1/14/22

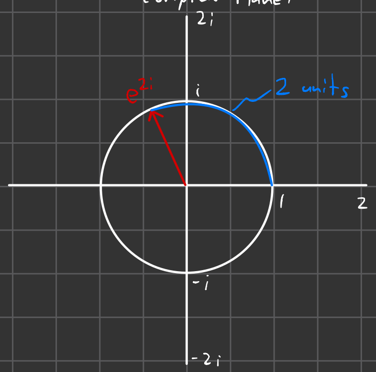
FT Notes

General Idea: decompose a complex signal into the pure frequencies that make it up

- unmixing a can of mixed up paint
- general strategy: build a mathematical machine that treats signals with a given frequency differently than it does for other signals
- center of mass is a complex number, with a real and imaginary part

Consider Euler's Formula:

Complex Plane:



one cycle per second

$e^{j2\pi t}$

frequency

$e^{j2\pi f t}$

1 cycle per 10 seconds

let $f = \frac{1}{10}$; $e^{j2\pi f t}$

For Fourier Transforms: rotate in clockwise direction: $e^{-2\pi f t}$

let $g(t) = \cos(t) \rightarrow$ * $g(t)e^{-2\pi f t}$ *: the rotating complex number is getting scaled according to the value of the function

$$\underbrace{\frac{1}{t_2 - t_1}}_{\text{do not consider}} \int_{t_1}^{t_2} g(t)e^{-j2\pi f t}\,dt$$

f is the winding frequency *

do not consider

in actual Fourier Transform: instead of looking at center of mass, scale it up by some amount

$$\hat{g}(f) = \int_{t_1}^{t_2} g(t)e^{-j2\pi f t}\,dt \quad ; \quad \text{in practice: } \hat{g}(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}\,dt$$

Key Points:

- exponentials here correspond to rotation
- multiplying that by $g(t)$ means drawing a wound up version of the graph
- an integral of a complex valued function can be interpreted in terms of a center of mass idea

Other Notes

- for any sum of waves, the period is the least common multiple of all the constituting wave's periods
- multiplying the base frequency by the winding frequency yields the number of petals *

1/15/22

Complex number - real & imaginary part

Magnitude $(Re + jIm) = |Re + jIm| = \sqrt{Re^2 + Im^2}$

Angle $(Re + jIm) = \arctan\left(\frac{Im}{Re}\right)$

$e^{jf} = \cos(f) + j\sin(f)$

Any complex number of the form $R + jI$ can be written as Ae^{jf}, where A is the magnitude and f is the angle

$$\mathcal{F}\{x(t)\} = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$\mathcal{F}^{-1}(x(t)) = x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}dt$$

Equations allow us to see what frequencies exist in the signal $x(t)$ (in other words, translate between time and frequency domain

note: $\underbrace{w}_{\text{angular frequency}} = 2\pi \underbrace{f}_{\text{frequency}}$

Ex: $X(f) = \int_{-\infty}^{\infty} 2e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} 2[\cos(-2\pi ft) + j\sin(-2\pi ft)]dt$

Intuitively, if $f = 0$:

$$X(0) = \int_{-\infty}^{\infty} 2[\cos(0) + j\sin(0)]dt = \int_{-\infty}^{\infty} 2dt$$

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

important things happen when input frequency $= \pm f$

for $x(t) = 2$, $X(f) = 2\infty$

☆ The transform of a DC signal will exist only at 0Hz ☆

Ex: $X(f) = \int_{-\infty}^{\infty} \sin(2\pi 1000t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} \sin(2000\pi t)[\cos(2\pi ft) - j\sin(2\pi ft)]$

Term goes to zero unless $f = \pm 1000$Hz

$$X(1000) = \int_{-\infty}^{\infty} \sin(2000\pi t)[\cos(2000\pi t) - j\sin(2000\pi t)]dt$$

$$= \int_{-\infty}^{\infty} \sin(2000\pi t)\cos(2000\pi t)dt - j\int_{-\infty}^{\infty} \sin^2(2000\pi t)dt$$

using some identities:

$$X(1000) = \int_{-\infty}^{\infty} \frac{1}{2}\sin(4000\pi t)dt - j\int_{-\infty}^{\infty} \frac{1}{2}dt + j\int_{-\infty}^{\infty} \frac{1}{2}\cos(4000\pi t)dt$$

$$= -j\int_{-\infty}^{\infty} \frac{1}{2}dt$$

similarly, $X(-1000) = j\int_{-\infty}^{\infty} \frac{1}{2}dt$

$$\boxed{\text{Dirac Delta Function: } \int_{-\infty}^{\infty} G(t)\,\delta(t-t_0)\,dx = G(t_0)}$$

other notes: $\lim_{t \to t_0} \delta(t-t_0) = \infty$, $\delta(t) = 0\,;\ t \neq 0$

$\int_0^{\infty} \delta(t-t_0)\,dt = 1$ $\quad \mathcal{F}\{\delta(t)\} = \frac{1}{2\pi} e^{i5t} = 1$

Using the Dirac function: $\mathcal{F}\{\sin(2000\pi t)\} = X(f) = \frac{-j\,\delta(f-1000)}{2} + \frac{j\,\delta(f+1000)}{2}$

Similarly: $\mathcal{F}\{\cos(4000\pi t)\} = X(f) = \frac{\delta(f-2000)}{2} + \frac{\delta(f+2000)}{2}$

