Continuous Space Fourier Transform (CSFT)

Forward CSFT:

frequency variable for horizontal direction

frequency variable for vertical direction

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\underbrace{e^{-j2\pi(ux+vy)}}_{=e^{-j2\pi ux}e^{-j2\pi vy}}dxdy$$

↳ can separate the integrals

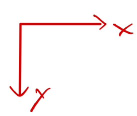
Inverse CSFT:

$$\underbrace{f(x,y)}_{\text{signal}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\underbrace{e^{j2\pi(ux+vy)}}_{\text{can be written in terms of sine waves}}dudv$$

decompose signal into multiple sine waves at different frequencies

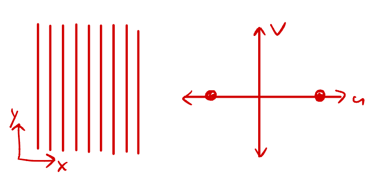
space domain ⟷ frequency domain

- Space coordinates:
  1. Usually, x is horizontal and y is vertical coordinate
  2. Usually, y points down

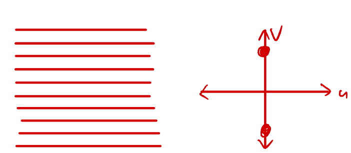


  3. Raster order - Television *scans* rapidly from left to right and more slowly from top to bottom.

- Frequency coordinates:
  1. u corresponds to horizontal frequency components (vertical strips).
  2. v corresponds to vertical frequency components (horizontal strips).



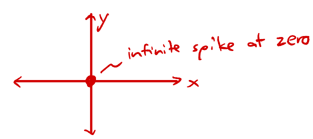
conversely:



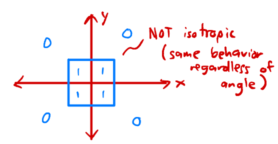
rotations in space = rotations in frequency

Useful Continuous Space Signal Definitions

$$\delta(x,y) \stackrel{\triangle}{=} \delta(x)\,\delta(y)$$



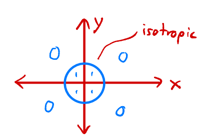
$$\text{rect}(x,y) \stackrel{\triangle}{=} \text{rect}(x)\,\text{rect}(y)$$



$$\text{sinc}(x,y) \stackrel{\triangle}{=} \text{sinc}(x)\,\text{sinc}(y)$$

$$\text{circ}(x,y) \stackrel{\triangle}{=} \text{rect}(\sqrt{x^2+y^2})$$

comes up a lot in optics



- A 2-D function $f(x,y)$ is said to be **separable** if it is formed by the product of two 1-D functions.

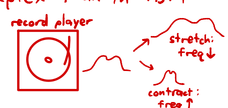
$$\boxed{f(x,y) = g(x)\,h(y)}$$

rect(x,y), sinc(x,y), and $\delta(x,y)$ are separable functions.

- Is circ(x,y) a separable function? No

CSFT Properties Inherited from CTFT

- Some properties of the CSFT are very similar to corresponding CTFT properties.

Property	Space Domain Function	CSFT
Linearity	$af(x,y)+bg(x,y)$	$aF(u,v)+bG(u,v)$
?? Conjugation	$f^*(x,y)$	$F^*(-u,-v)$ symmetry is more complex than in IDFT
Scaling	$f(ax,by)$ stretching a signal in time equates to contracting it in frequency, and vice versa (contract → stretch)	$\frac{1}{\lvert ab\rvert}F(u/a,v/b)$ 
Shifting	$f(x-x_0,y-y_0)$	$e^{-j2\pi(ux_0+vy_0)}F(u,v)$
Modulation	$e^{j2\pi(u_0x+v_0y)}f(x,y)$	$F(u-u_0,v-v_0)$
Convolution	$f(x,y)*g(x,y)$	$F(u,v)G(u,v)$
Multiplication	$f(x,y)g(x,y)$	$F(u,v)*G(u,v)$
Duality	$F(x,y)$	$f(-u,-v)$

- Inner product property

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)g^*(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)G^*(u,v)\,du\,dv
\end{aligned}
$$

Properties Specific to CSFT

- But some properties of the CSFT are quite unique to the 2-dimensional problem.

