Problem 3

$$\ddot{x} + 2\xi w_0 \dot{x} + w_0^2 x = u_0 \cos(wt)$$

$$\vec{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} ; \quad \dot{\vec{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -2\xi w_0 \dot{x} - w_0^2 x + u_0 \cos(wt) \end{bmatrix} ; \quad u =$$

$$\frac{dx}{dt} = \dot{x} \quad (1)$$

$$\frac{d\dot{x}}{dt} = -2\xi w_0 \dot{x} - w_0^2 x + u_0 \cos(wt) \quad (2)$$

Discretize Time (Transitions)

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = \dot{x}(t) \rightarrow x(t+\Delta t) = x(t) + \Delta t \dot{x}(t)$$

$$\frac{\dot{x}(t+\Delta t) - \dot{x}(t)}{\Delta t} = \ddot{x}(t) \rightarrow \dot{x}(t+\Delta t) = \dot{x}(t) + \Delta t \ddot{x}(t)$$

$$= \dot{x}(t) + \Delta t\left(-2\xi w_0 \dot{x}(t) - w_0^2 x(t) + u_0 \cos(wt)\right)$$

$$= -w_0^2 \Delta t\, x(t) + \dot{x}(t) - 2\Delta t \xi w_0 \dot{x}(t) + u_0 \Delta t \cos(wt)$$

↳ $\vec{x}_{j+1} = A\vec{x}_j + Bu_j + z_j$

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix}_{j+1} = \overset{A}{\begin{bmatrix} 1 & \Delta t \\ -w_0^2 \Delta t & 1 - 2\Delta t \xi w_0 \end{bmatrix}} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_j + \overset{B}{\begin{bmatrix} 0 \\ \Delta t \end{bmatrix}} u_j + z_j$$

$$u_j = u(j\Delta t)$$

$$= u_0 \cos(w(j\Delta t))$$

$$z_j \sim N(0, Q) ; \quad Q = \begin{bmatrix} \varepsilon & 0 \\ 0 & \sigma_q \end{bmatrix}$$

Discretize Time (Emissions):

$$y_j = C\vec{x}_j + w_j = \ddot{x}(j\Delta t) - u_0\cos(\omega t) + w_j$$

$$= -2\xi\omega_0\dot{x}(j\Delta t) - \omega_0^2 x(j\Delta t) + u_0\cos(\omega t) - u_0\cos(\omega t) + w_j$$

$$= \underbrace{-\omega_0^2 x(j\Delta t) - 2\xi\omega_0\dot{x}(j\Delta t)}_{C} + w_j$$

$$y_j = [-\omega_0^2 \quad -2\xi\omega_0]\vec{x}_j + w_j$$

$$w_j \sim N(0, R) \; ; \quad R = [\sigma_r]$$

$$\underset{1\times 2}{C}\;\overbrace{\underset{2\times 2}{\mathrm{cov}}\;\underset{2\times 1}{C^T}}^{X}$$

$$\underset{1\times 2\;2\times 1}{C X} = \underset{1\times 1}{\#}$$