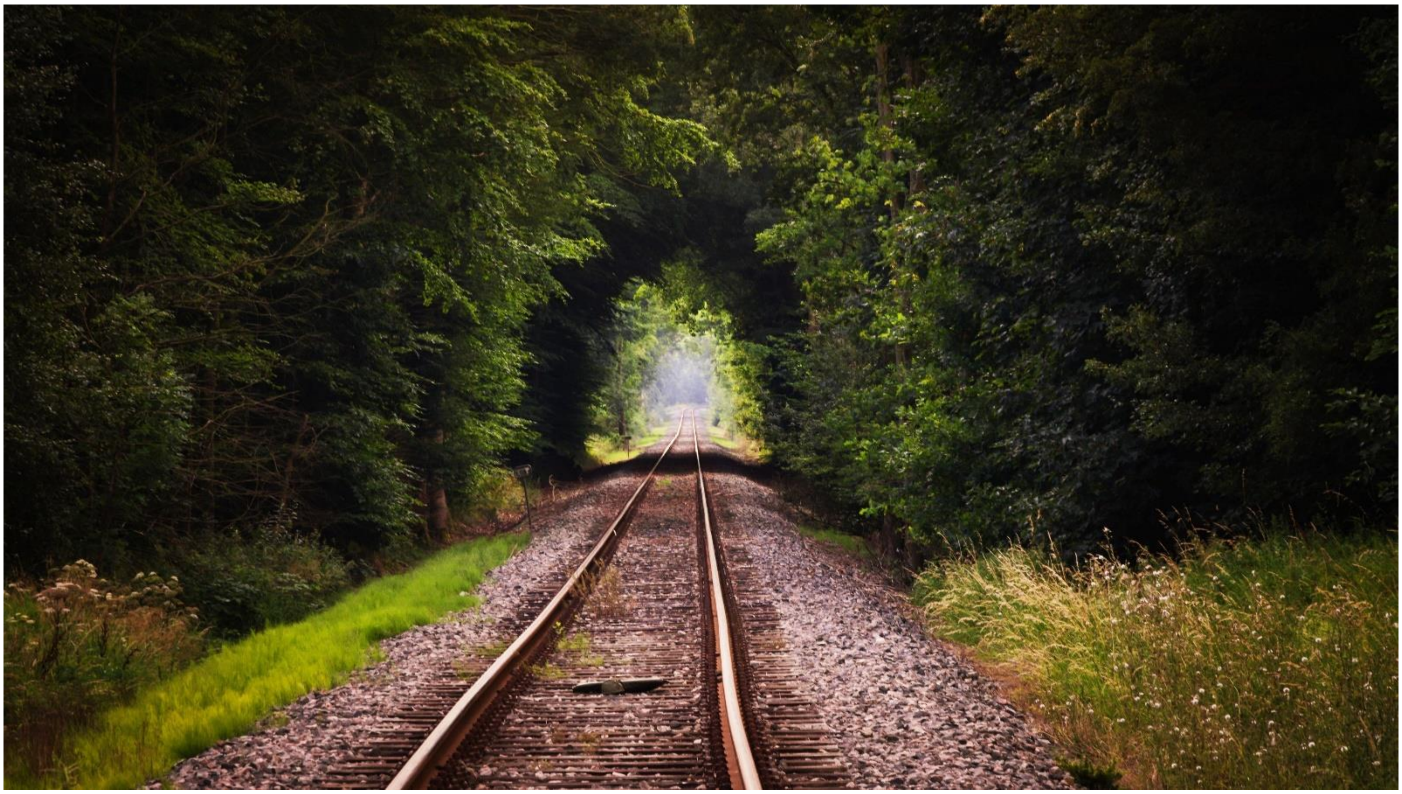

Mathematical Model and Simulation of a Train Accident

Control & Automation Mini-Project Report

Sean O'Hara and Jack Gibson

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Introduction & Background

Modelling the behaviour of a train crash is of significant relevance to many different stakeholders in a transport organisation. It is vital for the purposes of insurance, the mechanical design of the train itself, surrounding environment design (i.e. run-off area around the tracks), and the automation of such a rail network.

The practical simulation of a train crash is costly to replicate in the physical world as it involves qualified safety personnel, measurement equipment and vast volumes of materials to spend on testing (crash tests in cars may only require one car, whereas train crashes may involve many dozens of carriages). For the purposes of saving time and money, a method for mathematical modelling of such crashes is of critical importance. In recent years highly advanced models have been developed to even completely replace these expensive experimental crash tests.

For the development of this simulation, a method of Lumped Mass Damper modelling has been employed to create a coupled system of trains to make contact. This method of modelling is hugely popular with accident engineers due to its relative simplicity and accuracy. The simulation is to be exclusively based on a collision from behind where movement is assumed to be one-dimensional. One Train travels at a velocity of 50km/hr and impacts the back of another stationary train. This simulation monitors the changes in acceleration, velocity and displacement of the carriages as a function of time.

The project is carried out in several stages:

1. Starting with one train making an impact with another
2. Simplifying the mechanical model
3. Deciding on how many independent coordinate systems are needed
4. Putting together the differential equations of each section of each train
5. Expanding the mathematical model to include 2 coupled trains, each with an engine carriage and a passenger carriage.

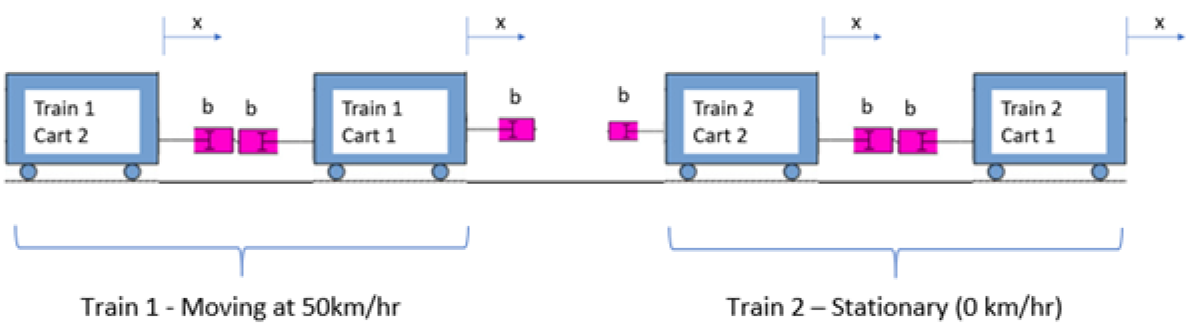

Figure 1 Fully Coupled Dynamics of two trains

The final model should be configured as shown in Figure 1 above. Each train has a damper connected in series to the points of impact and between carriages.

At the point of collision, it is vital to define the type of plasticity or elasticity of the impact. An elastic impact occurs most of the time at very low speeds with a small mass (E.g. a bee flying into a wall). The mass is affected elastically and displaced negatively by some force. As each train has a mass of 80 tonnes and impact occurs at 50km/hr this type of collision is unlikely to occur.

