American University of Armenia

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Numerical Analysis Project

Traffic Flow: Analysis and Numerical Solution of the Transport Equation

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Abstract

In this paper we aim to study traffic flow by deriving a Partial Differential Equation that models object flow on a one dimensional path by relating the variables of density and flux. After establishing the PDE we will utilize numerical methods to find a solution. We will perform a by-hand solution and develop a Python algorithm which will find solutions for variable scenarios.

Keywords: Traffic Flow, Path, PDE, Transport Equation

Contents

1 Introduction

1.1 Problem Setting and Description

The problem of traffic flow has been applicable through history whenver people, things, objects, etc. moved along a given path. In this paper, we will focus on vehicular traffic, namely cars along a straight, finite road. The first rigorous theoretical analysis of traffic flow took place in the early 20th century by Frank Knight, who analyzed traffic equilibrium. [1] Applied studies began in the 1930's, when Bruce D. Greenshields studied relationships between traffic volume and speed and behavior at intersections. In the 1950's theoretical analysis development grew through the works of Reuschel, Wardrop, Pipes, Lighthill and Whitham, Newell, Webster, Edie and Foote, Chander et al., and Herman et al. In 1959 the first International Symposium on Traffic Flow theory was held at General Motors Research Laboratories in Warren, Michigan. [2]

Today traffic flow is more important than ever, as the number of vehicles on roads continue to increase. Even with the use of computers there is no foolproof model of traffic flow due to the complexity of traffic behavior, which is dictated by human characteristics. Thus traffic flow analysis is a very broad sector with multiple theories aiming to reduce the problem to solveable standards.

We seek to relate two variables: density (quantity of vehicles per unit length at a given time) and flux (quantity of vehicles passing a point at a given time). We will use these variables and a general scenario to derive the following PDE:

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial J}{\partial x}\right)$$

- $\frac{\partial \rho}{\partial t}$ is the rate of density change with respect to time
- $\frac{\partial J}{\partial x}$ is the rate of flux change with respect to distance, x

We will also recognize 3 specific traffic scenarios and utilize original initial conditions to produce a solution which we will portray as an animated graph in our code. Our scenario will be a one lane road with a red traffic light at the half-way mark. We will analyze traffic density after the light has turned green.

[1] which were refined into John Wardrop's first and second principles of equilibrium

2 Theoretical Background

2.1 Modeling Motion

We consider a finite one lane roadway from left to right (along the x-axis with positive direction) under the assumption that cars do not enter or leave the road at any one of the points. The number of cars in our road will vary from zero to the maximum capacity. Assume the density is continuous, so ignore the individual cars' density. Assume all cars have the same length and distance from each other (this is the uniform distribution model).

We begin by formulating the density function $\rho = \rho(x_0, t_o)$.

Take some arbitrary point

$$x = x_0$$

Then a spatial interval around x_0 is

$$x_0 - \Delta x < x_0 < x_0 + \Delta x$$

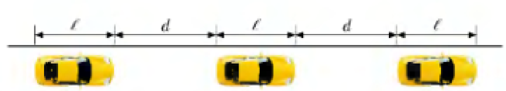
And the length of the interval is $2\Delta x$

To find the density, simply count the quantity of vehicles in this interval.

Then, $\rho(x_0, t_0) \approx \frac{number\ of\ cars\ in\ x_0 - \Delta x < x_0 < x_0 + \Delta x}{2\Delta x}$ at t_0

We assume that Δx is small enough such that only cars directly around x_0 fall into the interval, while being large enough that it covers more than one vehicle.

We now take into account the distance covered by individual vehicles. Take ℓ to be the length of individual (and all) vehicles. Assume all vehicles in our interval are spaced an equal distance d apart.



[1]

Then, the amount of vehicles that can fit in the interval $2\Delta x$ is $\left(\frac{2\Delta x}{\ell + d}\right)$ plugging this back into our $\rho(x_0, t_0)$ we get

$$\frac{2\Delta x}{\ell + d}\frac{1}{2\Delta x} = \frac{1}{\ell + d} = \rho(x_0, t_0)$$

We can now see that a max density $\rho_m = \frac{1}{\ell}$ is possible when d is 0.

We will now move on to our second prominent variable, flux, which we will denote as

$$J(x_0, t_0)$$

Flux measures the quantity of cars passing an aribitrary point x_0 at t_0 . Similar to our density evaluation, we take some interval (except now we will focus on time instead of space):

$$t_0 - \Delta t < t_0 < t_0 + \Delta t$$

Now we simply count the number of cars that pass x_0 in this time interval $2\Delta t$. So, our flux

$$J(x_0, t_0) = \frac{number\ of\ cars\ that\ pass\ x_0\ in\ 2\Delta t}{2\Delta t}$$

We assume that Δt is small enough that our flux is measured only by cars in the direct of vicinity of x_0.

