Lecture 21: Gaussian process regression

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The effect of the covariance function function

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The effect of the covariance function - The Squared exponential (SE)

$d=1$: $x \in \mathbb{R}$

$$c(x,x') = s^2 \exp\left\{-\frac{(x-x')^2}{2\ell^2}\right\}$$

s^2: signal variance

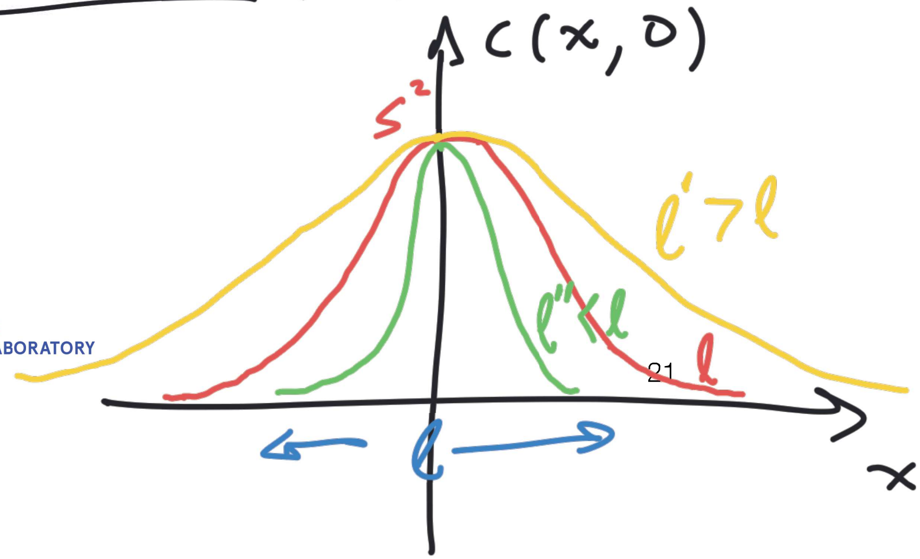
ℓ: lengthscale

this is not the traditional equation for covariance; it cannot be negative

$d>1$:

$$c(x,x') = s^2 \exp\left\{-\sum_{i=1}^{d}\frac{(x_i-x_i')^2}{2\ell_i^2}\right\}$$

x_i: i'th component of input



$c(0,0) = s^2$

$c(x,0) \to 0$, as $x \to \pm\infty$

the rate of which depends on the lengthscale ℓ

$$s^2: \mathbb{V}[f(x)] \overset{\text{def.}}{=} c(x,x) \overset{\text{SE}}{=} s^2$$