Property	Space Domain Function	CSFT
will use on exam - Separability	$f(x)g(y)$	$F(u)G(v)$ product of 1D FT's of each 1D function
Rotation	$f\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)$ 2 signals: 1 in space & 1 in frequency - they rotate together	$\lvert\mathbf{A}\rvert^{-1}F\left([u,v]\mathbf{A}^{-1}\right)$

$T(x,y) = f(x)g(y) = F(u)G(v)$

Separability of CSFT

Proof on 2D Separability

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\underbrace{e^{-j2\pi(ux+vy)}dxdy}$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\underbrace{e^{-j2\pi ux}dx}\right]\underbrace{e^{-j2\pi vy}dy}$$

separate

Define the CTFT of $f(x,y)$ in the variable x

$$\tilde{F}(u,y) = \int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx$$

Then the CSFT may be computed as the CTFT of $\tilde{F}(u,y)$ in y

$$F(u,v) = \int_{-\infty}^{\infty} \tilde{F}(u,y)e^{-j2\pi vy}dy$$

- Comment: 2-D CSFT can be computed as two 1-D CTFT's.

CSFT of Separable Functions

Let

$$g(t) \overset{CTFT}{\Leftrightarrow} G(f)$$
$$h(t) \overset{CTFT}{\Leftrightarrow} H(f)$$

Then

$$g(x)h(y) \overset{CSFT}{\Leftrightarrow} G(u)H(v)$$

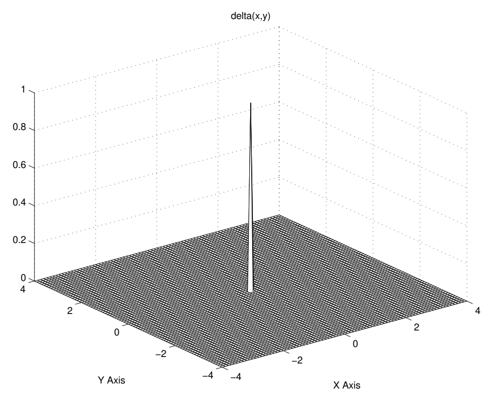
Proof:

$$\begin{aligned}
F(u,v) &= CSFT\{g(x)\,h(y)\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,h(y)\,e^{-j2\pi(ux+vy)}dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,h(y)\,e^{-j2\pi ux}e^{-j2\pi vy}dxdy \\
&= \left[\int_{-\infty}^{\infty} g(x)e^{-j2\pi ux}dx\right]\left[\int_{-\infty}^{\infty} h(y)e^{-j2\pi vy}dy\right] \\
&= G(u)H(v)
\end{aligned}$$

Useful CSFT Transform Pairs

- 2-D delta function:

$$
\begin{aligned}
CSFT\{\delta(x,y)\} &= CSFT\{\delta(x)\delta(y)\} \\
&= CTFT\{\delta(x)\} \cdot CTFT\{\delta(y)\} \\
&= 1 \cdot 1 = 1
\end{aligned}
$$

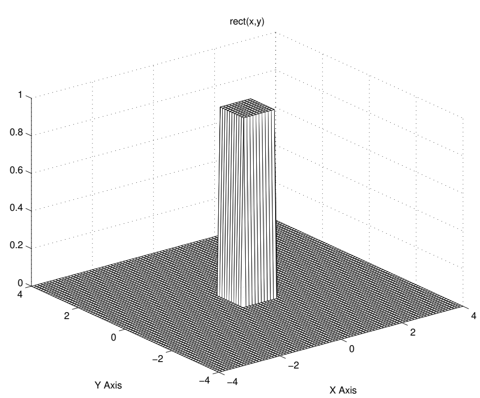


$\Rightarrow \quad 1$

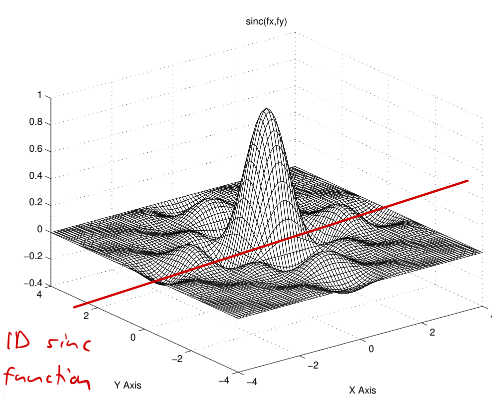
- 2-D rect function:

$$
\begin{aligned}
CSFT\{\text{rect}(x,y)\} &= CSFT\{\text{rect}(x)\text{rect}(y)\} \\
&= CTFT\{\text{rect}(x)\} \cdot CTFT\{\text{rect}(y)\} \\
&= \text{sinc}(u)\,\text{sinc}(v) \\
&= \text{sinc}(u,v)
\end{aligned}
$$

how to tell if a function is separable? ☆



$\Rightarrow$



1D sinc function is visualized

Rotated Functions

- Let the matrix $\mathbf{A}$ be an orthonormal rotation of angle θ

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

- Because $\mathbf{A}$ is an orthonormal transform

$$|\mathbf{A}| = 1$$
$$\mathbf{A}^{-1} = \mathbf{A}^t$$

- Then the CSFT of the function $g\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is given by

Try to work through proof on your own

$$\begin{aligned} CSFT\left\{ g\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)\right\} &= |\mathbf{A}|^{-1} G\left([u,v]\,\mathbf{A}^{-1}\right) \\ &= |\mathbf{A}|^{-1} G\left([u,v]\,\mathbf{A}^{t}\right) \\ &= G\left(\mathbf{A}\begin{bmatrix} u \\ v \end{bmatrix}\right) \end{aligned}$$

- So we have

$$g\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right) \overset{CSFT}{\Leftrightarrow} G\left(\mathbf{A}\begin{bmatrix} u \\ v \end{bmatrix}\right)$$

Rotated Rect Function

- Rotated 2-D rect function:

$$\mathrm{rect}\left(\frac{y+x}{\sqrt{2}}, \frac{y-x}{\sqrt{2}}\right) = \mathrm{rect}\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

where

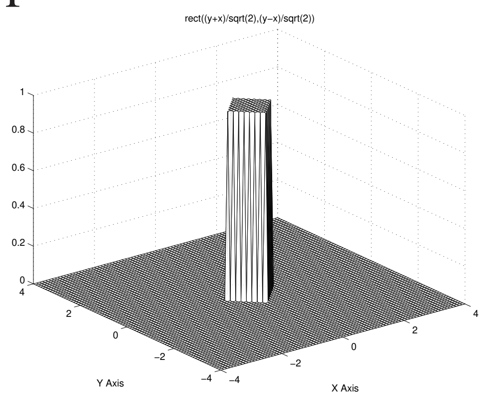
$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

- $\mathbf{A}$ is a 45° rotation, so it is and orthonormal transform

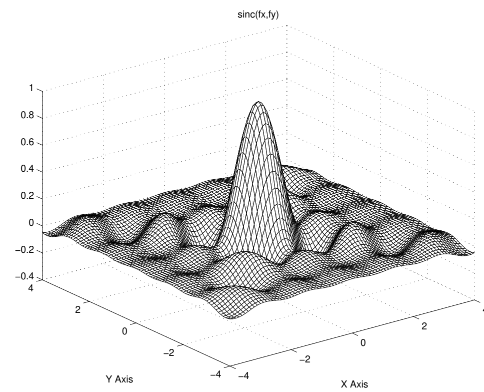
$$\begin{aligned} CSFT\left\{\mathrm{rect}\left(\frac{y+x}{\sqrt{2}}, \frac{y-x}{\sqrt{2}}\right)\right\} &= CSFT\left\{\mathrm{rect}\left(\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}\right)\right\} \\ &= \mathrm{sinc}\left(\mathbf{A}\begin{bmatrix} u \\ v \end{bmatrix}\right) \\ &= \mathrm{sinc}\left(\frac{v+u}{\sqrt{2}}, \frac{v-u}{\sqrt{2}}\right) \end{aligned}$$

