Lecture 5: Collections of Random Variables

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Independent random variables

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Independent random variables

- We say that the two random variables are independent conditional on I, and we write:

$$X \perp Y \mid I$$

if and only if conditioning on one does not tell you anything about the other, i.e.,

$$p(x \mid y, I) = p(x \mid I)$$

- When there is no ambiguity, we can drop I.

$$p(x \mid y) = p(x)$$

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Independent random variables

- It is easy to show using Bayes' rule that the definition is consistent.

$$X \perp Y \mid I \Rightarrow Y \perp X \mid I$$

- That is, if Y does not give you any information about X, then X does not give you any information about Y, i.e.,

$$\underline{p(x|y)} = p(x)$$

then it follows that $\Rightarrow$

$$p(y|x) = p(y).$$

Proof: $p(y|x) = \frac{p(x,y)}{p(x)} \underset{\text{Bayes rule}}{=} \dots = p(y)$

$$p(x|y) = \frac{p(x,y)}{p(y)} \rightarrow p(x,y) = p(x|y)p(y) = p(x)p(y)$$

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Properties of independent random variables

- Assume X and Y are independent.
- Then, the joint pdf factorizes:

$$\boxed{p(x,y) = p(x)p(y).}$$

Proof: $X \perp Y \Rightarrow p(x|y) = p(x) \Leftrightarrow p(y|x) = p(y)$

$X \perp Y$

Bays' rule: $p(x,y) = \underline{p(x|y)}\, p(y) = p(x)\, p(y)$ ∎

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Properties of independent random variables

- Assume X and Y are independent.

- The expectation of the product is the product of the expectation:

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

Proof:

$$\mathbb{E}[XY] = \iint xy\, \underline{p(x,y)}\, dx\, dy = \iint xy\, p(x)\, p(y)\, dx\, dy$$

$$= \left(\int x\, p(x)\, dx\right) \cdot \left(\int y\, p(y)\, dy\right) \quad \text{(Fubini's Theorem)}$$

$$= \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

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Properties of independent random variables

- Assume X and Y are independent.

- The covariance is zero:

$$
\mathbb{C}[X,Y] = 0.
$$

Proof: $\mathbb{C}[X,Y] = \mathbb{E}[(X-\mathbb{E}[X])\cdot(Y-\mathbb{E}[Y])]$

$$
= \mathbb{E}[XY - X\cdot\mathbb{E}[Y] - \mathbb{E}[X]\cdot Y + \mathbb{E}[X]\cdot\mathbb{E}[Y]]
$$

$$
= \mathbb{E}[XY] - \mathbb{E}[X\underbrace{\mathbb{E}[Y]}_{\text{const.}}] - \mathbb{E}[\underbrace{\mathbb{E}[X]}_{\text{const.}}\cdot Y] + \mathbb{E}[\underbrace{\mathbb{E}[X]\cdot\mathbb{E}[Y]}_{\text{const.}}]
$$

$$
= \underbrace{\cancel{\mathbb{E}[XY]}}_{\substack{? \\ \| \\ \mathbb{E}[X]\cdot\mathbb{E}[Y] \\ (X\perp Y)}} - \cancel{\mathbb{E}[Y]\cdot\mathbb{E}[X]} - \cancel{\mathbb{E}[X]}\cdot\mathbb{E}[Y] + \cancel{\mathbb{E}[X]\cdot\mathbb{E}[Y]}
$$

$$
= 0
$$

$X \perp Y \overset{\text{implies}}{\Rightarrow}$ uncorrelated.

