1-D Sampling

$$g(t)$$
Continuous–time input

$$\boxed{\begin{array}{c} \text{Sampler} \\ \text{Period T} \end{array}}$$
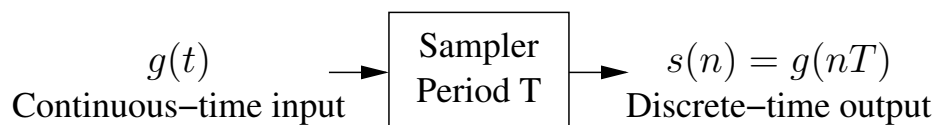
$$s(n) = g(nT)$$
Discrete–time output

fundamental method for converting continuous time to discrete time

- Let $f_s = 1/T$ be the sampling frequency.

- What is the relationship between $S(e^{j\omega})$ and $G(f)$?

L, not TI ☞

$$S(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G\left(\frac{\omega - 2\pi k}{2\pi T}\right)$$

discrete   *continuous*

- Intuition   *T terms compensate for changing energy when you sample*

  – Scale frequencies

$$\boxed{\begin{array}{c} \text{Frequency of CT signal??} \quad f = 0 \;\Leftrightarrow\; \omega = 0 \\[2mm] f = \frac{1}{2T} = \frac{1}{2}f_s \;\overset{\text{any } f \text{ beyond this leads to aliasing?}}{\Leftrightarrow}\; \omega = \pi \\[2mm] f = \frac{1}{T} = f_s \;\Leftrightarrow\; \omega = 2\pi \end{array}}$$

  – Replicate at period 2π $f_s = 10\,kHz$

  – Apply gain factor of $\frac{1}{T}$. $5kHz = \frac{1}{2} \cdot 10\,kHz$ ✓

$$3kHz = \frac{10}{2T}$$

DTFT always periodic with period 2π

$\frac{1}{2} f_s \longleftrightarrow w = \pi$

$G(f)$

$G\left(\frac{w}{2\pi}\right)$ - not periodic with period 2π

beyond $\frac{1}{2} f_s$: aliasing

Highest Signal Frequency

$f = \frac{1}{2} f_s \longrightarrow f_s = 2f \longrightarrow$ Nyquist $\longrightarrow w = \pi$