Rotated 2-D Rect and Sinc Transform Pairs

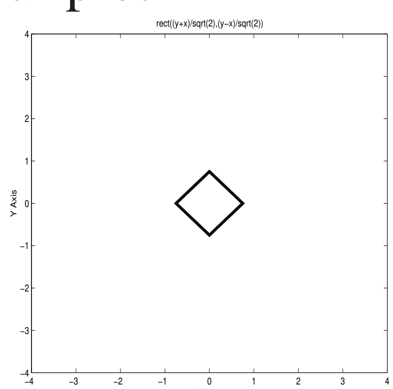
- Mesh plot



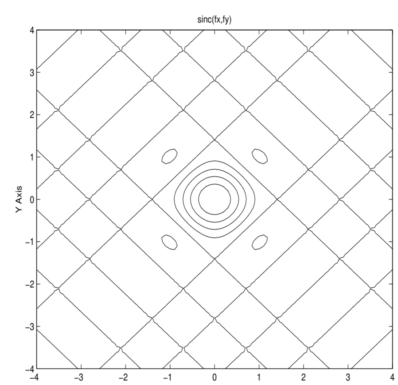
$\Rightarrow$



- Contour plot



$\Rightarrow$



More Useful CSFT Transform Pairs

- Circ function:

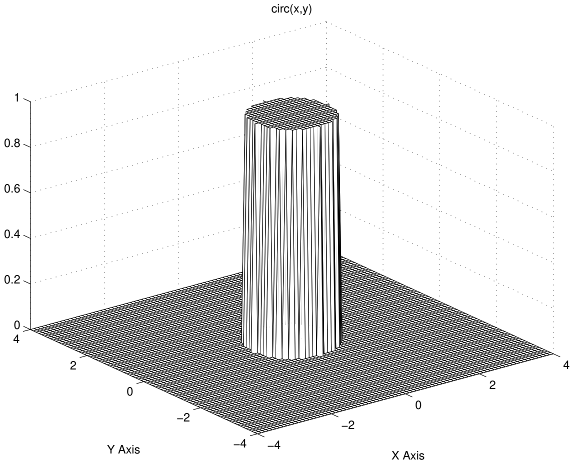
$$CSFT\left\{\text{circ}(x,y)\right\} = \text{jinc}(u,v)$$

beneficial in tomography

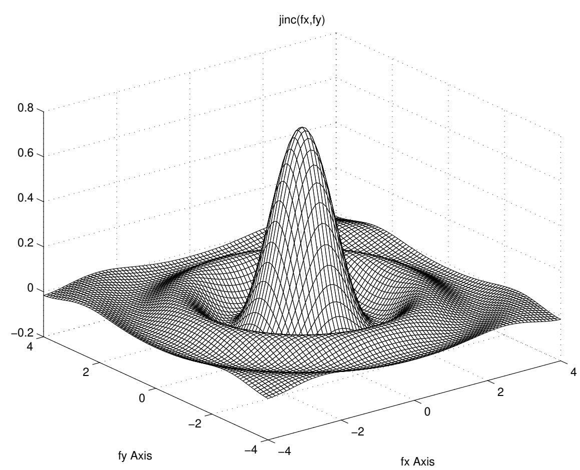
where

$$\text{jinc}(u,v) = \frac{J_1\left(\pi\sqrt{u^2+v^2}\right)}{2\sqrt{u^2+v^2}}$$

and $J_1(r)$ is the Bessel function of the first kind order 1.



$\Rightarrow$



- Notice that both functions are circularly symmetric bridge between image domain and frequency domain

CSFT of a Plane Wave

- Consider an impulse in the 2-D frequency domain.

sine wave at only one frequency – a point

$$F(u,v) = \delta(u-u_o, v-v_o)$$

- Its inverse transform is a 2-D plane wave.

