a)

$$p(x,0)=\psi(x,0)\cdot\psi^*(x,0)$$

$$\int_{-\infty}^{\infty} p(x,0)=1$$

$$\int_{-\infty}^{\infty} \psi(x,0)\cdot\psi^*(x,0)=1$$

$$\psi(x,0)=A\cdot e^{-\lambda|x|}$$

note that this is a purely real term. Therefore, the complex conjugate is the number itself, and the probability density function is just the square of the wave function.

$$\int_{-\infty}^{\infty} A\cdot e^{-\lambda|x|}\cdot A\cdot e^{-\lambda|x|}=1$$

Rather than integrate the absolute value of x from -∞ to ∞, we can drop the absolute value and double the integral from 0 to ∞, since the curve will be symmetric about the y-axis

$$2\cdot A^2\int_0^{\infty} e^{-2\lambda x}=2\cdot A^2\cdot\left.\frac{e^{-2\lambda x}}{-2\lambda}\right|_0^{\infty}=\frac{A^2}{\lambda}=1$$

$$A=\sqrt{\lambda}$$

b)

given that we know that there is no x-offset, we know that

$$\langle x \rangle = 0$$

however, working out the proof is good practice.

$$p(x,t) = \psi(x,t) \cdot \psi^*(x,t)$$

note here that I have not encountered the complex conjugate of $e^{-i \cdot \omega \cdot t}$

let's stop to figure out what that is

$$C = e^{-i \cdot \omega \cdot t} = \cos(-\omega \cdot t) + i \sin(-\omega \cdot t)$$

$$C^* = \cos(-\omega \cdot t) - i \sin(-\omega \cdot t)$$

If plotted on the complex plane, it could be seen that the angle of the conjugate vector is equal to 2*pi – the angle of the original vector. Alternatively, this is just the negative of the original angle (didn't realize this until I looked it up online; however, it gives the same answer which is good – I didn't do anything too dumb).

Therefore, the complex conjugate of $e^{-i \cdot \omega \cdot t}$ is $e^{i(2 \cdot \pi - [-\omega \cdot t])} = e^{i(2 \cdot \pi + \omega \cdot t)} = e^{i \cdot \omega \cdot t}$

Another way to understand this is that $e^{i(2 \cdot \pi + \omega \cdot t)} = e^{i \cdot 2 \cdot \pi} \cdot e^{i \cdot \omega \cdot t}$ but $e^{i \cdot 2 \cdot \pi} = 1$

$$p(x,t) = \psi(x,t) \cdot \psi^*(x,t) = \left(\sqrt{\lambda} \cdot e^{-\lambda \cdot |x|} \cdot e^{-i \cdot \omega \cdot t}\right) \cdot \left(\sqrt{\lambda} \cdot e^{-\lambda \cdot |x|} \cdot e^{i \cdot \omega \cdot t}\right)$$

$$p(x,t) = \lambda \cdot e^{-2 \cdot \lambda \cdot |x|} \cdot e^{-i \cdot \omega \cdot t + i \cdot 2 \cdot \pi + i \cdot \omega \cdot t} = \lambda \cdot e^{-2 \cdot \lambda \cdot |x|}$$

$$p(x,t) = \lambda \cdot e^{-2 \cdot \lambda \cdot |x|}$$

note that this function is independent of time. I'm sort of wondering if I made a mistake here?

$$p(x,t) = p(x)$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot p(x) \cdot dx = \int_{-\infty}^{\infty} x \cdot \lambda \cdot e^{-2 \cdot \lambda \cdot |x|} \cdot dx = \lambda \cdot \int_{0}^{\infty} x \cdot e^{-2 \cdot \lambda \cdot x} \cdot dx + \lambda \cdot \int_{-\infty}^{0} x \cdot e^{2 \cdot \lambda \cdot x} \cdot dx$$

We will continue this solution on the next page.

