The Multivariate Normal - Conditioning

Contents


```
import numpy as np
np.set_printoptions(precision=3)
import matplotlib.pyplot as plt
%matplotlib inline
import seaborn as sns
sns.set(rc={"figure.dpi":100, 'savefig.dpi':300})
sns.set_context('notebook')
sns.set_style("ticks")
```

Objectives

- To demonstrate conditioning of a multivariate normal.

The multivariate mormal - Conditioning

Consider the N-dimensional multivariate normal:

$$\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\mu}$ is a N-dimensional vector, $\boldsymbol{\Sigma}$ is a *positive-definite matrix*. Assume that $\boldsymbol{\mu}$ can be decomposed in two blocks of dimensions N_1 and N_2 $(N_1 + N_2 = N)$:

$$\boldsymbol{\mu} = \begin{pmatrix}\boldsymbol{\mu}_1\\ \boldsymbol{\mu}_2\end{pmatrix}.$$

Similarly for $\boldsymbol{\Sigma}$:

$$\boldsymbol{\Sigma} = \begin{pmatrix}\boldsymbol{\Sigma}_1 & \boldsymbol{\Sigma}_{12}\\ \boldsymbol{\Sigma}_{12}^T & \boldsymbol{\Sigma}_2\end{pmatrix},$$

where $\boldsymbol{\Sigma}_{ii}$ are $N_i \times N_i$ matrices, and $\boldsymbol{\Sigma}_{12}$ is a $N_1 \times N_2$ matrix. In lecture, we saw that when you observe that $\mathbf{X}_2 = \mathbf{x}_2$, then $\mathbf{X}_1$ is distributed according to:

$$\mathbf{X}_1|\mathbf{X}_2 = \mathbf{x}_2 \sim N(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_2^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}_{12}^T).$$

Let's demonstrate this first for the case of one-dimensional x_1 and x_2.

```
import scipy.stats as st

# This is the multivariate normal we are going to play with
X = st.multivariate_normal(
    mean=np.array([1.0, 2.0]),
    cov=np.array(
        [
            [2.0, 0.9],
            [0.9, 4.0]
        ]
    )
)

print("X ~ N(mu, Sigma),")
print(f"mu = {X.mean}")
print("Sigma = ")
print(X.cov)
print("")

x2_observed = -2.0
print(f"x_2 = {x2_observed:.2f} (hypothetical observation)")
```

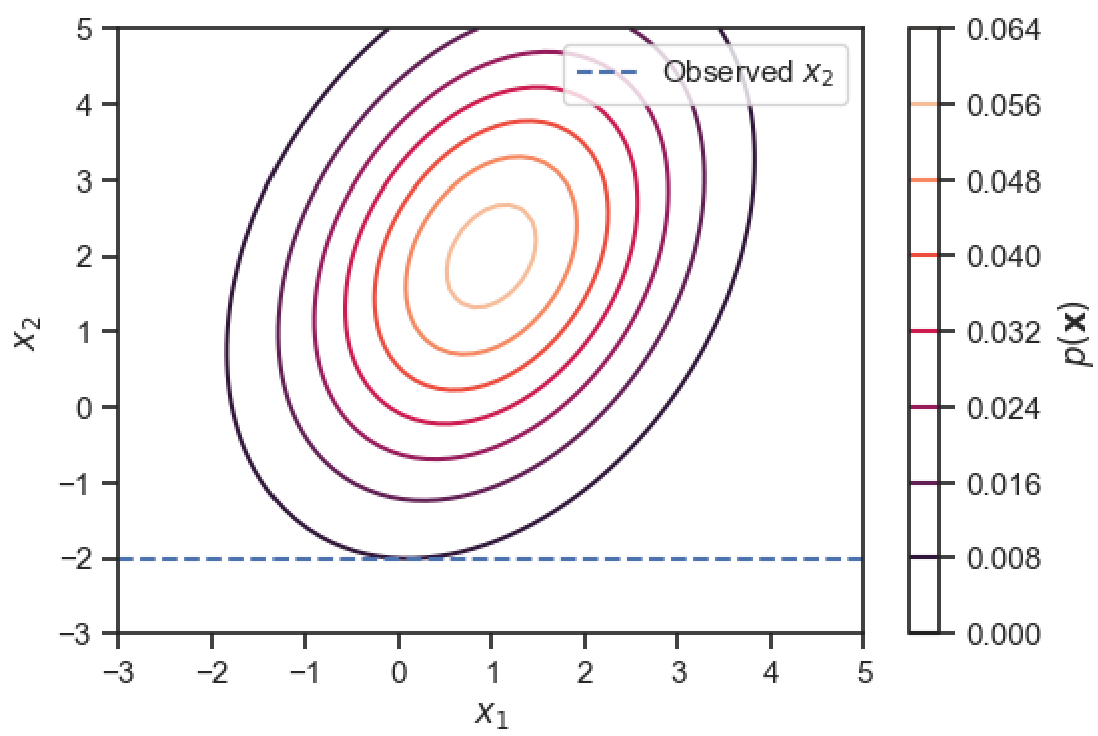
```
X ~ N(mu, Sigma),
mu = [1. 2.]
Sigma =
[[2.  0.9]
 [0.9 4. ]]

x_2 = -2.00 (hypothetical observation)
```

