Lecture 13: Linear Regression via Least Squares

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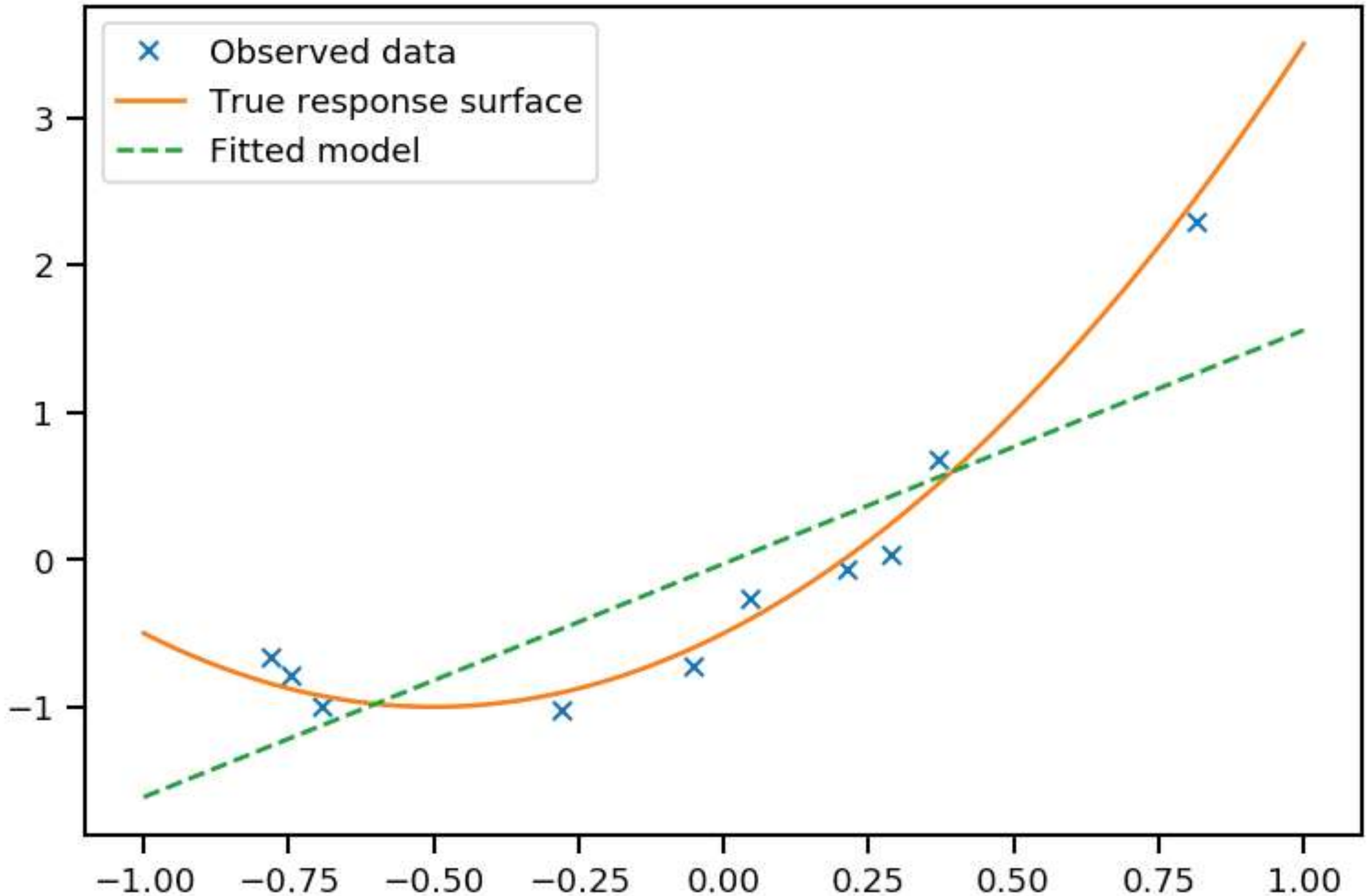
Polynomial regression

An example that doesn't work



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Regression model

$$y = w_0 + w_1 \cdot x + w_2 \cdot x^2$$

$$\underline{w} = (w_0, w_1, w_2)$$

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Least squares loss function

$$L(\underline{w}) = \sum_{i=1}^{n} \Big(\overset{\text{target}}{y_i} - \underbrace{w_0 - w_1 \cdot x_i - w_2 \cdot x_i^2}_{\text{prediction}}\Big)^2$$

$$= \| \underline{y} - \underline{\underline{X}}\, \underline{w} \|_2^2$$

$$\underline{\underline{X}} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{pmatrix} \quad (n \times \overset{\text{degree}+1}{\underline{\underline{3}}})$$

