1.1 FUNDAMENTAL OF THE PROBABILITY THEORY

1.1.1. Introduction : In Science and Technology we have to be concerned with every phenomena whose future behaviour is not predictable in a deterministic fashion. We have to depend on 'Chance' in every field. In theory of probability we are very much concerned with 'Chance'. In fact 'Probability' is nothing but a numerical measurement of this 'Chance'. Future can be guessed if this measurement is deduced. In this chapter we are going to deal with this measurement 'Probability'. Conception on Set-theory and Combinatorics Theory are the only prerequisistes for this chapter.

1.1.2. Introductory Definitions :

Random Experiment : An experiment or observation which may be repeated a large number of times under very nearly identical conditions and the possible outcome of any particular observation is unpredictable but all possible outcomes can be described prior to its performance, is known as Random Experiment.

For example, the experiment of tossing a coin is a random experiment, as the possible outcomes are 'tails' or 'heads' but the outcome of a particular tossing cannot be predicted.

Sample Points/ Event Points. The outcomes of a random experiment are called sample points or event points.

For example, the sample points in the experiment 'tossing a coin' are Head and Tail, in symbol H and T.

Sample Space/Event Space. The set of all sample points i.e. the set of all possible outcomes of a Random Experiment is called the sample space. It is denoted by S.

For example if we throw two coins once, then

$S = \{HH, HT, TH, TT\}$; if we roll a die once,

we have $S = \{1, 2, 3, 4, 5, 6\}$ as the event space.

Event. Any subset of the sample space S of a random experiment is called an Event.

For example, in the experiment of 'throwing two coins', $A = \{TH, HT\}$ is an event because $A \subset S$.

Certain Event. Since every set is a subset of itself, so the sample space is a subset of itself. So this is an event. This event is called a certain event.

Impossible Event. An event that contains no sample points is called impossible event. It is denoted by ϕ. For example in the experiment 'throwing a die' the event 'Face 7' $= \phi$.

Complementary Event. For any event A, there is an event containing all the sample points in the sample space which are not in A. This event is called the complementary event of A and is denoted by A' or $\bar{A}$ or A^c.

Obviously A' = 'Not A'.

For example, if $A = \{TH, HT\}$

where $S = \{HH, TT, TH, HT\}$, then $\bar{A} = \{HH, TT\}$.

Note. $\bar{S} = \phi$; $\bar{\phi} = S$; $\overline{(\bar{A})} = A$

Simultaneous Occurence of two Events.

Let A_1 and A_2 be two events. Then the set $A_1 \cap A_2$ represents the simultaneous occurence of the two events A_1 and A_2. This event is also denoted by $A_1 A_2$.

For example, in the experiment 'rolling a die' let

A_1 = 'Even face' A_2 = 'Multiple of three'.

Then $A_1 \cap A_2 = \{6\}$ is the event whose occurence shows the simultaneous occurence of A_1 and A_2.

At least one of Two Events.

Let A_1 and A_2 be two events. Then the set $A_1 \cup A_2$ represents 'at least one of A_1 and A_2'. This event is also denoted by $A_1 + A_2$.

For example, in the experiment rolling a die let

A_1 = Even face $= \{2, 4, 6\}$, A_2 = Multiple of three $= \{3, 6\}$.

Then $A_1 \cup A_2 = \{2, 4, 6, 3\}$ is the event whose occurence shows the occurence of at least one of 'even face' and 'Multiple of 3'.

Disjoint or Mutually Exclusive (m.e) Events. If two events A_1, A_2 have no common sample points i.e. if $A_1 \cap A_2 = \phi$, they are called Mutually Exclusive Events.

For example, in a previous example if $A_1 = \{HH, TT\}$ $A_2 = \{HT, TH\}$, then $A_1 \cap A_2 = \phi$. So, A_1 and A_2 are mutually exclusive events. Two m.e events cannot occur simultaneously.

Pairwise Disjoint Events. Let $A_1, A_2, \cdots, A_n$ be n number of events. Events $A_i (i = 1, 2, \cdots, n)$ are said to be pairwise disjoint if no two of them have any common event points i.e. if $A_i \cap A_j = \phi, i \neq j$ and $i, j = 1, 2, \cdots, n$.

Exhaustive Events. Two or more events are said to be exhaustive if at least one of them neccessarily occurs or in other words the events $A_1, A_2, \cdots \cdots$ are exhaustive if $A_1 \cup A_2 \cup A_3 \cup \cdots \cdots = S$.

For example, in the experiment of throwing two coins once, the events $A_1 = \{HH\}$, $A_2 = \{TT\}$ and $A_3 = \{HT, TH\}$ are exhaustive.

Elementary or Simple Event. An event containing exactly one sample point is called Elementary Event.

For example, $A_1 = \{2\}$, $A_2 = \{5\}$, $A_3 = \{3\}$ etc are simple event of an experiment of rolling a die.

Composite Event. The event which can be decomposed into simple events i.e. which can be expressed as union of two or more simple events is called Composite Event.

For example, in the experiment of 'rolling a die' $A_1 = \{2, 3, 4\}$, $A_2 = \{1, 5\}$ etc are composite events.

Equally Likely Sample Points. The sample points of a sample space are said to be equally likely if one of them may not be expected rather than the other.

***1.1.3. Classical Definition of Probability.**

Let us suppose that a random experiment E is such that its sample space S contains a finite number $n(S)$ of sample points, all of which are equally likely. Then the probability of an event A which contains $n(A)$ sample points, is defined by

$$P(A) = \frac{n(A)}{n(S)}. \quad \cdots (1)$$

* Beyond the syllabus

Illustration. A perfect die is rolled once and observed whether an odd number appears. If A denotes this event, then $A=\{1,3,5\}$ and the sample space $S=\{1,2,3,4,5,6\}$

$$\therefore n(A)=3 \text{ and } n(S)=6$$

$$\therefore P(A)=\frac{n(A)}{n(S)}=\frac{3}{6}=\frac{1}{2}$$

This tells that chance of occuring 'odd face' is one per two throws.

Criticism of Classical Definition.

(i) To define the probability, firstly we presumes that the sample points are equally likely which means equally probable i.e., probability of each sample point is same. So this definition is circular.

(ii) This defintion does not provide any criterion of deciding whether the possible outcomes of an experiment are equally likely.

(iii) In many experiments, the no. of possible outcomes is infinite and so this defintion is not suitable in those cases.

(iv) This definition can be used only in very simple and unimportant cases like games of chance.

(v) In some complicated problems, the calculation of possible outcomes and favourable cases are difficult, for example the sex of a newly born child and the throw of an untrue coin.

1.1.4. Theorems on Probability.

Some important properties of Probability are presented as the following theorems. Proofs are given considering the Frequency definition.

Theorem 1. $0\le P(A)\le 1$ for any event A.

