Lecture 4: Continuous Random Variables

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The Gaussian distribution

interchangable

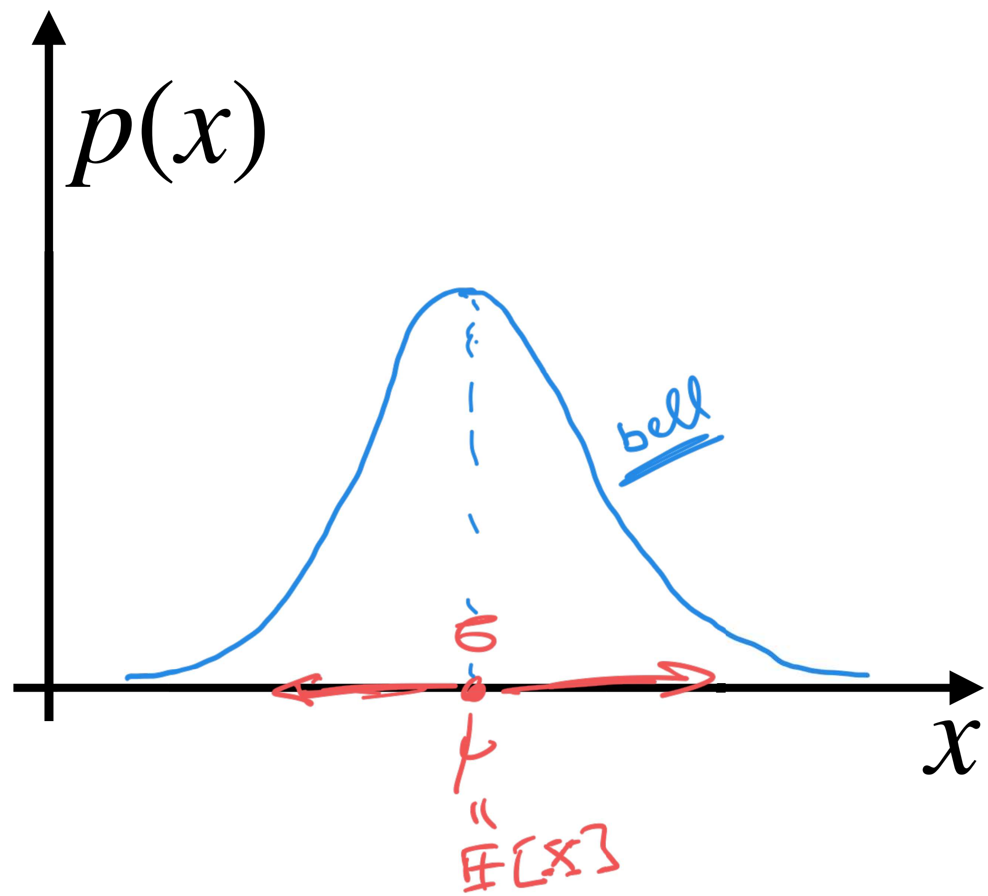
The Gaussian (or normal) distribution

- Models a random variable that takes values any real value.
- The values are concentrated around a value μ. (mean)
- The variance σ^2 determines how spread out the function values are around the mean.
- We write:

mean variance

$$X \sim N(\mu, \sigma^2)$$

Expectation $\equiv$ mean



The PDF and CDF of the normal distribution

- Consider:

$$X \sim N(\mu, \sigma^2)$$

- The PDF of the normal rises naturally from the central limit theory and the maximum entropy principle (later).

- The PDF is:

$$\boxed{p(x) = N(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

- The CDF is not analytically available.