$f_s = f_{max}$

-2π 2π w

1-D Sampling

$G(f)$

1

band-limited

$-\frac{1}{2} f_s$ 0 $\frac{1}{2} f_s$ f_s

CT frequency $[-\infty, \infty]$

$S(e^{j\omega})$

$\frac{1}{T}$

$\cdots$ -4π -3π -2π $-\pi$ 0 π 2π 3π 4π $\cdots$

DT Frequency $[-\pi, \pi]$

(then it gets repeated)
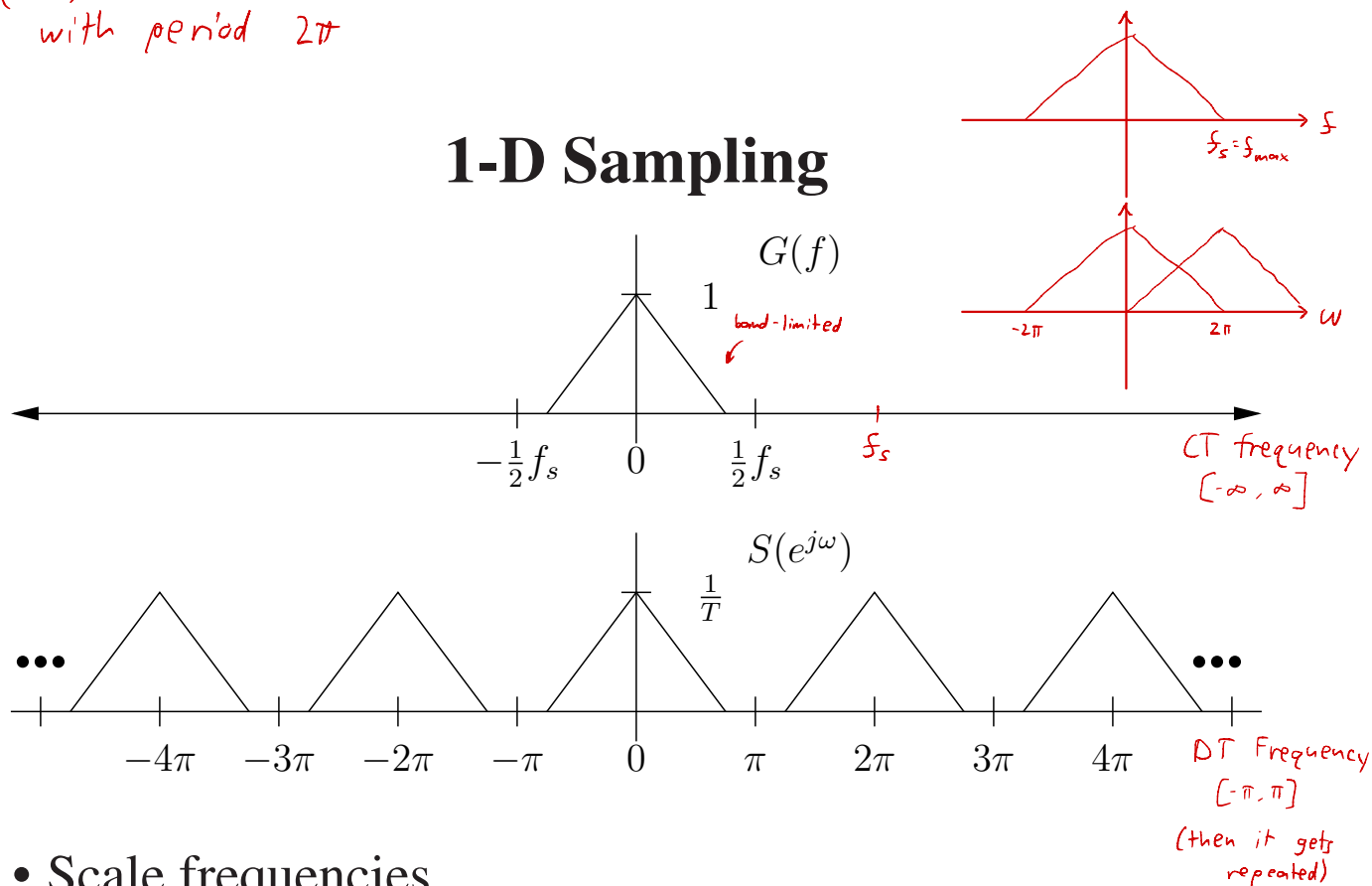
- Scale frequencies

CT Frequency	DT Frequency
0	0
$\frac{1}{2} f_s$	π
f_s	2π

- Replicate at period 2π

- Apply gain factor of $\frac{1}{T}$.

2-D Sampling

g(x,y) → | Sampler period (T_x, T_y) | → s(m,n) = g(mT$_x$,nT$_y$)
Continuous space input   Discrete space output
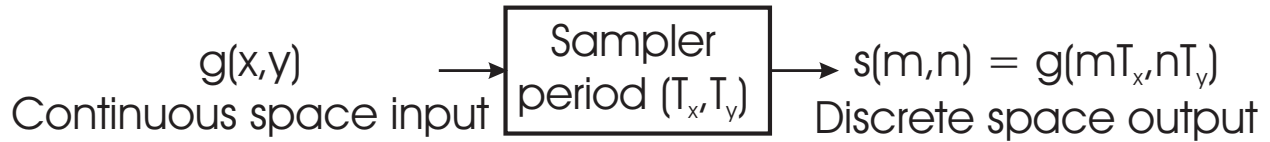
- Let T_x and T_y be the sampling period in the x and y dimensions.

