Continuous Random Variables

Contents

Something more on probability spaces

It turns out that when Ω is a continuous space, like $\mathbb{R}$ for example, it is not possible to take $\mathcal{F}$ to be all the subsets of Ω (you need to take a measure-theoretic probability theory class to understand why this is). However, many "nice" subsets of Ω are usually in $\mathcal{F}$. For example, in the case of $\Omega = \mathbb{R}$, $\mathcal{F}$ can include all intervals, and any countable unions and intersections of intervals. That's a lot of sets. Borel set

In any case, $\mathcal{F}$ must satisfy certain properties for everything to be well-defined. These properties are:

- $\Omega \in \mathcal{F}$
- For any A in $\mathcal{F}$, the complement A^c is in $\mathcal{F}$.
- For any $A_1, A_2, \dots$ in $\mathcal{F}$, the union $\cup_n A_n$ is in $\mathcal{F}$. When a set of subsets $\mathcal{F}$ satisfies these properties, we say that it forms a σ-algebra.

Continuous random variables

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A continuous random variable models the result of an experiment that can potentially take infinitely many values. That is, it is a function

$$X : \Omega \rightarrow \mathbb{R}.$$

The cumulative distribution function

Let X be a continuous random variable. Its cumulative distribution function (CDF) $F_X(x)$ gives the probability that X is smaller than x:

$$F_X(x) := \mathbb{P}(X \le x) = \mathbb{P}\left(\{\omega : X(\omega) \le x\}\right).$$

Properties of the cumulative distribution function

- $F_X(x)$ is an increasing function.
- $F_X(-\infty) = 0$.
- $F_X(+\infty) = 1$.
- $\mathbb{P}(a \le X \le b) = F_X(b) - F_X(a)$.

The probability density function

The probability density function (PDF) is a "function" $f_X(x)$ that can give us the probability that X is in any "good" subset A of $\mathbb{R}$ as follows:

$$\mathbb{P}(X \in A) = \int_A f_X(x) dx.$$

Note that certain random variables may not have a PDF that is a function. That's why I put the word "function" in quotes. However, if you allow the PDF to include Dirac's δ, then any random variable has a PDF. We will ignore this complication for the moment.

In this class, we will simplify the notation and we will be writing:

$$p(x) \equiv f_X(x),$$

when there is no ambiguity.

Properties of the probability density function

- $p(x) \geq 0$ for all x.
- $\int_{-\infty}^{\infty} p(x)dx = 1$.
- The derivative of the CDF is the PDF, i.e., $F_X'(x) = p(x)$.

Expectations of continuous random variables

The expectation of a continuous random variable is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xp(x)dx.$$

Its properties are the same as the expectation of a discrete random variable.

The expectation of a function of the random variable is:

$$\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)p(x)dx.$$

It satisfies the same properties as the expectation of discrete random variables.

Variance of continuous random variables

The variance of a continuous random variable is:

$$\mathbb{V}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 p(x)dx.$$

It satisfies the same properties as the expectaiton of discrete andom variables.

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