Proof: Let a random experiment E be repeated n times under identical conditions and A be an event which occurs $n(A)$ times. Then we have

$$0\le n(A)\le n \quad \text{or, } 0\le \frac{n(A)}{n}\le 1$$

$$\therefore 0\le P(A)\le 1.$$

Theorem 2. $P(S)=1$ and $P(\phi)=0$ where S is certain event, ϕ is impossible event.

Proof: Now $P(S)=\frac{n(S)}{n}=\frac{n}{n}=1$

$$P(\phi)=\frac{n(\phi)}{n}=\frac{0}{n}=0.$$

Theorem 3. If A_1 and A_2 be two mutually exclusive events, then $P(A_1\cup A_2)=P(A_1)+P(A_2)$.

Proof: Let a random experiment E be repeated n times under identical condition and A_1, A_2 be two events which occurs $n(A_1)$ and $n(A_2)$ times. Since A_1 and A_2 are mutually exclusive, so $n(A_1\cup A_2)=n(A_1)+n(A_2)$

$$\therefore \frac{n(A_1\cup A_2)}{n}=\frac{n(A_1)}{n}+\frac{n(A_2)}{n}$$

$$\therefore P(A_1\cup A_2)=P(A_1)+P(A_2)$$

Theorem 4. If the events $A_1, A_2, \cdots, A_n$ are mutually exclusive then $P(A_1\cup A_2\cup\cdots\cup A_n)=P(A_1)+P(A_2)+\cdots+P(A_n)$

i.e., $P\left(\bigcup_{i=1}^{n} A_i\right)=\sum_{i=1}^{n} P(A_i)$

Proof: Left as an exercise.

(Proceeding as Th. 3 and using induction theorem can be proved)

Theorem 5. (Total Probability Theorem). For any two events A_1 and A_2 (may not be mutually exclusive),

$P(A_1\cup A_2)=P(A_1)+P(A_2)-P(A_1\cap A_2)$ [W.B.U.Tech 2007]

Proof: $P(A_1\cup A_2)=P\left(A_1\cup\left(\overline{A}_1\cap A_2\right)\right)$, when $\overline{A}$ is complement of A.

$=P(A_1)+P(\overline{A}_1\cap A_2)$, since A_1 and $\overline{A}_1\cap A_2$ are m.e

Again, $A_2=(A_1\cap A_2)\cup(\overline{A}_1\cap A_2)$

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$\therefore\ P(A_2) = P((A_1 \cap A_2) \cup (\overline{A}_1 \cap A_2))$

$= P(A_1 \cap A_2) + P(\overline{A}_1 \cap A_2) \quad \because A_1 \cap A_2$ and $\overline{A}_1 \cap A_2$ are m.e

or, $P(\overline{A}_1 \cap A_2) = P(A_2) - P(A_1 \cap A_2)$

Therefore, from above

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Theorem 6. If A_1, A_2, A_3 are any three events (not necessarily m.e.) then

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3)$$

Proof: Left as an exercise.

Theorem 7. If $A_1, A_2, A_3, \cdots \cdots, A_n$ are any n events then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = \Sigma P(A_1) - \Sigma P(A_1)P(A_2) + \Sigma P(A_1)P(A_2)P(A_3) - \cdots \cdots + (-1)^n P(A_1)P(A_2)\cdots P(A_n)$$

Proof: Omitted.

Theorem 8. For any event A, $P(\overline{A}) = 1 - P(A)$ where $\overline{A}$ is the complementary event of A.

Proof: We have A and $\overline{A}$ are mutually exclusive events and $A \cup \overline{A} = S$, the certain event.

$\therefore\ P(A \cup \overline{A}) = P(S)$

or, $P(A) + P(\overline{A}) = 1$, since A and $\overline{A}$ are mutually exclusive.

$\therefore\ P(\overline{A}) = 1 - P(A)$.

Theoem 9. If $A_1, A_2, \cdots, A_n$ are mutually exclusive and exhaustive events, then $\sum_{i=1}^{n} P(A_i) = 1$.

Proof: Since the events are exhaustive, so $\bigcup_{i=1}^{n} A_i = S$, the certain event.

Hence by Theorem (4) we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

i.e., $\sum_{i=1}^{n} P(A_i) = P(S)$

$\therefore\ \sum_{i=1}^{n} P(A_i) = 1$, by Theorem 2.

Theorem 10. For any two events A_1 and A_2 where $A_1 \subset A_2$

(i) $P(A_1) \le P(A_2)$ (ii) $P(A_2 - A_1) = P(A_2) - P(A_1)$

Proof:

(i) $A_1 \subset A_2 \Rightarrow n(A_1) \le n(A_2) \Rightarrow \dfrac{n(A_1)}{n} \le \dfrac{n(A_2)}{n} \Rightarrow P(A_1) \le P(A_2)$

(ii) $A_1 \subset A_2 \Rightarrow A_2 = A_1 \cup (A_2 - A_1)$, where $A_1, A_2 - A_1$ become m.e.

$\therefore\ P(A_2) = P(A_1) + P(A_2 - A_1) \ \therefore\ P(A_2 - A_1) = P(A_2) - P(A_1)$

1.1.5. Axiomatic definition of probability.

[W.B.U.Tech 2005]

Let E be a random experiment and S be its sample space ; Σ be the class of all events (i.e., subsets of S). Let P be a function from Σ to the set of all real numbers satisfying the following axioms :

Axiom I. $P(A) \ge 0$, for every event A in Σ

Axiom II. $P(S) = 1$.

Axiom III. If $A_1, A_2, \cdots$ be a finite or infinite sequence of pairwise mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

Then for any event A the real number $P(A)$ is called its probability.

Illustration. Let a biased coin be tossed.

The event space $S = \{H, T\}$.

It has four events $\{H\}, \{T\}$ ϕ and S itself.

We define P such that $P(H) = 1/3$, $P(T) = 2/3$, $P(\phi) = 0$, $P(\{H, T\}) = 1$

Then we see $P(A) \geq 0$ for all event A i.e., Axiom I is satisfied. $P(S) = P(\{H,T\}) = 1$ i.e., Axiom II is satisfied. Consider the two m.e. events $\{H\}$ and $\{T\}$.

Then $\{H\} \cup \{T\} = \{H,T\}$

$\therefore \ P(\{H\} \cup \{T\}) = P(\{H,T\}) = 1 = \frac{1}{3} + \frac{2}{3} = P(\{H\}) + P(\{T\})$,

i.e., Axiom III is satisfied.

Thus this function P represents probability. In particular we can say probability of Head is $\frac{1}{3}$.

All the Theorems which were deduced for the Frequency definition of probability can be deduced for Axiomatic definition also. Some of thems are shown in the next page.

Some Deductions from the Axioms :

(i) ***For any*** $A \subset S$, $P(A) \leq 1$

Let $\overline{A} = S - A$. Then $A, \overline{A}$ are mutually exclusive and $A \cup \overline{A} = S$.

