Lecture 12: Analytical examples of Bayesian inference

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Test statistics and Bayesian p-values

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Test quantities

- Test quantities, characterize the discrepancy between the model and the data.
- They can help you zoom in into characteristics of the data that are of particular interest.
- Mathematically, they are just a scalar function of the data and the parameters:

$$T(x_{1:n}, \theta)$$

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What to do with test quantities

- We want to compare the observed test quantity: $\underline{T(x_{1:n})}$
- To the posterior probability density of test quantities of replicated data: $\underline{T(x_{1:n}^{rep})}$

Question:

- "If I repeated run the experiment how often would I see the observed value of the test quantity?"

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Bayesian p-values

- You can summarize a test quantity using the Bayesian p-value defined as:
- "The probability that replicated data will give you a test quantity larger than the observed test quantity value under the assumption that the model is correct."
- Mathematically:

$$p_B = \mathbb{P}\left[T(x_{1:n}^{rep}) > T(x_{1:n}) \mid x_{1:n}\right]$$

Interpretation of Bayesian p-values

$$p_B = \mathbb{P}(T(x_{1:n}^{\text{rep}}, \theta) > T(x_{1:n}, \theta) \,|\, x_{1:n})$$

- "The probability that replicated data will give you a test quantity larger than the observed test quantity value under the assumption that the model is correct."
- **They are not the probability that the model is correct!!!**
- Values close to 0 or 1 indicate some issue with the model with regards to that particular test quantity.
- Values close to 0.5 indicate no issue, but they do not mean that the model is correct.

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Example: Coin toss

- Let's try the test quantity "number of heads."
- Mathematically, it is:

$$T_h(x_{1:n}) = \sum_{i=1}^{n} x_i$$

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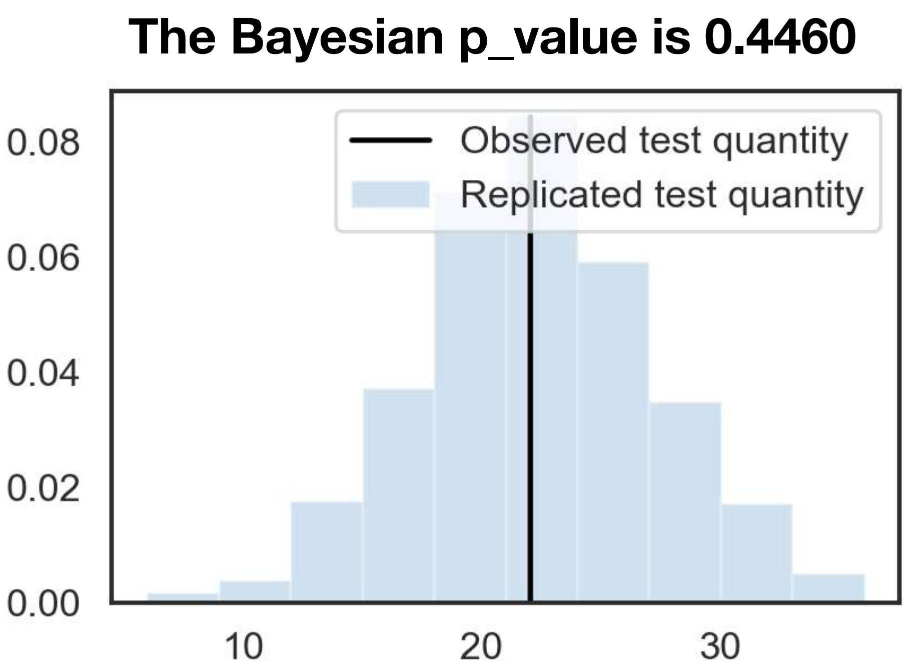
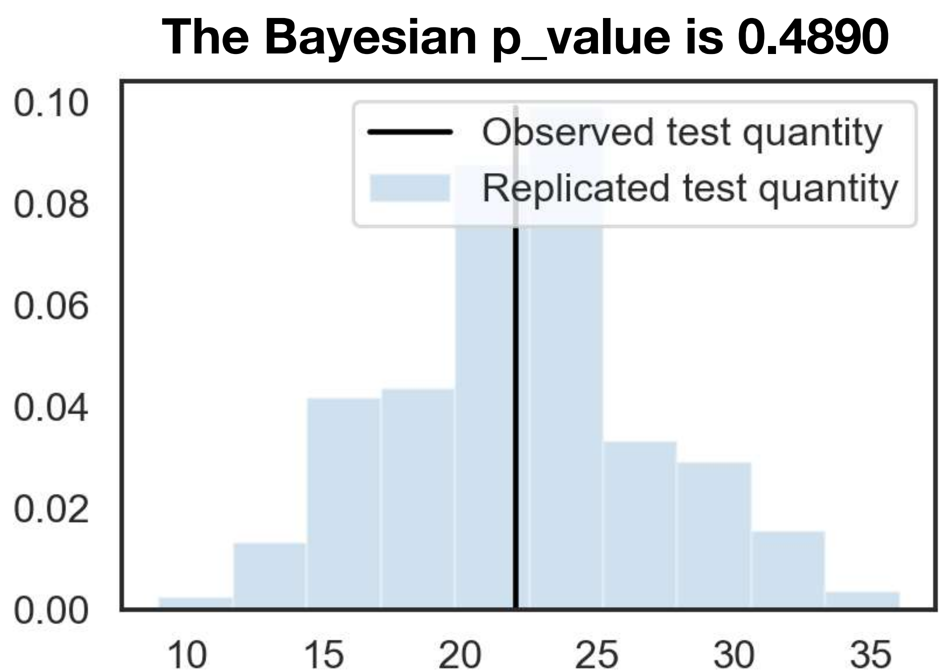
Example: Coin toss



