M-estimators

Find the centroid of a set of data

Chose some function $\rho(\theta)$, such that

$$\rho(A) = \rho(-A)$$

and

$$\frac{d}{d\theta}\rho(\Delta) = \rho'(A) \begin{cases} \text{best if} \\ \text{exists } \forall A \in \mathbb{R} \\ \text{is continuous } A \in \mathbb{R} \end{cases}$$

let $\rho(\Delta) = \sum_k |\theta - x(k)|$

$\rho'(\Delta) = \sum_k \text{sign}$

$\rho'(\theta)$ is known as the "influence" function

$$Y(m) = \sum_{k \in W(m)} \rho(\underbrace{x(k) - \theta}_{\Delta})$$

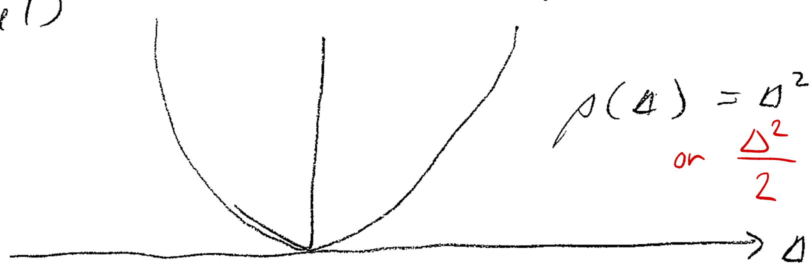
$$\frac{d}{d\theta} \sum_{k \in W(m)} \rho(x(k) - \theta) =$$

$$= \sum_{k \in W(m)} \rho'(x(k) - \theta) = 0$$

like a force balance equation

Example 1)

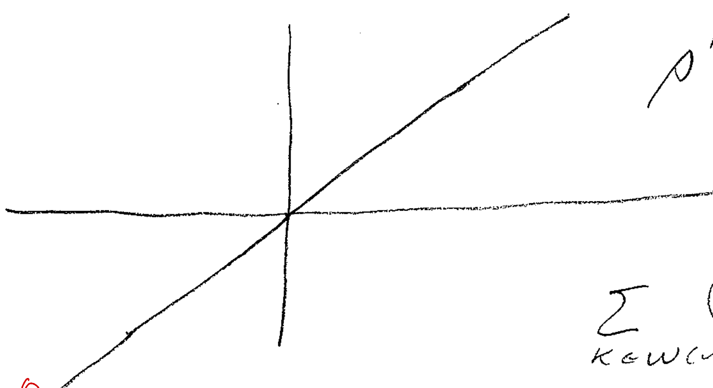
potential function



$\rho(\Delta) = \Delta^2$ or $\frac{\Delta^2}{2}$

$\Delta = x(k) - \theta$

influence function



$\rho'(\Delta) = 2\Delta = x_i - \theta$ or Δ

$\sum_k (x(k) - \theta) = 0$

$\sum_k x(k) - \sum_k \theta = 0$

$\sum_k x(k) = \sum_k \theta = p\theta$

$\hookrightarrow \theta = \frac{1}{p} \sum_k x(k)$

$$\sum_{k \in W(m)} (x(k) - \theta) = 0$$

$$\theta = \frac{1}{p} \sum_{k \in W(m)} x(k)$$

mean

outlier pixels have large effect.

$x_0, \ldots, x_{p-1}$

"Find centroid"

clustered data

quote a value that represents the centroid

$$\theta^* = \underset{\theta}{\arg\min} \sum_{i=1}^{p-1} \rho(x_i - \theta) \quad - \text{ M estimator}$$

Properties of ρ:

1) $\rho(\Delta) = \rho(-\Delta)$
2) $\rho(0) = 0$
3) $\rho(\Delta) \geq 0 \quad \forall \Delta$

sometimes $\rho(\Delta)$ is convex

$$\frac{d}{d\theta} \sum_{i=1}^{p-1} \rho(x_i - \theta) \Big|_{\theta = \theta^*} = 0$$

$$\hookrightarrow \sum_{i=1}^{p-1} \rho'(x_i - \theta^*) = 0$$

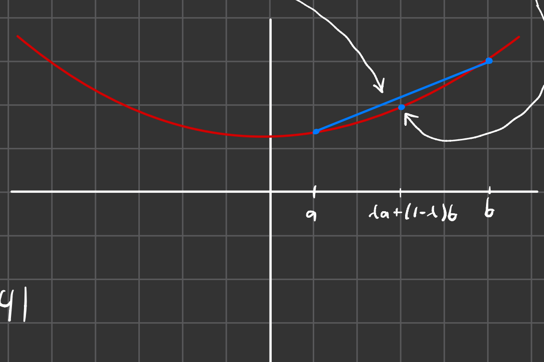
force balance

energy minimization

Convex Functions:

