CHAPTER

5

Electromagnetic Field Theory

5.1 INTRODUCTION

In previous chapters we have discussed the fundamentals of electrostatics and magnetostatics. The static electric and magnetic fields are produced by charges at rest and steady current, respectively, and they are independent of each other.

In the present chapter, we discuss the time-varying fields. A time-varying electric field produces a magnetic field and a time-varying magnetic field produces an electric field. Michael Faraday gave the fundamental postulate for electromagnetic induction that relates the time-varying magnetic field with an electric field.

In this chapter, we deal with the interaction between electric and magnetic fields and obtain the four Maxwell's equations. Maxwell provided a mathematical theory that showed a close relationship between all electrical and magnetic phenomena, and form the foundation of electromagnetic theory. The combined Maxwell's equations yield wave equations and predict the existence of electromagnetic waves.

5.2 MAGNETIC FLUX

The magnetic flux linked with a surface held in a magnetic field is defined as the number of magnetic field lines crossing the surface normally.

The magnetic flux linked with a surface ds [Fig. 5.1] held inside a magnetic field is given by

$$d\varphi = B_n\, ds \qquad ...(5.1)$$

where B_n is the normal component of the magnetic field B along the direction of a surface element ds.

Again from Fig. 5.1, $B_n = B\cos\theta$

so,
$$d\varphi = B\cos\theta\, ds = \vec{B}\cdot\vec{ds} \qquad ...(5.2)$$

If the magnetic field B is uniform over the surface area S, then total flux

$$\varphi = \vec{B}\cdot\vec{S} \qquad ...(5.3)$$

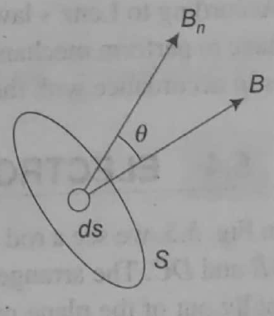

Fig. 5.1 Magnetic flux associated with a surface S.

If the normal is drawn in the direction of the magnetic field then the flux is taken as positive and if the normal is drawn opposite to the direction of the field then magnetic flux is taken as negative. The SI unit of magnetic flux is Weber (Wb) or Tesla m^2. Magnetic flux density of magnetic field induction is the magnetic flux per unit area, i.e., $B = \frac{\varphi}{A}$.

The SI unit of B is Tesla (T).

5.3 FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

Faraday observed that if a bar magnet moved either away or toward the axis of a conducting loop with no battery, then a current is produced in the loop. The current exists as long as the magnet is moving. The current flowed in the circuit when the flux through the circuit is altered. Faraday called this phenomenon *electromagnetic induction. Electromagnetic induction* is the process in which an emf is induced in a circuit placed in a magnetic field when the magnetic flux linked with the circuit changes. **Faraday's law tells us whenever the flux (φ) of magnetic field through the area bounded by a close conducting loop changes, an emf (ε) is produced in the loop.** Mathematically,

$$|\varepsilon| \propto \frac{d\varphi}{dt} \qquad ...(5.4)$$

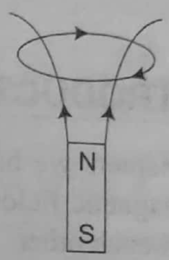

Fig. 5.2 The direction of induced emf according to Lenz's law.

The direction of the induced emf is provided by *Lenz's law*. This law states that *the direction of the induced emf is such that the magnetic flux associated with the current generated by it opposes the original change of flux causing emf.* To explain Lenz's law, we consider a magnet in the direction as shown in Fig. 5.2, i.e., towards the loop. As the magnet gets closer to the loop, the magnetic field increases and hence, the flux of the magnet field through the area of the loop increases. Thus increasing the magnetic flux through the coil, the induced current will flow in the direction shown, so that its own flux opposes the increase in the flux of the magnet. The induced current produces an induced emf. The induced emf is often called the back emf.

So, $$\varepsilon = -\frac{d\varphi}{dt} \qquad ...(5.5)$$

The direction of the current that produces a field towards the magnet can easily be obtained by using the right-hand thumb rule.

Lenz's Law and Conservation of Energy

According to Lenz's law, induced emf opposes the change that produces it. For change in magnetic flux, we have to perform mechanical work. So mechanical energy is converted into electrical energy. Thus, Lenz's law is in accordance with the law of conservation of energy.

5.4 ELECTROMOTIVE FORCE

In Fig. 5.3, we see a rod of length L sliding on a pair of parallel conducting tracks AB and DC. The arrangement is kept in a uniform magnetic field B which is normally out of the plane paper. Suppose, the rod moves parallel to the track with a velocity v making an angle θ with the magnetic flux (φ) link with the loop will change with time and we get an induced emf. Now applying Faraday's law, in unit time the area of the loop increases by the area of the parallelogram with sides L and v, the rate of change of flux is

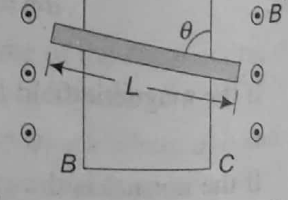

Fig. 5.3 Motional emf.

$$\frac{\partial\varphi}{\partial t} = BLv \sin\theta \qquad ...(5.6)$$

If the resistance of the loop is R, the current will be

$$I = \frac{Blv \sin\theta}{R}$$ in the clockwise direction.

The value of the induced emf will be $(\varepsilon) = BLv \sin\theta$ ...(5.7)

Note: *Fleming's right-hand rule* The direction of motional emf is given by either Lenz's law or Fleming's right-hand rule. **Fleming's right-hand rule states that if the thumb and the first two fingers of your right-hand are spread out in mutually perpendicular directions then the first finger points in the direction of the magnetic field and the thumb in the direction of motion of the conductor, and the central fingers points in the direction of the induced emf and thus the induced current.**

5.5 INTEGRAL FORM OF FARADAY'S LAW

If at any time t, the flux linked with the closed coil is φ then according to Faraday's law, the induced emf in the coil

$$\varepsilon = -\frac{d\varphi}{dt} \qquad ...(5.8)$$

If $\vec{E}$ be the field induced in space then the induced emf ε around the closed path C is given by integration of $\vec{E}$ and can be written as

$$\varepsilon = \oint_C \vec{E}\cdot\vec{dl} \qquad ...(5.9)$$

$$\therefore \quad \oint_C \vec{E}\cdot\vec{dl} = -\frac{d\varphi}{dt} \qquad ...(5.10)$$

The total flux through the circuit is equal to the integral of normal component of flux density $\vec{B}$ over the surface bounded by the circuit.

$$\varphi = \int_S \vec{B}\cdot\vec{dS} \qquad ...(5.11)$$

$$\therefore \quad \oint_C \vec{E}\cdot\vec{dl} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} \qquad ...(5.12)$$

This is the integral form of Faraday's law of electromagnetic induction. Here, we use $\frac{\partial\vec{B}}{\partial t}$ instead of $\frac{d\vec{B}}{dt}$, because $\vec{B}$ is a function of both position and time.

5.5.1 Differential Form of Faraday's Law

Using Stoke's theorem,

$$\oint_C \vec{E}\cdot\vec{dl} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{S} \qquad ...(5.13)$$

Now, from Eq. (5.12), we have

$$\int_S (\vec{\nabla}\times\vec{E})\cdot\vec{dS} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot\vec{dS} \qquad ...(5.14)$$

or, $$\int_S \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right) \cdot \vec{dS} = 0 \quad \text{...(5.15)}$$

Eq. (5.15) must hold for any arbitrary surface S

so, $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{...(5.16)}$$

This is the differential form of Faraday's law of electromagnetic induction.

The sources of the electromagnetic field are of two kinds. First one is the electrostatics field in which energy is conserved during a cyclic process and such a field has no curl. The second one is the magnetic field in which energy is transferred in a cyclic process and such a field is specified by the curl sources and has no divergence.

Taking divergence of Eq. (5.16), we have

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B}) \quad \text{...(5.17)}$$

Since divergence of any curl is always zero, this is possible if

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{...(5.18)}$$

So, $\vec{B}$ is solenoidal.

Thus, Faraday's law gives two important results:

(a) The electric field is no longer a conservative field when the magnetic field varies with time.

(b) Magnetic free pole does not exist. All magnetic poles occur in pairs.

The time-varying electric and magnetic fields are thus interrelated and these two fields combine to form a single field known as *electromagnetic field*.

According to **Helmholtz theorem, any vector field is uniquely determined if its divergence and curl sources are given.** Electromagnetic fields have both types of sources.

5.6 DISPLACEMENT CURRENT

The concept of displacement current was introduced by Maxwell to account for production of magnetic field in empty space. The current for production of magnetic field is called *displacement current*. The current carried by conductors due to flow of charges is called *conduction current*. In empty space, conduction current is zero.

The displacement current is different from the conduction current in the sense that the former exists only when the electric field varies with time. For steady electric field in a conducting wire, the displacement current is zero. The current arising due to time-varying electric field between the plates of a capacitor is known as the displacement current.

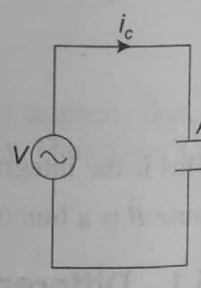

Fig. 5.4 **Charging of a capacitor.**

Figure 5.4 shows a circuit connecting a time-varying voltage source V to a pure capacitor (C). The current through a capacitor is called displacement current. Actually the displacement current does not flow through the capacitor, the displacement is only an apparent current representing the rate of transport of charge from one plate to another. When a voltage is applied to a capacitor the current through it is

$$I = \frac{dQ}{dt} = C\frac{dV}{dt} \quad \text{... (5.19)}$$

where Q is the instantaneous charge, equal to CV. Again we know that for a parallel-plate capacitor

$$C = \frac{\varepsilon_0 A}{d} \quad \text{... (5.20)}$$

where A is the area of the parallel plate, d is the separation between the plate and ε_0 is the free space permittivity.

So, $$I = \frac{\varepsilon_0 A}{d}\frac{dV}{dt} \quad \text{... (5.21)}$$

The relation between the electric field (ε_0) in the capacitor with potential

$$E = \frac{V}{d} \quad \text{... (5.22)}$$

Now from Eq. (5.21)

$$I = \varepsilon_0 A \frac{dE}{dt} \quad \text{... (5.23)}$$

or, $$\frac{I}{A} = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{\partial D}{\partial t} \quad \text{... (5.24)}$$

where $D = \varepsilon_0 E$ is known as electric displacement.

