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Pricing of geometric Asian options under Heston's stochastic volatility model

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In this work, it is assumed that the underlying asset price follows Heston's stochastic volatility model and explicit solutions for the prices of geometric Asian options with fixed and floating strikes are derived. This approach has to deal with the derivation of the generalized joint Fourier transform of a square-root process and of three different weighted integrals of the square-root process with constant, linear and quadratic weights. Numerical implementation results for the complicated expressions are presented, together with the computational stability and efficiency of the method.

Keywords: Stochastic volatility; Asian options; Options pricing; Quantitative finance techniques; Methodology of pricing derivatives

JEL Classification: C6, C63, C65

1. Introduction

Asian options are path-dependent options whose payoffs are determined by the difference between the average asset price over a certain time period and the strike price. Asian options are usually written and traded in the over-the-counter market to avoid price distortions near to maturity which often occur with European vanilla options. Generally speaking, there are distinct types of European-style Asian options depending on whether the geometric or arithmetic average is taken, whether the average is observed continuously or discretely and whether the strike price is floating or fixed.

Many valuation methods for Asian options have been developed over the past two decades. These methods have raised some interesting issues that have led mathematicians to develop meaningful results and invoke more active interactions between mathematical theory and quantitative methods in finance. Under the Black–Scholes assumption, no simple analytic solution has been found for the arithmetic Asian option price. The explicit expression for the geometric Asian option price has been derived by Kemna and Vorst (1990) and Angus (1999), due to the log normal distribution of the geometric average of an asset price. Although there is no exact formula available for the arithmetic Asian option price, there is still an extensive list of previous results on numerical techniques. Of these, a few references are cited, but are far from being exhaustive.

Turnbull and Wakeman (1991) proposed numerical approximations, and Monte Carlo simulations were employed by Boyle *et al*. (1997). Geman and Yor (1993) derived the Laplace transform for a continuous arithmetic Asian option analytically, but it is impractical to find the numerical inversion of the Laplace transform. In one notable previous work from the viewpoint of PDEs, Rogers and Shi (1995) improved the PDE approach by formulating the price of an arithmetic Asian option as the solution of a PDE of a one-dimensional space variable. Employing a similar space reduction method, Vecer (2001) derived an alternative one-dimensional PDE for an arithmetic Asian option.

In spite of its simple mathematical tractability, the Black–Scholes model has been proven by many empirical studies, but there are inconsistences between the model results and complex financial market data. As a result, many attempts have been made to propose substitute models for the Black–Scholes model. Of the huge variety of alternative models, empirical observations appear to confirm that stochastic volatility models are more suitable candidates to capture the implied volatility smile and fat-tailed distribution of the asset price return.

Typical stochastic volatility models commonly include Hull and White (1987), Scott (1987), Stein and Stein

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(1991) and Heston (1993). There are various extensions, the empirical performances of which have been presented and compared quite extensively by Bakshi *et al.* (1997). It is commonly recognized that Heston's stochastic volatility model empirically outperforms other stochastic volatility models in addition to generating rich mathematical results and enjoying the positivity of the volatility process.

In addition, Heston's model is popular amongst academic researchers and practitioners due its analytical tractability, in particular for admitting semi-closed-form solutions for European vanilla options in terms of a Fourier transform (e.g. Heston 1993). More recently, Dragulescu and Yakovenko (2004) reported explicit formulas for the joint transition density function of an asset price with its variance for Heston's model. In addition to the many analytical attempts to exploit the intrinsic properties of Heston's model, various numerical methods have been proposed to price exotic path-dependent options.

To begin with, it is quite natural to focus on a Monte Carlo simulation to determine the prices of path-dependent options. Several studies are listed, among the more recent being, for instance, Broadie and Kaya (2006), Anderson (2008) and Lord *et al.* (2010), where Monte Carlo simulations have been dealt with extensively. The reader may consult Ballestra *et al.* (2007) for a more extensive list of relevant references, particularly in the framework of Heston's model for various exotic path-dependent options.

Vecer (2002), Fouque and Han (2003) and Vecer and Xu (2004) focus on the valuation of arithmetic Asian options. It was shown that path dependence can be removed from the analysis for the valuation of arithmetic Asian options that are either continuously or discretely sampled, under general frameworks that include jump-diffusion models and mean-reverting stochastic volatility models. However, in general, the explicit closed-form solution for the arithmetic Asian option is not available and an additional numerical procedure is typically required, although the resulting PDE has been simplified significantly.

It seems that pricing geometric Asian option has received relatively less scholarly attention than its arithmetic counterpart. Wong and Cheung (2004) considered a fast mean-reverting stochastic volatility economy and derived approximate pricing formulae for a geometric Asian option based upon the asymptotic analysis used by Fouque and Han (2003).

In this paper, Heston's stochastic volatility model is considered rather than general stochastic volatility models and the aim is to derive exact and semi-analytic expressions for geometric Asian option prices. Quite successful analytic results are obtained in spite of the essential complexity of the two-dimensional diffusion process along with a mean-reverting square-root process for the variance. Explicit formulae are derived for the values of geometric Asian options with fixed and floating strike prices under Heston's stochastic volatility model. It should be stressed that the closed-form valuation formulae under the framework given in theorems 4.1 and 4.3 are entirely new in the existing literature.

Throughout this work, widely used techniques are employed such as change of measure and a Fourier transform inversion formula. First, the mathematically and technically challenging task of explicitly finding the generalized joint Fourier transform of the logarithm of the stock price and the logarithm of the geometric average of the stock price over a certain time period must be undertaken. This requires the evaluation of the generalized joint Fourier transform of a square-root process and its three differently weighted integrals with constant, linear and quadratic functions as weights, respectively. From a theoretical point of view, this result is an extension of a well-known result for the joint moment generating function of a square-root process and its temporal integral (e.g. Lamberton and Lapeyre 1995). This leads to the setup of complex-valued generalized Riccati equations with time-dependent coefficients that are solved explicitly in the form of a series expansion. To the authors' knowledge, explicit solutions to generalized Riccati equations are not available in the literature, in contrast to the well-known classical Riccati equation. This method can be a starting point for deriving the generalized Fourier transform under more general affine frameworks. It is the authors' belief that proposition 3.2 and theorem 3.3, in which explicit solutions for joint Fourier transforms of the relevant processes are provided, possess their own mathematical merits.

The rest of the paper is organized as follows. In section 2, Heston's model and geometric Asian options are briefly described. In section 3, theoretical results for the generalized joint Fourier transform of a log return of the asset and its geometric average are derived. In section 4, the prices of four different types of geometric Asian options are given. Section 5 describes the numerical results and compares this method with a Monte Carlo simulation. Section 6 concludes the paper.

2. Geometric Asian options under Heston's model

It is assumed that, under the risk-neutral measure Q, the evolution of the underlying asset price satisfies the following stochastic differential equation:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{v_t}S_t\,\mathrm{d}W_t, \tag{1}$$

$$\mathrm{d}v_t = \kappa(\theta - v_t)\mathrm{d}t + \sigma\sqrt{v_t}\,\mathrm{d}Z_t, \tag{2}$$

with positive initial values S_0 and v_0, where r is the interest rate, $\sqrt{v_t}$ is a volatility process, θ is the long-run average of v_t, κ is the rate of mean reversion and σ is the volatility of volatility. Here, W_t and Z_t are two Brownian motions with a correlation coefficient $\rho\in[-1,1]$, i.e. $\mathrm{d}W_t\,\mathrm{d}Z_t=\rho\,\mathrm{d}t$. The parameters θ, κ and σ are assumed to be positive constants. It is often convenient to write

$$W_t = \rho Z_t + \sqrt{1-\rho^2}\,\tilde{W}_t, \tag{3}$$

where $\tilde{W}_t$ is a standard Brownian motion independent of Z_t. We consider a filtration $\{\mathcal{F}_t : t \geq 0\}$, where $\mathcal{F}_t$ is the smallest σ-algebra generated by $\{Z_u, \tilde{W}_u : u \leq t\}$.

It is well known that a unique strong solution for (2) exists according to Yamada and Watanabe (e.g. Karatzas and Shreve 1997, proposition 2.13). In addition, it is noteworthy that the Feller condition (e.g. Karatzas and Shreve 1997, theorem 5.29) implies that $v_t > 0$ for all $t \geq 0$ with probability 1 if and only if

$$\frac{2\kappa\theta}{\sigma^2} \geq 1. \tag{4}$$

On the other hand, if

$$\frac{2\kappa\theta}{\sigma^2} < 1, \tag{5}$$

then $v_t = 0$ for some $t > 0$ with probability 1. Nevertheless, the set $\{t \geq 0 : v_t = 0\}$ has Lebesque measure zero with probability 1 and $v_t > 0$ with probability 1 for each fixed $t \geq 0$ (see the comments below of Andersen and Pitersburg 2007, proposition 2.1). We also mention that the results presented in this work hold in general regardless of the validity of the Feller condition (4). In order to confirm these results, in section 5 we present and test two numerical examples for which (4) and (5) hold, respectively.

In financial markets, there are four types of European-style continuously sampled geometric Asian option contracts: fixed-strike geometric Asian call, floating-strike geometric Asian call, fixed-strike geometric Asian put and floating-strike geometric Asian put. The payoffs for these options are now described in more detail.

Let $G_{[0,T]}$ be the geometric mean of S_t over time t during $[0, T]$, i.e.

$$G_{[0,T]} = \exp\left(\frac{1}{T}\int_0^T \ln S_u \mathrm{d}u\right).$$

The payoffs of fixed-strike geometric Asian call and put options with strike price K and maturity T are given, respectively, by

$$\max\{G_{[0,T]} - K, 0\}, \quad \max\{K - G_{[0,T]}, 0\}.$$

Floating-strike geometric Asian call and put options have respective payoffs of the form

$$\max\{S_T - G_{[0,T]}, 0\}, \quad \max\{G_{[0,T]} - S_T, 0\}.$$

In this work, two theorems are proven, theorems 4.1 and 4.3, in which the explicit closed-form formulae for the prices of a fixed-strike geometric Asian call and a floating-strike geometric Asian put option are presented. The prices of the other two types of geometric Asian options are given in corollaries 4.2 and 4.4 by put–call parity.

Before the option prices are exploited it is noted that, by Jensen's inequality,

$$\mathbb{E}^Q[G_{[0,T]}] \leq \mathbb{E}^Q\left[\frac{1}{T}\int_0^T S_u \, \mathrm{d}u\right] = \frac{S_0}{T}\int_0^T \mathrm{e}^{ru} \, \mathrm{d}u < \infty, \tag{6}$$

where the martingale property of $\{\mathrm{e}^{-rt}S_t\}$ under Q is used (e.g., proposition 2.5 of Andersen and Piterbarg (2007)). Thus, $G_{[0,T]}$ is integrable under the risk-neutral measure Q. The fixed-strike geometric Asian call option with payoff $\max\{G_{[0,T]} - K, 0\}$ at maturity T has price $C_{[0,T]}(t)$ at time $t \in [0, T]$, where

$$\begin{aligned} C_{[0,T]}(t) &= \mathbb{E}^Q[\mathrm{e}^{-r(T-t)} \max\{G_{[0,T]} - K, 0\} \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u \mathrm{d}u} \mathbb{E}^Q[\max\{G_{t,T} - K_{t,T}, 0\} \mid \mathcal{F}_t], \end{aligned} \tag{7}$$

where, for $0 \leq t \leq T$,

$$\begin{aligned} G_{t,T} &\equiv \mathrm{e}^{(1/T)\int_t^T \ln S_u \, \mathrm{d}u}, \\ K_{t,T} &\equiv K\mathrm{e}^{-(1/T)\int_0^t \ln S_u \, \mathrm{d}u}. \end{aligned} \tag{8}$$

The floating-strike geometric Asian put option with payoff $\max\{G_{[0,T]} - S_T, 0\}$ at maturity T has price $\tilde{P}_{[0,T]}(t)$ at time $t \in [0, T]$, where

$$\begin{aligned} \tilde{P}_{[0,T]}(t) &= \mathbb{E}^Q[\mathrm{e}^{-r(T-t)} \max\{G_{[0,T]} - S_T, 0\} \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u \mathrm{d}u} \\ &\quad \times \mathbb{E}^Q[\max\{G_{t,T} - S_T\mathrm{e}^{-(1/T)\int_0^t \ln S_u \mathrm{d}u}, 0\} \mid \mathcal{F}_t]. \end{aligned} \tag{9}$$

To shorten the argument, the derivation of analogous counterparts for other geometric Asian options is omitted.

