a)

We know from the definition of the Schrodinger function that the probability distribution function is given by the wave function multiplied by its complex conjugate (we're going to end with an exponentially decaying function, which we will normalize to ensure you never have $P(x,t)>1$)

$p(x,t)=\psi(x,t)\cdot\psi^*(x,t)$ Born's statistical interpretation of the wave function

$P(t)=\int_{-\infty}^{\infty} p(x,t)\cdot dx$ Definition of the probability distribution

$\frac{d}{dt}\int_{-\infty}^{\infty}[\psi\cdot\psi^*]dx=\int_{-\infty}^{\infty}\frac{\partial}{\partial t}[\psi\cdot\psi^*]dx$ We're going to derive a new $P(t)$ starting here

$\frac{\partial}{\partial t}[\psi\cdot\psi^*]=\psi\cdot\frac{\partial\psi^*}{\partial t}+\frac{\partial\psi}{\partial t}\cdot\psi^*$ Expanding the integrand

Now, the Schrodinger equation (with the left hand side isolated) says:

$$\frac{\partial\psi}{\partial t}=\frac{ih}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}-\frac{iV}{h}\cdot\psi$$

If V is complex, we have a complex number of the form $C_1=i\cdot C_2\cdot C_3$ whose complex conjugate is given by $C_1^*=-i\cdot C_2^*\cdot C_3^*$ **see note at the end for why this is true**

Taking the complex conjugate we get

$$\frac{\partial\psi^*}{\partial t}=-\frac{ih}{2m}\cdot\frac{\partial^2\psi^*}{\partial x^2}+\frac{iV^*}{h}\cdot\psi^*$$

Therefore, $\frac{\partial}{\partial t}[\psi\cdot\psi^*]=\psi\cdot(-\frac{ih}{2m}\cdot\frac{\partial^2\psi^*}{\partial x^2}+\frac{iV^*}{h}\cdot\psi^*)+(\frac{ih}{2m}\cdot\frac{\partial^2\psi}{\partial x^2}-\frac{iV}{h}\cdot\psi)\cdot\psi^*$

$$\frac{\partial}{\partial t}[\psi\cdot\psi^*]=\frac{ih}{2m}\cdot(\psi^*\cdot\frac{\partial^2\psi}{\partial x^2}-\psi\cdot\frac{\partial^2\psi^*}{\partial x^2})+(\frac{iV^*}{h}-\frac{iV}{h})\cdot\psi\cdot\psi^*$$

We have previously proven that $\psi^*\cdot\frac{\partial^2\psi}{\partial x^2}-\psi\cdot\frac{\partial^2\psi^*}{\partial x^2}=\frac{\partial}{\partial x}[\psi^*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi^*}{\partial x}]$

Therefore, $\frac{\partial}{\partial t}[\psi\cdot\psi^*]=\frac{ih}{2m}\cdot(\frac{\partial}{\partial x}[\psi^*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi^*}{\partial x}])+(\frac{iV^*}{h}-\frac{iV}{h})\cdot\psi\cdot\psi^*$

$$\frac{d}{dt}\int[\psi\cdot\psi^*]dx=\int\frac{\partial}{\partial t}[\psi\cdot\psi^*]dx=\int\{\frac{ih}{2m}\cdot(\frac{\partial}{\partial x}[\psi^*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi^*}{\partial x}])+(\frac{iV^*}{h}-\frac{iV}{h})\cdot\psi\cdot\psi^*\}dx$$

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So far, we have the following equality:

$$\frac{d}{dt}\int_{-\infty}^{\infty}[\psi\cdot\psi*]dx=\int_{-\infty}^{\infty}\{\frac{ih}{2m}\cdot(\frac{\partial}{\partial x}[\psi*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi*}{\partial x}])+(\frac{iV*}{h}-\frac{iV}{h})\cdot\psi\cdot\psi*\}dx$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}[\psi\cdot\psi*]dx=\frac{ih}{2m}\cdot\int_{-\infty}^{\infty}(\frac{\partial}{\partial x}[\psi*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi*}{\partial x}])dx+\int(\frac{iV*}{h}-\frac{iV}{h})\cdot\psi\cdot\psi*dx$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}[\psi\cdot\psi*]dx=\frac{ih}{2m}\cdot[\psi*\cdot\frac{\partial\psi}{\partial x}-\psi\cdot\frac{\partial\psi*}{\partial x}]\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}(\frac{iV*}{h}-\frac{iV}{h})\cdot\psi\cdot\psi*dx$$