ℓ: $\ell \uparrow$ $c(x,x') \uparrow$

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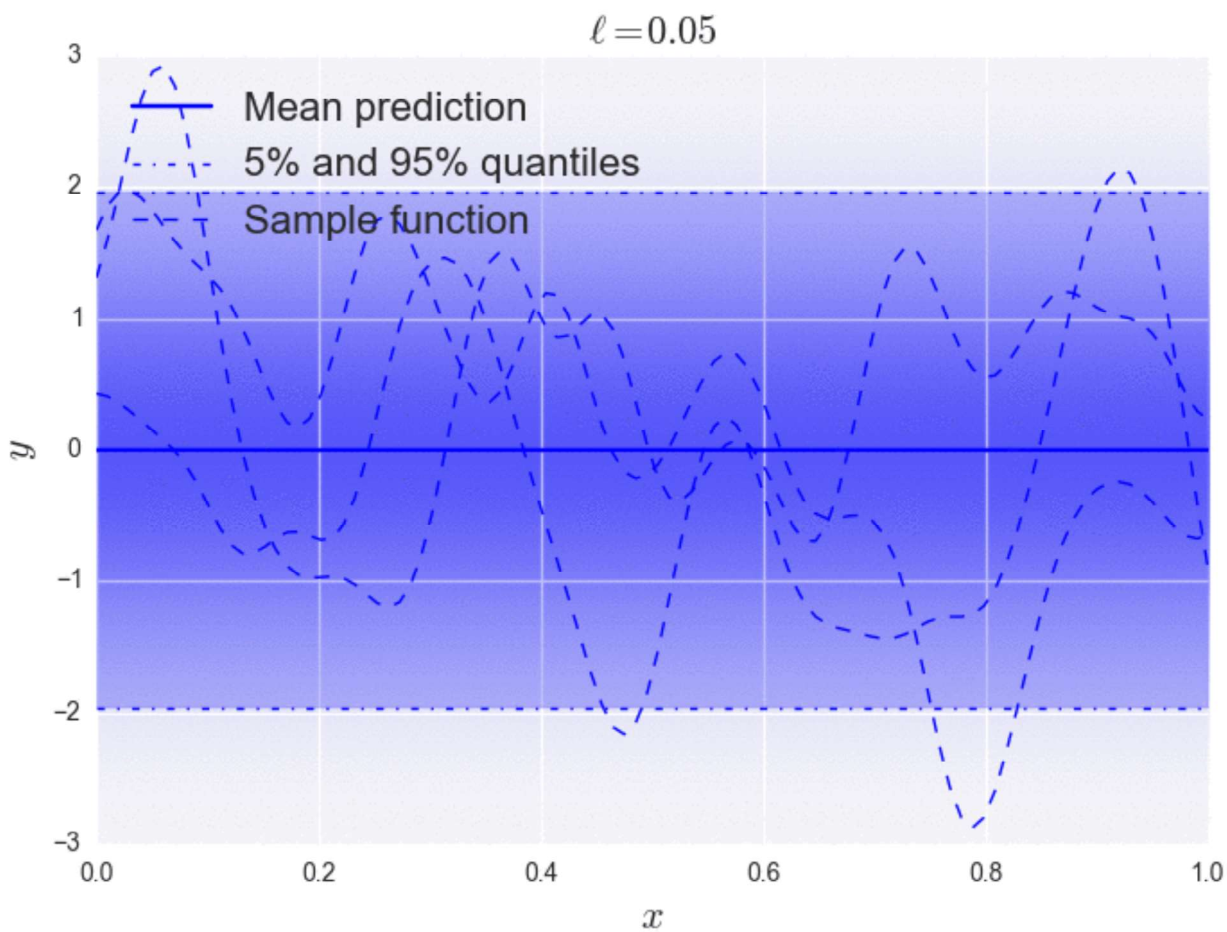
Changing the length scale



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The effect of the covariance function - Regularity

$f(\cdot) \sim GP(0, c(\cdot,\cdot))$

$\boxed{\text{Thm:}}$ Regularity of samples from the GP w/ cov. fun. c is the same as the regularity of $g(x) = c(x,x)$.

- f is continuous at x if $g(x) = c(x,x)$ is cont. at x.
- $\frac{\partial f}{\partial x_j}$ is continuous at x if $\frac{\partial^2 c(x,x)}{\partial x_j \partial x_j'}$ is cont. at x.
- $\frac{\partial^2 f}{\partial x_j \partial x_k}$ — // — if $\frac{\partial^4 c(x,x)}{\partial x_j \partial x_k \partial x_j' \partial x_k'}$ is cont. at x.

corresponding mixed derivative of the covariance function

SE cov. fun. : infinitely diff. $\Rightarrow$ f inf. diff.

