Tomography: take a whole sequence of views with specimen at different angles and allow computer to re-assemble the jigsaw puzzle in 3-dimensions

for a photon: $\nu \approx hE$ — ν frequency in cycles/time $= \frac{1}{\lambda}$; h Planck's constant; E energy

more photon energy: more penetrating power (generally speaking)

x-ray image capture film being slowly replaced by electronic detectors (CCD, cmos device)

x-rays need to be slowed down by a scintillating device before sensor capture, otherwise they would just pass through ↳ results in a single planar view

Tomography

→ reconstruct the 3D representation of the object

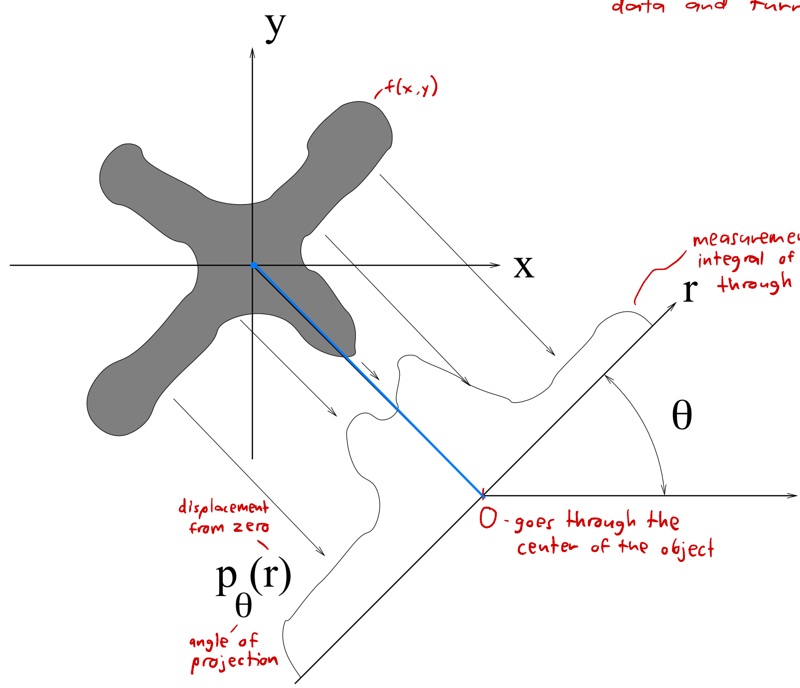
Comes up in a large number of applications

- Many medical imaging systems can only measure projections through an object with density $f(x, y)$.

  formal theory can be explained using continuous time

  – Projections must be collected at every angle θ and displacement r.

  – Forward projections $p_\theta(r)$ are known as a Radon transform. Question: Hard part is the signal processing, how do you take that data and turn it into an image?



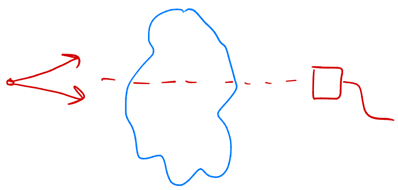
- Objective: reverse this process to form the original image $f(x, y)$. – reconstruct

  ↳ this is a function for density, as value gets larger the material gets denser

  – Fourier Slice Theorem is the basis of inverse

  – Inverse can be computed using convolution back projection (CBP)

Transmission

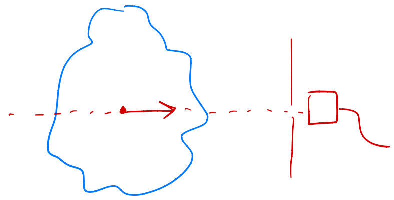


as material becomes denser, signal goes down

↳ brightfield microscopy: outside field is bright (larger signal)

$0 < \frac{\lambda_1}{\lambda_0} < 1 \rightarrow -\log\left(\frac{\lambda_1}{\lambda_0}\right)$

Emission, less common



↳ darkfield microscopy: outside field is dark (smaller signal)

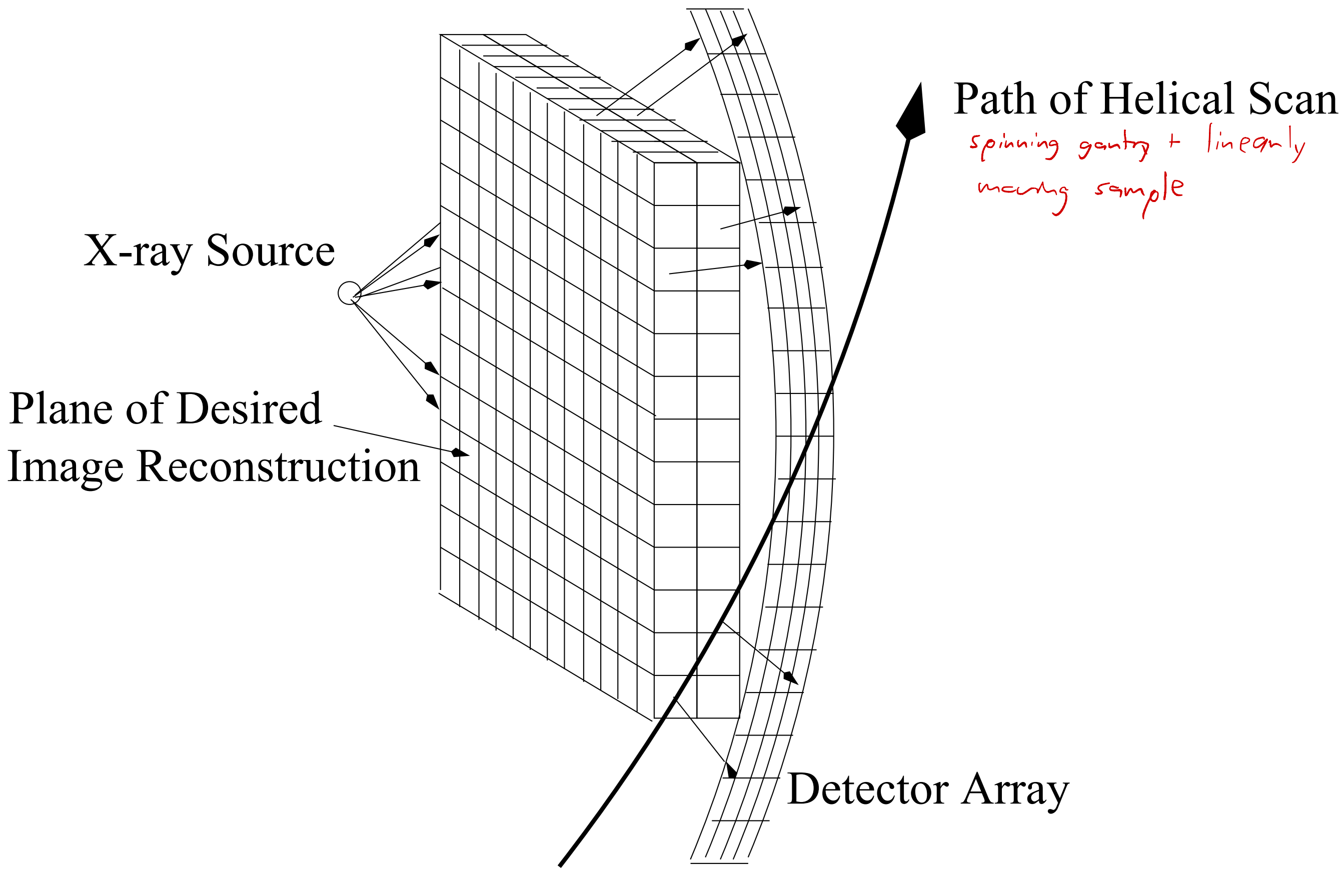
Medical Imaging Modalities

- Anatomical Imaging Modalities
  - Chest X-ray
  - Computed Tomography (CT)
  - Magnetic Resonance Imaging (MRI)
- Functional Imaging Modalities
  - Signal Photon Emission Tomography (SPECT)
  - Positron Emission Tomography (PET)
  - Functional Magnetic Resonance Imaging (fMRI)