$$F(x) = p(X \le x) = \int_{-\infty}^{x} N(\tilde{x}|\mu,\sigma^2)\,d\tilde{x}$$

dummy variable

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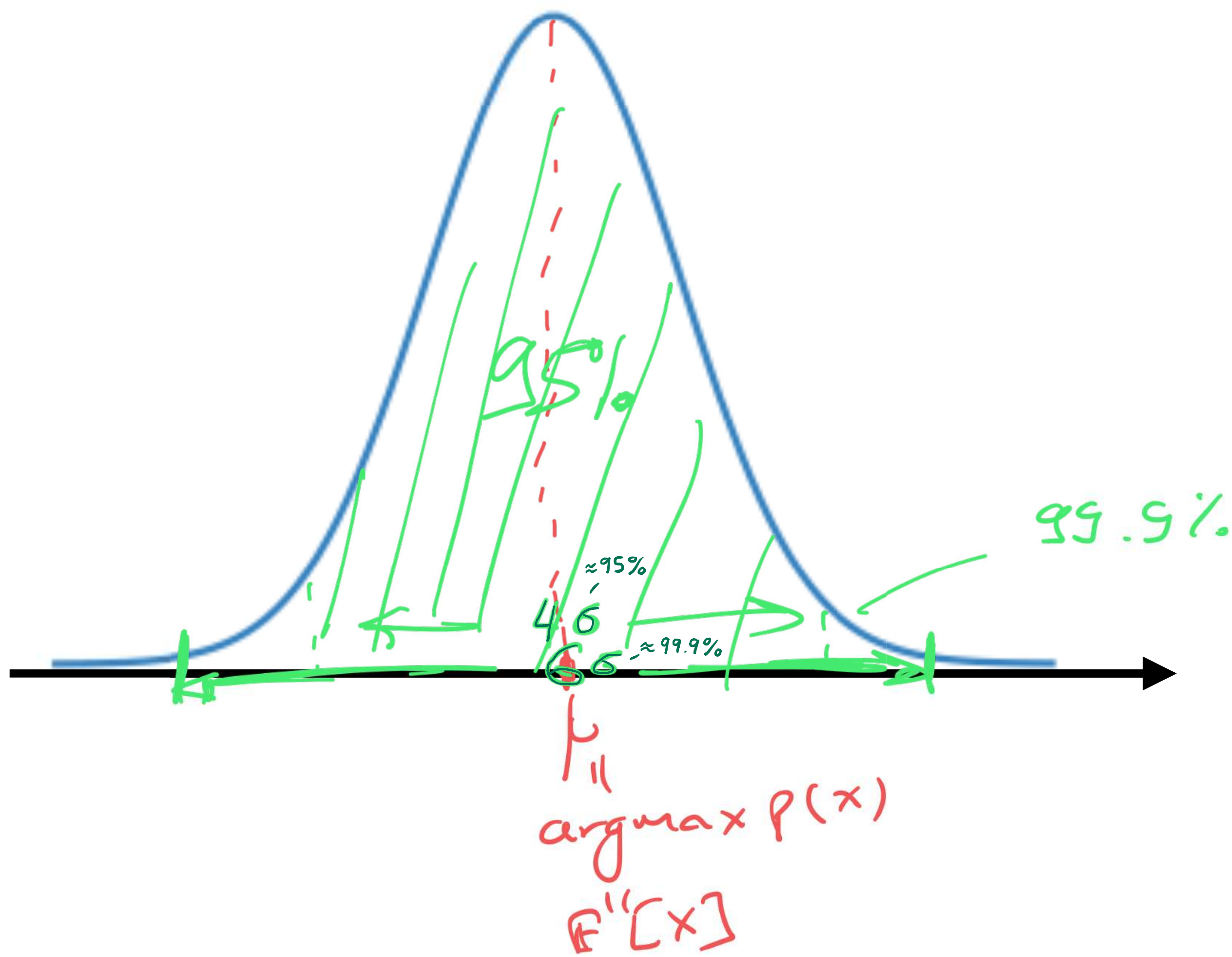
Some useful things to know about the normal



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The standard normal distribution

