Lecture 10: Quantifying uncertainties in Monte Carlo estimates

Professor Ilias Bilionis

Epistemic uncertainty of Monte Carlo estimates

Quantifying Epistemic Uncertainties in MC

- We wish to estimate: $I = \mathbb{E}[g(X)]$ via a sampling.
- We take iid random variables $X_1, X_2, \dots$
- Consider the also iid $Y_1 = g(X_1), Y_2 = g(X_2), \dots$
- And using the law of law large numbers we get:

sampling average

$$\bar{I}_N = \frac{g(X_1) + \dots + g(X_N)}{N} = \frac{Y_1 + \dots + Y_N}{N} \to I, \text{ a.s.}$$

sampling average can vary a lot if N is small

PREDICTIVE SCIENCE LABORATORY

Quantifying Epistemic Uncertainties in MC

- Note that $Y_i = g(X_i)$ are iid with mean:

$$\mathbb{E}[Y_i] = \mathbb{E}[g(X_i)] = I$$

- Assume their variance is finite:

$$\mathbb{V}[Y_i] = \sigma^2 < +\infty$$

- Then, the CLT holds and it gives:

$$\bar{I}_N = \frac{Y_1 + \dots + Y_N}{N} \sim N\left(I, \frac{\sigma^2}{N}\right),$$ N large

sampling-based estimate of the expectation

PREDICTIVE SCIENCE LABORATORY

Quantifying Epistemic Uncertainties in MC

- The CLT gives:

$$\bar{I}_N \sim N\left(I, \frac{\sigma^2}{N}\right).$$

- We can rewrite as:

* more useful:

$$\boxed{\bar{I}_N = I + \frac{\sigma}{\sqrt{N}} \cdot Z}, \quad Z \sim N(0,1)$$

we know: $\left\{ Z \sim N(0,1),\ W = \mu + \sigma \cdot Z \Rightarrow W \sim N(\mu, \sigma^2) \right\}$

- Solve for I:

$$I = \bar{I}_N - \frac{\sigma}{\sqrt{N}} \cdot Z$$

$$\boxed{I \sim N\left(\bar{I}_N, \frac{\sigma^2}{N}\right)}$$

PREDICTIVE SCIENCE LABORATORY

Quantifying Epistemic Uncertainties in MC

- We have shown that:

?

$$I \sim N\left(\bar{I}_N, \frac{\sigma^2}{N}\right)$$

$$\left\{ V[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 \right.$$

- We need the variance. We can estimate it by:

$$\bar{\sigma}_N^2 = \frac{1}{N}\sum_{i=1}^{N} Y_i^2 - \bar{I}_N^2$$

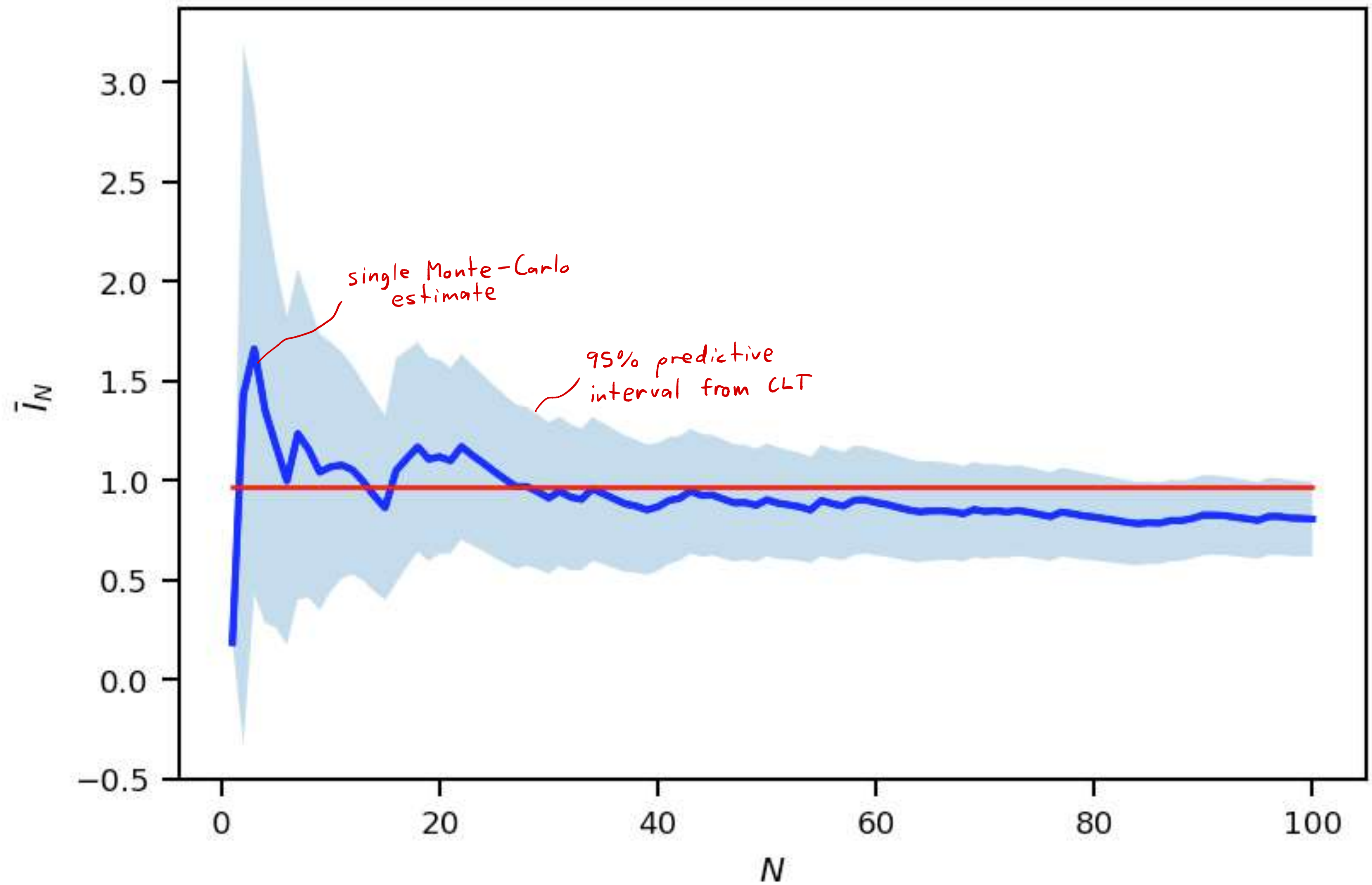
we know how to find

- And we end up with:

$$I \sim N\left(\bar{I}_N, \frac{\bar{\sigma}_N^2}{N}\right)$$

large N

Example



PREDICTIVE
SCIENCE LABORATORY