Image Restortation

- Problem:
  - You want to know some image X.
  - But you only have a corrupted version Y.
  - How do you determine X from Y?
- Corruption may result from:
  - Additive noise
  - Nonadditive noise
  - Linear distortion
  - Nonlinear distortion

Optimum Linear FIR Filter

- Find an "optimum" linear filter to compute X from Y.
- Filter uses input window of Y to estimate each output pixel X_s.
- Filter can be designed to be minimize mean squared error (MMSE).
- The estimate of X_s is denoted by $\hat{X}_s$.
- $W(s)$ denotes the window about s.
- The estimate, $\hat{X}_s$, is a function of $Y_{W(s)}$.

Empirical approach

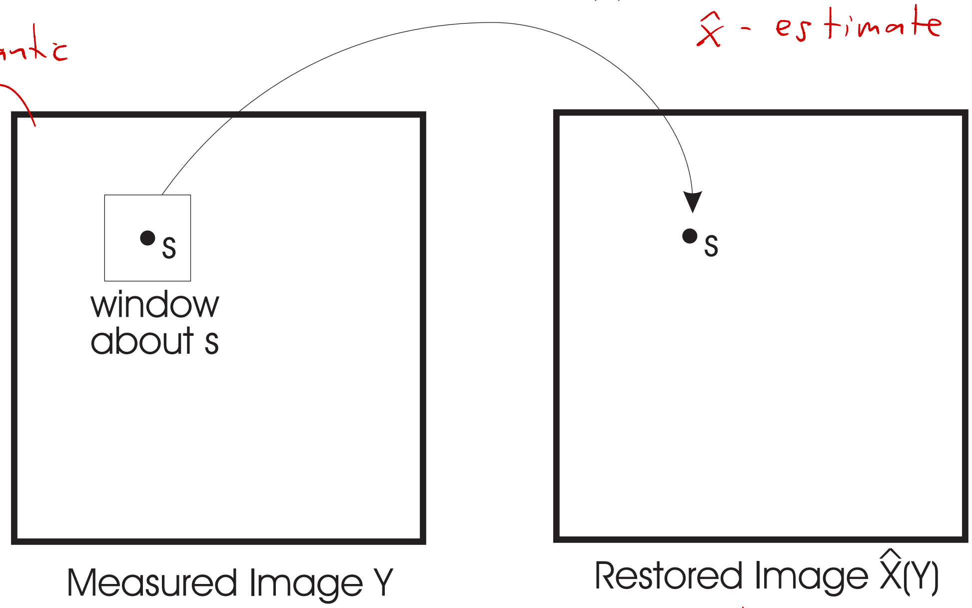
Gaussian models are like linear systems, they have an intimate relationship
↳ can get closed form solutions

Application of Optimum Filter

$$\hat{X}_s = f(Y_{W(s)})$$

X - ground truth
Y - data
$\hat{X}$ - estimate

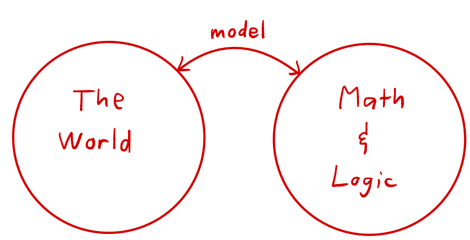
may contain structural semantic info on improving the image quality



Low quality to better quality

- The function $f(Y_{W(s)})$ is designed to produce a MMSE estimate of X.
- If $f(Y_{W(s)})$ is:
  - Linear $\Rightarrow$ linear space invariant filter.
  - Nonlinear $\Rightarrow$ nonlinear space invariant filter.
- This filter can reduce the effects of all types of corruption.

Images are NOT random variables, but they can be modeled as such



$X \rightarrow$ measurement system $\rightarrow Y$ contains noise and/or errors

$Y = AX + w$ — A: some system; w: noise/errors

$Y \rightarrow$ reconstruction $\rightarrow \hat{X} = E[X|Y]$

Optimality Properities of Linear Filter

- If both images are jointly Gaussian: typically an assumption
  - Then MMSE filter is linear.

first M for minimum

$$\hat{X}_s = E[X_s|Y_{W(s)}] = f(Y_{W(s)})$$
$$= \mathbf{A}\, Y_{W(s)} + b \qquad E[X] = E[Y] = 0 \rightarrow b = 0$$

- If images are not jointly Gaussian:
  - Then MMSE filter is generally not linear.

$$\hat{X}_s = E[X_s|Y_{W(s)}]$$
$$= f(Y_{W(s)})$$

  - However, the MMSE linear filter can still be very effective!

