The Central Limit Theorem

Contents


```
import numpy as np
np.set_printoptions(precision=3)
import matplotlib.pyplot as plt
%matplotlib inline
import seaborn as sns
sns.set(rc={"figure.dpi":100, "savefig.dpi":300})
sns.set_context("notebook")
sns.set_style("ticks")
```

Objectives

- Demonstrate the central limit theorem and the natural rise of the Gaussian distribution.

The central limit theorem (CLT)

Consider, $X_1, X_2, \dots$ be iid random variables with mean μ and variance σ^2. Define their average:

$$S_N = \frac{X_1 + \dots + X_N}{N}.$$

The Central Limit Theorem (CLT), states that:

$$S_N \sim N(S_N|\mu, \frac{\sigma^2}{N}),$$

for large N. That is, they start to look like Gaussian.

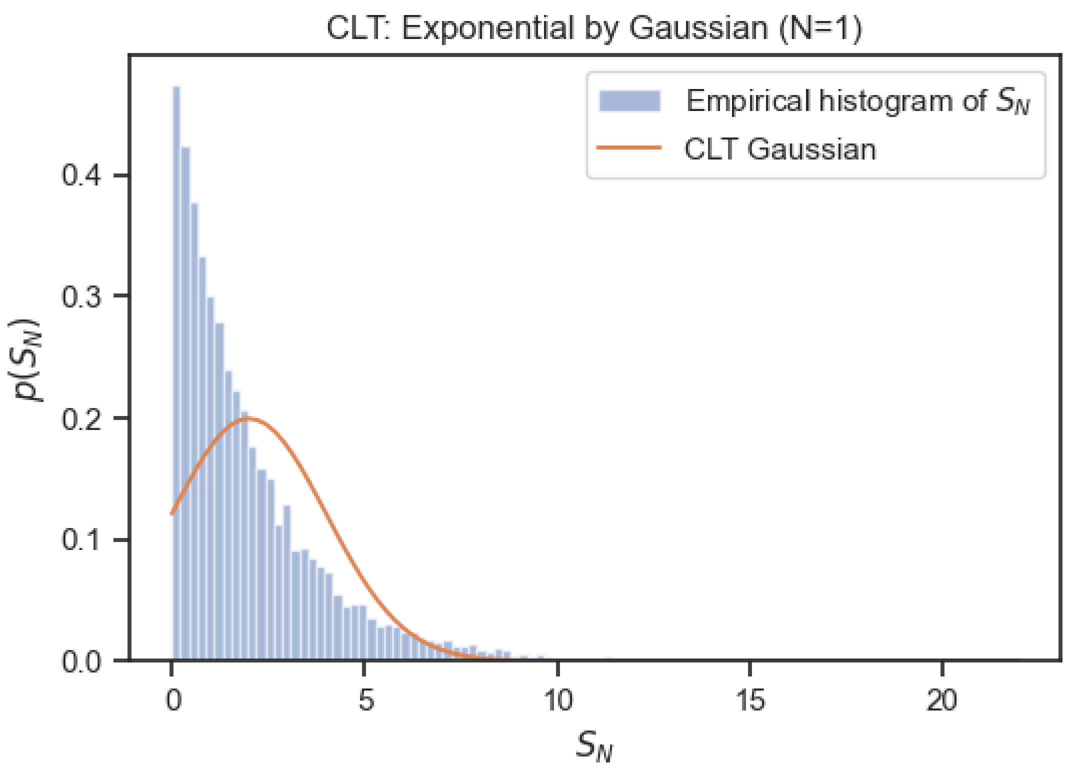
Example: Sum of Exponentials

Let's test it for the Exponential distribution. We will use `numpy.random.exponential` to sample from the exponential. Here $X_i \sim \mathrm{Exp}(r)$, for some fixed r, are independent. Let's take their average S_N and see how it is distributed.

```
import scipy.stats as st

r = 0.5
N = 1    # How many iid variables are we going to sum
M = 10000 # How many times do you want to sample
# The random variable to sample from
X = st.expon(scale=1.0 / r)  # THIS IS THE ONLY LINE YOU NEED TO CHANGE TO TRY OUT
DIFFERENT RVs
# The mean of the random variable
mu = X.expect()
# The variance of the random variable
sigma2 = X.var()
# The CLT standard deivation:
sigma_CLT = np.sqrt(sigma2 / N)

# Sample from X, N x M times.
x_samples = X.rvs(size=(N, M))
# Think of each column of x_samples as a sample from X1, X2, ..., XN.
# Take the average of each column:
SN = np.mean(x_samples, axis=0)

# Now you have M samples of SN
# Let's do their histogram
fig, ax = plt.subplots()
ax.hist(
    SN,
    bins=100,
    density=True,
    alpha=0.5,
    label='Empirical histogram of $S_N$'
)
# Let's depict in the same plot the PDF of the CLT Gaussian:
Ss = np.linspace(SN.min(), SN.max(), 100)
ax.plot(
    Ss,
    st.norm(
        loc=mu,
        scale=sigma_CLT
    ).pdf(Ss),
    label='CLT Gaussian'
)
ax.set_xlabel('$S_N$')
ax.set_ylabel('$p(S_N)$')
ax.set_title(f'CLT: Exponential by Gaussian (N={N})')
plt.legend(loc='best');
```

We have a collection of N i.i.d. random variables, but we also have a set of M samples/observations from each

average across the number of rv's that were summed



Questions

- Start increase N and observe the convergence.
- This holds for any random variable that satisfies the assumptions of the central limit theorem. Even discrete random variables! Modify the code above to so that $X_i \sim \text{Bernoulli}(\theta)$ for some θ. You may use

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