Now $\frac{I}{A}$ gives the current density (J_d)

So, $$J_d = \frac{\partial D}{\partial t} \quad \text{... (5.25)}$$

J_d is called the displacement current density. The displacement current $\frac{\partial D}{\partial t}$ is zero outside the plates but has a definite value between the plates. This definite value is exactly equal to the value of conduction current outside the plates.

5.7 MODIFIED AMPERE'S LAW

The differential form of Ampere's circuital law is

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} \quad \text{... (5.26)}$$

which is applied to a steady magnetic field only. Since the divergence of any curl is zero, we have from Eq. (5.26)

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = \mu_o \vec{\nabla} \cdot \vec{J} = 0 \quad \text{... (5.27)}$$

or, $$\vec{\nabla} \cdot \vec{J} = 0$$

However, the equation of continuity

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad \text{... (5.28)}$$

shows that Eq. (5.28) is true only if

$$\frac{\partial \rho}{\partial t} = 0, \rho = \text{constant of time.}$$

So, Ampere's circuital law is valid only for static charge density. Ampere's circuital law in case of time-varying field does not hold good. Maxwell added another term to Ampere's law and ensured that it is valid for a time-varying field.

Let us add an unknown M to Eq. (5.26)

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{M}) \quad ...(5.29)$$

Taking divergence on both sides of Eq. (5.29), we have

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \cdot (\vec{J} + \vec{M}) = 0$$

or,

$$\vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{M} = 0$$

or,

$$\vec{\nabla} \cdot \vec{M} = -\vec{\nabla} \cdot \vec{J} = \frac{\partial \rho}{\partial t} \quad \text{from Eq. (5.28)} \quad ...(5.30)$$

Again, from Gauss' law in electrostatics

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

or,

$$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E} \quad ...(5.31)$$

Now from Eqs. (5.30) and (5.31),

$$\vec{\nabla} \cdot \vec{M} = \varepsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E})$$

we get

$$\vec{M} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\partial \vec{D}}{\partial t} \quad ...(5.32)$$

After modification, Ampere's law becomes

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \quad ...(5.33)$$

$$= \mu_0 (\vec{J} + \vec{J}_d) \quad ...(5.34)$$

The term $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ is known as the displacement current density.

Thus, displacement current density is entirely different from conduction current density. Displacement current is taken as a current only because it produces a magnetic field. Even in perfect vacuum, displacement current exists although there is no charge of any type. The presence of the term $\vec{J}_d$ enabled Maxwell to predict that an electromagnetic field should propagate through space in form of waves.

5.8 CONTINUITY PROPERTY OF CURRENT

From the conservation of charge, it is necessary that the total current density $(\vec{J} + \vec{J}_d)$ should obey the continuity equation. The individual term may or may not be continuous. As an example, in the case of charging of a parallel-plate capacitor, there is a conduction current in the region outside the plates of the capacitor. In empty space between the plates of the capacitor, conduction current is zero, but displacement current has a definite value between the plates. Thus, the individual terms are discontinuous but the sum of conduction current and displacement has the same value both inside and outside the plates.

5.9 MAXWELL'S EQUATIONS

5.9.1 Maxwell's Equations in Differential Form

Maxwell's equations represent the four basic laws of electricity and magnetism. These four laws are (i) Gauss' law in electrostatics, (ii) Gauss' law in magnetostatics, (iii) Faraday's law of electromagnetic induction, and (iv) Ampere's law with Maxwell's correction. All the four Maxwell's equations along with their salient features are being discussed here.

(i) Maxwell's first equation

The first equation represents Gauss' law in electrostatics, which may be written as

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \qquad ...(5.35)$$

This is a time-independent steady-state equation which relates the spatial variation or divergence of an electric field with charge density. This relation is true both for stationary and moving charges.

(ii) Maxwell's second equation

The second Maxwell's equation represents Gauss' law in magnetostatics. Mathematically,

$$\vec{\nabla}\cdot\vec{B}=0 \qquad ...(5.36)$$

Equation (5.36) states that an isolated magnetic pole does not exist. This is also a time-independent or steady-state equation which gives the spatial variation or divergence of magnetic induction.

(iii) Maxwell's third equation

The third Maxwell's equation represents Faraday's law of electromagnetic induction. Mathematically,

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad ...(5.37)$$

This is a time-dependent equation. Equation (5.37) shows that a time-varying magnetic field acts as a source of electric field. It relates the spatial variation of electric field with time variation of a magnetic field.

(iv) Maxwell's fourth equation

The fourth Maxwell's equation represents modified Ampere's (Ampere's law with Maxwell's correction). Mathematically,

$$\vec{\nabla}\times\vec{B}=\mu_0\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right) \qquad ....(5.38)$$

$$=\mu_0\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

This is a time-independent equation. Equation (5.38) shows that a time-varying electric field acts as a source of magnetic field. The equation relates the spatial variation of a magnetic field with conduction current density and displacement current density.

Maxwell's equations are the basic equations for electromagnetism.

5.9.2 Maxwell's Equation in Integral Form

The four Maxwell's equations (5.35, 5.36, 5.37, 5.38) can be converted into an integral form.

(i) Maxwell's first equation

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$$

By taking the volume integral of both sides of $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ over a volume V with a closed surface S and using divergence theorem,

$$\left.\begin{aligned} \int_V (\vec{\nabla} \cdot \vec{E})\, dV &= \frac{1}{\varepsilon_0} \int_V \rho\, dV \\ \text{or,} \quad \oiint_S \vec{E} \cdot \vec{dS} &= \frac{q}{\varepsilon_0} \text{ where } q = \int_V \rho\, dV \end{aligned}\right\} \qquad \text{...(5.39)}$$

We can infer from this equation that the electric lines of force do not constitute continuous close path.

(ii) Maxwell's second equation

$$\vec{\nabla} \cdot \vec{B} = 0$$

By taking the volume integral of both sides of $\vec{\nabla} \cdot \vec{B} = 0$ over a volume V with a closed surface S and using divergence theorem,

$$\left.\begin{aligned} \int_V (\vec{\nabla} \cdot \vec{B})\, dV &= 0 \\ \text{or,} \quad \oiint_S \vec{B} \cdot \vec{dS} &= 0 \end{aligned}\right\} \qquad \text{...(5.40)}$$

We can infer from this equation that there is no magnetic flux sources, and magnetic flux lines always close upon themselves. It is the law of conservation of magnetic flux, i.e., magnetic monopole does not exist.

(iii) Maxwell's third equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

By taking the surface integral of both sides of $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ over an open surface S with a contour C and applying Stoke's theorem

$$\left.\begin{aligned} \int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{dS} &= -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{dS} \\ \text{or,} \quad \oint_C \vec{E} \cdot \vec{dl} &= -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \vec{dS} \end{aligned}\right\} \qquad \text{...(5.41)}$$

Equation (5.41) is the expression of Faraday's law, which states that a changing magnetic field $\vec{B}$ produces an electric field $\vec{E}$ such that line integral of $\vec{E}$ around a closed curve equals the negative rate of change of magnetic flux of a surface bounded by C.

(iv) Maxwell's fourth equation

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

By taking the surface integral of both sides of $\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$ over an open surface S with a contour C and applying Stoke's theorem

$$\left.\begin{aligned} \int_S (\vec{\nabla} \times \vec{B}) \cdot \vec{dS} &= \mu_0 \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{dS} \\ \text{or,} \quad \oint_C \vec{B} \cdot \vec{dl} &= \mu_0 \left[\int_S \vec{J} \cdot \vec{dS} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot \vec{dS} \right] \end{aligned}\right\} \qquad \text{...(5.42)}$$

Equation (5.42) is the expression of modified Ampere's circuital law which states that the circulation of the magnetic field intensity around any closed path is equal to μ_o times the sum of conduction current and displacement current.

5.9.3 Physical Significance of Maxwell's Equations

(i) The first equation

The first Maxwell's equation known as Gauss' law in electrostatics, states that "The total electric flux through any closed surface is equal to the total charge enclosed by the surface divided by free space permittivity." Mathematically,

$$\oint_S \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0} \quad \text{[integral form]}$$

or,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{[differential form]}$$

If a closed surface does not include any charge, then obviously the total flux over the surface is zero. The equation represents that the electric lines of force are not closed lines. The electric field lines start on positive charges (sources) and end on negative charges (sink). Gauss' law is valid not only for static charges but also for charges in motion.

(ii) The second equation

The second Maxwell's equation is known as Gauss' law in magnetostatics and states that there are no magnetic flux sources and magnetic flux lines always close upon themselves. Mathematically,

$$\oint_S \vec{B} \cdot \vec{dS} = 0 \quad \text{[integral form]}$$

or,

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{[differential form]}$$

Magnetic field is solenoidal means, it has no sources or sinks. The total magnetic flux through a closed surface is equal to zero, i.e., the magnetic flux entering into the volume is equal to the magnetic flux leaving the volume. The magnetic monopole does not exist, magnetic poles exist in pairs.

(iii) The third equation

The third Maxwell's equation is known as Faraday's law of electromagnetic induction and states that the line integral of $\vec{E}$ around a closed circuit is equal to the negative rate of change of magnetic flux linking the circuit. Mathematically,

$$\oint_C \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t}\int_S \vec{B} \cdot \vec{dS} \quad \text{[integral form]}$$

or,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{[differential form]}$$

It shows that the time variation of a magnetic field generates the electric field. So Faraday's law of electromagnetic induction shows how the electric and magnetic fields are interrelated. The time-varying magnetic fields acts as a source of electric field.

(iv) The fourth equation

The fourth Maxwell's equation is known as Ampere's law with Maxwell's correction. It relates the spatial variation of magnetic field with conduction current density and displacement current density. Mathematically,

$$\oint_C \vec{B} \cdot \vec{dl} = \mu_0 \left[\int_S \vec{J} \cdot \vec{dS} + \frac{\partial}{\partial t}\int_S \vec{D} \cdot \vec{dS}\right] \quad \text{[integral form]}$$

or, $$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right]$$ [differential form]

We see that both the conduction current density and displacement current density are two possible sources of magnetic field. The term $\frac{\partial \vec{D}}{\partial t}$ which arises from the variation of electric displacement with time is known as displacement current density and its introduction in $\vec{\nabla} \times \vec{B}$ equation was one of the major contributions of Maxwell.

Thus we see that the interrelation between electric and magnetic field generates a single field known as electromagnetic field which gives rise to propagation of electromagnetic waves.

