Nonlinear Filtering

real images are not well-modeled as Gaussian

- Linear filters
  - Tend to blur edges and other image detail.
  - Perform poorly with non-Gaussian noise.
  - Result from Gaussian image and noise assumptions.
  - Images are not Gaussian.
- Nonlinear filter - more common in practice
  - Can preserve edges
  - Very effective at removing impulsive noise
  - Result from non-Gaussian image and noise assumptions.
  - Can be difficult to design.

Linear Filters

- Definition: A system $y = T[x]$ is said to be linear if for all $\alpha, \beta \in \mathbb{R}$

$$\alpha y_1 + \beta y_2 = T[\alpha x_1 + \beta x_2]$$

  where $y_1 = T[x_1]$ and $y_2 = T[x_2]$.

- Any filter of the form

$$y_s = \sum_r h_{s,r} x_r$$

negate there exists

$\neg \left[\forall \alpha \; \exists \beta \text{ s.t. } P_{\alpha,\beta} \right]$

$\exists \alpha \text{ s.t. } \forall \beta \; \neg P_{\alpha,\beta}$

and

$P \,\&\, Q \rightarrow$

$P \vee Q \rightarrow$

or

Homogeneous Filters

- Definition: A filter $y = T[x]$ is said to be homogeneous if for all $\alpha \in \mathbb{R}$
$$\alpha y = T[\alpha x]$$
- This is much weaker than linearity.
- Homogeneity is a natural condition for scale invariant systems.

Median Filter

- Let W be a window with an odd number of points.
- Then the median filter is given by

$$y_s = \text{median}\left\{x_{s+r} : r \in W\right\}$$

- Is the median filter:
  - Linear?
  - Homogeneous?

review definition

- Consider the 1-D median filter with a 3-point window.

outlier

x(m)	0	0	1	1,000	1	1	2	2
y(m)	?	0	1	1	1	1	2	?

non-sensitive → robust

sensitive → fragile

Median Filter: Optimization Viewpoint

- Consider the median filter

$$y_s = \text{median}\left\{x_{s+r} : r \in W\right\}$$

and consider the following functional.

$$F_s(\theta) \stackrel{\triangle}{=} \sum_{r \in W} |\theta - x_{s+r}|$$

- Then y_s solves the following optimization equation.

$$y_s = \arg\min_{\theta} F_s(\theta)$$

here: median is solution to an optimization problem

- Differentiating, we have

subgradient

$$\begin{aligned}\frac{d}{d\theta}F(\theta) &= \frac{d}{d\theta}\sum_{r \in W} |\theta - x_{s+r}| \\ &= \sum_{r \in W} \text{sign}(\theta - x_{s+r}) \\ &\stackrel{\triangle}{=} f(\theta)\end{aligned}$$

not continuous

This expression only holds for $\theta \neq x_{s+r}$ for all $r \in W$.

- So the solution falls at $\theta = x_{s*}$ such that

$$0 = \sum_{\substack{r \in W \\ r \neq (s* - s)}} \text{sign}(\theta - x_{s+r})$$