emission processes

Multislice Helical Scan CT

- Multislice CT has a cone-beam structure



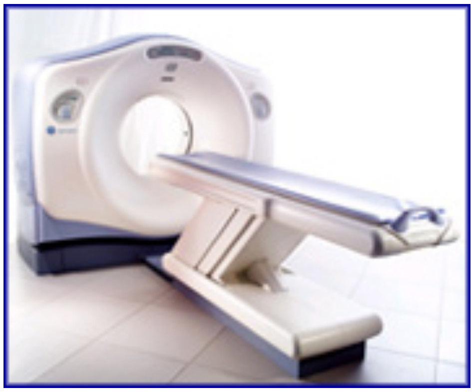
Discrete Space

projections $\vec{y} = A\vec{x}$ image

$\vec{x} = A^{-1}\vec{y}$ ← not practical : A may not be square, may be very large, etc

Example: CT Scan

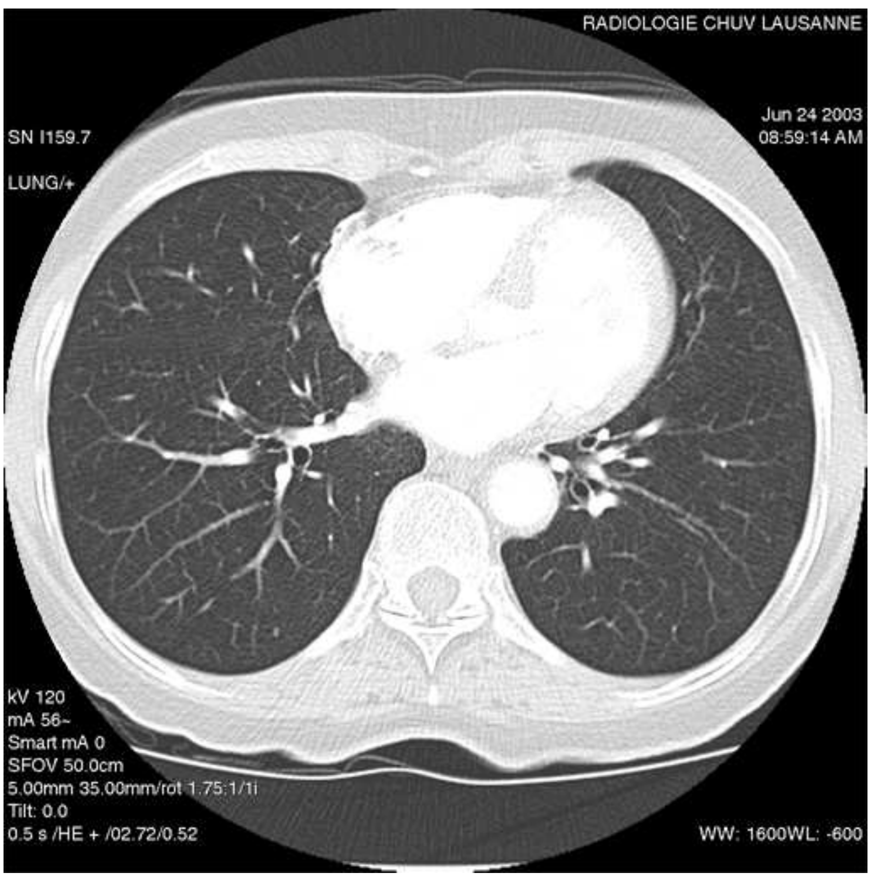
CT : Computed Tomography
CAT : Computed Axial Tomography - no longer used, now the process is helical



machine : 0.5-1 million $

- Gantry rotates under fiberglass cover
- 3D helical/multislice/fan beam scan

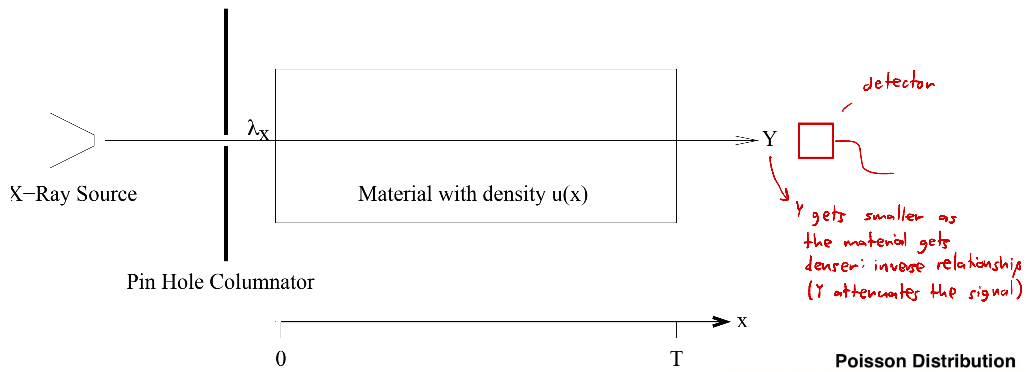
Downside : exposure to X-Rays
not good for diagnosing certain things

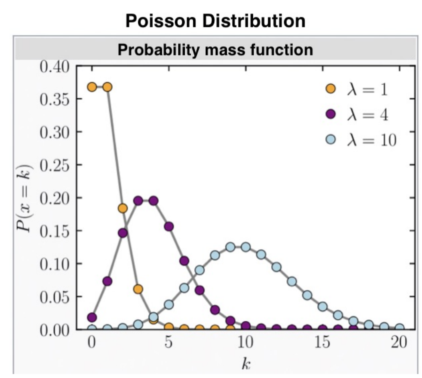


applications in manufacturing, ✱ security, science

CT reconstruction using different captured views

Photon Attenuation





x - depth into material measured in cm

Y_x - Number of photons at depth x

$\lambda_x = E[Y_x]$ - mean number of photons at depth x

Number of photons is a Poisson random variable

$$P\{Y_x = k\} = \frac{e^{-\lambda_x}\lambda_x^k}{k!} \, .$$

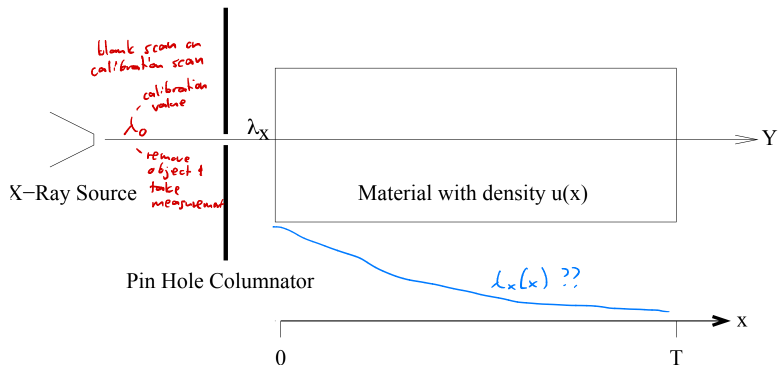
- As photons pass through material, they are absorbed.
- The rate of absorption is proportional to the number of photons and the density of the material.

