Filtered Random Processes

- Consider the 2-D linear system

$$Y(m,n) = h(m,n) * X(m,n)$$

  where $X(m,n)$ is a 2-D wide sense stationary random process.

- It may be easily shown that

$$\begin{aligned} R_y(m,n) &= h(m,n) * h(-m,-n) * R_x(m,n) \\ S_y(e^{j\mu}, e^{j\nu}) &= \left| H(e^{j\mu}, e^{j\nu}) \right|^2 S_x(e^{j\mu}, e^{j\nu}) \end{aligned}$$

"White Noise"

$X_n \sim N(0, \sigma^2)$ i.i.d

$$R(k) = E[X_n X_{n+k}] = \begin{cases} \sigma^2, & k = 0 \\ 0, & k \neq 0 \end{cases} = \sigma^2 \delta(k)$$

Autocorrelation doesn't make sense if process is not stationary

$$S(e^{j\omega}) = \sum_{n=-\infty}^{\infty} R(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sigma^2 \delta(n) e^{-j\omega n} = \sigma^2$$

LTI

$$\underset{\text{i.i.d.}}{X_n \sim N(0, \sigma^2)} \longrightarrow \boxed{h_n} \longrightarrow y_n$$

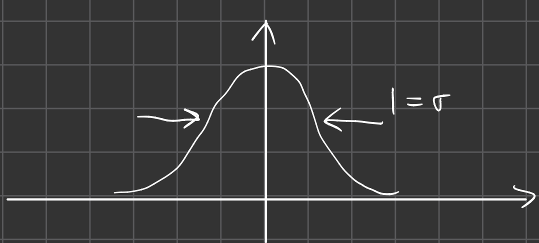
$H(e^{j\omega}) \Longleftrightarrow DTFT(h_n)$

Energy is square of amplitude

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$S_y(e^{j\omega}) = S_x(e^{j\omega}) |H(e^{j\omega})|^2$$

$$= \sigma^2 |H(e^{j\omega})|^2$$



standard deviation - easier to understand intuitively

variance - more helpful mathematically

2D Gaussian White Noise

- Definition:
  - $X(m,n)$ - independent identically distributed (i.i.d.) Gaussian random variables with distribution $N(0,1)$.
- Then
  - $X(m,n)$ is wide sense stationary with

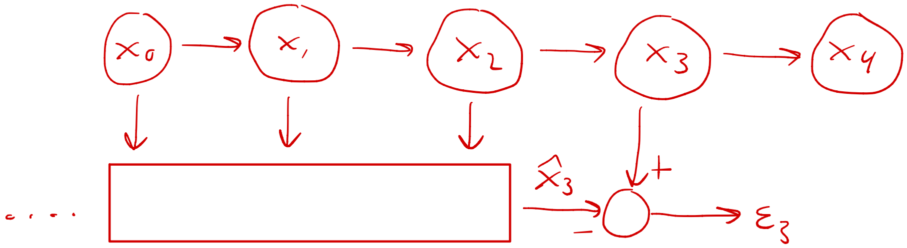
$$
\begin{aligned}
\mu(m,n) &= E[X(m,n)] = 0 \\
R_x(k,l) &= E[X(0,0)X(k,l)] = \delta(k,l) \\
S_x(e^{j\mu}, e^{j\nu}) &= DSFT\left\{R_x(k,l)\right\} \\
&= 1
\end{aligned}
$$

Filtered White Noise

- Definitions:
  - $X(m,n)$ - independent identically distributed (i.i.d.) Gaussian random variables with distribution $N(0,1)$.
  - $Y(m,n) = h(m,n) * X(m,n)$.
- Then
  - $Y(m,n)$ is wide sense stationary with

$$\begin{aligned} S_y(e^{j\mu}, e^{j\nu}) &= \left| H(e^{j\mu}, e^{j\nu}) \right|^2 S_x(e^{j\mu}, e^{j\nu}) \\ &= \left| H(e^{j\mu}, e^{j\nu}) \right|^2 \cdot 1 \\ R_y(k,l) &= h(m,n) * h(-m,-n) \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(m,n) h(m+k, n+l) \end{aligned}$$

  - $R_y(k,l)$ is the autocorrelation of $h(m,n)$ with itself.



x_n
- zero mean
- Gaussian
- stationary

Causal Prediction

$\hat{X}_n = E[X_n | X_k \; n > k \geq n-p] = f(X_{n-1}, X_{n-p}) = \sum_{k=1}^{p} a_{n,k} X_{n-k}$ for some $a_{n,k}$

Because system is stationary: $\hat{X}_n = \sum_{k=1}^{p} a_k X_{n-1}$

prediction error (zero mean):

$\varepsilon = X_n - \hat{X}_n$

Causal Prediction

Ex: predict stock market prices

- Y_s is a 2-D wide sense stationary zero mean Gaussian random process.
- Define
  - The past values are $Y_{<s} = \{Y_r : r < s\}$.
  - The minimum mean squared error (MMSE) predictor of Y_s given the past is

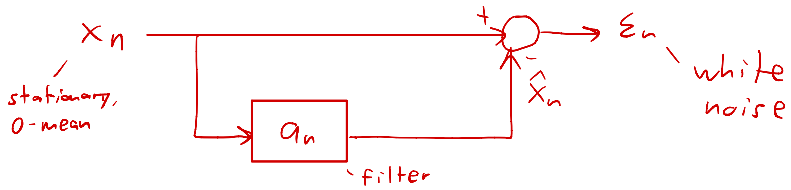
$$\hat{Y}_s = E[Y_s | Y_{<s}]$$

  - The prediction error is $X_s = Y_s - \hat{Y}_s$.

