Energy $\rightarrow$ Electric Field^2

Humans perceive photons

2-D Finite Impulse Response (FIR) Filters

Convolution: Flip & Shift, Integrate

- Difference equation

Filter   flip

$$y(m,n) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} h(k,l)x(m-k, n-l)$$

↳ N index corresponds to multiplies

- For $N = 2$ - $\circ$ input points; $\times$ output point

Impulse response ≡ point spread function

$$\begin{array}{ccccc}
\underbrace{\circ \quad \circ}^{N} & \circ & \underbrace{\circ \quad \circ}^{N} \rightarrow (2N+1)^2 \text{ Multiplies} \\
\circ \quad \circ & \circ & \circ \quad \circ \; \text{-5} \\
\circ \quad \circ & \circ & \circ \quad \circ \; \text{-5} \\
\circ \quad \circ & \times & \circ \quad \circ \; \text{-5} \\
\circ \quad \circ & \circ & \circ \quad \circ \; \text{-5} \\
\circ \quad \circ & \circ & \circ \quad \circ \; \text{-5}
\end{array}$$

- Number of multiplies per output point

$$\text{Multiplies} = (2N+1)^2$$

Ex: $N=2$ - Mult. $= (2(2)+1)^2 = 25$

- Transfer function

$$H(z_1, z_2) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} h(k,l) z_1^{-k} z_2^{-l}$$

$$H(e^{j\mu}, e^{j\nu}) = \sum_{k=-N}^{N} \sum_{l=-N}^{N} h(k,l) e^{-j(k\mu + l\nu)}$$

Spatial FIR Smoothing Filtering

- Filter point spread function (PSF) or impulse response: The box, $\boxed{X}$, indicates the center element of the filter.

DC Gain = Area under curve/sum

↳ when it sums to 1, the image does not get brighter or darker

$$\begin{array}{ccc} 1 & 2 & 1 \\ 2 & \boxed{4} & 2 \\ 1 & 2 & 1 \end{array} \cdot \frac{1}{16}$$

- Apply filter using free boundary condition: Assume that pixels outside the image are 0.

$$\underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \end{array}}_{\text{Input Image}} \Rightarrow \underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 & 4 & 3 \\ 0 & 0 & 3 & 9 & 12 & 12 & 9 \\ 0 & 0 & 4 & 12 & 16 & 16 & 12 \\ 0 & 0 & 4 & 12 & 16 & 16 & 12 \\ 0 & 0 & 3 & 9 & 12 & 12 & 9 \end{array}}_{\text{Output Image}}$$

PSF for FIR Smoothing Filter

$$\begin{array}{ccc} 1 & 2 & 1 \\ 2 & \boxed{4} & 2 \\ 1 & 2 & 1 \end{array} \cdot \frac{1}{16}$$

Spatial FIR Horizontal Derivative Filtering

- Filter point spread function (PSF) or impulse response: The box, $\boxed{\text{X}}$, indicates the center element of the filter.

Symmetric vertically

Asymmetric horizontally

$$\begin{array}{ccc} 2 & 0 & -2 \\ 4 & \boxed{0} & -4 \\ 2 & 0 & -2 \end{array} \cdot \frac{1}{16}$$

Discrete derivative in horizontal direction

- Apply filter using free boundary condition: Assume that pixels outside the image are 0.

$$\underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \end{array}}_{\text{Input Image}} \Rightarrow \underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & -2 \\ 0 & 0 & 6 & 6 & 0 & 0 & -6 \\ 0 & 0 & 8 & 8 & 0 & 0 & -8 \\ 0 & 0 & 8 & 8 & 0 & 0 & -8 \\ 0 & 0 & 6 & 6 & 0 & 0 & -6 \end{array}}_{\text{Output Image}}$$

PSF of FIR Horizontal Derivative Filter

$$
\begin{array}{ccc}
2 & 0 & -2 \\
4 & \boxed{0} & -4 \\
2 & 0 & -2
\end{array}
\cdot \frac{1}{16}
$$

Spatial FIR Vertical Derivative Filtering

- Filter point spread function (PSF) or impulse response: The box, $\boxed{\text{X}}$, indicates the center element of the filter.

