Lecture 8: The Monte Carlo method for estimating expectations

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The uncertainty propagation problem

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The uncertainty propagation problem

– characterize the statistics of Y, the model output

- You are given a function $g(x)$ representing a physical model.

  x: input , $g(x)$: output

- The inputs of the model are uncertain.
- You represent this uncertainty with a random variable:

$$X \sim p(x)$$

X has $p(x)$ as its pdf

- You would like to quantify your uncertainty about the model output:

$$\boxed{Y = g(X)}$$

may be epistemic

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- We would like to estimate the expected value of the output:

$$\mathbb{E}[Y] = \mathbb{E}[g(X)] = \int g(x)\, p(x)\, dx$$

Now, how to compute this integral?

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- We would like to estimate the variance of the output:

$$\mathbb{V}[Y] = \int (g(x) - \mathbb{E}[Y])^2 p(x) dx$$

$$= \mathbb{E}\left[[g(X)]^2\right] - \left(\mathbb{E}[g(X)]\right)^2$$

already have

$$\mathbb{E}\left[[g(X)]^2\right] = \int g^2(x)\, p(x)\, dx$$

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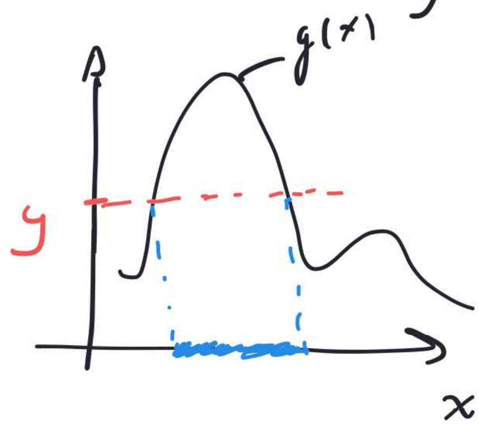
- Or maybe the probability that the output exceeds a threshold:

$$p(Y \geq y) = \int 1_{[y,\infty)}(\underbrace{g(x)}_{\in ?})\,p(x)\,dx = \mathbb{E}\left[1_{[y,\infty)}(g(X))\right]$$

$$= \mathbb{E}\left[1_{[y,\infty)}(Y)\right]$$



$$1_{[y,\infty)}(g(x)) = \begin{cases} 1, & \text{if } g(x) \geq y \\ 0, & \text{otherwise} \end{cases}$$

The <u>indicator function</u> of the set $[y, \infty)$

The uncertainty propagation problem

- Notice that all these statistics are essentially expectations of functions of X.
- We must learn how to do such integrals!