- Dirac function has an offset, which means we have spikes at the signal frequency and the negative of the signal frequency (remember: $\sin(-\theta) = -\sin(\theta)$, $\cos(-\theta) = -\cos(\theta)$)
- Fourier transform of the sine function is completely imaginary, while the Fourier transform of the cosine function only has real parts
- The angle of the transform of the sine function, which is the arctan of the imaginary over the real component, is $90°$ off from the angle of the transform of the cosine (just like sine and cosine functions are $90°$ off from one another)

Convolution & Linearity

$Ax(t) + By(t) = AX(f) + BY(f)$ - linearity property

One property the Fourier Transform does not have is that the transform of the product of functions is not the same as the product of the individual transforms

$$\mathcal{F}\{x(t)y(t)\} \neq X(f)Y(f)$$

$$\int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt \neq \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt$$

The Fourier transform of the product of two signals is the convolution of the two signals, denoted by an asterisk (*), and is defined as:

$$\mathcal{F}\{x(t)y(t)\} = X(f) * Y(f) = \int_{-\infty}^{\infty} X(f')Y(f-f')\,df' = \int_{-\infty}^{\infty} Y(f')X(f-f')\,df'$$

Ex: $\mathcal{F}\{\cos(1000\pi t)\cos(3000\pi t)\} = \mathcal{F}\{\cos(1000\pi t)\} * \mathcal{F}\{\cos(3000\pi t)\}$

compute individual Fourier Transforms $= \left[\frac{\delta(\xi-500)}{2} + \frac{\delta(\xi+500)}{2}\right] * \left[\frac{\delta(\xi-1500)}{2} + \frac{\delta(\xi+1500)}{2}\right]$

apply the equation for convolution $= \int_{-\infty}^{\infty} \left[\frac{\delta(\xi'-500)}{2} + \frac{\delta(\xi'+500)}{2}\right]\left[\frac{\delta(\xi-\xi'-1500)}{2} + \frac{\delta(\xi-\xi'+1500)}{2}\right] \delta\xi'$

FOIL $= \int_{-\infty}^{\infty} \left[\frac{\delta(\xi'-500)\delta(\xi-\xi'-1500)}{4} + \frac{\delta(\xi'+500)\delta(\xi-\xi'-1500)}{4} + \frac{\delta(\xi'-500)\delta(\xi-\xi'+1500)}{4} + \frac{\delta(\xi'+500)\delta(\xi-\xi'+1500)}{4}\right] \delta\xi'$

remember that $\delta(x) = 0$ where $x \neq 0$, and look for ξ values in which $x = 0$ in $\delta(x)$

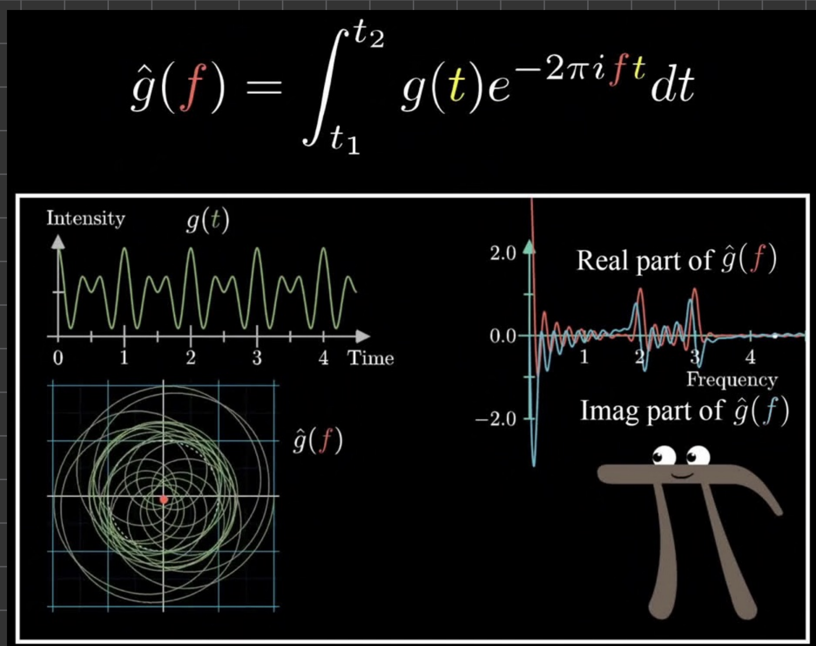
looking at first term: $\frac{\delta(\xi'-500)\,\delta(\xi-\xi'-1500)}{4}$: need $\xi' = 500$, $\xi = 2000$

repeating with other terms, we get non-zero results at
$f = 2000, 1000, -1000,$ and 2000

What happened? We multiplied two frequencies together and the result is that we essentially re-centered the response for one frequency at the frequency of the other

$\cos(1000\pi t)$ gives spikes at 500 and -500 Hz
$\cos(3000\pi t)$ gives spikes at 1500 and -1500 Hz
↳ so, we took the ±500 Hz spikes and re-centered them at ±1500 Hz

If we take any waveform and multiply it by a sine or cosine, the transform of the resulting signal is the original re-centered at the frequencies of the sine wave



1. The value $e^{-2\pi i f t}$ describes a value with length 1, rotating at a constant rate so that it makes f full cycles per unit time.
2. Multiplying that by the function $g(t)$ means drawing a wound up version of the graph.
3. The integral can be interpreted in terms of the center of mass of the wound-up graph, scaled up by the size of the time interval.