Discrete derivative in vertical direction

$$\begin{array}{ccc} 2 & 4 & 2 \\ 0 & \boxed{0} & 0 \\ -2 & -4 & -2 \end{array} \cdot \frac{1}{16}$$

- Apply filter using free boundary condition: Assume that pixels outside the image are 0.

$$\underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{16} & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \\ 0 & 0 & 0 & 16 & 16 & 16 & 16 \end{array}}_{\text{Input Image}} \Rightarrow \underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 & 8 & 8 & 6 \\ 0 & 0 & 2 & 6 & 8 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -6 & -8 & -8 & -6 \end{array}}_{\text{Output Image}}$$

PSF of FIR Vertical Derivative Filter

$$
\begin{array}{ccc}
2 & 4 & 2 \\
0 & \boxed{0} & 0 \\
-2 & -4 & -2
\end{array}
\cdot \frac{1}{16}
$$

Example 1: 2-D FIR Filter

- Consider the impulse response $h(m,n) = h_1(m)h_1(n)$ where

$$
\begin{aligned}
4h_1(n) &= (\cdots, 0, 1, 2, 1, 0, \cdots) \\
h_1(n) &= (\delta(n+1) + 2\delta(n) + \delta(n-1))/4
\end{aligned}
$$

separable ☆

$h(m,n) = h_1(m)h_1(n)$

Then $h(m,n)$ is a separable function with

case 1: $m=0, n=1$

$16h(0,1) = 16 \cdot \frac{1}{16}\left[(\delta(1)+2\delta(0)+\delta(-1))(\delta(2)+2\delta(1)+\delta(0))\right] = 2 \cdot 1 = 2$ ✓

case 2: $m=1, n=1$

$16h(1,1) = 16 \cdot \frac{1}{16}\left[(\delta(2)+2\delta(1)+\delta(0))(\delta(2)+2\delta(1)+\delta(0))\right] = 1 \cdot 1 = 1$ ✓

$$
16h(m,n) = \begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 0 & 0 & 0 & 0 & 0 & \\
\cdots & 0 & 1 & 2 & 1 & 0 & \cdots \\
\cdots & 0 & 2 & 4 & 2 & 0 & \cdots \\
\cdots & 0 & 1 & 2 & 1 & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Here: DC Gain = 1

case 3: $m=0, n=0$

$16h(0,0) = 16 \cdot \frac{1}{16}\left[(\delta(1)+2\delta(0)+\delta(-1))(\delta(1)+2\delta(0)+\delta(-1))\right]$

$= 2 \cdot 2 = 4$ ✓

- The DTFT of $h_1(n)$ is

$$
\begin{aligned}
H_1(e^{j\omega}) &= \frac{1}{4}\left(e^{j\omega} + 2 + e^{-j\omega}\right) \\
&= \frac{1}{2}(1 + \cos(\omega))
\end{aligned}
$$

- The DSFT of $h(m,n)$ is

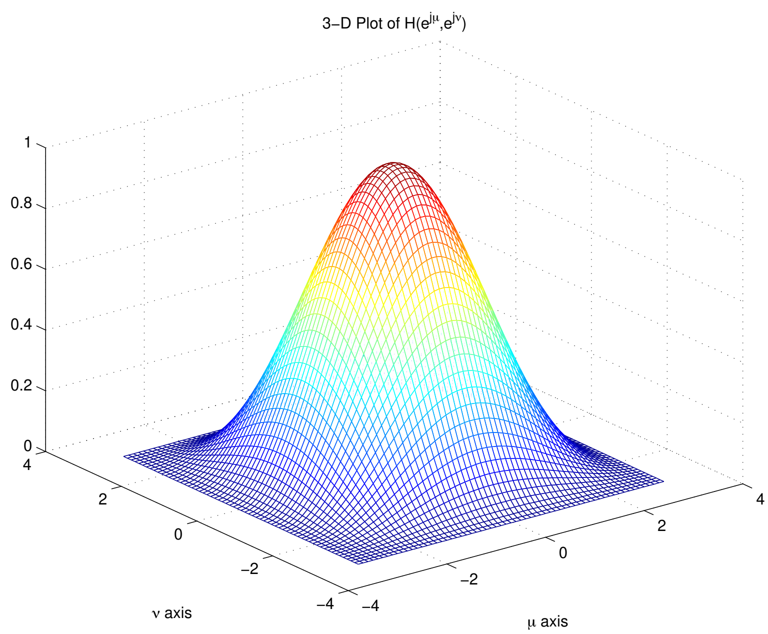
$$
\begin{aligned}
H(e^{j\mu}, e^{j\nu}) &= H_1(e^{j\mu})H_1(e^{j\nu}) \\
&= \frac{1}{4}(1 + \cos(\mu))(1 + \cos(\nu))
\end{aligned}
$$

