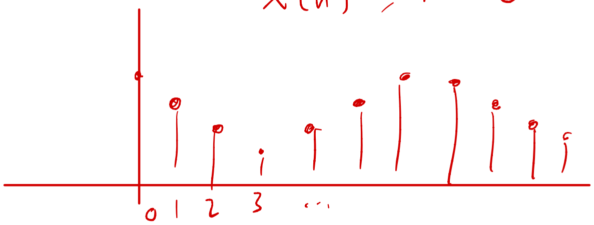
$x(n)$; $n \in \{\text{Integers}\}$, or subspace of integers



Discrete Time Fourier Transform (DTFT)

- integrals replaced with sum
- for DT: frequency in units of rad/sample (for CT: frequency in Hz)

* Always periodic with period 2π

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = X(e^{j(\omega+2\pi)})$$

FT is continuous in ω

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

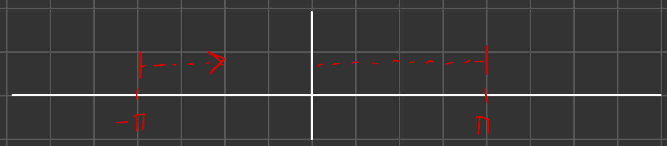
- Note: The DTFT is periodic with period 2π.

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

- Therefore functions such as $\text{rect}(\omega)$ are not valid DTFT's.

Maximum possible frequency of a discrete time signal: π

$x(n) = e^{j\omega n}$



largest variation is $+1, -1, +1, -1, \ldots$
and that corresponds to a frequency of π

- aliasing : frequency starts to go down once frequency passes π

- Hakam's Razor - simplest explanation

$$X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \Big|_{z=e^{j\omega}}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (e^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Useful Discrete Time Functions

step: $$u(n) \triangleq \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Dirac Delta: $$\delta(n) \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\text{pulse}_N(n) \triangleq u(n) - u(n-N)$$

$$\text{psinc}_N(\omega) \triangleq \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

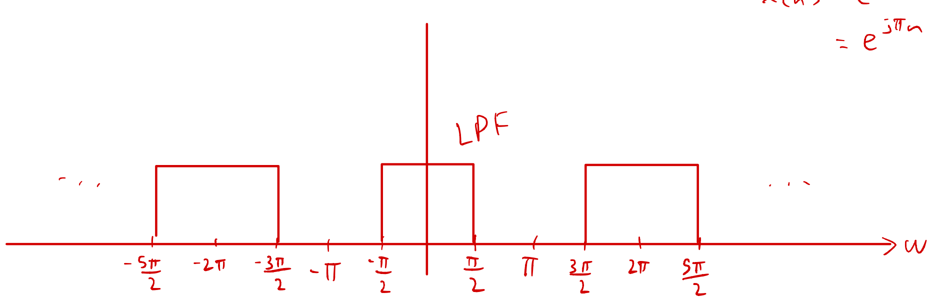
$$\text{prect}_{2\omega_o}(\omega) \triangleq \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\omega - 2\pi k}{2\omega_o}\right)$$

- $\text{psinc}_N(\omega)$ and $\text{prect}_{2\omega_o}(\omega)$ are periodic with period 2π.
- $\text{prect}_{2\omega_o}(\omega)$ has cut-off frequency ω_o.

when $\omega = \pi$

$x(n) = e^{j\omega n}$

$= e^{j\pi n} = (-1)^n$



$w = \pi + dw \longleftrightarrow w' = -\pi + \delta w$

$\delta = 0.1$

$w = 3.14 + 0.1 = 3.24$ — top frequency aliases as bottom one

$w' = -3.14 + 0.1 = -3.04$

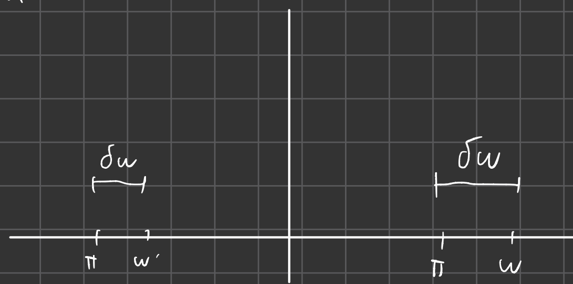
$\pi < w < 2\pi$ $\qquad w = \pi + \delta w$