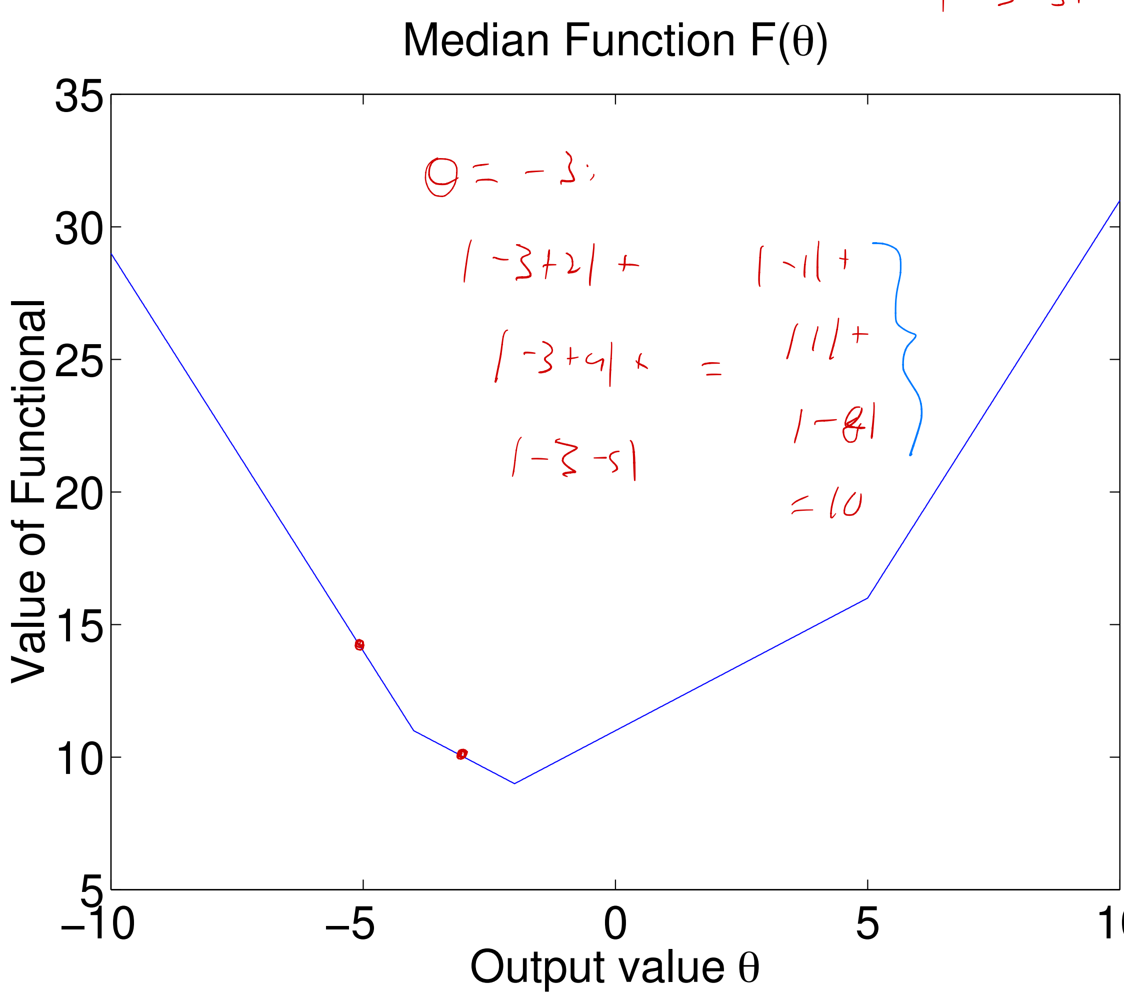
Example: Median Filter Function

- Consider a 1-D median filter
  - Three point window of $W = \{-1, 0, 1\}$
  - Inputs $[x(n-1), x(n), x(n+1)] = [-2, -4, 5]$.

$$F(\theta) = \sum_{k=-1}^{1} |\theta - x_{n+k}|$$

$\theta = -5$:

$|-5+2| + |-5+4| + |-5-5| = |-3| + |-1| + |-10| = 14$



$\theta = -3$:

$|-3+2| + |-3+4| + |-3-5| = |-1| + |1| + |-8| = 10$

Example: Derivative of Median Filter Function

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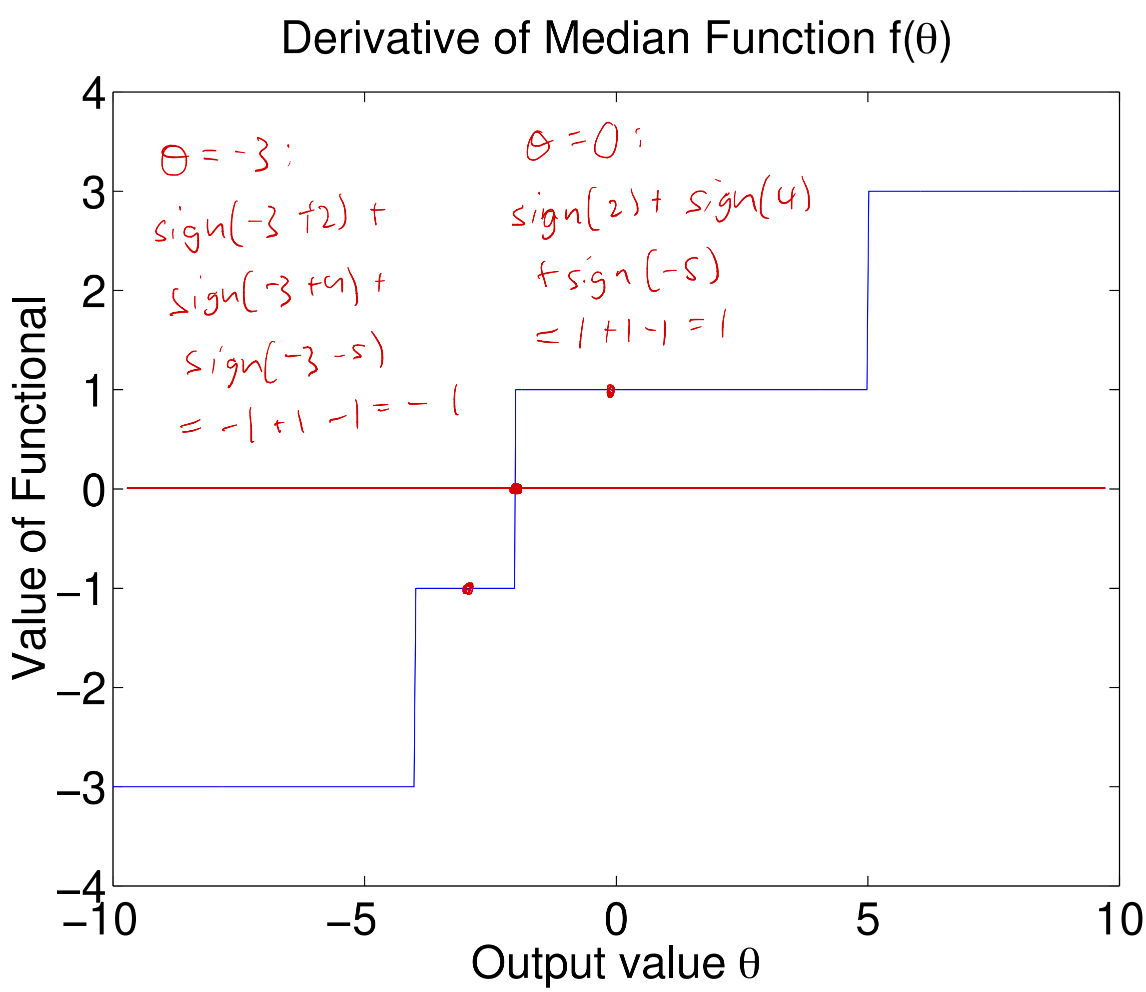
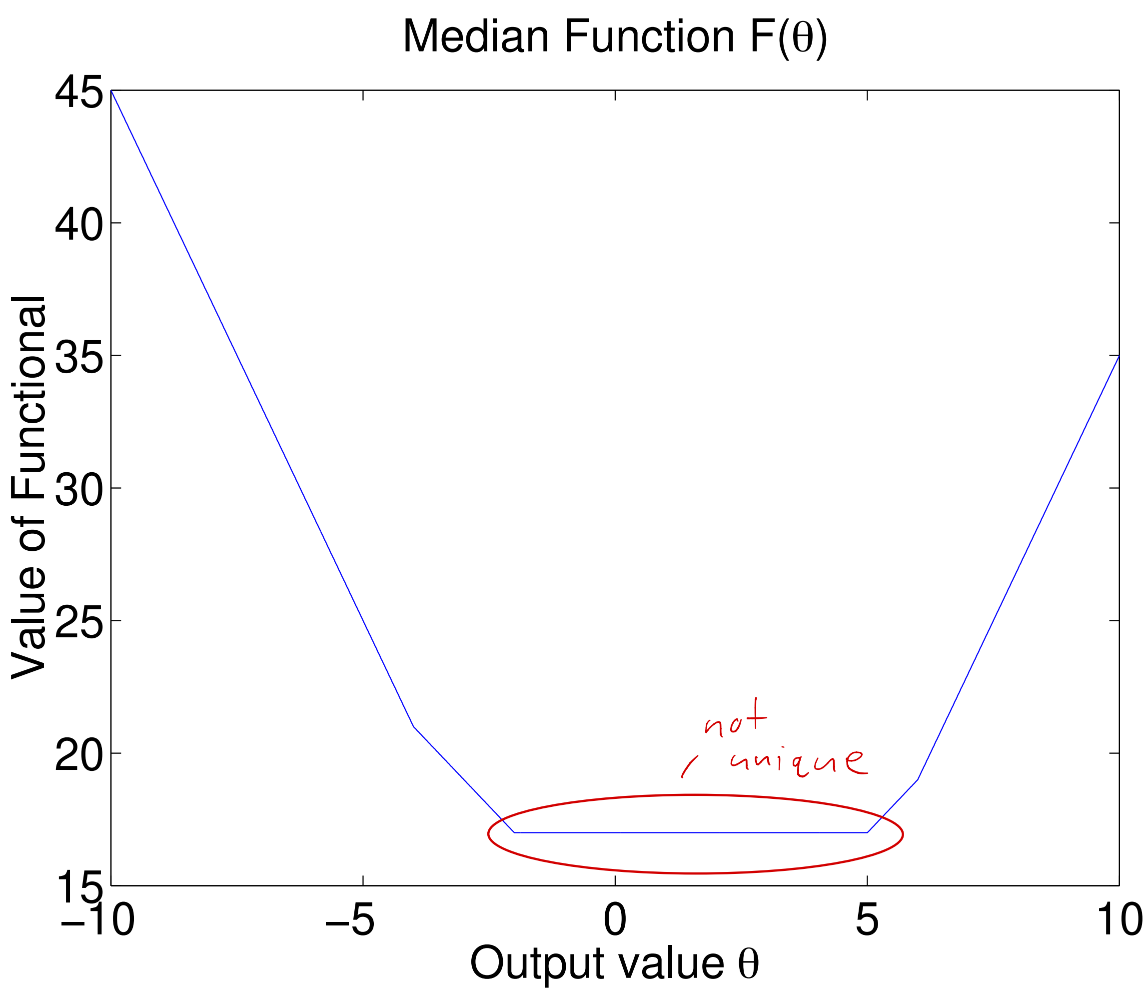