$$\begin{array}{ccccc} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \end{array}$$ - checkerboard

Example 1: Frequency Response of 2-D FIR Filter

- Plot of frequency response

$$H(e^{j\mu}, e^{j\nu}) = \frac{1}{4}\left(1 + \cos(\mu)\right)\left(1 + \cos(\nu)\right)$$



- This is a low pass filter with $H(e^{j0}, e^{j0}) = 1$

Example 2: 2-D FIR Filter

- Consider the impulse response $h(m,n) = h_1(m)h_1(n)$ where

$$
\begin{aligned}
4h_1(n) &= (\cdots, 0, 1, -2, 1, 0, \cdots) \\
h_1(n) &= (\delta(n+1) - 2\delta(n) + \delta(n-1))/4
\end{aligned}
$$

Then $h(m,n)$ is a separable function with

$$
16h(m,n) = \begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 0 & 0 & 0 & 0 & 0 & \\
\cdots & 0 & 1 & -2 & 1 & 0 & \cdots \\
\cdots & 0 & -2 & 4 & -2 & 0 & \cdots \\
\cdots & 0 & 1 & -2 & 1 & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Here: DC Gain = 0

- The DTFT of $h_1(n)$ is

$$
\begin{aligned}
H_1(e^{j\omega}) &= \frac{1}{4}\left(e^{j\omega} - 2 + e^{-j\omega}\right) \\
&= -\frac{1}{2}(1 - \cos(\omega))
\end{aligned}
$$

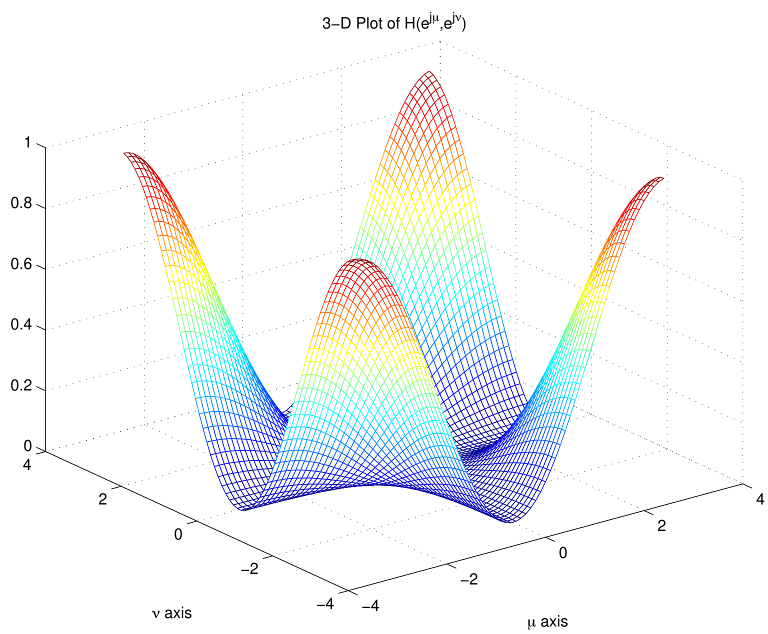
- The DSFT of $h(m,n)$ is

$$
\begin{aligned}
H(e^{j\mu}, e^{j\nu}) &= H_1(e^{j\mu})H_1(e^{j\nu}) \\
&= \frac{1}{4}(1 - \cos(\mu))(1 - \cos(\nu))
\end{aligned}
$$