Minimizing the loss function

$$L(\underline{w}) = \| \underline{y} - \underline{\underline{X}} \cdot \underline{w} \|_2^2$$

solve for

$$\nabla_{\underline{w}} L(\underline{w}) = 0 \Rightarrow \underbrace{\left(\underline{\underline{X}}^T \underline{\underline{X}} \right) \underline{w} = \underline{\underline{X}}^T \underline{y}}_{\text{Linear System}}$$

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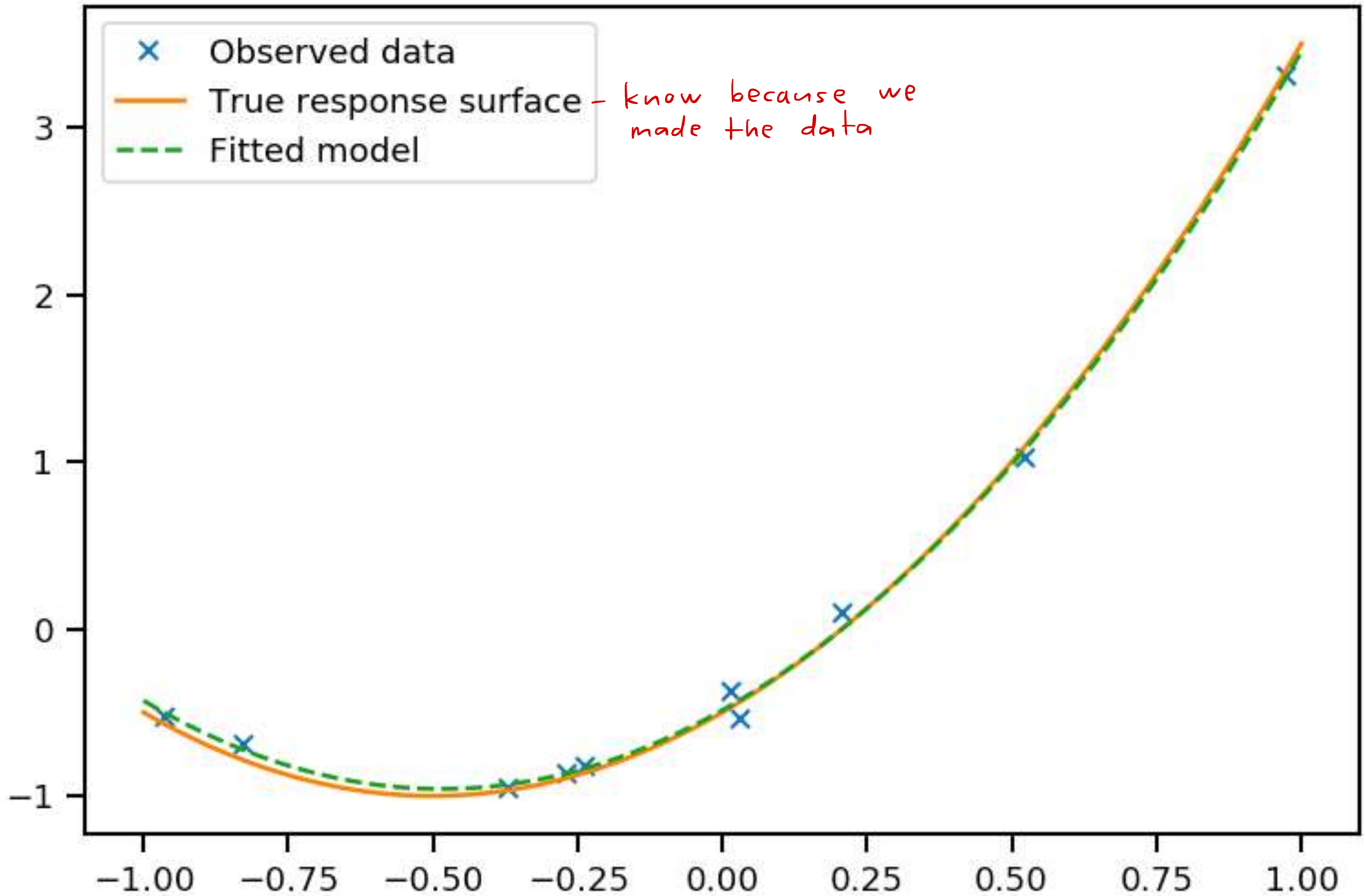
Example