5.9.4 Maxwell's Equations in Free Space

In free space, the following physical conditions are satisfied

$$\rho = 0, J = \sigma E = 0 \text{ as conductivity } \sigma = 0$$

Under these conditions, Maxwell's equations take the following form:

(i) $\vec{\nabla} \cdot \vec{E} = 0$ ...(5.43)

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ ...(5.44)

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ...(5.45)

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$ ...(5.46)

5.10 WAVE EQUATIONS IN FREE SPACE

The time-varying electric and magnetic fields give rise to the phenomenon of electromagnetic wave propagation. Here we deduce the relevant wave equation.

In free space, Maxwell's equations are:

(i) $\vec{\nabla} \cdot \vec{E} = 0$

(ii) $\vec{\nabla} \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$

$$= \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (\because \vec{D} = \varepsilon_0 \vec{E})$$

- *For Electric Field*

Take curl on both sides of Eq. (iii)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$$

or, $$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$ ...(5.47)

But $$\vec{\nabla} \cdot \vec{E} = 0 \text{ and } \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

So, from Eq. (5.47)

$$-\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

or, $$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad ...(5.48)$$

This is the three-dimensional wave equation for the vector field $\vec{E}$ in free space.

- *For Magnetic Field*

Taking curl on both sides of Eq. (iv)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \vec{\nabla} \times \left(\frac{\partial \vec{E}}{\partial t}\right)$$

or, $$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) \qquad ...(5.49)$$

But $$\vec{\nabla} \cdot \vec{B} = 0 \text{ and } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

So, from Eq. (5.49)

$$-\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

or, $$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \qquad ...(5.50)$$

This is the three-dimensional wave equation for the vector field $\vec{B}$ in free space.

Thus the fields satisfy the same formal partial differential equations for waves

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \qquad ...(5.51)$$

The three-dimensional wave function ψ depends on x, y, z, t and c is the velocity of the wave.

Thus, we conclude that the field vectors $\vec{E}$ and $\vec{B}$ are propagated in free space as waves whose speed is given by

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad [\text{here, } \mu_0 = 4\pi \times 10^{-7} \text{ weber/amp}, \varepsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/\text{Nm}^2]$$

$$= 2.9978 \times 10^8 \text{m/s}$$

which is the velocity of light in free space. So, we may conclude that light waves are electromagnetic waves.

From Eqs (5.49) and (5.50), we see that the electric field and the magnetic field satisfy the same wave equation, so they oscillate exactly in the same phase.

5.11 TRANSVERSE NATURE OF ELECTROMAGNETIC WAVE

The electromagnetic wave equations in free space are:

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \text{ and } \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

We assume that the plane wave fields are of the form,

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad ...(5.52)$$

and

$$\vec{B}(\vec{r}, t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad ...(5.53)$$

where $\vec{E}_0$ and $\vec{B}_0$ are vector constant in time and $\vec{k}$ is the propagation vector.

From Maxwell's first equation in free space

$$\vec{\nabla} \cdot \vec{E} = 0$$

or,

$$\vec{\nabla} \cdot \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} = 0$$

which gives the relation

$$\vec{k} \cdot \vec{E} = 0 \quad ...(5.54)$$

where $\vec{\nabla}$ is the operator, after operation on $\vec{E}$ its value (ik). Similarly, from Maxwell's second equation in free space

$$\vec{\nabla} . \vec{B} = 0$$

or,

$$\vec{\nabla} \cdot \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

which gives

$$\vec{k} \cdot \vec{B} = 0 \quad ...(5.55)$$

Equations (5.54) and (5.55) show that both $\vec{E}$ and $\vec{B}$ are perpendicular to the propagation vector $\vec{k}$.

Again from Maxwell's third equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

or,

$$\vec{\nabla} \times \vec{E}_0 e^{i(\vec{k}.\vec{r} - \omega t)} = -\frac{\partial}{\partial t} \vec{B}_0 e^{i(\vec{k}.\vec{r} - \omega t)}$$

or,

$$i\,\vec{k} \times \vec{E} = -(-i\omega)\,\vec{B}$$

Therefore,

$$\vec{k} \times \vec{E} = \omega \vec{B} \quad ...(5.56)$$

So, $\vec{B}$ is perpendicular to both $\vec{k}$ and $\vec{E}$. Therefore, from Eqs (5.54), (5.55) and (5.56), $\vec{E}$, $\vec{B}$ and $\vec{k}$ are mutually perpendicular to each other.

Now, considering only the magnitude of E, B and k, we have

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c \quad ...(5.57)$$

where c is the velocity of light.

Thus, we may conclude that (i) electromagnetic waves travel with the speed of light, and (ii) electromagnetic waves are transverse waves. The ratio of electric to magnetic field in electromagnetic waves equal to the speed of light.

From Fig. 5.5 (a, b) we see that both electric field and magnetic field are perpendicular to the direction of motion of the wave. Thus an electromagnetic wave is a transverse wave.

Figure 5.5 (b) is a graphical representation of the sinusoidal wave showing $\vec{E}$ and $\vec{B}$ vectors. At any fixed point, the electrical and magnetic field vectors vary sinusoidally with time. The energy flow in the +ve x direction (i.e., $\vec{E} \times \vec{B}$). Radio waves, light waves, x-rays, γ-rays are examples of electromagnetic waves.

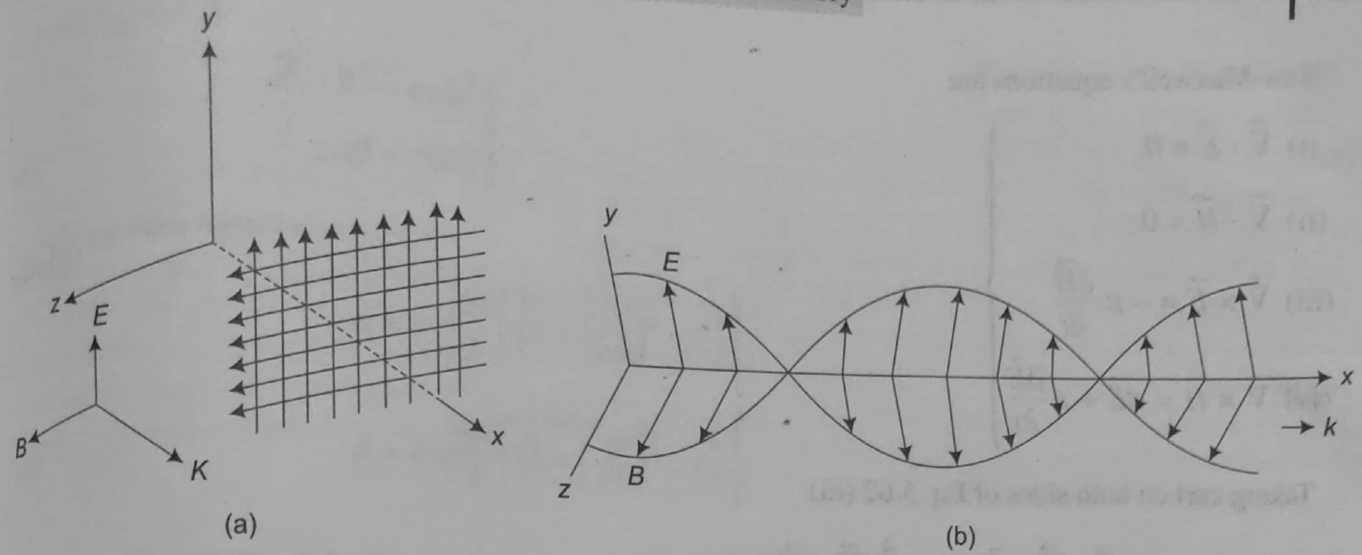

Fig. 5.5 **(a) Transverse nature of electromagnetic waves. (b) Representation of the sinusoidal wave showing $\vec{E}$ and $\vec{B}$ vectors.**

5.12 POTENTIALS OF ELECTROMAGNETIC FIELD

We know that magnetic field is solenoidal, i.e., $\vec{\nabla} \cdot \vec{B} = 0$, so $\vec{B}$, in terms of vector potential $\vec{B} = \vec{\nabla} \times \vec{A}$.

Again from Faraday's laws of electromagnetic induction

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) \quad [\text{Taking } \vec{B} = \vec{\nabla} \times \vec{A}]$$

Therefore,
$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \quad ...(5.58)$$

Since the curl of the gradient of a scalar function is zero, so,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\varphi \quad ...(5.59)$$

or,
$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \quad ...(5.60)$$

Here, $\vec{A}$ and φ are magnetic vector potential and scalar potential. The electric field $\vec{E}$ and magnetic field $\vec{B}$ can be found if we determine $\vec{A}$ and φ.

5.13 ELECTROMAGNETIC WAVES IN A CHARGE-FREE CONDUCTING MEDIA AND SKIN DEPTH

Inside the conductor $\rho = 0$. Because there is no permanent charge inside the conductor, it can only be redistributed on the surface of the conductor. The propagation of EM waves through conducting, homogeneous, isotropic medium of permittivity ε, permeability μ and conductivity σ hold the relations

$$\left.\begin{aligned} \vec{D} &= \varepsilon \vec{E} \\ \vec{B} &= \mu \vec{H} \\ \vec{J} &= \sigma \vec{E} \end{aligned}\right\} \quad ...(5.61)$$

Now Maxwell's equations are:

$$\left.\begin{aligned}
&\text{(i)}\ \vec{\nabla}\cdot\vec{E}=0\\
&\text{(ii)}\ \vec{\nabla}\cdot\vec{H}=0\\
&\text{(iii)}\ \vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}\\
&\text{(iv)}\ \vec{\nabla}\times\vec{H}=\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}
\end{aligned}\right\} \quad ...(5.62)$$

Taking curl on both sides of Eq. 5.62 (iii)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E}=-\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

or,

$$-\nabla^2\vec{E}=-\mu\sigma\frac{\partial\vec{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{[Using Eqs. (i) and (iv)]}$$

After arranging, we get

$$\nabla^2\vec{E}-\mu\sigma\frac{\partial\vec{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}=0 \quad ...(5.63)$$

Similarly taking curl of Eq. (5.62) (iv), we get

$$\nabla^2\vec{H}-\mu\sigma\frac{\partial\vec{H}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}=0 \quad ...(5.64)$$

Both Eqs. (5.63) and (5.64) are known as Helmholtz equations for electric field and magnetic field.