$\therefore P(A \cup \overline{A}) = P(S)$

or, $P(A) + P(\overline{A}) = 1$, by Axiom II and III

or, $P(A) = 1 - P(\overline{A})$ ... (1)

Again by axiom I, $P(\overline{A}) \geq 0$

Hence, $P(A) = 1 - P(\overline{A}) \leq 1$

(ii) $P(\phi) = 0$

Since $\overline{\phi} = S$. So from (1)

$P(\phi) = 1 - P(\overline{\phi}) = 1 - P(S) = 1 - 1$ by Axiom (II)

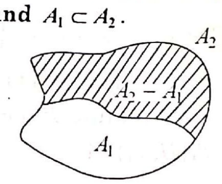
$= 0$

(iii) For any two events $A_1, A_2 \subset S$ **and** $A_1 \subset A_2$.

We have $P(A_1) \leq P(A_2)$

and $P(A_2 - A_1) = P(A_2) - P(A_1)$

Since $A_1 \subset A_2$, $A_2 = A_1 \cup (A_2 - A_1)$.



Also since A_1 and $A_2 - A_1$ are mutually exclusive, so by Axiom III, we have,

$P(A_2) = P(A_1) + P(A_2 - A_1)$

$\therefore P(A_2 - A_1) = P(A_2) - P(A_1)$

By Axiom I, $P(A_2 - A_1) \geq 0 \quad \therefore P(A_2) - P(A_1) \geq 0$

i.e., $P(A_2) \geq P(A_1)$.

(iv) (Boole's inequality)

For n events $A_1, A_2, \cdots, A_n$

$P(A_1 \cup A_2 \cup \cdots \cup A_n) \leq P(A_1) + P(A_2) + \cdots + P(A_n)$

i.e., $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$

Proof: For $n = 1$, the theorem is obviously true. We have

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$\therefore \ P(A_1 \cup A_2) \leq P(A_1) + P(A_2) \quad [\because \ P(A_1 \cap A_2) \geq 0]$ ... (1)

Thus the theorem is true for $n = 2$.

Let the theorem holds for $n = m$

$\therefore \ P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i)$ ... (2)

Now $P\left(\left(\bigcup_{i=1}^{m} A_i\right) \cup A_{m+1}\right) = P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left(\left(\bigcup_{i=1}^{m} A_i\right) \cap A_{m+1}\right)$

or, $P\left(\bigcup_{i=1}^{m+1} A_i\right) \leq P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1})$, by Axiom I,

$\leq \sum_{i=1}^{m} P(A_i) + P(A_{m+1})$, since $P\left(\left(\bigcup_{i=1}^{m} A_i\right) \cap A_{m+1}\right) \geq 0$

$= \sum_{i=1}^{m+1} P(A_i)$

$\therefore \ P\left(\bigcup_{i=1}^{m+1} A_i\right) \leq \sum_{i=1}^{m+1} P(A_i)$.

Thus the theorem holds for $n=m+1$ whenever it is hold for $n=m$. But the theorem holds for $n=1,2$.

Hence by induction, the theorem is true for any positive integral values of n.

(v) For any events $A_1, A_2, \cdots, A_n$,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq 1 - \sum_{i=1}^{n} P(\overline{A}_i)$$

Proof. By Boole's inequality we have

$$P(\overline{A}_1 + \overline{A}_2 + \cdots + \overline{A}_n) \leq P(\overline{A}_1) + P(\overline{A}_2) + \cdots + P(\overline{A}_n)$$

or, $P(\overline{A_1 \cap A_2 \cap \cdots \cap A_n}) \leq \sum_{i=1}^{n} P(\overline{A}_i)$, by De Morgan's law.

or, $1 - P(A_1 \cap A_2 \cap \cdots \cap A_n) \leq \sum_{i=1}^{n} P(\overline{A}_i)$

[$\because$ for any event A, $P(\overline{A}) = 1 - P(A)$]

$$\therefore P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq 1 - \sum_{i=1}^{n} P(\overline{A}_i)$$

1.1.6. Conditional Probability.

We consider two events A and B connected with a random experiment E. Let us make the hypothesis that the event A has occured $n(A)$ times and B occurs simultaneously with A $n(A \cap B)$ times in the n repetations of experiment E. . Then ratio $\frac{n(A \cap B)}{n(A)}$ is called the conditional probability of B on the hypothesis that A has already occured and is denoted by $P\left(B/A\right)$.

Thus $P\left(B/A\right) = \frac{n(A \cap B)}{n(A)}$

$= \frac{n(A \cap B)/n}{n(A)/n} = \frac{P(A \cap B)}{P(A)}$, provided $P(A) \neq 0$ and n - should be very large.

Similarly the conditional probability of A on the hypothesis that B has already occured is $P\left(A/B\right) = \frac{P(A \cap B)}{P(B)}$, provided $P(B) \neq 0$.

Illustration : Let one card be drawn from a full pack. A = 'spade' B = 'king'

$\therefore$ Probability of King supposing Spade occurs $= P\left(B/A\right)$

$= \frac{n(A \cap B)}{n(A)} = \frac{1}{13}$

Multiplication Rule of Probability.

Thus we have

$$P(A \cap B) = P(A) \cdot P\left(B/A\right) = P(B)P\left(A/B\right) \quad \cdots (1)$$

if $P(A) \neq 0, P(B) \neq 0$

Theorem. (Generalization of the Multiplication Rule)

For n events $A_1, A_2, \cdots, A_n$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P\left(A_2/A_1\right)P\left(A_3/(A_1 \cap A_2)\right) \cdots P\left(A_n/\bigcap_{i=1}^{n-1} A_i\right)$$

provided $P(\bigcap A_i) \neq 0$, $i = 1, 2, \cdots, n-1$

Proof: Left as an exercise.

1.1.7. Independent Events.

If for two events A and B, $P(A/B) = P(A)$ [i.e., the chance of occurrence of the event A is not affected by the occurrence of the event B], the event A is said to be independent of the event B.

The following theorem is the most important characterisation of being two events independent.

Theorem. Two events A and B are stochastically independent or statistically independent or independent if and only if $P(A \cap B) = P(A) \cdot P(B)$

Proof: $P(A/B) = P(A) \Leftrightarrow \frac{P(A \cap B)}{P(B)} = P(A) \Leftrightarrow P(A \cap B) = P(A)P(B)$

Hence the theorem.

Note : When $P(A \cap B) \neq P(A) \cdot P(B)$, the events A and B are said to be dependent.

Mutually Independent Events. The n events $A_1, A_2, \cdots, A_n$ are said to be mutually independent if the following conditions are satisfied

$$P(A_i \cap A_j) = P(A_i)P(A_j) \text{ for all } i \neq j;\ i, j = 1, 2, \cdots, n$$

$$P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k) \text{ for all } i \neq j \neq k$$

... ...

... ...