For this model, a fully plastic collision where impact happens at high speeds with a large object mass is assumed. During this type of collision, there is no negative displacement due to a lack of spring force and without relevant safety measures often results in fatalities. Therefore, modelling such a system and its behaviour is of vital importance for safety engineers.

A key factor when modelling any dynamic system is to model the loss of kinetic energy as a result of friction. Such a model is referred to as a Dissipative System. In this case, a combination of Static and Kinetic Coulomb Friction is considered in the model. For each carriage, the Static Friction Force must be overcome to initiate movement after which the Kinetic Friction takes effect on the system.

Modelling of Single Cart Collison

The first step of this project is to model two single carriages colliding as seen in Figure 2. As it is expected from this model that we can simulate a more extensive coupled system, a high-powered simulation tool is required to handle the complex mathematics. MATLAB / SIMULINK shall be used as the method of simulation for this model.

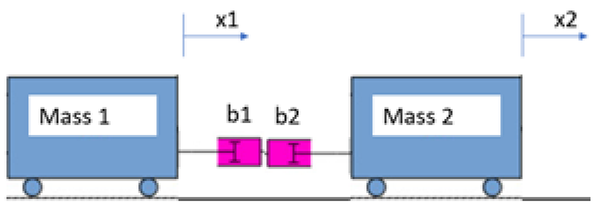

Figure 2 Lumped Mass Damper Collision of 2 Carts

To achieve consistency throughout the model, several parameters must be defined which describe the system. These are defined in the MATLAB script below.

```
% Model Coefficents
m = 80000;          % 80 tonne Mass of each carriage
l = 50;             % Length of each carriage is 50m
b = 200000;         % the damping coefficant is 200,000 N/(m/s)
u_st = 0.15;        % Static Coulomb Friction coeffecent 0.15
u_kin = 0.05;       % Kinetic Coulomb Friction coeffecent 0.05
v= 13.8889;         % Innitial Velocity of Train 1 in m/s
normal = 9.81*m;    %Normal Force Mass * Gravity(9.81) used for Friction
```

Before simulation, it is to be assumed that Train 1 moves at a constant velocity and makes an impact with Train 2 which has zero velocity. Simulation in Simulink commences at precisely the moment both trains collide. At the point where both trains crash the engine in Train 1 cuts out and loses power. From this moment on, the system loses Kinetic energy through friction and both sections stop at a final velocity and acceleration of 0.

Non-Dissipative Model

The most appropriate approach to implement this single cart collision is to try and calculate the acceleration of the carts with no effect of friction. As both carts come together and make contact on b1 and b2, there is a damping force between them. It is assumed that $\ddot{x}$ represents acceleration, $\dot{x}$ represents velocity and x represents displacement with a subscript for either Train 1 or Train 2. By implementing

Newtons Second Law, an equation for acceleration and its change over time can be calculated for both train sections, as seen below.

$$Force = Mass\ .\ Acceleration \tag{1}$$

$$Damping\ Force = -Damping\ Coefficent\ .\ Difference\ in\ Velocity \tag{2}$$

$$\ddot{x_1} = \frac{-b.(\dot{x_1} - \dot{x_2}) - b\ .(\dot{x_1} - \dot{x_2})}{m} \tag{3}$$

$$\ddot{x_2} = \frac{b.(\dot{x_1} - \dot{x_2}) + b.(\dot{x_1} - \dot{x_2})}{m} \tag{4}$$

From these equations, a basic Simulink model can be developed where friction can be added later. The Simulink model shown below in Figure 3 demonstrates this.

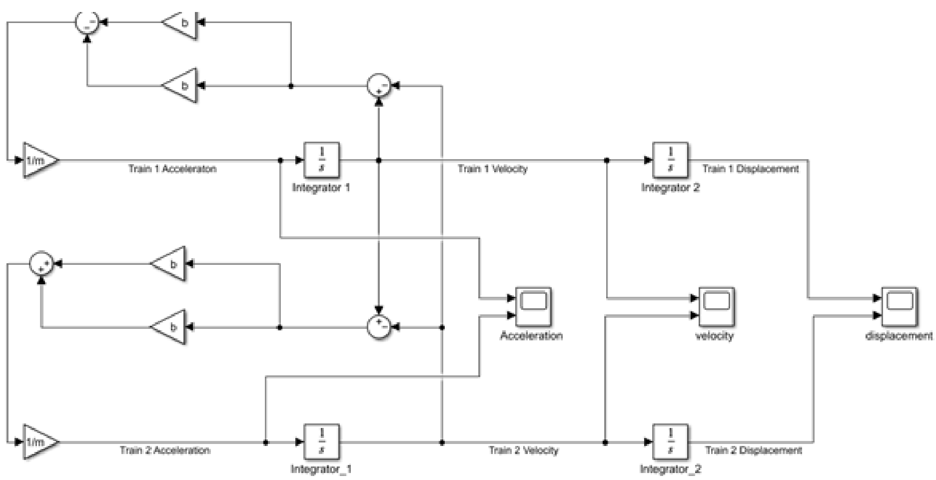

Figure 3 Basic Two Train Collision No Friction

Initial conditions must be set to the integrators to describe the system best. Train 1 Integrator 1 has an initial condition 'v' (defined in MATLAB script) to represent the initial velocity of the first Train. Train 2 Integrator_1 has an initial condition '0' to show its starting velocity is from stationary. Both Train 1 Integrator 2 and Train 2 Integrator 2 have been set to 'l' to show the length of each train carriage.

However, this model is a Non-Dissipative System due to its retention of kinetic energy through its lack of friction. This can be observed in the displacement scope Figure 4 where the trains' values increase towards infinity as there is no loss of energy from the system. It can also be observed from Figure 5 that after impact Train 1's velocity decreases as Train 2's increases and that they both continue along the track at the same velocity until infinity. The velocity in Figure 6 also looks very similar to the velocity in Figure 5 only reversed as Train 1 dips in acceleration on impact and increases back to constant. Train 2 spikes in acceleration from zero after impact where it falls after until a final value of zero.