Problem with an Even Number of Points

- Consider a 1-D median filter
  - Four point window of $W = \{-1, 0, 1, 2\}$
  - Inputs $[x(n-1), x(n), x(n+1), x(n+2)] = [-2, -4, 5, 6]$.
- Solution is not unique.

$$F(\theta) = \sum_{k=-1}^{2} |\theta - x_{n+k}|$$



Only parameter for median is the size of the window

Weighted Median Filter

- Defined the functional

$$F(\theta) \stackrel{\triangle}{=} \sum_{r \in W} a_r |\theta - x_{s+r}|$$

where a_r are weights assigned to each point in the window W.

- Weighted median is computed by

$$y_s = \arg\min_{\theta} \sum_{r \in W} a_r |\theta - x_{s+r}|$$

- Differentiating, we have

$$\begin{aligned} \frac{d}{d\theta} F(\theta) &= \frac{d}{d\theta} \sum_{r \in W} a_r |\theta - x_{s+r}| \\ &= \sum_{r \in W} a_r \text{sign}(\theta - x_{s+r}) \\ &\stackrel{\triangle}{=} f(\theta) \end{aligned}$$

This expression only holds for $\theta \neq x_r$ for all $r \in W$.

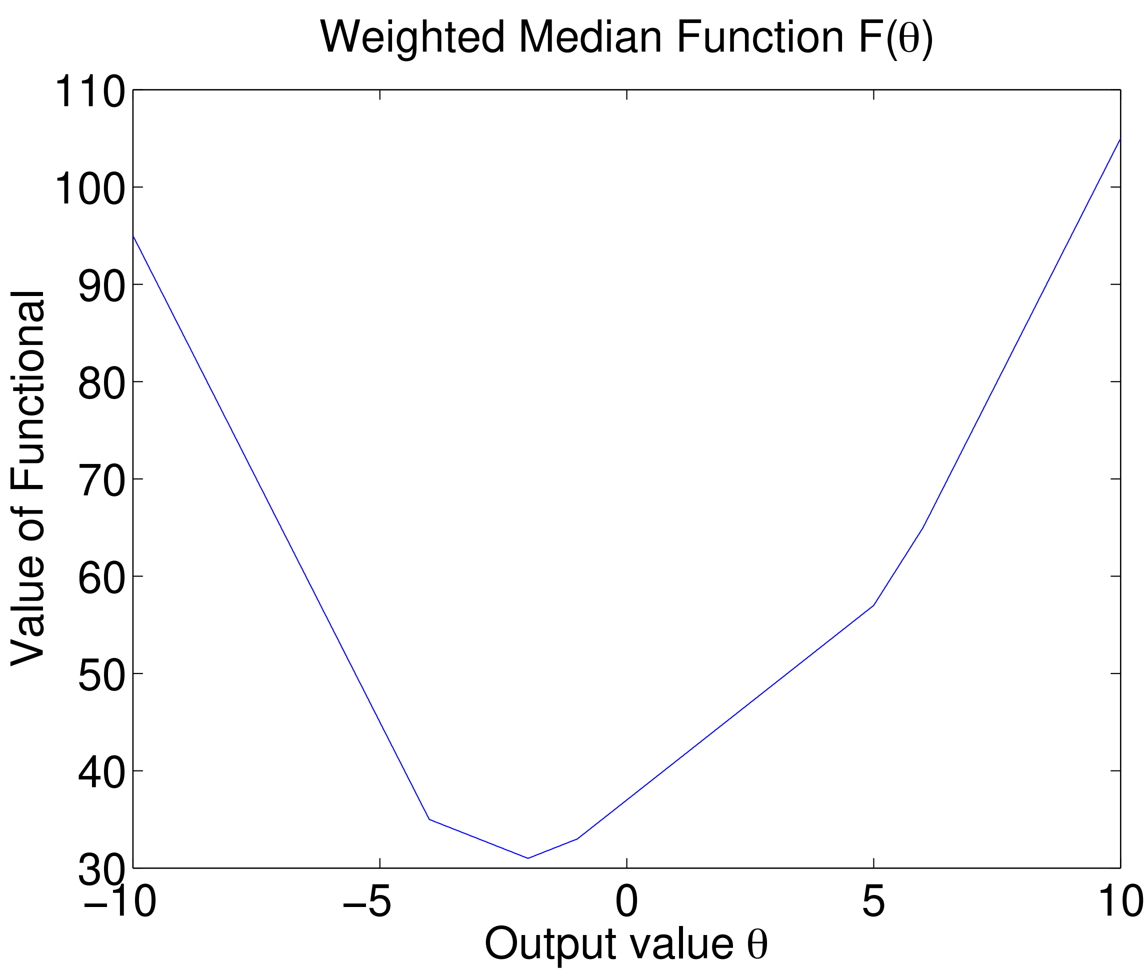
- Need to find $s*$ such that $f(\theta)$ is “nearly” zero.

Example: Weighted Median Filter Function

- Consider a 1-D median filter
  - Five point window of $W = \{-2, -1, 0, 1, 2\}$
  - Inputs $[x(n-2), \cdots, x(n+2)] = [6, -2, -4, 5, -1]$.
  - Weights $[a(-2), a(-1), a(0), a(1), a(2)] = [1, 2, 4, 2, 1]$.