- Then

$$S(e^{j\mu}, e^{j\nu}) = \frac{1}{T_x T_y} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G\left(\frac{\mu - 2\pi k}{2\pi T_x}, \frac{\nu - 2\pi l}{2\pi T_y}\right)$$

- Intuition

  – Scale frequencies

$$(u, v) = (0, 0) \iff (\mu, \nu) = (0, 0)$$

$$(u, v) = (\frac{1}{2T_x}, 0) \iff (\mu, \nu) = (\pi, 0)$$

$$(u, v) = (0, \frac{1}{2T_y}) \iff (\mu, \nu) = (0, \pi)$$

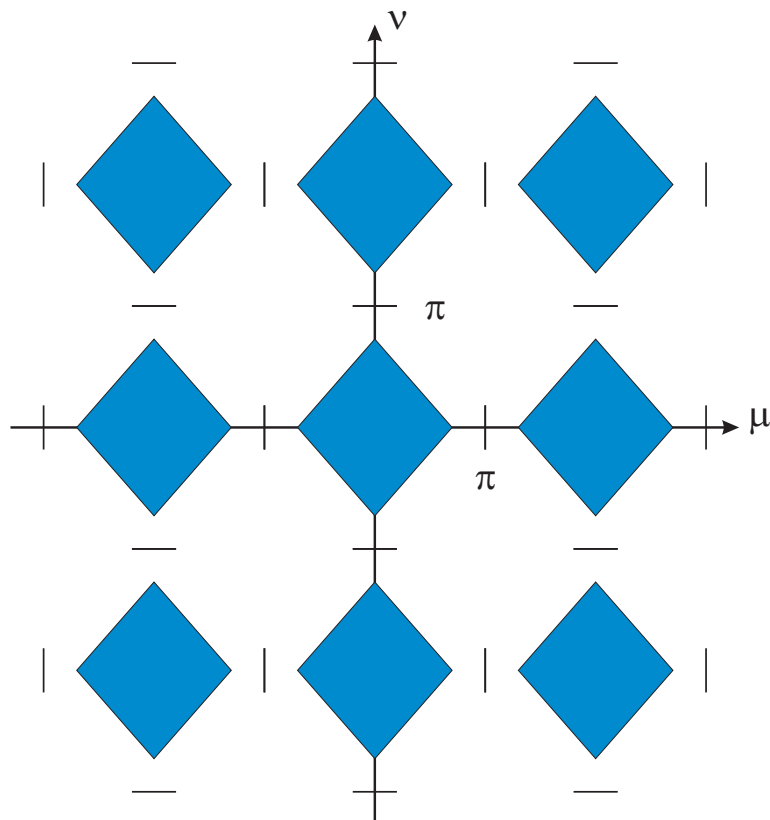$$(u, v) = (\frac{1}{2T_x}, \frac{1}{2T_y}) \iff (\mu, \nu) = (\pi, \pi)$$

  – Replicate along both μ and ν with period 2π
  – Apply gain factor of $\frac{1}{T_x T_y}$.

