Object Tracking Example

Contents


```python
import numpy as np
np.set_printoptions(precision=3)
import matplotlib.pyplot as plt
%matplotlib inline
import seaborn as sns
sns.set(rc={"figure.dpi":100, "savefig.dpi":300})
sns.set_context("notebook")
sns.set_style("ticks")
```

Objectives

- Demonstrate the filtering problem in the context of object tracking

Consider an object of mass m and position vector $\vec{r} = r_x\hat{i} + r_y\hat{j}$, where $\hat{i}$ and $\hat{j}$ are the unit vectors in the x- and y-direction, respectively. As we saw in the lecture video, the dynamics are given by Newton's law:

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F} = u_x\hat{i} + u_y\hat{j}.$$

u_x and u_y are force values

We also saw that these 2nd order differential equations can be written as four first order differential equations:

$$\frac{d\vec{r}}{dt} = \vec{v},$$

and

$$\frac{d\vec{v}}{dt} = \frac{u_x}{m}\hat{i} + \frac{u_y}{m}\hat{j}.$$

Then, we used the Euler scheme with a timestep Δt to numerically solve these equations, yielding:

$$\vec{r}((n+1)\Delta t) = \vec{r}(n\Delta t) + \Delta t\vec{v}(n\Delta t),$$

new position = prior position + time step * slope = prior position + time step * rise/run = prior position + rise = prior position + change in position

and

https://en.wikipedia.org/wiki/Euler_method

$$\vec{v}((n+1)\Delta t) = \vec{v}(n\Delta t) + \Delta t\left(u_x\hat{i} + u_y\hat{j}\right).$$

new velocity = prior velocity + time step * slope = prior velocity + time step * rise/run = prior velocity + rise = prior velocity + change in change in velocity

Writing the state vector as:

$$\mathbf{x}_n = \begin{bmatrix} r_x(n\Delta t)\\ r_y(n\Delta t)\\ v_x(n\Delta t) \end{bmatrix},$$

[v_y(ndt)]

shouldn't there be a 4th row here?

and the control vector as:

$$\mathbf{u}_n = \begin{bmatrix} u_x(n\Delta t)\\ u_y(n\Delta t) \end{bmatrix},$$

then we see that the system satisfies the linear transition equation:

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n + \mathbf{B}\mathbf{u}_n + \mathbf{z}_n,$$

with transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

control matrix:

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\Delta t}{m} & 0 \\ 0 & \frac{\Delta t}{m} \end{bmatrix}$$

and we assume that

$$\mathbf{z}_n \sim N(\mathbf{0}, \mathbf{Q}),$$

is some process noise with covariance matrix $\mathbf{Q}$. Notice that the process noise does not appear in the original system. We have included it by hand and it is a modeling choice. We take the process covariance matrix to be:

$$Q = \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \sigma_q^2 & 0 \\ 0 & 0 & 0 & \sigma_q^2 \end{bmatrix},$$

where we have included a small $\epsilon > 0$ is very small and captures the discretization error of the Euler scheme. The variance σ_q^2 can be larger as it captures both the discretization error and any external forces to the system (forces that are not captured in $\mathbf{u}_n$).

Now let's talk about the measurements. Let's assume that we measure a noisy version of the position of the object. This is typically of GPS measurements. Mathematically, we have:

$$\mathbf{y}_n = \mathbf{C}\mathbf{x}_n + \mathbf{w}_n,$$

with

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

with measurement covariance:

$$R = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}.$$

This is it. Now, let's define all the necessary quantities:

```
# The timestep
Dt = 0.5
# The mass
m = 1.0
# The variance for the process noise for position
epsilon = 1e-6
# The standard deviation for the process noise for velocity
sigma_q = 1e-2
# The standard deviation for the measurement noise for position
sigma_r = 0.1

# INITIAL CONDITIONS
# initial mean
mu0 = np.zeros((4,))
# initial covariance
V0 = np.array([0.1**2, 0.1**2, 0.1**2, 0.1**2]) * np.eye(4)

# TRANSITION MATRIX
A = np.array(
    [
        [1.0, 0, Dt, 0],
        [0.0, 1.0, 0.0, Dt],
        [0.0, 0.0, 1.0, 0.0],
        [0.0, 0.0, 0.0, 1.0]
    ]
)
# CONTROL MATRIX
B = np.array(
    [
        [0.0, 0.0],
        [0.0, 0.0],
        [Dt / m, 0.0],
        [0.0, Dt / m]
    ]
)
# PROCESS COVARIANCE
Q = (
    np.array(
        [epsilon, epsilon, sigma_q ** 2, sigma_q ** 2]
    )
    * np.eye(4)
)
# EMISSION MATRIX
C = np.array(
    [
        [1.0, 0.0, 0.0, 0.0],
        [0.0, 1.0, 0.0, 0.0]
    ]
)
# MEASUREMENT COVARIANCE
R = (
    np.array(
        [sigma_r ** 2, sigma_r ** 2]
    )
    * np.eye(2)
)
```

