Lecture 20: State-space models - Kalman filters

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Derivation of Kalman filter - Predict

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Derivation of Kalman filter - Predict

$\underline{0}$: $p(x_0) = \mathcal{N}(\mu_0, V_0)$

$\underline{1}$: $p(x_1 \mid u_0) \overset{\text{sum rule}}{=} \int \underbrace{p(x_1 \mid x_0, u_0)}_{G}\, \underbrace{p(x_0)}_{G}\, dx_0$ both known

$\mathcal{N}(Ax_0 + Bu_0, Q)$ $\quad \mathcal{N}(\mu_0, V_0)$

$= \mathcal{N}(x_1 \mid A\mu_0 + Bu_0 \mid AV_0A^T + Q)$

$x_1 = Ax_0 + Bu_0 + z_1, \quad z_1 \sim \mathcal{N}(0, Q)$

$\mathbb{E}[x_1] = \mathbb{E}[Ax_0 + Bu_0] = \mathbb{E}[Ax_0] + \mathbb{E}[\overset{Bu_0}{Bu_0}]$

$= A \cdot \mathbb{E}[x_0] + Bu_0 = A\mu_0 + Bu_0$

$\mathbb{C}[x_1] = \mathbb{C}[Ax_0 + \underbrace{Bu_0}_{\text{scalar}} + z_1]$

$= \mathbb{C}[Ax_0 + z_1] = \underbrace{\mathbb{C}[Ax_0]}_{AV_0A^T} + \underbrace{\mathbb{C}[z_1]}_{Q} + \underbrace{\mathbb{C}[Ax_0, z_1]}_{0}$ independent

can predict first state before we observe anything

. . .

at state n: $\quad p(x_{n-1} \mid y_{1:n-1}, u_{0:n-2}) = \mathcal{N}(x_{n-1} \mid \mu_{n-1}, V_{n-1})$

Assume you know this!

$p(x_n \mid \underbrace{y_{1:n-1}, u_{1:n-2}}_{\text{everything we have}}, \underbrace{u_{n-1}}_{\text{control we're about to apply}}) \overset{\text{sum rule}}{=} \int \underbrace{p(x_n \mid x_{n-1}, u_{n-1})}_{G,\ \text{transition}}\, \underbrace{p(x_{n-1} \mid y_{1:n-1}, u_{0:n-2})}_{G}\, dx_{n-1}$

$= \mathcal{N}(x_n \mid A\mu_{n-1} + Bu_{n-1} \mid AV_{n-1}A^T + Q)$

$x_n = Ax_{n-1} + Bu_{n-1} + z_n, \quad z_n \sim \mathcal{N}(0, Q)$

$\mathbb{E}[x_n] = A\mu_{n-1} + Bu_{n-1}; \quad \mathbb{C}[x_n] = AV_{n-1}A^T + Q$

If $p(x_{n-1} \mid y_{1:n-1}, u_{0:n-2}) = \mathcal{N}(x_{n-1} \mid \mu_{n-1}, V_{n-1})$

Then <u>Predict</u>:

$$p(x_n \mid y_{1:n-1}, u_{0:n-2}, u_{n-1}) = \mathcal{N}(x_n \mid A\mu_{n-1} + Bu_{n-1}, AV_{n-1}A^T + Q)$$

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