does not imply

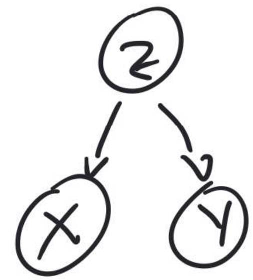
$\mathbb{C}[X,Y] = 0$

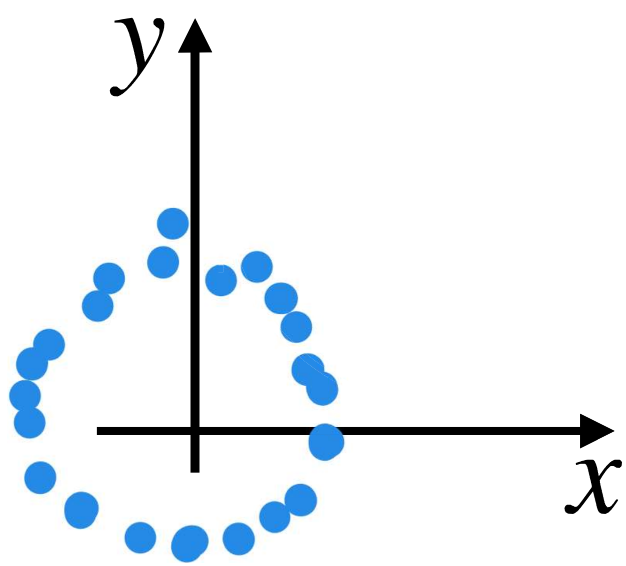
$\boxed{\text{The reverse is not true!}}$ Uncorrelated variables do not have to be independent

$Z \sim U([0, 2\pi])$

$X = \cos Z$

$Y = \sin Z$





$\mathbb{C}[X, Y] = 0$,

but X and Y are very dependent

Properties of independent random variables

- Assume X and Y are independent.

- The variance of the sum of two independent random variables is the sum of the variance of the random variables:

Must remember

$$\boxed{\mathbb{V}[X+Y] = \mathbb{V}[X] + \mathbb{V}[Y].}$$

Proof: $\mathbb{V}[X+Y] = \mathbb{V}[X] + \mathbb{V}[Y] + 2\,\underbrace{\mathbb{C}[X,Y]}_{0}$

$= \mathbb{V}[X] + \mathbb{V}[Y]. \blacksquare$

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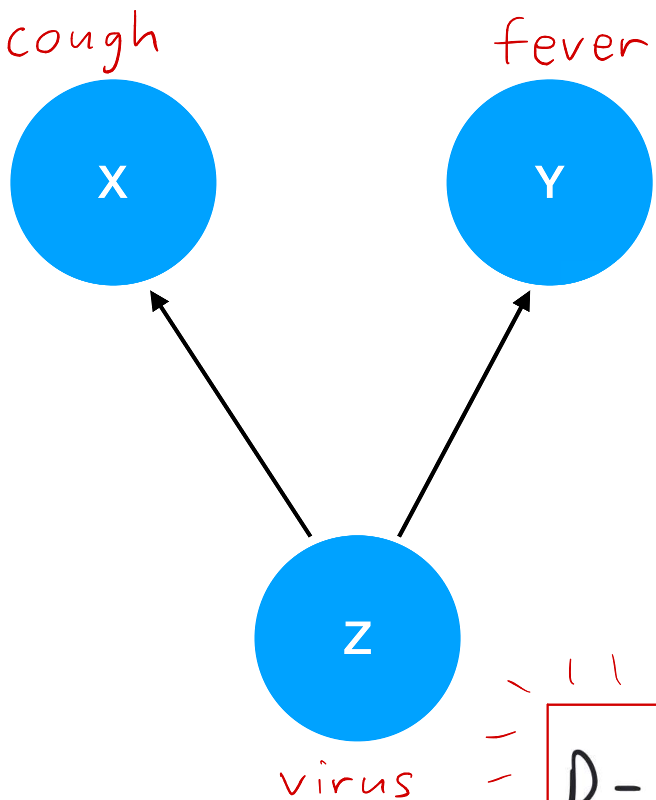
Reading independence from causal graphs



- X and Y are not independent in general.

  $p(x,y) \neq p(x)p(y)$

- But X is independent of Y conditioned on Z.

  $p(x,y|z) = p(x|z)p(y|z)$

$\boxed{\text{D-separation (advanced)}}$

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$$p(X=0, Y=0) = p(X=0)p(Y=0) = (0.3)(0.3) = 0.09$$

$$p(X=0, Y=1) = p(X=0)p(Y=1) = (0.3)(0.7) = 0.21$$

$$p(X=1, Y=1) = p(X=1)p(Y=1) = (0.7)(0.7) = 0.49$$

$$E[XY] = E[X]E[Y]$$

$$= \sum_x x p(x) \sum_y y p(y)$$

$$= \left[0(0.3) + 1(0.7)\right]\left[0(0.3) + 1(0.7)\right]$$

$$= (0.7)(0.7) = 0.49$$