$$\langle x\rangle=\lambda\cdot\int_0^\infty x\cdot e^{-2\cdot\lambda\cdot x}\cdot dx+\lambda\cdot\int_{-\infty}^0 x\cdot e^{2\cdot\lambda\cdot x}\cdot dx$$

$$u=2\lambda x;\, x=\frac{u}{2\lambda};\, dx=\frac{du}{2\lambda}$$

$$\langle x\rangle=\lambda\int_0^\infty \frac{u}{2\lambda}\cdot e^{-u}\cdot\frac{du}{2\lambda}+\lambda\int_{-\infty}^0 \frac{u}{2\lambda}\cdot e^{u}\cdot\frac{du}{2\lambda}=\frac{1}{4\lambda}\int_0^\infty u\cdot e^{-u}\cdot du+\frac{1}{4\lambda}\int_{-\infty}^0 u\cdot e^{u}\cdot du$$

we will do integration by parts on each half of the integral separately. For the integral from 0 to ∞:

$$v=u;\, dw=e^{-u}\cdot du \qquad dv=du;\, w=-e^{-u}$$

$$\frac{1}{4\lambda}\int_0^\infty u\cdot e^{-u}\cdot du=\frac{1}{4\lambda}\int_0^\infty v\cdot dw=\frac{1}{4\lambda}\cdot v\cdot w\Big|_0^\infty-\frac{1}{4\lambda}\cdot\int_0^\infty w\cdot dv=\frac{1}{4\lambda}\cdot u\cdot(-e^{-u})\Big|_0^\infty-\frac{1}{4\lambda}\cdot\int_0^\infty -e^{-u}\cdot du$$

$$\frac{1}{4\lambda}\int_0^\infty u\cdot e^{-u}\cdot du=-\frac{u}{4\cdot\lambda\cdot e^u}\Big|_0^\infty-\frac{1}{4\cdot\lambda\cdot e^u}\Big|_0^\infty=-\frac{u+1}{4\cdot\lambda\cdot e^u}\Big|_0^\infty$$

we'll now have to use L'Hopital's rule to evaluate

$$\frac{1}{4\lambda}\int_0^\infty u\cdot e^{-u}\cdot du=-\frac{1}{4\cdot\lambda\cdot e^u}\Big|_0^\infty=\frac{1}{4\cdot\lambda}$$

For the integral from -∞ to 0:

$$v=u;\, dw=e^{u}\cdot du \qquad dv=du;\, w=e^{u}$$

$$\frac{1}{4\lambda}\int_{-\infty}^0 u\cdot e^{u}\cdot du=\frac{1}{4\lambda}\int_{-\infty}^0 v\cdot dw=\frac{1}{4\lambda}\cdot v\cdot w\Big|_{-\infty}^0-\frac{1}{4\lambda}\cdot\int_{-\infty}^0 w\cdot dv=\frac{1}{4\lambda}\cdot u\cdot(e^{u})\Big|_{-\infty}^0-\frac{1}{4\lambda}\cdot\int_{-\infty}^0 e^{u}\cdot du$$

$$\frac{1}{4\lambda}\int_{-\infty}^0 u\cdot e^{u}\cdot du=\frac{u\cdot e^u}{4\lambda}\Big|_{-\infty}^0-\frac{e^u}{4\lambda}\Big|_{-\infty}^0=\frac{e^u}{4\lambda}\cdot(u-1)\Big|_{-\infty}^0=-\frac{1}{4\lambda}-\frac{e^u}{4\lambda}\cdot(u-1)\Big|_{-\infty}=-\frac{1}{4\lambda}$$

I'm only, like, pretty sure that I did that last part right. I first evaluated the function at 0, and then used a L'Hopital-esque rule, where because $e^{-\infty}$ ends up in the denominator, I differentiated it as if it was in the denominator even though we started with e^u in the numerator... It's not exactly valid but I'm pretty sure I could substitute another variable, say, w=-u, and then evaluate this at positive infinity, and that would be correct.

$$\langle x\rangle=\frac{1}{4\lambda}\int_0^\infty u\cdot e^{-u}\cdot du+\frac{1}{4\lambda}\int_{-\infty}^0 u\cdot e^{u}\cdot du=\frac{1}{4\lambda}+\left(-\frac{1}{4\lambda}\right)=0$$

I got the answer I expected so...I guess I got it right. Maybe.

b)