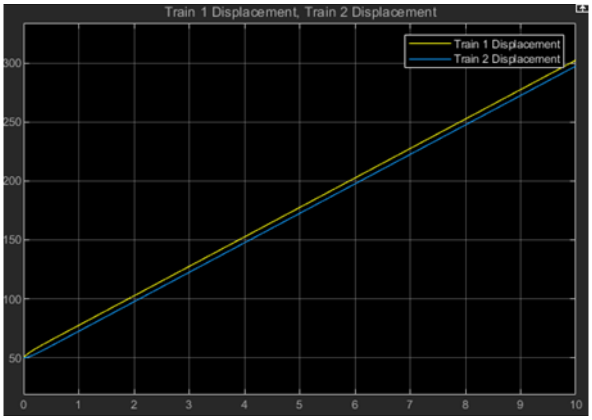

Figure 4 Displacement of Both Trains No Friction

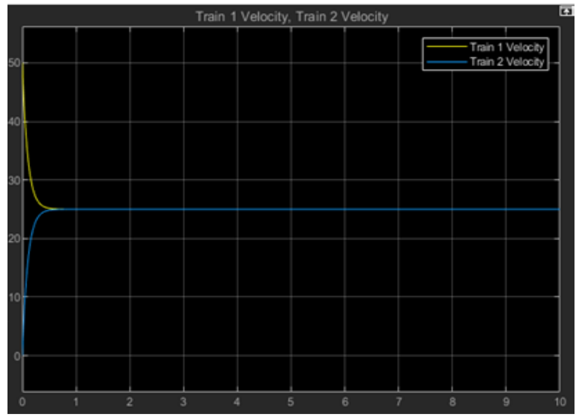

Figure 5 Velocity of Both Trains No Friction

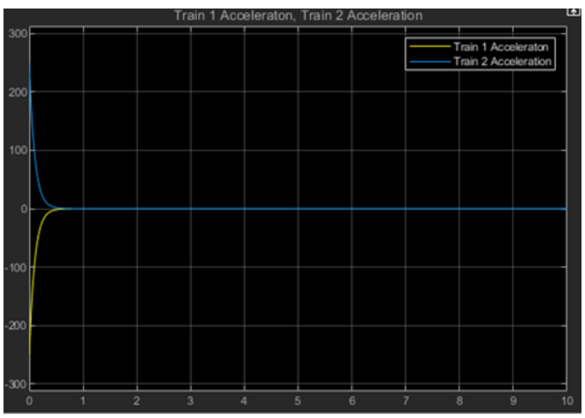

Figure 6 Acceleration of Both Trains No Friction

Dissipative Model

The next step in order to more accurately represent the response of such a system is to add friction to the model in Figure 3. Instead of using constant friction for the train a Coulomb Friction is to be used which shall give a more accurate result of how such a system would behave in the real world with stick-slip characteristics.

To use Coulomb friction accurately in any system there must first be a clear understanding and definition of when to switch between static and kinetic friction. The static Coulomb coefficient is 0.15 whereas the kinetic coefficient is 0.05.

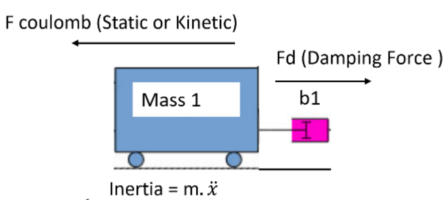

Figure 7 Free Body Diagram of Train 1 (One Carriage)

Both trains experience a stick for as long as the damping force shown as Fd in Figure 7 is less than the Static Coulomb Friction Force given in the equations below. Once this Damping Force Fd exceeds the Static Coulomb Friction Force, the F Coulomb changes from a stick to a slip momentum. At this point, the model should use the value of Kinetic Coulomb Friction Force for Friction n the simulation.

$$m\ddot{x} + F_{Coulomb} - Fd = 0 \tag{5}$$

$$Normal = Mass * Gravity \tag{6}$$

$$F_{Coulomb} = \begin{cases} F_{Kinetic} = \mu_{Kinetic} . Normal & : When\ in\ Motion \\ F_{Static} = \mu_{Static} . Normal & : When\ Static \end{cases} \tag{7}$$

$$\ddot{x} = \frac{Fd - F_{Kinetic}}{m}\ When\ in\ motion \tag{8}$$

$$\ddot{x} = \begin{cases} Fd : Fd \leq F_{static}\ AND\ velocity = 0 \\ \dfrac{Fd - F_{Kinetic}}{m} \end{cases} \tag{9}$$

These Equations can then be incorporated into the model in Figure 3 to enable the finished system to include friction. Figure 8 below shows such a model with simplification to the damper blocks. As serial damping is considered in the collision damping can be given as:

$$(\frac{1}{b} + \frac{1}{b})' = serial\ damping \tag{10}$$

$$\frac{b}{2} = simplified\ serial\ damping \tag{11}$$

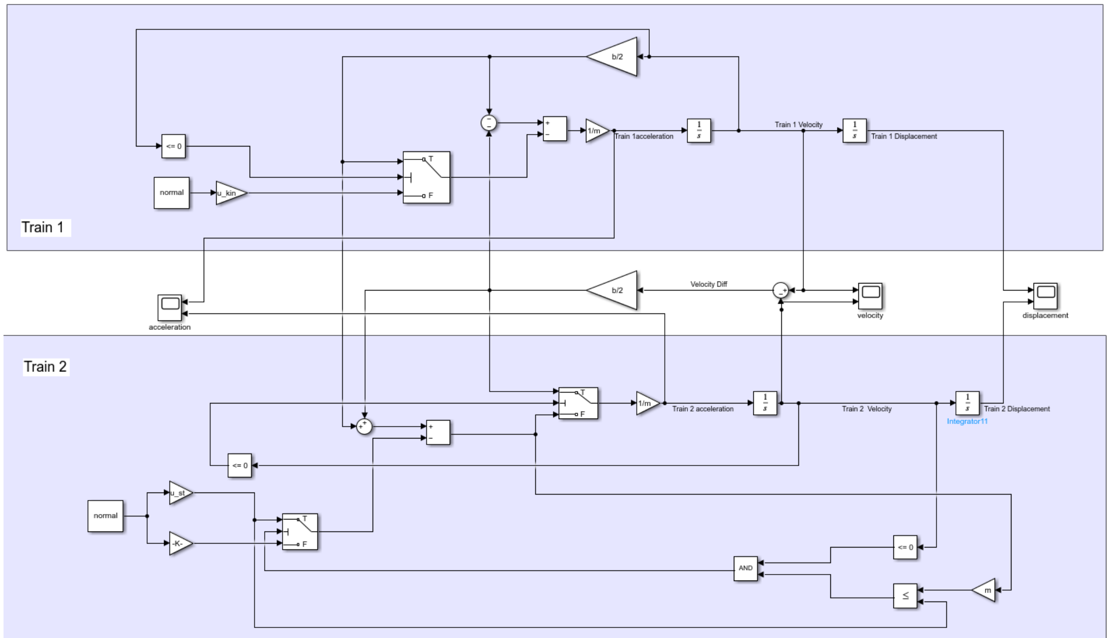

Figure 8 Single Train Crash including Friction

Areas have been drawn around elements in Figure 8 to show the blocks corresponding to each train. It is evident that Train 1 does not use a Static Friction Force. This is because its initial state is in motion and until it stops at Velocity zero, only Kinetic Friction Force is used.

Train 2 however does start from stationary, where velocity is at zero. Therefore, if there is to be movement in Train 2 the Static Friction Force must be overcome by the Damper Force and only at this point may the Kinetic Friction Force be used.

