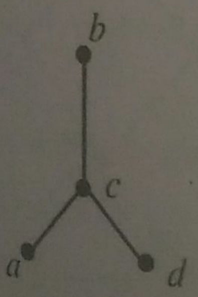
21. Prove that the following Hasse-diagram is a lattice. Find all the sublattices with five elements of the given lattice. Is it bounded, complemented ?



22. Consider $D=\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$ the factors of 60. Prove that $(D, /)$ is a lattice. Find complements of 2 and 10.

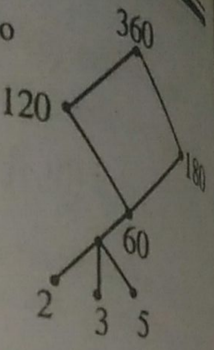
ANSWERS

1. (i) no, (ii) no, (iii) yes (iv) no (ρ is not antisymmetric)

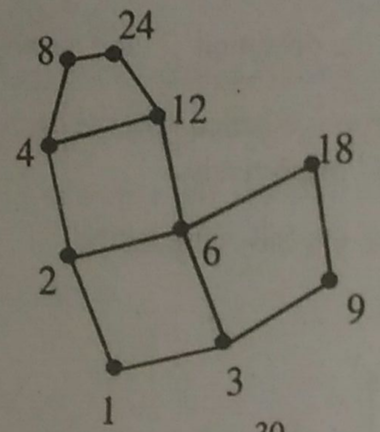
2. 

Max element is 360, min elements are 2, 3, 5

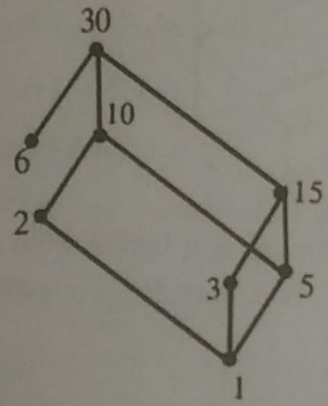
3. $l.b$ does not exists, $u.b$ is 300, Supremum is 360. No Infimum



4.

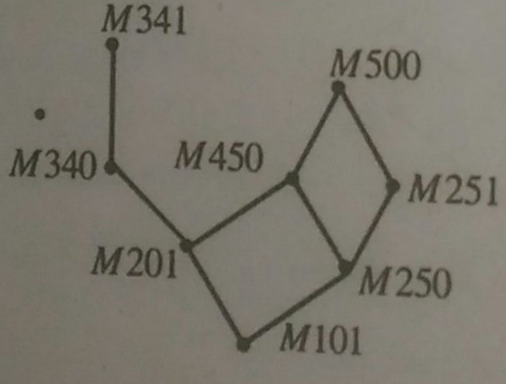


5. (a) (i)



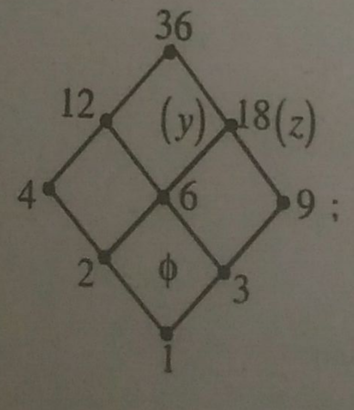
(ii)

6.

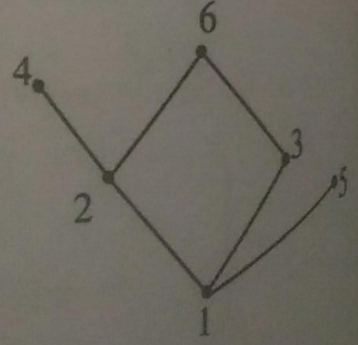


7. not a PO set.

8.



9.



10. (i) Maximal elements are 10, 9, 8, 7, 6. Minimal element is 1 ; No $u.b$, $l.b$ is 1, No Sup ; The Inf of $S = 1$.

