# Problem 1

## Problem 1

Disclaimer: This example is a modified version of the one found in a 2013 lecture on Bayesian Scientific Computing taught by Prof. Nicholas Zabaras. I am not sure where the original problem is coming from.

We are tasked with assessing the usefulness of a tuberculosis test. The prior information I is:

> The percentage of the population infected by tuberculosis is 0.4%. We have run several experiments and determined that:

- If a tested patient has the disease, then 80% of the time the test comes out positive.
- If a tested patient does not have the disease, then 90% of the time the test comes out negative.

To facilitate your analysis, consider the following logical sentences concerning a patient:

> A: The patient is tested and the test is positive.

> B: The patient has tuberculosis.

A) $p(B|I) = p(B) = 0.4\% = \boxed{0.004}$

B) $p(A|B, I) = p(A|B) = 80\% = \boxed{0.8}$

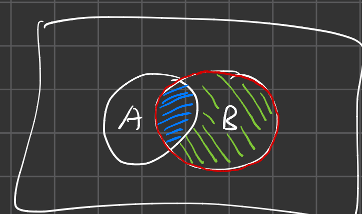
C) $p(A|\neg B, I) = p(A|\bar{B})$

$p(\bar{A}|\bar{B}) = 90\% = 0.9$

Using the obvious rule:

$p(\bar{A}|\bar{B}) + p(A|\bar{B}) = 1$

$\hookrightarrow p(A|\bar{B}) = 1 - p(\bar{A}|\bar{B})$
$= 1 - 0.9 = \boxed{0.1}$




$P(A|B) = \dfrac{P(A \cap B)}{P(B)} =$

$P(\bar{A}|B) = \dfrac{P(\bar{A} \cap B)}{P(B)} =$

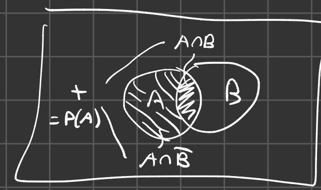
do sum to 1, same if $\bar{B}$ was used instead

D) $p(B|A, I) = p(B|A)$

$p(B|A) = \dfrac{p(A|B)\,p(B)}{p(A)} = \dfrac{0.8 \cdot 0.004}{p(A)}$

$\cancel{p(B|A,I) = p(B|A)}$

&




$$p(A) = p(A \cap I) = p(A \cap (B \cup \bar{B}))$$
$$= p((A \cap B) \cup (A \cap \bar{B}))$$
$$= p(A \cap B) + p(A \cap \bar{B}) \quad \text{(disjoint)}$$
$$= p(A|B)\,p(B) + p(A|\bar{B})\,p(\bar{B})$$
$$= 0.8 \cdot 0.004 + 0.1 \cdot 0.996 = 0.1028$$

$\hookrightarrow p(B|A) = \boxed{0.031128}$ ✓

E) $p(B|\neg A, I) = p(B|\bar{A})$

$$p(B|\bar{A}) = \frac{p(\bar{A}|B)\,p(B)}{p(\bar{A})} = \frac{(1 - p(A|B)) \cdot 0.004}{1 - p(A)}$$
$$= \boxed{0.000891663 \approx 0.09\%}$$

Yes, this does change our prior state of knowledge about the patient, because $p(B|\bar{A}) < p(B)$. Therefore, the test is useful

F) dropping the I:

$$p(B|A) = \frac{p(A|B)\,p(B)}{p(A)} = \frac{p(A|B)\,p(B)}{p(A|B)\,p(B) + p(A|\bar{B})\,p(\bar{B})} = 0.99$$
$$= \frac{x\,p(B)}{x\,p(B) + y\,p(\bar{B})} = \frac{0.004x}{0.004x + 0.996y}$$

$$0.99(0.004x + 0.996y) = 0.004x$$

$$0.00396x + 0.98604y = 0.004x \longrightarrow \boxed{0.98604y = 0.00004x}$$

1 equation relating 2 unknowns: multiple (infinite) solutions

$$\begin{bmatrix} 0.00004 & -0.98604 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [0] = \vec{0}$$

define what a good test looks like

$$\begin{bmatrix} 1 & -24651 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [0] \longrightarrow \begin{matrix} x_1 = 24651x_2 \\ x_2 = x_2 \end{matrix} \longrightarrow \vec{x} = \left\{ \begin{bmatrix} 24651 \\ 1 \end{bmatrix} \right\}$$