3. Generalized joint Fourier transforms

This section is devoted to the derivation of the conditional joint Fourier transform of the logarithm of the stock price at maturity, and the logarithm of the geometric mean of the stock price over a certain time period. This is done before the main results are presented in the next section. Recall the definition of $G_{t,T}$ from (8) and denote by ψ_t the conditional joint Fourier transform of $\ln G_{t,T}$ and $\ln S_T$, conditioned on $\mathcal{F}_t$ under Q. Let

$$\psi_t(s, w) \equiv \mathbb{E}^Q[\mathrm{e}^{s \ln G_{t,T} + w \ln S_T} \mid \mathcal{F}_t], \tag{10}$$

for complex numbers s and w for which the right-hand side of (10) is well defined. In order to derive ψ_t, two procedures are performed, each of which has an independent interest of its own. The first step is to transform ψ_t into an expression as the joint Fourier transform of a square-root process and of its three different integrals with constant, linear and quadratic weights as stated in proposition 3.1. Next, the joint Fourier transform of v_t and its three integrals are represented explicitly as an exponential affine function of v_0, as described in proposition 3.2.

From a mathematical point of view, proposition 3.2 is non-trivial and a meaningful extension of the well-known result concerning the joint moment generating function of a square-root process and its temporal integral (e.g. Lamberton and Laperyre 1995). The task is accomplished by solving the accompanying complex-valued Riccati equation with time-dependent coefficients in the form of a series expansion. In addition, it is also necessary to settle the issue of convergence of the integrals. The logarithm of complex-valued functions has to be properly defined in order to avoid discontinuities, which turns out to be technically complicated as demonstrated in the proof of proposition 3.2 (deferred to the appendix). Expression (23) can be understood such that the exponential affine structure of a square-root process and its temporal integral is passed on when two more weighted integrals are added. Since a square-root process on its own is an important theoretical model for the equity price, interest rate and credit risk, it is expected that the joint Fourier transform along with the explicit expression has its own mathematical merits and will provide a useful tool for pricing options under those models.

Finally, in theorem 3.3, propositions 3.1 and 3.2 are combined to express ψ_t as an exponential affine function of ln S_t and v_t, as demonstrated in (25). This can be rearranged into a different form to emphasize the affine property clearly, but is left as in (25) for computational purposes. The next proposition serves as a starting point of this section. Let

$$\mathcal{D} \equiv \{(s,w) \in \mathbb{C}^2 : \mathrm{Re}(s) \geq 0,\ \mathrm{Re}(w) \geq 0,\\ 0 \leq \mathrm{Re}(s) + \mathrm{Re}(w) \leq 1\}. \tag{11}$$

Proposition 3.1: *If* $t \in [0,T]$ *and* $(s,w) \in \mathcal{D}$, *then* $\mathbb{E}^Q|e^{s \ln G_{t,T} + w \ln S_T}| < \infty$ *and*

$$\psi_t(s,w) = e^{z_0}\mathbb{E}^Q[e^{z_1\int_t^T (T-\tau)^2 v_\tau d\tau + z_2\int_t^T (T-\tau)v_\tau d\tau + z_3\int_t^T v_\tau d\tau + z_4 v_T} \mid v_t], \tag{12}$$

where

$$\begin{aligned} z_0 &\equiv z_0(s,w) \\ &= s\left(\frac{T-t}{T}\ln S_t + \frac{(r\sigma - \kappa\theta\rho)(T-t)^2}{2\sigma T} - \frac{\rho(T-t)}{\sigma T}v_t\right) \\ &\quad + w\left(\ln S_t - \frac{\rho}{\sigma}v_t + \left(r - \frac{\rho\kappa\theta}{\sigma}\right)(T-t)\right), \\ z_1 &\equiv z_1(s,w) = \frac{s^2(1-\rho^2)}{2T^2}, \\ z_2 &\equiv z_2(s,w) = \frac{s(2\rho\kappa - \sigma)}{2\sigma T} + \frac{sw(1-\rho^2)}{T}, \\ z_3 &\equiv z_3(s,w) = \frac{s\rho}{\sigma T} + \frac{w(2\rho\kappa - \sigma)}{2\sigma} + \frac{w^2(1-\rho^2)}{2}, \\ z_4 &\equiv z_4(s,w) = \frac{w\rho}{\sigma}. \end{aligned} \tag{13}$$

Proof: Suppose that $(s,w) \in \mathcal{D}$. If $0 \leq t < T$, then

$$\begin{aligned} \mathbb{E}^Q[|e^{s\ln G_{t,T} + w\ln S_T}|] &\leq \mathbb{E}^Q\left[\frac{1}{T-t}\int_t^T S_u\, du + S_T\right] \\ &= \left(\frac{1}{T-t}\int_t^T e^{ru}\, du + e^{rT}\right)S_0. \end{aligned}$$

We omit $t = T$ due to its triviality. To derive (12), we start from (1) and (2) to obtain

$$\begin{aligned} \ln S_T &= \ln S_t - \frac{\rho v_t}{\sigma} + \left(r - \frac{\rho\kappa\theta}{\sigma}\right)(T-t) + \left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right)\int_t^T v_\tau\, d\tau \\ &\quad + \frac{\rho}{\sigma}v_T + \sqrt{1-\rho^2}\int_t^T \sqrt{v_\tau}\, d\tilde{W}_\tau. \end{aligned} \tag{14}$$

On the other hand, we have

$$\begin{aligned} \ln G_{t,T} &= \frac{1}{T}\int_t^T \ln S_u\, du \\ &= \frac{1}{T}\left((T-t)\ln S_t + \frac{r(T-t)^2}{2} - \frac{1}{2}\int_t^T (T-\tau)v_\tau\, d\tau \right.\\ &\quad + \rho\int_t^T (T-\tau)\sqrt{v_\tau}\, dZ_\tau \\ &\quad \left. + \sqrt{1-\rho^2}\int_t^T (T-\tau)\sqrt{v_\tau}\, d\tilde{W}_\tau\right). \end{aligned} \tag{15}$$

Using

$$d(\tau v_\tau) = (v_\tau + \tau\kappa(\theta - v_\tau))\, d\tau + \tau\sigma\sqrt{v_\tau}\, dZ_\tau,$$

we have

$$\begin{aligned} &\int_t^T (T-\tau)\sqrt{v_\tau}\, dZ_\tau \\ &= \frac{1}{\sigma}\left(\kappa\int_t^T (T-\tau)v_\tau\, d\tau + \int_t^T v_\tau\, d\tau - (T-t)v_t - \frac{\kappa\theta(T-t)^2}{2}\right). \end{aligned} \tag{16}$$

Substituting (16) into (15) yields

$$\begin{aligned} \ln G_{t,T} = \frac{1}{T}\Bigg[&(T-t)\ln S_t + \frac{(r\sigma - \kappa\theta\rho)(T-t)^2}{2\sigma} \\ &+ \frac{2\rho\kappa - \sigma}{2\sigma}\int_t^T (T-\tau)v_\tau\, d\tau \\ &+ \frac{\rho}{\sigma}\int_t^T v_\tau\, d\tau - \frac{\rho}{\sigma}(T-t)v_t \\ &+ \sqrt{1-\rho^2}\int_t^T (T-\tau)\sqrt{v_\tau}\, d\tilde{W}_\tau\Bigg]. \end{aligned} \tag{17}$$

Let $\mathcal{G}$ be the σ-field generated by $\mathcal{F}_t$ and $Z_u : t < u \leq T\}$. By (14) and (17), we have, for $(s,w) \in \mathcal{D}$,

$$\begin{aligned} \psi_t(s,w) &= \mathbb{E}^Q[\mathbb{E}^Q[e^{s\ln G_{t,T} + w\ln S_T} \mid \mathcal{G}] \mid \mathcal{F}_t] \\ &= A_1\mathbb{E}^Q[A_2\,\mathbb{E}^Q[A_3 \mid \mathcal{G}] \mid \mathcal{F}_t], \end{aligned} \tag{18}$$

where

$$\begin{aligned} A_1 &= \exp\left(s\left(\frac{T-t}{T}\ln S_t + \frac{(r\sigma - \kappa\theta\rho)(T-t)^2}{2\sigma T} - \frac{\rho(T-t)}{\sigma T}v_t\right)\right. \\ &\qquad \left. + w\left(\ln S_t - \frac{\rho}{\sigma}v_t + \left(r - \frac{\rho\kappa\theta}{\sigma}\right)(T-t)\right)\right), \\ A_2 &= \exp\left(s\left(\frac{2\rho\kappa - \sigma}{2\sigma T}\int_t^T (T-\tau)v_\tau\, d\tau + \frac{\rho}{\sigma T}\int_t^T v_\tau\, d\tau\right)\right. \\ &\qquad \left. + w\left(\frac{\rho}{\sigma}v_T + \left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right)\int_t^T v_\tau\, d\tau\right)\right), \\ A_3 &= \exp\left(\sqrt{1-\rho^2}\int_t^T \left(\frac{s}{T}(T-\tau) + w\right)\sqrt{v_\tau}\, d\tilde{W}_\tau\right). \end{aligned}$$

Substituting

$$\begin{aligned}&\mathbb{E}^{Q}[A_3 \mid \mathcal{G}]\\ &= \exp\left(\frac{1-\rho^2}{2}\int_t^T\left(\frac{s^2}{T^2}(T-\tau)^2+\frac{2sw}{T}(T-\tau)+w^2\right)v_\tau\,\mathrm{d}\tau\right)\end{aligned}$$

into (18) and applying the Markov property of $\{v_u : 0 \le u \le T\}$ leads to (12), which completes the proof. □

Before proceeding to investigate ψ_t, it is obvious from proposition 3.1 that we need to search for an exact formula for the generalized joint Fourier transform of v_t and of the three different integrals of v_t appearing in (12). This turns out to be essential in the proof of theorem 3.3 and may also be of independent interest. For this purpose, two series of functions are introduced as follows. Define F_τ and $\tilde{F}_\tau : \mathbb{C}^4 \to \mathbb{C}$ as

$$F_\tau(z_1, z_2, z_3, z_4) = \sum_{n=0}^{\infty} f_n, \tag{19}$$

$$\tilde{F}_\tau(z_1, z_2, z_3, z_4) = \sum_{n=1}^{\infty} \frac{n}{\tau} f_n, \tag{20}$$

where f_n, $n = 0, 1, 2, \ldots$, are functions of z_1, z_2, z_3, and z_4 defined as

$$\begin{aligned}&f_{-2} = f_{-1} = 0, \quad f_0 = 1,\\ &f_1 = \frac{(\kappa - z_4\sigma^2)\tau}{2},\\ &f_n = -\frac{\sigma^2\tau^2}{2n(n-1)}\left(z_1\tau^2 f_{n-4} + z_2\tau f_{n-3} + \left(z_3 - \frac{\kappa^2}{2\sigma^2}\right)f_{n-2}\right),\\ &\quad n \ge 2.\end{aligned} \tag{21}$$

Let

$$\begin{aligned}\mathcal{D}_\tau \equiv \{&(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 :\\ &\mathbb{E}^{Q}[\mathrm{e}^{\mathrm{Re}(z_1)\int_0^\tau(\tau-t)^2 v_t\,\mathrm{d}t + \mathrm{Re}(z_2)\int_0^\tau(\tau-t)v_t\,\mathrm{d}t + \mathrm{Re}(z_3)\int_0^\tau v_t\,\mathrm{d}t + \mathrm{Re}(z_4)v_\tau}]\\ &< \infty\}.\end{aligned} \tag{22}$$

Proposition 3.2:

(a) *For every $(z_1, z_2, z_3, z_4) \in \mathcal{D}_\tau$, $F_\tau(z_1, z_2, z_3, z_4) \ne 0$. The argument of F_τ, $\arg F_\tau$, is uniquely determined on $\mathcal{D}_\tau$ with the following properties.*

(1) *If z_1, z_2, z_3, and z_4 are real numbers in $\mathcal{D}_\tau$, then* $\arg F_\tau(z_1, z_2, z_3, z_4) = 0$.