Let us now find the plane wave solutions of Maxwell's equations for a conducting medium. We assume that the field vector $\vec{E}(\vec{r}, t)$ and $\vec{H}(\vec{r}, t)$ vary harmonically with time,

i.e.,

$$\left.\begin{aligned}
&E(\vec{r}, t)=E_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\\
\text{and}\quad &H(\vec{r}, t)=H_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}
\end{aligned}\right\} \quad ...(5.65)$$

Substituting Eq. (5.65) in Eq. (5.63)

$$-k^2E(r, t)+\varepsilon\mu\omega^2\vec{E}(r, t)+i\sigma\mu\omega\vec{E}(r, t)=0$$

i.e.,

$$k^2=\varepsilon\mu\omega^2\left(1+\frac{i\sigma}{\varepsilon\omega}\right) \quad ...(5.66)$$

The propagation vector is complex, and may be expressed as

$$k=\alpha+i\beta$$

$$=\left[\varepsilon\mu\omega^2\left(1+\frac{i\sigma}{\varepsilon\omega}\right)\right]^{1/2} \quad ...(5.67)$$

From Eq. (5.67)

$$\left.\begin{aligned}\alpha^2-\beta^2&=\varepsilon\mu\omega^2\\ 2\alpha\beta&=\sigma\mu\omega\end{aligned}\right\}\qquad ...(5.68)$$

and

Solving these equations,

$$\left.\begin{aligned}\alpha&=\omega\sqrt{\frac{\varepsilon\mu}{2}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]}\\ \beta&=\omega\sqrt{\frac{\varepsilon\mu}{2}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]}\end{aligned}\right\}\qquad ...(5.69)$$

and

For good conductor, if the frequency is not too high, $\frac{\sigma}{\varepsilon\omega} >> 1$.

$\therefore$ propagation vector,

$$k=\sqrt{\mu\sigma\omega i}=\sqrt{\mu\sigma\omega}\,(\cos 45° + i\sin 45°)$$

$$=\sqrt{\mu\sigma\omega}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)$$

$$=\alpha+i\beta$$

So

$$\alpha=\beta=\sqrt{\frac{\mu\sigma\omega}{2}}\qquad ...(5.70)$$

$$=\frac{1}{\delta}\quad\text{where}\quad \delta=\sqrt{\frac{2}{\mu\sigma\omega}}$$

Since $k=\alpha+i\beta$, Eq. (5.65) can be written as

$$\left.\begin{aligned}\vec{E}&=\vec{E}_0\,e^{-\beta r}\,e^{i(\alpha r-\omega t)}\\ \vec{H}&=\vec{H}_0\,e^{-\beta r}\,e^{i(\alpha r-\omega t)}\end{aligned}\right\}\qquad ...(5.71)$$

These equation indicate that a plane wave cannot propagate in a conducting medium without attenuation.

5.14 SKIN DEPTH OR DEPTH OF PENETRATION (δ)

From Eq. (5.71)

$$\vec{E}=\vec{E}_0\,e^{-r/\delta}\,e^{i\left(\frac{r}{\delta}-\omega t\right)}$$

At $r=\delta$ the amplitude decreases in magnitude to $\frac{1}{e}$ times its value at the surface which is called **skin depth** or **penetration depth** [Fig. 5.6].

The phenomenon that the alternating fields and hence currents are confined within a small region of a conducting medium inside the surface is known as the ***skin effect*** and the small distance from the surface of the conductor is known as ***skin depth***.

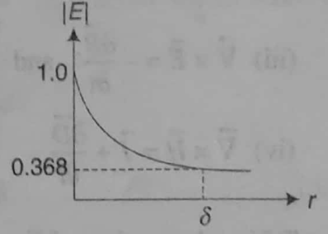

Fig. 5.6 Variation of intensity with distance in a change free conducting medium.

Significance of skin depth The attenuation of electromagnetic waves in a conduction medium is due to the conversion of the electromagnetic energy of the wave into Joule's heat because the electric field of the wave induces currents in a conducting medium which produce the heat. The energy in the form of electromagnetic waves carried by a current propagates in the space surrounding the conductors that partially penetrates the conductor surface to maintain the motion of the electrons. So, the current is maintained in the parts of the conductor which receive electromagnetic energy from the surrounding space. This energy can penetrate the conductor only by such small distance, called the skin depth (δ) and current may exist near the surface of the conductor only within the limits of this depth. The skin depth in copper for 1 mm microwaves is 10^{-4} m and for visible light 10^{-6} m. A poor conductor can be made a good conductor with a thin coating of copper or silver. A silver coating on a piece of glass may be an excellent conductor at microwave frequency. The performance of gold and gold-coated brass waveguides will be same if the coating thickness is equal to skin depth. This method is useful to reduce the cost of the material.

5.14.1 Electromagnetic Shielding

We may enclose a volume with a thin layer of good conductor to act as an electromagnetic shield. Depending on the application, the electromagnetic shield may be necessary to prevent waves from radiating out of the shielded volume or to prevent waves from penetrating into the shielded volume.

5.14.2 Phase Velocity

The phase velocity in the conducting medium is given by

$$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}} \qquad ...(5.72)$$

In the conductor, α and β are large. The wave attenuates greatly as it progresses and the phase shift per unit length is also large. The phase velocity of the wave is small. Phase velocity depends on frequency, so dispersion takes place in the conducting medium.

5.15 ELECTROMAGNETIC ENERGY FLOW AND POYNTING VECTOR

The electromagnetic waves carry energy when they propagate and there is an energy density associated with both the electric and magnetic fields. As electromagnetic waves propagate through the space from the source to the receiver, there exists a simple and direct relationship between the rate of energy transfer and the amplitude of electric and magnetic field strengths. The relation may be obtained from Maxwell's equations

$$\left.\begin{aligned}
&\text{(i)}\quad \vec{\nabla}\cdot\vec{D} = 0\\
&\text{(ii)}\quad \vec{\nabla}\cdot\vec{B} = 0\\
&\text{(iii)}\quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\quad\text{and}\\
&\text{(iv)}\quad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}
\end{aligned}\right\} \qquad ...(5.73)$$

Taking dot product of Eqs. (5.73) (iii) and (iv) with $\vec{H}$ and $\vec{E}$ respectively, we have

$$\vec{H} \cdot \vec{\nabla} \times \vec{E} = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \quad ...(5.74)$$

and
$$\vec{E} \cdot \vec{\nabla} \times \vec{H} = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad ...(5.75)$$

Form Eqs. (5.74) and (5.75)

$$\vec{H} \cdot \vec{\nabla} \times \vec{E} - \vec{E} \cdot \vec{\nabla} \times \vec{H} = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad ...(5.76)$$

$$= -\left(\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right) - \vec{E} \cdot \vec{J}$$

or,
$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\left(\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right) - \vec{E} \cdot \vec{J} \quad ...(5.77)$$

$$[\because \vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \vec{\nabla} \times \vec{A} - \vec{A} \cdot \vec{\nabla} \times \vec{B}]$$

From relations $\vec{B} = \mu \vec{H}$ and $\vec{D} = \varepsilon \vec{E}$, we have

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\left[\vec{H} \cdot \frac{\partial}{\partial t}(\mu \vec{H}) + \vec{E} \cdot \frac{\partial}{\partial t}(\varepsilon \vec{E})\right] - \vec{E} \cdot \vec{J}$$

$$= -\left[\frac{1}{2}\mu \frac{\partial}{\partial t}(H^2) + \frac{1}{2}\varepsilon \frac{\partial}{\partial t}(E^2)\right] - \vec{E} \cdot \vec{J}$$

$$= -\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{H} \cdot \vec{B}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E} \cdot \vec{D}\right)\right] - \vec{E} \cdot \vec{J}$$

or,
$$\vec{E} \cdot \vec{J} = \frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H} \cdot \vec{B} + \vec{E} \cdot \vec{D})\right] - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) \quad ...(5.78)$$

Integrating over the volume V

$$\int_V (\vec{E} \cdot \vec{J})\, dV = -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\vec{H} \cdot \vec{B} + \vec{E} \cdot \vec{D})\right\} dV - \int_V \vec{\nabla} \cdot (\vec{E} \times \vec{H})\, dV$$

or,
$$\int_V (\vec{E} \cdot \vec{J})\, dV = -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\vec{H} \cdot \vec{B} + \vec{E} \cdot \vec{D})\right\} dV - \oint_s (\vec{E} \times \vec{H}) \cdot d\vec{S} \quad ...(5.79)$$

$$\left[\because \int_V \vec{\nabla} \cdot (\vec{E} \times \vec{H})\, dV = \oint_s (\vec{E} \times \vec{H}) \cdot d\vec{S}\right]$$

or,
$$\oint (\vec{E} \times \vec{H}) \cdot d\vec{S} = -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\vec{H} \cdot \vec{B} + \vec{E} \cdot \vec{D})\right\} dV - \int_V (\vec{E} \cdot \vec{J})\, dV$$

$$= -\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\mu H^2 + \varepsilon E^2)\right\} dV - \int_V (\vec{E} \cdot \vec{J})\, dV \quad ...(5.80)$$

Equation (5.80) is known as **Poynting theorem**. This is also known as the **work-energy theorem** of electrodynamics.

Interpretation of each term

(a) $\frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\mu H^2 + \varepsilon E^2)\, dV\right.$: The terms $\frac{1}{2}\mu H^2$ and $\frac{1}{2}\varepsilon E^2$ represent the energy stored in electric and magnetic fields respectively and their sum will be equal to the total energy stored in electromagnetic field. This expression represents the rate of decrease of energy stored within volume V due to electric and magnetic fields.

(b) $\int_V (\vec{E}\cdot\vec{J})\, dV$ or $\int \sigma E^2 dV$: This term represents the total ohmic power dissipated within the volume. This is a generalisation of Joule's law.

(c) $\oint_S (\vec{E}\times\vec{H})\cdot\vec{dS}$: This term represents the rate at which electromagnetic energy is leaving the volume V through the closed surface S.

The vector $(\vec{E}\times\vec{H})$ is known as the Poynting vector $\vec{P}$ or $\vec{P} = (\vec{E}\times\vec{H})$.

Poynting Vector: The amount of energy flowing through unit area, perpendicular to the direction of energy propagation per unit time, i.e., the rate of energy transport per unit area, is called the Poynting vector.

Poynting Theorem: It states that the vector product $\vec{P} = (\vec{E}\times\vec{H})$ at any point is a measure of the rate of energy flow per unit area at that point. The direction of energy flow is in the direction of the vector represented by the product $(\vec{E}\times\vec{H})$ and is perpendicular to both $\vec{E}$ and $\vec{H}$.