Example 1: 2-D Sampling Without Aliasing
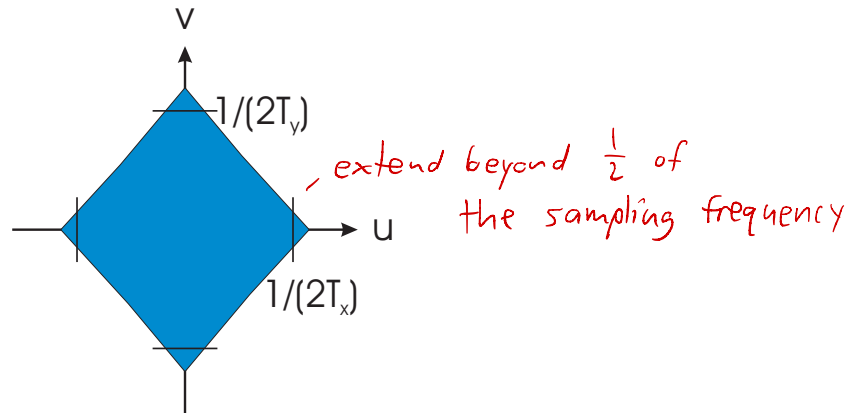
$G(u, v)$ - Spectrum of continuous space image.
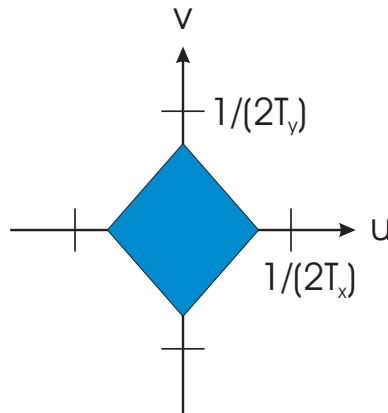
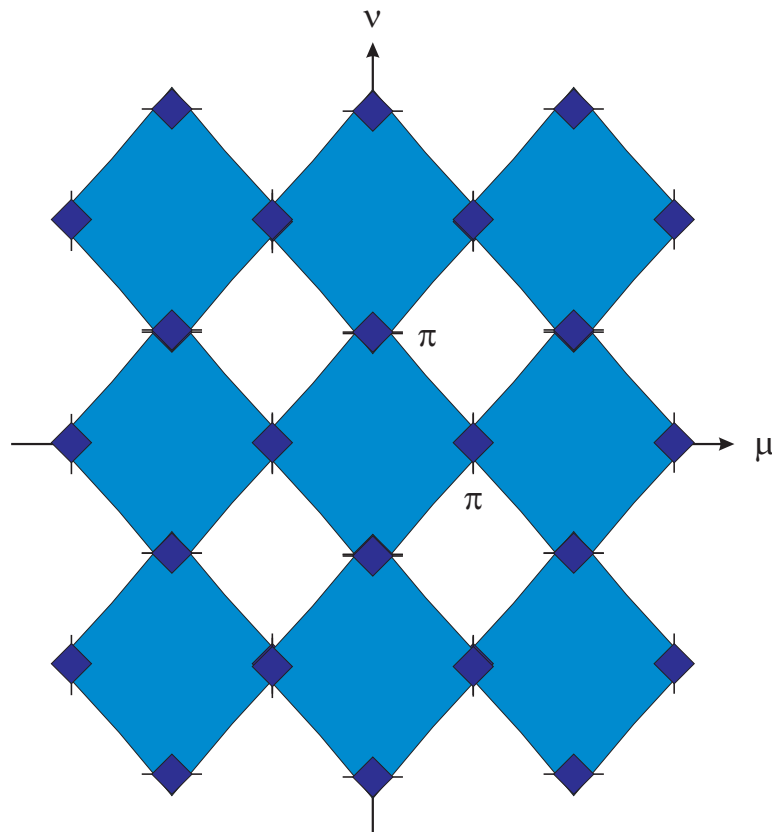$S(e^{j\mu}, e^{j\nu})$ - Spectrum of sampled image.

Example 2: 2-D Sampling With Aliasing

$G(u, v)$ - Spectrum of continuous space image.

, extend beyond $\frac{1}{2}$ of the sampling frequency

$S(e^{j\mu}, e^{j\nu})$ - Spectrum of sampled image.

Nyquist Condition

- A continuous-space signal, $g(x, y)$, may be uniquely reconstructed from its sampled version, $s(m, n)$, if $G(u, v) = 0$ for all $|u| > \frac{1}{2T_x}$ and $|v| > \frac{1}{2T_y}$.

$$|u| > \frac{1}{2} f_x \qquad |v| > \frac{1}{2} f_y$$

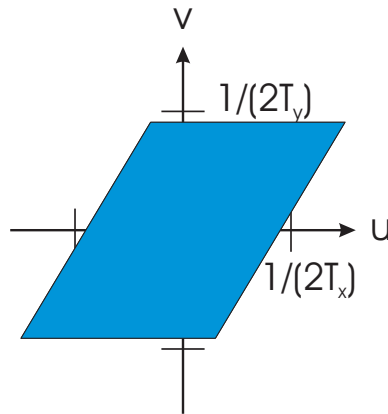or:

max allowable non-zero u, v:

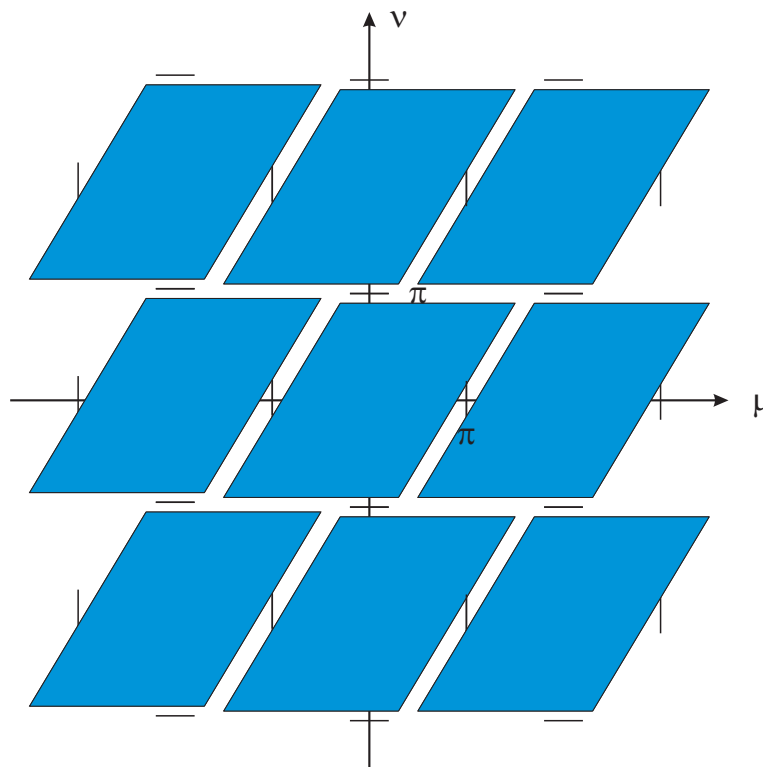$$|u| \leq \frac{1}{2} f_x \quad , \quad |v| \leq \frac{1}{2} f_y$$

$$f_N > 2 f_{max}$$

- This condition is sufficient, but not necessary.

Example 3: Nonrectangular Spectral Support
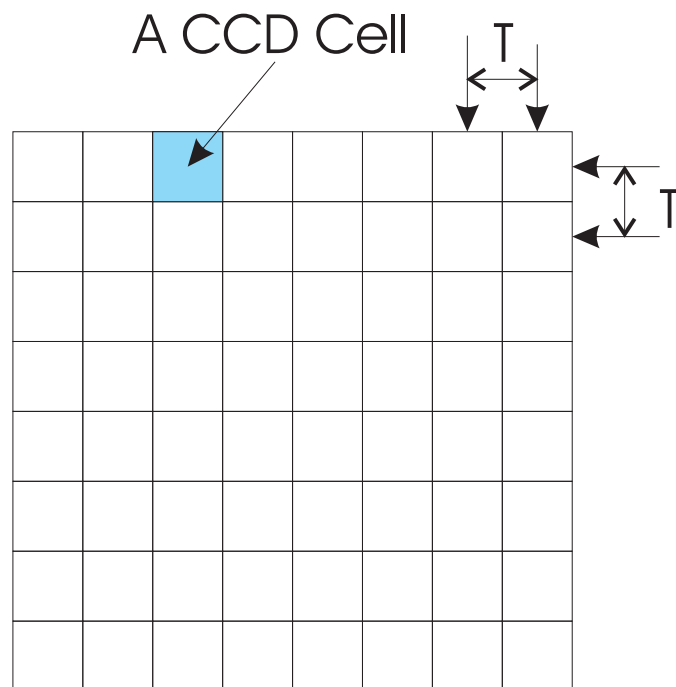
$G(u, v)$ - Spectrum of continuous space image.

$S(e^{j\mu}, e^{j\nu})$ - Spectrum of sampled image.