Example 2: Frequency Response of 2-D FIR Filter

- Plot of frequency response

$$H(e^{j\mu}, e^{j\nu}) = \frac{1}{4}\left(1 - \cos(\mu)\right)\left(1 - \cos(\nu)\right)$$



- This is a **high pass** filter with $H(e^{j0}, e^{j0}) = 0$

Ordering of Points in a Plane

causal ordering - all circles occur before the "x"

- Recursive filter implementations require the ordering of points in the plane.
- Let $s = (s_1, s_2) \in \mathcal{Z}^2$ and $r = (r_1, r_2) \in \mathcal{Z}^2$.
- Quarter plane - then $s < r$ means:

$$(s_2 < r_2) \text{ and } (s_1 < r_1) \text{ and } s \neq r$$

$$\begin{array}{ccc} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \times \end{array}$$

- Symmetric half plane - then $s < r$ means:

$$(s_2 < r_2)$$

$$\begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ & & \times & & \end{array}$$

- Nonsymmetric half plane - then $s < r$ means:

$$(s_2 < r_2) \text{ or } ((s_2 = r_2) \text{ and } (s_1 < r_1))$$

$$\begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \times & & \end{array}$$

Filter's response is anisotropic : non-symmetric visually

Recursive Filtering

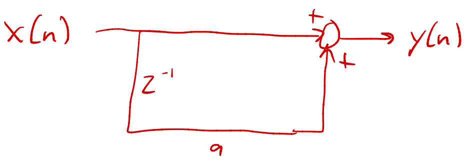
2-D Infinite Impulse Response (IIR) Filters

Power filtering, low computation (compared to FIR - large kernels aka multiplies per pixel)

- Difference equation

$$y(m,n) = \underbrace{\sum_{k=-N}^{N}\sum_{l=-N}^{N} b(k,l)x(m-k,n-l)}_{\text{FIR Part}}$$

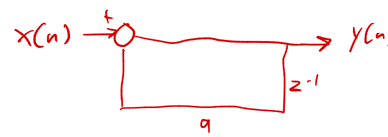
$$\underbrace{+ \sum_{k=-P}^{P}\sum_{l=1}^{P} a(k,l)y(m-k,n-l) + \sum_{k=1}^{P} a(k,0)y(m-k,n)}_{\text{IIR Part}}$$



$y(n) = x(n) + a\,x(n-1)$

$Y(z) = X(z) + az^{-1}X(z)$

$H(z) = \frac{Y(z)}{X(z)} = 1 + az^{-1}$



$y(n) = x(n) + a\,Y(n-1)$

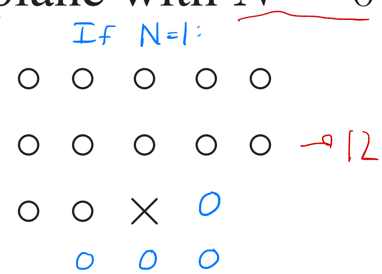
$Y(z) = X(z) + az^{-1}Y(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}}$

Simplified notation

$$y_s = \sum_r b_r x_{s-r} + \sum_{r>(0,0)} a_r y_{s-r}$$

- For nonsymetric half plane with $N = 0$ and $P = 2$



- Number of multiplies per output point

$$\text{Multiplies} = \underbrace{(2N+1)^2}_{\text{FIR Part}} + \underbrace{2(P+1)P}_{\text{IIR Part}}$$

$(2(0)+1)^2 + 2(2+1)2$

$1 + 2(3)(2) = 1 + 12 = 13$

not LTI → no TF??

$H(z)\big|_{z=e^{j\omega}} = H(e^{j\omega})$

$-\pi \quad \pi$

amplitude (gain) & phase can change

$e^{j\omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow Ae^{j\omega n}$

A is complex

$A = H(e^{j\omega})$

embodies gain & phase

magic, characterizes the LTI system

characterize LTI system by transfer function or impulse response

$x(n) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(n)$

$H(e^{j\omega}) = \text{DTFT}\{h(n)\}$

2-D IIR Filter Transfer Functions

- Transfer function in Z-transform domain is

$$H(z_1,z_2) = \frac{\sum_{k=-N}^{N}\sum_{l=-N}^{N} b(k,l) z_1^{-k} z_2^{-l}}{1-\sum_{k=-P}^{P}\sum_{l=1}^{P} a(k,l) z_1^{-k} z_2^{-l} - \sum_{k=1}^{P} a(k,0) z_1^{-k}}$$