$\forall \lambda \in (0,1) \quad \forall a, b \in \mathbb{R}^n$

$$\lambda f(a) + (1-\lambda) f(b) \geq f(\lambda a + (1-\lambda) b)$$



$$\sum_{i=0}^{p-1} (x_i - \theta) = 0 \quad \text{when } \rho = 2$$

$$= \sum_{i=0}^{p-1} x_i - p\theta^k = 0$$

did not finish writing

this simplifies to the average

$\rho(\Delta) = \Delta^p$

$p = 1$ - medean

$p = 2$ - average

$p = 1.2$ - something inbetween

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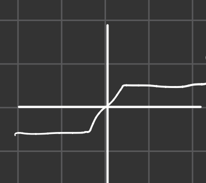
$f(x) = ax + b$ — both convex and concave

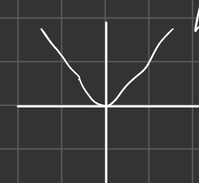
f, g are both convex, then

$f(x) + g(x)$ are convex

$f(\vec{x})$ convex

$f(A\vec{x})$ also convex





Huber Function

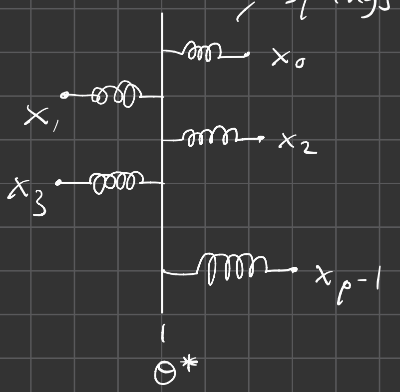
$$L(\theta) = \sum_{k=0}^{p-1} \rho(x_r - \theta)$$

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}}\, L(\theta)$$

$$\sum_{k=0}^{p-1} \rho'(x_r - \theta) = 0$$

$\rho(\Delta)$ - represents energy as a function of displacement
$\rho'(\Delta)$ - usually pick this, then integrate to get ρ (represents force)

springs not necessarily linear

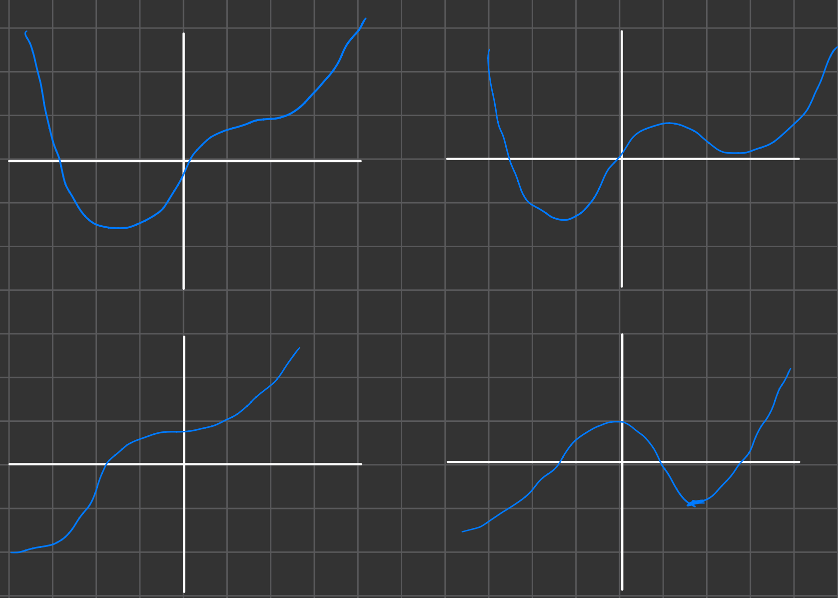


energy, distance

$$E = F \cdot d$$

$$\frac{dE}{dd} = F$$

resting state - minimum energy - Force balance



$\rho(\Delta)$

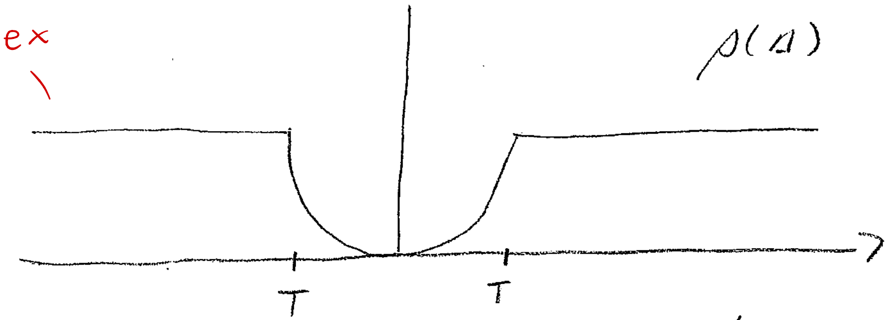
$\rho'(\Delta)$

convexity is a sufficient condition to justify that all local minimums are global