Focal Plain Arrays
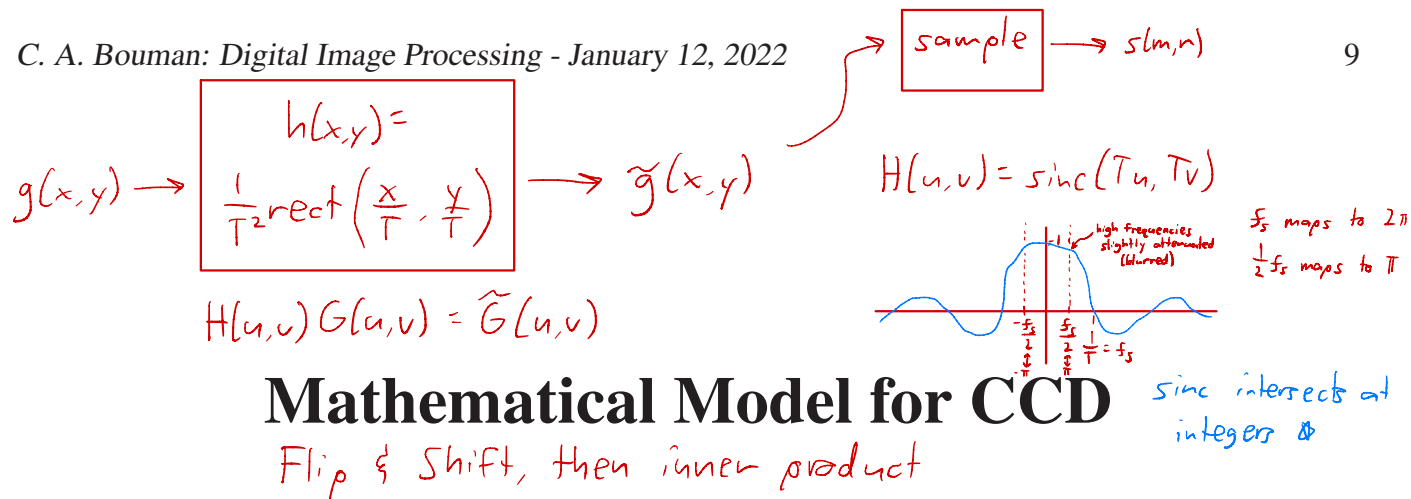
CMOS Devices

- Typical Charged Coupled Devices (CCD) Imaging array

A CCD Cell

- Solid state device used in video and still cameras.

- Each cell collects photons in a square $T \times T$ region.

- Response of each cell is linear with energy (photons).

- Signal is "shifted out" after capture.

- Cell should be large for best sensitivity.

- Finite cell size violates sampling assumptions.

$$\text{sample} \longrightarrow s(m,n)$$

$$h(x,y) =$$

$$g(x,y) \longrightarrow \boxed{\frac{1}{T^2}\text{rect}\left(\frac{x}{T}, \frac{y}{T}\right)} \longrightarrow \tilde{g}(x,y)$$

$$H(u,v) = \text{sinc}(Tu, Tv)$$

high frequencies slightly attenuated (blurred)

f_s maps to 2π

$\frac{1}{2}f_s$ maps to π

$-\frac{f_s}{2}$ $\frac{f_s}{2}$ $\frac{1}{T} = f_s$

$$H(u,v)\,G(u,v) = \widetilde{G}(u,v)$$

Mathematical Model for CCD

sinc intersects at integers $\emptyset$

Flip & Shift, then inner product

- Let $s(m,n)$ be the output of cell (m,n), then

$$s(m,n) = \int_{\mathbb{R}^2} h(x - mT, y - nT)\, g(x,y)\, dxdy$$

  where $h(x,y)$ is the rectangular window for each cell.

$$h(x,y) = \frac{1}{T^2}\text{rect}(x/T, y/T)$$

- Define $\tilde{g}(x,y)$ so that

$$\tilde{g}(\xi, \eta) = \int_{\mathbb{R}^2} h(x - \xi, y - \eta)\, g(x,y)\, dxdy$$

$$= h(-x, -y) * g(x,y)$$

  and then we have that

$$s(m,n) = \tilde{g}(mT, nT)$$

CCD Model in Space Domain

- Filter signal with space reversed cell profile

$$
\begin{aligned}
\tilde{g}(x, y) &= h(-x, -y) * g(x, y) \\
&= \frac{1}{T^2}\text{rect}(x/T, y/T) * g(x, y)
\end{aligned}
$$

- Sample filtered image

$$
s(m, n) = \tilde{g}(mT, nT)
$$

- Cell aperture blurs image.