As pointed out in the original derivation of the normalize-ability (?) of the wave function,

$$\psi(\infty)=\psi(-\infty)=0$$

So we are left with

$$\frac{dP(t)}{dt}=\frac{d}{dt}\int_{-\infty}^{\infty}[\psi\cdot\psi*]dx=(\frac{iV*}{h}-\frac{iV}{h})\int_{-\infty}^{\infty}\psi\cdot\psi*dx$$

But $P(t)=\int_{-\infty}^{\infty}\psi\cdot\psi*dx$

So $\frac{dP(t)}{dt}=(\frac{iV*}{h}-\frac{iV}{h})\cdot P(t)$

Now, if $V=V_0-i\Gamma$ then $V*=V_0+i\Gamma$ so $\frac{iV*}{h}-\frac{iV}{h}=\frac{i}{h}[(V_0+i\Gamma)-(V_0-i\Gamma)]=\frac{-2\Gamma}{h}$

$$\frac{dP(t)}{dt}=\frac{-2\Gamma}{h}\cdot P(t)$$

b)

In part a), we found $\frac{dP(t)}{dt} = \frac{-2\Gamma}{h} \cdot P(t)$

This is a differential equation whose solution is given by

$$P(t) = C_0 \cdot e^{\frac{-2\Gamma}{h}t}$$

However, $P(0)=1$ so $C_0=1$

$$P(t) = e^{\frac{-2\Gamma}{h}t}$$

The half-life of this particle is given by $P(t)=0.5$

$$0.5 = e^{\frac{-2\Gamma}{h}t}$$

$$t_{1/2} = -\frac{\ln(0.5) \cdot h}{2\Gamma}$$

explanation of complex conjugate in part a)

How do we take the complex conjugate of such a number? The only way I can wrap my brain around this is by writing it out explicitly

$$C_1 = i\cdot(R_2 + i\cdot I_2)\cdot(R_3 + i\cdot I_3) = i\cdot(R_2\cdot R_3 + i\cdot I_2\cdot R_3 + i\cdot I_3\cdot R_2 - I_2\cdot I_3)$$

$$C_1 = (i\cdot R_2\cdot R_3 - I_2\cdot R_3 - I_3\cdot R_2 - i\cdot I_2\cdot I_3)$$

$$C_1 = -(I_2\cdot R_3 + I_3\cdot R_2) + i(R_2\cdot R_3 - I_2\cdot I_3)$$

$$C_1{}^* = -(I_2\cdot R_3 + I_3\cdot R_2) - i(R_2\cdot R_3 - I_2\cdot I_3)$$

We now need to make sense of this mess in terms of $i, C_2{}^*, \text{and}\, C_3{}^*$

Let's assume that $C_1{}^* = C_2{}^*\cdot C_3{}^*$ => possibly multiplied by i, -1, or -i

$$C_1{}^* \stackrel{?}{=} (R_2 - i\cdot I_2)\cdot(R_3 - i\cdot I_3) \quad \text{(x some stuff)}$$

$$C_1{}^* \stackrel{?}{=} (R_2\cdot R_3 - i\cdot I_2\cdot R_3 - i\cdot I_3\cdot R_2 - I_2\cdot I_3) \quad \text{(x some stuff)}$$

$$C_1{}^* \stackrel{?}{=} (R_2\cdot R_3 - I_2\cdot I_3) - i\cdot(I_2\cdot R_3 + I_3\cdot R_2) \quad \text{(x some stuff)}$$

From here, we can see that $C_1{}^* = -i\cdot C_2{}^*\cdot C_3{}^*$ as if you multiply the right side of the above equation by -i, you get the originally derived value for $C_1{}^*$