Figures 9, 10 and 11 below show the response of the two-train collision with Friction Force for acceleration, velocity and displacement. The simulation time has also been increased to allow for the system to come to a stop. It is apparent for these figures when compared with those shown in Figures 4, 5 and 6 that the values of displacement do not climb to infinity. Instead they rise to a steady value slowly until they stop at around 14 seconds. The velocity differs from its frictionless counterpart as both trains' velocities settle at zero. In Figure 11 the acceleration settles at around 1 second to a value of around -0.5 where there is a constant deceleration of both trains. At time 14.8 seconds, the value of acceleration jumps to zero. As the Damper Force has no longer been able to pass the Static Friction Force, the Coulomb Friction switches from Kinetic to Static causing the train to grind to an abrupt halt (in around 0.1 seconds).

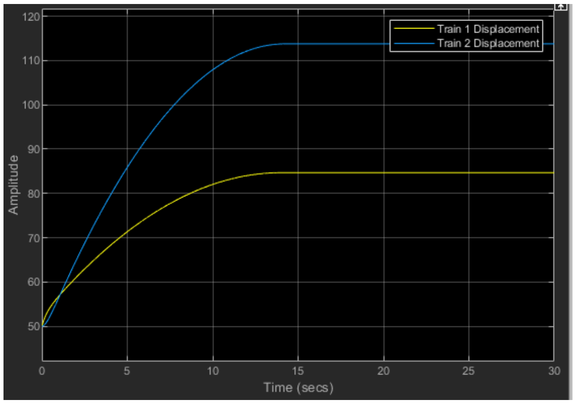

Figure 9 Displacement of Both Trains With Friction

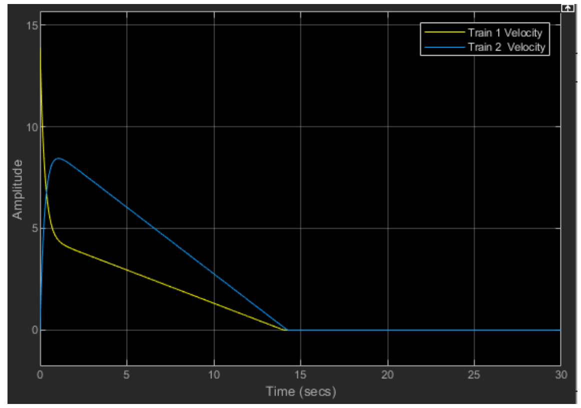

Figure 10 Velocity of Both Trains With Friction

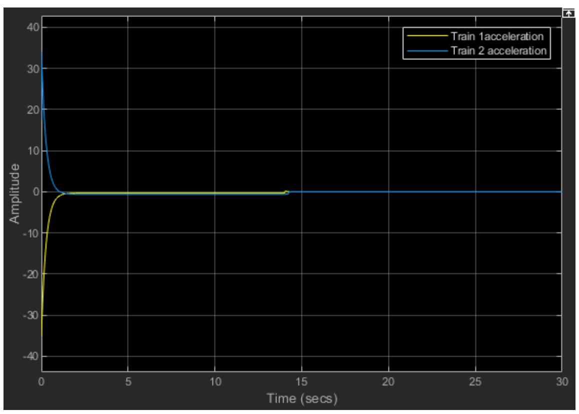

Figure 11 Acceleration of Both Trains With Friction

Modelling of Coupled Cart Collision

The fully coupled system should look like that shown in Figure 1, one train, with two carriages moving at a velocity of 50km/hr colliding with a static train with two carriages. The most logical approach to model these coupled systems is first to model the second carriage separately and then attach it to the already built collision shown in Figure 8. This section shall be broken up into three parts, firstly the modelling of the extra carriage, secondly simulation of a 2-carriage train impact on a 1 carriage train (Coupling Two Carts on Train 1) and finally the fully coupled four carriage train.

Modelling of Extra Carriage

The modelling of an extra carriage is coupled with the already existing Train 1 to complete Train 1 with both carriages. The mathematical equations regarding this extra carriage are similar to that defined for the Coulomb Friction for the Dissipative Model. If Figure 7 is used again to represent the extra carriage to the left of Train 1, the same forces act on the new carriage. If the coupling is removed for the moment and the carriage is modelled purely on a pulling force on the damper, then a modified Equation (9) holds where:

$$\ddot{x} = \begin{cases} 0 : \ Fd \ \leq F_{static} \ AND \ velocity = 0 \\ \dfrac{Fd - F_{Kinetic}}{m} \end{cases} \tag{12}$$

From Equation 12 a Simulink model can be derived as shown below in Figure 12

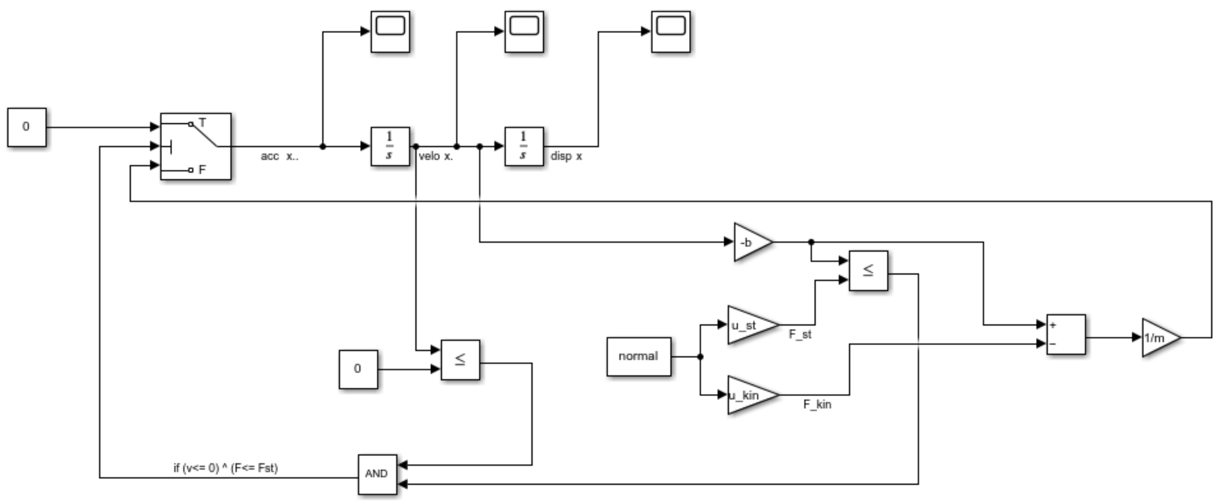

Figure 12: Model of Extra Cart

The acceleration of the cart is assumed to be damping force (Fd) minus Kinetic Friction over Mass (m), whilst the cart is in motion. The conditions of motion are similar to those calculated for the single cart collision. The cart experiences a stick every time the velocity is less than or equal to zero and the Damping Force is less than the Static Coulomb Friction Force. If there is a stick, then the acceleration of the cart jumps to zero. The responses to this model are seen in Figure 13, Figure 14 and Figure 15.

