Lecture 8: The Monte Carlo method for estimating expectations

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The law of large numbers

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The strong law of large numbers

- Take an infinite series of independent random variables $X_1, X_2, \ldots$ with the same distribution (it doesn't matter what distribution).

- The *sample average*: $\frac{X_1 + \ldots + X_N}{N} \longrightarrow \mu$

$$\bar{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i \to \mu \text{ a.s.}$$

almost surely
(measure theory)

where $\boxed{\mu = \mathbb{E}[X_i]}$ (as $N \to \infty$).

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The Monte Carlo method for estimating integrals

- Take a random variable $X \sim p(x)$ and some function $g(x)$.

- We want to estimate the expectation:

$$\boxed{I = \mathbb{E}[g(X)] = \int g(x)p(x)dx}$$

- Make independent identical copies of X: $X_1, X_2, \dots \sim p(x)$

- Consider the also the independent iden. dist.:

$Y_1 = g(X_1), Y_2 = g(X_2), \dots$

- By the strong law of large numbers:

$$I_N = \frac{Y_1 + \dots + Y_N}{N} \longrightarrow \mathbb{E}[Y_i] = \mathbb{E}[g(X_i)] = I \quad \text{a.s.}$$

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Example: 1D expectation

(This is Example 3.4 of Robert & Casella (2004))