5.16 AVERAGE POWER CALCULATION USING POYNTING VECTOR

The Poynting vector $\vec{P} = \vec{E}\times\vec{H}$ gives the instantaneous rate of energy flow. Since the vector $\vec{E}$ and $\vec{H}$ vary harmonically with time, the average flow can be found by taking the average of $\vec{P} = \vec{E}\times\vec{H}$ over a complete period, i.e.,

$$\langle P\rangle = \langle R_e E \times R_e H\rangle \qquad ...(5.81)$$

where R_e stands for the real part.

Here E and H are complex quantities, so

$$\left.\begin{aligned} E &= (E_1 + iE_2)\, e^{-i\omega t} \\ \text{and} \qquad H &= (H_1 + iH_2)\, e^{-i\omega t}\end{aligned}\right\} \qquad ...(5.82)$$

where E_1, E_2, H_1 and H_2 are real

Now $$R_e E = E_1 \cos\omega t + E_2 \sin\omega t$$

and $$R_e H = H_1 \cos\omega t + H_2 \sin\omega t$$

So $$R_e E\times R_e H = (E_1\times H_1)\cos^2\omega t + (E_1\times H_2)\cos\omega t \sin\omega t$$
$$+ (E_2\times H_1)\sin\omega t\cos\omega t + (E_2\times H_2)\sin^2\omega t$$

Now, over a complete period of oscillation

$$\langle\cos^2\omega t\rangle = \langle\sin^2\omega t\rangle = \frac{1}{2}$$

and $$\langle \sin \omega t \cos \omega t \rangle = 0$$

Therefore,
$$\langle R_e E \times R_e H \rangle = \frac{1}{2}[(E_1 \times H_1) + (E_2 \times H_2)] \quad \text{...(5.83)}$$

Let as now compute $R_e(E \times H^*)$

$$E = (E_1 + iE_2)\, e^{-i\omega t} = (E_1 + iE_2)(\cos \omega t - i \sin \omega t)$$

$$H = (H_1 + iH_2)\, e^{-i\omega t} = (H_1 + iH_2)(\cos \omega t - i \sin \omega t)$$

or,
$$H^* = (H_1 - iH_2)(\cos \omega t + i \sin \omega t)$$

Therefore,
$$R_e(E \times H^*) = (E_1 \times H_1)\cos^2 \omega t + (E_1 \times H_2)\cos \omega t \sin \omega t$$
$$+ (E_2 \times H_2)\cos^2 \omega t - (E_2 \times H_1)\cos \omega t \sin \omega t$$
$$- (E_1 \times H_2)\cos \omega t \sin \omega t + (E_1 \times H_1)\sin^2 \omega t$$
$$+ (E_2 \times H_1)\cos \omega t \sin \omega t + (E_2 \times H_2)\sin^2 \omega t$$

$$R_e(E \times H^*) = (E_1 \times H_1) + (E_2 \times H_2) \quad \text{...(5.84)}$$

Hence,
$$\langle R_e E \times R_e H \rangle = \frac{1}{2} R_e(E \times H^*) \quad \text{...(5.85)}$$

So, average Poynting vector

$$\langle P \rangle = \frac{1}{2} R_e(E \times H^*)$$

The average power
$$\langle P \rangle = \frac{1}{2} R_e(E \times H^*) \quad \text{...(5.86)}$$

Worked Out Problems

Example 5.1 A rectangular loop of sides 8 cm and 2 cm having a resistance of 1.6 Ω is placed in a magnetic field of 0.3 Tesla directed normal to the loop. The magnetic field is gradually reduced at the end of 0.02 Ts^{-1}. Find out the induced current.

Sol. Induced emf $e = \dfrac{d\varphi}{dt} = \dfrac{d}{dt}(BA) = A\dfrac{dB}{dt}$

$$= (8 \times 2 \times 10^{-4}) \times 0.02$$

$$= 3.2 \times 10^{-5}\ \text{V}$$

Now induced current $I = \dfrac{e}{R} = \dfrac{3.2 \times 10^{-5}}{1.6} = 2.0 \times 10^{-5}\ \text{A}$

Example 5.2 A metal bar slides without friction on two parallel conducting rails at distance l apart. A resistor R is connected across the rails and a uniform magnetic field B, pointing into this plane fills the entire region. If the bar moves to the right at a constant speed v then what is the current in the resistor?

[WBUT 2008]

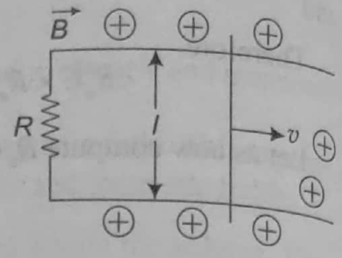

Fig. 5.1W

Sol. Induced emf $|e| = \frac{d\varphi}{dt} = \frac{d}{dt}(BA) = B\frac{dA}{dt}$

If the wire of length l moves a distance dx in time dt then $A = l\,dx$

or, $$|e| = B\frac{d}{dt}(l\,dx) = Bl\frac{dx}{dt} = Blv$$

and induced current $$i = \frac{e}{R} = \frac{Blv}{R}$$

Example 5.3 Flux φ (in Weber) in a closed circuit of resistance 10 Ω varies with time t (in seconds) according to the equation

$$\varphi = 6t^2 - 5t + 1$$

Find induced current at $t = 0.25$ second

Sol. The induced emf $e = -\frac{d\phi}{dt} = -\frac{d}{dt}(6t^2 - 5t + 1) = -12t + 5$

At $t = 0.25$ s, $e = -12 \times 0.25 + 5 = -3 + 5 = \mathbf{2V}$

Now induced current $I = \frac{e}{R} = \frac{2}{10} = 0.2$ A

Example 5.4 A metallic wheel with 6 metallic spokes, each 0.5 m long is rotating at a speed of 120 revolutions per minute in a plane perpendicular to a magnetic field of strength 0.2×10^{-4} Tesla. Find the magnitude of the induced emf between the axle and rim of the wheel.

Sol. If a conductor of length l is rotating perpendicularly to a magnetic field (B) about the fixed point with a constant angular velocity ω, then we can easily calculate the induced emf in the conductor.

Let dl be a small element of the conductor and its velocity be v. Then induced emf in the element is

$$de = Bv\,dl$$

Now, total emf induced in the conductor of length l is

$$e = \int de = \int_0^l Bv\,dl = \int_0^l Bl\omega\,dl = B\omega\frac{l^2}{2} \quad [v = \omega l]$$

$$= \frac{1}{2}B\omega l^2$$

In our problem $\omega = 2\pi \times \frac{120}{60} = 4\pi$ rad/s

and induced emf $e = \frac{1}{2}Bl^2\omega$

$$= \frac{1}{2} \times 0.2 \times 10^{-4} \times (0.5)^2 \times 4\pi = 3.14 \times 10^5\text{ V.}$$

Example 5.5 An ac voltage source $V = V_0 \sin \omega t$ is connected across a parallel-plate capacitor C. Verify that the displacement current in the capacitor is the same as the conduction current in the wire.

Sol. For a parallel-plate capacitor, if A is the area and d is the separation between the plates then

$$C = \frac{\varepsilon_0 A}{d}$$

Again electric field $E = \frac{V}{d}$

So, $D = \varepsilon_0 E = \frac{\varepsilon_0 V}{d} = \frac{\varepsilon_0 V_0 \sin \omega t}{d}$

The displacement current

$$I_d = \int_s \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot \vec{ds} = \varepsilon_0 \frac{A}{d} V_0 \omega \cos \omega t$$

$$= C V_0 \omega \cos \omega t$$

Again conduction current $I = \frac{dQ}{dt} = \frac{d}{dt}(CV) = C\frac{dV}{dt}$

$$= C\frac{d}{dt}(V_0 \sin \omega t)$$

$$= C V_0 \omega \cos \omega t$$

So, both currents are same.

Example 5.6 A parallel-plate capacitor with circular plates of 10 cm radius separated by 5 mm is being charged by an external source. The charging current is 0.2 A. Find (i) the rate of change of potential difference between the plates, and (ii) obtain the displacement current.

Sol. Here capacitance $C = \frac{\varepsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times \pi (0.1)^2}{5 \times 10^{-3}}$

$$= 5.56 \times 10^{-11}\ F$$

Given $I = \frac{dQ}{dt} = C\frac{dV}{dt} = 0.2\ A$

So, $\frac{dV}{dt} = \frac{0.2}{C} = \frac{0.2}{5.56 \times 10^{-11}} = 3.6 \times 10^{11}\ V/s.$

Again displacement current $I_d = \varepsilon_0 \frac{d\varphi}{dt} = \varepsilon_o A \frac{dE}{dt} = \frac{dQ}{dt} = I$

So displacement current is equal to 0.2 A.

Example 5.7 Show that $\frac{\sigma}{\varepsilon}\rho + \frac{\partial \rho}{\partial t} = 0$, where σ is the electric conductivity and ε is the electric permittivity of the medium.

Sol. From equation of continuity, $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$ and from Ohm's law, $\vec{J} = \sigma \vec{E}$.

Now $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

or,
$$\vec{\nabla}\cdot(\sigma\vec{E})+\frac{\partial\rho}{\partial t}=0$$

or,
$$\sigma\vec{\nabla}\cdot\vec{E}+\frac{\partial\rho}{\partial t}=0$$

Again we know
$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon} \quad \text{so,} \quad \frac{\sigma\rho}{\varepsilon}+\frac{\partial\rho}{\partial t}=0 \quad \text{(Proved)}$$

Example 5.8 Given $\vec{E}=\hat{i}E_0\cos\omega\left(\frac{z}{c}-t\right)+\hat{j}E_0\sin\omega\left(\frac{z}{c}-t\right)$, determine the magnetic field $\vec{B}$.