Columnator - essentially a pinhole array

also need a lot of emission

↳ issue is resolution, but x-Rays have a very small wavelength

Differential Equation for Photon Attenuation



The attenuation of photons obeys the following equation Beer's Law

rate at which photons are absorbed

$$\frac{d\lambda_x}{dx} = -\mu(x)\lambda_x$$

wavelength

photons is decreasing

Beer's Law

where $\mu(x)$ is the density in units of cm^{-1}.

- The solution to this equation is given by

measurement

$$\lambda_x = \lambda_0 e^{-\int_0^x \mu(t)dt}$$

obtained through calibration

re-visit how this solution is obtained (separation of variables?)

- So we see that - solve for $\mu(t)$

$$\int_0^x \mu(t)dt = -\log\left(\frac{\lambda_x}{\lambda_0}\right) = y$$

$$\approx -\log\left(\frac{Y_x}{\lambda_0}\right) \geq 0$$

<1

measurement

natural log

dosage

projection

take measurements you get into what you need

contrast reversal

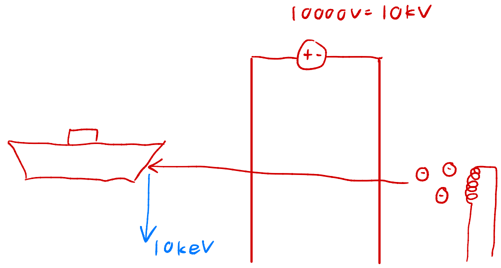
negative log of the attenuation through the material

∴ Tomography is a non-linear problem that we linearize

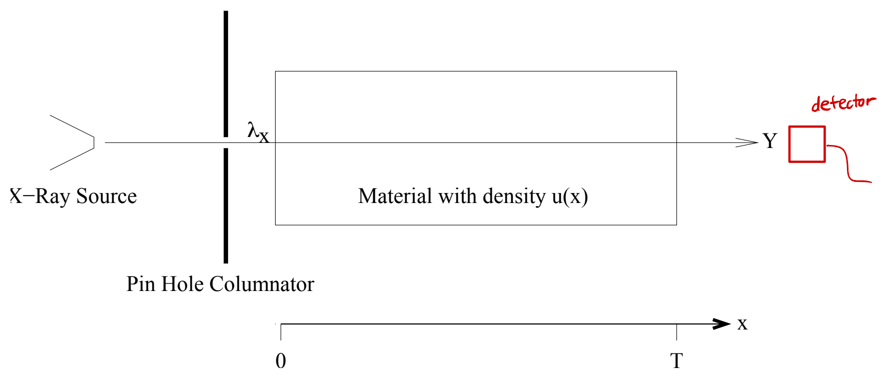
This is a common exam problem

$E = h\nu$

energy keV — Planck's constant — frequency



Estimate of the Projection Integral



A commonly used estimate of the projection integral is

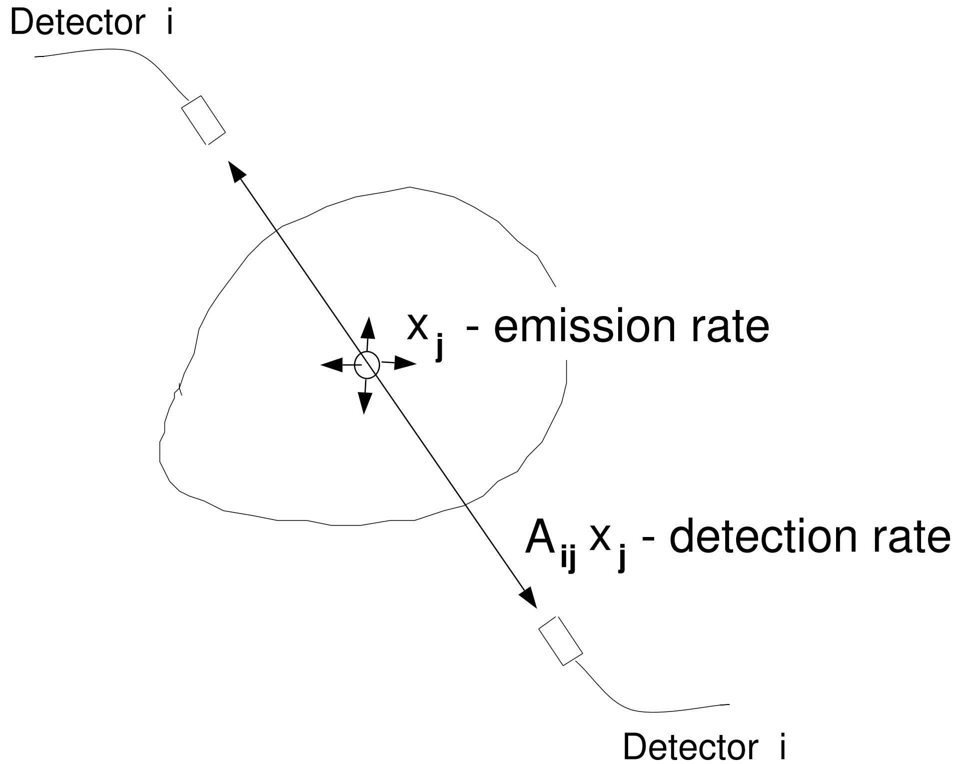
$$\int_0^T \mu(t)dt \cong -\log\left(\frac{Y_T}{\lambda_0}\right)$$ → how $p_\theta(r)$ is determined

where:

λ_0 is the dosage : blank scan measurement, no sample

Y_T is the photon count at the detector

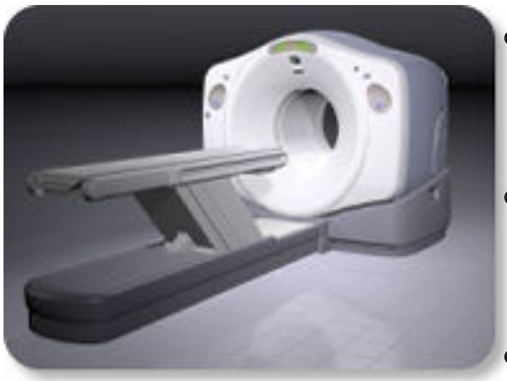
Positron Emission Tomography (PET)