With data from fair coin.



With made-up data.

Observed test quantity is just a single value
(the number of observed heads here)

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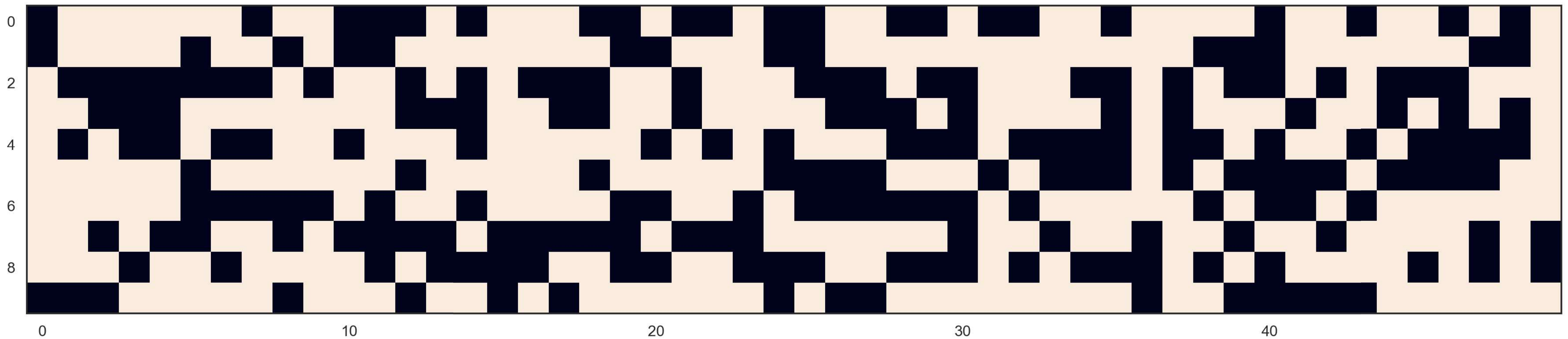
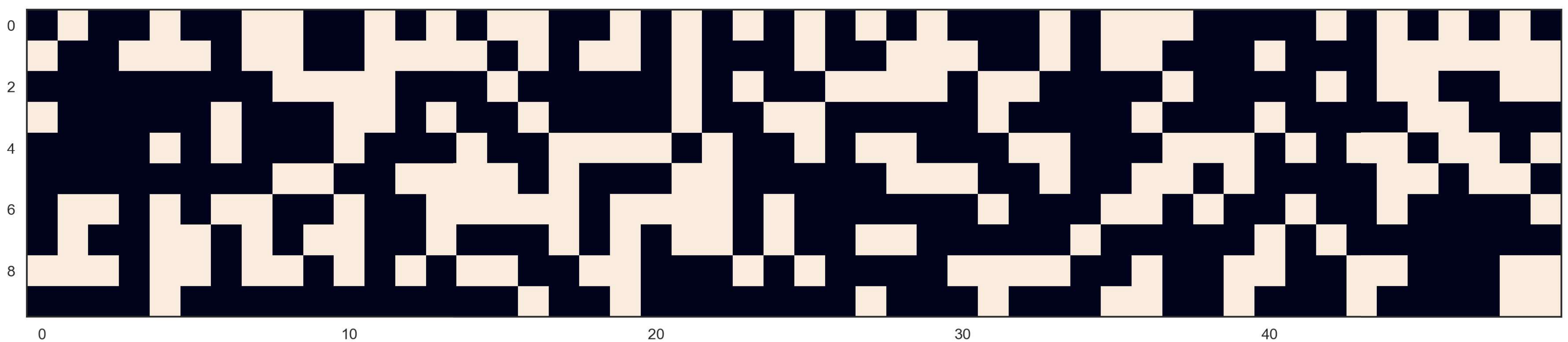
Posterior Predictive Checking



With data from fair coin.