(2) $\arg F_\tau$ *is continuous on $\mathcal{D}_\tau$.*

(b) *With the argument of F_τ on $\mathcal{D}_\tau$ defined as above, we have, for $(z_1, z_2, z_3, z_4) \in \mathcal{D}_\tau$,*

$$\begin{aligned}&\mathbb{E}^{Q}\left[\mathrm{e}^{z_1\int_0^\tau(\tau-t)^2 v_t\,\mathrm{d}t + z_2\int_0^\tau(\tau-t)v_t\,\mathrm{d}t + z_3\int_0^\tau v_t\,\mathrm{d}t + z_4 v_\tau}\right]\\ &= \exp\left(\frac{\kappa v_0 + \kappa^2\theta\tau}{\sigma^2} - \frac{2v_0}{\sigma^2}\frac{\tilde{F}_\tau(z_1, z_2, z_3, z_4)}{F_\tau(z_1, z_2, z_3, z_4)}\right.\\ &\qquad\left. - \frac{2\kappa\theta}{\sigma^2}\ln F_\tau(z_1, z_2, z_3, z_4)\right).\end{aligned} \tag{23}$$

Proof: The rigorous and technical proof is deferred to the appendix. □

It is noted from (23) that $\mathcal{D}_\tau$ is not affected by the initial state v_0 and the expectation is expressed as an exponential affine function of v_0. It is also noted that the series in (19) and (20) converge for every $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4$. In fact, it can be shown that the radii of convergence for the power series $\sum_{n=0}^{\infty} f_n x^n$ and $\sum_{n=1}^{\infty} n f_n x^n$ are infinity. Besides, F_τ and $\tilde{F}_\tau$ are analytic functions of four complex variables on $\mathbb{C}^4$.

Applying proposition 3.2 to proposition 3.1 leads to the expression of ψ_t explicitly as in theorem 3.3. In order to depict the result briefly, new functions need to be introduced. Define $H_{t,T}$ and $\tilde{H}_{t,T} : \mathbb{C}^2 \to \mathbb{C}$ as

$$\begin{aligned}H_{t,T}(s, w) &= F_{T-t}(z_1(s, w), z_2(s, w), z_3(s, w), z_4(s, w)),\\ \tilde{H}_{t,T}(s, w) &= \tilde{F}_{T-t}(z_1(s, w), z_2(s, w), z_3(s, w), z_4(s, w)),\end{aligned} \tag{24}$$

where $z_i(s, w)$, $i = 1, 2, 3, 4$, were defined in (13). Recall the definitions of $\mathcal{D}$ from (11) and ψ_t from (10) and (12).

Theorem 3.3:

(a) *If $(s, w) \in \mathcal{D}$, then $H_{t,T}(s, w) \ne 0$. The argument of $H_{t,T}$, $\arg H_{t,T}$, is uniquely determined on $\mathcal{D}$ with the following properties.*

(1) $\arg H_{t,T}(s, w) = 0$ *if s and w are real.*

(2) $\arg H_{t,T}$ *is continuous on $\mathcal{D}$.*

(b) *For $(s, w) \in \mathcal{D}$, with $\arg H_{t,T}$ defined as above, $\psi_t(s, w)$ is given by*

$$\begin{aligned}\psi_t(s, w) = \exp\Bigg(&-a_1\frac{\tilde{H}_{t,T}(s, w)}{H_{t,T}(s, w)} - a_2 \ln H_{t,T}(s, w)\\ &+ a_3 s + a_4 w + a_5\Bigg),\end{aligned} \tag{25}$$

where

$$\begin{aligned}a_1 &= \frac{2v_t}{\sigma^2},\\ a_2 &= \frac{2\kappa\theta}{\sigma^2},\\ a_3 &= \frac{T-t}{T}\ln S_t + \frac{(r\sigma - \kappa\theta\rho)(T-t)^2}{2\sigma T} - \frac{\rho(T-t)}{\sigma T}v_t,\\ a_4 &= \ln S_t - \frac{\rho}{\sigma}v_t + \left(r - \frac{\rho\kappa\theta}{\sigma}\right)(T-t),\\ a_5 &= \frac{\kappa v_t + \kappa^2\theta(T-t)}{\sigma^2}.\end{aligned}$$

Proof: *Suppose that* $(s, w) \in \mathcal{D}$ and let $z_i = z_i(s, w)$, $i = 1, 2, 3, 4$, be given by (13). By closely examining proposition 3.1 and the definition of $\mathcal{D}_{T-t}$ in (22), it is noted that $(z_1, z_2, z_3, z_4) \in \mathcal{D}_T \subset \mathcal{D}_{T-t}$ for any $0 \le t \le T$. It is also observed that (23) remains valid for any positive values of v_0. Now the time homogeneous Markov property of v_t implies that (25) is well established when propositions 3.1 and 3.2 are applied. □

The expressions for $H_{t,T}$ and $\tilde{H}_{t,T}$ are now rewritten in a form ready for use later. Substituting the definition of z_i described in (13) into (19) and (20), $H_{t,T}$ and $\tilde{H}_{t,T}$ are expressed as follows. For $(s, w) \in \mathbb{C}^2$,

$$H_{t,T}(s, w) = \sum_{n=0}^{\infty} h_n(s, w), \tag{26}$$

$$\tilde{H}_{t,T}(s, w) = \sum_{n=1}^{\infty} \frac{n}{T-t} h_n(s, w), \tag{27}$$

where

$$\begin{aligned} h_{-2}(s,w) &= h_{-1}(s,w) = 0, \quad h_0(s,w) = 1, \\ h_1(s,w) &= \frac{(T-t)(\kappa - w\rho\sigma)}{2}, \\ h_n(s,w) &= \frac{(T-t)^2}{4n(n-1)T^2}(-s^2\sigma^2(1-\rho^2)(T-t)^2 h_{n-4}(s) \\ &\quad + (s\sigma T(\sigma - 2\rho\kappa) - 2sw\sigma^2 T(1-\rho^2))(T-t)h_{n-3}(s) \\ &\quad + T(\kappa^2 T - 2s\rho\sigma - w(2\rho\kappa - \sigma) \\ &\quad \times \sigma T - w^2(1-\rho^2)\sigma^2 T)h_{n-2}(s)), \quad n \geq 2. \end{aligned} \tag{28}$$

In the same manner as we have inferred infinite radii of convergence for the power series $\sum_{n=0}^{\infty} f_n x^n$ and $\sum_{n=0}^{\infty} n f_n x^n$ previously, the power series $\sum_{n=0}^{\infty} h_n(s, w)x^n$ and $\sum_{n=1}^{\infty} n h_n(s, w)x^n$ also have infinite radii of convergence. In addition, $H_{t,T}$ and $\tilde{H}_{t,T}$ are analytic functions of two complex variables on $\mathbb{C}^2$.

4. Geometric Asian option prices

In this section, analytic formulae are derived for the prices of fixed-strike and floating-strike geometric Asian options as a consequence of theorem 3.3. Throughout this section, two well-known methods are used repeatedly, numeraire change and the Fourier inversion formula, which go back to, for instance, Geman *et al.* (1995) and Gil-Pelaez (1951), respectively. Recall the definitions of $G_{t,T}$ and $K_{t,T}$ from (8) and the expression for ψ_t given in (25).

4.1. Fixed-strike Asian options

As defined earlier, the payoff for a fixed-strike Asian call is written as $\max\{G_{[0,T]} - K, 0\}$, the price of which at $t \in [0.T]$ is given by the following.

Theorem 4.1: *The price $C_{[0,T]}(t)$ of a fixed-strike geometric Asian call option at time $t \in [0, T]$ is given by*

$$\begin{aligned} C_{[0,T]}(t) &= \mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u \, \mathrm{d}u} \\ &\quad \times \left[\frac{\psi_t(1,0) - K_{t,T}}{2} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}((\psi_t(1+i\xi, 0) \right. \\ &\quad \left. - K_{t,T}\psi_t(i\xi, 0)) \frac{\mathrm{e}^{-i\xi \ln K_{t,T}}}{i\xi} \Big) \mathrm{d}\xi \right], \end{aligned} \tag{29}$$

where $K_{t,T}$ and ψ_t are given by (8) *and* (25).

Proof: From (7),

$$\begin{aligned} C_{[0,T]}(t) &= \mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u \, \mathrm{d}u}(\mathbb{E}^Q[G_{t,T} 1_{\{G_{t,T} > K_{t,T}\}} \mid \mathcal{F}_t] \\ &\quad - K_{t,T} Q(G_{t,T} > K_{t,T} \mid \mathcal{F}_t)). \end{aligned} \tag{30}$$

From (6), we can introduce a probability measure Q^* on $\mathcal{F}_T$ by

$$\mathrm{d}Q^* \equiv \frac{G_{[0,T]}}{\mathbb{E}^Q[G_{[0,T]}]} \mathrm{d}Q. \tag{31}$$

Since

$$\begin{aligned} Q^*(G_{t,T} > K_{t,T} \mid \mathcal{F}_t) &= \frac{\mathbb{E}^Q[1_{\{G_{t,T} > K_{t,T}\}} G_{[0,T]} \mid \mathcal{F}_t]}{\mathbb{E}^Q[G_{[0,T]} \mid \mathcal{F}_t]} \\ &= \frac{\mathbb{E}^Q[1_{\{G_{t,T} > K_{t,T}\}} G_{t,T} \mid \mathcal{F}_t]}{\mathbb{E}^Q[G_{t,T} \mid \mathcal{F}_t]}, \end{aligned}$$

equation (30) can be written as

$$\begin{aligned} C_{[0,T]}(t) &= \mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u \, \mathrm{d}u} \\ &\quad \times (Q^*(G_{t,T} > K_{t,T} \mid \mathcal{F}_t)\mathbb{E}^Q[G_{t,T} \mid \mathcal{F}_t] \\ &\quad - K_{t,T} Q(G_{t,T} > K_{t,T} \mid \mathcal{F}_t)). \end{aligned} \tag{32}$$

It is now necessary to compute the three conditional expectations appearing on the right-hand side of (32) in terms of ψ_t. First, it is noted that

$$\mathbb{E}^Q[G_{t,T} \mid \mathcal{F}_t] = \psi_t(1, 0). \tag{33}$$

Second, to deal with $Q(G_{t,T} > K_{t,T} | \mathcal{F}_t)$, is it observed that $\psi_t(i\xi, 0)$, $\xi \in \mathbb{R}$, is the conditional characteristic function of $\ln G_{t,T}$, given $\mathcal{F}_t$, under the probability measure Q. The inversion formula of the characteristic functions yields

$$Q(G_{t,T} > K_{t,T} \mid \mathcal{F}_t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left(\psi_t(i\xi, 0) \frac{\mathrm{e}^{-i\xi \ln K_{t,T}}}{i\xi} \right) \mathrm{d}\xi. \tag{34}$$

