Continuous Time Fourier Transform (CTFT)

complex number representing
the amplitude & phase of component

decompose a
signal into individual
frequencies

$$F(f) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ft} dt$$

causes resulting
F(f) to have
units of $\frac{\text{cycles}}{\text{second}}$

frequency domain

time domain

signal can also
be reconstructed
by doing inverse

$$f(t) = \int_{-\infty}^{\infty} F(f) e^{j2\pi ft} df$$

take the
complex conjugate

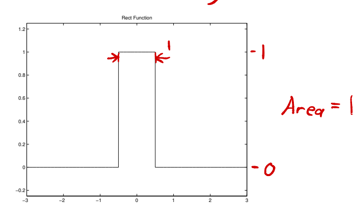
- $f(t)$ is continuous time. (Also known as continuous parameter.)
- $F(f)$ is a continuous function of frequency $-\infty < f < \infty$.

Useful Continuous Time Signal Definitions

usually take these functions & integrate them against something

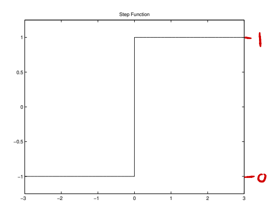
- Rect function: $\text{rect}(t) = \begin{cases} 1 & \text{for } |t| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$

  rectangle

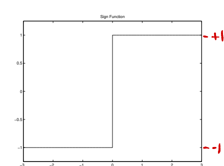


- Step function: $\text{u}(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

  used commonly to analyze the behavior of linear systems

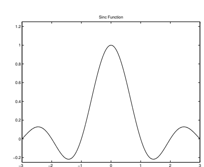


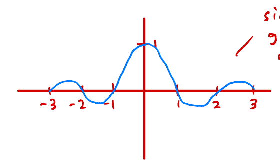
- Sign function: $\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ -1 & \text{for } t < 0 \end{cases}$



- Sinc function: $\text{sinc}(t) = \dfrac{\sin(\pi t)}{\pi t}$

  crosses zero at the integer values, except at zero

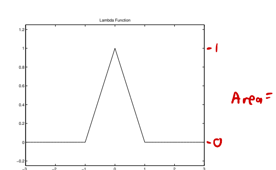




side lobes get smaller as you get further out

- Lambda function: $\Lambda(t) = \begin{cases} 1 - |t| & \text{for } |t| \leq 1 \\ 0 & \text{for } |t| > 1 \end{cases}$

  results from a convolution of a rect with itself



Continuous Time Delta Function

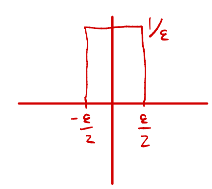
- The "function" $\delta(t)$ is actually **not** a function.

  really, it is a shorthand notation for a limiting behavior

- $\delta(t)$ is defined by the property that for all continuous functions $g(t)$

Area $= \epsilon \cdot \frac{1}{\epsilon} = 1$

$\delta_\epsilon(t)$



$$g(0) = \int_{-\infty}^{\infty} \delta(t) g(t) dt$$

$$g(t_0) = \int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt$$

- Intuitively, we may think of $\delta(t)$ as a very short pulse with unit area.

becomes a large scaling factor

term converges on 1 @ t=0 and 0 everywhere else as ε gets large

$$g(0) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{\epsilon} \text{rect}(t/\epsilon) \right]}_{\delta(t)} g(t) dt$$

Intuitively (but not rigorously)

$$1 = \int_{-\infty}^{\infty} \delta(t) \cancel{=} \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{rect}(t/\epsilon) = 0$$

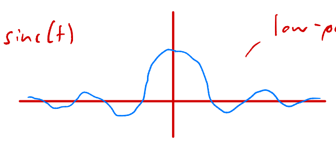
for the rest of class, can pretend it is a function although it really isn't

Useful CTFT Relations

$$\boxed{\begin{array}{rcl} \delta(t) & \overset{CTFT}{\Leftrightarrow} & 1 \\[2em] 1 & \overset{CTFT}{\Leftrightarrow} & \delta(f) \end{array}}$$

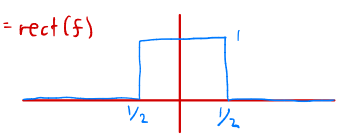
impulse response of a perfect low-pass filter

f(t)=sinc(t)



$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(f)$$

F(f)=rect(f)



$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(f)$$

$$\Lambda(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}^2(f)$$

convolution in the time domain is equal to multiplication in the frequency domain

$$F^{-1}\left(\frac{N_0}{2}\,\mathrm{rect}\left(\frac{w}{2W}\right)\right) = F^{-1}(F(w))$$

$$\frac{W}{W}\cdot\frac{N_0}{2}\,\mathrm{rect}\left(\frac{w}{2W}\right) = N_0W\cdot\frac{1}{2W}\,\mathrm{rect}\left(\frac{w}{2W}\right) \xrightarrow{F^{-1}} N_0W\,\mathrm{sinc}(2Wt)$$

CTFT Properties

Property	Time Domain Function	CTFT
Linearity	$af(t) + bg(t)$	$aF(f) + bG(f)$
magnitude vs phase Conjugation	? $f^*(t)$	$F^*(-f)$ For real $f(t) \rightarrow F(f) = F^*(-f)$
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F(f/a)$
Shifting	$f(t-t_0)$ shifting a signal in time applies a linear phase modulation	$\exp\{-j2\pi f t_0\}F(f)$ (frequency)
Modulation	$\exp\{j2\pi f_0 t\}f(t)$ modulation onto a carrier – this is why we have wireless transmission –	$F(f-f_0)$ multiplying something by a linear phase in the time domain shifts the result in frequency
Convolution	$f(t) * g(t)$	$F(f)\,G(f)$ filtering: $f(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = f(t) * h(t)$
Multiplication	$f(t)g(t)$	$F(f) * G(f)$
Duality	$F(t)$	$f(-f)$

- Inner product property

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(f)G^*(f)df$$

inner products in time correspond to inner products in frequency

energy measured in space = energy measured in frequency