With made-up data.

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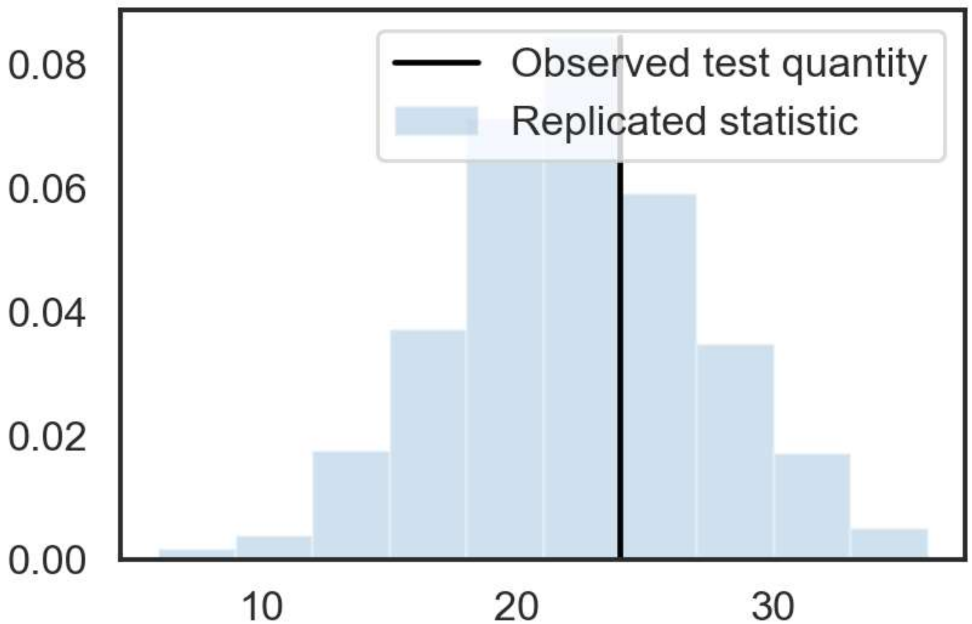
Example: Coin Toss

$T_s(x_{1:n}) =$ **# number of switches from 0 and 1 in the sequence** $x_{1:n}$

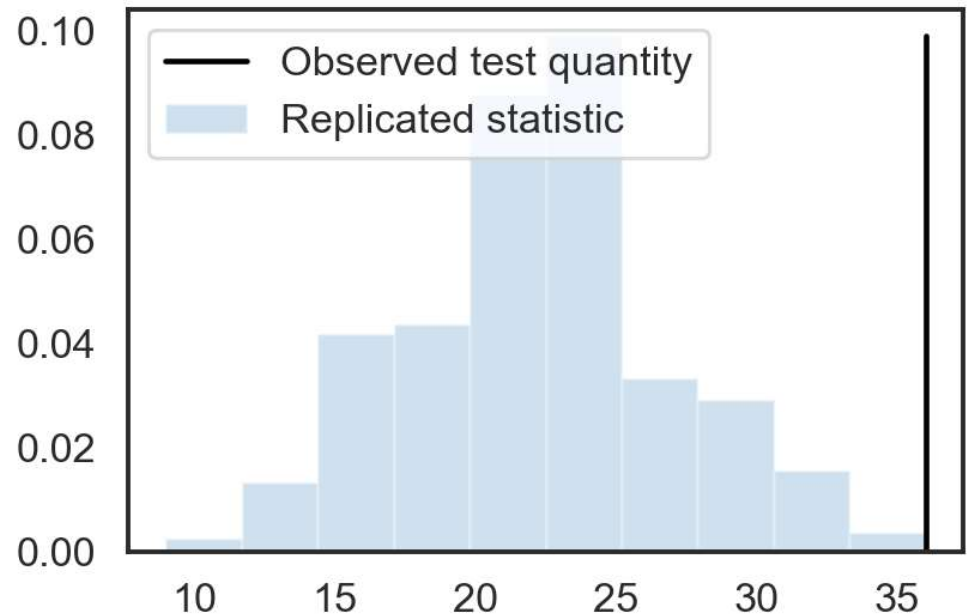
Example: Coin toss

The observed test quantity is 24
The Bayesian p_value is 0.4340



With data from fair coin.

The observed test quantity is 36
The Bayesian p_value is 0.0010



With made-up data.

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