(ii) Maximal = 7, Minimal = 2, no $u.b$, $l.b$ is 1, Inf. is 1, No supremum.

(iii) Maximal elements are 2, 3, Minimal is 1. $u.b.$ is 6, 1 is $l.b$, Sup is 6, Inf is 1

(iv) Maximal is 4, minimal is 1; $u.b$ are 4 and 8, $l.b$ is 1, Sup is 4, Inf is 1.

(v) 2 is not a $l.b$ of A because 2/7 does not hold ; 3 is not $u.b$ of B because 2/3 is not true.

11. (i) minimal elements are a_4, a_6 ; Maximal elements are a_1, a_2

(ii) no l.b. ; no u.b. (iii) no supremum, no infimum

(iv) $\{a_1, a_3, a_4\}, \{a_2, a_3, a_4\}, \{a_2, a_5, a_6\}$

12. (i) no l.b, no u.b (ii) no Supremum, No Infimum.

(iii) $\{x_1, x_3, x_5\}, \{x_1, x_3, x_6\}, \{x_2, x_3, x_5\}, \{x_2, x_3, x_6\}, \{x_2, x_4, x_6\}$

13. (i) $\{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}$ (ii) $\{1\}, \{2\}, \{4\}$

(iii) $\{2, 4\}, \{2, 3, 4\}$ supremum is $\{2, 4\}$ (iv) $\{3, 4\}, \{4\}$, infimum is $\{3, 4\}$

16. (i) is not distributive, bounded complemented (ii) is distributive.

17. no, because complement of $\{a\}$ does not exist.

20. $\{A, a, b, d, B\}, \{A, a, e, d, B\}, \{A, a, d, e, B\}, \{A, b, c, e, B\}, \{A, a, c, e, B\}, \{A, c, d, e, B\}$

21. $\{0, a, b, d, 1\}, \{0, a, c, d, 1\}, \{0, a, d, e, 1\}, \{0, b, c, e, 1\}, \{0, a, c, e, 1\}$ $\{0, c, d, e, 1\}$. Bounded but not complemented

22. 2 and 10 have no complement.

III. MULTIPLE CHOICE QUESTIONS

1. In the set of real numbers the relation < (less than) is not

(a) transitive (b) reflexive

2. In the set of integers the relation $\neq$ (not equal to) is

(a) reflexive (b) antisymmetric

(c) transitive (d) none

3. In the set of integers the relation $>$ (greater than) is

(a) reflexive (b) antisymmetric

(c) transitive (d) none

4. In the set of integers the relation $\geq$ (greater or equal to) is

(a) reflexive (b) antisymmetric

(c) transitive (d) none

5. In the set of integers the relation ρ is defined by $a\rho b$ hold if $a-b=$ integer. Then ρ is

(a) reflexive (b) antisymmetric

(c) transitive (d) none

6. In the set of integers ρ is defined by $a\rho b$ hold if b is multiple of a. Then ρ is

(a) reflexive (b) antisymmetric

(c) transitive (d) none

7. In the set of real numbers the relation ρ is defined as $a\rho b$ hold if $a-b<3$. Then ρ is

(a) reflexive (b) antisymmetric

(c) transitive (d) none.

8. Let $S=\{1,2,3,4,5\}$ and $P(S)$ be its power set. In the PO set $(P(S),\subseteq)$ indicate which one of the following is true for $A=\{2,3,4\}$, $B=\{1,2,3,4\}$

(a) A precedes B (b) B precedes A

(c) A suceeds B (d) A and B are not related

9. Let $S=\{a,b,c,d,e\}$; $A=\{a,b\}$, $B=\{b,c,d\}$. In the PO set $(S,\subseteq)$

(a) A precedes B (b) B precedes A

(c) A and B are non-comparable (d) A suceeds B

10. In the set $S=\{1,2,3,4,6,9\}$ the relation R, defined as aRb hold whenever b is multiple of a. Then which of the following statements is correct