We must now regard the motion of the vehicles in order to compute the quantity of cars. As we assumed that all cars move to the right (positive direction) with a constant velocity v. From physics, distance = velocity x time; cars a distance of $2\Delta tv$ from x_0 will pass x_0 in our given time interval. So, the amount of cars that pass x_0 in our time interval is equal to $\frac{2\Delta tv}{\ell+d}$. Plugging this into our $J(x_0, t_0)$ we get:

$$J(x_0, t_0) = \frac{\frac{2\Delta tv}{\ell+d}}{2\Delta t} = \frac{v}{\ell+d}$$

Note that $J(x_0, t_0) = \frac{1}{\ell+d}\mathrm{v} = \rho\mathrm{v}$

We now have enough formation to begin the formulation of our PDE! We will do so by relating our Density and Flux functions through a volume control scenario; we will show how both the Flux and Density functions can compute the number of cars that come and go over our intervals.

Consider our spatial interval of $x_0 - \Delta x < x_0 < x_0 + \Delta x$ and our time interval of $t_0 - \Delta t < t_0 < t_0 + \Delta t$

$$number\ of\ cars\ in\ our\ space\ interval\ at\ t = t_0 + \Delta t$$

$$-number\ of\ cars\ in\ our\ space\ interval\ at\ t = t_0 - \Delta t$$

(volume control for Density)

$$=$$

$$number\ of\ cars\ that\ pass\ x_0 - \Delta x\ from\ t_0 - \Delta t\ to\ t_0 + \Delta t$$

$$-number\ of\ cars\ that\ pass\ x_0 + \Delta x\ from\ t_0 - \Delta t\ to\ t_0 + \Delta t$$

(volume control for Flux)

We can re-write both using our previously derived equations for Density and Flux:

$$2\Delta x[\rho(x_0, t_0 + \Delta t) \text{ - } \rho(x_0, t_0 - \Delta t)] = 2\Delta t[J(x_0 - \Delta x, t_0) - J(x_0 + \Delta x, t_0)]$$

In order to relate these volume control equations with derivatives we expand both using Taylor's Theorem:

$$2\Delta x \left(\rho + \Delta t \rho_t + \tfrac{1}{2}(\Delta t)^2 \rho_{tt} + \tfrac{1}{6}(\Delta t)^3 \rho_{ttt} + ... \text{ - } \rho + \Delta t \rho_t \text{ -} \tfrac{1}{2}(\Delta t)^2 \rho_{tt} + \tfrac{1}{6}(\Delta t)^3 \rho_{ttt} + ...\right)$$

$$= 2\Delta t \left(\text{J - } \Delta x J_x + \tfrac{1}{2}(\Delta x)^2 J_{xx} \text{ - } \tfrac{1}{6}(\Delta x)^3 J_{xxx} + ... \text{ - J - } \delta x J_x \text{ -} \tfrac{1}{2}(\Delta x)^2 J_{xx} \text{ - } \tfrac{1}{6}(\Delta x)^3 J_{xxx} + ...\right)$$

Which re-writes to...

$$\rho_t + \mathrm{O}\big((\Delta t)^2\big) = -J_x + \mathrm{O}\big((\Delta x^2)\big)$$

We let $\Delta x \to 0$ and $\Delta t \to 0$

And we are left with

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial J}{\partial x}\right)$$

According to both theoretical and experimental findings we will assume:

- Rather than considering individual cars on the entire road which is difficult, we will analyize the density of cars on a given interval.
- The average velocity of the cars at any point depends on the density of the cars.

3 Numerical Solution

3.1 Numerical Background

We will now demonstrate the general method for solving our PDE. We start by assuming all vehicles have a constant linear velocity portrayed by the following:

$$\text{v} = v_{max}(1\text{-}\frac{\rho}{\rho_{max}})$$

This implies that when our density is at a max then velocity is 0 (imagine bumper to bumper traffic) and our velocity is at a max when our density is 0 (or near 0, in the real world).