$\mu = E[X]$

$E[\|X - \mu\|^2]$

Formulation of MMSE Linear Filter: Definitions

- $W(s)$ - window about the pixel s.
- p - number of pixels in $W(s)$
- z_s - row vector containing pixels of $Y_{W(s)}$.
- θ - column containing filter parameters

- Detailed definitions:
  - Definition of $W(s)$

    $$W(s) = [s, s + r_1, \ldots, s + r_{p-1}]$$

    where $r_1, \ldots, r_{p-1}$ index neighbors.
  - Definition of z_s

    $$z_s = [y_s, y_{s+r_1}, \ldots, y_{s+r_{p-1}}]$$

  - Definition of θ

    $$\theta = [\theta_0, \ldots, \theta_{p-1}]$$

Formulation of MMSE Linear Filter: Objectives

- Linear filter is given by

$$\hat{x}_s = z_s\theta$$

- Mean squared error is given by

$$\begin{aligned} MSE &= E[|x_s - \hat{x}_s|^2] \\ &= E[|x_s - z_s\theta|^2] \end{aligned}$$

- The MMSE filter parameters θ^* are given by

$$\underbrace{\theta^*}_{\text{not random}} = \arg\min_{\theta} E[|x_s - z_s\theta|^2] \; . \quad = E[\|f(\theta)\|^2]$$

- How do we solve this problem?

More Matrix Notation

- Define the subset S_0 of image pixels.
  1. $S_0 \subset S$
  2. S_0 contains $N_0 < N$ pixels
  3. S_0 usually does not contain pixels on the boundary of the image.
  4. $S_0 = [s_1, \ldots, s_{N_0}]$
- Define the $N_0 \times p$ matrix Z

$$Z = \begin{bmatrix} z_{s_1} \\ z_{s_2} \\ \vdots \\ z_{s_{N_0}} \end{bmatrix} .$$

x

- Define the $N_0 \times 1$ column vectors X and $\hat{X}$

$$X = \begin{bmatrix} x_{s_1} \\ x_{s_2} \\ \vdots \\ x_{s_{N_0}} \end{bmatrix} \quad \text{and} \quad \hat{X} = \begin{bmatrix} \hat{x}_{s_1} \\ \hat{x}_{s_2} \\ \vdots \\ \hat{x}_{s_{N_0}} \end{bmatrix} .$$

$N_0 \ll N$

- Then

N_0 $\quad$ $N_0 \times p$ $\quad$ $p \times 1$

$$X \approx \hat{X} = Z\theta$$

may need to calculate for an exam

MSE: $L^1(\theta) = E[|x_s - z_s\theta|^2]$ - deterministic

LS: $L^2(\theta) = \frac{1}{N_0}\sum_s |x_s - z_s\theta|^2$ - random

MMSE: $\theta^* = \arg\min_\theta L^1(\theta)$ - deterministic

LSE: $\hat{\theta} = \arg\min_\theta L^2(\theta)$ - random

differences between two approaches

Least Squares Linear Filter

- We expect that

$$
\begin{aligned}
MSE &= E[|x_s - z_s\theta|^2] \\
&\approx \frac{1}{N_0}\sum_{s\in S_0} |x_s - z_s\theta|^2 \\
&= \frac{1}{N_0}||X - Z\theta||^2
\end{aligned}
$$

- good if $N_0 >> p$

parameters = window-dimension^2

- So we may solve the equation

$$
\theta^* = \arg\min_\theta ||X - Z\theta||^2
$$

↳ random - an estimate of the true minimum

- The solution θ^* is the least squares estimate, of θ, and the estimate

$$
\hat{X} = Z\theta^*
$$

is known as the least squares filter.