CCD Model in Frequency Domain

- Filter signal with cell profile

$$\tilde{G}(u,v) = H^*(u,v)G(u,v)$$

$$= \text{sinc}(uT, vT)G(u,v)$$

- Sample filtered image

$$S(e^{j\mu}, e^{j\nu}) = \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \tilde{G}\left(\frac{\mu - 2\pi k}{2\pi T}, \frac{\nu - 2\pi l}{2\pi T}\right)$$

- Complete model

$$S(e^{j\mu}, e^{j\nu}) = \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \text{sinc}\left(\frac{\mu - 2\pi k}{2\pi}, \frac{\nu - 2\pi l}{2\pi}\right) \cdot$$

$$G\left(\frac{\mu - 2\pi k}{2\pi T}, \frac{\nu - 2\pi l}{2\pi T}\right)$$

Some unknown function

- Sinc function filters image.

Sampled Image Display or Rendering (Reconstruction)

- CRT's and LCD displays convert discrete-space images to continuous-space images.

- Notation:

  $s(m, n)$ - sampled image

  $p(x, y)$ - point spread function (PSF) of display

  $f(x, y)$ - displayed image

- Model:

  – In space domain:

  acquired discrete space signal

  $$f(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s(m, n)p(x - mT, y - nT)$$
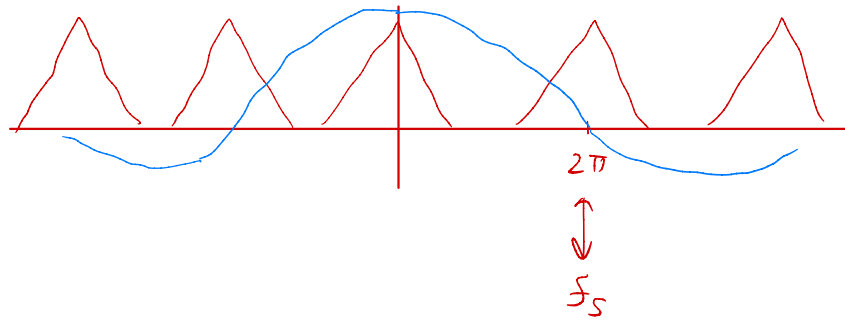
  – In frequency domain:

  periodic with period 2π

  $$F(u, v) = P(u, v)S(e^{j2\pi Tu}, e^{j2\pi Tv})$$

  $$\mu \rightarrow 2\pi Tu$$

  $$\nu \rightarrow 2\pi Tv$$

- Monitor PSF further "softens" image.