defined above

↳ which $c(\cdot,\cdot)$ to use depends on underlying knowns

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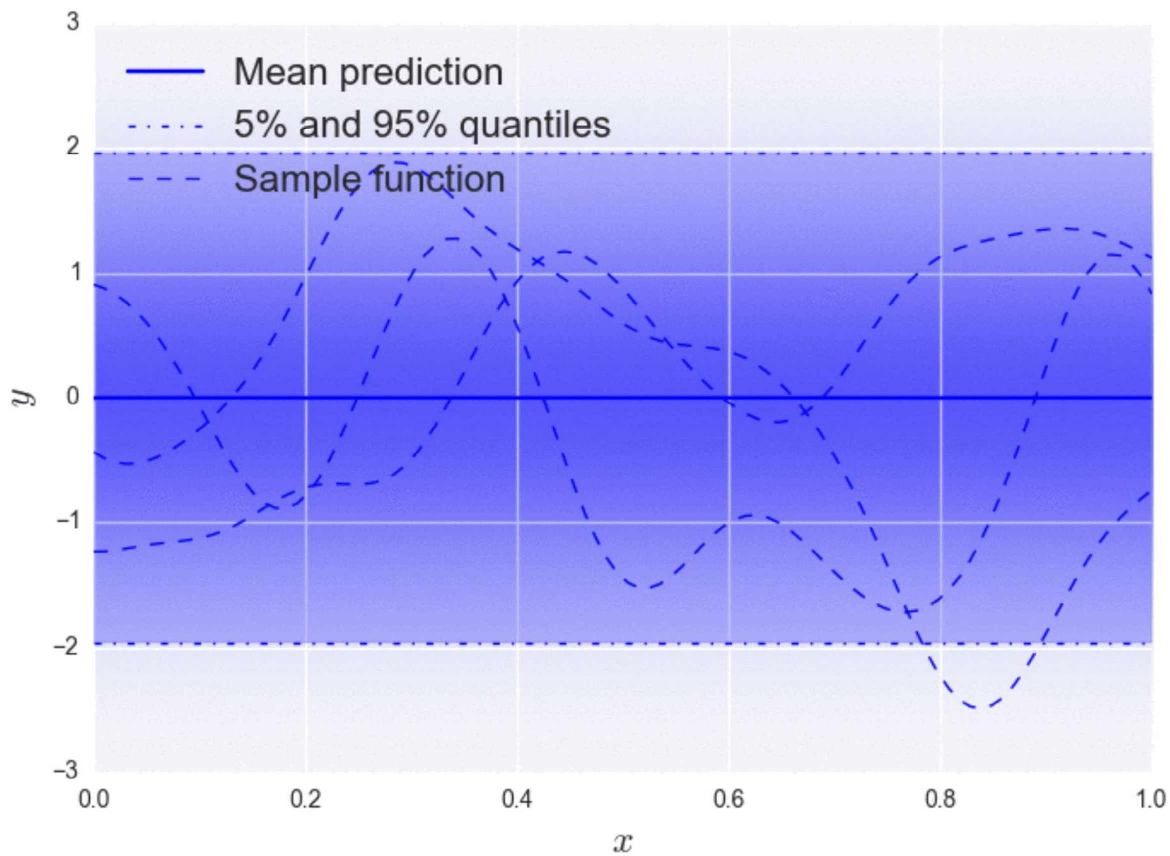
The samples are as smooth as the covariance

Infinitely smooth SE covariance



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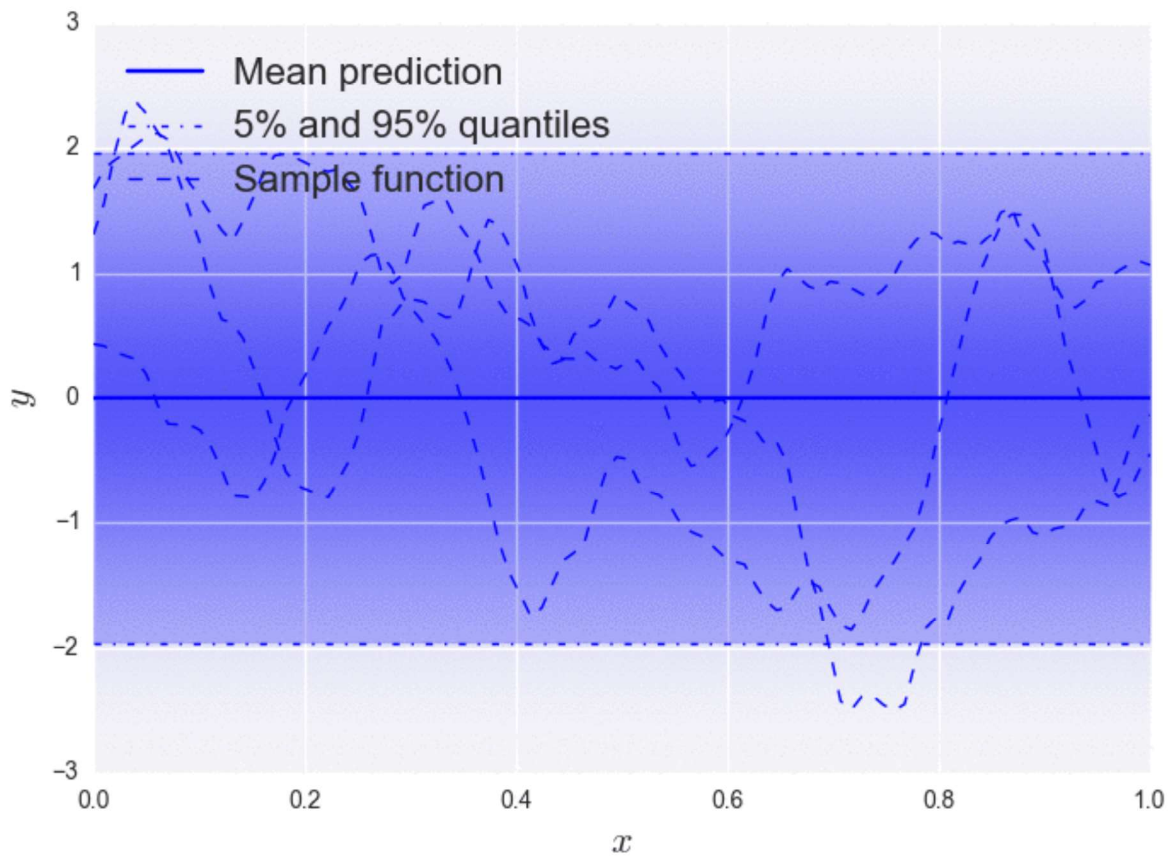
The samples are as smooth as the covariance

