Solution to the minimum-error

$$\min_{\underline{V}, \underline{W}, \underline{x}_0} \sum_{i=1}^{n} \| \underline{V}(\underline{W}^T \underline{x}_i) - \underline{x}_0 - \underline{x}_i \|^2$$

subject to $\underline{V}$ and $\underline{W}$ orthogonal matrices — additional constraints allows for a unique solution to exist

$\underline{v}_i^T \underline{v}_j = 0, \; i \neq j, \quad \underline{w}_i^T \underline{w}_j = 0, \; i \neq j$

Solution : Necessary con. : $\underset{\text{w.r.t. param}}{\text{grad}}$ (gradient) loss $= 0$

$\Rightarrow \underline{x}_0 = \frac{1}{n} \sum_{i=1}^{n} \underline{x}_i$ (empirical mean)

$\underline{V} = \underline{W}$ → same

Empirical Covariance Matrix : $\underline{\underline{C}} = \frac{1}{n} \sum_{i=1}^{n} (\underline{x}_i - \underline{x}_0)(\underline{x}_i - \underline{x}_0)^T$

$\underline{\underline{C}}$ $D \times D$ pos. def. $\Rightarrow$ positive eigenvalues $\lambda_i > 0$ and orthogonal eigenvectors. $\underline{u}_i \in \mathbb{R}^D$

$\lambda_1 > \lambda_2 > \dots > \lambda_D > 0$ - eigenvalues ordered

$\underline{\underline{C}} \, \underline{u}_i = \lambda_i \underline{u}_i$ - eigenvector-eigenvalue relationship

$$\underline{W} = [\underline{w}_1, \dots, \underline{w}_d]$$

$$= [\sqrt{\lambda_1} \underline{u}_1, \sqrt{\lambda_2} \underline{u}_2, \dots, \sqrt{\lambda_d} \underline{u}_d]$$

$$= \underset{D \times d}{\underline{\underline{U}}} \cdot \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}) \rightarrow \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_d \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \; \sqrt{\lambda_d} \end{bmatrix}$$

; $d \ll D$

Projection map :

$$\boxed{\begin{aligned} \underline{z}_i = \underline{W}^T \underline{x}_i &= (\underline{\underline{U}} \cdot \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}))^T \underline{x}_i \\ &= \underset{d}{\text{diag}}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}) \cdot \underline{\underline{U}}^T \cdot \underline{x}_i \\ &= \sum_{j=1}^{d} \sqrt{\lambda_j} \cdot \underline{u}_j^T \underline{x}_i \end{aligned}}$$

matrix math

get set of orthogonal unit vectors in $\mathbb{R}^d$ (d many)

Reconstruction map :

$$\tilde{\underline{x}}_i = \underline{V} \underline{z}_i + \underline{x}_0 = \underline{W} \underline{z}_i + \underline{x}_0$$

$$= \underline{\underline{U}} \cdot \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}) \cdot \underline{z}_i + \underline{x}_0$$

PREDICTIVE SCIENCE LABORATORY

10