(a) 3 and 4 are comparable (b) 9 suceeds 3

(c) 3 suceeds 9 (d) 4 and 6 are comparable

11. The set of all real numbers with respect to the relation $\leq$, is a

(a) *PO* set (b) Totally ordered set

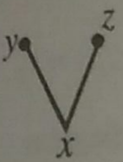
(c) none of the two (d)

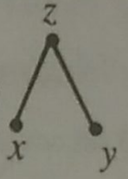
12. If $S = \{2,9,7,8,0\}$ then the power set of S, with respect to the relation $\subseteq$, is

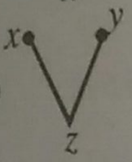
(a) totally ordered (b) partially ordered

(c) none of the two (d)

13. Let $S = \{x, y, z\}$ and ρ be a partially ordered relation such that $x\rho z$ and $y\rho z$ hold. Then the Hasse-diagram of this relation is

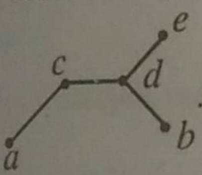
(a) 

(b) 

(c) 

(d) none

14. The *PO* set $(S, \prec)$ is represented by the Hasse-diagram

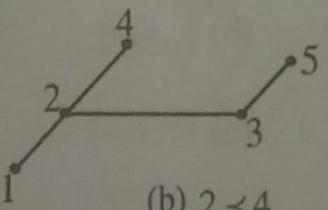


. Then

(a) $a \prec d$ (b) $d \prec b$

(c) $c \prec b$ (d) $d \prec e$

15. In the Hasse diagram



(a) $2 \prec 5$ (b) $2 \prec 4$

(c) $1 \prec 4$ (d) $1 \prec 5$

16. In the set of rational number under the order relation $\leq$ (less or equal) every element has immediate predecessor and immediate successor

(a) Yes (b) No

17. Consider the P O set $S = \{1,2,3,4,6,9\}$ w. r. t. divide relation. The maximal element of S is/are

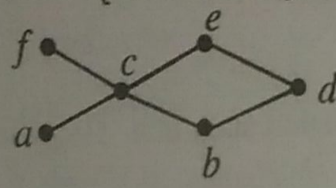
(a) 9 (b) 6

(c) 6,9 (d) 4, 6, 9

18. Consider the P O set $\{1,2,3,4,6,9\}$ w. r. t. 'divide' relation. The minimal element of the set is/are

(a) 1
(b) $1,2$
(c) 2
(d) $1,2,3$

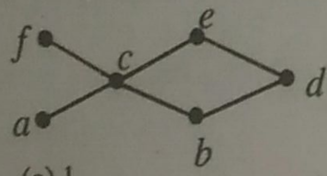
19. $S=\{a,b,c,d,e,f\}$ is a P O set as shown in the Hasseian diagram



. The number of maximal element in S is

(a) 1
(b) 2
(c) 3
(d) 0

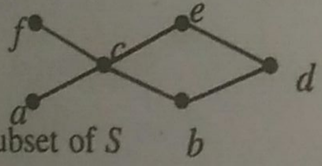
20. $S=\{a,b,c,d,e,f\}$ is a P O set according to the Hasseian diagram



The number of minimal element in S is

(a) 1
(b) 0
(c) 2
(d) 3

21. $S=\{a,b,c,d,e,f\}$ is a P O set according to the Hasseian diagram



. Which of the following set is a totally ordered subset of S

(a) $\{b,c,e,d\}$
(b) $\{c,d,e\}$
(c) $\{b,c,e\}$
(d) $\{f,c,e\}$

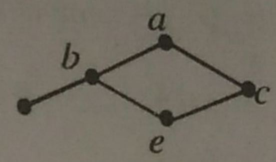
22. The number of minimal element of the PO set $(Z,\leq)$ is

(a) 0
(b) 1
(c) 2
(d) 3

23. In the P O set $(Z^+,/)$, where Z^+ represents set of all positive integers and $/$ represents 'divides', which of the following pairs are not comparable