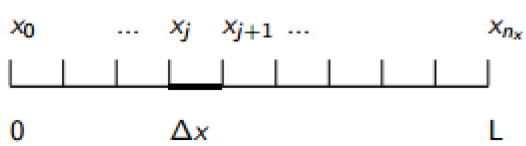
Recall that $J(x_0, t_0) = \frac{1}{l+d}\text{v} = \rho\text{v}$

Our PDE can then be re-written as

$$\frac{\partial \rho}{\partial t} + \left(\frac{\partial \rho v}{\partial x}\right) = 0$$

And our flux can be re-written as J = $\rho v_{max}(1\text{-}\frac{\rho}{\rho_{max}})$

We will use the method of finite differences to reduce our PDE to a solveable form. To do so, we discretize our spatial and time intervals into small intervals of Δx and Δt, respectively.Our discretized nodes will be labelled x_j and t_i



[3]

Take [0,L] be the length of some arbitrary road.

Our method requires us to use the definition of the derivative to form difference equations which we will solve using finite differences.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

let x = x_j and h = Δx

We omit the limit, and are left with a forward difference of:

$$f'(x_j) \approx \frac{f(x_{j+1})-f(x_j)}{\Delta x}$$

And a backwards difference of:

$$f'(x_j) \approx \frac{f(x_j)-f(x_{j-1})}{\Delta x}$$

Recall our re-written PDE:

$$\frac{\partial \rho}{\partial t} + \left(\frac{\partial \rho v}{\partial x}\right) = 0$$

Re-write our $\rho(x,t)$ as $\rho(x_j, t_i) = \rho_j^i$

We use the forward derivative as an approximation of time and the backwards derivative as an approximation of space.
Expanding our PDE with the definition of the derivative using these standards we get:

$$\frac{\rho_j^{i+1}-\rho_j^i}{\Delta t} + \text{v}\frac{\rho_j^i-\rho_{j-1}^i}{\Delta x} = 0$$

Solve for ρ_j^{i+1} and our final scheme is:

$$\rho_j^{i+1} = \rho_j^i - \text{v}\frac{\Delta t}{\Delta x}\ (\rho_j^i - \rho_{j-1}^i)$$

3.2 The Algorithm

Recall our planned traffic scenario of a one lane finite road with a traffic light at the halfway mark. We will establish an initial density condition and then use our numerical scheme as an algorithm to compute an array of densities. We now establish a general initial density condition with the following parameters:

- road length = L
- max velocity = v_{max}
- max density = ρ_{max}

There is a traffic light at x = $\frac{L}{2}$, where traffic has a density of $\frac{\rho_{max}}{2}$, for simplicity's sake assume density increases linearly up to this point.
Then, we can represent this mathematically by:

$$\rho(x,0) = \left\{ \begin{array}{ll} \frac{x(\rho_{max})}{L}, & 0 \leq x < \frac{L}{2} \\ 0, & \frac{L}{2} \leq x < L \end{array} \right\}$$

Our algorithm utilizes the previously derived forward time backwards space numerical scheme to compute a density array. In order to utilize the scheme we must provide the parameters of:

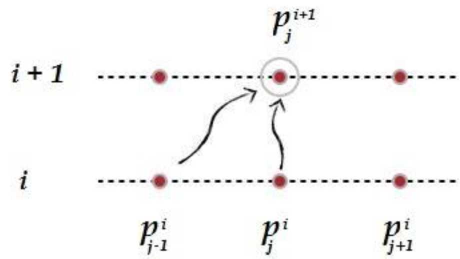
- nt: number of time-steps
- Δt: time step size
- Δx: mesh size
- ρ: array of density values at initial time
- ρ_{max}: maximum allowed density
- v_{max}: maximum allowed velocity
- x_i: number of grid points on the road

Recall that v = v_{max}(1-$\frac{\rho}{\rho_{max}}$) and that J = ρv = ρv_{max}(1-$\frac{\rho}{\rho_{max}}$)

Recall our numerical scheme of $\rho_j^{i+1} = \rho_j^i$ - v$\frac{\Delta t}{\Delta x}$ (ρ_j^i - ρ_{j-1}^i)

Distribute the v and we get: $\rho_j^{i+1} = \rho_j^i$ - $\frac{\Delta t}{\Delta x}$ (J_j^i -J_{j-1}^i)

We calculate the densities of each time-step from 1 to nt starting from the initial density values taken as a parameter. Since we are using a forward time backwards space scheme we must start with the second grid point (j = 1) on our x-axes (road) because if we take j = 0, then our backwards scheme will be undefined (we will get a negative value).