Matern 2-3, 2 times differentiable



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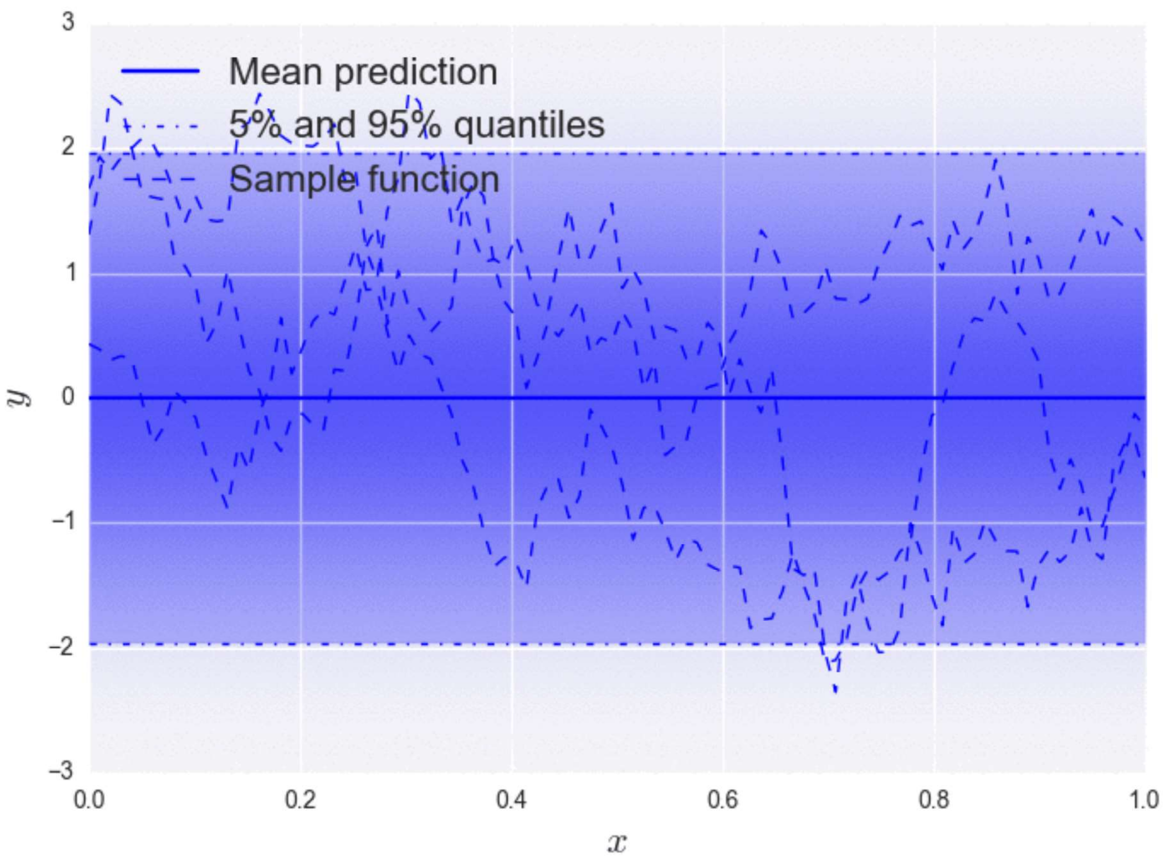
The samples are as smooth as the covariance

Exponential, continuous, nowhere differentiable



- Fractals

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