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2) \cdots P(A_n)$$

Pairwise Independent Events : The n events $A_1, A_2, \cdots, A_n$ are said to be pairwise independent if

$$P(A_i \cap A_j) = P(A_i)\,P(A_j) \text{ for all } i \neq j;\ i, j = 1, 2, \cdots, n$$

Illustrations. Let one card be drawn from a full pack. A = 'spade',

B = 'King'. Then $P(B/A) = \frac{n(B \cap A)}{n(A)} = \frac{1}{13}$.

Again $P(B) = \frac{n(B)}{n(S)} = \frac{4}{52} = \frac{1}{13}$.

Thus $P(B/A) = P(B)$. So in this experiment 'king' and 'spade' are independent.

1.1.8. Baye's Theorem. If $A_1, A_2, \cdots, A_n$ be a given set of n pairwise mutually exclusive and exhaustive events then for any event A where $P(A) \neq 0$

(i) $P(A) = P(A_1)P(A/A_1) + P(A_2)P(A/A_2) + \cdots + P(A_n)P(A/A_n)$

$$= \sum_{i=1}^{n} P(A_i)P(A/A_i)$$

(ii) $P(A_i/A) = \frac{P(A_i)P(A/A_i)}{P(A)}$, for $i = 1, 2, \cdots, n$ [W.B.U.Tech.2008]

Proof : Beyond the scope of the book

Illustration. There are three urns. First urn contains 3 red, 4 black balls ; second urn contains 6 black, 2 red balls ; third urn contains 3 black balls and 5 red balls. One urn is chosen and then a ball is drawn from the urn. Let A_1 = '1st urn is chosen' A_2 = '2nd urn is chosen' A_3 = 3rd urn is chosen ; A = The ball is red.

According to Baye's theorem

$$P(A) = P(A_1)P(A/A_1) + P(A_2)P(A/A_2) + P(A_3)P(A/A_3)$$

$$= \frac{1}{3} \cdot \frac{3}{7} + \frac{1}{3} \cdot \frac{2}{8} + \frac{1}{3} \cdot \frac{5}{8} = \frac{1}{7} + \frac{1}{12} + \frac{5}{24} = \frac{73}{168}.$$

On the other hand the probability that the 3rd urn was chosen supposing that the ball is red

$$= P(A_3/A) = \frac{P(A_3)P(A/A_3)}{P(A)}$$

$$= \frac{\frac{1}{3} \cdot \frac{5}{8}}{\frac{73}{168}} = \frac{5}{24} \times \frac{168}{73} = \frac{35}{73}.$$

1.1.9. Illustrative Examples.

Ex. 1. *What is the chance that a leap year selected at random will contain 53 wednesdays ?* [W.B.U.Tech 2002]

A leap year contains 366 days that is 52 full weeks and two days extra. The extra two days will be either (i) Sunday, Monday or (ii) Monday, Tuesday or (iii) Tuesday, Wednesday or (iv) Wednesday, Thursday or (v) Thursday, Friday or (vi) Friday, Saturday or (vii) Saturday, Sunday.

So a leap year will contain 53 Wednesdays if one of the two extra days is Wednesday. Therefore out of the above seven cases two are favourable.

Hence the required probability is $\frac{2}{7}$.

Ex. 2. *The integers x and y are chosen at random with replacement from nine natural numbers 1, 2, ... 8, 9. Find the probability that $(x^2 - y^2)$ is divisible by 2.*

$x^2-y^2=(x-y)(x+y)$ will be divisible by 2 iff x, y are either both even or both odd. Now two even numbers can be chosen from $\{1,2,\cdots,9\}$ with replacement in 4×4 ways. Similarly both odd number can be selected in 5×5 ways. So the total no. of favourable cases is $16+25=41$.

Again two integers x, y can be chosen at random with replacement from $\{1,2,\cdots,9\}$ is $9\times9=81$.

Hence the required probability is $\frac{41}{81}$.

Ex. 3. *Given* $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}, P(AB)=\frac{1}{4}$. (*AB* means $A\cap B$)

(a) find the values of the following probabilities

$$P(\bar{A}), P(A\cup B), P(A/B), P(\bar{A}B), P(\bar{A}\,\bar{B}), P(\bar{A}\cup B)$$

(b) State whether the events A and B are
(i) mutually exclusive (ii) exhaustive (iii) equally likely (iv) independent.

(a) $P(\bar{A})=1-P(A)=1-\frac{1}{2}=\frac{1}{2}$

$$P(A\cup B)=P(A)+P(B)-P(AB)=\frac{1}{2}+\frac{1}{3}-\frac{1}{4}=\frac{7}{12}$$

$$P(A/B)=\frac{P(AB)}{P(B)}=\frac{\frac{1}{4}}{\frac{1}{3}}=\frac{3}{4}$$

$$P(\bar{A}B)=P\{(S-A)B\}=P(SB-AB)$$

$$=P(B-AB)=P(B)-P(AB)=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$

$$P(\bar{A}\bar{B})=1-P(A\cup B)=1-\frac{7}{12}=\frac{5}{12}$$

$$P(\bar{A}\cup B)=P(\bar{A})+P(B)-P(\bar{A}B)=\frac{1}{2}+\frac{1}{3}-\frac{1}{12}=\frac{3}{4}.$$

(b) (i) No ; because $P(AB)=\frac{1}{4}\neq0$, i.e., $A\cap B\neq\phi$

(ii) No ; because $P(A\cup B)=\frac{7}{12}\neq1$, i.e., $A\cup B\neq S$

(iii) No; because $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}$ i.e., $P(A)\neq P(B)$

(iv) No ; because $P(AB)\neq P(A)\cdot P(B)$, here

$$P(AB)=\frac{1}{4} \text{ but } P(A)\cdot P(B)=\frac{1}{6}.$$

Ex. 4. *In an examination 30% of the students failed in Physics, 25% in Mathematics and 12% in both Physics and Mathematics. A student is selected at random. Find the probability that (i) the student has failed in Physics, if it is known that he has failed in Mathematics.*

(ii) the student has failed at least one of the two subjects

(iii) the student has passed at least one of the two subjects.

(iv) the student has passed in Mathematics if he failed in Physics.

Let A and B denote the events " a student failed in Physics" and "a student failed in Mathematics" respectively. Then

$$P(A)=0\cdot30,\ P(B)=0\cdot25,\ P(A\cap B)=0\cdot12$$

Now (i) probability that a student has failed in Physics if it is known that he has failed in Mathematics is

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{0\cdot12}{0\cdot25}=0\cdot48.$$

(ii) the probability that a student has failed at least one of the subjects is

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=0\cdot30+0\cdot25-0\cdot12=0\cdot43$$

(iii) $\bar{A}$ = The student passed in Mathematics, $\bar{B}$ = he passed in Physics.