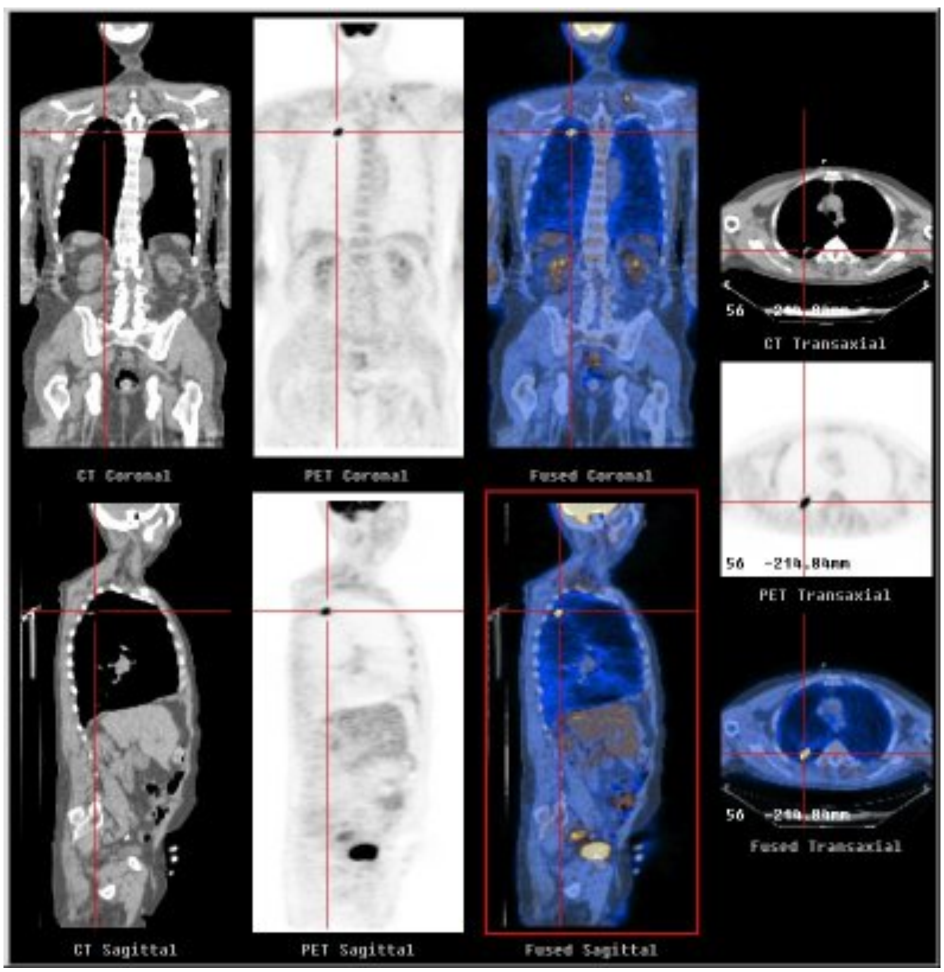
$$E[y_i] = \sum_j A_{ij} x_j$$

- Subject is injected with radio-active tracer
- Gamma rays travel in opposite directions
- When two detectors detect a photon simultaneously, we know that an event has occurred along the line connecting detectors.
- A ring of detectors can be used to measure all angles and displacements

Example: PET/CT Scan



- Generally low space/time resolution
- Little anatomical detail $\Rightarrow$ couple with CT
- Can detect disease



Coordinate Rotation

Beginning of Reconstruction Process

- Define the counter-clockwise rotation matrix

$A_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$A_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

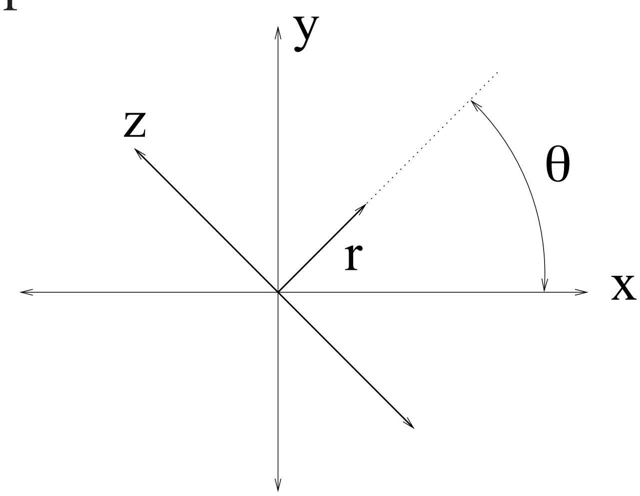
$$\mathbf{A}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- Define the new coordinate system (r, z)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}$$

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

- Geometric interpretation

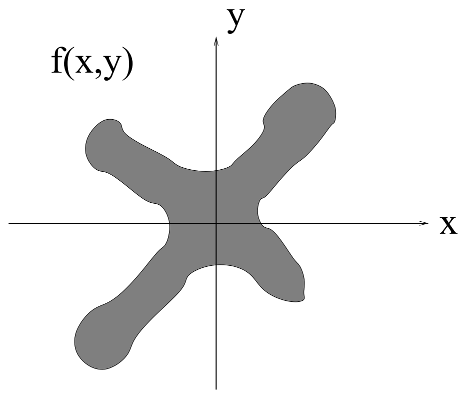


- Inverse transformation

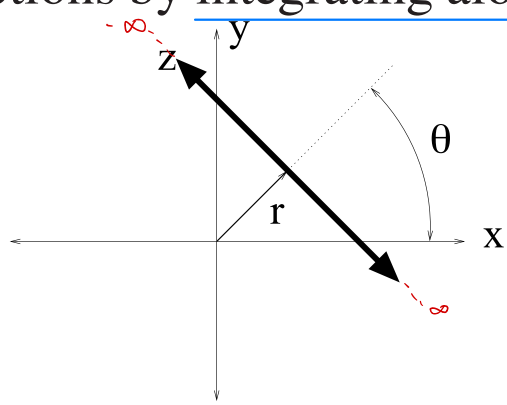
$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix} = A_\theta^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

Integration Along Projections

- Consider the function $f(x, y)$. density function may or may not be density?



- We compute projections by integrating along z for each r.



- The projection integral for each r and θ is given by

$$
\begin{aligned}
p_\theta(r) &= \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz \\
&= \int_{-\infty}^{\infty} f\left(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)\right) dz
\end{aligned}
$$

structure is based on A_θ definition for linear transformation

* each point on $p_\theta(r)$ is an integral of $f(x,y)$ through z at particular r and θ

$$\begin{array}{ccc} f(x,y) & \xleftrightarrow{\text{CSFT}} & F(u,v) \\ \updownarrow \text{RT} & & \updownarrow \text{rectangular} \to \text{polar} \\ p_\theta(r) & \xleftrightarrow{\text{CFT}} & P_\theta(\rho) \end{array}$$

related by Fourier Slice Theorem

obtain via transmission or emission imaging, etc.

The Radon Transform

MEASURED with instrument that needs calibration

- The Radon transform of the function $f(x, y)$ is defined as

↳ takes you from the image to the sinogram

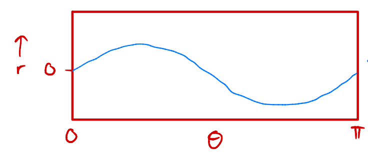
$$p_\theta(r) = \int_{-\infty}^{\infty} f\left(r\cos(\theta) - z\sin(\theta), r\sin(\theta) + z\cos(\theta)\right) dz$$

linear transformation to a new coordinate system

every θ has a unique function

not an orthonormal transformation

2D Array:

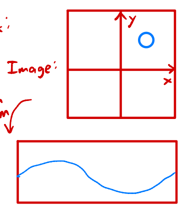


specific object projection data as θ is changed (location along r changes)

