Lecture 6: Random Vectors

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The multivariate normal - full covariance case

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Multivariate normal - full covariance case

- The random vector $\mathbf{X}$ follows a multivariate normal with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, and we write:

$$\mathbf{X} \sim N(\underline{\mu}, \underset{\text{matrix}}{\underline{\underline{\Sigma}}})$$

if the joint PDF is given by:

Remember

$$\boxed{p(\mathbf{x}) = (2\pi)^{-N/2} |\boldsymbol{\Sigma}|^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}}$$

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Multivariate normal - full covariance case

- Of course, if we carried out the appropriate integrals we would find:

$$\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$$

and covariance:

$$\mathbb{C}[\mathbf{X}, \mathbf{X}] = \mathbf{\Sigma}$$

must be symmetric

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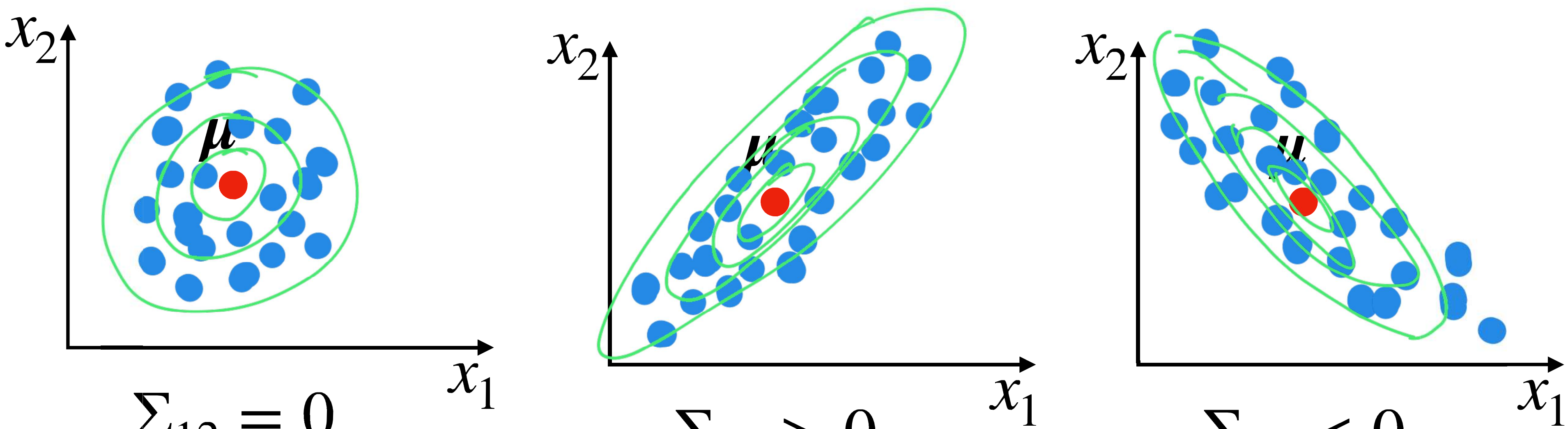
Visualizing the joint PDF of the multivariate normal with diagonal covariance

σ^2 - keep constant for this example

$$X \sim N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\right)$$

Σ_{12}



$\Sigma_{12} = 0$ uncorrelated.

$\Sigma_{12} > 0$ pos. corr.

$\Sigma_{12} < 0$ neg. corr.

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Restrictions of the covariance matrix

$$p(\mathbf{x}) = (2\pi)^{-N/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

Hessian

$$\nabla_x^2 \log p(x) \underset{\text{proportional to}}{\propto} -\Sigma$$

is negative definite

- The covariance matrix has to be positive definite, i.e., for any $\mathbf{v} \neq \mathbf{0}$, we must have:

$$\underbrace{\underset{1\times N}{\mathbf{v}^T}\underset{N\times N}{\boldsymbol{\Sigma}}\underset{N\times 1}{\mathbf{v}}}_{1\times 1} > 0$$

- This is so that $p(\mathbf{x})$ has a global maximum. - single peak

- Equivalently, $\boldsymbol{\Sigma}$ must have positive eigenvalues.

Connection to the standard normal

- Let $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$ be a collection of independent standard normal random variables.

$z_i \sim N(0, 1)$

python does this internally

- Define the random vector:

$$\mathbf{X} = \overbrace{\boldsymbol{\mu} + \mathbf{A}\mathbf{Z}}^{\text{affine transformation}}$$

$N \times N$

- Then:

$$\mathbf{X} \sim N(\underline{\mu}, \underline{\underline{A}} \cdot \underline{\underline{A}}^T) \quad ①$$

$$\mathbb{E}[\underline{X}] = \mathbb{E}[\underline{\mu} + \underline{\underline{A}} \cdot \underline{Z}] = \underline{\mu} + \underline{\underline{A}} \cdot \mathbb{E}[\underline{Z}] = \underline{\mu}$$

$$\mathbb{C}[\underline{X}, \underline{X}] = \mathbb{E}[(\underline{X} - \mathbb{E}[\underline{X}]) \cdot (\underline{X} - \mathbb{E}[\underline{X}])^T] = \dots = \underline{\underline{A}} \cdot \underline{\underline{A}}^T$$

substitute with $\underline{\mu}$

$\mu + AZ$