$x(n) = e^{jwn} = e^{jwn}e^{j2\pi n}$

$= e^{j(w-2\pi)n}$

$= e^{j\underbrace{(\delta w - \pi)}_{w'}n}$

aliasing



$x[n] = e^{jwn}$

let $w = \pi$

$x[n] = e^{j\pi n} = (-1)^n \rightarrow$ when $w = \pi$, frequency alternates

why can't go over 2π?

now let $w > \pi$; use $w = \pi + \Delta w$ with $0 < \Delta w < \pi$

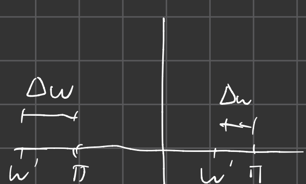
$x[n] = e^{j(\pi + \Delta w)n} = e^{j(\pi + \Delta w - 2\pi)n} = e^{j(\Delta w - \pi)}$

$\Delta w = 0$: max

$\Delta w = \pi$: $w = 0$

$w < \pi$, use $w = \pi - \Delta w$ with $0 \le \Delta w \le \pi$

$x[n] = e^{j(\pi - \Delta w)n}$

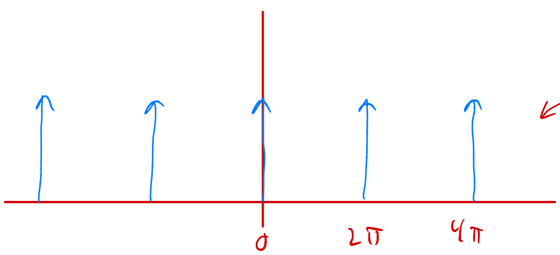


$x[n] = e^{j3\pi n} = e^{j\pi n}$

$x[n] = e^{j2\pi n} = e^{0n} = 1^n$??

Useful DTFT Transform Pairs

$$\delta(n) \stackrel{DTFT}{\Leftrightarrow} 1$$



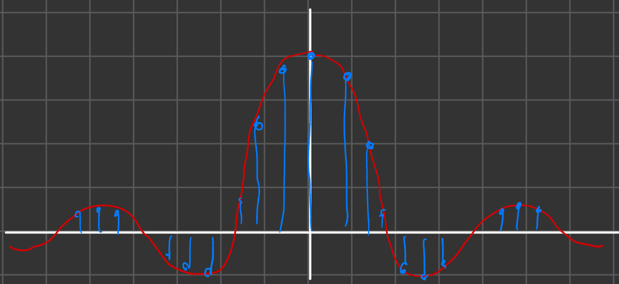
$$1 \stackrel{DTFT}{\Leftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$\text{pulse}_N(n) \stackrel{DTFT}{\Leftrightarrow} \text{psinc}_N(\omega) e^{-j\omega \frac{N-1}{2}}$$

$$\text{sinc}(Tn) \stackrel{DTFT}{\Leftrightarrow} \frac{1}{T}\text{prect}_{2\pi T}(\omega)$$

$$\frac{\omega_o}{\pi}\text{sinc}\left(\frac{\omega_0 n}{\pi}\right) \stackrel{DTFT}{\Leftrightarrow} \text{prect}_{2\omega_o}(\omega)$$

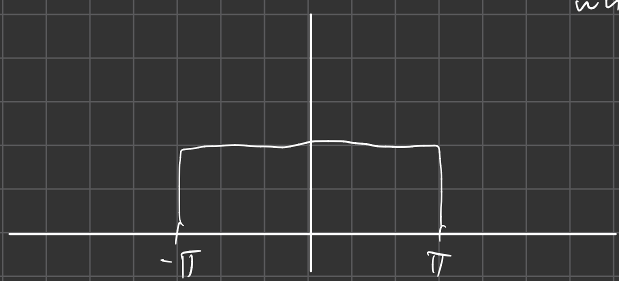
sampled
sinc

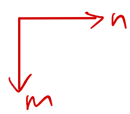


LTI
LPF

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = h(n) * x(n) = x(n) * h(n)$$