Then the probability that the student has passed at least one of the subject is

$$P(\bar{A}\cup\bar{B})=1-P(\overline{\bar{A}\cup\bar{B}})=1-P(A\cap B)\text{ [by D' Morgans law]}$$

$$=1-0\cdot12=0\cdot88$$

and (iv) the probability that the student has passed in Mathematics if he failed in Physics is

$$P(\bar{B}/A)=1-P(B/A)=1-\frac{P(AB)}{P(A)}=1-\frac{0\cdot12}{0\cdot30}=0\cdot60.$$

Ex. 5. *Two urns contain respectively 2 red, 5 black, 7 green and 1 red, 4 black, 9 green balls. One ball is drawn from each urn. Find the probability that both the balls are of the same colour.*

Let A_1, A_2, A_3 be the event that both drawn balls are red, black and green respectively.

Then the required event is $A_1 \cup A_2 \cup A_3$, where A_1, A_2, A_3 are pairwise exclusive.

$\therefore P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$

Now, $P(A_1) = \frac{2}{14} \times \frac{1}{14} = \frac{2}{196}$

$P(A_2) = \frac{5}{14} \times \frac{4}{14} = \frac{20}{196}$

$P(A_3) = \frac{7}{14} \times \frac{9}{14} = \frac{63}{196}$

So the required probability is $\frac{2}{196} + \frac{20}{196} + \frac{63}{196} = \frac{85}{196}$.

Ex.6. *Two cards are drawn from a well-shuffled pack. Find the probability that at least one of them is diamond.*

Let A be the event that at least one of the drawn cards is diamond. Then $\bar{A}$ be the event that the drawn cards is not diamond.

$\therefore P(\bar{A}) = \frac{^{39}C_2}{^{52}C_2} = \frac{19}{34} \quad \therefore P(A) = 1 - P(\bar{A}) = 1 - \frac{19}{34} = \frac{15}{34}$

So, the required probability is $\frac{15}{34}$.

Ex.7. *A card is drawn at random from an ordinary deck of 52 playing cards. Find the probability that it is (i) an ace (ii) a heart (iii) a nine or a club (iv) neither a spade nor a ten.*

Let H, D, C, and S be the event that the drawn balls are hearts, diamonds, clubs and spades respectively. Also let us use the numbers 1, 2, 3, ... 10 for ace, two, three, .. ten respectively.

Then (i) P (an ace) $= P(1) = \frac{^4C_1}{^{52}C_1} = \frac{4}{52} = \frac{1}{13}$

(ii) P (a heart) $= P(H) = \frac{^{13}C_1}{^{52}C_1} = \frac{13}{52} = \frac{1}{4}$

(iii) P (a nine or a club)

$= P(9 \cup C) = P(9) + P(C) - P(9 \cap C)$

$= \frac{^4C_1}{^{52}C_1} + \frac{^{13}C_1}{^{52}C_1} - \frac{1}{52} = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$

(iv) P (neither a spade nor a ten)

$= P(S' \cap 10') = P\{(S \cup 10)'\} = 1 - P(S \cup 10)$

$= 1 - \{P(S) + P(10) - P(S \cap 10)\}$

$= 1 - \left(\frac{13}{52} + \frac{4}{52} - \frac{1}{52}\right) = 1 - \frac{4}{13} = \frac{9}{13}$.

Ex. 8. *Two dice are thrown n times in succession. What is the probability of obtaining double six at least once. Hence find the minimum number of throws so that the probability of obtaining double six at least once is less than* $\frac{1}{2}$.

Let A be the event that there is at least once double six in n throws of two dice in succession. Then $\bar{A}$ be the event that there is no double six in n throws of two dice. So

$P(\bar{A}) = \left(\frac{35}{36}\right)^n \quad \therefore P(A) = 1 - P(\bar{A}) = 1 - \left(\frac{35}{36}\right)^n$

So the required probability is $1 - \left(\frac{35}{36}\right)^n$.

Again when $P(A) < \frac{1}{2}$ then $1 - \left(\frac{35}{36}\right)^n < \frac{1}{2}$

or, $\left(\frac{35}{36}\right)^n > \frac{1}{2}$

or, $n \log\left(\frac{35}{36}\right) > -\log 2$

or, $n > \frac{\log 2}{\log 36 - \log 35} \simeq 24.6 \quad \therefore n \geq 25$

Hence the minimum number of throws is 2?

Ex. 9. *A can hit a target 4 times in 5 shots ; B 3 times in 4 shots ;C twice in 3 shots. They fire a target. What is the probability that at least two shots hit ?*

Let A_1, A_2, A_3 be the event that A, B, C hit the target respectively. Then

$$P(A_1)=\frac{4}{5} \qquad \therefore P(\overline{A_1})=1-\frac{4}{5}=\frac{1}{5}$$

$$P(A_2)=\frac{3}{4} \qquad \therefore P(\overline{A_2})=1-\frac{3}{4}=\frac{1}{4}$$

$$P(A_3)=\frac{2}{3} \qquad \therefore P(\overline{A_3})=1-\frac{2}{3}=\frac{1}{3}$$

For at least two hits, we may have

(i) A, B, C all hit the target, the probability for which is

$$P(A_1 A_2 A_3)=P(A_1)P(A_2)P(A_3)=\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}=\frac{2}{5}$$

(ii) A, B hit the target and C misses it, the probability for which is

$$P(A_1 A_2 \overline{A_3})=P(A_1)P(A_2)P(\overline{A_3})=\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{1}{3}=\frac{1}{5}$$

(iii) A, C hit the target and B misses it, the probability for which is

$$P(A_1\overline{A_2}A_3)=P(A_1)P(\overline{A_2})P(A_3)=\frac{4}{5}\cdot\frac{1}{4}\cdot\frac{2}{3}=\frac{2}{15}.$$

(iv) B, C hit the target and A misses it, the probability for which is

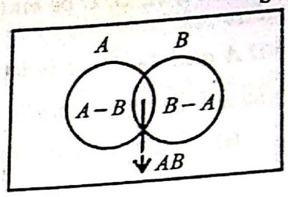
$$P(\overline{A_1}A_2A_3)=P(\overline{A_1})P(A_2)P(A_3)=\frac{1}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}=\frac{1}{10}.$$

So the required prob is $\frac{2}{5}+\frac{1}{5}+\frac{2}{15}+\frac{1}{10}=\frac{5}{6}.$

Ex. 10. *Show that the probability of occurrence of only one of the events A and B is* $P(A)+P(B)-2P(AB)$. [W.B.U.Tech 2006]

Let C be the event of occurence of only one of the events A and B. Then

$$C=(A\cup B)-(A\cap B)$$
$$=(A-B)\cup(B-A)$$



$\therefore P(C)=P(A-B)+P(B-A)$ ... (1)

[$\because A-B$ and $B-A$ are disjoint]

Now $A=(A-B)+AB$

$\therefore P(A)=P(A-B)+P(AB)$

$\therefore P(A-B)=P(A)-P(AB)$ ... (2)

Again $B=(B-A)+AB$

$\because P(B)=P(B-A)+P(AB)$ ... (3)

$\because P(B-A)=P(B)-P(AB)$

In virtue of (1), (2), and (3) we get

$$P(C)=P(A)-P(AB)+P(B)-P(AB)=P(A)+P(B)-2P(AB)$$

Ex. 11. *If A and B are independent events, then show that the following pairs are independent :*

(i) $\overline{A}$ and $\overline{B}$ [W.B.U.Tech 2002,2005,2007]

(ii) A and $\overline{B}$ (iii) $\overline{A}$ and B.