Sol. We know
$$\vec{B}=\frac{\vec{k}\times\vec{E}}{\omega}=\frac{k}{\omega}(\hat{k}\times\vec{E}) \quad [\because \vec{k}=k\hat{k}]$$

So,
$$\vec{B}=\frac{k}{\omega}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 1\\ E_0\cos\omega\left(\frac{z}{c}-t\right) & E_0\sin\omega\left(\frac{z}{c}-t\right) & 0\end{vmatrix}$$

$$=\frac{k}{\omega}\left[-\hat{i}\,E_0\sin\omega\left(\frac{z}{c}-t\right)+\hat{j}\,E_0\cos\omega\left(\frac{z}{c}-t\right)\right]$$

$$=-\hat{i}\frac{E_0}{c}\sin\omega\left(\frac{z}{c}-t\right)+\hat{j}\frac{E_0}{c}\cos\omega\left(\frac{z}{c}-t\right) \quad \left[\because \frac{k}{\omega}=c\right]$$

Hence, magnetic field
$$\vec{B}=-\hat{i}\frac{E_0}{c}\sin\omega\left(\frac{z}{c}-t\right)+\hat{j}\frac{E_0}{c}\cos\omega\left(\frac{z}{c}-t\right)$$

Example 5.9 A wave has a wavelength of 4 mm and the electric field associated with it has an amplitude of 40 V/m. Determine the amplitude and frequency of oscillations of the magnetic field.

Sol. The relation between electric and magnetic field
$$B_0=\frac{E_0}{c}=\frac{40}{3\times10^8}=13.3\times10^{-8}\ \text{Tesla}$$

Frequency of oscillation
$$f=\frac{c}{\lambda}=\frac{3\times10^8}{4\times10^{-3}}=0.75\times10^{11}\ \text{Hz}$$

Example 5.10 Calculate the skin depth for radio waves of 3 m wavelength (in free space) in copper, the electrical conductivity of which is 6×10^7 S/m. [Given permeability of free space $\mu_0=4\pi\times10^{-7}$ H/m]

Sol. Given
$$\lambda=3\ \text{m},\ \sigma=6\times10^7\ \text{S/m},\ \mu_0=4\pi\times10^{-7}\ \text{H/m}$$

Skin depth
$$\delta=\sqrt{\frac{2}{\omega\sigma\mu}}$$

$$\omega=2\pi f=2\pi\frac{c}{\lambda}=\frac{2\pi\times3\times10^8}{3}=2\pi\times10^8\ \text{rad/s}$$

Now, skin depth
$$\delta=\sqrt{\frac{2}{\omega\mu\sigma}}=\sqrt{\frac{2}{2\pi\times10^8\times6\times10^7\times4\pi\times10^{-7}}}$$
$$=6.5\times10^{-6}\ \text{m.}$$

Example 5.11 The earth is considered to be a good conductor when $\frac{\omega\varepsilon}{\sigma} << 1$. Calculate the highest frequencies for which the earth can be considered a good conductor if << 1 means less than 0.1.

[Assume $\sigma = 5 \times 10^{-3}$ mho/m, $\varepsilon = 10\ \varepsilon_0$]

Sol. Here $\frac{\omega\varepsilon}{\sigma} < 0.1$

or,
$$\omega < \frac{0.1\sigma}{\varepsilon} < \frac{0.1 \times 5 \times 10^{-3}}{8.854 \times 10^{-12} \times 10}$$

$\therefore\ \omega < 5.65 \times 10^6$

Highest frequency for which the earth can be considered as a good conductor is

$$f = \frac{\omega}{2\pi} = \frac{5.65 \times 10^6}{2\pi} = 0.9 \text{ MHz.}$$

Example 5.12 Find the skin depth δ at a frequency of 1.6 MHz in *Al*, where $\sigma = 38.2$ Ms/m and $\mu_r = 1$. Also find the propagation constant and wave velocity.

Sol.
$$\sigma = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi \times 1.6 \times 10^6 \times 4\pi \times 10^{-7} \times 38.2 \times 10^6}}$$

$$= 6.43 \times 10^{-2} \text{ mm}$$

The propagation constant $k = \alpha + i\beta$

But
$$\alpha = \beta = \frac{1}{\delta} = 15.53 \times 10^3 \text{ m}^{-1}$$

$\therefore$
$$k = 15.53 \times 10^3 + i\ 15.53 \times 10^3$$
$$= 21.96 \times 10^3 < 45° \text{ m}^{-1}$$

Wave velocity
$$v = \frac{\omega}{\beta} = \omega\delta = 2\pi \times 1.6 \times 10^6 \times 6.43 \times 10^{-5} \text{ m/s}$$
$$= 647.2 \text{ m/s}$$

Example 5.13 Calculate the value of Poynting vector at the surface of the sun if the power radiated by the sun is 3.8×10^{26} W and its radius is 7×10^8 m.

Sol. Here, Power = 3.8×10^{26} W and $r = 7 \times 10^8$ m

If P is the average Poynting vector at the surface of the sun then

$$P = \frac{\text{Power}}{4\pi r^2} = \frac{3.8 \times 10^{26}}{4 \times 3.14 \times (7 \times 10^8)^2}$$
$$= 6.17 \times 10^7 \text{ W/m}^2$$

Example 5.14 Calculate the strength of the electric and magnetic field of radiation if the earth's surface receives sunlight of energy per unit time per unit area is 3 cal/min cm^2.

Sol. Here, solar energy which the earth receives is 3 cal/min cm^2

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i.e., $I = 3 \text{ cal/(min cm}^2)$

or, $$I = \frac{3 \times 4.2 \times 10^4}{60} = 2100 \text{ J/m}^2\text{s}$$

$\therefore$ the poynting vector, $\vec{P} = \vec{E} \times \vec{H}$

$$= EH \sin 90°$$

$$= 2100 \text{ J/m}^2\text{s}$$

Again $$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} = 377$$

$\because$ $$EH \times \frac{E}{H} = 2100 \times 377$$

$$E^2 = 7917 \times 10^2$$

or, $$E = 890 \text{ V/m}$$

and $$H^2 = \frac{2100}{377} = 5.57$$

or, $$H = 2.36 \text{ A/m}$$

Example 5.15 Find the magnetic field B and Poynting vector P of electromagnetic waves in free space if the components of the electric fields are $E_x = E_y = 0$ and $E_z = E_0 \cos kx \sin \omega t$.

Sol. From Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

But $$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z \end{vmatrix} = \hat{i}\frac{\partial E_z}{\partial y} - \hat{j}\frac{\partial E_z}{\partial x}.$$

Now $E_z = E_0 \cos kx \sin \omega t$

So $$\frac{\partial E_z}{\partial y} = 0 \text{ and } \frac{\partial E_z}{\partial x} = \frac{\partial}{\partial x}(E_0 \cos kx \sin \omega t) = -E_0 k \sin kx \sin \omega t$$

$\therefore$ $$\vec{\nabla} \times \vec{E} = +E_0 k \sin kx \sin \omega t\, \hat{j} = -\frac{\partial \vec{B}}{\partial t}$$

$\therefore$ $$\vec{B} = +\frac{E_0 k}{\omega} \sin kx \cos \omega t\, \hat{j}$$

Now Poynting vector $\vec{P} = \vec{E} \times \vec{H}$

$$= E_0 \cos kx \sin \omega t\, \hat{k} \times \frac{E_0 k}{\mu_0 \omega} \sin kx \cos \omega t\, \hat{j}$$

$$= \frac{E_0^2 k}{\mu_0 \omega} \times \frac{1}{4} \sin 2kx \sin 2\omega t\, (\hat{k} \times \hat{j})$$

Example 5.16 Consider a monochromatic plane wave, where the electric field is given by

$$\vec{E} = E_0\, e^{i(kz-\omega t)}\, \hat{i}$$

where E_0 is an arbitrary constant vector.

(i) Show that the electric field vector lies in a direction perpendicular to the propagation.

(ii) Determine the corresponding magnetic field.

Sol. From Maxwell's equation

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

or

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\mu_0 \left[\hat{i}\frac{\partial H_x}{\partial t} + \hat{j}\frac{\partial H_y}{\partial t} + \hat{k}\frac{\partial H_z}{\partial t} \right]$$

Comparing both sides

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_x}{\partial z} \quad \text{and} \quad H_x = 0, H_z = 0$$

$$\therefore \quad \frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\frac{\partial}{\partial z}\left[E_0\, e^{i(kz-\omega t)}\right]$$

$$= -\frac{ikE_0}{\mu_0} e^{i(kz-\omega t)}$$

$$\therefore \quad H_y = -i\frac{k}{\mu_0}E_0 \int e^{i(kz-\omega t)}\, dt = \frac{E_0 k}{\mu_0 \omega} e^{i(kz-\omega t)}$$

Here, the electric field propagates in the x direction and the magnetic field propagates in the y direction, whereas the wave propagates in the z direction. So, we can say that the electric field vector lies in a direction perpendicular to the propagation.

(ii) The corresponding magnetic field

$$\vec{H} = \frac{E_0 k}{\mu_0 \omega} e^{i(kz-\omega t)}\, \hat{j}$$

Example 5.17 Show that for frequency $\leq 10^9$ Hz, a sample of silicon will act like a good conductor. For silicon, one may assume $\frac{\varepsilon}{\varepsilon_0} = 12$ and $\sigma = 2$ mho/cm. Also calculate the penetration depth for this sample at frequency 10^6 Hz.

Sol. A material will be good conductor if $\frac{\sigma}{\omega\varepsilon} >> 1$

Here $\quad \sigma = 2$ mhos/cm = 200 mhos/m

$$\omega = 2\pi f = 2\pi \times 10^9$$

$$\varepsilon = 12\, \varepsilon_0$$

Now $$\frac{\sigma}{\omega\varepsilon} = \frac{200 \times 2}{2 \times \pi \times 10^9 \times 12\, \varepsilon_0} = \frac{400 \times 9 \times 10^9}{12 \times 10^9} = 300$$

So $\frac{\sigma}{\omega\varepsilon} >> 1$; a sample of silicon will act like a conductor at frequency $\leq 10^9$ Hz.

The penetration depth for good conductor

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$= \sqrt{\frac{2}{2\pi \times 10^6 \times 4\pi \times 10^{-7} \times 200}} = 3.6 \times 10^{-2} \text{ m}$$

$$= 3.6 \text{ cm}$$

Example 5.18 Calculate the skin depth for a frequency 10^{10} Hz for silver.