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(u-u_o, v-v_o) e^{j2\pi(ux+vy)} du dv$$

$$= e^{j2\pi(u_o x+v_o y)}$$

- We know that

$$\cos(2\pi(u_o x + v_o y)) = \frac{1}{2}\left[e^{j2\pi(u_o x+v_o y)} + e^{-j2\pi(u_o x+v_o y)} \right]$$

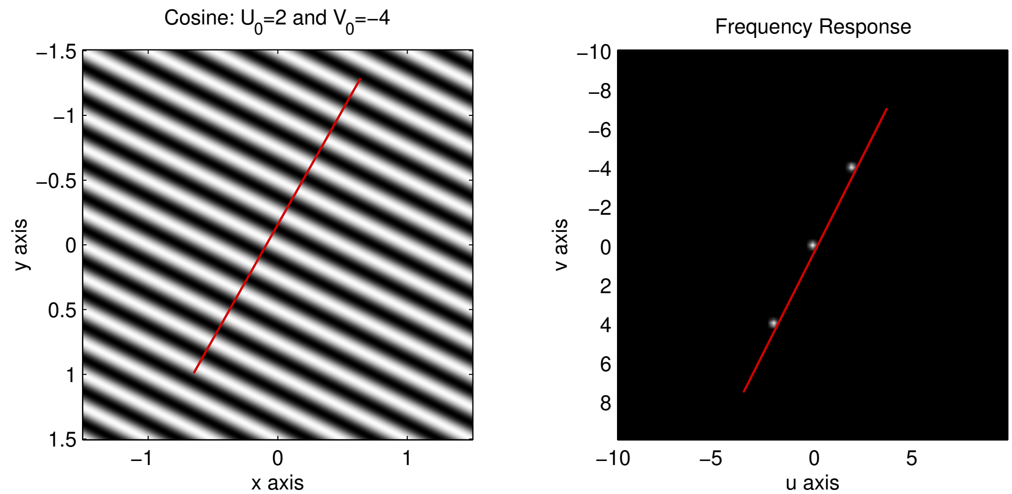
impulses in pairs

- So we have that

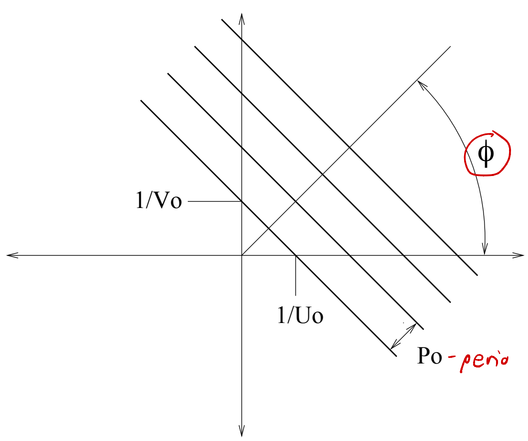
$$\cos(2\pi(u_o x + v_o y)) \overset{CSFT}{\Leftrightarrow} \frac{1}{2}\left[\delta(u-u_o, v-v_o) + \delta(u+u_o, v+v_o)\right]$$

2-D Plane Wave Example

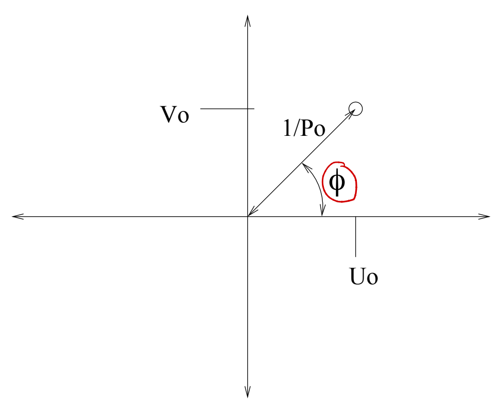
- Example transform pair computed with Matlab [1]