Since A and B are independent, so

$P(A\cap B)=P(A)P(B)$ ... (1)

(i) Now $P(\overline{A}\cap\overline{B})=P(\overline{A\cup B})$, by D' Morgan's law

$=1-P(A\cup B)=1-[P(A)+P(B)-P(A\cap B)]$

$=1-P(A)-P(B)+P(A\cap B)=1-P(A)-P(B)+P(A)\cdot P(B)$ by (1)

$=(1-P(A))(1-P(B))=P(\overline{A})P(\overline{B})$.

$\therefore \overline{A}$ and $\overline{B}$ are independent.

(ii) Again $A\cap B$ and $A\cap\overline{B}$ are mutually exclusive and

$(A\cap B)\cup(A\cap\overline{B})=A$

$\therefore P(A)=P(A\cap B)+P(A\cap\overline{B})$

$\therefore P(A\cap\overline{B})=P(A)-(A\cap B)=P(A)-P(A)\cdot P(B)$ by (1)

$=P(A)(1-P(B))=P(A)P(\overline{B})$

$\therefore A$ and $\overline{B}$ are independent.

(iii) Taking $B=(A\cap B)\cup(\overline{A}\cap B)$, we can prove the result as (ii)

Ex.12. *If A, B, C be mutually independent events, then prove that*

(i) A and $B+C$ are independent [W.B.U.Tech 2003]

(ii) $\bar{A},\bar{B},\bar{C}$ are mutually independent

(i) $P\{A(B+C)\}=P(AB+AC)=P(AB)+P(AC)-P(ABC)$

$$=P(A)P(B)+P(A)P(C)-P(A)P(BC)$$

[$\because A,B,C$ are mutually independent]

$$=P(A)[P(B)+P(C)-P(BC)]=P(A)P(B+C)$$

Hence A and $B+C$ are independent.

(ii) Now $\bar{A},\bar{B}$ are independent as A, B are independent (by previous example). Similarly $\bar{B},\bar{C}$ and $\bar{C},\bar{A}$ are independent. Also since A and $B+C$ are independent so $\bar{A}$ and $\overline{B+C}$ i.e., $\bar{A}$ and $\bar{B}\bar{C}$ are independent $\left[\because \overline{B+C}=\bar{B}\bar{C}\right]$

$$\therefore P\{\bar{A}(\bar{B}\bar{C})\}=P(\bar{A})P(\bar{B}\bar{C})=P(\bar{A})P(\bar{B})P(\bar{C})$$

Hence $\bar{A},\bar{B},\bar{C}$ are mutually independent.

Ex.13. *The face cards are removed from a pack of 52 cards. Then 4 cards are drawn one by one from the remaining 40 cards. What is the probability that 4 cards belong to different suits and different denomination.*

4 cards can be drawn out of 40 cards one by one is $40\times39\times38\times37$ ways. So the total numbers of possible outcomes is $40\times39\times38\times37$.

Now the number of ways in which 4 cards belong to different suits and different denominators is ${}^{10}C_1\times{}^{9}C_1\times{}^{8}C_1\times{}^{7}C_1=10\times9\times8\times7$.

So the total number of favourable cases is $10\times9\times8\times7$

Hence the required prob. $=\dfrac{10\times9\times8\times7}{40\times39\times38\times37}=\dfrac{21}{9139}$.

Ex.14. *A and B throw alternatively with a pair of dice. A wins if he throws 8 before B throws 5 and B wins if he throws 5 before A throws 8. Find the probability that A wins.*

[W.B.U.Tech 2006]

Here the total no. of event pts. is 36 in a trial of throwing two dice.

Let X, Y be the event that A throws 8 and B throws 5 with a pair of dice. Then

$$X=\{(3,5),(5,3),(2,6),(6,2),(4,4)\}\quad \therefore n(X)=5$$

$$Y=\{(2,3),(3,2),(4,1),(1,4)\}\quad \therefore n(Y)=4$$

$$\therefore P(X)=\frac{5}{36},\ P(Y)=\frac{4}{36}=\frac{1}{9}$$

$$\therefore P(\bar{X})=1-\frac{5}{36}=\frac{31}{36}$$

$$P(\bar{Y})=1-\frac{1}{9}=\frac{8}{9}.$$

So, the probability of A wins in the game is given by

$$P(X)+P(\bar{X}\bar{Y}X)+P(\bar{X}\bar{Y}\bar{X}\bar{Y}X)+\cdots$$

$=P(X)+P(\bar{X})P(\bar{Y})P(X)+\cdots$ [$\because \bar{X},\bar{Y},X$ are all independent]

$$=\frac{5}{36}+\frac{31}{36}\cdot\frac{8}{9}\cdot\frac{5}{36}+\left(\frac{31}{36}\right)^2\left(\frac{8}{9}\right)^2\cdot\frac{5}{36}+\cdots$$

$$=\frac{\frac{5}{36}}{1-\frac{8}{9}\cdot\frac{31}{36}}=\frac{45}{76}.$$

Ex.15. *A pack of 2n cards, n of which are red and another n are black. It is divided into two equal parts and a card is drawn from each. Find the probability that the cards drawn are of the same colour.*

Let $2n$ cards be divided into such way that first part contains k red cards and $(n-k)$ black cards where $k=1,2,\cdots,n-1$. Then the 2nd part contains $(n-k)$ red cards and k black cards.

Then the probability for both the drawn cards are of black is $\dfrac{n-k}{n}\cdot\dfrac{k}{n}$ and that of red is $\dfrac{k}{n}\cdot\dfrac{n-k}{n}$.

So the probability for both the drawn cards are of the same colour is

$$\frac{n-k}{n}\cdot\frac{k}{n}+\frac{k}{n}\cdot\frac{n-k}{n}=2\frac{(n-k)k}{n^2},\ k=1,2,\cdots,n-1$$

Hence the required probability $=\sum_{k=1}^{n-1}\frac{2(n-k)k}{n^2}$

$$=\frac{2}{n^2}\left(n\sum_{k=1}^{n-1}k-\sum_{k=1}^{n-1}k^2\right)=\frac{2}{n^2}\left\{n\cdot\frac{(n-1)n}{2}-\frac{n(n-1)(2n-1)}{6}\right\}=\frac{n^2-1}{3n}.$$

Ex.16. *An urn contains a white and b black balls, from which k balls are drawn one by one and they laid aside without noticeing their colours. Then one more ball is drawn. Find the probability that it is white.*

Here the total number of cases of drawing $(k+1)$ balls from $(a+b)$ balls is $(a+b)(a+b-1)\cdots(a+b-k)$.