Let's plot the contour of the joint and see where x_2 falls:

```
fig, ax = plt.subplots()
x1 = np.linspace(-3, 5, 64)
x2 = np.linspace(-3, 5, 64)
Xg1, Xg2 = np.meshgrid(x1, x2)
Xg_flat = np.hstack(
    [
        Xg1.flatten()[:, None],
        Xg2.flatten()[:, None]
    ]
)
Z = X.pdf(Xg_flat).reshape(Xg1.shape)
c = ax.contour(Xg1, Xg2, Z)
plt.colorbar(c, label=r"$p(\mathbf{x})$")
ax.plot(
    x1,
    [x2_observed] * np.ones(x1.shape[0]),
    "--",
    label=r"Observed $x_2$")
plt.legend(loc="best")
ax.set_xlabel('$x_1$')
ax.set_ylabel('$x_2$');
```

Follows same steps for plotting contours as before

Plotting line corresponding to observed X2 value



So, intuitively, conditioned on known $X_2 = x_2$ the values of X_1 must have higher probability in the intersection of the dashed line with the contours of the joint.

Let's see what is the answer we get from the theory. We need to calculate the mean and variance of x_1 conditional on observing x_2. Because x_1 is one dimensional, it is very simple to implement the formula we have above.

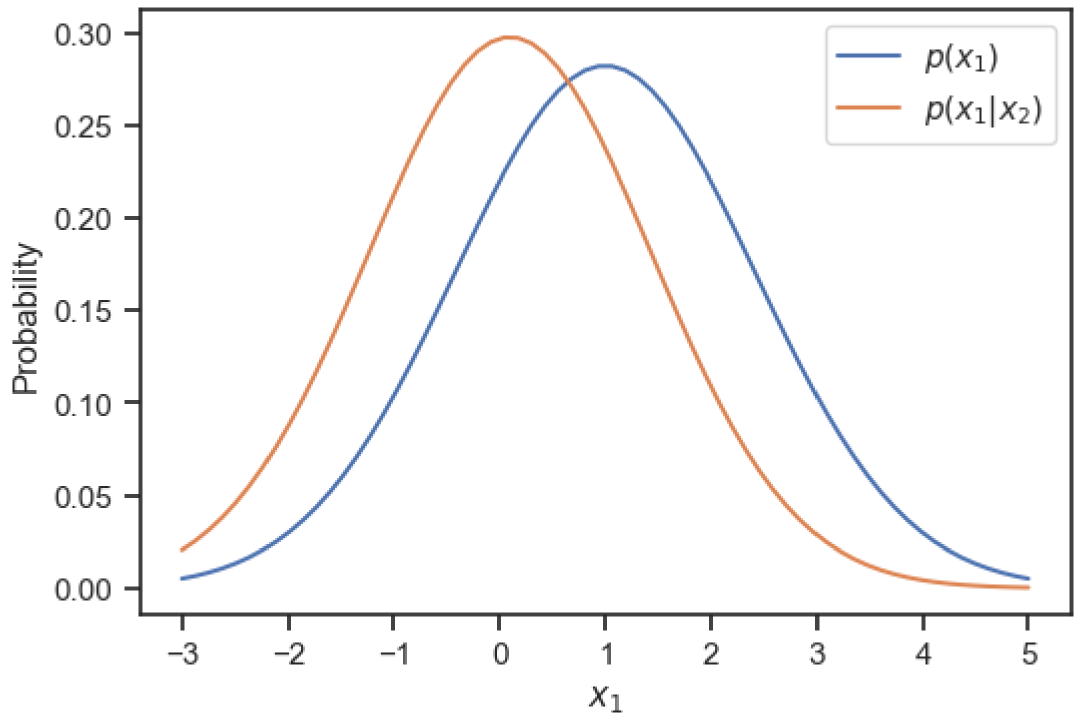
```
Sigma11 = X.cov[0, 0]
Sigma12 = X.cov[0, 1]
Sigma22 = X.cov[1,1]

mu1 = X.mean[0]
mu2 = X.mean[1]

mu1_cond = mu1 + Sigma12 * (x2_observed - mu2) / Sigma22

Sigma11_cond = Sigma11 - Sigma12 ** 2 / Sigma22

print(f"x_1 | x_2 ~ N(mu = {mu1_cond:.2f}, sigma^2 = {Sigma11_cond:.2f})")
```

Using equation at the top of this document

```
x_1 | x_2 ~ N(mu = 0.10, sigma^2 = 1.80)
```

Let's plot this conditional pdf for x_1 and compare it to its marginal pdf:

```
X1_cond = st.norm(
    loc=mu1_cond,
    scale=np.sqrt(Sigma11_cond)
)
X1_marg = st.norm(
    loc=X.mean[0],
    scale=np.sqrt(Sigma11)
)
fig, ax = plt.subplots()
ax.plot(
    x1,
    X1_marg.pdf(x1),
    label=r"$p(x_1)$"
)
ax.plot(
    x1,
    X1_cond.pdf(x1),
    label=r"$p(x_1|x_2)$"
)
ax.set_xlabel(r"$x_1$")
ax.set_ylabel("Probability")
plt.legend(loc="best");
```



This is our first example of how Bayes' rule can be used to condition on observations. In the plot above, you can think of $p(x_1)$ as the best think you can say about x_1 before you see any data. Then, you see an observation of x_2. Because x_1 and x_2 are correlated, your state of knowledge about x_1 changes. This is captured by the conditional $p(x_1|x_2)$.

because their covariance is non-zero

Questions

- Rerun the code above multiple times to see how the conditinal PDF moves around as other points are picked randomly.
- Modify the code so that you get the conditional PDF of X_2 given $X_1 = x_1$.

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