Called a sinogram

- The geometric interpretation is

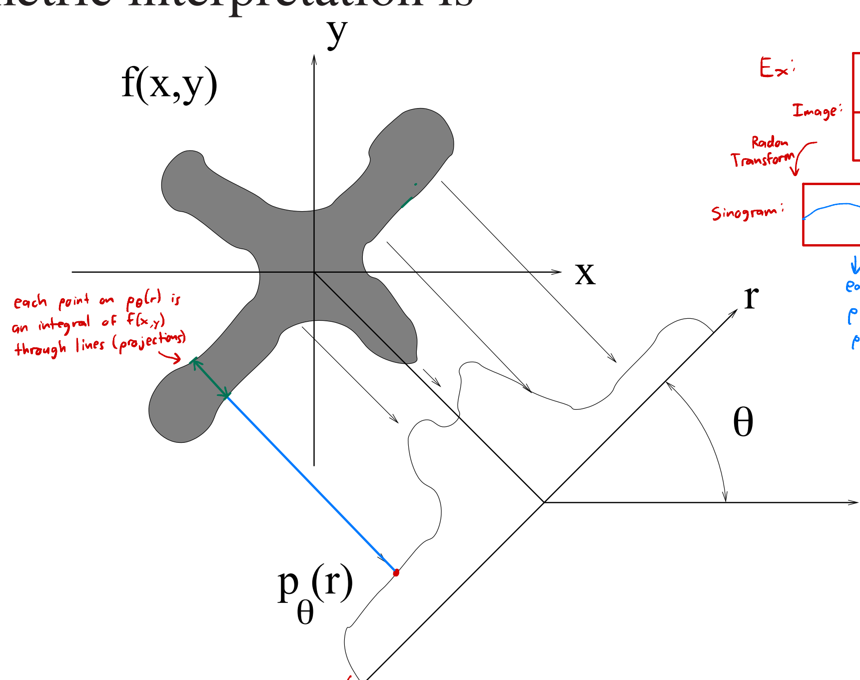
Ex:



* Phase of sine wave is determined by angle of the object w.r.t. x-y coordinate system

* Amplitude of sine wave is determined by the distance from the center of rotation

↓ each point is projection data particular angle



↳ projections need to be linear integrals of the unknown object

Notice that the projection corresponding to $r = 0$ goes through the point $(x, y) = (0, 0)$.

$$\begin{tikzcd} p_\theta(r) \arrow[r, "CTFT"] & P_\theta(\rho) \arrow[d, "\text{polar to rectangular}"] \\ f(x,y) \arrow[u, "\text{Radon Transform}"] & F(u,v) \arrow[l, "CSFT"] \end{tikzcd}$$

related by Fourier Slice Theorem

The Fourier Slice Theorem

- Let

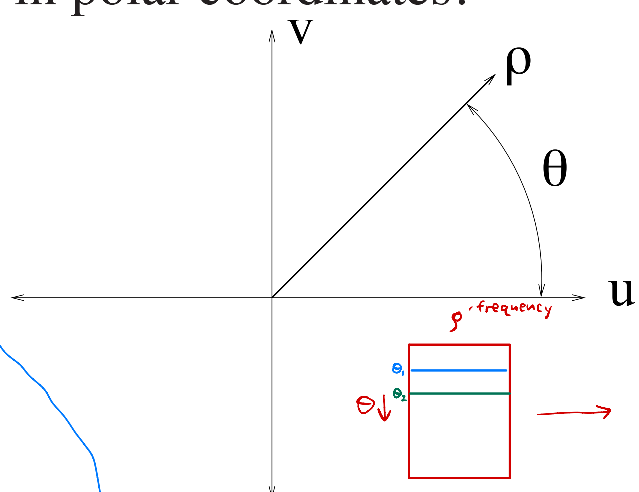
$$\begin{aligned} P_\theta(\rho) &= CTFT\{p_\theta(r)\} \\ F(u,v) &= CSFT\{f(x,y)\} \end{aligned}$$

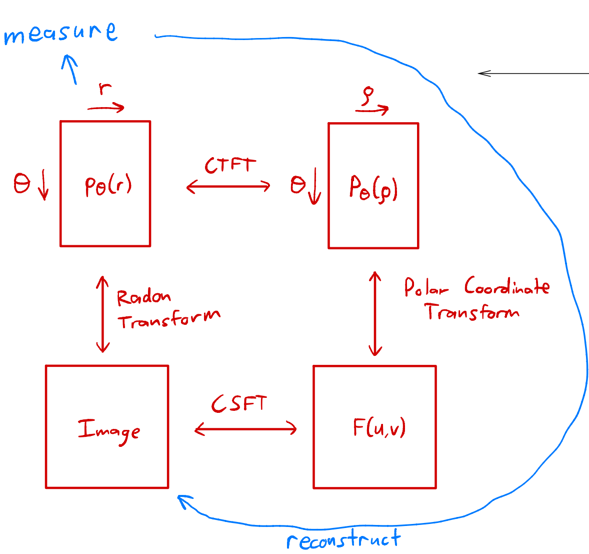
Then

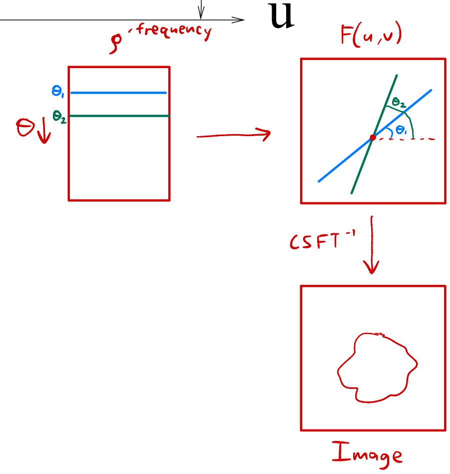
$$P_\theta(\rho) = F(\rho\cos(\theta), \rho\sin(\theta))$$

– polar coordinates

- $P_\theta(\rho)$ is $F(u,v)$ in polar coordinates!







Proof of the Fourier Slice Theorem

- By definition

$$p_\theta(r) = \int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz$$

- The CTFT of $p_\theta(r)$ is then given by

$$\begin{aligned} P_\theta(\rho) &= \int_{-\infty}^{\infty} p_\theta(r) e^{-j2\pi\rho r} dr \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) dz\right] e^{-j2\pi\rho r} dr \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\mathbf{A}_\theta \begin{bmatrix} r \\ z \end{bmatrix}\right) e^{-j2\pi\rho r} dz dr \end{aligned}$$

- We next make the change of variables

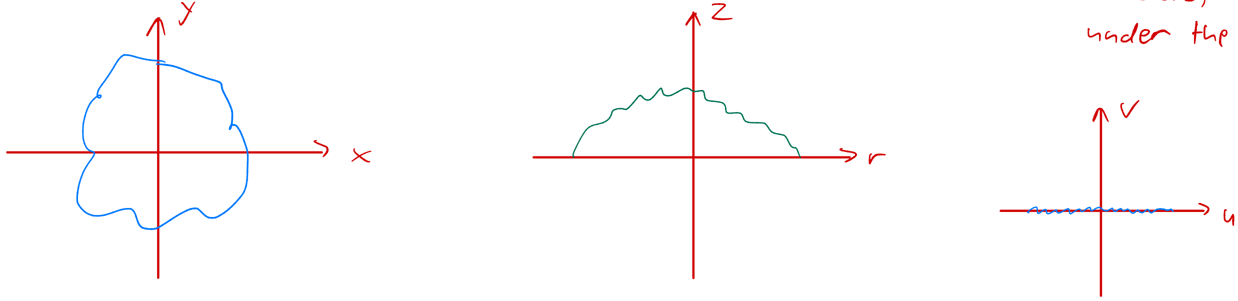
$$\begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{A}_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix} .$$

Notice that the Jacobian is $|\mathbf{A}_\theta| = 1$, and that $r = x\cos(\theta) + y\sin(\theta)$. This results in