Since the last drawn ball shall be white, we choose one white ball from a white balls in a ways. Then k balls can be drawn from the rest $(a+b-1)$ balls is $(a+b-1)(a+b-2)\cdots(a+b-k)$ways.

So the total number of favourable cases in which the last drawn ball is white in $(k+1)$ drawn is

$$(a+b-1)(a+b-2)\cdots(a+b-k)a.$$

Hence the required probability

$$=\frac{(a+b-1)(a+b-2)\cdots(a+b-k)\cdot a}{(a+b)(a+b-1)\cdots(a+b-k)}=\frac{a}{a+b}.$$

Ex.17. *15 new students are to be evenly distributed among 3 classes. Suppose that there are 3 whiz-kids among the fifteen. What is the probability that each class gets one whiz-kid and one class gets them all ?*

15 students can be evenly distributed among 3 classes in

$${}^{15}C_5\times{}^{10}C_5\times{}^{5}C_5\text{ ways }=\frac{(15)!}{(5!)^3}\text{ ways.}$$

So the total number of distributions is $\frac{(15)!}{(5!)^3}$.

(i) We can allot one whiz-kid to each of three classes in 3! ways. Then the other 12 students can be evenly distributed among 3 classes in

$${}^{12}C_4\times{}^{8}C_4\times{}^{4}C_4\text{ ways }=\frac{(12)!}{(4!)^3}\text{ ways.}$$

So the probability that each class gets one whiz-kid is

$$\frac{3!\frac{(12)!}{(4!)^3}}{\frac{(15)!}{(5!)^3}}=\frac{25}{91}.$$

(ii) We can allot all the three whiz-kids to one class in 3 ways and the rest 12 students in

$${}^{12}C_5\times{}^{7}C_5\times{}^{2}C_2=\frac{(12)!}{(5!)^2\ 2!}\text{ ways.}$$

So the prob. that one class gets all 3 whiz-kids is

$$\frac{3\times\frac{(12)!}{(5!)^2\ 2!}}{\frac{(15)!}{(5!)^3}}=\frac{6}{91}.$$

Ex.18. *An urn contains 4 white and 6 black balls. Two balls are successively drawn from the urn without replacement of the first ball. If the first ball is seen to be white, what is the probability that the 2nd ball is also white ?*

Let A_1 be the event that the first drawn ball is white and A_2 be the event that the second drawn ball is white. Then $A_1\cap A_2$ be the event that the both drawn ball is white.

$$\therefore\ P(A_1\cap A_2)=\frac{4}{10}\times\frac{3}{9}=\frac{2}{15}$$

Also $P(A_1)=\frac{4}{10}=\frac{2}{5}$.

Then by definition of conditional probability,

$$P(A_2/A_1)=\frac{P(A_1\cap A_2)}{P(A_1)}=\frac{\frac{2}{15}}{\frac{2}{5}}=\frac{1}{3}$$

Hence the required probability $=\frac{1}{3}$.

Ex.19. *There are two identical urns containing respectively 4 white, 3 red balls and 3 white, 7 red balls. An urn is chosen at random and a ball is drawn from it. Find the probability that the ball is white. If the ball drawn is white, what is the probability that it is from the first urn ?*

Let A_1, A_2 be the event that the ball is drawn from the first and second urn respectively. Clearly the events A_1, A_2 are mutually exclusive and exhaustive events.

$\therefore\ P(A_1)=P(A_2)=\frac{1}{2}$

Also let A be the event that the drawn ball is white. Then we have

$$P(A/A_1)=\frac{4}{7},\ P(A/A_2)=\frac{3}{10}$$

Now $P(A)=P(A_1)\cdot P(A/A_1)+P(A_2)\cdot P(A/A_2)$

$$=\frac{1}{2}\cdot\frac{4}{7}+\frac{1}{2}\cdot\frac{3}{10}=\frac{61}{140}.$$

So the probability that the drawn ball is white is $\frac{61}{140}$.

Now by Baye's theorem we have

$$P(A_1/A)=\frac{P(A_1)P(A/A_1)}{P(A)}=\frac{\frac{1}{2}\cdot\frac{4}{7}}{\frac{61}{140}}=\frac{40}{61}.$$

So the prob. that the white ball is drawn from the first urn is $\frac{40}{61}$.

Ex.20. *Two urns contain respectively 5 white, 7 black balls and 4 white, 2 black balls. One of the urns is selected by the toss of a fair coin and then 2 balls are drawn without replacement from the selected urn. If both balls drawn are white, what is the probability that the first urn is selected ?*

Let A_1, A_2 be the event that the ball is drawn from the first and second urn respectively. Then as the urn is chosen by coin-tossing, so we have $P(A_1)=P(A_2)=\frac{1}{2}$.

Now let A be the event that the drawn two balls are white. Then

$$P(A/A_1)=\frac{{}^5C_2}{{}^{12}C_2}=\frac{\frac{5\cdot 4}{2\cdot 1}}{\frac{12\cdot 11}{2\cdot 1}}=\frac{5}{33}$$

$$P(A/A_2)=\frac{{}^4C_2}{{}^6C_2}=\frac{\frac{4\cdot 3}{2\cdot 1}}{\frac{6\cdot 5}{2\cdot 1}}=\frac{2}{5}$$

$\therefore$ By Baye's Theorem $P(A)=P(A_1)P(A/A_1)+P(A_2)P(A/A_2)$

$$=\frac{1}{2}\cdot\frac{5}{33}+\frac{1}{2}\cdot\frac{2}{5}=\frac{91}{330}$$

Again by Baye's theorem, the required probability

$$P(A_1/A)=\frac{P(A_1)P(A/A_1)}{P(A)}=\frac{\frac{1}{2}\cdot\frac{5}{33}}{\frac{91}{330}}=\frac{25}{91}.$$

Ex.21. *A speaks the truth 3 out of 4 times and B 7 times out of 10. They agree in their statement that from a bag contaning 6 balls of different colours a white ball has been drawn. Find the probability that the statement is true.*

Let A_1 and A_2 be the events that the joint statement of A and B is true and false respectively

Then $P(A_1)=\frac{1}{6},\ P(A_2)=\frac{5}{6}$

Now let X be the event that both A and B agreed in their statement. Then

$$P(X/A_1)=\frac{3}{4}\times\frac{7}{10}=\frac{21}{40}.$$

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$$P(X/A_2)=\left(\frac{1}{4}\times\frac{1}{5}\right)\times\left(\frac{3}{10}\times\frac{1}{5}\right)=\frac{3}{1000}$$

$$\therefore\ P(X)=P(A_1)\cdot P(X/A_1)+P(A_2)\cdot P(X/A_2)$$

$$=\frac{1}{6}\times\frac{21}{40}+\frac{5}{6}\times\frac{3}{1000}=\frac{7}{80}+\frac{1}{400}=\frac{9}{100}$$