$h(n) = \text{sinc}(n) = \delta(n)$
when sampling rate is 1





μ & ν used vs u, v in CSFT *

Discrete Space Fourier Transform (DSFT)

$$F(e^{j\mu}, e^{j\nu}) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(m,n) e^{-j(\mu m + \nu n)}$$

$$f(m,n) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(e^{j\mu}, e^{j\nu}) e^{j(\mu m + \nu n)} d\mu d\nu$$

- Note: The DSFT is a 2-D periodic function with period 2π in in both the μ and ν dimensions.

$$F(e^{j(\mu+2\pi)}, e^{j(\nu+2\pi)}) = F(e^{j\mu}, e^{j\nu})$$

- Note: In matrix notation rows and columns are usually transposed.

$$f_{n,m} = f(m,n)$$

row $\rightarrow n$; column $\rightarrow m$

DSFT Properties Inherited from DTFT

- Some properties of the DSFT are directly inherited from the DTFT.

Property	Space Domain	DSFT
Linearity	$af(m,n)+bg(m,n)$	$aF(e^{j\mu},e^{j\nu})+bG(e^{j\mu},e^{j\nu})$
Conjugation	$f^*(m,n)$	$F^*(e^{-j\mu},e^{-j\nu})$
Shifting	$f(m-m_0,n-n_0)$	$e^{-j(\mu m_0+\nu y_0)}F(e^{j\mu},e^{j\nu})$
Modulation	$e^{j2\pi(u_0m+v_0n)}f(m,n)$	$F\left(e^{j(\mu-\mu_0)},e^{j(\nu-\nu_0)}\right)$
Convolution	$f(m,n)*g(m,n)$	$F(e^{j\mu},e^{j\nu})G(e^{j\mu},e^{j\nu})$
Multiplication	$f(m,n)g(m,n)$	$\frac{1}{(2\pi)^2}F(e^{j\mu},e^{j\nu})*G(e^{j\mu},e^{j\nu})$

- Inner product property

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(m,n)g^*(m,n)$$

$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(e^{j\mu},e^{j\nu})G^*(e^{j\mu},e^{j\nu})d\mu d\nu$$

Properties Specific to DSFT

- Some properties are distinct to the DSFT.

Property	Space Domain	DSFT
Separability	$f(m)g(n)$	$F(e^{j\mu})G(e^{j\nu})$

- Notice that there are no scaling or rotation properties for the DSFT.

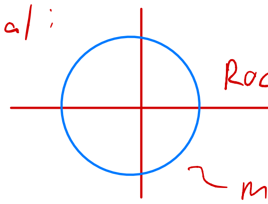
Absolutely Convergent if:

$$\sum_n |x(n) z^{-1}| < \infty$$

$$\sum_n |x(n)||z|^{-n} < \infty$$

z gets larger → improves convergence

causal:



Roc - region of convergence

may or may not contain unit circle

absolute convergence

2-D Z-Transform Transform

closely related to DTFT

- 1-D Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int_{C \in ROC} X(z) z^{n-1} dz$$

region of convergence

define by ensuring that denominator > 0 ??

- 2-D Z-transform

fundamental theorem of algebra

$$F(z_1, z_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m,n) z_1^{-m} z_2^{-n}$$

$$f(m,n) = \frac{1}{(2\pi j)^2} \int_{C^2 \in ROC} F(z_1, z_2) z_1^{m-1} z_2^{n-1} dz_1 dz_2$$

If it converges on unit circle

$$\sum_n |x(n)| < \infty \text{ - stable}$$

DTFT's : $\sum_n x(n) e^{-j\omega n}$

$\sum_n x(n) z^{-n}$ → easier to write

causal:

stable if and only if Z-transform converges on unit circle.

this is true if and only if the poles < 1

when denominator → 0 in T.F.

Relationship Between Fourier and Z-Transforms

- In 1-D, the DTFT is the 1-D Z-transform evaluated on the unit circle.

$$F(e^{j\omega}) = F(z)\big|_{z=e^{j\omega}}$$

- In 2-D the DSFT is the 2-D Z transform evaluated on the unit sphere.

$$F(e^{j\mu}, e^{j\nu}) = F(z_1, z_2)\big|_{z_1=e^{j\mu}, z_2=e^{j\nu}}$$