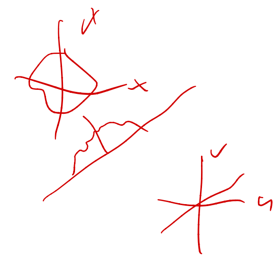
$$\begin{aligned} P_\theta(\rho) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi\rho[x\cos(\theta)+y\sin(\theta)]} dx dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi[x\rho\cos(\theta)+y\rho\sin(\theta)]} dx dy \\ &= F(\rho\cos(\theta), \rho\sin(\theta)) \end{aligned}$$

$g(t) \xrightarrow{\text{CTFT}} G(f) \qquad G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \rightarrow G(0) = \int_{-\infty}^{\infty} g(t)\, dt$ DC component

frequency is zero, area under the curve



Alternative Proof of the Fourier Slice Theorem



- First let $\theta = 0$, then

$$p_0(r) = \int_{-\infty}^{\infty} f(r, y)\, dy$$

take the DC term in that direction

Then

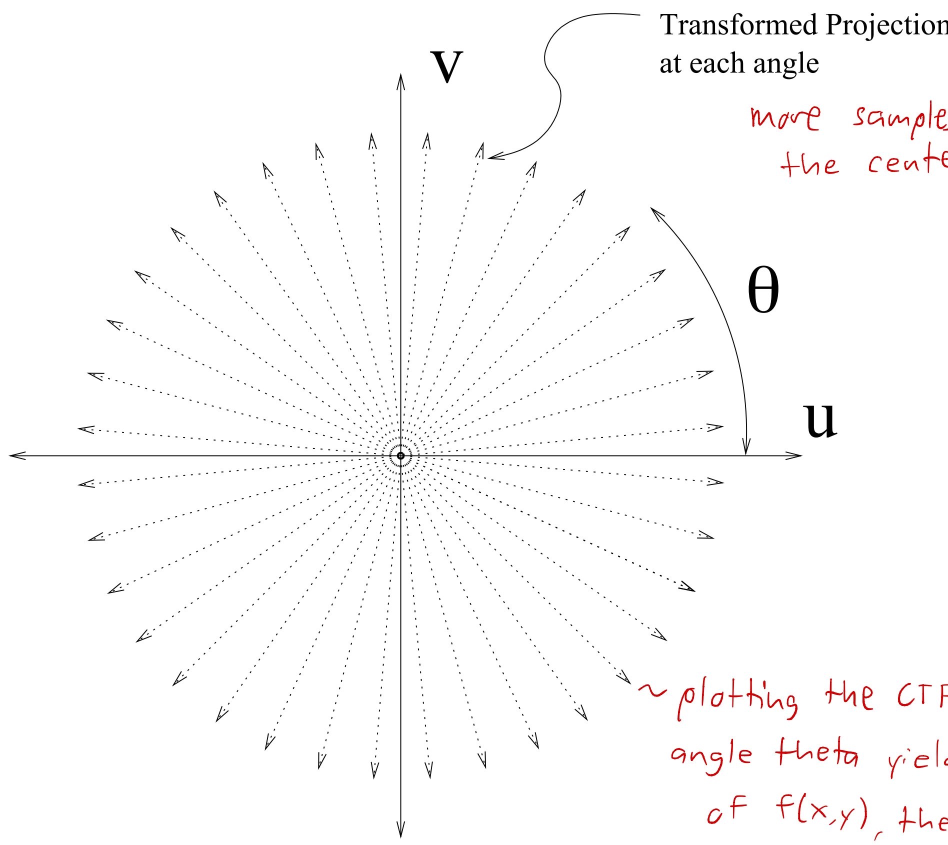
$$\begin{aligned}
P_0(\rho) &= \int_{-\infty}^{\infty} p_0(r) e^{-2\pi j r \rho}\, dr \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(r, y)\, dy \right] e^{-2\pi j r \rho}\, dr \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, y) e^{-2\pi j (r\rho + y0)}\, dr\, dy \\
&= F(\rho, 0)
\end{aligned}$$

= DC component = area under curve at specific theta

- By rotation property of CSFT, it must hold for any θ.

Inverse Radon Transform

- Physical systems measure $p_\theta(r)$.
- From these, we compute $P_\theta(\rho) = CTFT\{p_\theta(r)\}$.



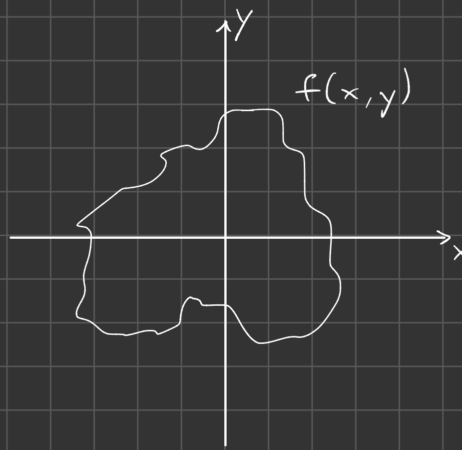
more samples near the center

~ plotting the CTFT at each angle theta yields the CSFT of f(x,y), then take inverse CSFT to get back the original image

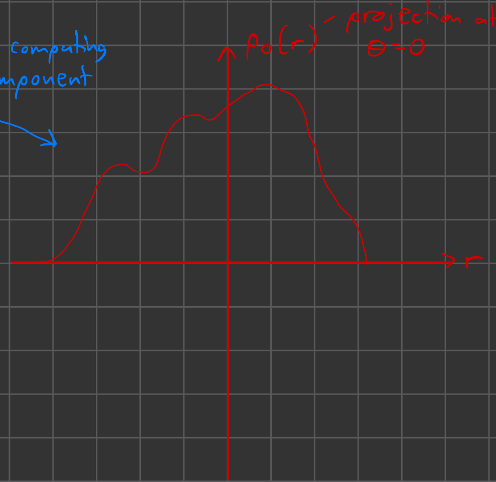
- Next we take an inverse CSFT to form $f(x, y)$.

  Problem: This requires polar to rectagular conversion.

  Solution: Convolution backprojection



effectively computing the DC component



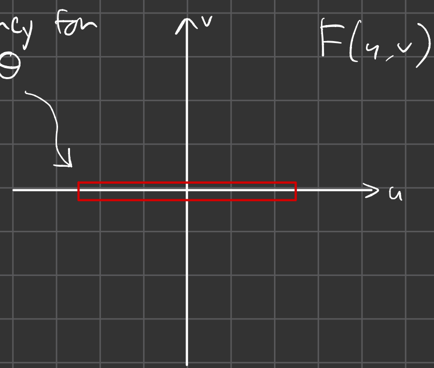
Computing the sum, or integral, through lines

$$p_\theta(r) = \int_{-\infty}^{\infty} f(x,y)\,dy \xrightarrow[\text{to rectangular}]{\text{CTFT, polar}} F(u,0)$$

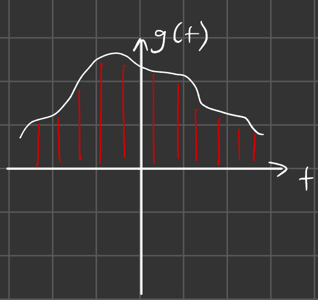
- evaluates at zero frequency for each θ

↳ this is done for all possible angles

- area under the curve is plotted at each θ in 2D $F(u,v)$ plane, then back-projected