In deriving $\langle x \rangle$ we found $p(x,t)=p(x)=\lambda \cdot e^{-2\cdot\lambda\cdot|x|}$

$$\langle x^2 \rangle = \lambda \cdot \int_0^\infty x^2 \cdot e^{-2\cdot\lambda\cdot x} \cdot dx + \lambda \cdot \int_{-\infty}^0 x^2 \cdot e^{2\cdot\lambda\cdot x} \cdot dx$$

$$u = 2\lambda x ; x = \frac{u}{2\lambda} ; dx = \frac{du}{2\lambda}$$

$$\langle x^2 \rangle = \lambda \cdot \int_0^\infty \left(\frac{u}{2\cdot\lambda}\right)^2 \cdot e^{-u} \cdot \frac{du}{2\cdot\lambda} + \lambda \cdot \int_{-\infty}^0 \left(\frac{u}{2\cdot\lambda}\right)^2 \cdot e^{u} \cdot \frac{du}{2\cdot\lambda} = \frac{1}{8\cdot\lambda^2} \cdot \int_0^\infty u^2 \cdot e^{-u} \cdot du + \frac{1}{8\cdot\lambda^2} \cdot \int_{-\infty}^0 u^2 \cdot e^{u} \cdot du$$

For the integral from 0 to ∞:

$$v = u^2 ; dw = e^{-u} \cdot du \qquad dv = 2 \cdot u \cdot du ; w = -e^{-u}$$

$$\int_0^\infty u^2 \cdot e^{-u} \cdot du = \int_0^\infty v \cdot dw = v \cdot w \Big|_0^\infty - \int_0^\infty w \cdot dv = u^2 \cdot (-e^{-u}) \Big|_0^\infty - \int_0^\infty (-e^{-u}) \cdot (2 \cdot u \cdot du)$$

$$\int_0^\infty u^2 \cdot e^{-u} \cdot du = -\frac{u^2}{e^u}\Bigg|_0^\infty + 2\int_0^\infty e^{-u} \cdot u \cdot du$$

in the derivation for <x> we found that

$$\frac{1}{4\lambda}\int_0^\infty u \cdot e^{-u} \cdot du = \frac{1}{4\cdot\lambda} \quad \text{which implies that} \quad \int_0^\infty u \cdot e^{-u} \cdot du = 1$$

$$\int_0^\infty u^2 \cdot e^{-u} \cdot du = -\frac{u^2}{e^u}\Bigg|_0^\infty + 2 = 2$$

the integral from -∞ to 0 is evaluated on the next page

BUT HOLY SHIT MARK SHOWED ME SOMETHING SO COOL

$$\int_0^\infty u^n \cdot e^{-u} \cdot du = n!$$

this agrees with what we've shown so far; when u=1, we got 1, when u=2, we got 2. We can see that if u=3, we would get 6...which is sort of how the proof goes, anyway.

The proof for this is shown in the picture "fast integration by parts – proof of the gamma function" (this is the gamma function)

For the integral from -∞ to 0:

$$v=u^2; dw=e^u\cdot du \qquad dv=2\cdot u\cdot du; w=e^u$$

$$\int_{-\infty}^{0} u^2\cdot e^u\cdot du=\int_{-\infty}^{0} v\cdot dw=v\cdot w\Big|_0^{\infty}-\int_{-\infty}^{0} w\cdot dv=u^2\cdot(e^u)\Big|_{-\infty}^{0}-\int_{-\infty}^{0}(e^u)\cdot(2\cdot u\cdot du)$$

$$\int_{-\infty}^{0} u^2\cdot e^u\cdot du=u^2\cdot e^u\Big|_{-\infty}^{0}-2\int_{-\infty}^{0} e^u\cdot u\cdot du$$

in the derivation for <x> we found that

$$\frac{1}{4\lambda}\int_{-\infty}^{0} u\cdot e^u\cdot du=-\frac{1}{4\lambda} \quad \text{which implies that} \quad \int_{-\infty}^{0} u\cdot e^u\cdot du=-1$$

$$\int_{-\infty}^{0} u^2\cdot e^u\cdot du=-u^2\cdot e^u\Big|_{-\infty}^{0}-2\cdot(-1)=2$$

$$\langle x^2\rangle=\frac{1}{8\cdot\lambda^2}\cdot\int_{0}^{\infty} x^2\cdot e^{-2\cdot\lambda\cdot x}\cdot dx+\frac{1}{8\cdot\lambda^2}\cdot\int_{-\infty}^{0} x^2\cdot e^{2\cdot\lambda\cdot x}\cdot dx=\frac{1}{8\cdot\lambda^2}\cdot 2+\frac{1}{8\cdot\lambda^2}\cdot 2=0$$

$$\langle x^2\rangle=\frac{1}{2\cdot\lambda^2}$$