Third, an expression is derived for $Q^*(G_{t,T} > K_{t,T} \mid \mathcal{F}_t)$. Under the probability measure Q^*, the conditional characteristic function of $\ln G_{t,T}$, given $\mathcal{F}_t$, is given by

$$\begin{aligned} \mathbb{E}^{Q^*}[\mathrm{e}^{i\xi \ln G_{t,T}} \mid \mathcal{F}_t] &= \frac{\mathbb{E}^Q[\mathrm{e}^{i\xi \ln G_{t,T}} G_{[0,T]} \mid \mathcal{F}_t]}{\mathbb{E}^Q[G_{[0,T]} \mid \mathcal{F}_t]} \\ &= \frac{\mathbb{E}^Q[\mathrm{e}^{i\xi \ln G_{t,T}} G_{t,T} \mid \mathcal{F}_t]}{\mathbb{E}^Q[G_{t,T} \mid \mathcal{F}_t]} \\ &= \frac{\psi_t(1+i\xi, 0)}{\psi_t(1, 0)}. \end{aligned}$$

Hence, the inversion formula of the characteristic functions yields

$$\begin{aligned} &Q^*(G_{t,T} > K_{t,T} \mid \mathcal{F}_t) \\ &\quad = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left(\frac{\psi_t(1+i\xi, 0)}{\psi_t(1,0)} \frac{\mathrm{e}^{-i\xi \ln K_{t,T}}}{i\xi} \right) \mathrm{d}\xi. \end{aligned} \tag{35}$$

Plugging (33), (34) and (35) into (32) yields (29). □

The payoff of a fixed-strike geometric Asian put is written as $\max\{K - G_{[0,T]}, 0\}$. By put–call parity the price of the fixed-strike geometric Asian put at time t is given as follows.

Corollary 4.2: *The price $P_{[0,T]}(t)$ of the fixed-strike geometric Asian put option at time $t \in [0,T]$ is given by*

$$P_{[0,T]}(t) = \mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u\,\mathrm{d}u} \times \left[\frac{K_{t,T}-\psi_t(1,0)}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left((\psi_t(1+i\xi,0) - K_{t,T}\psi_t(i\xi,0))\frac{\mathrm{e}^{-i\xi\ln K_{t,T}}}{i\xi}\right)\mathrm{d}\xi\right],$$

where $K_{t,T}$ and ψ_t are given by (8) *and* (25).

Proof: As is clear from the payoffs of the fixed-strike geometric Asian put and call at maturity T, we have the put–call parity

$$\begin{aligned} C_{[0,T]}(t) - P_{[0,T]}(t) &= \mathrm{e}^{-r(T-t)}\mathbb{E}^Q[G_{[0,T]} - K \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u\,\mathrm{d}u}\psi_t(1,0) \\ &\quad - \mathrm{e}^{-r(T-t)}K, \end{aligned}$$

where the last equality is due to (33). This completes the proof. □

4.2. Floating-strike Asian options

As defined in an earlier section, the floating-strike geometric Asian put option has payoff $\max\{G_{[0,T]} - S_T, 0\}$. The price at time t is derived in the following theorem by employing the analogous methodology as in theorem 4.1.

Theorem 4.3: *The price $\tilde{P}_{[0,T]}(t)$ at time $t \in [0,T]$ of the floating-strike geometric Asian put is given by*

$$\begin{aligned} \tilde{P}_{[0,T]}(t) = \mathrm{e}^{-r(T-t)}\Bigg[&\frac{1}{2}\left(\mathrm{e}^{(1/T)\int_0^t \ln S_u\,\mathrm{d}u}\psi_t(1,0) - \mathrm{e}^{r(T-t)}S_t\right) \\ &+ \frac{1}{\pi}\int_0^\infty \mathrm{Re}\Bigg((\mathrm{e}^{(1/T)\int_0^t \ln S_u\,\mathrm{d}u}\psi_t(1+i\xi,-i\xi) \\ &- \psi_t(i\xi,1-i\xi))\frac{\mathrm{e}^{i(\xi/T)\int_0^t \ln S_u\,\mathrm{d}u}}{i\xi}\Bigg)\mathrm{d}\xi\Bigg], \end{aligned} \tag{36}$$

where ψ_t is given by (25).

Proof: From (9),

$$\begin{aligned} \tilde{P}_{[0,T]}(t) &= \mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u\,\mathrm{d}u} \\ &\quad \times \mathbb{E}^Q[\max\{G_{t,T} - S_T\mathrm{e}^{-(1/T)\int_0^t \ln S_u\,\mathrm{d}u}, 0\} \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u\,\mathrm{d}u} \\ &\quad \times \mathbb{E}^Q[G_{t,T}\,1_{\{\ln(G_{t,T}/S_T) > -(1/T)\int_0^t \ln S_u\,\mathrm{d}u\}} \mid \mathcal{F}_t] \\ &\quad - \mathrm{e}^{-r(T-t)}\mathbb{E}^Q[S_T\,1_{\{\ln(G_{t,T}/S_T) > -(1/T)\int_0^t \ln S_u\,\mathrm{d}u\}} \mid \mathcal{F}_t]. \end{aligned} \tag{37}$$

The conditional expectation in the first term on the right-hand side of (37) is given by

$$\begin{aligned} &\mathbb{E}^Q[G_{t,T}\,1_{\{\ln(G_{t,T}/S_T) > -(1/T)\int_0^t \ln S_u\,\mathrm{d}u\}} \mid \mathcal{F}_t] \\ &\quad = \mathbb{E}^Q[G_{t,T} \mid \mathcal{F}_t]\,Q^*\left(\ln\frac{G_{t,T}}{S_T} > -\frac{1}{T}\int_0^t \ln S_u\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right) \\ &\quad = \psi_t(1,0)Q^*\left(\ln\frac{G_{t,T}}{S_T} > -\frac{1}{T}\int_0^t \ln S_u\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right), \end{aligned} \tag{38}$$

where Q^*, the probability measure given by (31) and (33), has been used for the last equality. Under the probability measure Q^*, the conditional characteristic function of ln $(G_{t,T}/S_T)$, given $\mathcal{F}_t$, is given by

$$\begin{aligned} \mathbb{E}^{Q^*}[\mathrm{e}^{i\xi\ln(G_{t,T}/S_T)} \mid \mathcal{F}_t] &= \frac{\mathbb{E}^Q[G_{t,T}\mathrm{e}^{i\xi\ln(G_{t,T}/S_T)} \mid \mathcal{F}_t]}{\mathbb{E}^Q[G_{t,T} \mid \mathcal{F}_t]} \\ &= \frac{\psi_t(1+i\xi,-i\xi)}{\psi_t(1,0)}. \end{aligned}$$

Hence, the Fourier inversion formula yields

$$\begin{aligned} &Q^*\left(\ln\frac{G_{t,T}}{S_T} > -\frac{1}{T}\int_0^t \ln S_u\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right) \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\psi_t(1+i\xi,-i\xi)}{\psi_t(1,0)}\frac{\mathrm{e}^{i\xi(1/T)\int_0^t \ln S_u\,\mathrm{d}u}}{i\xi}\right)\mathrm{d}\xi. \end{aligned} \tag{39}$$

The conditional expectation in the second term on the right-hand side of (37) is given by

$$\begin{aligned} &\mathbb{E}^Q[S_T\,1_{\{\ln(G_{t,T}/S_T) > -(1/T)\int_0^t \ln S_u\,\mathrm{d}u\}} \mid \mathcal{F}_t] \\ &\quad = \mathbb{E}^Q[S_T \mid \mathcal{F}_t]\,Q^{**}\left(\ln\frac{G_{t,T}}{S_T} > -\frac{1}{T}\int_0^t \ln S_u\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right) \\ &\quad = \mathrm{e}^{r(T-t)}S_tQ^{**}\left(\ln\frac{G_{t,T}}{S_T} > -\frac{1}{T}\int_0^t \ln S_u\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right), \end{aligned} \tag{40}$$

where Q^{**} is the probability measure of $\mathcal{F}_T$ defined as

$$\mathrm{d}Q^{**} \equiv \frac{\mathrm{e}^{-rT}S_T}{S_0}\,\mathrm{d}Q.$$

Q^{**} is well-defined due to the martingale property of $\mathrm{e}^{-rt}S_t$ under Q. Under the probability measure Q^{**}, the conditional characteristic function of $\ln(G_{t,T}/S_T)$, given $\mathcal{F}_t$, is given by

$$\begin{aligned} \mathbb{E}^{Q^{**}}[\mathrm{e}^{i\xi\ln(G_{t,T}/S_T)} \mid \mathcal{F}_t] &= \frac{\mathbb{E}^Q[S_T\mathrm{e}^{i\xi\ln(G_{t,T}/S_T)} \mid \mathcal{F}_t]}{\mathbb{E}^Q[S_T \mid \mathcal{F}_t]} \\ &= \frac{\psi_t(i\xi,1-i\xi)}{\mathrm{e}^{r(T-t)}S_t}. \end{aligned}$$

Hence, the Fourier inversion formula yields

$$\begin{aligned} &Q^{**}\left(\ln\frac{G_{t,T}}{S_T} > -\frac{1}{T}\int_0^t \ln S_u\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right) \\ &\quad = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\psi_t(i\xi,1-i\xi)}{\mathrm{e}^{r(T-t)}S_t}\frac{\mathrm{e}^{i\xi(1/T)\int_0^t \ln S_u\,\mathrm{d}u}}{i\xi}\right)\mathrm{d}\xi. \end{aligned} \tag{41}$$

Plugging (38), (39), (40) and (41) into (37) yields (36). □

Table 1. Fixed-strike geometric Asian call option prices of different maturities and strike prices, varying n for I; $S_0=100$, $v_0=0.09$, $t=0$, $r=0.05$, $\kappa=1.15$, $\theta=0.348$, $\rho=-0.64$, $\sigma=0.39$, CPU times in seconds.

T	K	n	Option value	CPU	T	K	n	Option value	CPU	T	K	n	Option value	CPU
0.2	90	10	10.6571	0.024	1.5	90	10	16.5062	0.027	3	90	10	20.5727	0.027
		20	10.6571	0.039			20	16.5030	0.042			20	20.5102	0.044
		30	10.6571	0.053			30	16.5030	0.058			30	20.5102	0.060
		40	10.6571	0.067			40	16.5030	0.073			40	20.5102	0.076
	95	10	6.5871	0.020		95	10	13.7653	0.026		95	10	18.3664	0.027
		20	6.5871	0.032			20	13.7625	0.041			20	18.3060	0.041
		30	6.5871	0.044			30	13.7625	0.056			30	18.3060	0.056
		40	6.5871	0.056			40	13.7625	0.071			40	18.3060	0.071
	100	10	3.4478	0.019		100	10	11.3395	0.026		100	10	16.3467	0.027
		20	3.4478	0.030			20	11.3374	0.040			20	16.2895	0.042
		30	3.4478	0.041			30	11.3374	0.055			30	16.2895	0.058
		40	3.4478	0.052			40	11.3374	0.069			40	16.2895	0.074
	105	10	1.4551	0.021		105	10	9.2254	0.026		105	10	14.5061	0.029
		20	1.4552	0.034			20	9.2245	0.042			20	14.4531	0.042
		30	1.4552	0.047			30	9.2245	0.057			30	14.4531	0.058
		40	1.4552	0.060			40	9.2245	0.074			40	14.4531	0.073
	110	10	0.4724	0.023		110	10	7.4117	0.027		110	10	12.8360	0.029
		20	0.4724	0.036			20	7.4122	0.042			20	12.7882	0.047
		30	0.4724	0.050			30	7.4122	0.058			30	12.7882	0.065
		40	0.4724	0.063			40	7.4122	0.073			40	12.7882	0.083

n denotes the number of terms we take in the series expansions of H and $\tilde{H}$ in (26) and (27).