change in derivative not uniform anymore

$$F(\theta) = \sum_{k=-1}^{1} a(k)|\theta - x_{n+k}|$$



Example: Derivative of Median Filter Function

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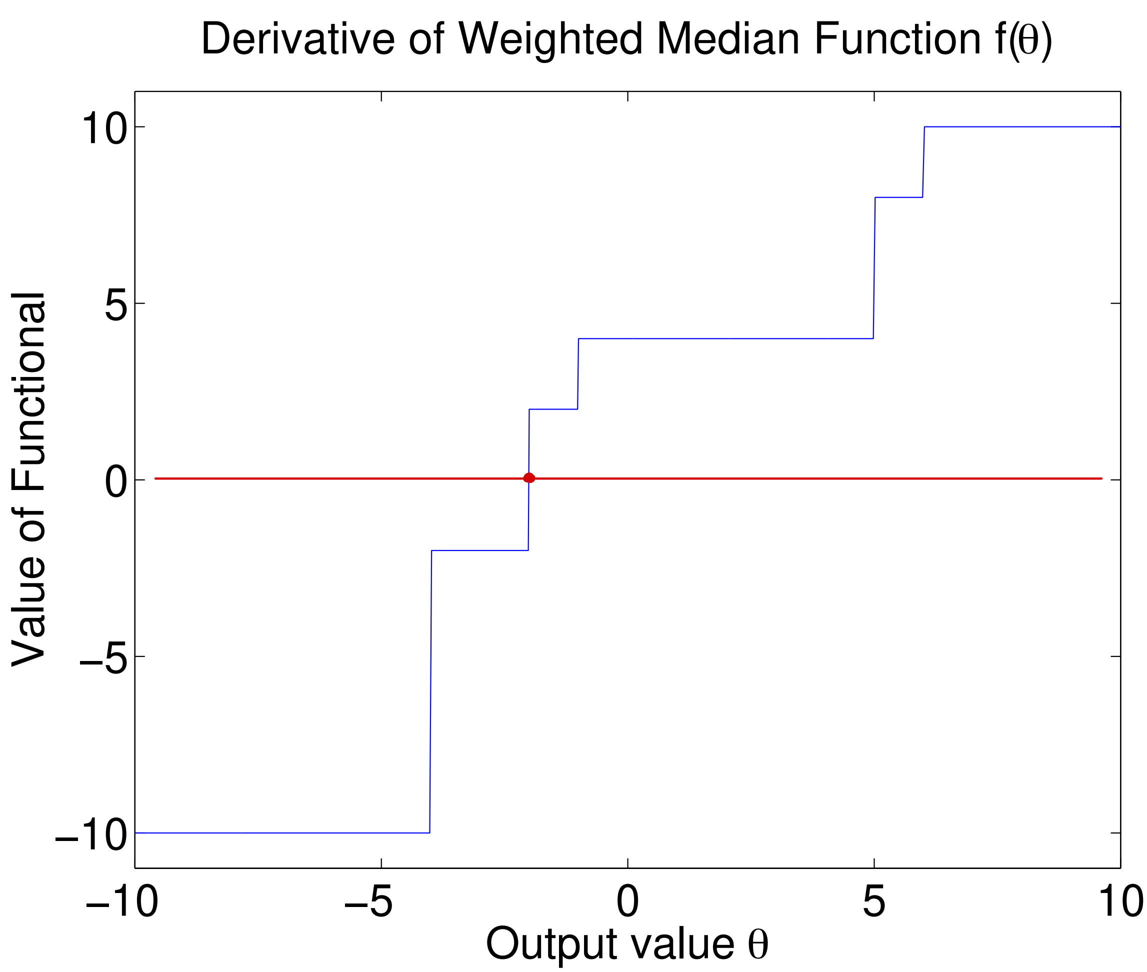
  - Weights $[a(-2), a(-1), a(0), a(1), a(2)] = [1, 2, 4, 2, 1]$.

$$f(\theta) = \sum_{k=-1}^{1} a(k)\, \mathrm{sign}(\theta - x_{n+k})$$



$\begin{pmatrix} x_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ a_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_p \\ a_p \end{pmatrix}$ - sort, but keep pairs together

$\begin{pmatrix} x_{(1)} \\ a_{(1)} \end{pmatrix}, \begin{pmatrix} x_{(2)} \\ a_{(2)} \end{pmatrix}, \ldots, \begin{pmatrix} x_{(p)} \\ a_{(p)} \end{pmatrix}$ - determine the index of the weighted median such that the sum of the weights is "balanced" or meets some sort of mathematical criteria

then calculate x_3

$a_{(1)} \quad a_{(2)} \quad a_{(3)} \quad \ldots \quad a_{(p)}$

Computation of Weighted Median

1. Sort points in window.
   - Let $x_{(1)} \le x_{(2)} \le \cdots \le x_{(p)}$ be the sorted values.
   - These values are known as order statistics.
   - Let $a_{(1)}, a_{(2)}, \cdots, a_{(p)}$ be the **corresponding** weights.
2. Find $i*$ such that the following equations hold

$$a_{i*} + \sum_{i=1}^{i*-1} a_{(i)} \ge \sum_{i=i*+1}^{p} a_{(i)}$$

$$\sum_{i=1}^{i*-1} a_{(i)} \le \sum_{i=i*+1}^{p} a_{(i)} + a_{i*}$$

3. Then the value $x_{(i*)}$ is the weighted median value.

This method not super commonly used

Comments on Weighted Median Filter

- Weights may be adjusted to yield the “best” filter.
- Largest and smallest values are ignored.
- Same as median filter for $a_r = 1$.