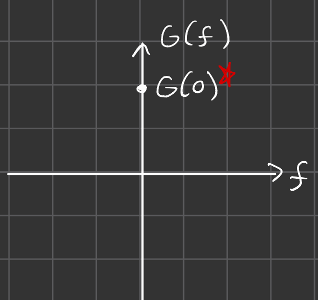
Offshoot:



$$g(t) \xrightarrow{\text{CTFT}} G(f)$$

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\,dt$$

$$G(0) = \int_{-\infty}^{\infty} g(t) e^{0}\,dt = \int_{-\infty}^{\infty} g(t)\,dt$$



$G(0)$ is the area under the curve of $g(t)$ or the DC component ☆

↳ for a filter, to preserve the DC component, you want to make sure that the impulse response sums to 1

Convolution Back Projection (CBP) Algorithm

for infinite number of angles

- In order to compute the inverse CSFT of $F(u,v)$ in polar coordinates, we must use the Jacobian of the polar coordinate transformation. want to do the polar coordinate inverse Fourier Transform

$$du\,dv = |\rho| d\theta\, d\rho$$

$(u,v) = (\rho\cos\theta, \rho\sin\theta)$

- This results in the expression

$$\begin{aligned} f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{2\pi j(xu+yv)}\, du dv \\ &= \underbrace{\int_{-\infty}^{\infty}\int_{0}^{\pi}}_{\text{entire plane is covered}} P_\theta(\rho) e^{2\pi j(x\rho\cos(\theta)+y\rho\sin(\theta))}\, |\rho| d\theta\, d\rho \\ &= \int_0^\pi \underbrace{\left[\int_{-\infty}^{\infty} |\rho| P_\theta(\rho) e^{2\pi j\rho(x\cos(\theta)+y\sin(\theta))}\, d\rho\right]}_{g_\theta(x\cos(\theta)+y\sin(\theta))} d\theta \end{aligned}$$

polar coordinate integration

- Then $g(t)$ is given by

multiplication in frequency

$$\begin{aligned} g_\theta(t) &= \int_{-\infty}^{\infty} |\rho| P_\theta(\rho) e^{2\pi j\rho t}\, d\rho \\ &= CTFT^{-1}\{|\rho| P_\theta(\rho)\} \\ &= h(t) * p_\theta(r) \end{aligned}$$

convolution in time

↳ filtered version of projections

where $h(t) = CTFT^{-1}\{|\rho|\}$, and (this is the filter component)

$$f(x,y) = \int_0^{\pi} g_\theta\left(x\cos(\theta) + y\sin(\theta)\right) d\theta$$

time variable

filter

$$p_\theta(r) \longrightarrow \boxed{h(r)} \longrightarrow g_\theta(t)$$

convolution in time
multiplication in frequency

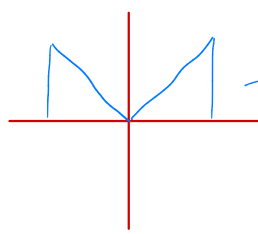
represent LTI filter

$$P_\theta(\rho) \longrightarrow \boxed{H(\rho)} \longrightarrow G_\theta(\rho) = H(\rho)\cdot P_\theta(\rho)$$

$$h(t) \xrightarrow{CTFT} H(\rho)$$

$\rho = 0$: DC component

$H(\rho) = |\rho| \longrightarrow$



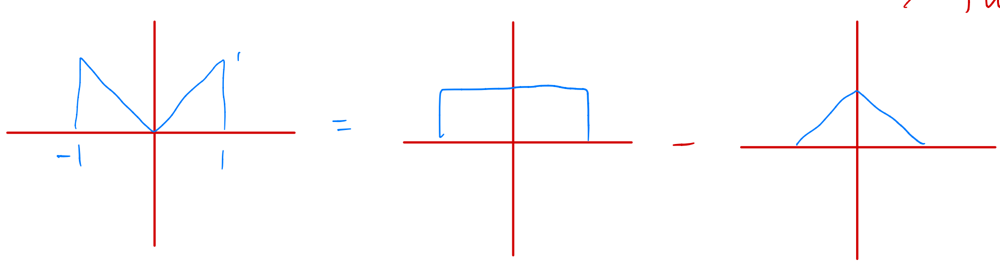
assume band limited
sample at twice the highest frequency: Nyquist

As frequency goes up: higher gain

amplify vs attenuation*

DC component: average value

convolution of two rects

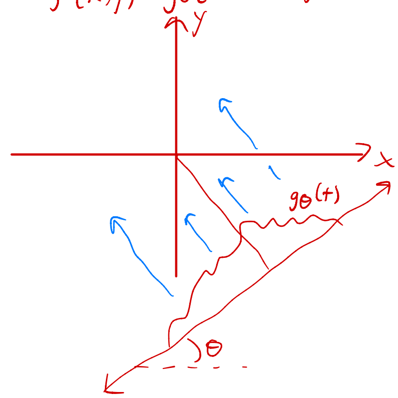


convolution in frequency
multiplication in time

↳ sinc − sinc²

need to be sufficient in intuition & analytical approaches to this class

Back Projection $f(x,y) = g_\theta(x\cos\theta + y\sin\theta)$



Forward Projection: Radon Transform

Backward Projection: "Smear" the projection back into 2D space

more samples in middle → blurry

↳ so: multiply by high pass filter

Summary of CBP Algorithm

1. Measure projections $p_\theta(r)$. - some physical process, like transmission tomography
2. Filter the projections $g_\theta(r) = h(r) * p_\theta(r)$.

   ↳ frequency response → increase gain at higher frequencies

   aka: accentuate high frequencies, make image "edgier"
3. Back project filtered projections

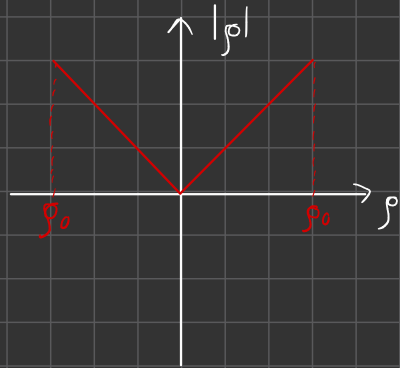
$$f(x,y) = \int_0^\pi g_\theta\left(x\cos(\theta) + y\sin(\theta)\right) d\theta$$

$\vec{y} = A\vec{x}$

$z = A^T\vec{y}$

adjoint ≡ transpose

complex conjugation



- can't go to ∞, or integral wouldn't converge
- assume band limited

$\divideontimes$ inverse fourier transform will look like sinc - sinc^2

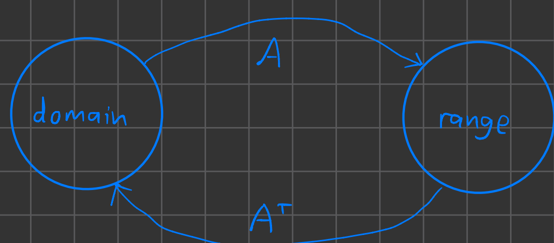
$$\underbrace{g_\theta(r)}_{\text{filtered projections}} = \overbrace{h(r)}^{\text{high-pass filter}} * p_\theta(r)$$

- Backprojection is the adjoint operator for Forward Projection

Adjoint (Transpose)

(sinogram) $\underset{\text{in range}}{y} = A\underset{\text{in domain}}{x}$ (image)

$$z = A^T y$$



- Backprojection: projections at each θ are "smeared" back into 2D space, like with a paintbrush

Note: adjoint only equal to inverse when transform is orthonormal

↳ When you filter the projections, then it does become the inverse

A Closer Look at Projection Filter

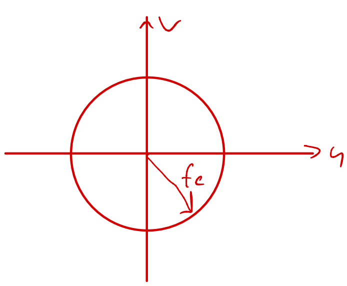
1. At each angle, projections are filtered.