can transpose a scalar

$$E[|x_s - z_s\theta|^2] = E[(x_s - z_s\theta)^t(x_s - z_s\theta)] = L^1(\theta)$$

$$= E[x_s^2 + \theta^t z_s^t z_s\theta - 2x_s z_s\theta]$$

$$= a + \theta^t R\theta - 2b^t\theta$$

$$R \triangleq E[z_s^t z_s]$$

$$b = E[x_s z_s], \quad a = E[x_s^2]$$

$$\nabla_\theta L'(\theta) = 0 + 2R\theta - 2b = 0$$

$$R\theta = b$$

$$\theta^* = R^{-1}b$$

→ good exam question

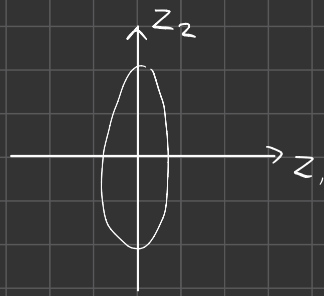
Computing Least Squares Linear Filter

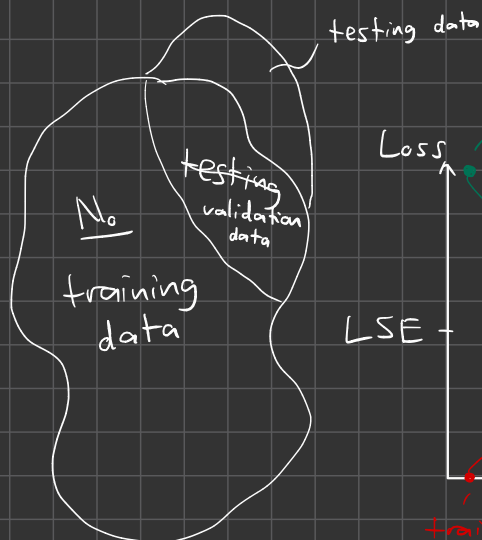
$$\theta^* = \arg\min_\theta \frac{1}{N_0}||X - Z\theta||^2$$

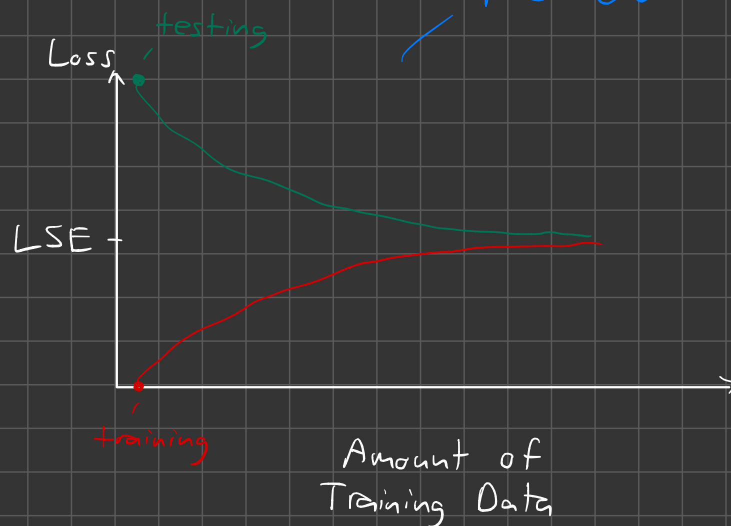
- So

$$\begin{aligned}
\theta^* &= \arg\min_\theta \left(\frac{1}{N_0}||X - Z\theta||^2 \right) \\
&= \arg\min_\theta \left(\frac{1}{N_0}(X - Z\theta)^t(X - Z\theta) \right) \\
&= \arg\min_\theta \left(\frac{1}{N_0}(X^tX - 2\theta^tZ^tX + \theta^tZ^tZ\theta) \right) \\
&= \arg\min_\theta \left(\frac{X^tX}{N_0} - 2\theta^t\frac{Z^tX}{N_0} + \theta^t\frac{Z^tZ}{N_0}\theta \right) \\
&= \arg\min_\theta \left(\theta^t\underbrace{\frac{Z^tZ}{N_0}}_{\substack{\text{matrix} \\ \hat{R}_{zz}}}\theta - 2\theta^t\underbrace{\frac{Z^tX}{N_0}}_{\substack{\text{row vector} \\ \hat{r}_{zx}}} \right)
\end{aligned}$$