Hence the probability of the statement being true is

$$P(A_1/X)=\frac{P(A_1)P(X/A_1)}{P(A)} \text{ (by Baye's theorem)}$$

$$=\frac{\frac{1}{6}\times\frac{21}{40}}{\frac{9}{100}}=\frac{35}{36}.$$

Ex.22. *Assuming that each child is as likely to be a boy as it is to be a girl, what is the conditional probability that in a family of two children both are boys, given that (i) the older child is a boy (ii) at least one of the children is a boy ?*

Let A_1 and A_2 be the event that the older child is a boy and the younger child is a boy

Then $P(A_1)=P(A_2)=\frac{1}{2}$

Also $A_1 \cup A_2 =$ at least one of the children is a boy

and $A_1 \cap A_2 =$ both children are boys. Since A_1, A_2 are independent

$$\therefore\ P(A_1 \cap A_2)=P(A_1)P(A_2)=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

$$P(A_1 \cup A_2)=P(A_1)+P(A_2)-P(A_1 \cap A_2)=\frac{1}{2}+\frac{1}{2}-\frac{1}{4}=\frac{3}{4}.$$

(i) Thus the probability that both children are boys given that the older is a boy is

$$P((A_1 \cap A_2)/A_1)=\frac{P(A_1 \cap A_2 \cap A_1)}{P(A_1)}=\frac{P(A_1 \cap A_2)}{P(A_1)}=\frac{1/4}{1/2}=\frac{1}{2}.$$

(ii) The probability that both the children are boys, given that at least one of them is a boy, is

$$P\{(A_1 \cap A_2)/(A_1 \cup A_2)\}$$

$$=\frac{P[(A_1 \cap A_2)\cap(A_1 \cup A_2)]}{P(A_1 \cup A_2)}$$

$$=\frac{P(A_1 \cap A_2)}{P(A_1 \cup A_2)}=\frac{1/4}{3/4}=\frac{1}{3}.$$

Ex.23. *In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total of their output. 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probability that it was manufactured by machines A, B and C ?* [W.B.U.T. 2012,2003]

Let X_1, X_2 and X_3 be the events that a bolt is manufactured by A, B and C respectively and X be the event that a bolt is defective. Then

$$P(X_1)=\frac{25}{100}=\frac{1}{4},\ P(X_2)=\frac{35}{100}=\frac{7}{20},\ P(X_3)=\frac{40}{100}=\frac{2}{5}$$

$$\therefore\ P(X/X_1)=\frac{5}{100}=\frac{1}{20}$$

$$P(X/X_2)=\frac{4}{100}=\frac{1}{25},\ P(X/X_3)=\frac{2}{100}=\frac{1}{50}$$

$$\therefore\ P(X)=P(X_1)P(X/X_1)+P(X_2)P(X/X_2)+P(X_3)P(X/X_3)$$

$$=\frac{1}{4}\times\frac{1}{20}+\frac{7}{20}\times\frac{1}{25}+\frac{2}{5}\times\frac{1}{50}=\frac{1}{80}+\frac{7}{500}+\frac{1}{125}=\frac{69}{2000}$$

Then by Baye's theorem, we have

$$P(X_1/X)=\frac{P(X_1)P(X/X_1)}{P(X)}=\frac{\frac{1}{4}\cdot\frac{1}{20}}{\frac{69}{2000}}=\frac{25}{69}$$

Similarly $P(X_2/X)=\frac{28}{69},\ P(X_3/X)=\frac{16}{69}.$

So, the required probability that defective bolt was manufactured by machines A,B,C are $\frac{25}{69}, \frac{28}{69}, \frac{16}{69}$ respectively.

Ex.24. *An urn contains 10 white and 3 black balls, while another urn 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second arn and then a ball is drawn from the latter. What is the probability that it is a white ball ?*

Let A,B,C be the events that the drawn two balls from the first unr are both white, both black and one white and one black respectively. Then

$$P(A) = \frac{{}^{10}C_2}{{}^{13}C_2} = \frac{10\times 9}{13\times 12} = \frac{15}{26}$$

$$P(B) = \frac{{}^{3}C_2}{{}^{13}C_2} = \frac{3\times 2}{13\times 12} = \frac{1}{26}$$

$$P(C) = \frac{{}^{10}C_1 \times {}^{3}C_1}{{}^{13}C_2} = \frac{5}{13}$$

When two balls are transferred into the 2nd arn, it will contain either 5 white, 5 black balls. or, 3 white, 7 black balls or 4 white and 6 black balls according to the events A,B,C respectively.

Let W denote the event of drawing a white ball from the second urn.

Then $P(W/A) = \frac{5}{10} = \frac{1}{2}, P(W/B) = \frac{3}{10}$

$$P(W/C) = \frac{4}{10} = \frac{2}{5}.$$

So the required probability,

$$P(W) = P(A)\cdot P(W/A) + P(B)P(W/B) + P(C)\cdot P(W/C)$$

$$= \frac{15}{26}\cdot\frac{1}{2} + \frac{1}{26}\cdot\frac{3}{10} + \frac{5}{13}\cdot\frac{2}{5} = \frac{75+3+40}{260} = \frac{59}{130}.$$

Ex.25. *A student has to answer a multiple choice question with 5 alternatives. What is the probability that the student know the answer, given that he answered it correctly.*

Let B_1 and B_2 be the events that the student knew right answer and guesses the right answer respectively. Also let A be event that he gets the right answer. Again let p be the probability that he knew the corrcet answer.

$$\therefore \quad P(B_1) = p, P(B_2) = 1-p$$

Also $P(A/B_1) = 1, P(A/B_2) = \frac{1}{5}$

$\therefore$ By Baye's theorem,

$$P(B_1/A) = \frac{P(B_1)P(A/B_1)}{P(B_1)P(A/B_1) + P(B_2)P(A/B_2)}$$

$$= \frac{p-1}{p\cdot 1 + (1-p)\frac{1}{5}}$$

$$= \frac{5p}{4p+1}.$$

EXERCISES

[I] **SHORT ANSWER QUESTIONS**

1. Twenty balls in an urn are numbered 1 through 20. A blind folded contestant draws five balls from the win, with the order of the draw recorded. What is the probability that the number 3 ball was selected ?

   [Hints : The required probability $= \frac{{}^{19}C_4}{{}^{20}C_5} = \frac{1}{4}$.]

2. Prove that for any two events A,B with

   $$B \subset A, P(A\cap\overline{B}) = P(A) - P(B).$$

   [Hints : As $B \subset A$, so B and $A\cap\overline{B}$ are disjoint and $B\cup(A\cap\overline{B}) = A$

   $\therefore \quad P(B)\cup P(A\cap\overline{B}) = P(A)$.]