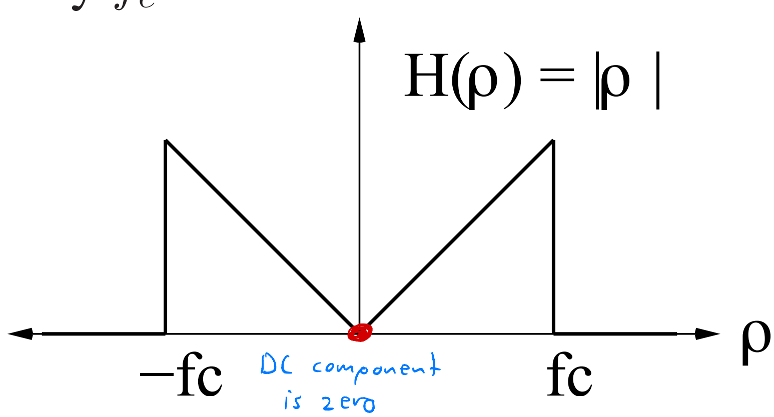
$$g_\theta(r) = h(r) * p_\theta(r)$$

2. The frequency response of the filter is given by

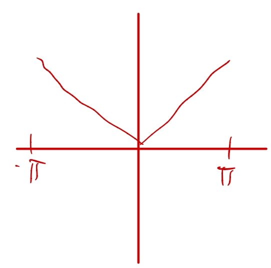
$$H(\rho) = |\rho|$$

3. But real filters must be bandlimited to $|\rho| \leq f_c$ for some cut-off frequency f_c.





DC component is zero
∴ area under $h(r)=0$



maximum frequency in discrete time is π

So

$$\text{frequency response: } H(\rho) = f_c\left[\text{rect}\left(f/(2f_c)\right) - \Lambda\left(f/f_c\right)\right]$$

+1 −1 +1 −1
↳ corresponds to frequency of π

$$\text{impulse response: } \int_{-\infty}^{\infty} h(r) = f_c^2\left[2\text{sinc}(t2f_c) - \text{sinc}^2(tf_c)\right] = 0$$

A Closer Look at Back Projection

- Back Projection function is

$$f(x, y) = \int_0^{\pi} b_\theta(x, y) \, d\theta$$

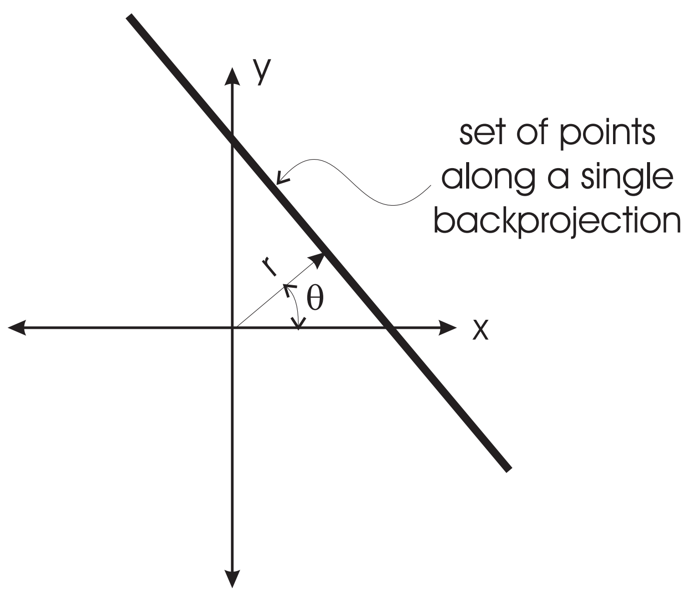
where

$$b_\theta(x, y) = g_\theta(x\cos(\theta) + y\sin(\theta))$$

- Consider the set of points (x, y) such that

$$r = x\cos(\theta) + y\sin(\theta)$$

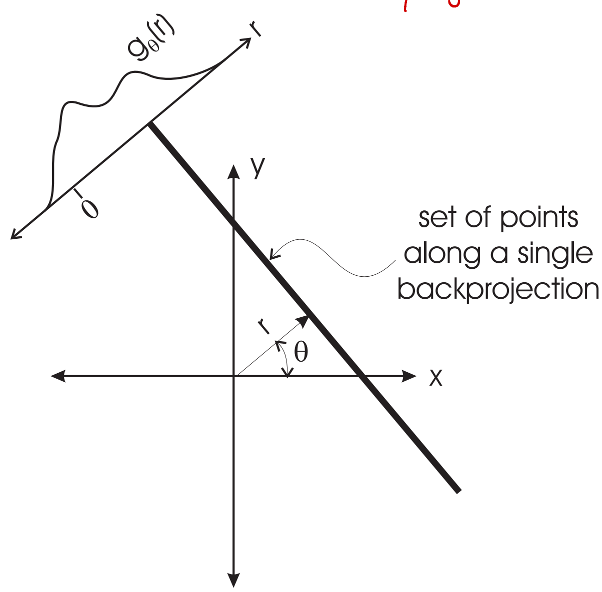
This set looks like



- Along this line $b_\theta(x, y) = g_\theta(r)$.

Back Projection Continued

- For each angle θ back projection is constant along lines of angle θ and takes on value $g_\theta(r)$. In reality: finite number of projections are captured (discrete sampling)

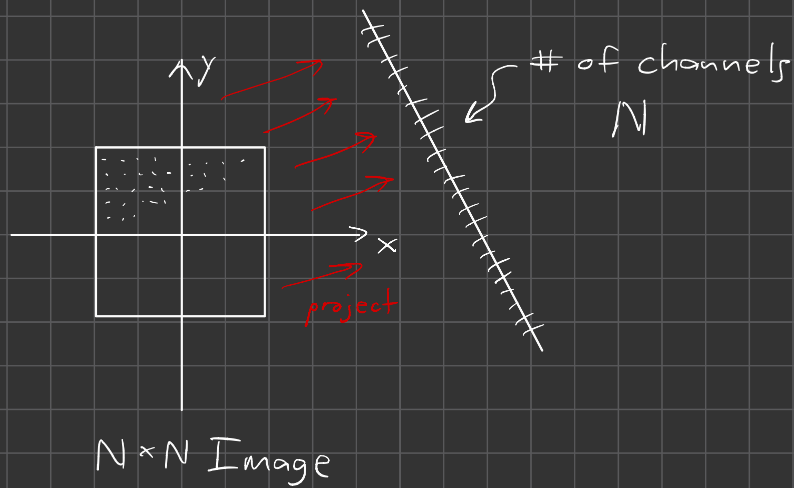


- Complete back projection is formed by integrating (summing) back projections for angles ranging from 0 to π.

$$f(x,y) = \int_0^\pi b_\theta(x,y)\, d\theta$$

$$\approx \underbrace{\frac{\pi}{M}}_{\text{correction factor}} \sum_{m=0}^{M-1} b_{\frac{m\pi}{M}}(x,y)$$

- Back projection "smears" values of $g(r)$ back over image, and then adds smeared images for each angle.



Assuming N channels per projection, need ≈ N unique projections in order to reconstruct the N×N image

☆ Area under curve of impulse response is zero if the frequency response at DC is zero