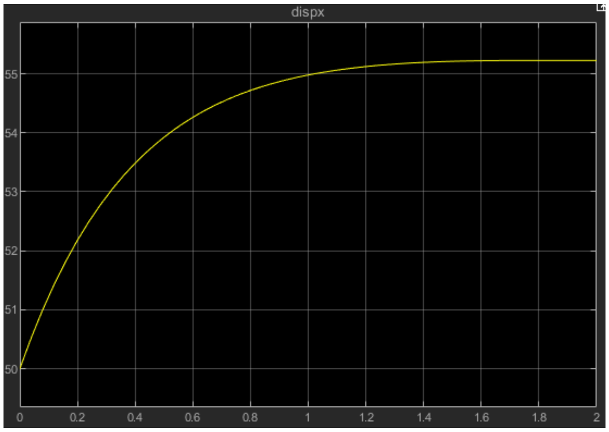

Figure 13 Displacement of single Extra Cart

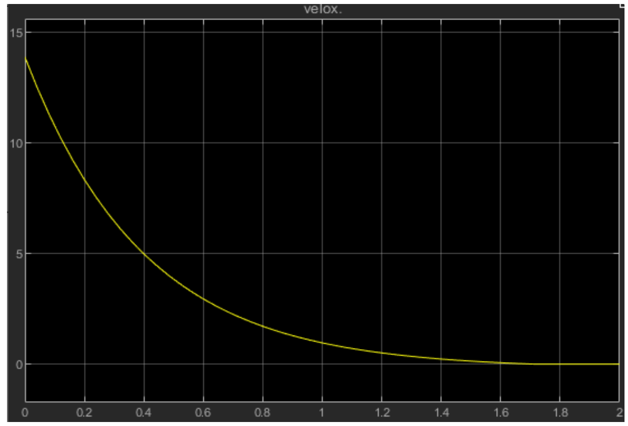

Figure 14 Velocity of Single Extra Cart

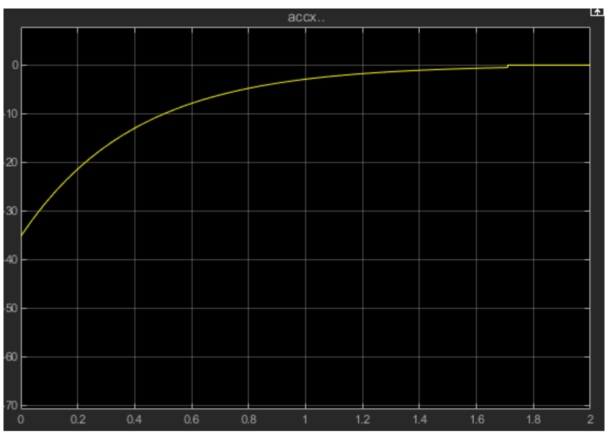

Figure 15 Acceleration of Single Extra Cart.

As expected, the displacement of the cart moves only around 5m. As the force Fd is stopped after the simulation starts, the system gets no further input of force and so comes to rest in around 1.8 seconds. Velocity also acts as expected, initial velocity starts at 13.9 m/s and falls with no other force input to the system until its final value of zero.

In the acceleration scope, at around 1.7 seconds, the effect of Coulomb friction can be observed. There is a deceleration until the point at which the velocity is less than or equal to zero and the Damping Force (Fd) is not able to overcome the Static Coulomb Friction Force. It can be seen in Figure 15 that after this switch to Static Friction, deceleration of the cart happens rapidly.

Coupling Two Carts on Train 1

The model shown in Figure 8 and Figure 12 must be combined to create a partially coupled system as shown in Figure 16 below.

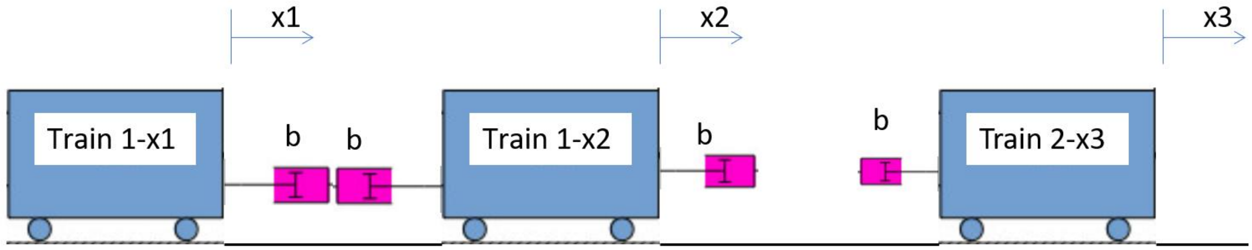

Figure 16 Coupling of Train 1 with Single Train 2

Train 1 comprises two carts, whilst Train 2 shall be left with one in order to test simulation results are as expected. As both systems are joined as one, each is directly linked to the other. However, as they are their own entity, they each require their own set of reference coordinates (set of integrators). As there are now three sets of integrators each shall be given its own subscript x_1 for Train 1 Carriage 1, x_2 for Train 1 Carriage 2 and x_3 for Train 2. The coupling of the systems uses differential equations. For Train 1, these are shown as Equations (13) and (14) below and as equation (15) for Train 2.

$$\ddot{x}_1 = \begin{cases} \frac{-\frac{b}{2}(\dot{x}_1 - \dot{x}_2)}{m} : \ \dot{x}_1 \leq 0 \ AND \ -\frac{b}{2}(\dot{x}_1 - \dot{x}_2) \leq F_{Static} \\ \frac{-\frac{b}{2}(\dot{x}_1 - \dot{x}_2) - F_{Kinetic}}{m} : \qquad else \end{cases} \tag{13}$$

$$\ddot{x}_2 = \begin{cases} \dfrac{-\frac{b}{2}(\dot{x}_2) - \frac{b}{2}(\dot{x}_2 - \dot{x}_3) - \frac{b}{2}(\dot{x}_2)}{m} : & \dot{x}_2 \leq 0 \\ \dfrac{-\frac{b}{2}(\dot{x}_2) - \frac{b}{2}(\dot{x}_2 - \dot{x}_3) - F_{Kinetic}}{m} : & else \end{cases} \tag{14}$$

$$\ddot{x}_3 = \begin{cases} \dfrac{\frac{b}{2}(\dot{x}_2 - \dot{x}_3)}{m} : \ \dot{x}_3 \leq 0 \\ \dfrac{\frac{b}{2}(\dot{x}_2) + \frac{b}{2}(\dot{x}_2 - \dot{x}_3) - F_{Static}}{m} : & \dot{x}_3 \leq 0 \ AND \ F_d \leq F_{static} \\ \dfrac{\frac{b}{2}(\dot{x}_2) + \frac{b}{2}(\dot{x}_2 - \dot{x}_3) - F_{Kinetic}}{m} : & else \end{cases} \tag{15}$$

The resulting Simulink model from Equations (13) , (14) and (15) is shown in Figure 17