- Graphical representation of space-frequency domain



Plane Wave in Space Domain

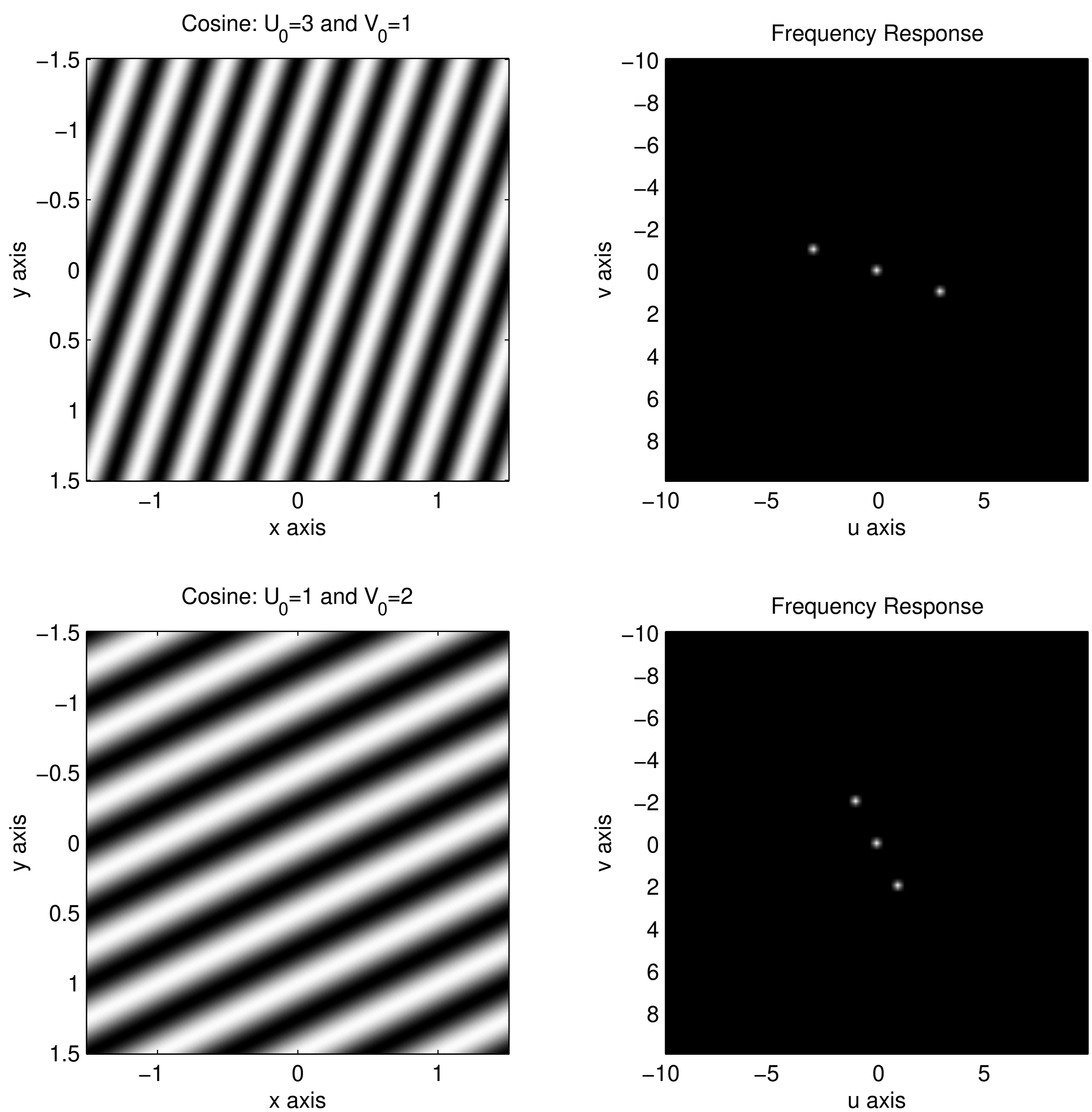


Impulse in Frequency Domain

  - $1/P_0 = \sqrt{V_0^2 + U_0^2}$
  - Rotations in space and frequency domains are the same.

[1] $f(x, y) = \cos(u_0\, x + v_0\, y) + 0.5$

More Examples 2-D Plane Waves



More Examples 2-D Plane Waves