The payoff of the fixed-strike Asian put is written as $\max\{K-G_{[0,T]},\ 0\}$. By put–call parity the price of the fixed-strike Asian put at time t is given as follows.

Corollary 4.4: *The price $\tilde{C}_{[0,T]}(t)$ of the floating-strike geometric Asian call option at time $t\in[0,T]$ is given by*

$$\begin{aligned}\tilde{C}_{[0,T]}(t) = \mathrm{e}^{-r(T-t)}\Bigg[&\frac{1}{2}(\mathrm{e}^{r(T-t)}S_t - \mathrm{e}^{(1/T)\int_0^t \ln S_u\,\mathrm{d}u}\psi_t(1,0))\\ &+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Bigg((\mathrm{e}^{(1/T)\int_0^t \ln S_u\,\mathrm{d}u}\psi_t(1+i\xi,\,-i\xi)\\ &-\psi_t(i\xi,1-i\xi))\frac{\mathrm{e}^{i(\xi/T)\int_0^t \ln S_u\,\mathrm{d}u}}{i\xi}\Bigg)\,\mathrm{d}\xi\Bigg],\end{aligned}$$

where ψ_t is given by (25).

Proof: Comparing with the payoffs for the floating-strike geometric Asian put and call at maturity T, the put–call parity is given by

$$\begin{aligned}\tilde{C}_{[0,T]}(t)-\tilde{P}_{[0,T]}(t) &= \mathrm{e}^{-r(T-t)}\mathbb{E}^Q[S_T-G_{[0,T]}\,|\,\mathcal{F}_t]\\ &= S_t-\mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u\,\mathrm{d}u}\mathbb{E}^Q[G_{t,T}\,|\,\mathcal{F}_t]\\ &= S_t-\mathrm{e}^{-r(T-t)+(1/T)\int_0^t \ln S_u\,\mathrm{d}u}\psi_t(1,0).\end{aligned}$$

This completes the proof. □

5. Numerical results

In this section, we present numerical results for fixed-strike geometric Asian call option prices at time 0. Two different sets of parameters are used. The first set is denoted I, and $\kappa=1.15$, $\theta=0.348$, $\sigma=0.39$, and $\rho=-0.64$, which correspond to the daily averages of the respective implied parameters reported in table IV of Bakshi *et al*. (1997). The second set is denoted II, and $\kappa=2.0$, $\theta=0.09$, $\sigma=1.0$, and $\rho=-0.3$, which were chosen from Brodie and Kaya (2006) and Ballestra *et a*l. (2007). For both sets, the following values are taken: $r=0.05$ and $v_0=0.09$.

These two contrasting sets of parameters were chosen purposively from the perspective of the Feller condition. The first set satisfies $2\kappa\theta/\sigma^2\geq 1$, the Feller condition mentioned earlier in (4), whilst the second set satisfies $2\kappa\theta/\sigma^2<1$. Recall the relevant remarks made earlier after (4) and (5) in section 2. Tables 1, 3 and 5 consider the first set labeled I and tables 2, 4 and 6 the second set, labeled II. Generally speaking, numerical implementation results are shown to be quite stable and efficient for both sets as presented in tables 1–6. However, one can observe that there exists a slight difference between the two sets from the viewpoint of computational efficiency. Comparing CPU times for corresponding cases under I and II indicates that these numerical implementations are less efficient for II than for I irrespective of the circumstances. It appears that this phenomenon is related to the convergence rates of the integrals appearing in the analytic formulae.

To implement the analytic formulae given in theorem 4.1 numerically, it is required to carry out two different types of truncation procedures. First, it is necessary to determine the number of terms taken for the computation of the infinite series expansions for $H_{0,T}$ and $\tilde{H}_{0,T}$ in (26) and (27). It is required that this is performed whilst maintaining considerable computational accuracy. Secondly, it is necessary to truncate the infinite range to a finite interval for the integral appearing in (29).

We start by investigating the influence of the number of terms taken in (26) and (27). Tables 1 and 2 present numerical values for fixed-strike geometric Asian call option prices when we take varying numbers of terms

Table 2. Fixed-strike geometric Asian call option prices of different maturities and strike prices, varying n for II; $S_0 = 100$, $v_0 = 0.09$, $t = 0$, $r = 0.05$, $\kappa = 2.0$, $\theta = 0.09$, $\rho = -0.3$, $\sigma = 1.0$, CPU times in seconds.

T	K	n	Price	CPU	T	K	n	Price	CPU	T	K	n	Price	CPU
0.2	90	10	10.6437	0.032	1.5	90	20	14.9984	0.052	3	90	30	18.1232	0.068
		20	10.6425	0.043			30	14.9954	0.060			40	18.1218	0.077
		30	10.6425	0.059			40	14.9955	0.075			50	18.1219	0.095
		40	10.6425	0.074			50	14.9955	0.090			60	18.1219	0.112
	95	10	6.4385	0.025		95	20	11.6763	0.046		95	30	15.2031	0.065
		20	6.4362	0.034			30	11.6710	0.057			40	15.2009	0.079
		30	6.4362	0.047			40	11.6707	0.073			50	15.2009	0.096
		40	6.4362	0.059			50	11.6707	0.088			60	15.2009	0.112
	100	10	3.1549	0.021		100	20	8.7805	0.041		100	30	12.5729	0.061
		20	3.1578	0.032			30	8.7772	0.055			40	12.5709	0.075
		30	3.1578	0.044			40	8.7767	0.071			50	12.5707	0.091
		40	3.1578	0.055			50	8.7767	0.084			60	12.5707	0.106
	105	10	1.1896	0.022		105	20	6.3776	0.042		105	30	10.2545	0.057
		20	1.1936	0.035			30	6.3816	0.053			40	10.2542	0.073
		30	1.1936	0.047			40	6.3818	0.069			50	10.2539	0.088
		40	1.1936	0.060			50	6.3818	0.083			60	10.2539	0.104
	110	10	0.3632	0.025		110	20	4.5007	0.043		110	30	8.2590	0.059
		20	0.3609	0.037			30	4.5110	0.059			40	8.2612	0.073
		30	0.3609	0.050			40	4.5118	0.074			50	8.2611	0.089
		40	0.3609	0.063			50	4.5118	0.088			60	8.2611	0.104

n denotes the number of terms we take in the series expansions of H and $\tilde{H}$ in (26) and (27).

Table 3. Fixed-strike geometric Asian call option prices of different maturities and strike prices, varying M for I; $S_0 = 100$, $v_0 = 0.09$, $t = 0$, $r = 0.05$, $\kappa = 1.15$, $\theta = 0.348$, $\rho = -0.64$, $\sigma = 0.39$, CPU times in seconds.

T	K	M	Option value	CPU	T	K	M	Option value	CPU	T	K	M	Option value	CPU
0.2	90	10^1	9.7057	0.006	1.5	90	10^1	16.5751	0.027	3	90	10^1	20.5097	0.033
		10^2	10.6571	0.039			10^2	16.5030	0.041			10^2	20.5102	0.045
		10^3	10.6571	0.042			10^3	16.5030	0.046			10^3	20.5102	0.051
		10^4	10.6571	0.049			10^4	16.5030	0.054			10^4	20.5102	0.051
		10^5	10.6571	0.053			10^5	16.5030	0.057			10^5	20.5102	0.060
	95	10^1	5.3025	0.005		95	10^1	13.7970	0.019		95	10^1	18.3068	0.032
		10^2	6.5871	0.032			10^2	13.7625	0.038			10^2	18.3060	0.045
		10^3	6.5871	0.036			10^3	13.7625	0.044			10^3	18.3060	0.050
		10^4	6.5871	0.041			10^4	13.7625	0.051			10^4	18.3060	0.052
		10^5	6.5871	0.043			10^5	13.7625	0.055			10^5	18.3060	0.056
	100	10^1	2.1142	0.005		100	10^1	11.3257	0.021		100	10^1	16.2914	0.031
		10^2	3.4478	0.026			10^2	11.3374	0.041			10^2	16.2895	0.044
		10^3	3.4478	0.033			10^3	11.3374	0.045			10^3	16.2895	0.048
		10^4	3.4478	0.035			10^4	11.3374	0.044			10^4	16.2895	0.051
		10^5	3.4478	0.040			10^5	11.3374	0.054			10^5	16.2895	0.057
	105	10^1	0.1748	0.007		105	10^1	9.1699	0.023		105	10^1	14.4556	0.030
		10^2	1.4552	0.032			10^2	9.2245	0.041			10^2	14.4531	0.044
		10^3	1.4552	0.033			10^3	9.2245	0.044			10^3	14.4531	0.050
		10^4	1.4552	0.037			10^4	9.2245	0.049			10^4	14.4531	0.051
		10^5	1.4552	0.046			10^5	9.2245	0.057			10^5	14.4531	0.058
	110	10^1	−0.6628	0.008		110	10^1	7.3272	0.022		110	10^1	12.7907	0.031
		10^2	0.4724	0.035			10^2	7.4122	0.044			10^2	12.7882	0.050
		10^3	0.4724	0.036			10^3	7.4122	0.044			10^3	12.7882	0.053
		10^4	0.4724	0.045			10^4	7.4122	0.048			10^4	12.7882	0.051
		10^5	0.4724	0.050			10^5	7.4122	0.057			10^5	12.7882	0.064

M denotes the upper limit for the integral in (29).

in (26) and (27) under the first and second set of parameters, respectively. In both tables, the upper limit is designated as 10^5 for the infinite integral in (29), which is considered to be sufficiently large to reduce the effects from cutting down the integral range to a finite interval. In addition, tables 1 and 2 include the corresponding option prices with various maturities and strike prices and the required CPU times. It is noted that the numerical values tend to be quite stable as the number of terms, denoted n, increases under the various circumstances

Table 4. Fixed-strike geometric Asian call option prices of different maturities and strike prices, varying M for II; $S_0=100$, $v_0=0.09$, $t=0$, $r=0.05$, $\kappa=2.0$, $\theta=0.09$, $\rho=-0.3$, $\sigma=1.0$, CPU times in seconds.