Observed data
True response surface
Fitted model
know because we made the data
3
2
1
0
−1
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

Higher degree polynomials

column in design matrix for each

$$y = w_0 + w_1 x + w_2 x^2 + \dots + w_p \cdot x^p$$

$$L(\underline{w}) = \sum_{i=1}^{n} \Big(\underbrace{y_i}_{\text{target}} - \underbrace{w_0 - w_1 x_i - w_2 x_i^2 - \dots - w_p x_i^p}_{\text{prediction}} \Big)^2$$

$$= \| \underline{y} - \underline{\underline{X}} \cdot \underline{w} \|_2^2$$

$$\underline{w} = (w_0, w_1, \dots, w_p)$$

$$\underline{\underline{X}} = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^p \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^p \end{pmatrix} \quad (n \times (p+1))$$

$$\nabla_{\underline{w}} L(\underline{w}) = 0 \Rightarrow \underbrace{\underline{\underline{X}}^T \underline{\underline{X}} \cdot \underline{w} = \underline{\underline{X}}^T \cdot \underline{y}}_{\text{same form}}$$

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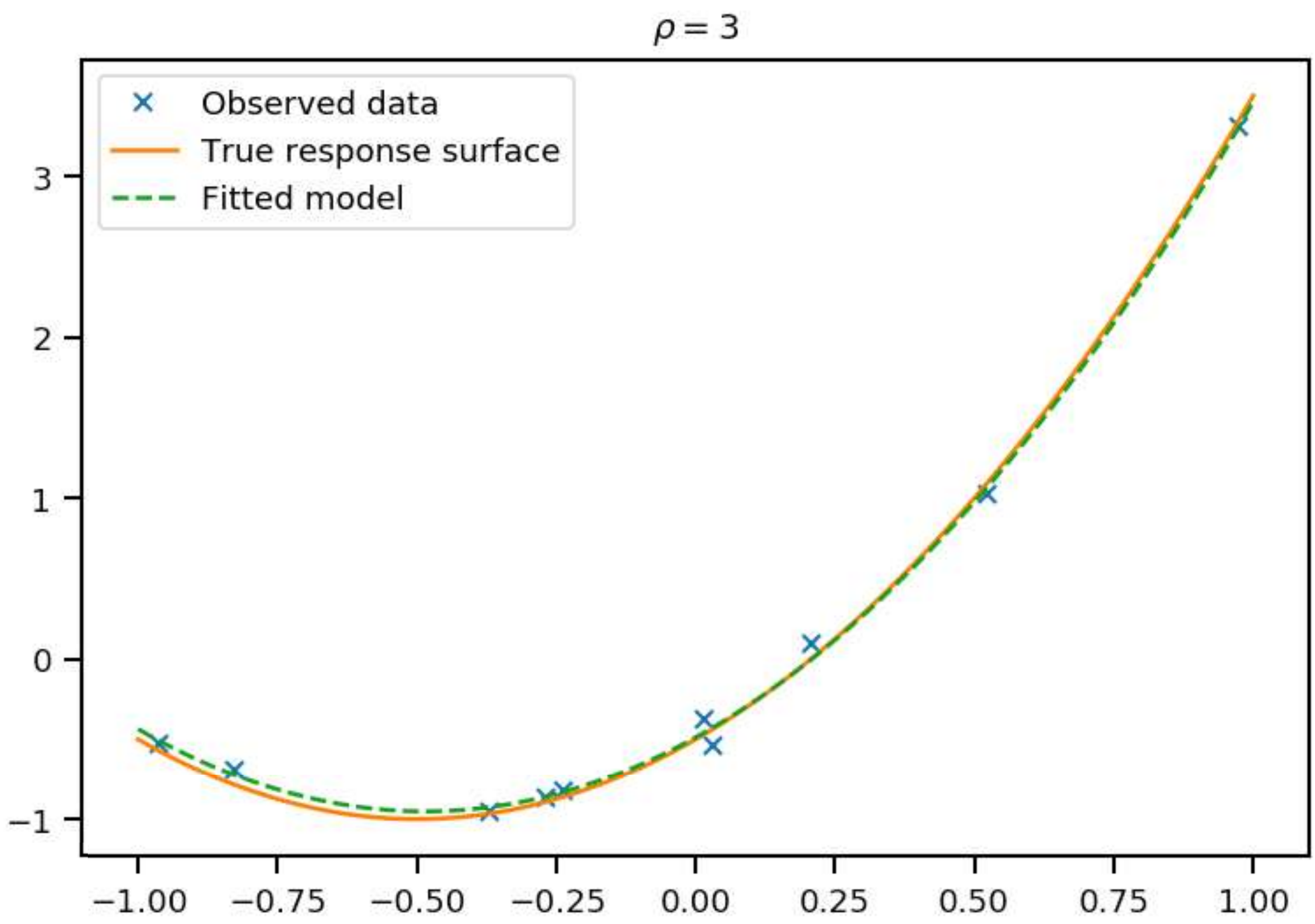
Example