Now we are going to simulate a trajectory of this particle.

```
np.random.seed(12345)

# The number of steps in the trajectory
num_steps = 50
# Space to store the trajectory (each state is 4-dimensional)
true_trajectory = np.ndarray((num_steps + 1, 4))
# Space to store the observations (each observation is 2-dimensional)
observations = np.ndarray((num_steps, 2))

# Sample the initial conditions
x0 = mu0 + np.sqrt(np.diag(V0)) * np.random.randn(4)
true_trajectory[0] = x0

# Pick a set of pre-determined forces to be applied to the object
# so that it does something interesting
force = .1
omega = 2.0 * np.pi / 5
times = Dt * np.arange(num_steps + 1)
us = np.zeros((num_steps, 2))
us[:, 0] = force * np.cos(omega * times[1:])
us[:, 1] = force * np.sin(omega * times[1:])

# Sample the trajectory
for n in range(num_steps):
    x = (
        A @ true_trajectory[n]
        + B @ us[n]
        + np.sqrt(np.diag(Q)) * np.random.randn(4)
    )
    true_trajectory[n+1] = x
    y = (
        C @ x
        + np.sqrt(np.diag(R)) * np.random.randn(2)
    )
    observations[n] = y
```

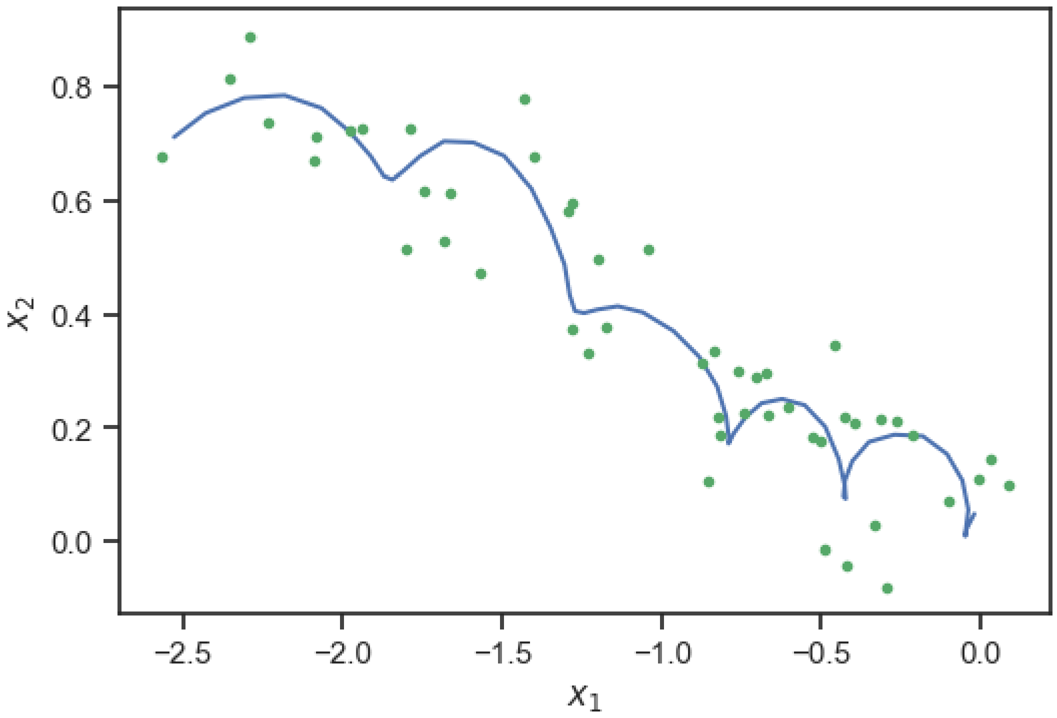
position and velocity in x and y directions

position in x and y directions

Here is a plot of the true trajectory along with the noisy GPS measurements:

```
fig, ax = plt.subplots()
ax.plot(true_trajectory[:, 0], true_trajectory[:, 1], '-')
ax.plot(observations[:, 0], observations[:, 1], 'g.')
ax.set_xlabel("$x_1$")
ax.set_ylabel("$x_2$");
```

Plot of the x and y positions



And here are the timeseries data of the states:

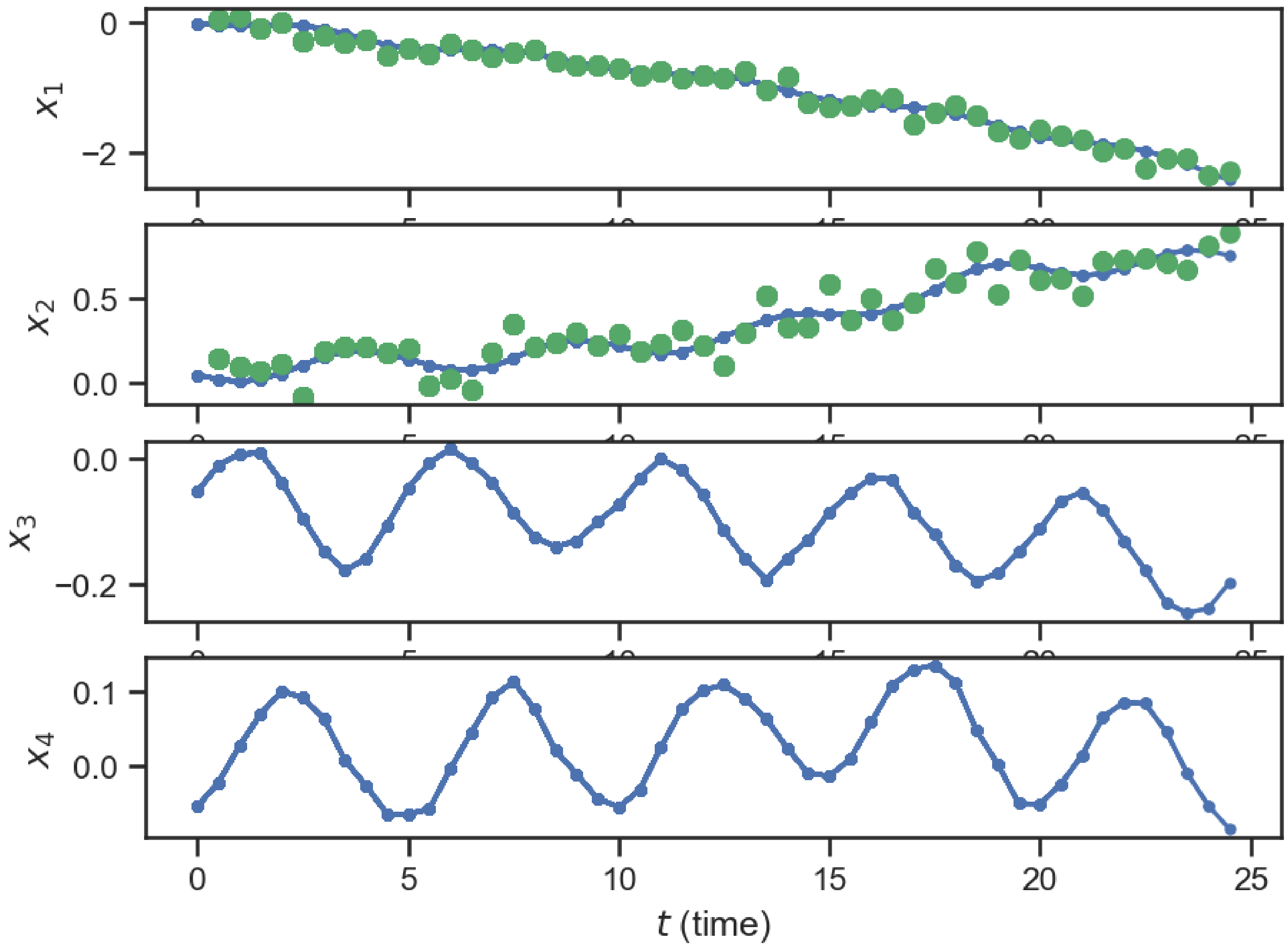
```
y_labels = ['$x_1$', '$x_2$', '$x_3$', '$x_4$']

res_x = 1024
res_y = 768
dpi = 150
w_in = res_x / dpi
h_in = res_y / dpi
fig, ax = plt.subplots(4, 1, dpi=dpi)
fig.set_size_inches(w_in, h_in)

for j in range(4):
    ax[j].set_ylabel(y_labels[j])
ax[-1].set_xlabel('$t$ (time)')

for n in range(1, num_steps):
    for j in range(4):
        ax[j].plot(times[:n+1], true_trajectory[:n+1, j], 'b.-')
        if j < 2:
            ax[j].plot(times[1:n+1], observations[:n, j], 'go')
```

x position

y position

z velocity

y velocity



Questions

- Rerun the code a couple of times to observe different trajectories.
- Double the process noise variance σ_q^2. What happens?
- Double the measurement noise variance σ_r^2. What happens?
- Zero-out the control vector $\mathbf{u}_{0:n-1}$. What happens?

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