T	K	M	Option value	CPU	T	K	M	Option value	CPU	T	K	M	Option value	CPU
0.2	90	10^1	9.5875	0.012	1.5	90	10^1	15.0021	0.033	3	90	10^1	18.2717	0.047
		10^2	10.6425	0.071			10^2	14.9955	0.079			10^2	18.1219	0.085
		10^3	10.6425	0.082			10^3	14.9955	0.087			10^3	18.1219	0.093
		10^4	10.6425	0.090			10^4	14.9955	0.103			10^4	18.1219	0.101
		10^5	10.6425	0.106			10^5	14.9955	0.106			10^5	18.1219	0.112
	95	10^1	5.0991	0.009		95	10^1	11.3739	0.030		95	10^1	15.2324	0.042
		10^2	6.4362	0.055			10^2	11.6707	0.071			10^2	15.2009	0.085
		10^3	6.4362	0.068			10^3	11.6707	0.082			10^3	15.2009	0.093
		10^4	6.4362	0.071			10^4	11.6707	0.098			10^4	15.2009	0.093
		10^5	6.4362	0.085			10^5	11.6707	0.104			10^5	15.2009	0.112
	100	10^1	1.8634	0.009		100	10^1	8.2754	0.031		100	10^1	12.4810	0.041
		10^2	3.1579	0.047			10^2	8.7767	0.074			10^2	12.5707	0.079
		10^3	3.1578	0.060			10^3	8.7767	0.076			10^3	12.5707	0.082
		10^4	3.1578	0.068			10^4	8.7767	0.093			10^4	12.5707	0.093
		10^5	3.1578	0.079			10^5	8.7767	0.088			10^5	12.5707	0.106
	105	10^1	−0.0794	0.011		105	10^1	5.7912	0.031		105	10^1	10.0681	0.041
		10^2	1.1936	0.058			10^2	6.3818	0.076			10^2	10.2539	0.079
		10^3	1.1936	0.060			10^3	6.3818	0.082			10^3	10.2539	0.084
		10^4	1.1936	0.074			10^4	6.3818	0.092			10^4	10.2539	0.095
		10^5	1.1936	0.085			10^5	6.3818	0.099			10^5	10.2539	0.104
	110	10^1	−0.8810	0.014		110	10^1	3.9317	0.028		110	10^1	8.0193	0.041
		10^2	0.3608	0.060			10^2	4.5118	0.080			10^2	8.2611	0.079
		10^3	0.3609	0.071			10^3	4.5118	0.082			10^3	8.2611	0.085
		10^4	0.3609	0.087			10^4	4.5118	0.093			10^4	8.2611	0.098
		10^5	0.3609	0.090			10^5	4.5118	0.104			10^5	8.2611	0.104

M denotes the upper limit for the integral in (29).

considered here. In fact, it was checked that the option values stay unchanged after taking $n=20$ and 50 terms, at most, under the first and second set of parameters, respectively.

We now turn to examining the possible errors that occur from curtailing the range of the infinite integral in (29). Tables 3 and 4 show the numerical values for fixed-strike geometric Asian call option prices when the upper limit, denoted M, of the infinite integral in (29) is designated as 10^1, 10^2, 10^3, 10^4 and 10^5. Numerical results, together with the required CPU times, are listed according to varying maturities and strike prices, while $n=30$ and $n=60$ terms are kept throughout in tables 3 and 4, respectively. It is observed that fairly stable numerical results are achieved. To be more specific, it was found that the numerical values remain unchanged after $M=10^2$ and $M=10^3$ are taken under the first and second set of parameters, respectively.

Tables 5 and 6 present numerical results obtained by the analytic pricing formulae and by the standard Monte Carlo method for fixed-strike geometric Asian call options with varying maturities and strike prices under the two sets of parameters. The first 30 and 60 terms in the series expansions of $H_{0,T}$ and $\tilde{H}_{0,T}$ in (26) and (27) are taken for the first and second sets of parameters, respectively. The upper limit of the infinite integral in (29) for both sets is set at 10^5.

The main reason that standard Monte Carlo simulations are performed in addition to the method detailed in this paper is to confirm the validity and accuracy of the analytic formulae and the accompanying implementation process rather than to compare their computational efficiency. Therefore, it is intended to reduce the errors of standard Monte Carlo simulations that may occur from an insufficient number of random samples and sparse time intervals at the expense of efficiency.

For the Monte Carlo results in tables 5 and 6, 10^5 replications are generated for each result. Each replication is generated using a Euler method discretized with 3000 points for the time axis. More precisely, a replication $\{(\hat{S}_{k\delta}, \hat{v}_{k\delta}) : k=0,1,\ldots,3000\}$ of $\{(S_t, v_t) : 0 \le t \le T\}$ at discrete time points $t=k\delta$, $k=0,1,\ldots,3000$, with $\delta=T/3000$, is obtained by the evolution equations

$$\log \hat{S}_{k\delta} = \log \hat{S}_{(k-1)\delta} + \left(r - \frac{\hat{v}_{(k-1)\delta}}{2}\right)\delta + \sqrt{\hat{v}_{(k-1)\delta}\delta}\, W_k^1,$$

$$\hat{v}_{k\delta} = \max\left\{\hat{v}_{(k-1)\delta} + \kappa(\theta - \hat{v}_{(k-1)\delta})\delta + \sigma\sqrt{\hat{v}_{(k-1)\delta}\delta}\,(\rho W_k^1 + \sqrt{1-\rho^2}\, W_k^2), 0\right\}, \tag{42}$$

with initial values $\hat{S}_0 = S_0$ and $\hat{v}_0 = v_0$. Here $\{W_k^1 : k=1,2,\ldots,3000\}$ and $\{W_k^2 : k=1,2,\ldots,3000\}$ are independent sequences of independent and identically distributed standard normal random vectors. The maximum on the right-hand side of (42) is taken to avoid negative values of the variance process. See, for example, Lord *et al.* (2010) for further explanation of this scheme.

As a consequence, it is found that the Monte Carlo results have computational accuracy and stability comparable to the analytic results, although there is a

Table 5. Comparison of our method and the Monte Carlo for fixed-strike geometric Asian call options for I; $S_0=100$, $v_0=0.09$, $t=0$, $r=0.05$, $\kappa=1.15$, $\theta=0.348$, $\rho=-0.64$, $\sigma=0.39$, CPU times in seconds. Confidence intervals of the 95% confidence level.

		Our method*		Simulation†		
T (years)	K	Option value	CPU	Option value	Confidence interval	CPU
0.2	90	10.6571	0.052	10.6887	(10.6426, 10.7347)	90.340
	95	6.5871	0.043	6.5879	(6.5482, 6.6276)	90.576
	100	3.4478	0.040	3.4537	(3.4237, 3.4838)	90.478
	105	1.4552	0.046	1.4634	(1.4438, 1.4830)	90.314
	110	0.4724	0.049	0.4734	(0.4627, 0.4840)	90.424
0.4	90	11.7112	0.050	11.6855	(11.6225, 11.7484)	90.543
	95	8.0894	0.049	8.8446	(7.9810, 8.0906)	90.469
	100	5.1616	0.043	5.1528	(5.1077, 5.1980)	90.505
	105	3.0018	0.045	2.9916	(2.9570, 3.0261	90.521
	110	1.5715	0.046	1.5902	(1.5652, 1.6153)	90.339
0.5	90	12.2329	0.048	12.2131	(12.1431,12.2831)	90.589
	95	8.7553	0.046	8.7521	(8.6905, 8.8137)	90.407
	100	5.8971	0.045	5.9188	(5.8669, 5.9706)	90.504
	105	3.7072	0.047	3.7187	(3.6771, 3.7602)	90.291
	110	2.1589	0.052	2.1349	(2.1035, 2.1662)	90.403
1	90	14.5779	0.057	14.5905	(14.4900, 14.6910)	90.563
	95	11.5551	0.049	11.5024	(11.4112, 11.5937)	90.402
	100	8.9457	0.046	8.9749	(8.8930, 9.0569)	90.400
	105	6.7559	0.049	6.7160	(6.6446, 6.7873)	90.284
	110	4.9722	0.053	4.9723	(4.9104, 5.0341)	90.211
1.5	90	16.5030	0.057	16.6320	(16.5054, 16.7585)	90.676
	95	13.7625	0.055	13.7193	(13.6020, 13.8365)	90.512
	100	11.3374	0.054	11.3762	(11.2681, 11.4843)	90.330
	105	9.2245	0.057	9.2524	(9.1538, 9.3510)	90.342
	110	7.4122	0.056	7.4073	(7.3187, 7.4960)	90.471
2	90	18.0914	0.056	18.0014	(17.8530, 18.1498)	90.438
	95	15.5640	0.055	15.5382	(15.3976, 15.6789)	90.292
	100	13.2933	0.056	13.3507	(13.2185, 13.4829)	90.258
	105	11.2728	0.056	11.2313	(11.1092, 11.3534)	90.309
	110	9.4921	0.055	9.4202	(9.3076, 9.5328)	90.395
3	90	20.5102	0.060	20.4411	(20.2512, 20.6311)	90.625
	95	18.3060	0.055	18.2506	(18.0690, 18.4322)	90.238
	100	16.2895	0.056	16.3119	(16.1376, 16.4862)	90.138
	105	14.4531	0.057	14.4497	(14.2843, 14.6151)	90.452
	110	12.7882	0.064	12.7261	(12.5705, 12.8817)	90.369

*We take the first 30 terms in our series expansions of H and $\tilde{H}$ in (26) and (27), and the upper limit of the infinite integral in (29) is set at 10^5.
†We generates 10^5 replications for each result, with 3000 time points for each replication.

significant loss of computational efficiency on the basis of the CPU times required. The analytic method is generally expected to outperform the Monte Carlo simulation method. It is more desirable to compare the analytic method with more elaborate Monte Carlo methods (e.g. Broadie and Kaya 2006, Andersen 2008), which is postponed for future research. Finally, it is noted that all of the necessary computations, including numerical integrations, throughout this section are readily executed with the intrinsic functions built into MATLAB on a Pentium 2.4 GHz personal computer.

6. Conclusion

We consider Heston's stochastic volatility model and derive exact analytic expressions for the prices of fixed-strike and floating-strike geometric Asian options with continuously sampled averages. The pricing formulae involve functions given by series expansions. Widely used techniques are adopted, such as the change of numeraire and the Fourier transform inversion methods. This work has three different merits. First of all, in spite of the well-known methodology, all the formulae presented in this paper, particularly geometric option prices in this framework, are believed to be entirely new entries on the list of previous results for option price formulae. Secondly, we find the joint Fourier transform of the square-root process and three weighted temporal integrals with constant, linear and square functions as weights. This is a non-trivial extension of a previous well-known result for the joint moment generating function of the square-root process and its temporal integral. Finally, although the price formulae appear to be quite complicated, they are very easy to implement utilizing common commercial software. Furthermore, it turns out that the numerical implementation is very accurate and computationally efficient. It is expected that this methodology can be

Table 6. Comparison of our method and the Monte Carlo for fixed-strike geometric Asian call options for II; $S_0 = 100$, $v_0 = 0.09$, $t = 0$, $r = 0.05$, $\kappa = 2.0$, $\theta = 0.09$, $\rho = -0.3$, $\sigma = 1.0$, CPU times in seconds. Confidence intervals of the 95% confidence level.

		Our method*		Simulation†		
T (years)	K	Option value	CPU	Option value	Confidence interval	CPU
0.2	90	10.6425	0.107	10.6331	(10.5904, 10.6758)	90.442
	95	6.4362	0.084	6.4424	(6.4057, 6.4792)	90.471
	100	3.1578	0.078	3.1646	(3.1368, 3.1923)	90.300
	105	1.1936	0.084	1.1913	(1.1736, 1.2090)	90.165
	110	0.3609	0.089	0.3638	(0.3539, 0.3738)	90.339
0.4	90	11.5186	0.106	11.5421	(11.4865, 11.5976)	90.508
	95	7.6207	0.084	7.6061	(7.5576, 7.6545)	90.368
	100	4.4638	0.081	4.4799	(4.4401, 4.5198)	90.420
	105	2.2865	0.085	2.2965	(2.2666, 2.3263)	90.527
	110	1.0573	0.092	1.0626	(1.0414, 1.0839)	90.622
0.5	90	11.9208	0.101	11.8762	(11.8157, 11.9367)	90.473
	95	8.1147	0.087	8.1211	(8.0679, 8.1742)	90.481
	100	4.9926	0.084	4.9859	(4.9418, 5.0300)	90.375
	105	2.7576	0.092	2.7600	(2.7253, 2.7947)	90.416
	110	1.4046	0.092	1.4139	(1.3879, 1.4398)	90.480
1	90	13.6212	0.109	13.5715	(13.4918, 13.6512)	90.416
	95	10.1043	0.097	10.0866	(10.0148, 10.1584)	90.538
	100	7.1073	0.100	7.1272	(7.0635, 7.1910)	90.405
	105	4.7406	0.097	4.7168	(4.6625, 4.7711)	90.418
	110	3.0328	0.094	3.0759	(3.0302, 3.1217)	90.456
1.5	90	14.9955	0.107	14.9491	(14.8544,15.0438)	90.632
	95	11.6707	0.103	11.7032	(11.6158, 11.7907)	90.407
	100	8.7767	0.098	8.8340	(8.7543, 8.9136)	90.442
	105	6.3818	0.097	6.3696	(6.3000, 6.4392)	90.442
	110	4.5118	0.103	4.5653	(4.5037, 4.6270)	90.466
2	90	16.1713	0.107	16.2136	(16.1056, 16.3216)	90.402
	95	13.0009	0.097	13.0609	(12.9603, 13.1614)	90.357
	100	10.2009	0.100	10.2536	(10.1616, 10.3456)	90.463
	105	7.8166	0.100	7.8824	(7.7987, 7.9661)	90.367
	110	5.8676	0.103	5.9558	(5.8796, 6.0321)	90.507
3	90	18.1219	0.112	18.0921	(17.9625, 18.2216)	90.552
	95	15.2009	0.111	15.2964	(15.1728, 15.4200)	90.504
	100	12.5707	0.105	12.7383	(12.6214, 12.8551)	90.471
	105	10.2539	0.103	10.3284	(10.2217, 10.4350)	90.360
	110	8.2611	0.103	8.3260	(8.2269, 8.4252)	90.504

*We take the first 60 terms in our series expansions of H and $\tilde{H}$ in (26) and (27), and the upper limit of the infinite integral in (29) is set at 10^5.
†We generates 10^5 replications for each result, with 3000 time points for each replication.

extended to determine geometric Asian option prices with a discretely sampled average, which is postponed for future research.