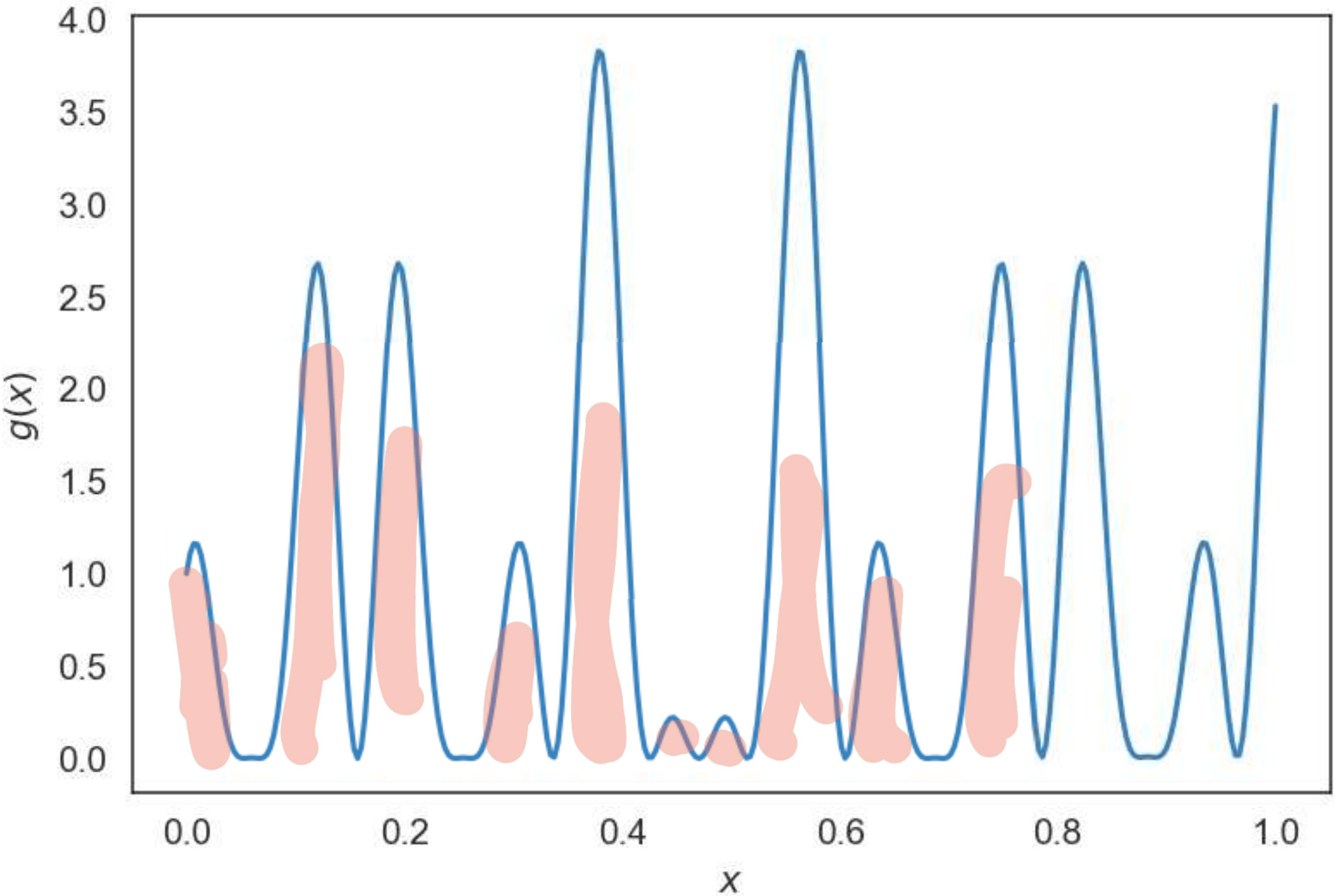
$$X \sim \mathcal{U}([0,1])$$

$$g(x) = \big(\cos(50x) + \sin(20x)\big)^2$$

The correct value for the integral is:

$$\mathbb{E}[g(X)] = 0.965$$

Example: 1D expectation



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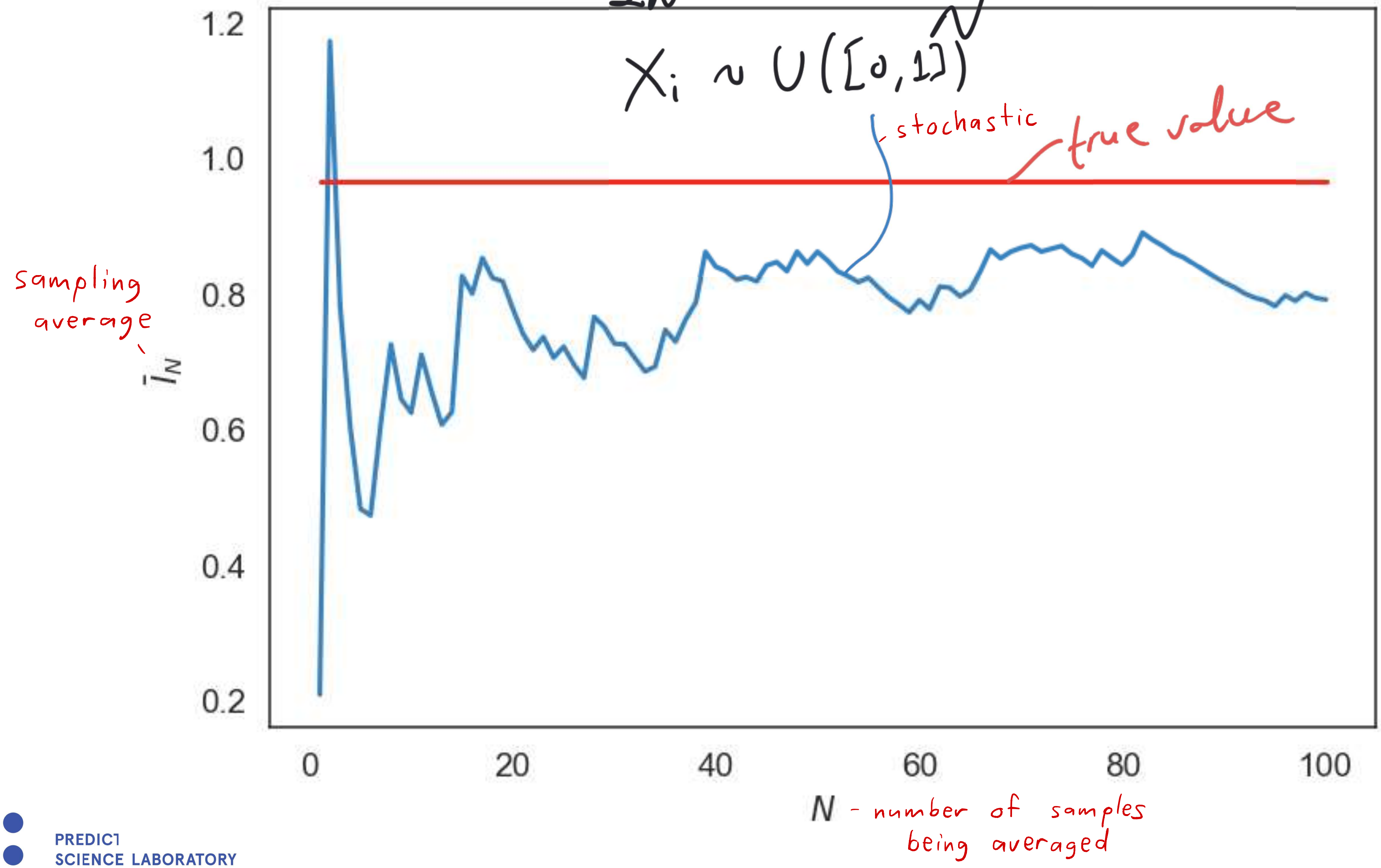
Example: 1D expectation