Given $$\sigma = 2 \times 10^7 \text{ Sm}^{-1} \quad \text{and} \quad \mu = 4\pi \times 10^{-7} \text{ Hm}^{-1}$$

Sol. Here $\omega = 2\pi f = 2\pi \times 10^{10}$, $\sigma = 2 \times 10^7$ Sm^{-1} and $\mu = 4\pi \times 10^{-7}$ Hm^{-1}

Now skin depth $$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}$$

$$= \left(\frac{2}{2\pi \times 10^{10} \times 4\pi \times 10^{-7} \times 2 \times 10^7}\right)^{1/2}$$

$$= 1.12 \times 10^{-6} \text{ m}$$

Example 5.19 Discuss the behavior of copper to electromagnetic waves of frequency 0.5×10^{16} Hz and 7.0×10^{20} Hz. Given the conductivity of copper $\sigma = 5.8 \times 10^7$ mho m^{-1} and permittivity $\varepsilon = 9 \times 10^{-12}$ C^2/Nm^2

Sol. Here $$\omega = 2\pi \times 0.5 \times 10^{16}$$

$$\frac{\sigma}{\omega\varepsilon} = \frac{5.8 \times 10^7}{2\pi \times 0.5 \times 10^{16} \times 9 \times 10^{-12}} = 205$$

Since $\frac{\sigma}{\omega\varepsilon} > 100$, the conduction current dominates. Hence for frequency 0.5×10^{16} Hz copper is a conductor.

Now for frequency $f = 7 \times 10^{20}$ Hz, $\omega = 2\pi \times 7 \times 10^{20}$

$$\therefore \quad \frac{\sigma}{\omega\varepsilon} = \frac{5.8 \times 10^7}{2\pi \times 7 \times 10^{20} \times 9 \times 10^{-12}} = 1.47 \times 10^{-3}$$

Since $\frac{\sigma}{\omega\varepsilon} < 100$, the displacement current dominates. Hence for frequency 7×10^{20} Hz, copper is a dielectric.

Example 5.20 Calculate the value of Poynting vector for a 60 W lamp at a distance of 0.5 m from it.

Sol. Total average power emitted by the lamp = 60 W

The light emitted by the lamp will spread out in the form of a sphere, the radius of which is equal to the distance from it.

∴ radius of the sphere $R = 0.5$ m

Let P be the average Poynting vector over the surface of the sphere, then

$$P = \frac{\text{Power}}{4\pi R} = \frac{60}{4\pi(0.5)^2} = 19.1 \text{ W/m}^2.$$

Review Exercises

Part 1: Multiple Choice Questions

1. The magnetic flux linked with a coil at any instant 't' is given by $\varphi_t = 5t^3 - 100\,t + 200$, the emf induced in the coil at $t = 2$ seconds is
   (a) 200 V (b) 40 V (c) 20 V (d) –20 V
2. A cylindrical conducting rod is kept with its axis along the positive z axis, where a uniform magnetic field exists parallel to the z axis. The current induced in the cylinder is
   (a) clockwise as seen from the + z axis (b) zero
   (c) anticlockwise as seen from the – z axis (d) None of these
3. In an electromagnetic wave in a free space, the electric and magnetic fields are [WBUT 2008]
   (a) parallel to each other (b) perpendicular to each other
   (c) inclined at an acute angle (d) inclined at an obtuse angle
4. If $\vec{E}$ and $\vec{B}$ are the electric field and the magnetic field of electromagnetic waves traveling in vacuum with propagation vector then [WBUT 2006]
   (a) $\vec{k} \cdot \vec{E} = 0$ (b) $\vec{k} \times \vec{E} = 0$ (c) $\vec{B} \times \vec{E} = 0$ (d) $\vec{k} \times \vec{E} = -\vec{B}$
5. The velocity of a plane electromagnetic wave is given by
   (a) $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ (b) $c = \frac{1}{\mu_0 \varepsilon_0}$ (c) $c = \mu_0 \varepsilon_0$ (d) $\frac{\varepsilon_0}{\mu_0}$
6. $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ represent
   (a) Ampere's law (b) Laplace's equation
   (c) Gauss' law in electrostatics (d) Faraday's law of electromagnetic induction
7. The differential form of Faraday's law of electromagnetic induction is
   (a) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (b) $\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$
   (c) $\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t}$ (d) $\vec{\nabla} \times \vec{B} = -\frac{\partial \vec{E}}{\partial t}$

8. Maxwell's electromagnetic wave equations in terms of an electric field vector $\vec{E}$ in free space is [WBUT 2005]

(a) $\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$ (b) $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

(c) $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ (d) $\vec{\nabla} E = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

9. Displacement current arises due to [WBUT 2004, 2007]
   (a) positive charge only (b) negative charge only
   (c) time-varying electric field (d) None of these

10. In electromagnetic induction
   (a) mechanical energy is converted into magnetic energy
   (b) mechanical energy is converted into electrical energy
   (c) magnetic energy is converted into mechanical energy
   (d) magnetic energy is converted into electrical energy

11. Waves originating from a point source and traveling in an isotropic medium are described as [WBUT 2007]

(a) $\varphi = \varphi_0 \exp i\,(kr - \omega t)$ (b) $\varphi = \varphi_0 \exp i\,(kr - \omega t)/r$

(c) $\varphi = \varphi_0 \exp i\,(kr - \omega t)/r^2$ (d) $\varphi = \varphi_0 \exp i\,(kr + \omega t)/r$

12. Electromagnetic wave is propagated through a region of vacuum, which does not contain any charge or current. If the electric vector is given by $\vec{E} = \vec{E}_o \exp i\,(kx - \omega t)\,\hat{j}$ then the magnetic vector is [WBUT 2007]
   (a) in the x direction (b) in the y direction
   (c) in the z direction (d) rotating uniformly in the xy plane

13. Steady current produces
   (a) magnetostatic field (b) electrostatic field
   (c) time varying electric field (d) time-varying magnetic field

14. A conducting rod is moved with a constant velocity v in a magnetic field. A potential difference appears across the two ends
   (a) if $\vec{v} \parallel \vec{l}$ (b) if $\vec{v} \parallel \vec{B}$ (c) if $\vec{l} \parallel \vec{B}$ (d) None of these

15. A bar magnet is released from rest along the axis of a very long vertical copper tube. After some time, the magnet
   (a) will stop in the tube (b) will move with almost constant speed
   (c) will move with an acceleration g (d) will oscillate

16. The dimension of $\mu_0 \varepsilon_0$ is
   (a) $L^{-2}T^{-2}$ (b) $L^{-2}T^{2}$ (c) LT^{-1} (d) $L^{-1}T^{-1}$

17. The modified Ampere's circuital law is

(a) $\oint \vec{B} \cdot \vec{dl} = \mu_0 (I + I_d)$ (b) $\oint \vec{B} \cdot \vec{dl} = \mu_0 I$

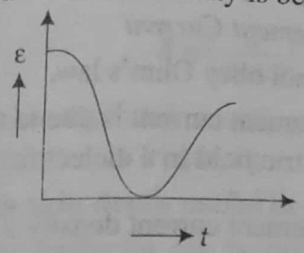
(c) $\oint \vec{B} \cdot \vec{dl} = \varepsilon_0 I$ (d) $\oint \vec{B} \cdot \vec{dl} = \varepsilon_0 (I + I_d)$

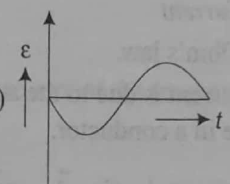
18. The electromagnetic wave is called transverse wave because [WBUT 2008]
   (a) the electric field and magnetic field are perpendicular to each other
   (b) the electric field is perpendicular to the direction of propagation
   (c) the magnetic field is perpendicular to the direction of propagation
   (d) both the electric field and magnetic field are perpendicular to the direction of propagation

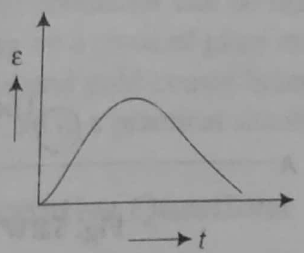
19. The solution of a plane electromagnetic wave $\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$ is
   (a) $\vec{B} = \vec{B}_0\, e^{i(\omega t - \vec{k}\cdot\vec{r})}$
   (b) $\vec{B} = \vec{B}_0\, e^{i(\omega t + \vec{k}\cdot\vec{r})}$
   (c) $\vec{B} = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$
   (d) $B = \vec{B}_0\, e^{-i(\omega t + \vec{k}\cdot\vec{r})}$

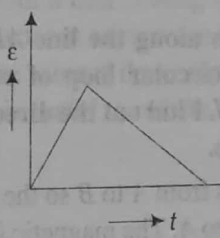
20. When a magnet is being moved towards a coil, the induced emf does not depend upon
   (a) the number of turns of the coil
   (b) the motion of the magnet
   (c) the magnetic moment of the magnet
   (d) the resistance of the coil

21. The variation of induced emf (ε) with time (t) in a coil if a short bar magnet is moved along its axis with a constant velocity is best represented as

(a)


(b)


(c)


(d)


22. The SI unit of Poynting vector is
   (a) Wm
   (b) Wm^{-1}
   (c) Wm^{2}
   (d) Wm^{-2}

23. The Poynting vector is given by the expression
   (a) $\vec{E} \times \vec{H}$
   (b) $\vec{H} \times \vec{E}$
   (c) $\vec{E} \cdot \vec{H}$
   (d) None of these

24. Skin depth for a conductor in reference to electromagnetic wave varies
   (a) inversely as frequency
   (b) directly as frequency
   (c) inversely as square root of frequency
   (d) directly as square of frequency

25. The ratio of the phase velocity and velocity of light is
   (a) one
   (b) less than one
   (c) greater than one
   (d) zero

26. The value of skin depth (δ) in a conducting medium is
   (a) $\delta = \sqrt{\frac{2\sigma}{\mu\omega}}$
   (b) $\delta = \sqrt{\frac{2}{\mu\omega\sigma}}$
   (c) $\delta = \sqrt{\frac{1}{\mu\omega\sigma}}$
   (d) $\delta = \sqrt{\frac{2\mu}{\omega\sigma}}$

27. Skin depth is proportional to

(a) ω (b) μ (c) $\frac{1}{\sqrt{\sigma}}$ (d) $\sqrt{\sigma}$

[**Ans.** 1 (b), 2 (b), 3 (b), 4 (a), 5 (a), 6 (c), 7 (a), 8 (b), 9 (c), 10 (d), 11 (a), 12 (a), 13 (a), 14 (d), 15 (b), 16 (b), 17(a), 18 (d), 19 (c), 20 (d), 21 (b), 22 (d), 23 (a), 24 (c), 25 (b), 26 (b), 27 (c)]

Short Questions with Answers

1 State Faraday's laws of electromagnetic induction.

Ans. (i) Whenever there is a change in the magnetic flux linked with a coil an emf is set up in it and stays as long as the magnetic flux linked with it is changing.

(ii) The magnitude of the induced emf is proportional to the rate of change of magnetic flux linked with the coil, i.e.,

$$\varepsilon \, \alpha \frac{d\varphi}{dt}$$

2 What is the difference between conduction current and the displacement current?

Ans.

