Lie Group & Lie Algebra

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1 Review of Group Theory

Definition 1.1. *A group G is a set of element $\{g\}$ which defines the product with the following properties:*

1. $(g_1 g_2) g_3 = g_1 (g_2 g_3)$*;*
2. $\exists$ *the identity element* $e \in G$*,s. t.* $eg = ge = g$*,* $\forall\ g \in G$*;*
3. $\forall g \in G,\ \exists g^{-1}$ *s. t.* $g^{-1} g = g g^{-1} = e$*.*

Definition 1.2. *If subset H with the identical product of G forms a group, then H is the **subgroup** of G.*

Example 1.1. $\mathbb{R}$ *is a group, which performs the product of addition:*

$$x \circ y := x + y;$$

the identity element is 0. One of its subgroup is 0.

Definition 1.3. *Given two groups G and G', if there exists a mapping $\mu : G \to G'$ which conserves the group's product, i.e. $\mu(g_1 g_2) = \mu(g_1)\mu(g_2)\ \forall g_1, g_2 \in G$, then the two groups are **homomorphism**.*

Definition 1.4. *If the mappping μ between two groups G and G' is **a one-one onto mapping**(surjection), then the two groups are **isomorphism**.*

Example 1.2. ***Adjoined isomorphism**: given a group G and a certain element g, if there exists a mapping*

$$I_g : G \to G,$$
$$I_g(h) := g h g^{-1},\ \forall h \in G,$$

*then the this mapping is an **adjoined isomorphism**.*

Proof. 1. *Homomorphism.* By definition,

$$I_g(h_1 h_2) = g(h_1 h_2) g^{-1} = g h_1 g^{-1} g h_2 g^{-1} = I_g(h_1) I_g(h_2)$$

2. *Isomorphism.* The inverse mapping writes

$$I_g^{-1}(h') = I_g^{-1}(g h g^{-1}) = h = g^{-1} h' g;$$

it tramsforms in the same way, thus this mapping is isomorphism.

□

Definition 1.5. *Given two group G and G', the direct product group $G \times G'$ is formed by the set $\{(g, g')\}$, where $g \in G,\ g \in G'$. The product in the new group can define with the help of the product of the original group:*

$$(g_1, g_1')(g_2, g_2') := (g_1 g_2, g_1' g_2')$$

2 Basic Definition of Lie Group

Definition 2.1. *A Lie group is a group which is also a manifold satisfying the following two conditions:*

(a) Group product $G \times G \to G$ is C^∞;

(b) The Inverse mapping $G \to G$ is C^∞.

Example 2.1. $\mathbb{R}$ *is a 1-dimensional Lie Group.*

Example 2.2. $\mathbb{R}^2 \equiv R \times R$ *is a 2 dimensional Lie Group.* $\mathbb{R}^n$ *is a nth-dimensional Lie Group.*

Example 2.3. *Diffeomorphism group of a single parameter: a mapping $\phi :\ \mathbb{R} \times M \to M$ forms a one-dimensional Lie group:*

$$G \equiv \{\phi_t :\ M \xrightarrow{diff} M | t \in \mathbb{R}\}.$$

Definition 2.2. *If a homomorphism mapping μ is C^∞, then this mapping is lie group homomorphism.*

Definition 2.3. *Lie group isomorphism is a diffeomorphism.*

Definition 2.4 (Left Translation)**.** *A* ***left translation*** *mapping $L_g :\ G \to G$ is defined as*

$$L_g(h) := gh,\ \forall h \in G,$$

Some properties are listed below:

1. L_e is an identical mapping.
2. $L_{gh} = L_g \circ L_h$.

   Proof.

   $$L_{gh}(h') = ghh' = gL_h(h') = L_g(L_h(h')).$$

   □

3. $L_g^{-1} = L_{g^{-1}}$.
4. L_g is a diffeomorphism.

Definition 2.5 (Left invariant vector field)**.** *Given a vector field $\bar{A}$ in group G, if there exists a mapping $L_{g*}\bar{A} = \bar{A},\ \forall g \in G$ and $\bar{A}|_h$, then this vector field is* ***left invariant****. More especially, this can be described as a diffeomorphism:*

$$(L_{g*}\bar{A})_{gh} = L_{g*}\bar{A}_h = \bar{A}_{gh},$$

where $(L_{g*}\bar{A})_{gh}$ is the left translated field $\bar{A}$ at point gh and $\bar{A}_h$ is the vector field at point h.

The set of left invariant vector field is a linear vector space $\mathscr{L}$.

Theorem 2.1. *The set of tangant vector A of the identical element V_e and the left invariant space $\mathscr{L}$ are isomorphism.*

Proof. Define the mapping $\eta: \ A \to \bar{A} \equiv \eta(A)$. Thus, $\bar{A}_g := L_{g*}A, \ \forall g \in G$. There is no doubt that $\bar{A}_e = A$.

Since two vector fields are isomorphism when they are linear and the mapping is one-one onto, we can easily see that the operator L_{g*} is linear.

One-one: $\bar{A} = \bar{B} \Rightarrow \bar{A}_e = \bar{B}_e \Rightarrow A = B$.

Onto: $\forall \bar{A} \in \mathscr{L}$, fix this vector in identical element e, then $\bar{A}_e = A$ is an identical mapping. □

3 Lie Algebras

Definition 3.1. *An algebra is a vector space $\mathscr{V}$ with a product $\mathscr{V} \times \mathscr{V} \to \mathscr{V}$. In Lie algebra, the product is **Lie Bracket**, which satisfies the following conditions:*

1. *Antisymmetric condition:* $[A, B] = -[B, A]$*;*
2. *Jacobi identity:* $[A, [B, C]] + [C, [A, B]] + [B, [A, C]] = 0$*;*

where $A, B, C \in \mathscr{V}$.