Model for Sampling and Reconstruction

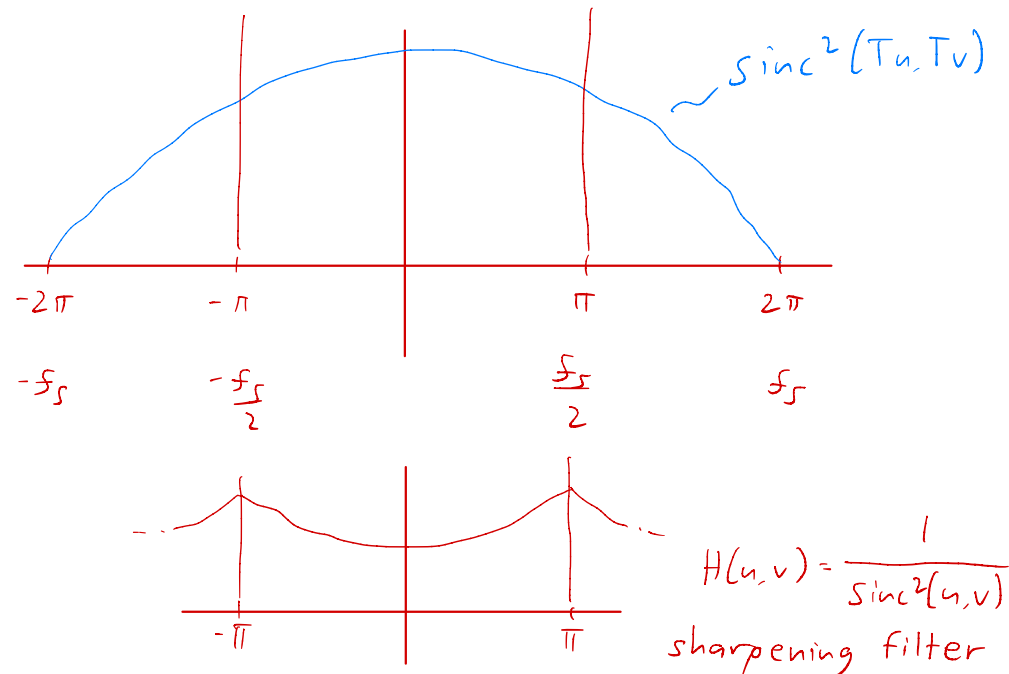- Combining models for sampling and reconstruction results in:

$$F(u, v) =$$

$$\frac{P(u,v)}{T^2} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} H^*\left(u - \frac{K}{T}, v - \frac{l}{T}\right) G\left(u - \frac{K}{T}, v - \frac{l}{T}\right)$$

- When no aliasing occurs, this reduces to

$$F(u, v) = \frac{P(u,v)H^*(u,v)}{T^2} G(u,v)$$

$$= \frac{P(u,v)\operatorname{sinc}(uT, vT)}{T^2} G(u,v)$$

$$F(u,v) = \frac{\operatorname{sinc}^2(uT, vT)}{T \cdot G(u,v)}$$

$sinc^2(Tu, Tv)$

-2π $-\pi$ π 2π

$-f_s$ $-\dfrac{f_s}{2}$ $\dfrac{f_s}{2}$ f_s

$-\pi$ π

$H(u,v) = \dfrac{1}{sinc^2(u,v)}$

sharpening filter

Effect of Sampling and Reconstruction

- The image is effectively filtered by the transfer function

$$\frac{1}{T^2}P(u,v)H^*(u,v) = \frac{1}{T^2}P(u,v)\mathrm{sinc}(uT,vT)$$

- Scanned image normally must be "sharpened" to remove the effect of softening produced in the scanning and display processes.

Raster Scan Ordering

- Specific scan pattern for mapping 2-D images to 1-D.

- Order pixels from top to bottom and left to right.

- Example: Consider the discrete-space image $f(m, n)$

$$\begin{bmatrix} f(0,0) & \cdots & f(M-1,0) \\ \vdots & \ddots & \vdots \\ f(0, N-1) & \cdots & f(M-1, N-1) \end{bmatrix}$$

- Raster ordering produces a 1-D signal x_n

$$\begin{bmatrix} x_0 & x_1 & \cdots & x_{M-1} \\ x_M & x_{M+1} & \cdots & x_{2*M-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(N-1)*M} & x_{(N-1)*M+1} & \cdots & x_{N*M-1} \end{bmatrix}$$

Vector Representation of Images

- An image is **not a matrix**. (A Matrix specifies a linear function.)

- Vectorizing images

  - Often image must be converted to a vector (data).
  - Vector looks like

  $$x = \begin{bmatrix} f(0,0) \\ \vdots \\ f(M-1,0) \\ \vdots \\ f(0,N-1) \\ \vdots \\ f(M-1,N-1) \end{bmatrix}$$

  - Mapping from vector to image is $f(m,n) = x_{n*M+m}$.