Interesting Facts:

- ε_n values are i.i.d
- ε_n are zero mean
- ε_n are Gaussian

$\varepsilon_n \sim N(0, \sigma_\varepsilon^2)$



MMSE Linear Predictor

Properties of Causal Predictors

- Fact 1: (WLOG, assume $r < s$.)

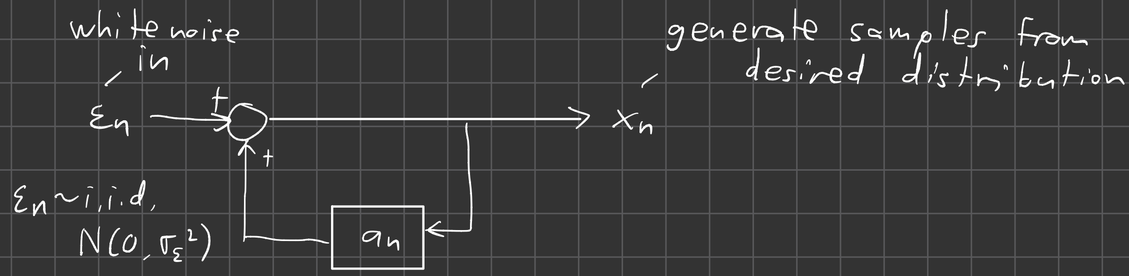
$$\begin{aligned}
E[X_s X_r] &= E[E[X_s X_r | Y_{<s}]] \\
&= E[E[(Y_s - \hat{Y}_s)(Y_r - \hat{Y}_r) | Y_{<s}]] \\
&= E[E[(Y_s - \hat{Y}_s) | Y_{<s}](Y_r - \hat{Y}_r)] \\
&= E[(E[Y_s | Y_{<s}] - \hat{Y}_s)(Y_r - \hat{Y}_r)] \\
&= E[(\hat{Y}_s - \hat{Y}_s)(Y_r - \hat{Y}_r)] \\
&= E[0(Y_r - \hat{Y}_r)] = 0
\end{aligned}$$

- Fact 2: $\sigma^2 \triangleq E[X_s^2]$ is the prediction variance.
- Fact 3: The predictor must be space invariant and linear.

$$\hat{Y}_s = \sum_{r>(0,0)} h_r Y_{s-r}$$

$$\varepsilon_n = x_n - \underbrace{a_n * x_n}_{\hat{x}_n}$$

$$x_n = \varepsilon_n + a_n * x_n$$ - IIR Filter



Autoregressive (AR) Processes

- Definitions:
  - Y_s - 2-D wide sense stationary zero mean Gaussian random process.
  - h_s - MMSE linear predictor for Y_s.
  - $X_s = Y_s - h_s * Y_s$ - predictor error.

- If h_s is FIR, then Y_s is known as an autoregressive (AR) process.

Properites of AR Processes

- Remember that

$$X_s = Y_s - h_s * Y_s$$

- Then
  - We know that

$$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

  - Since X_s is white noise,

$$\begin{aligned} R_x(s) &= \sigma^2 \delta(s) \\ S_x(e^{j\mu}, e^{j\nu}) &= \sigma^2 \end{aligned}$$

  - So the power spectrum of Y_s is given by

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\sigma^2}{\left|1 - H(e^{j\mu}, e^{j\nu})\right|^2}$$

Spectral Estimate Using AR Processes

- Compute MMSE linear predictor $\hat{h}_s$ for Y_s.
- Compute the prediction variance

$$\hat{\sigma}^2 = \frac{1}{|S|} \sum_{s \in S} |Y_s - h_s * Y_s|^2$$

where S is a finite set of points in plain, and $|S|$ is the number of points in S.

- Estimate the power spectrum

$$S_y(e^{j\mu}, e^{j\nu}) = \frac{\hat{\sigma}^2}{\left|1 - \hat{H}(e^{j\mu}, e^{j\nu})\right|^2}$$

- Can produce a more accurate estimate of the power spectrum.

Generating AR Processes

- Select a causal prediction filter h_s.
- Apply IIR filter to white noise random process X_s

$$Y(e^{j\mu}, e^{j\nu}) = \frac{1}{1 - H(e^{j\mu}, e^{j\nu})} X(e^{j\mu}, e^{j\nu})$$

- Y_s is sometimes referred to as a white noise driven process.
- Do linear FIR prediction filters $\hat{h}_s$ always form a stable IIR filter?
  - In 1-D, yes.
  - In 2-D, not always!
- Other problems:
  - Causal ordering of points may cause asymmetric artifacts in results.
  - Complexity increases rapidly with IIR filter order P.