$$Z = [z_1 \ z_2]$$
$$z_1 = 0$$
$$z_2 = N(0,1)$$







Review

p-hacking

Covariance Estimates

- Define the $p \times p$ matrix

$$\begin{aligned}
\hat{R}_{zz} &\triangleq \frac{Z^t Z}{N_0} \\
&= \frac{1}{N_0} \left[z_{s_1}^t, z_{s_2}^t, \ldots, z_{s_{N_0}}^t \right] \begin{bmatrix} z_{s_1} \\ z_{s_2} \\ \vdots \\ z_{s_{N_0}} \end{bmatrix} \\
&= \frac{1}{N_0} \sum_{i=1}^{N_0} z_{s_i}^t z_{s_i}
\end{aligned}$$

- Define the $p \times 1$ vector

$$\begin{aligned}
\hat{r}_{zx} &\triangleq \frac{Z^t X}{N_0} \\
&= \frac{1}{N_0} \left[z_{s_1}^t, z_{s_2}^t, \ldots, z_{s_{N_0}}^t \right] \begin{bmatrix} x_{s_1} \\ x_{s_2} \\ \vdots \\ x_{s_{N_0}} \end{bmatrix} \\
&= \frac{1}{N_0} \sum_{i=1}^{N_0} z_{s_i}^t x_{s_i}
\end{aligned}$$

- So

$$\theta^* = \arg\min_{\theta} \left(\theta^t \hat{R}_{zz} \theta - 2\theta^t \hat{r}_{zx} \right)$$

↳ differentiate w.r.t θ ☆

Interpretation of $\hat{R}_{zz}$ and $\hat{r}_{zx}$

- $\hat{R}_{zz}$ is an estimate of the covariance of z_s.

$$
\begin{aligned}
E\left[\hat{R}_{zz}\right] &= E\left[\frac{1}{N_0}\sum_{s=1}^{N} z_s^t z_s\right] \\
&= E[z_s^t z_s] \\
&= R_{zz}
\end{aligned}
$$

- $\hat{r}_{zx}$ is an estimate of the cross correlation between z_s and x_s.

$$
\begin{aligned}
E\left[\hat{r}_{zx}\right] &= E\left[\frac{1}{N_0}\sum_{s=1}^{N} z_s^t x_s\right] \\
&= E[z_s^t x_s] \\
&= r_{zx}
\end{aligned}
$$

Solution to Least Squares Linear Filter

- We need

$$\theta^* = \arg\min_\theta \frac{1}{N_0} ||X - Z\theta||^2$$

We have shown this is equivalent to

$$\theta^* = \arg\min_\theta \left(\theta^t \hat{R}_{zz} \theta - 2\theta^t \hat{r}_{zx} \right)$$

- Taking the gradient of the cost functional

$$\begin{aligned} 0 &= \nabla_\theta \left(\theta^t \hat{R}_{zz} \theta - 2\theta^t \hat{r}_{zx} \right)\Big|_{\theta=\theta^*} \\ &= \left(2\hat{R}_{zz}\theta - 2\hat{r}_{zx} \right)\Big|_{\theta=\theta^*} \end{aligned}$$

Solving for θ^* yeilds

$$\theta^* = \left(\hat{R}_{zz} \right)^{-1} \hat{r}_{zx}$$

↳ random

$\nabla_\theta \, \theta^t B \theta = 2B\theta$ assuming θ is symmetric

Summary of Solution to Least Squares Linear Filter

- First compute

$$\hat{R}_{zz} = \frac{1}{N_0} \sum_{s=1}^{N} z_s^t z_s$$

$$\hat{r}_{zx} = \frac{1}{N_0} \sum_{s=1}^{N} z_s^t x_s$$

- Then compute

$$\theta^* = \left(\hat{R}_{zz} \right)^{-1} \hat{r}_{zx}$$

- The vector θ^* then contains the values of the filter coefficients.