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References

Andersen, L., Simple and efficient simulation of the Heston stochastic volatility model. *J. Comput. Financ.*, 2008, **11**(3), 1–42.

Andersen, L. and Piterbarg, V.V., Moment explosions in stochastic volatility models. *Finance Stochast.*, 2007, **11**(1), 1432–1122.

Angus, J., A note on pricing derivatives with continuous geometric averaging. *J. Fut. Mkts*, 1999, **19**, 823–839.

Bakshi, G.S., Cao, C. and Chen, Z.W., Empirical performance of alternative option pricing models. *J. Financ.*, 1997, **2**, 2003–2049.

Ballestra, L.V., Pacelli, G. and Zirilli, F., A numerical method to price exotic path-dependent options on an underlying described by the Heston stochastic volatility model. *J. Bank. Financ.*, 2007, **31**, 3420–3437.

Boyle, P., Broadie, M. and Glasserman, P., Monte Carlo methods for security pricing. *J. Econ. Dynam. Control*, 1997, **21**, 1267–1322.

Broadie, M. and Kaya, O., Exact simulation of stochastic volatility and other affine jump diffusion processes. *Oper. Res.*, 2006, **54**, 217–231.

Dragulescu, A.A. and Yakovenko, V.M., Probability distribution of returns in the Heston model with stochastic volatility. *Quant. Financ.*, 2004, **2**, 443–453.

Fouque, J.-P. and Han, C.H., Pricing Asian options with stochastic volatility. *Quant. Financ.*, 2003, **3**, 353–362.
Geman, H., Karoui, N. and Rochet, J.-C., Changes of numeraire, changes of probability measure and option pricing. *J. Appl. Probab.*, 1995, **32**, 443–458.
Geman, H. and Yor, M., Bessel processes, Asian options and perpetuities. *Math. Financ.*, 1993, **3**, 349–375.
Gil-Pelaez, J., Note on the inversion theorem. *Biometrika*, 1951, **38**, 481–482.
Heston, S.L., A closed-form solution for options with stochastic volatility with applications to bonds and currency options. *Rev. Financ. Stud.*, 1993, **6**, 327–343.
Hull, J. and White, A., The pricing of options on assets with stochastic volatilities. *J. Financ.*, 1987, **42**, 281–300.
Karatzas, I. and Shreve, E., *Brownian Motion and Stochastic Calculus*, 2nd ed., 1997 (Springer: Berlin).
Kemna, A.G.Z. and Vorst, A.C.F., A pricing method for options based on average asset values. *J. Bank. Financ.*, 1990, March, 113–129.
Lamberton, D. and Lapeyre, B., *Introduction to Stochastic Calculus Applied to Financ.*, 1995 (Chapman & Hall: London).
Lord, R., Koekkooek, R. and Van Dijk, D., A comparison of biased simulation schemes for stochastic volatility models. *Quant. Financ.*, 2010, **10**(2), 177–194.
Rogers, L. and Shi, Z., The value of an Asian option. *J. Appl. Probab.*, 1995, **32**, 1077–1088.
Scott, L., Option pricing when the variance changes randomly: Theory, estimation, and an application. *J. Financ. Quant. Anal.*, 1987, **22**, 419–438.
Stein, E. and Stein, J., Stock price distributions with stochastic volatility: an analytic approach. *Rev. Financ. Stud.*, 1991, **4**, 727–752.
Turnbull, S.M. and Wakeman, L.M., A quick algorithm for pricing European average options. *J. Financ. Quant. Anal.*, 1991, 377–389.
Vecer, J., A new PDE appraoch for pricing arithmetic average Asian options. *J. Comput. Financ.*, 2001, **4**(4), 105–113.
Vecer, J., Unified pricing of Asian options. *Risk*, 2002, June, 113–116.
Vecer, J. and Xu, M., Pricing Asian options in a semimartingale model. *Quant. Financ.*, 2004, **4**, 170–175.
Wong, H.Y. and Cheung, Y.L., Geometric Asian options: Valuation and calibration with stochastic volatility. *Quant. Financ.*, 2004, **4**, 301–314.

Appendix A: Proof of Proposition 3.2

The proof proceeds in three steps.

Step A.1: Proposition 3.2 holds when $\mathcal{D}_\tau$ is replaced by

$$\mathcal{D}' \equiv \{(z_1, z_2, z_3, z_4) \in \mathbb{R}^4 : z_k < 0,\ k = 1, 2, 3, 4\}.$$

Proof: Suppose that $(z_1, z_2, z_3, z_4) \in \mathcal{D}'$. Using induction on n, it can be shown that $f_n > 0$, $n = 0, 1, 2, \ldots$. This implies that $F_\tau(z_1, z_2, z_3, z_4) > 0$, and thus $\arg F_\tau(z_1, z_2, z_3, z_4)$ can be defined as 0. It now remains to show that (23) holds for $(z_1, z_2, z_3, z_4) \in \mathcal{D}'$. To verify this, we introduce a process $X(t)$, for $0 \le t \le \tau$,

$$X(t) = z_1 \int_0^t (t-u)^2 v_u \,\mathrm{d}u + A(t) \int_0^t (t-u) v_u \,\mathrm{d}u + B(t) \int_0^t v_u \,\mathrm{d}u + C(t) v_t + D(t), \tag{A1}$$

where $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are continuously differentiable functions that will be determined below. Now

$$\begin{aligned} \mathrm{d}e^{X(t)} = e^{X(t)} \bigg[&(2z_1 + A'(t)) \int_0^t (t-u) v_u \,\mathrm{d}u \\ &+ (A(t) + B'(t)) \int_0^t v_u \,\mathrm{d}u \\ &+ \left(B(t) + C'(t) - \kappa C(t) + \frac{\sigma^2}{2} (C(t))^2 \right) v_t \\ &+ \kappa\theta C(t) + D'(t) \bigg] \mathrm{d}t \\ &+ e^{X(t)} C(t) \sigma \sqrt{v_t} \,\mathrm{d}Z_t. \end{aligned} \tag{A2}$$

The next step is to impose conditions on $A(t)$, $B(t)$, $C(t)$ and $D(t)$ such that $\{e^{X(t)}, 0 \le t \le \tau\}$ becomes a martingale and yet it holds that

$$X(\tau) = z_1 \int_0^\tau (\tau-u)^2 v_u \,\mathrm{d}u + z_2 \int_0^\tau (\tau-u) v_u \,\mathrm{d}u + z_3 \int_0^\tau v_u \,\mathrm{d}u + z_4 v_\tau,$$

which is the exponent appearing on the left-hand side of (23). These conditions are written as the following system of equations:

$$2z_1 + A'(t) = 0, \tag{A3}$$

$$A(t) + B'(t) = 0, \tag{A4}$$

$$B(t) + C'(t) - \kappa C(t) + \frac{\sigma^2}{2} (C(t))^2 = 0, \tag{A5}$$

$$\kappa\theta C(t) + D'(t) = 0, \tag{A6}$$

$$A(\tau) = z_2, \tag{A7}$$

$$B(\tau) = z_3, \tag{A8}$$

$$C(\tau) = z_4, \tag{A9}$$

$$D(\tau) = 0. \tag{A10}$$

From (A3) and (A7), it is established that

$$A(t) = 2z_1(\tau - t) + z_2. \tag{A11}$$

Also, from (A4), (A8) and (A11),

$$B(t) = z_1(\tau - t)^2 + z_2(\tau - t) + z_3. \tag{A12}$$

It is worth noting that equation (A5) is analogous to a Riccati ODE appearing in the computation of bond prices in the CIR model of interest rates, except that $B(t)$ is now expressed as a time-dependent function instead of a constant. Thus we mimic the well-known method of substitution employed to solve for the Riccati equation. Making the substitution

$$C(t) = -\frac{2}{\sigma^2\tau}\frac{f'[1-(t/\tau)]}{f[1-(t/\tau)]} + \frac{\kappa}{\sigma^2}, \tag{A13}$$

$0 \le t \le \tau$, establishes

$$f''\Big(1-\frac{t}{\tau}\Big) + \frac{\sigma^2\tau^2}{2} f\Big(1-\frac{t}{\tau}\Big)\Big(B(t) - \frac{\kappa^2}{2\sigma^2}\Big) = 0. \tag{A14}$$

Due to the non-constant coefficients in contrast to the Riccati equation, we look for a solution of (A14) in the form of a series expansion. For brevity, let $f(0)=1$ and for $0 \le u \le 1$,

$$f(u) = \sum_{n=0}^{\infty} f_n u^n. \tag{A15}$$

It is established with help of (A9) that

$$f''(u) + \frac{\sigma^2\tau^2}{2}\Big(z_1\tau^2u^2 + z_2\tau u + z_3 - \frac{\kappa^2}{2\sigma^2}\Big) f(u) = 0, \quad 0 \le u \le 1, \tag{A16}$$

$$f(0) = 1, \tag{A17}$$

$$f'(0) = \frac{(\kappa - z_4\sigma^2)\tau}{2}. \tag{A18}$$

By direct substitution, equation (21) is obtained. Here it is noted that $f(u) > 0$ for $0 \le u \le 1$ due to $f_n > 0$, $n = 0, 1, 2, \ldots$. After we have expressed $C(t)$ in terms of f and solved for f in the series expansion, it remains to solve for $D(t)$. Since $f(u) > 0$ for $0 \le u \le 1$, $\ln f(1 - t/\tau)$ can be taken as a real-valued logarithm for $0 \le t \le \tau$. Using (A6), (A10) and (A13), the terms can be rearranged to obtain

$$D(t) = \frac{\kappa^2\theta\tau}{\sigma^2}\Big(1-\frac{t}{\tau}\Big) - \frac{2\kappa\theta}{\sigma^2}\ln f\Big(1-\frac{t}{\tau}\Big), \quad 0 \le t \le \tau. \tag{A19}$$