need to spend more time trying to understand this equation

- Transfer function in DSFT domain is

$$H(e^{j\mu},e^{j\nu}) = \frac{\sum_{k=-N}^{N}\sum_{l=-N}^{N} b(k,l) e^{-j(k\mu+l\nu)}}{1-\sum_{k=-P}^{P}\sum_{l=1}^{P} a(k,l) e^{-j(k\mu+l\nu)} - \sum_{k=1}^{P} a(k,0) e^{-j(k\mu)}}$$

$\delta(n) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(n)$

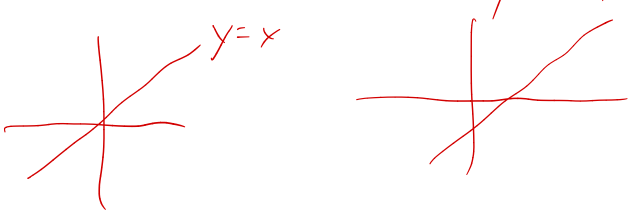
impulse response

* measure h(n): sweeping sine wave in, measure gain & phase, plot as function of ω, calculate inverse FT

* real systems are <u>approximated</u> as LTI

Network Analyzer

@ x = x+1 → y = x on original



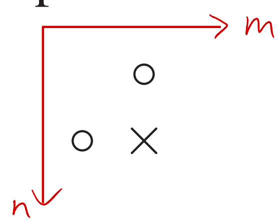
Example 3: 2-D IIR Filter

- Consider the difference equation

Be cognizant of plot shift vs argument shift

$$y(m,n) = x(m,n) + ay(m-1,n) + ay(m,n-1)$$

- Spatial dependencies - $\circ$ previous value; $\times$ curent value



- Taking the Z-transform of the difference equation

$$Y(z_1,z_2) = X(z_1,z_2) + az_1^{-1}Y(z_1,z_2) + az_2^{-1}Y(z_1,z_2)$$

The transfer functions is then

$$\boxed{H(z_1,z_2) = \frac{Y(z_1,z_2)}{X(z_1,z_2)}} = \frac{1}{1-az_1^{-1}-az_2^{-1}}$$

$$H(e^{j\mu},e^{j\nu}) = \frac{1}{1-ae^{-j\mu}-ae^{-j\nu}}$$

impulse response not isotropic *

Example 3: 2-D IIR Filter in Space Domain

- For $a = 1/2$

$$y(m,n) = x(m,n) + \frac{1}{2}y(m-1,n) + \frac{1}{2}y(m,n-1)$$

- Looks like

$$\begin{array}{cc} & 1/2 \\ 1/2 & \times \end{array}$$

- Apply filter in raster scan order.

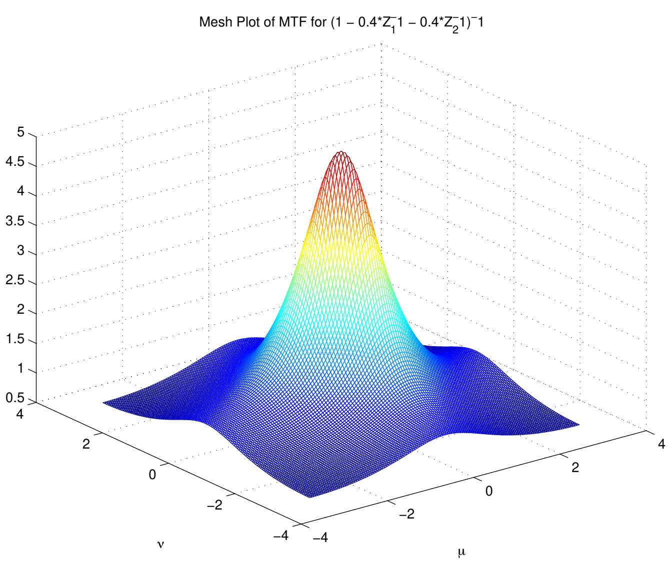
$$\underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}}_{\text{Input Image}} \Rightarrow \underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64 & 32 & 16 & 8 \\ 0 & 0 & 0 & 32 & 32 & 24 & 16 \\ 0 & 0 & 0 & 16 & 24 & 24 & 20 \\ 0 & 0 & 0 & 8 & 16 & 20 & 20 \end{array}}_{\text{Output Image}}$$