Training

- θ^* is usually estimated from "training" data.
- Training data
  - Generally consists of image pairs (X, Y) where Y is the measured data and X is the undistorted image.
  - Should be typical of what you might expect.
  - Can often be difficult to obtain.
- Testing data
  - Also consists of image pairs (X, Y).
  - Is used to evaluate the effectiveness of the filters.
  - Should never be taken from the training data set.
- Training versus Testing
  - Performance on training data is always better than performance on testing data.
  - As the amount of training data increases, the performance on training and testing data both approach the best achievable performance.

Comments

- Wiener filter is the MMSE **linear** filter.
- Wiener filter may be optimal, but it isn't always good.
  - Linear filters blur edges
  - Linear filters work poorly with non-Gaussian noise.
- Nonlinear filters can be designed using the same methodologies.

Is MMSE a Good Quality Criteria for Images?

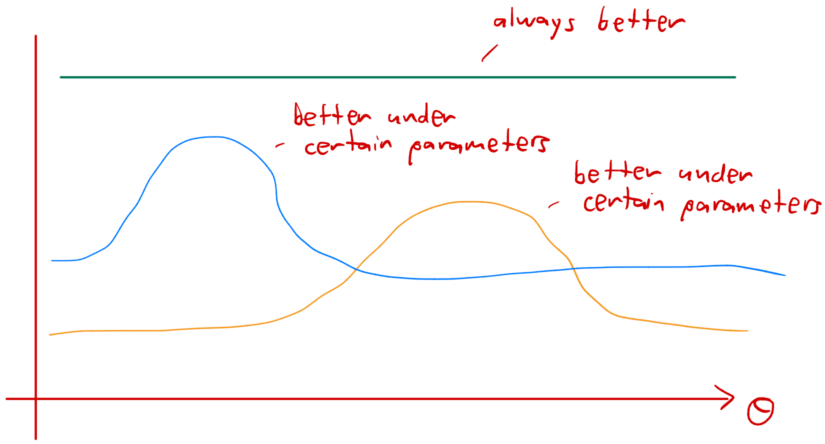
- In general, NO! ... But sometimes it is OK.
- For achromatic images, it is best to choose X and Y in L^* or gamma corrected coordinates.
- Let H be a filter that implements the CSF for the human visual system.
  - Then a better metric of error is

$$\begin{aligned} HVSE &= \|H(X - \hat{X})\|^2 \\ &= \left(X - \hat{X}\right)^t H^t H \left(X - \hat{X}\right) \\ &= \|X - \hat{X}\|_B^2 \end{aligned}$$

    where $B = H^t H$.
  - $\|X - \hat{X}\|_B^2$ is a quadratic norm.
- What is the minimum HVSE estimate $\hat{X}$?

$$\left.\begin{array}{l}\text{conditional mean vs}\\ \text{conditional median}\end{array}\right\}\ \text{different estimator types}$$

Example: Maximization



Answer

- The answer is $\hat{X} = E[X|Y]$.
  - This is the same as for mean squared error!
  - The conditional expectation minimizes any quadratic norm of the error.
  - This is also true for non-Gaussian images.
- Let $\hat{X} = AY_{W(s)} + b$ be the MMSE **linear** filter.
  - This filter is also the minimum HVSE **linear** filter.
  - This is also true for non-Gaussian images.

Proof

- Define $V \stackrel{\triangle}{=} HX$ and $B = H^t H$

$$
\begin{aligned}
& \min_{\hat{X}} E\left[||X - \hat{X}||_B^2 \right] \\
= & \min_{\hat{X}} E\left[||H(X - \hat{X})||^2 \right] \\
= & \min_{\hat{V}} E\left[||V - \hat{V}||^2 \right] \\
= & \ E\left[||V - E[V|Y]||^2 \right] \\
= & \ E\left[||HX - E[HX|Y]||^2 \right] \\
= & \ E\left[||H(X - E[X|Y])||^2 \right] \\
= & \ E\left[||X - E[X|Y]||_B^2 \right]
\end{aligned}
$$

- So, $\hat{X} = E[X|Y]$ minimizes the error measure.

$$
HVSE = ||X - \hat{X}||_B^2 \ .
$$