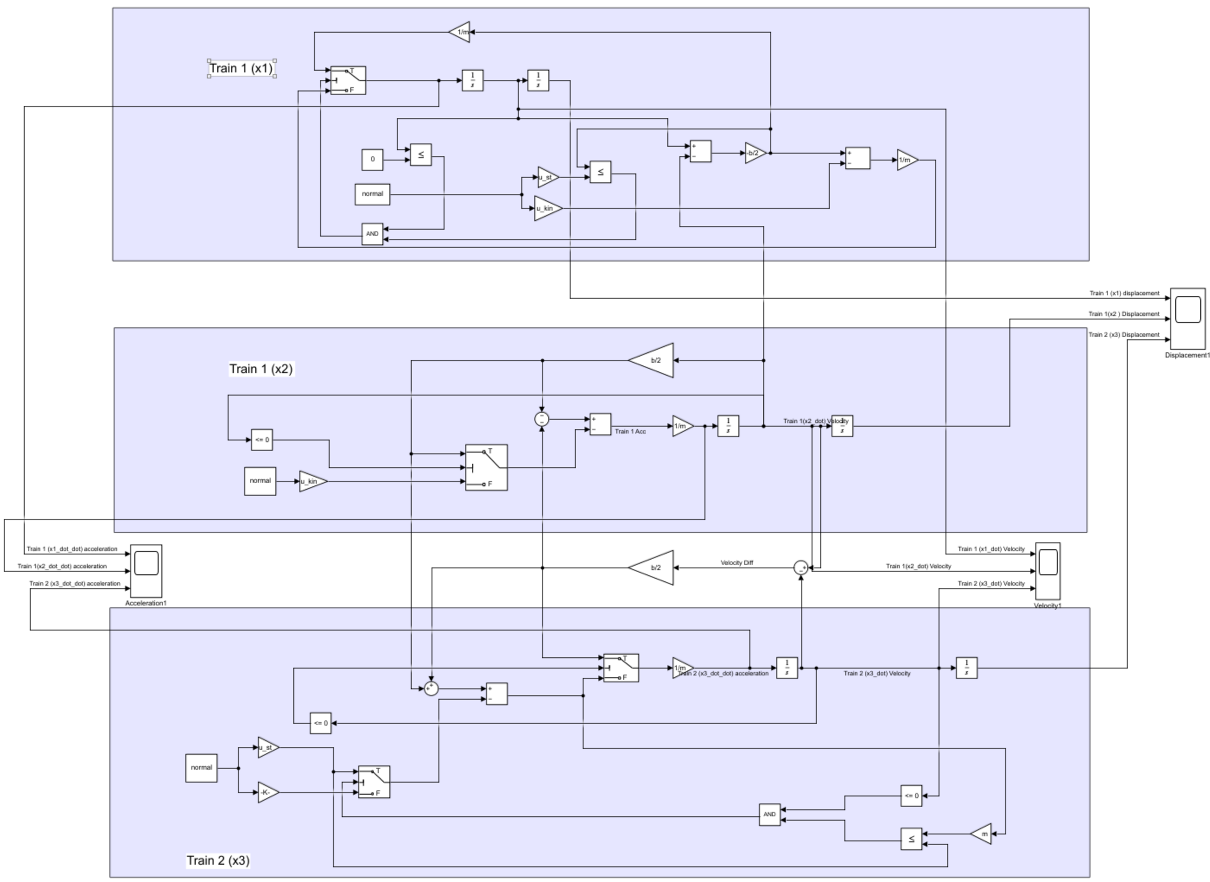

Figure 17: Simulink model for Coupling of Two Carts on Train 1

Figures 18, 19 and 20 show the acceleration, velocity and displacement responses of two carriage Train 1 as it makes an impact with stationary single carriage Train 2.

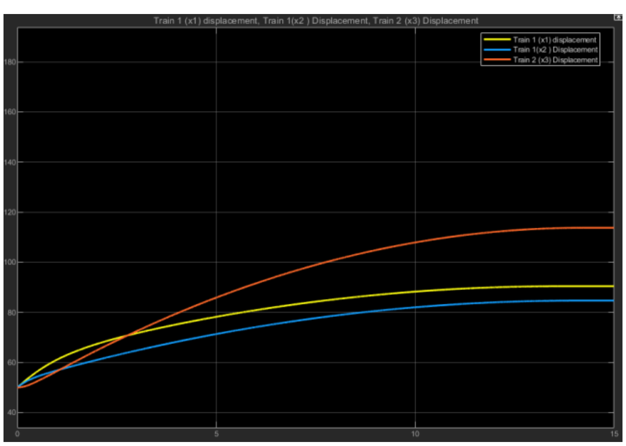

Figure 18 Displacement for Coupling of Two Carts on Train 1

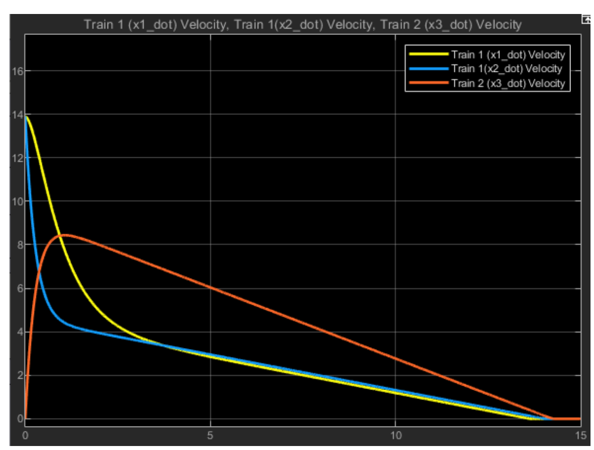

Figure 19 Velocity for Coupling of Two Carts on Train 1

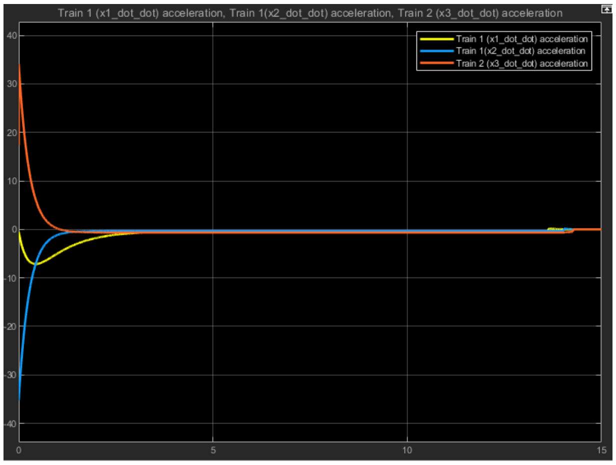

Figure 20 Acceleration for Coupling of Two Carts on Train 1

Figures 18, 19 and 20 were simulated using the same parameters as Figures 9, 10 and 11 however the additional curve reflects the addition of a second carriage coupled to Train 1 (Train 1 x_1) . Train 2 exhibits identical behaviour when comparing results from Figures 18, 19 and 20 with Figures 9, 10 and 11 as it remains uncoupled with no additional forces having been applied.