- Let

$$Z \sim N(0,1).$$

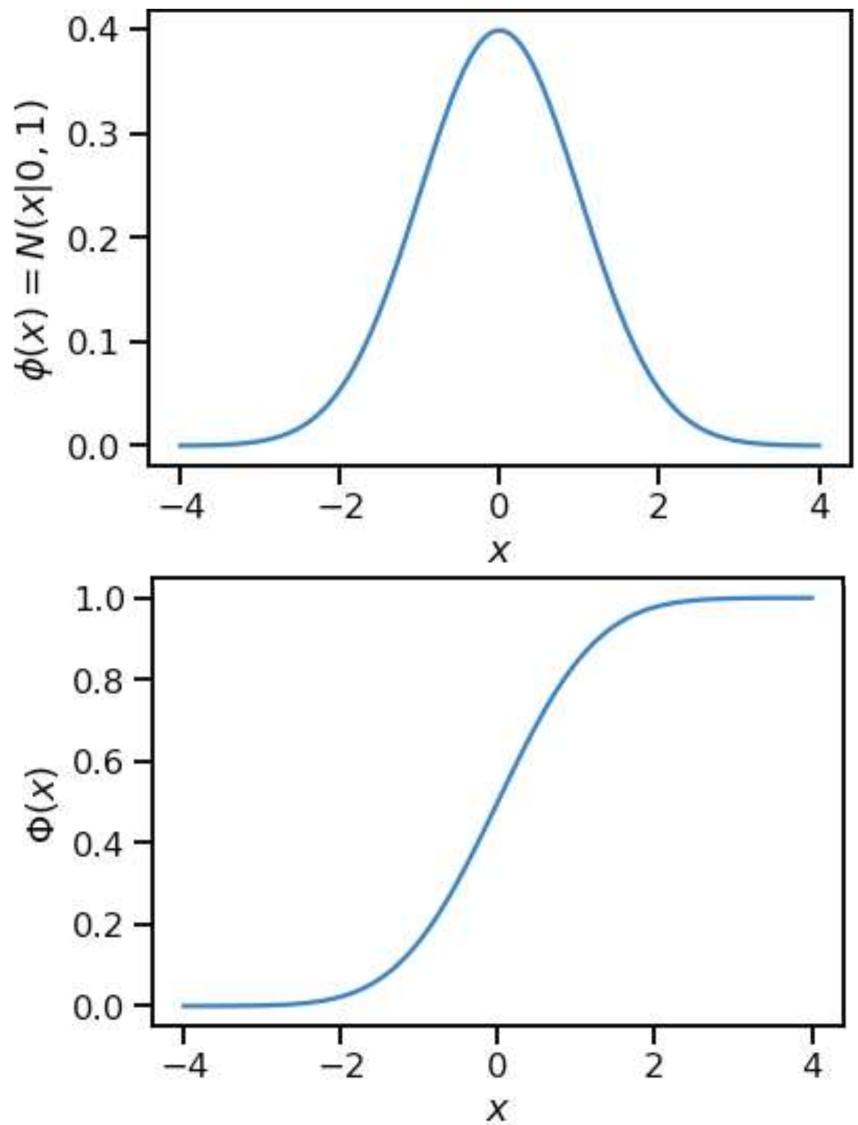
- The PDF of the standard normal is:

$$\phi(z) := N(z|0,1) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\}$$

- The CDF of the standard normal is:

$$\Phi(z) := \mathbb{P}(Z \le z) = \int_{-\infty}^{z} \phi(z')dz',$$

is also not analytically available.



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Connections between the normal and the standard normal

- Take a standard normal $Z \sim N(0,1)$ and two numbers μ and σ^2.
- Make the random variable $\boxed{X = \mu + \sigma Z.}$

can sample Z, then construct X

- Then $X \sim N(\mu, \sigma^2)$.

useful for generating arbitrary random samples

- Proof requires the change variables formula, but note:

$$\mathbb{E}[X] = \mathbb{E}[\mu + \sigma Z] = \mu + \mathbb{E}[\sigma Z] = \mu + \sigma \cancelto{0}{\mathbb{E}[Z]} = \mu$$

$$\mathbb{V}[X] = \mathbb{V}[\mu + \sigma Z] = \mathbb{V}[\sigma Z] = \sigma^2 \underbrace{\mathbb{V}[Z]}_{1} = \sigma^2$$

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Connections between the normal and the standard normal

can also go the other direction

- Take a normal $X \sim N(\mu, \sigma^2)$.

- Make the random variable $Z = \dfrac{X - \mu}{\sigma}$.

- Then $Z \sim N(0,1)$.

- Proof requires the change variables formula, but note:

$$\mathbb{E}[Z] = \mathbb{E}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\mathbb{E}[X-\mu] = \frac{1}{\sigma}\left\{\cancel{\mathbb{E}[X]}^{\mu} - \mu\right\} = 0.$$

$$\mathbb{V}[Z] = \mathbb{V}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma^2}\mathbb{V}[X-\mu] = \frac{1}{\sigma^2}\cancel{\mathbb{V}[X]}^{\sigma^2} = 1.$$

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Connections between the normal and the standard normal

- Take a normal $X \sim N(\mu, \sigma^2)$.
- We can now write:

CDF of $N(\mu,\sigma^2)$

$= $ CDF of $N(0,1)$

$$p(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- The proof is good practice of probability laws:

$$p(X \leq x) = p(X-\mu \leq x-\mu) = p\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$$

$\sim N(0,1)$

$$= p\left(Z \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

cdf of standard normal

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