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in textbook

PROBLEM SET 11.9

1. Review in complex. Show that $1/i = -i$, $e^{-ix} = \cos x - i \sin x$, $e^{ix} + e^{-ix} = 2\cos x$, $e^{ix} - e^{-ix} = 2i \sin x$, $e^{ikx} = \cos kx + i \sin kx$.

2–11 FOURIER TRANSFORMS BY INTEGRATION

Find the Fourier transform of $f(x)$ (without using Table III in Sec. 11.10). Show details.

2. $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

5. $f(x) = \begin{cases} e^{x} & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = e^{-|x|} \quad (-\infty < x < \infty)$

7. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

8. $f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$

9. $f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

10. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

11. $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

12–17 USE OF TABLE III IN SEC. 11.10. OTHER METHODS

12. Find $\mathcal{F}(f(x))$ for $f(x) = xe^{-x}$ if $x > 0$, $f(x) = 0$ if $x < 0$, by (9) in the text and formula 5 in Table III (with $a = 1$). *Hint.* Consider xe^{-x} and e^{-x}.

13. Obtain $\mathcal{F}(e^{-x^2/2})$ from Table III.

14. In Table III obtain formula 7 from formula 8.

15. In Table III obtain formula 1 from formula 2.

16. TEAM PROJECT. Shifting (a) Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and $\mathcal{F}\{f(x - a)\} = e^{-iwa}\mathcal{F}\{f(x)\}$.

(b) Using (a), obtain formula 1 in Table III, Sec. 11.10, from formula 2.

(c) Shifting on the *w*-Axis. Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.

(d) Using (c), obtain formula 7 in Table III from 1 and formula 8 from 2.

17. What could give you the idea to solve Prob. 11 by using the solution of Prob. 9 and formula (9) in the text? Would this work?

18–25 DISCRETE FOURIER TRANSFORM

18. Verify the calculations in Example 4 of the text.

19. Find the transform of a general signal $f = [f_1 \;\; f_2 \;\; f_3 \;\; f_4]^T$ of four values.

20. Find the inverse matrix in Example 4 of the text and use it to recover the given signal.

21. Find the transform (the frequency spectrum) of a general signal of two values $[f_1 \;\; f_2]^T$.

22. Recreate the given signal in Prob. 21 from the frequency spectrum obtained.

23. Show that for a signal of eight sample values, $w = e^{-i/4} = (1 - i)/\sqrt{2}$. Check by squaring.

24. Write the Fourier matrix **F** for a sample of eight values explicitly.

25. CAS Problem. Calculate the inverse of the 8×8 Fourier matrix. Transform a general sample of eight values and transform it back to the given data.

The expression in brackets is a function of w, is denoted by $\hat{f}(w)$, and is called the **Fourier transform** of f; writing $v = x$, we have

(6) $$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx.$$

Table III. Fourier Transforms

See (6) in Sec. 11.9.

	$f(x)$	$\hat{f}(w) = \mathcal{F}(f)$
1	$\begin{cases} 1 & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin bw}{w}$
2	$\begin{cases} 1 & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$
3	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}\,\frac{e^{-a\lvert w\rvert}}{a}$
4	$\begin{cases} x & \text{if } 0 < x < b \\ 2x - b & \text{if } b < x < 2b \\ 0 & \text{otherwise} \end{cases}$	$\frac{-1 + 2e^{ibw} - e^{-2ibw}}{\sqrt{2\pi}w^2}$
5	$\begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (a > 0)$	$\frac{1}{\sqrt{2\pi}(a + iw)}$
6	$\begin{cases} e^{ax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{(a-iw)c} - e^{(a-iw)b}}{\sqrt{2\pi}(a - iw)}$
7	$\begin{cases} e^{iax} & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin b(w - a)}{w - a}$
8	$\begin{cases} e^{iax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{i}{\sqrt{2\pi}}\,\frac{e^{ib(a-w)} - e^{ic(a-w)}}{a - w}$
9	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}}e^{-w^2/4a}$
10	$\frac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}$ if $\lvert w\rvert < a$; 0 if $\lvert w\rvert > a$

2) $$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{2ix}e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{(2i-iw)x}\,dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{(2i-iw)x}}{2i-iw}\Big|_{-1}^{1}\right]$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{(2i-iw)}}{2i-iw} - \frac{e^{-(2i-iw)}}{2i-iw}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(2-w)} - e^{-i(2-w)}}{i(2-w)}\right] = \boxed{\frac{2}{(2-w)\sqrt{2\pi}}\sin(2-w)}$$

3) $$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{a}^{b} e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{-iw}\Big|_{a}^{b}\right] = \boxed{\frac{1}{\sqrt{2\pi}}\left[\frac{e^{-ibw} - e^{-iaw}}{-iw}\right]}$$

8) $$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{0} xe^{-x}e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{0} xe^{-x(1+iw)}\,dx$$

$v' = e^{-x(1+iw)}$

$$= \frac{1}{\sqrt{2\pi}}\left[x\left(\frac{e^{-x(1+iw)}}{-(1+iw)}\right)\Big|_{-1}^{0} - \int_{-1}^{0}\frac{e^{-x(1+iw)}}{-(1+iw)}\,dx\right] = \ldots = \boxed{\frac{1}{\sqrt{2\pi}}\left[\frac{1}{(1+iw)^2}\left(e^{(1+iw)} - 1\right) - \frac{1}{(1+iw)}e^{(1+iw)}\right]}$$