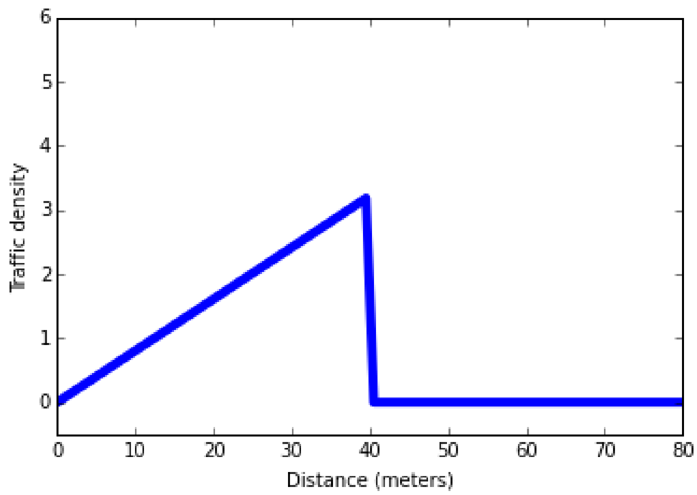


3.3 Numerical Solution and Results

We will now establish some traffic scenarios in order to compute a numerical solution. In all our scenarios, we take a one lane road of length 80m, with a traffic light at the halfway mark. The max density in all our cases is 6. We will run our algorithm and compute density values after the light has turned green.

Case 1:

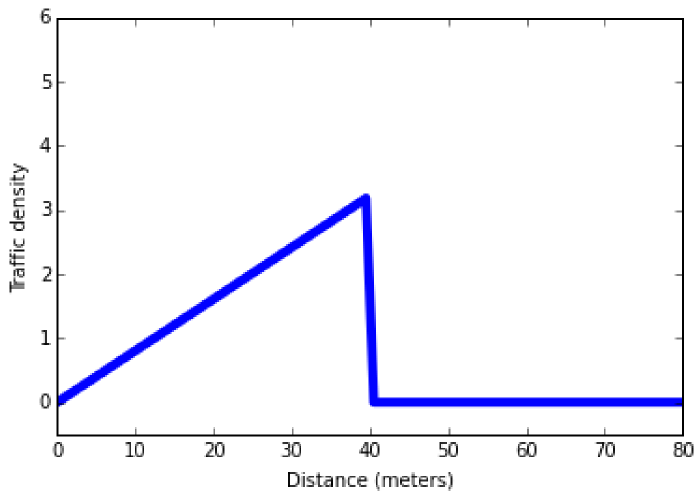
- $\rho_{atlight} = 3$
- L = 80
- $v_{max} = speed_{max}(1000/3600)(0.1)$ where $speed_{max} = 50$ Km/h (converted to m/s(0.1) to match our units of calculation)
- $\rho_{max} = 6$
- nt = 120
- $x_i = 81$
- Δ x = $\frac{L}{x_i - 1}$
- Δ t = $\frac{\Delta x}{v_{max}}$



X/T	0	10	20	30	40	50	60	70	80	90	100	110
0	0	0	0	0	0	0	0	0	0	0	0	0
10	10	0.74074	0.01321	0	0	0	0	0	0	0	0	0
20	1.48148	0.98815	0.04537	0	0	0	0	0	0	0	0	0
30	2.22222	1.9631	1.46452	0.15526	0	0	0	0	0	0	0	0
40	0	2.56839	2.70844	2.63982	1.24211	0	0	0	0	0	0	0
50	0	0	1.4357	1.89815	2.14626	2.30128	0.00009	0	0	0	0	0
60	0	0	0	1.01154	1.46568	1.7481	1.94165	2.07741	0	0	0	0
70	0	0	0	0	0.78309	1.19595	1.4765	1.68059	1.83598	0.01325	0	0
80	0	0	0	0	0	0.63956	1.01089	1.27874	1.48212	1.64208	1.75615	0

Case 2:

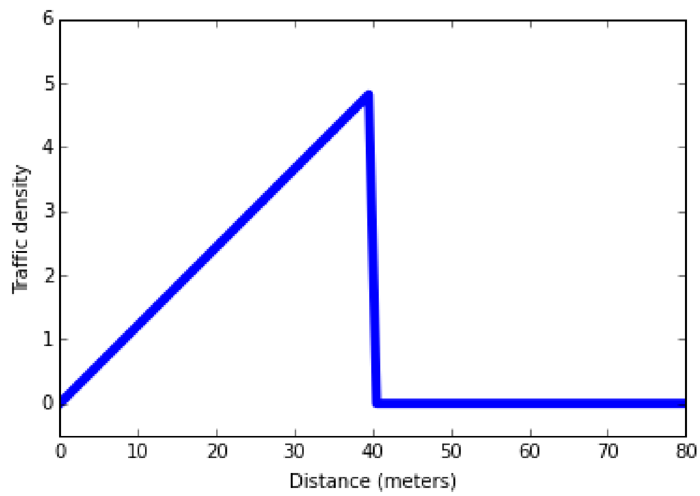
- $\rho_{atlight} = 3.3$
- L = 80
- $v_{max} = speed_{max}(1000/3600)(0.1)$ where $speed_{max} = 50$ Km/h (converted to m/s(0.1) to match our units of calculation)
- $\rho_{max} = 6$
- nt = 120
- $x_i = 81$
- Δ x = $\frac{L}{x_i - 1}$
- Δ t = $\frac{\Delta x}{v_{max}}$



X/T	0	10	20	30	40	50	60	70	80	90	100	110
0	0	0	0	0	0	0	0	0	0	0	0	0
10	0.81481	0.01665	0	0	0	0	0	0	0	0	0	0
20	1.62963	1.12288	0.06216	0	0	0	0	0	0	0	0	0
30	2.44444	2.22910	1.77202	0.26316	0	0	0	0	0	0	0	0
40	0	2.55116	2.67430	2.60879	1.74044	0	0	0	0	0	0	0
50	0	0	1.43492	1.89675	2.14413	2.29818	2.14535	0	0	0	0	0
60	0	0	0	1.01127	1.46520	1.74743	1.94076	2.08162	0.00014	0	8.487983163861089260e-314	0
70	0	0	0	0	0.78294	1.19570	1.47616	1.68015	1.83543	1.93945	0	0
80	0	0	0	0	0	0.63947	1.01073	1.27853	1.48185	1.64175	1.77089	0.00003

Case 3:

- $\rho_{atlight} = 5$
- L = 80
- $v_{max} = speed_{max}(1000/3600)(0.1)$ where $speed_{max} = 50$ Km/h (converted to m/s(0.1) to match our units of calculation)
- $\rho_{max} = 6$
- nt = 120
- $x_i = 81$
- Δ x = $\frac{L}{x_i - 1}$
- Δ t = $\frac{\Delta x}{v_{max}}$



X/T	0	10	20	30	40	50	60	70	80	90	100	110
0	0	0	0	0	0	0	0	0	0	0	0	0
10	1.23457	0.04916	0	0	0	0	0	0	0	0	0	0
20	2.46914	2.07831	0.36688	0	0	0	0	0	0	0	0	0
30	3.70370	4.10745	5.63720	6365.46039	nan	nan	nan	nan	nan	nan	nan	nan
40	0	0.25526	-3.89090	-9132.32522	nan	nan	nan	nan	nan	nan	nan	nan
50	0	0	0.62490	-5672.08686	nan	nan	nan	nan	nan	nan	nan	nan
60	0	0	0	0.80598	nan	nan	nan	nan	nan	nan	nan	nan
70	0	0	0	0	0.70773	nan	nan	nan	nan	nan	nan	nan
80	0	0	0	0	0	0.60081	nan	nan	nan	nan	nan	nan

As we can see, our results are instable for cases 2 and 3. When our initial density is greater than $\frac{\rho_{max}}{2}$, the forwards time backwards space scheme is becoming unstable. A small error is produced somewhere in our calculation, and due the the nature of numerical calculations, the error is blown out of proportion in the algorithm.

4 Conclusion

Traffic flow model is an essential tool to design effective highways and streets. Traffic flow helps engineers in the implementation of road necessities such as spotlights, control devices or markings. It also contributes to the prevention of accidents on roadways by aiding in the prediction of safe speed limits and maximum vehicle densities. Traffic flow will improve indefinitely, because the number of cars in the world continues to increase so we continuously need new models.

Our goal was to calculate the derived PDE using numerical solutions. In doing so we found that the finite difference scheme, while a good initial model, is not the best scheme for complex traffic scenarios. It performed okay for our simple scenarios. Furthermore, the numerical concept of instability is very well highlighted in traffic flow analysis.

4.1 What more can be done

In the development of new numerical schemes, stability analyis can be performed to test schemes before they are put to use. Stability analysis can help determine whether there will be asymptotic regions before the algorithm is run. Furthermore, one can use fast fourier transforms as a quantitative scheme for larger scenarios. More on fast fourier methods can be found in [4].

5 Appendices

5.1 B. Algorithm Implementation in Python

See attached files

6 Bibliography

References

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