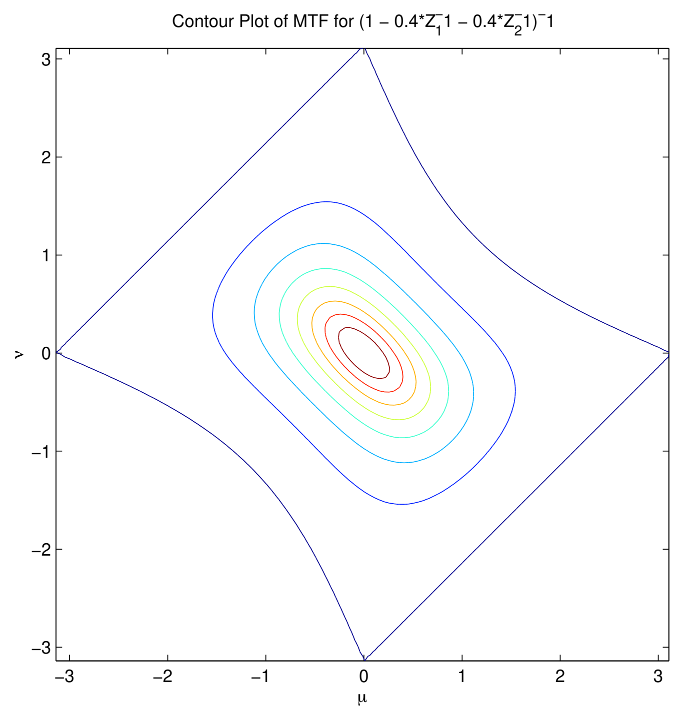
Example 3: Frequency Response of 2-D IIR Filter

- Plot of frequency response

$$H(z_1, z_2) = \frac{1}{1 - az_1^{-1} - az_2^{-1}}$$

for $a = 0.4$.





Example 4: 2-D IIR Filter

- Consider the difference equation

$$\begin{aligned} y(m,n) &= x(m,n) + ay(m-1,n) + ay(m,n-1) \\ &\quad +2ay(m+1,n-1) \end{aligned}$$

- Spatial dependencies - $\circ$ previous value; $\times$ curent value

$$\begin{array}{ccc} & \circ & \circ \\ \circ & \times & \end{array}$$

- The transfer functions is then

$$\begin{aligned} H(z_1,z_2) &= \frac{1}{1 - az_1^{-1} - az_2^{-1} - 2az_1^{+1}z_2^{-1}} \\ H(e^{j\mu}, e^{j\nu}) &= \frac{1}{1 - ae^{-j\mu} - ae^{-j\mu} - 2ae^{+j\mu - j\nu}} \end{aligned}$$

Example 4: 2-D IIR Filter in Space Domain

- For $a = 1/4$

$$\begin{aligned} y(m,n) &= x(m,n) + \frac{1}{4}y(m-1,n) + \frac{1}{4}y(m,n-1) \\ &\quad + \frac{1}{2}y(m+1,n-1) \end{aligned}$$

- Looks like

$$\begin{array}{ccc} & 1/4 & 1/2 \\ 1/4 & \times & \end{array}$$

- Apply filter in raster scan order.

$$\underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}}_{\text{Input Image}} \Rightarrow \underbrace{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64 & 16 & 4 & 1 \\ 0 & 0 & 32 & 32 & 14 & 5 & 1\frac{1}{2} \\ 0 & 16 & 28 & 22 & 11\frac{1}{2} & 4\frac{7}{8} & \frac{45}{32} \end{array}}_{\text{Output Image}}$$