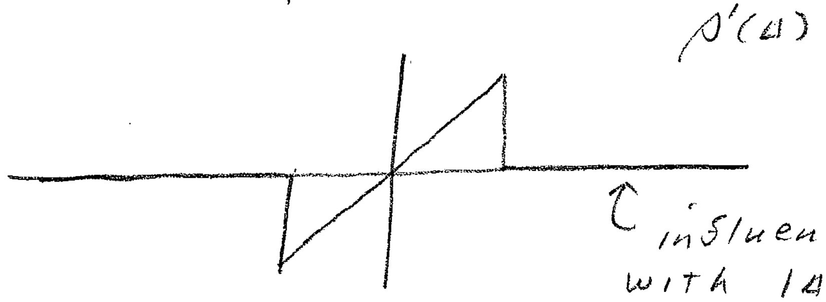
Example 2)

$$\rho(\Delta) = \begin{cases} \Delta^2 & |\Delta| < T \\ T^2 & |\Delta| > T \end{cases}$$

not convex



broken spring model



influence of pixels with $|\Delta| > T$ is zero!

- Problem: $\rho(\Delta)$ is not convex $\Rightarrow$ optimization is difficult.

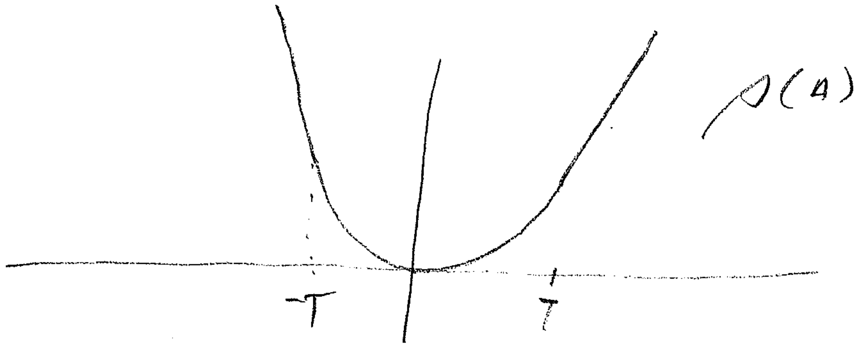
"spring snaps back"

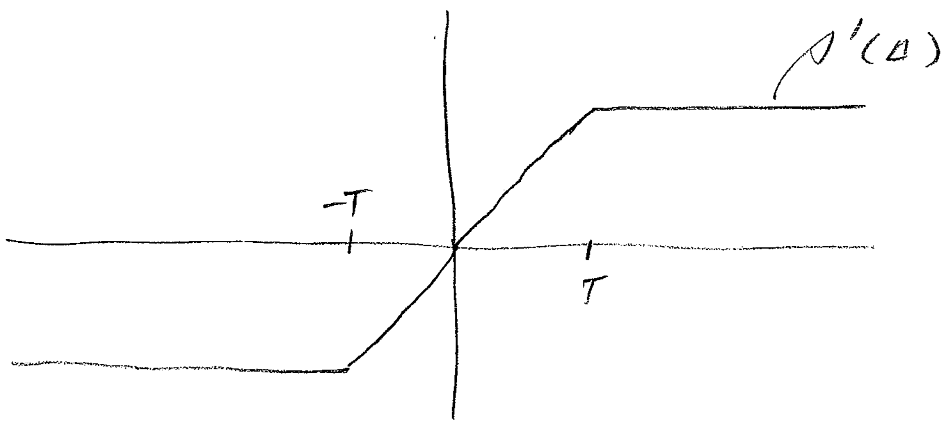
Example 3)

$$\rho(\Delta) = \begin{cases} \Delta^2 & |\Delta| < T \\ \cancel{2T - T} \; 2T|\Delta| - T^2 & |\Delta| \geq T \end{cases}$$

convex

- Known as Huber function





"spring has no further influence after some point"