$$\bar{I}_N = \frac{g(X_1) + \dots + g(X_N)}{N}$$

$$X_i \sim U([0,1])$$



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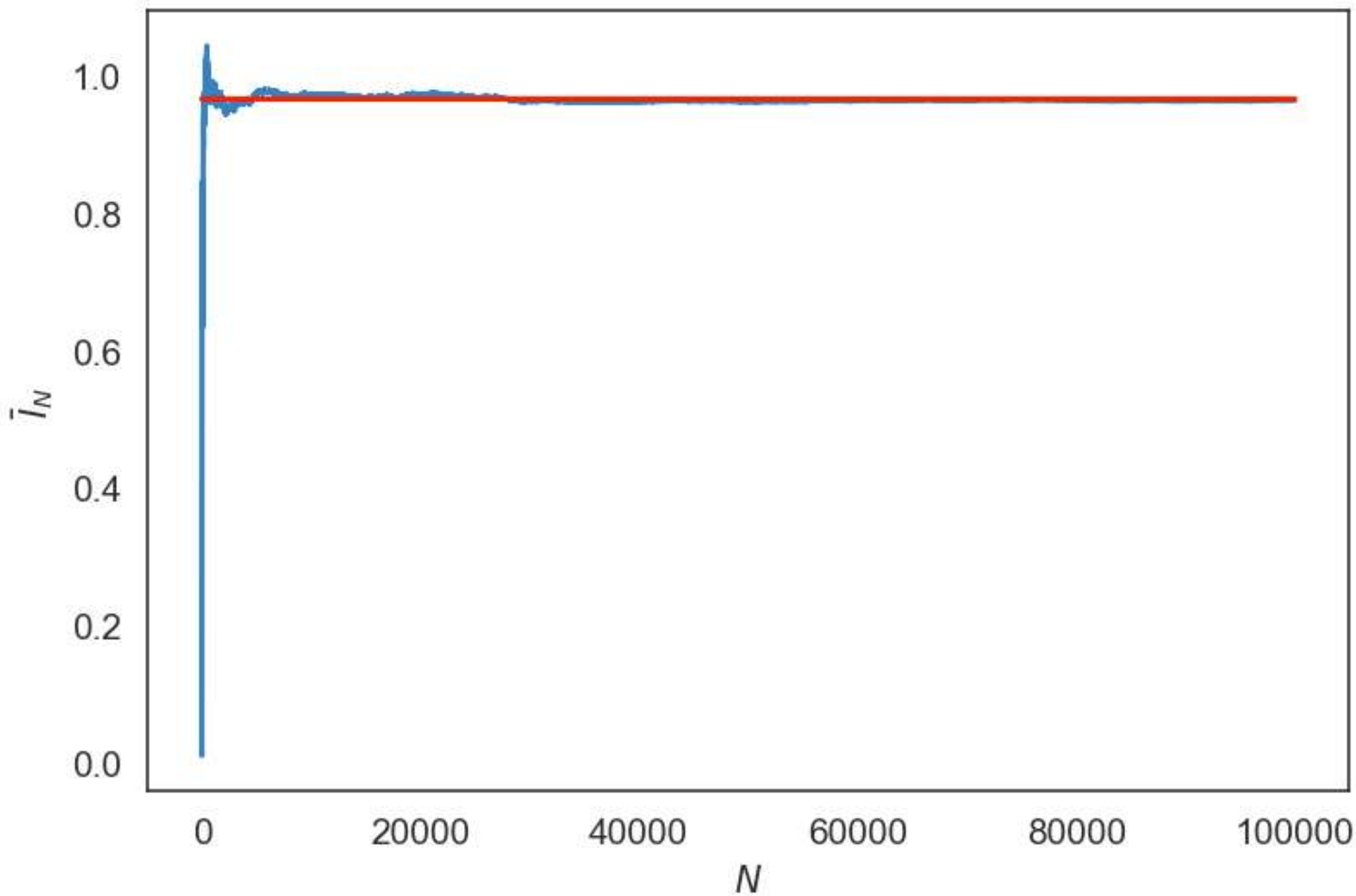
Example: 1D expectation



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