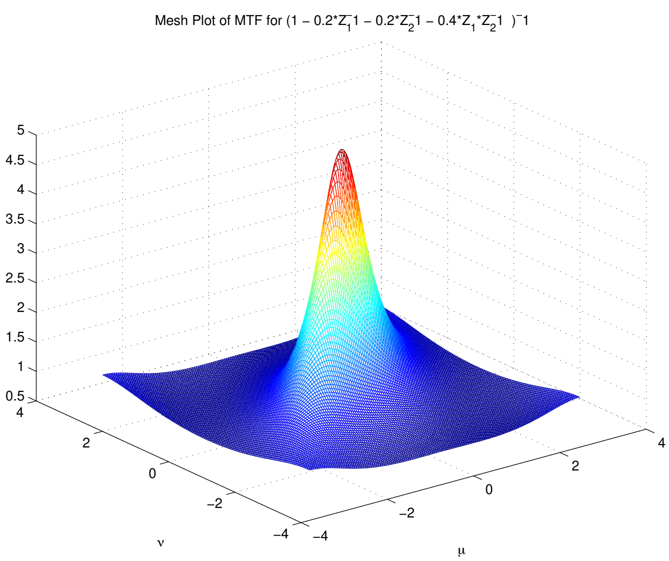
Example 4: Frequency Response of 2-D IIR Filter

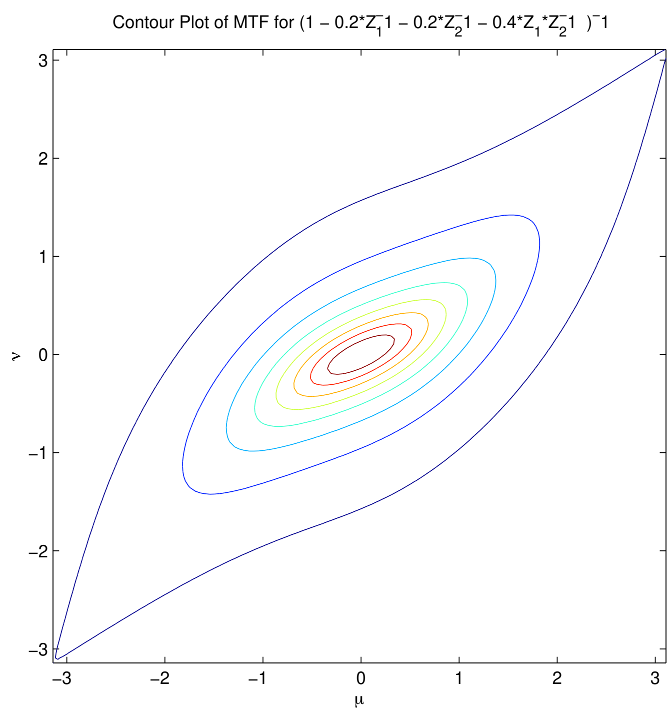
Good calcs to know for exam ☆

- Plot of frequency response

$$H(z_1, z_2) = \frac{1}{1 - az_1^{-1} - az_2^{-1} - 2az_1z_2^{-1}}$$

for $a = 0.2$.





- Notice that transfer function has a diagonal orientation.

$\vec{y} = \vec{x} + H\vec{y}$

$(I - H)\vec{y} = \vec{x}$

$\vec{y} = \underbrace{(I - H)^{-1}}\vec{x}$

if causal:
matrix is
upper triangular

Example 5: 2-D IIR Filter

- Consider the difference equation

$$
\begin{aligned}
y(m,n) \;=\; & x(m,n) + ay(m-1,n) + ay(m,n-1) \\
& + ay(m+1,n) + ay(m,n+1)
\end{aligned}
$$

- Spatial dependencies - $\circ$ previous value; $\times$ curent value

$$
\begin{array}{ccc}
 & \circ & \\
\circ & \times & \circ \\
 & \circ &
\end{array}
$$

- Theoretically, the transfer functions is then

$$
\begin{aligned}
H(z_1, z_1) \;&=\; \frac{1}{1 - az_1^{-1} - az_2^{-1} - az_1 - az_2} \\
H(e^{j\mu}, e^{j\nu}) \;&=\; \frac{1}{1 - ae^{-j\mu} - ae^{-j\mu} - ae^{j\mu} - ae^{j\nu}}
\end{aligned}
$$

- **THIS DOESN'T WORK**