Lecture 5: Collections of Random Variables

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Joint probability density function

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Discrete rvs: Joint probability mass function

- Consider two discrete random variables X and Y.

- The **joint probability mass function** of the pair (X, Y) is the function $p(x, y)$ giving the probability that $X = x$ and $Y = y$:

$$p(x,y) = p(X=x, Y=y)$$

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Properties of the joint pmf

- It is nonnegative:

$$p(x,y) \geq 0$$

- If you sum over all the possible values of all random variables, you should get one:

$$\sum_x \sum_y p(x,y) = 1$$

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Properties of the joint pmf

- If you marginalize over the values of one of the random variables you get the pmf of the other:

$$\sum_y p(x,y) = p(x)$$

(Sum rule: $p(A) = \sum_i p(A, B_i)$, where $p(B_1 \text{ or } B_2 \text{ or} \dots) = 1$, $p(B_i, B_j) = 0$, $i \neq j$.)

$A = \{X = x\}$, $B_i = \{Y = y_i\}$

- and

$$\sum_x p(x,y) = p(x)$$

$A_i = \{X = x_i\}$, $B = \{Y = y\}$

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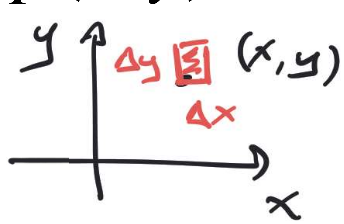
Continuous rvs: Joint probability density function

- Consider two continuous random variables X and Y.
- The **joint probability density function** of the pair (X, Y) is the function $p(x, y)$ giving the probability that $X = x$ and $Y = y$:

$$p(x, y) \approx \frac{p(x \leq X \leq x + \Delta x, y \leq Y \leq y + \Delta y)}{\Delta x \Delta y}$$



$$, \quad p(x,y) \geq 0 \quad , \quad \iint p(x,y)\,dx\,dy = 1$$

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Properties of the jpdf

- If you **marginalize** over the values of one of the random variables you get the pdf of the other:

$$\int p(x,y)\,dy = p(x)$$

and

$$\int p(x,y)\,dx = p(y)$$

Note on notation

- We will not distinguish between the notation of discrete and continuous random variables.

$$p(x)\begin{cases}\text{pdf}\\ \text{pmf}\end{cases},\quad p(x,y)\begin{cases}\text{jpdf}\\ \text{jpmf}\\ \ldots\text{or a mix}\end{cases}$$

- We will always use the integral sign to indicate marginalization understanding that it is a summation over all possible values if we have a discrete random variable.

$$\int \cdot\, dx \begin{cases}\sum_x & \text{if } X \text{ disrete}\\ \int \cdot\, dx & \text{if } X \text{ continuous.}\end{cases}$$

- We will only say joint pdf instead of joint pmf.

$$\text{pdf}\begin{cases}\text{pmf} & \text{if } X \text{ discrete}\\ \text{pdf} & \text{if } X \text{ is cont.}\end{cases}$$

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Conditioning random variables on one another

- Take two random variables X and Y with joint pdf $p(x, y)$.

- Suppose that you observe $Y = y$ and you want to update your state of knowledge about X.

$$p(A|B) = \frac{p(A,B)}{p(B)}$$

- The **conditional pdf** gives you this info:

$A = \{X = x\}, \quad B = \{Y = y\}$ – select events to assign to A and B

$$p(X=x \mid Y=y) = \frac{p(X=x, Y=y)}{p(Y=y)}$$

$$\boxed{p(x \mid y) = \frac{p(x,y)}{p(y)}}$$

The expectation of a sum of random variables

- Take two random variables X and Y with joint pdf $p(x,y)$.

- The expectation of their sum is:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Proof: $\mathbb{E}[X+Y] = \iint (x+y)\,p(x,y)\,dx\,dy$

$$= \iint x\,p(x,y)\,dx\,dy + \iint y\,p(x,y)\,dx\,dy$$

$$= \int x\Big(\underbrace{\int p(x,y)\,dy}_{p(x)}\Big)dx + \int y\Big(\underbrace{\int p(x,y)\,dx}_{p(y)}\Big)dy$$

$$= \int x\,p(x)\,dx + \int y\,p(y)\,dy = \mathbb{E}[X] + \mathbb{E}[Y]$$

	$X=0$	$X=1$
$Y=0$	0.1	0.2
$Y=1$	0.2	0.5

$$\sum_{xy} p(x,y) = 1$$

$$\sum_{y} p(x,y) = p(x) = \{0.3, 0.7\} \rightarrow E[X] = 0(0.3) + 1(0.7) = 0.7$$

$$\sum_{x} p(x,y) = p(y) = \{0.3, 0.7\} \rightarrow E[Y] = 0(0.3) + 1(0.7) = 0.7$$

$$E[X + 2Y] = E[X] + 2E[Y] = 2.1$$

$$p(x|y) = \frac{p(x,y)}{p(y)}$$

$$p(X=x|Y=0) = \frac{p(X=x, Y=0)}{p(Y=0)}$$

$$\hookrightarrow p(X=0|Y=0) = \frac{p(X=0, Y=0)}{p(Y=0)} = \frac{0.1}{0.3} = 0.333$$

$$\hookrightarrow p(X=1|Y=0) = \frac{p(X=1, Y=0)}{p(Y=0)} = \frac{0.2}{0.3} = 0.667$$

$$E[X|Y=0] = \sum_{x} x p(x|Y=0) = 0(0.333) + 1(0.667) = 0.667$$