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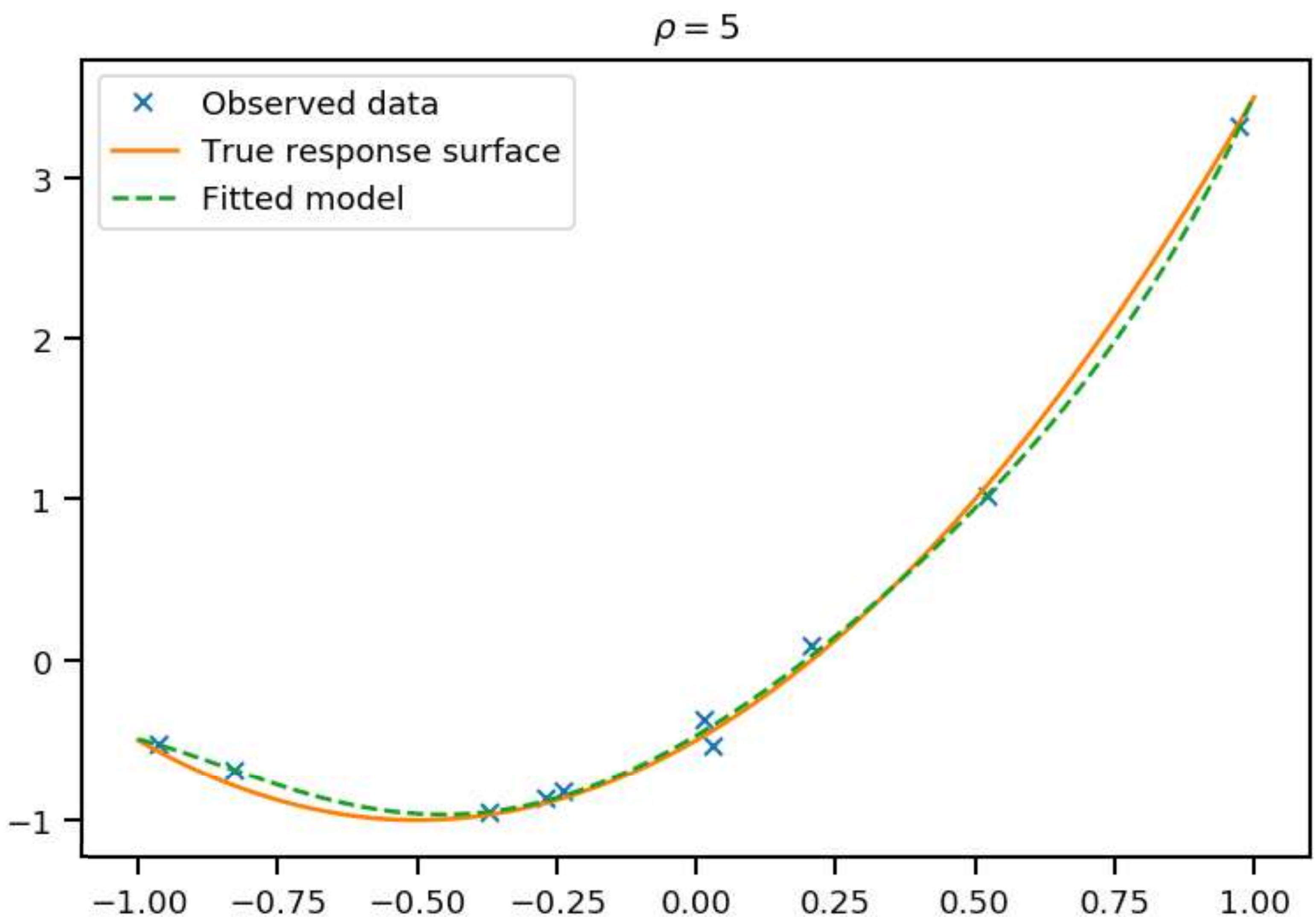
Example



