a)

$$p(x,t)=\psi(x,t)\cdot\psi^*(x,t)$$

$$\int p(x,t)\cdot dx=P(x,t)$$

$$\int_a^b p(x,t)\cdot dx=P_{ab}(t)$$

Note that the above definite integral is exactly equivalent to taking the indefinite integral and then evaluating $P(a,t)-P(b,t)$

$$P_{ab}(t)=P(a,t)-P(b,t)$$

$$\frac{dP(x,t)}{dt}=\frac{d}{dt}\int[\psi\cdot\psi^*]dx=\int\frac{\partial}{\partial t}[\psi\cdot\psi^*]dx$$

$$\frac{\partial}{\partial t}[\psi\cdot\psi^*]=\psi\cdot\frac{\partial\psi^*}{\partial t}+\frac{\partial\psi}{\partial t}\cdot\psi^*$$

Now, the Schrodinger equation says:

$$ih\frac{\partial\psi}{\partial t}=-\frac{h^2}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}+V\cdot\psi$$

Or, re-written:

$$\frac{\partial\psi}{\partial t}=\frac{ih}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}-\frac{iV}{h}\cdot\psi$$

Taking the complex conjugate we get

$$\frac{\partial\psi^*}{\partial t}=-\frac{ih}{2m}\cdot\frac{\partial^2\psi^*}{\partial x^2}+\frac{iV}{h}\cdot\psi^*$$

This relies on the fact that if you have a complex number of the form $C_1=i\cdot C_2+i\cdot C_3$ the complex conjugate will be of the form $C_1{}^*=-i\cdot C_2{}^*-i\cdot C_3{}^*$ - because if $C_x=i\cdot C_{xi}$ then $C_{xi}=-i\cdot C_x$ and therefore $C_x{}^*=[i\cdot C_{xi}]^*=-i\cdot C_{xi}{}^*$ (this makes more sense if you look at it in terms of $C=R+i\cdot I$ but I'm including it here to remind myself – see "Complex Conjugate of iC" for a more explicit proof)

Continued on the next page

So far, we have the following equalities:

$$\frac{dP}{dt}=\int[\psi\cdot\frac{\partial\psi*}{\partial t}+\frac{\partial\psi}{\partial t}\cdot\psi*]\cdot dx$$

$$\frac{\partial\psi}{\partial t}=\frac{ih}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}-\frac{iV}{h}\cdot\psi$$

$$\frac{\partial\psi*}{\partial t}=-\frac{ih}{2m}\cdot\frac{\partial^2\psi*}{\partial x^2}+\frac{iV}{h}\cdot\psi*$$

Combining these, we get:

$$\frac{dP}{dt}=\int[\psi\cdot(-\frac{ih}{2m}\cdot\frac{\partial^2\psi*}{\partial x^2}+\frac{iV}{h}\cdot\psi*)+(\frac{ih}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}-\frac{iV}{h}\cdot\psi)\cdot\psi*]\cdot dx$$

the $\frac{iV}{h}\cdot\psi*\cdot\psi$ terms are equal and opposite and therefore cancel out and we are left with:

$$\frac{dP}{dt}=\frac{ih}{2m}\cdot\int[\psi*\cdot\frac{\partial^2\psi}{\partial x^2}-\psi\cdot\frac{\partial^2\psi*}{\partial x^2}]\cdot dx$$

Now, $\frac{\partial}{\partial x}[\psi*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi*}{\partial x}]=\psi*\cdot\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi*}{\partial x}\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial^2\psi*}{\partial x^2}-\frac{\partial\psi}{\partial x}\cdot\frac{\partial\psi*}{\partial x}=\psi*\cdot\frac{\partial^2\psi}{\partial x^2}-\psi\cdot\frac{\partial^2\psi*}{\partial x^2}$

Therefore, $\psi*\cdot\frac{\partial^2\psi}{\partial x^2}-\psi\cdot\frac{\partial^2\psi*}{\partial x^2}=\frac{\partial}{\partial x}[\psi*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi*}{\partial x}]$

So $\frac{dP}{dt}=\frac{ih}{2m}\cdot\int\{\frac{\partial}{\partial x}[\psi*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi*}{\partial x}]\}\cdot dx=\frac{ih}{2m}\cdot[\psi*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi*}{\partial x}]$

$$\frac{dP}{dt}=\frac{ih}{2m}\cdot[\psi*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi*}{\partial x}]$$

Therefore, since $J(x,t)=\frac{ih}{2m}\cdot[\psi*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi*}{\partial x}]$

$$\frac{dP}{dt}=J(x,t)$$

And since $P_{ab}(t)=P(a,t)-P(b,t)$

$$\frac{dP_{ab}(t)}{dt}=J(a,t)-J(b,t)$$

The units of $J(x,t)$ must be $\frac{1}{t}$ because $P(x,t)$ is unitless (it's a probability) b)

If my previous answer was correct, $J(x,t)=0$ because $\frac{\partial\psi}{\partial t}=0$

Let's do the proof to see if we got the previous answer wrong

$$J(x,t)=\frac{dP}{dt}=\int[\psi\cdot\frac{\partial\psi^*}{\partial t}+\frac{\partial\psi}{\partial t}\cdot\psi^*]\cdot dx$$

$$\psi=\sqrt{\lambda}\cdot e^{-\lambda\cdot|x|}\cdot e^{-i\cdot\omega\cdot t} \qquad \frac{d\psi}{dt}=-i\cdot\omega\cdot\sqrt{\lambda}\cdot e^{-\lambda\cdot|x|}\cdot e^{-i\cdot\omega\cdot t}$$

$$\psi^*=\sqrt{\lambda}\cdot e^{-\lambda\cdot|x|}\cdot e^{i\cdot\omega\cdot t} \qquad \frac{d\psi^*}{dt}=i\cdot\omega\cdot\sqrt{\lambda}\cdot e^{-\lambda\cdot|x|}\cdot e^{i\cdot\omega\cdot t}$$

$$J(x,t)=\frac{dP}{dt}=\int[(\sqrt{\lambda}\cdot e^{-\lambda\cdot|x|}\cdot e^{-i\cdot\omega\cdot t})\cdot(i\cdot\omega\cdot\sqrt{\lambda}\cdot e^{-\lambda\cdot|x|}\cdot e^{i\cdot\omega\cdot t})+(-i\cdot\omega\cdot\sqrt{\lambda}\cdot e^{-\lambda\cdot|x|}\cdot e^{-i\cdot\omega\cdot t})\cdot(\sqrt{\lambda}\cdot e^{-\lambda\cdot|x|}\cdot e^{i\cdot\omega\cdot t})]\cdot dx$$

$J(x,t)=0$ as expected