(a) $(4,6)$
(b) $(5,5)$
(c) $(2,4)$
(d) $(3,15)$

24. Let $A = \{a, b, c, d, e\}$ be ordered by the Hasse diagram



Which one of the following is not valid

(a) $a \succ e$ (b) $b \| c$

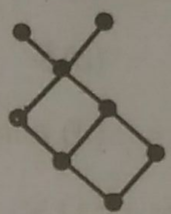
(c) $d \prec a$ (d) $e \prec b$

25. Which one of the following is not true in the ordered set $(N, /)$

(a) $18 \| 24$ (b) $2/8$

(c) $6/8$ (d) $6//24$

26. Let $S = \{1,2,3,4,5,6,7,8\}$ be a P O set defined by the Hasse diagram

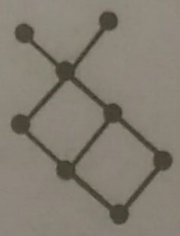


All the upper bounds of the set $\{4,5,7\}$ is/are

(a) 3 (b) 1, 2, 3

(c) 4, 5 (d) 1, 2

27. Let $S = \{1,2,3,4,5,6,7,8\}$ be a PO set defined by the Hasse diagram

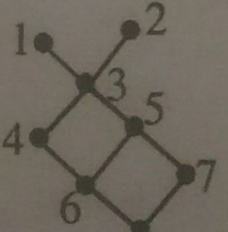


All the lower bounds of the set $\{4,5,7\}$ is/are

(a) 4, 5 (b) 6

(c) 7 (d) 8

28. Let $S = \{1,2,3,4,5,6,7,8\}$ be a P O set defined by the Hassé diagram

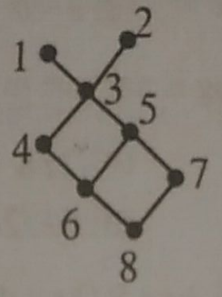


The supremum of the set $\{4,5,7\}$ is

(a) 4 (b) 5

(c) 3 (d) 2

29. Let $S=\{1,2,3,4,5,6,7,8\}$ be a P O set defined by the Hasse diagram

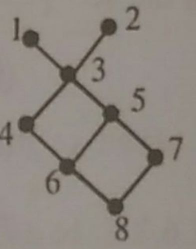


Then Inf $\{4,5,7\}=$

(a) 6 (b) 8

(c) 7 (d) none

30. Let $S=\{1,2,3,4,5,6,7,8\}$ be a P O set defined by the Hasse diagram



Then the upper bound of the set $\{1,2,5\}$ is/are

(a) 1 (b) 2

(c) both 1 and 2 (d) none

31. Let $X=\{x : x\in R \text{ and } 5<x^3<27\}$ where R is the set of real numbers. X is bounded

(a) Yes (b) No

32. Let $X=\{x : x\in R \text{ and } 5<x^3<27\}$ where R is the set of real numbers. Then supremum of $X=$

(a) 3 (b) 3.5

(c) 2.999 (d) none

33. Let $X = \{x : x \in R \text{ and } 5 < x^3 < 27\}$ where R is the set of reals. Then Infimum of $X=$

(a) 5
(b) $5^{\frac{1}{3}}$
(c) $5^{\frac{2}{3}}$
(d) none of these

34. If $N = \{1,2,3,........\}$ be ordered by divisibility. Which one of the following subsets of N is not totally ordered

(a) $\{7\}$
(b) $\{5,15,30\}$
(c) $\{24,2,6\}$
(d) $\{3,15,5\}$

35. If N be P O set of all positive integers w. r. t. the relation 'divisibility' then which one of the following subsets of N is not totally ordered