Example 3.1. *3-dimensional Euclidean Space $(\mathbb{R}^3, \delta_{ab})$ is a Lie Algebra in which the product writes*

$$[\boldsymbol{v_1}, \boldsymbol{v_2}] := \boldsymbol{v_1} \times \boldsymbol{v_2},$$

where $\boldsymbol{v_1}, \boldsymbol{v_2} \in (\mathbb{R}^3, \delta_{ab})$.

Proof. Antisymmetric condition: $[\boldsymbol{v_2}, \boldsymbol{v_1}] := \boldsymbol{v_2} \times \boldsymbol{v_1} = -\boldsymbol{v_1} \times \boldsymbol{v_2} = -[\boldsymbol{v_1}, \boldsymbol{v_2}]$.

Jacobi identity:

$$\begin{aligned}
&[\boldsymbol{u}, [\boldsymbol{v}, \boldsymbol{w}]] + [\boldsymbol{w}, [\boldsymbol{u}, \boldsymbol{v}]] + [\boldsymbol{v}, [\boldsymbol{w}, \boldsymbol{u}]] \\
=&\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{w} \times (\boldsymbol{u} \times \boldsymbol{v}) + \boldsymbol{v} \times (\boldsymbol{w} \times \boldsymbol{u}) \\
=&\boldsymbol{v}(\boldsymbol{u} \cdot \boldsymbol{w}) - \boldsymbol{w}(\boldsymbol{u} \cdot \boldsymbol{v}) + \boldsymbol{u}(\boldsymbol{w} \cdot \boldsymbol{v}) - \boldsymbol{v}(\boldsymbol{w} \cdot \boldsymbol{u}) + \boldsymbol{w}(\boldsymbol{v} \cdot \boldsymbol{u}) - \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = 0
\end{aligned}$$

□

Example 3.2. *The set of all the m-dimensional square matrices $\mathscr{M} = \{M\}$ is a Lie Algebra in which the product writes*

$$[A, B] := AB - BA, \ \forall A, B \in \mathscr{M}.$$

Example 3.3. *The set of all the left invariant vector fields $\mathscr{L}$ is a Lie Algebra in which the product writes*

$$[\bar{A}, \bar{B}] = \bar{A}\bar{B} - \bar{B}\bar{A}, \ \forall \bar{A}, \bar{B} \in \mathscr{M}.$$

A property: $L_{g*}[\bar{A}, \bar{B}] = [L_{g*}\bar{A}, L_{g*}\bar{B}] = [\bar{A}, \bar{B}]$.

Definition 3.2. *Lie Algebra* ***homomorphism*** *is the mapping β which conserves lie bracket, i.e.*

$$\beta[A, B] = [\beta(A), \beta(B)]$$

Lie Algebra Isomorphism defines similarly.

Noted that the tangent vector set of identical element V_e and the left invariant space $\mathscr{L}$ are isomorphism, we can show that they are lie algebra isomorphism by defining the Lie Bracket in V_e:

$$[A, B] := [\bar{A}, \bar{B}]_e$$

then we can claim that mapping η conserves lie bracket.

Proof.

$$\eta([A, B]) = \eta([\bar{A}, \bar{B}]_e) = L_{e*}[\bar{A}, \bar{B}] = [\bar{A}, \bar{B}] = [\eta(A), \eta(B)].$$

□

Theorem 3.1. *Given a Lie Group G, we can define a Lie Algebra $\mathscr{G}$ as the tangent vector space V_e of identical element e or the left-invariant vector space $\mathscr{L}$. Given a Lie Algebra $\mathscr{G}$, we can find several Lie Groups, among which there is a unique simply connected group.*

Example 3.4. *The Lie Algebra of the Lie Group $\mathbb{R}$ is also $\mathbb{R}$. Another Group is Group $S(1)$ with the product defined as*

$$\exp(\phi_1) \cdot \exp(\phi_2) = \exp(\phi_1 + \phi_2).$$

Nevertheless, $\mathbb{R}$ and $S(1)$ are not isomorphism, for the different topological form, more specifically, the former one is simply connected, but the latter is not. The Lie Algebra of $S(1)$ is also $\mathbb{R}$.

4 One Parameter Subgroups & Exponential Maps

Definition 4.1. *Given a Lie Group G, if there exists a C^∞ mapping satisfying $\gamma : \mathbb{R} \to G$ as well as the condition $\gamma(s+t) = \gamma(s)\gamma(t)$, $\forall s, t \in \mathbb{R}$, then this mapping γ is the one parameter subgroup of Lie Group G.*

Actually, γ is definitely violating the definition of group. Thus, the very subgroup is the set satisfying $\{\gamma(t) | t \in \mathbb{R}\}$. In addition, the condition $\gamma(s + t) = \gamma(s)\gamma(t)$ resemble the definition of homomorphism $\mu(g_1 g_2) = \mu(g_1)\mu(g_2)$ and $\mathbb{R}$ is also a special Lie group.

Some properties of one param. subgroups:

1. Identical element: $\gamma(0)$;
2. Inverse element: $\gamma(t)^{-1} = \gamma(-t)$.

We denote the tangent vector at point s of a curve $\gamma(t)$ as

$$\bar{A}|_{\gamma(s)} = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=s}\gamma(t),$$

i.e., $\gamma(t)$ is the **integral curve** of $\bar{A}|_{\gamma(t)}$. Through a point in group G, there exists a unique integral curve that is not continuable. If the domain of the param of a integral curve is $\mathbb{R}$, then the vector field $\bar{A}$ is **uniform**.

Definition 4.2. *Any integral curves of left invariant vector field is a one param. subgroup, and vice versa.*