Displacement of Train 1 in Figure 18 indicates that a constant distance is maintained between both carriages as it moves down the track, indicating both carriages are attached. The velocity of Train 2 remains the same as that shown in Figure 10. However, the impact of coupling an extra carriage can also be seen here. Figure 20 shows acceleration for Train 1 and Train 2 where the most drastic differences for x_1 and x_2 can be seen. When the collision occurs x_2 immediately decelerates towards 0 where as x_1 accelerates. When x_2 initiates contact, its momentum immediately reverses whereas x_1 continues to carry its momentum forward until the Damper Force takes effect at around 1 second after impact, pushing it back before it decelerates to 0 alongside x_2.

Coupling of all Train Elements

After it has been determined that the coupling of the second carriage onto Train 1 has been modelled correctly, the same method of coupling can be reliably used for the second carriage on Train 2. The differential equation for the coupling of Train 2's second carriage is given in equation (16) below. As a new carriage is added to the model, another coordinate system must be used for x_4. As this equation is based on the first coupled carriage, the equation for its acceleration is the same as equation (13), with slight changes made to the coordinate systems used.

$$\ddot{x}_4 = \begin{cases} \dfrac{-\frac{b}{2}(\dot{x}_4 - \dot{x}_3)}{m} : & \dot{x}_4 \leq 0 \; AND \; -\frac{b}{2}(\dot{x}_4 - \dot{x}_3) \leq F_{Static} \\ \dfrac{-\frac{b}{2}(\dot{x}_4 - \dot{x}_3) - F_{Kinetic}}{m} : & else \end{cases} \tag{16}$$

Once Equation 16 has been implemented into Figure 17 the complete Coupled system can be created as shown in the opening picture Figure 1. The Simulink model of such a system is seen below in Figure 21.

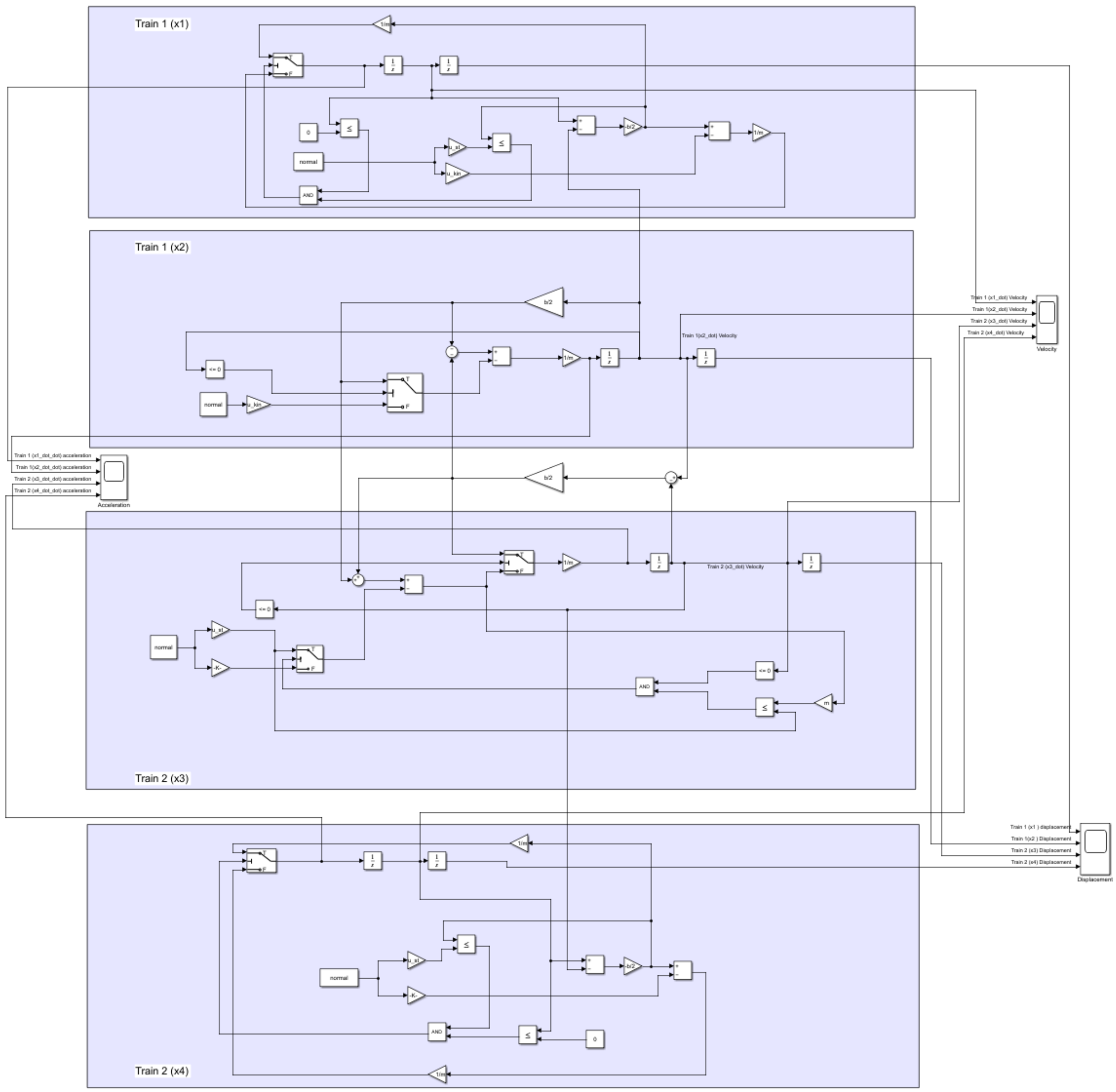

Figure 21 Fully Coupled Four Carriage System

The behaviour of such a system as shown in Figure 21 can be seen in Figure 22, Figure 23 and Figure 24.