(a) $\{1,2,3,4,.....\}$
(b) $\{24,2,6\}$
(c) $\{5\}$
(d) $\{2,4,8,16\}$

36. The lower bound of the P O set $(Z^+,/)$ is, (Z^+ is set of all + ve integers)

(a) 0
(b) 1
(c) 2
(d) none

37. The upper bound of the P O set $(Z^+, /)$ is,

(a) 100
(b) 1000
(c) ∞
(d) none

38. In the P O set represented by the following Hasse diagram the maximal element which is not upper bound is

(a) e and f both
(b) c and f both
(c) b
(d) none

39. $B=\{1,2,3\}$ is a subset of the PO set $(S,/)$ where $S=\{1,2,3,4,5,6,7,8,9,10\}$. Then a maximal element and the supremum of B are respectively

(a) 3 and 3
(b) 1 and 6
(c) 2 and 6
(d) 2 and 3

40. In a lattice which one of the following statement is correct

(a) The set $\{a,b\}$ has supremum though it may not have infimum for all a, b in the lattice

(b) the set $\{a,b\}$ has infimum though it may not have supremum for all a, b in the lattice.

(c) The set $\{a,b\}$ has supremum and infimum for all a, b in the lattice

(d) the set $\{a,b\}$ has supremum and infimum for some a, b in the lattice

41. The P O set $\{1,3,6,9,12\}$ w. r. t. to the order relation 'divisibility' is a lattice

(a) Yes (b) No

42. The P O set $\{1,5,25,125\}$ w. r. t. to the order relation 'divisibility' is a lattice

(a) Yes (b) No

43. In the lattice $\{1,5,25,125\}$ the complement of 25 is

(a) 1 (b) 5

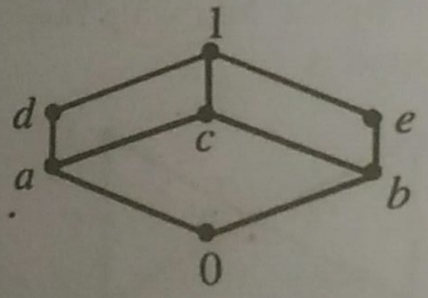
(c) 25 (d) 125

44. In the lattice $\{1,5,25,125\}$ w. r. t. to the order relation 'divisibility' the complement of 1 is

(a) 1 (b) 5

(c) 25 (d) 125

45. Which of the following subsets of the lattice L representcd by the Hasse diagram. are sublattice



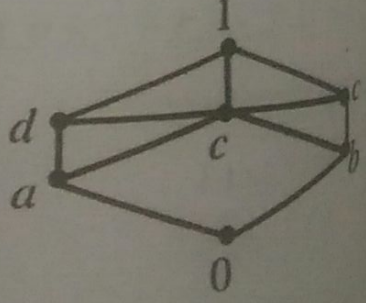
(a) $\{0,a,b,1\}$ (b) $\{0,a,e,1\}$

(c) $\{a,c,d,1\}$ (d) $\{0,c,d,1\}$

46. Which of the following subsets of the lattice L represented by the Hasse diagram are sub-lattice

(a) $\{0,a,b,1\}$ (b) $\{0,a,e,1\}$

(c) $\{a,c,d,1\}$ (d) $\{0,c,d,1\}$



47. The lattice $D_{42} = \{1,2,3,6,7,14,21,42\}$ w. r. t. the order relation 'divisibility' is a complemented lattice

(a) Yes (b) No

48. In the lattice D_{60} = set of divisors of 60 w. r. t. the order relation divisibility complement of 2 is

(a) 60 (b) 30

(c) 1 (d) does not exist

ANSWERS

1.b	2.c	3.c	4.a,b,c	5.a, c	6.a, c
7.a	8.a	9.c	10.b	11.a,b	12.b
13.b	14.d	15.b	16.a	17. d	18.a
19.b	20.c	21.c	22.a	23.a	24.d
25.c	26.b	27.d	28. c	29.b	30.d
31. a	32.a	33.b	34.d	35.a	36.b
37.d	38.b	39.c	40.c	41.b	42.a
43.b	44.d	45.b, c	46. b, c	47.a	48.d