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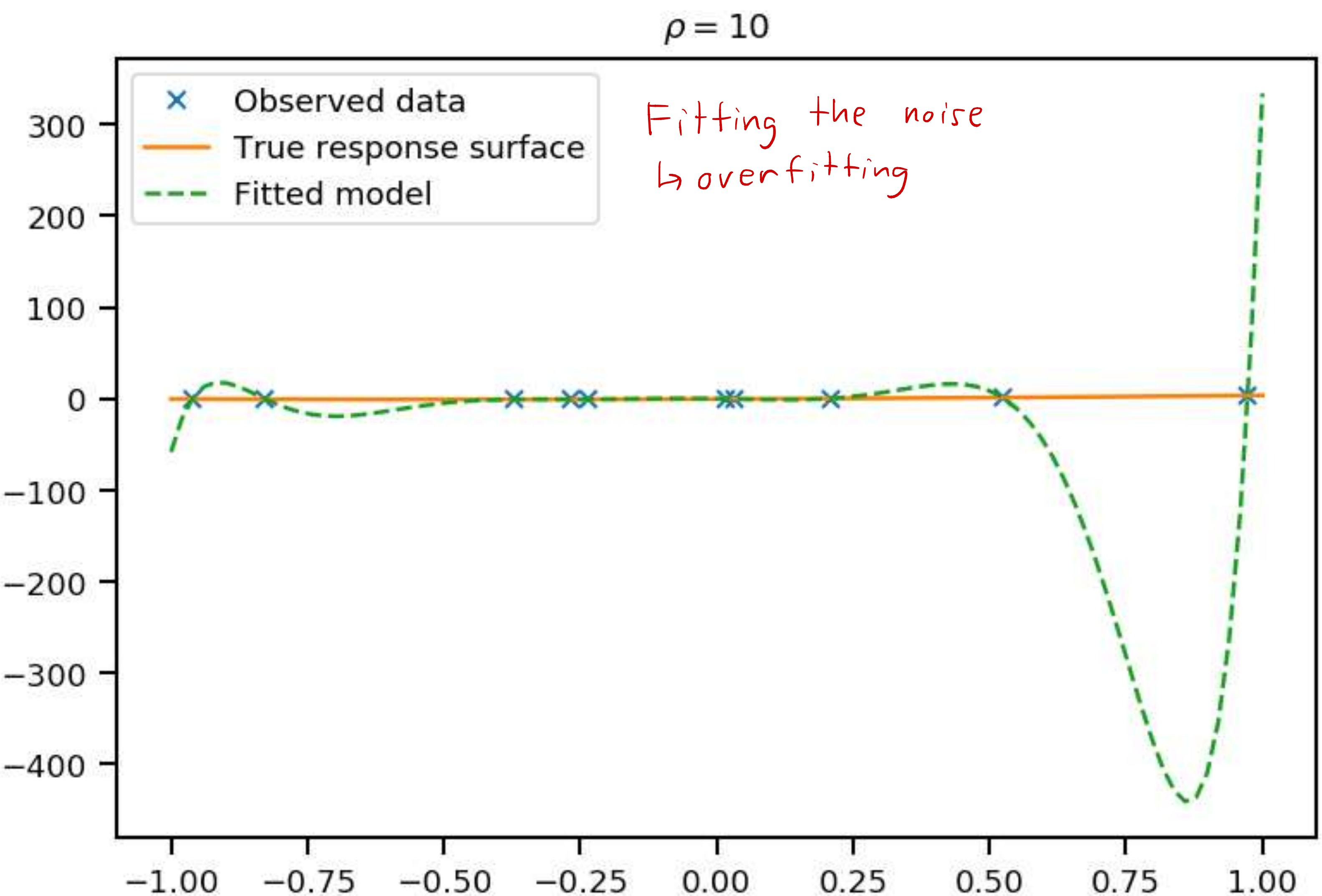
Example



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