Sampling Estimates of the Probability Density via Histograms

Contents


```
import numpy as np
np.set_printoptions(precision=3)
import matplotlib.pyplot as plt
%matplotlib inline
import seaborn as sns
sns.set(rc={"figure.dpi":100, 'savefig.dpi':300})
sns.set_context('notebook')
sns.set_style("ticks")
```

Objectives

- To estimate probability density via histrograms.

Estimating the probability density function via histograms

As before take X to be a random variable and $Y = g(X)$ a function of X. We wish to approximate from samples the probability density $p(y)$ of $Y = g(X)$. We start by spliting the domain of y into M small bins. Assume that these bins have bounds $b_0, b_1, \dots, b_M$. That is, the first bin is $[b_0, b_1]$, the second one is $[b_1, b_2]$, etc. We will approximate $p(y)$ with a constant inside its bin. That is, the approximation is:

$$\hat{p}_M(y) = \sum_{j=1}^M c_j 1_{[b_{j-1}, b_j]}(y),$$

where the c_j's are constants to be determined. Each one of these constants is the probability that a sample of Y falls inside the bin, i.e.,

$$c_j = p(b_{j-1} \le Y \le b_j).$$

Of course, this probability can be written as

$$c_j = F(b_j) - F(b_{j-1}),$$

where $F(y)$ is the CDF of Y. Therefore, we can approximate the constants using our estimate of the CDF. In the notation of the previous section, we have that:

$$\bar{c}_{j,N} := \bar{F}_N(b_j) - \bar{F}_N(b_{j-1}) \rightarrow c_j \text{ a.s.}$$

Of course, this is nothing more but:

$$\bar{c}_{j,N} = \frac{\text{number of samples that fall in bin } [b_{j-1}, b_j]}{N}$$

Putting everything together, our estimate for the PDF $p(y)$ is:

$$\hat{p}_{M,N}(y) = \sum_{j=1}^M \bar{c}_{j,N} 1_{[b_{j-1}, b_j]}(y),$$

which does converge to $p(y)$ (in some sense) as both N and M go to infinity.

Example: 1D CDF

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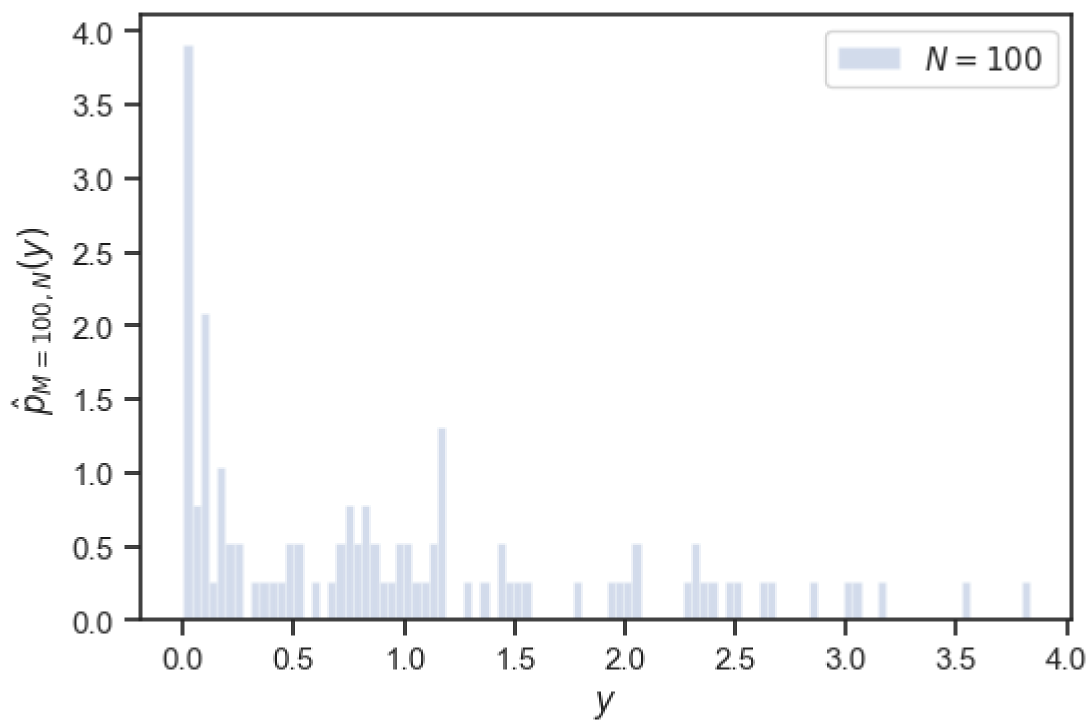
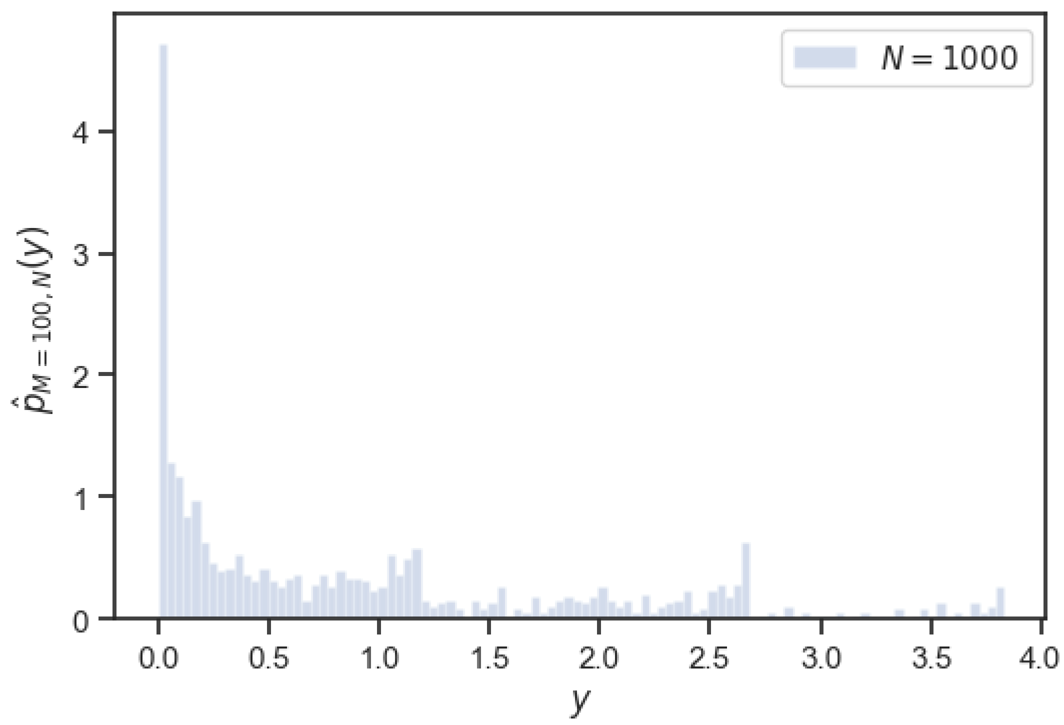
We will continue using the 1D test function of Example 3.4 of Robert & Casella (2004). Assume that $X \sim \mathcal{U}([0,1])$ and pick:

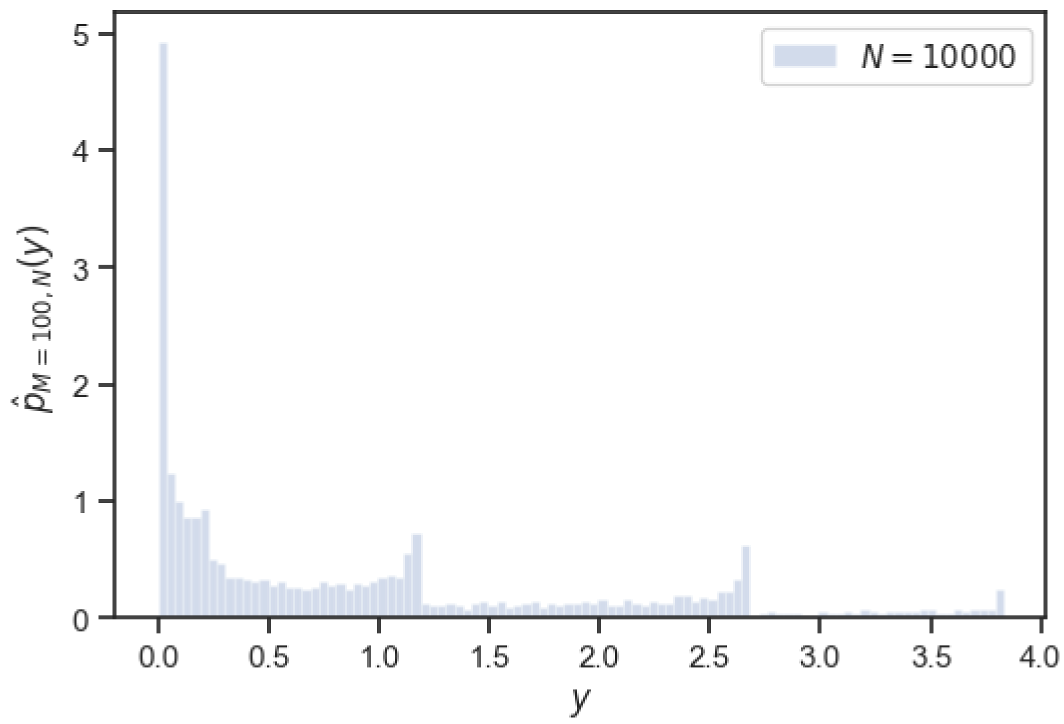
$$g(x) = (\cos(50x) + \sin(20x))^2.$$

```
# define the function here:
g = lambda x: (np.cos(50 * x) + np.sin(20 * x)) ** 2

# Again, we do not need to write any code for the histogram
# It's already implemented in several packages.
# We will use the matplotlib implementation

# Maximum number of samples to take
max_n = 10000
# The number of bins
num_bins = 100
# Generate samples from X
x_samples = np.random.rand(max_n)
# Get the corresponding Y's
y_samples = g(x_samples)

# Make the plot
for N in [100, 1000, max_n]:
    fig, ax = plt.subplots()
    ax.hist(
        y_samples[:N],
        label=f"$N={N:d}$",
        bins=num_bins,
        density=True,
        alpha=0.25
    )
    ax.set_xlabel(r"$y$")
    ax.set_ylabel(r"$\hat{{p}}_{{M={0:d},N}}(y)$".format(num_bins))
    plt.legend(loc="best");
```

For each value in the set of 3 elements







Questions

- Experiment with the number of bins M. Repeat the code above with $M = 5, 10$ and 1000. What do you observe? What happens when you have to few bins? What happens when you have to many bins? You should pick the number of bins and N together. As a rule of thumb N should be about ten times M. For a given choice of M, it is possible to pick how many N's you need using what we will learn in lecture 10.

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