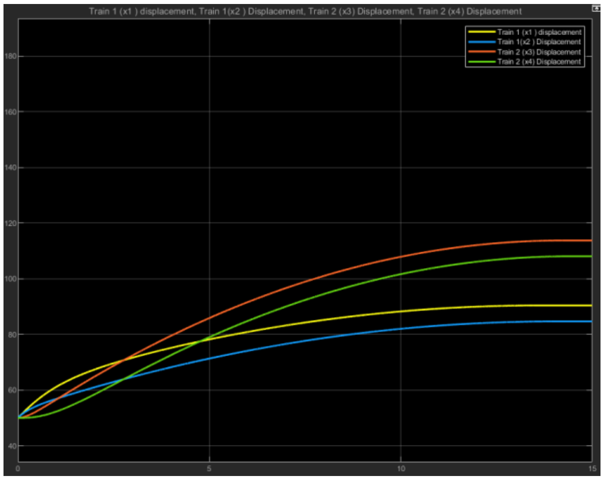

Figure 22 Displacement of Fully Coupled System

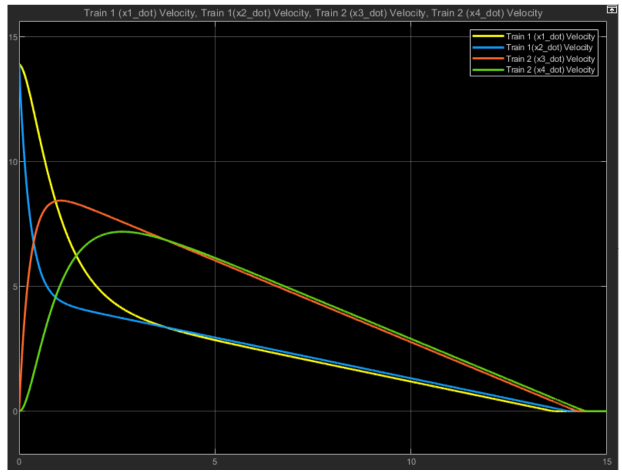

Figure 23 Velocity of Fully Coupled System

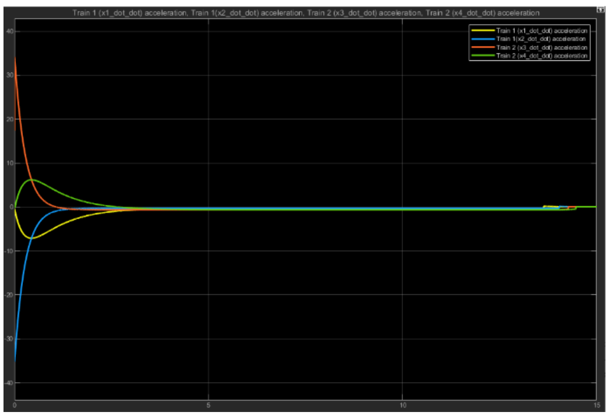

Figure 24 Acceleration of Fully Coupled System

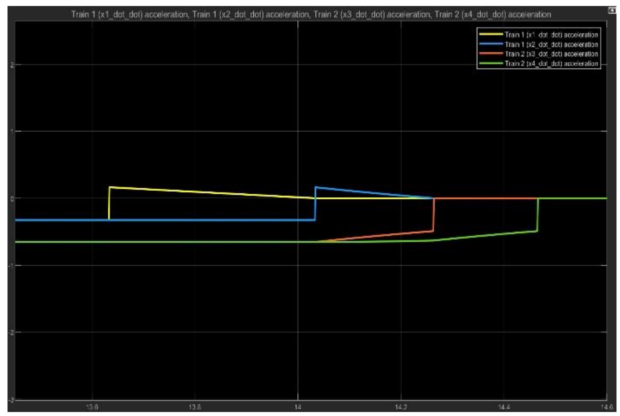

Figure 25. Fully Coupled Acceleration Rest Point (13.5 secs to 14.6 secs sim time)

The above figures (Figure 22, Figure 23, Figure 24 and Figure 25) indicate that coupling of carriage x_4 adopts a similar behavior to x_3 in the same way x_1 and x_2 have been correlated as discussed in the previous section.

In Figure 22, displacement of the fully coupled system can be seen. The influence of the Damper Force between carriages in both trains is evident, as both carts of each train stay similarly displaced from one another until they rest at their final value. The gap between carts in Train 1 (x_1 and x_2) and carts in Train 2 (x_3 and x_4) stays in the range of around 50m. This value is due to the initial condition of the displacement integrator and in real-world terms, shows the length of each carriage, thereby indicating that both carriages remain attached through the simulation.

Velocity observed in Figure 23 shows the carriage x_3 taking an initial impact. x_3 has a higher peak velocity than x_4 and reaches its peak faster. The x_4 curve peaks around 7 m/s, it takes longer to get to this value due to most of the force in the collision being exerted onto the first carriage x_3 before the force is then dissipated onto the second carriage.

In Figure 24 the acceleration almost looks like the two trains mirror each other along the x axis. Carriage x_4 accelerates rapidly in a similar way to x_3, although x_3 takes longer to reach its peak and has a much lower value; it also takes longer to reach 0 acceleration. The most exciting part of the acceleration curve occurs when the Coulomb friction starts to take effect on the system at around 14 seconds where the friction switches from Kinetic to Static. This response has been shown in Figure 25. From this, it can be seen that the first carriage to come to rest is Train 1 x_1, followed by x_2 and then finally by Train 2 x_3 and x_4. At the point at which each cart spikes from its value in the negative range (deceleration) to zero is the point where Static Friction takes over the system as the trains are slowed to a halt.

Equations (13), (14), (15) and (16) can be simplified to show Equations (17), (18), (19) and (20) below. Each differential equation has been simplified to eliminate the top fraction. The model, however, is not able to be simplified considerably as the top fraction shown in Equations (13), (14), (15) and (16) is required for the conditions of the switches.

$$\ddot{x}_1 = \begin{cases} \dfrac{-b(\dot{x}_1 - \dot{x}_2)}{2m} : & \dot{x}_1 \leq 0 \ AND - \dfrac{b}{2}(\dot{x}_1 - \dot{x}_2) \leq F_{Static} \\ \dfrac{-b(\dot{x}_1 - \dot{x}_2) - 2.F_{Kinetic}}{2m} : & else \end{cases} \tag{17}$$

$$\ddot{x}_2 = \begin{cases} \dfrac{-b(3\dot{x}_2 - \dot{x}_3)}{2m} : & \dot{x}_2 \leq 0 \\ \dfrac{-b(2\dot{x}_2 + \dot{x}_3) - 2F_{Kinetic}}{2m} : & else \end{cases} \tag{18}$$

$$\ddot{x}_3 = \begin{cases} \dfrac{b(\dot{x}_2 - \dot{x}_3)}{2m} : & \dot{x}_3 \leq 0 \\ \dfrac{b(2\dot{x}_2 - \dot{x}_3) - 2F_{Static}}{2m} : & \dot{x}_3 \leq 0 \ AND \ F_d \leq F_{static} \\ \dfrac{b(2\dot{x}_2 - \dot{x}_3) - 2F_{Kinetic}}{2m} : & else \end{cases} \tag{19}$$

$$\ddot{x}_4 = \begin{cases} \dfrac{-b(\dot{x}_4 - \dot{x}_3)}{2m} : & \dot{x}_4 \leq 0 \ AND - \dfrac{b}{2}(\dot{x}_4 - \dot{x}_3) \leq F_{Static} \\ \dfrac{-b(\dot{x}_4 - \dot{x}_3) - 2F_{Kinetic}}{2m} : & else \end{cases} \tag{20}$$

Conclusion

This investigation has modelled the behaviour of a Nonlinear train accident, making use of inter carriage Damper and Coulomb Friction Forces that act on the system. The main findings of this project are relating to the correlation between each train as it moves from impact towards a static rest in terms of acceleration, velocity and displacement. Analysis of the behaviour of the project has been possible by deriving conditional differential equations for each carriage. Each equation has then been modelled and coupled to create a complex differential model. From this model, it has been possible to analyse the behaviour of this crash graphically. Further development of this model such as assumption that deformation not only single-dimensional, implementation of material analysis to predict failures or other additions to simulate with a higher degree of accuracy. Analysis of findings would be helpful to many stakeholders in the transport industry to aid the design and implemention of safety features to future trains and safety requirements for their enviornments.

References

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