The solutions for $A(t)$, $B(t)$, $C(t)$ and $D(t)$ ensure that $e^{X(t)}$ satisfies

$$de^{X(t)} = e^{X(t)} C(t)\sigma\sqrt{v_t}\, dZ_t, \quad 0 \le t \le \tau.$$

Since $(z_1, z_2, z_3, z_4) \in \mathcal{D}'$, we have, for $0 \le t \le \tau$,

$$A(t) < 0,$$
$$B(t) < 0,$$
$$C(t) < \frac{\kappa}{\sigma^2},$$

which implies

$$\int_0^\tau \mathbb{E}^Q[e^{2X(t)}(C(t))^2\sigma^2 v_t]\, dt < \infty.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}^Q[e^{z_1\int_0^\tau(\tau-t)^2 v_t\,dt + z_2\int_0^\tau(\tau-t)v_t\,dt + z_3\int_0^\tau v_t + z_4 v_\tau}] \\
&\quad = \mathbb{E}^Q[e^{X(\tau)}] \\
&\quad = \mathbb{E}^Q[e^{X(0)}] \\
&\quad = \exp\Bigg(\frac{\kappa v_0 + \kappa^2\theta\tau}{\sigma^2} - \frac{2v_0}{\sigma^2}\frac{\tilde{F}_\tau(z_1,z_2,z_3,z_4)}{F_\tau(z_1,z_2,z_3,z_4)} \\
&\qquad - \frac{2\kappa\theta}{\sigma^2}\ln F_\tau(z_1,z_2,z_3,z_4)\Bigg),
\end{aligned}$$

and the proof for step 1 is complete. □

Step A.2: Proposition 3.2 holds when $\mathcal{D}_\tau$ is replaced by

$$\mathcal{D}'' \equiv \{(z_1,z_2,z_3,z_4) \in \mathbb{C}^4 : \mathrm{Re}(z_k) < 0,\ k = 1,2,3,4\}.$$

Proof: First, it is shown that $F_\tau(z_1,z_2,z_3,z_4) \neq 0$ for all $(z_1,z_2,z_3,z_4) \in \mathcal{D}''$. Suppose, on the contrary, that there is

$$(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4) \in \mathcal{D}'' \quad \text{such that } F_\tau(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4) = 0.$$

We denote for $k = 1, 2, 3, 4$

$$\hat{z}_k = \alpha_k + i\beta_k.$$

It is noted that $F_\tau(\alpha_1 + w\beta_1, \alpha_2 + w\beta_2, \alpha_3 + w\beta_3, \alpha_4 + w\beta_4)$ can be regarded as an entire function of $w \in \mathbb{C}$, and satisfies $F_\tau(\alpha_1,\alpha_2,\alpha_3,\alpha_4) > 0$. An application of the well-known identity theorem implies that $\{w \in \mathbb{C} : F_\tau(\alpha_1 + w\beta_1, \alpha_2 + w\beta_2, \alpha_3 + w\beta_3, \alpha_4 + w\,\beta_4) = 0\}$ has no limit point. By choosing different $(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4) \in \mathcal{D}'$ if necessary, it may be assumed, without loss of generality, that there is $\epsilon > 0$ such that $\alpha_k + |\beta_k|\,\epsilon < 0$, $1 \le k \le 4$, and

$$F_\tau(\alpha_1 + w\beta_1, \alpha_2 + w\beta_2, \alpha_3 + w\beta_3, \alpha_4 + w\beta_4) \neq 0, \quad \text{for all } w \in \mathcal{R},$$

where

$$\mathcal{R} \equiv \{w \in \mathbb{C} : |\mathrm{Re}(w)| < \epsilon,\ -\epsilon < \mathrm{Im}(w) \le 1\} \setminus \{i\}.$$

Since $\mathcal{R}$ is simply connected and $F_\tau(\alpha_1 + w\beta_1, \alpha_2 + w\beta_2, \alpha_3 + w\beta_3, \alpha_4 + w\beta_4)$ is a non-zero complex-valued continuous function of w in $\mathcal{R}$, the argument of $F_\tau(\alpha_1 + w\beta_1, \alpha_2 + w\beta_2, \alpha_3 + w\beta_3, \alpha_4 + w\beta_4)$ is uniquely determined for $w \in \mathcal{R}$ by the following properties.

(i) If w is a real number in $\mathcal{R}$, then $\arg F_\tau(\alpha_1 + w\beta_1, \alpha_2 + w\beta_2, \alpha_3 + w\beta_3, \alpha_4 + w\beta_4) = 0$.
(ii) $\arg F_\tau(\alpha_1 + w\beta_1, \alpha_2 + w\beta_2, \alpha_3 + w\beta_3, \alpha_4 + w\beta_4)$ is continuous in w on $\mathcal{R}$.

By step A.1, (23) holds with $z_k = \alpha_k + w\beta_k$, $1 \le k \le 4$, if $w \in \mathcal{R} \cap \mathbb{R}$. The identity theorem implies that (23) holds with $z_k = \alpha_k + w\beta_k$, $1 \le k \le 4$, for all $w \in \mathcal{R}$. Since

$$\frac{\tilde{F}_\tau(\alpha_1 + w\beta_1, \alpha_2 + w\beta_2, \alpha_3 + w\beta_3, \alpha_4 + w\beta_4)}{F_\tau(\alpha_1 + w\beta_1, \alpha_2 + w\beta_2, \alpha_3 + w\beta_3, \alpha_4 + w\beta_4)},$$

as a function of w, has either a pole or removable singularity at $w = i$ and $\mathrm{Re}(\ln F_\tau(\alpha_1 + w\beta_1, \alpha_2 + w\beta_2, \alpha_3 + w\beta_3, \alpha_4 + w\beta_4))$ tends to $-\infty$ as $w \to i$ in $\mathcal{R}$, the right-hand side of (23) with $z_k = \alpha_k + w\beta_k$, $1 \le k \le 4$, is unbounded in w on $\mathcal{R}$. On the other hand, the left-hand side of (23) with $z_k=\alpha_k+w\beta_k$, $1\le k\le 4$, is bounded in w on $\mathcal{R}$, which is a contradiction. Therefore, $F_\tau(z_1,z_2,z_3,z_4)\neq 0$ for all $(z_1,z_2,z_3,z_4)\in\mathcal{D}''$.

Since $\mathcal{D}''$ is simply connected and $F_\tau(z_1,z_2,z_3,z_4)$ is a non-zero complex-valued continuous function of (z_1,z_2,z_3,z_4) in $\mathcal{D}''$, the argument of $F_\tau(z_1,z_2,z_3,z_4)$ is uniquely determined on $\mathcal{D}''$ with the two properties described in part (a) of proposition 3.2. According to step A.1, both sides of (23) have the same complex partial derivatives of all orders for $(z_1,z_2,z_3,z_4)\in\mathcal{D}'$. This implies that every point in $\mathcal{D}'$ is an interior point of the subset of $\mathcal{D}''$ on which (23) holds. By the identity theorem for analytic functions of several complex variables, (23) holds for all $(z_1,z_2,z_3,z_4)\in\mathcal{D}''$. □

Step A.3: Proposition 3.2 holds.

Proof: First, it is shown that $F_\tau(z_1,z_2,z_3,z_4)\neq 0$ for all $(z_1,z_2,z_3,z_4)\in\mathcal{D}_\tau$. Suppose that there is $(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4)\in\mathcal{D}_\tau$ such that $F_\tau(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4)=0$. According to step A.2, $F_\tau(\hat{z}_1+w,\hat{z}_2+w,\hat{z}_3+w,\hat{z}_4+w)\neq 0$ if $\mathrm{Re}(\hat{z}_k)+\mathrm{Re}(w)<0$, $1\le k\le 4$, since $F_\tau(\hat{z}_1+w,\hat{z}_2+w,\hat{z}_3+w,\hat{z}_4+w)$ is an entire function of w and the identity theorem implies that $\{w\in\mathbb{C}: F_\tau(\hat{z}_1+w,\hat{z}_2+w,\hat{z}_3+w,\hat{z}_4+w)=0\}$ has no limit point. By choosing different $(\hat{z}_1,\hat{z}_2,\hat{z}_3,\hat{z}_4)\in\mathcal{D}_\tau$ if necessary, it may be assumed, without loss of generality, that there is $\epsilon>0$ such that

$$F_\tau(\hat{z}_1+w,\hat{z}_2+w,\hat{z}_3+w,\hat{z}_4+w)\neq 0,\quad \text{for all } w\in\mathcal{R}^*,$$

where

$$\mathcal{R}^*\equiv\{w\in\mathbb{C}:\mathrm{Re}(w)\le 0, |\mathrm{Im}(w)|<\epsilon\}\setminus\{0\}.$$

If $\mathrm{Re}(\hat{z}_k)+\mathrm{Re}(w)<0$, $1\le k\le 4$, then $\arg F_\tau(\hat{z}_1+w,\hat{z}_2+w,\hat{z}_3+w,\hat{z}_4+w)$ is defined in step A.2. Since $\mathcal{R}^*$ is simply connected and $F_\tau(\hat{z}_1+w,\hat{z}_2+w,\hat{z}_3+w,\hat{z}_4+w)$ is a non-zero complex-valued continuous function of w in $\mathcal{R}^*$, $\arg F_\tau(\hat{z}_1+w,\hat{z}_2+w,\hat{z}_3+w,\hat{z}_4+w)$ is uniquely determined for all $w\in\mathcal{R}^*$ by continuity. By step A.2, if $\mathrm{Re}(\hat{z}_k)+\mathrm{Re}(w)<0$, $1\le k\le 4$, then (23) holds by replacing z_k with $\hat{z}_k+w$, $1\le k\le 4$. According to the identity theorem, if $w\in\mathcal{R}^*$, then (23) holds by replacing z_k with $\hat{z}_k+w$, $1\le k\le 4$. Since

$$\frac{\tilde{F}_\tau(\hat{z}_1+w,\hat{z}_2+w,\hat{z}_3+w,\hat{z}_4+w)}{F_\tau(\hat{z}_1+w,\hat{z}_2+w,\hat{z}_3+w,\hat{z}_4+w)},$$

as a function of w, has either a pole or removable singularity at $w=0$ and $\mathrm{Re}(\ln F_\tau(\hat{z}_1+w,\hat{z}_2+w,\hat{z}_3+w,\hat{z}_4+w))$ tends to $-\infty$ as $w\to 0$ in $\mathcal{R}^*$, the right-hand side of (23) on replacing z_k with $\hat{z}_k+w$, $1\le k\le 4$, is unbounded in w on $\mathcal{R}^*$. On the other hand, the left-hand side of (23) on replacing z_k with $\hat{z}_k+w$, $1\le k\le 4$, is bounded in w on $\mathcal{R}^*$, which is a contradiction. Therefore, $F_\tau(z_1,z_2,z_3,z_4)\neq 0$ for all $(z_1,z_2,z_3,z_4)\in\mathcal{D}_\tau$.

Since $\mathcal{D}_\tau\cap\mathbb{R}^4$ is connected, $F_\tau(0,0,0,0)>0$ and $F_\tau(z_1,z_2,z_3,z_4)\in\mathbb{R}$ for all $(z_1,z_2,z_3,z_4)\in\mathcal{D}_\tau\cap\mathbb{R}^4$, we have $F_\tau(z_1,z_2,z_3,z_4)>0$ for all $(z_1,z_2,z_3,z_4)\in\mathcal{D}_\tau\cap\mathbb{R}^4$ by continuity of F_τ. Since $\mathcal{D}_\tau$ is simply connected and $F_\tau(z_1,z_2,z_3,z_4)$ is a non-zero complex-valued continuous function of (z_1,z_2,z_3,z_4) in $\mathcal{D}_\tau$, the argument of $F_\tau(z_1,z_2,z_3,z_4)$ is uniquely determined on $\mathcal{D}_\tau$ with the two properties of proposition 3.2(a). According to step A.2 and the identity theorem for analytic functions of several complex variables, (23) holds for all $(z_1,z_2,z_3,z_4)\in\mathcal{D}_\tau$. □