Bayesian Parameter Estimation

Contents


```
import numpy as np
np.set_printoptions(precision=3)
import matplotlib.pyplot as plt
%matplotlib inline
import seaborn as sns
sns.set(rc={"figure.dpi":100, "savefig.dpi":300})
sns.set_context("notebook")
sns.set_style("ticks")
```

Objective

- Introduce Bayesian parameter estimation via an analytical example

Example: Inferring the probability of a coin toss from data

A coin with unknown probability of heads θ is tossed N times independently and you observe the result:

$$x_{1:N} := (x_1, \dots, x_N).$$

Assume that we have coded the result so that heads corresponds to a "0" and tails to a "1." Our goal is to estimate the probability of heads θ from this dataset.

Assuming that we know nothing, we set:

$$\theta \sim U([0,1]).$$

In terms of probability densities this:

$$p(\theta) = \begin{cases} 1, & \text{if } \theta \in [0,1], \\ 0, & \text{otherwise} \end{cases} = 1_{[0,1]}(\theta),$$ --> the prior

where we used the indicator function to simplify the notation.

Now, let's write down the likelihood of the data. Because of the independence assumption, we have:

$$p(x_{1:N}|\theta) = \prod_{n=1}^N p(x_n|\theta).$$

Then, each measurement is a Bernoulli with probability of success θ, i.e.,

$$x_n|\theta \sim \text{Bernoulli}(\theta).$$

In terms of probability densities, we have:

standard notation for pmf of a Bernoulli rv: $$p(x_n|\theta) = \begin{cases} \theta, & \text{if } x_n = 0, \\ 1-\theta, & \text{otherwise.} \end{cases}$$ which corresponds to heads as specified above --> the likelihood

Using a common mathematical trick, we can rewrite this as:

$$p(x_n|\theta) = \theta^{x_n}(1-\theta)^{1-x_n}.$$

Work out the cases $x_n = 0$ and $x_n = 1$ to convience yourself.

Now we can find the expression for the likelihood of the entire dataset. It is

$$
\begin{aligned}
p(x_{1:N}|\theta) = & \prod_{n=1}^N p(x_n|\theta) \\
= & \prod_{n=1}^N \theta^{x_n}(1-\theta)^{1-x_n} \\
= & \theta^{\sum_{n=1}^N x_n}(1-\theta)^{N-\sum_{n=1}^N}.
\end{aligned}
$$

This has the intuitive meaning that it is the probability of getting $\sum_{n=1}^N x_n$ heads and the rest $N - \sum_{n=1}^N x_n$ tails.

We can now find the posterior. It is:

$$
\text{posterior} \propto \text{likelihood} \times \text{prior}.
$$

In our problem:

$$
\begin{aligned}
p(\theta|x_{1:N}) \propto & \; p(x_{1:N}|\theta)p(\theta) \\
= & \; \theta^{\sum_{n=1}^N x_n}(1-\theta)^{N-\sum_{n=1}^N} 1_{[0,1]}(\theta) \\
= & \begin{cases} \theta^{\sum_{n=1}^N x_n}(1-\theta)^{N-\sum_{n=1}^N}, & \text{if } \theta \in [0,1] \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

And this is just the density corresponding to a [Beta distribution](https://en.wikipedia.org/wiki/Beta_distribution): https://en.wikipedia.org/wiki/Beta_distribution

$$
p(\theta|x_{1:N}) = \text{Beta}\left(\theta\middle|\underbrace{1+\sum_{n=1}^N x_n}_{\text{alpha}}, \underbrace{1+N-\sum_{n=1}^N x_n}_{\text{beta}}\right).
$$

Let's try this out with some fake data.

Take a fake coin which is a little bit biased:

```
import scipy.stats as st
theta_true = 0.8
X = st.bernoulli(theta_true)
```

Sample from it a number of times to generate our data = (x1, ..., xN):

```
N = 5
data = X.rvs(size=N)
data
```

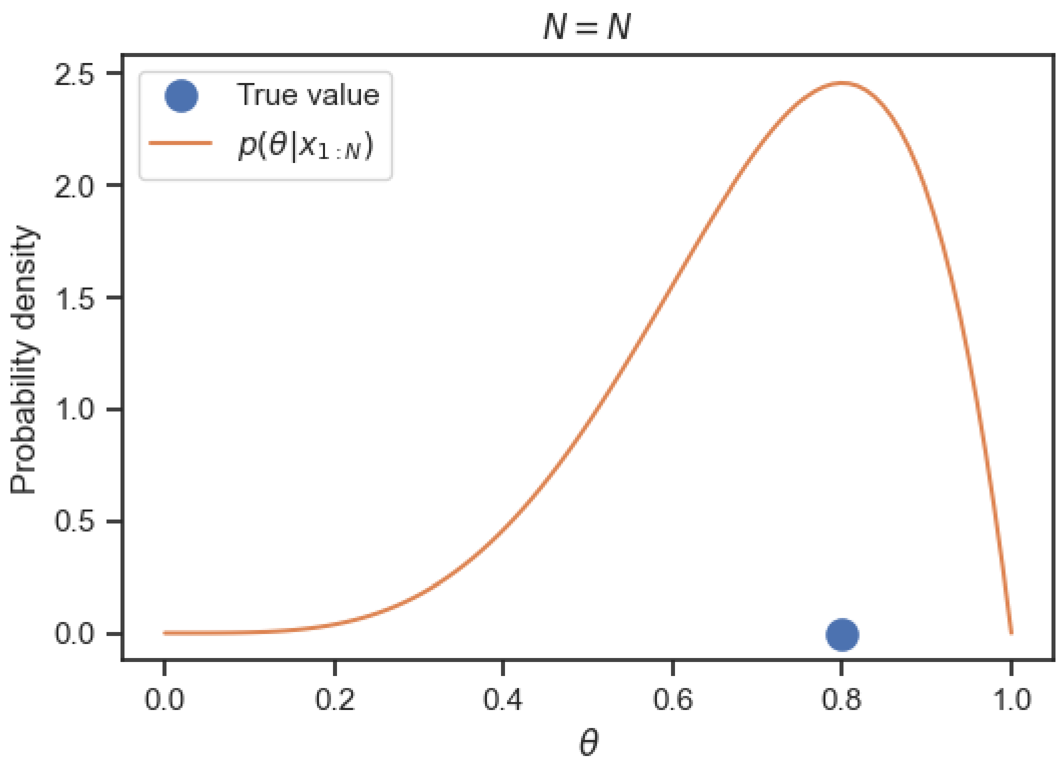
```
array([1, 1, 0, 1, 1])
```

Now we are ready to calculate the posterior which the Beta we have above:

```
alpha = 1.0 + data.sum()
beta = 1.0 + N - data.sum()
Theta_post = st.beta(alpha, beta)
```

And we can plot it:

```
fig, ax = plt.subplots()
thetas = np.linspace(0, 1, 100)
ax.plot(
    [theta_true],
    [0.0],
    'o',
    markeredgewidth=2,
    markersize=10,
    label='True value'
)
ax.plot(
    thetas,
    Theta_post.pdf(thetas),
    label=r'$p(\theta|x_{1:N})$'
)
ax.set_xlabel(r'$\theta$')
ax.set_ylabel('Probability density')
ax.set_title('$N={N}$')
plt.legend(loc='best');
```



Questions

- Try $N = 0, 5, 10, 100$ and see what happens.

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