Conduction Current	*Displacement Current*
(i) It does obey Ohm's law.	(i) It does not obey Ohm's law.
(ii) Conduction current is due to the actual flow of charge in a conductor.	(ii) Displacement current is due to the time-varying electric field in a dielectric.
(iii) Conduction current density $\vec{J} = \sigma \vec{E}$.	(iii) Displacement current density $\vec{J}_D = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

3. An electron moves along the line *AB* which lies in the same plane as a circular loop of conduction wire, as shown in Fig. 5.2W. Find out the direction of the induced current in the loop.

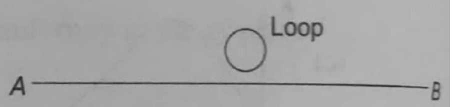


Fig. 5.2W

Ans. The electron moves from *A* to *B* so the direction of the current will be from *B* to *A*. The magnetic field generated in the loop due to the motion of the current will be directed into the plane of the paper. To oppose this, the current in the coil must be anticlockwise, in accordance with Lenz's law.

4. Why is electromagnetic wave called transverse wave?

Ans. An electromagnetic wave is called a transverse wave because by the direction of the propagation, the electric field and magnetic field are mutually perpendicular to each other.

5. What is displacement current?

Ans. See Section 5.6.

6. Why do birds fly off a high-tension wire when current is switched on?

Ans. When current begins to increase from zero to maximum value, a current is induced in the body of the bird. This produces a repulsive force and the bird flies off.

7. Two similar circular coaxial loops carry equal currents in the same direction. If the loops be brought nearer, what will happen to the currents in them?

Ans. When the loops are brought closer, there is an increase of magnetic flux. An induced emf is produced. According to Lenz's law, the induced emf has to oppose the change of magnetic flux. So, the current in each loop will decrease.

8. Two coils are being moved out of a magnetic field. One coil is moved rapidly and the other slowly. In which case is more work done and why?

Ans. More work will be done in the case of a rapidly moving coil. This is because the induced emf will be more in this coil as compared to slow moving coil.

9. Define skin depth.

Ans. Skin depth is defined as the distance in the conductor over which the electric field vector of the wave propagating in the medium decays to $1/e$ times its value at the surface.

10. What is Poynting vector?

Ans. The cross product of the electric vector $\vec{E}$ and the magnetic field vector $\vec{H}$ is known as a Poynting vector. Mathematically Poynting vector $\vec{P} = \vec{E} \times \vec{H}$

11. What is the effect of frequency on skin depth?

Ans. We know that skin depth

$$\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}$$

Thus skin depth is inversely proportional to the square root of the frequency. So, skin depth decreases with increase in frequency.

12. How is skin depth useful in practical situation?

Ans. A poor conductor can be made a good conductor with a thin coating of copper or silver. A silver coating on a piece of glass may be an excellent conductor at microwave frequency. The performance of gold and gold-coated brass wave guides will be the same if the coating thickness is equal to skin depth. So in a practical situation, skin depth method is useful to reduce the cost of the material.

Part 2: Descriptive Questions

1. (a) Write down Faraday's law of electromagnetic induction. [WBUT 2004]
   (b) Express it in differential form. [WBUT 2006]
2. (a) Write down Maxwell's equations in differential form and explain the physical significance of each equation. [WBUT 2002, 2004]
   (b) Show that the wave equation in free space for electric field $\vec{E}$ is given by $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$. [WBUT 2004]
3. (a) State Maxwell's equations. From these equations, derive the wave equations for an electromagnetic wave. What is the velocity of this wave?
   (b) Assuming a plane wave solution, establish the relation between the propagation vector ($\vec{k}$), electric field ($\vec{E}$) and magnetic field ($\vec{B}$). [WBUT 2008]
4. (a) Distinguish between the conduction current and displacement current.
   (b) Write down Faraday's law of electromagnetic induction.

5. (a) Write down Maxwell's field equations, explaining the term used. Show that in vacuum, both electric and magnetic vectors obey wave equation. Assuming a plane wave solution show that magnetic field is always orthogonal to the electric field.

   (b) Find the displacement current within a parallel-plate capacitor in series with a resistor which carries current I. Area of the capacitor plates are A and the dielectric is vacuum. [WBUT 2006]

6. (a) Write and explain differential and integral forms of Maxwell's equations.

   (b) Explain the significance of displacement current.

7. (a) Write down Maxwell's field equations.

   (b) From those equations identify Gauss' law, Ampere's law and Faraday's law.

   (c) How does velocity of light depend on the properties of vacuum? [WBUT 2005]

8. Use Faraday's law of electromagnetic induction and the fact that magnetic induction $\vec{B}$ can be derived from a vector potential $\vec{A}$. Show that the electric field can be expressed as

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \quad \text{where } \varphi \text{ is the scalar potential}$$ [WBUT 2007]

9. (a) What is displacement current? Distinguish between conduction and displacement current.

   (b) Show that $\vec{E} = \vec{E}_0\, e^{i(\vec{k}.\vec{r} - \omega t)}$ and $\vec{B} = \vec{B}_0\, e^{i(\vec{k}.\vec{r} - \omega t)}$ are possible solutions of Maxwell's electromagnetic wave equations in terms of electric and magnetic field.

10. Starting from Maxwell's equations in free space, show that the magnetic field $\vec{B}$ and the electric field $\vec{E}$ in an electromagnetic wave travel with the same speed.

11. Write down Maxwell's equations in integral form and explain the physical significance of each equation.

12. Show that $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ and $\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J} + \varepsilon_o \dfrac{\partial \vec{E}}{\partial t}\right]$

13. Show that in free space, the electric field $\vec{E}$, magnetic field $\vec{B}$ and propagation vector $\vec{k}$ are perpendicular to each other.

14. The electric field associated with an electromagnetic wave is $\vec{E} = \hat{i}\, E_0 \cos(kz - \omega t) + \hat{j}\, E_0 \sin(kz - \omega t)$, where E_0 is a constant. Find the corresponding magnetic field $\vec{B}$.

15. (a) State Lenz's law. Explain it from the principle of conservation of energy.

    (b) A wire is rotated about one of its ends at right angles to a magnetic field. Deduce the expression for induced emf.

16. Define skin depth. Show that in case of a semi-infinite solid conductor, the skin depth δ is given by

$$\delta = \frac{1}{\sqrt{\omega\mu\sigma}}$$

where symbols have their usual meanings.

17. What is Poynting vector? Show that Poynting vector measures the flow of energy per unit area per second in an electromagnetic wave.

18. State and prove Poynting theorem.

19. Show that average power $\langle P \rangle = \dfrac{1}{2} R_e(E \times H^*)$

20. What is Poynting vector? Find the expression of Poynting vector. What is the physical interpretation of this vector?

21. A plane electromagnetic wave is incident normally on a metal of electrical conductivity σ. Show that the electromagnetic wave is damped inside the conductor and find the skin depth.

Part 3: Numerical Problems

1. A parallel-plate capacitor with plate area A and separation d between the plates is charged by a constant current I. Consider a plane surface of area $\frac{A}{4}$ parallel to the plates and drawn symmetrically between the plates. Calculate the displacement current through this area. $\left[\textbf{Ans. } I_D = \frac{I}{4}\right]$

2. A parallel-plate capacitor with circular plates of radius $a = 5.5$ cm is being charged at a uniform rate so the electric field between the plates changes at a constant rate $\frac{dE}{dt} = 1.5 \times 10^{12}$ V/ms. Find the displacement current for the capacitor. [**Ans.** $I_D = 0.13$A]

3. The magnetic field in a plane electromagnetic wave is given by $B_y = 2 \times 10^{-7} \sin(0.5 \times 10^3 x + 1.5 \times 10^{11} t)$ Tesla. (a) What is the wavelength of the wave? (b) Write an expression for the electric field. [**Ans.** $\lambda = 1.26$ cm, $E_0 = 60$ Vm^{-1}]

4. Capacitance of a parallel-plate capacitor is $2\mu F$. Calculate the rate at which the potential difference between the two plates must charge to get a displacement current of 0.4 A. [**Ans.** $\frac{dV}{dt} = 2 \times 10^5$ V/s]

5. A current of 5 A is passed through a solenoid of 50 cm length, 3.0 cm radius and having 200 turns. When the switch is open, the current becomes zero within 10^{-3} s. Calculate the emf induced across the switch. [**Ans.** $e = 1.42$ volt] $\left[\textbf{Hints: } e = N\frac{d\varphi}{dt}, \varphi = BA = \mu_0 ni A, n = \frac{N}{l}\right]$

6. A rectangular loop of 8 cm side and 2 cm having a resistance of 1.6 Ω is placed in a magnetic field and gradually reduced at the rate of 0.02 Tesla/s. Find out the induced current. [**Ans.** 2×10^{-5} A]

7. A 50 cm long bar PQ is moved with a speed of 4 ms^{-1} in a magnetic field $B = 0.01$ Tesla as shown in Fig. 5.3W. Find out the induced emf. [**Ans.** 0.02 V]

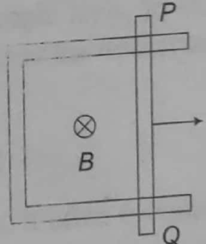


Fig. 5.3W

8. Calculate the skin depth for a frequency 10^{10} Hz for silver. Given $\sigma = 2 \times 10^7$ Sm^{-1} and $\mu = 4\pi \times 10^{-7}$ Hm^{-1}. [**Ans.** $\delta = 1.12 \times 10^{-6}$ m]

9. Find the skin depth δ at a frequency of 1.6 MHz in aluminium, where $\sigma = 38.2$ mS/m and $\mu_r = 1$. Also find wave velocity. [**Ans.** $\delta = 6.4 \times 10^{-5}$ m, $v = 6.47 \times 10^2$ m/s]

10. A laser beam has a diameter of 2 mm, what is the amplitude of the electric and magnetic field in the beam in vacuum if the power of the laser is 1.5 mW? [Given $\mu_0 = 4\pi \times 10^{-}$ H/m. $\varepsilon_0 = 8.85 \times 10^{-}$ F/m]. [**Ans.** $E_0 = 600$ V/m $H_0 = 1.59$ amp/m]

11. Find the depth of penetration of a megacycle wave into copper which has conductivity of $\sigma = 5.8 \times 10^7$ mho/m and a permeability equal to that of free space. [**Ans.** $\delta = 6.6 \times 10^{-5}$ m]