Lecture 12: Analytical examples of Bayesian inference

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Bayesian parameter estimation

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Example: Coin toss

- We run a coin toss experiment N times and we wish to figure out the probability of heads.

- The data we have observe are: $x_1, x_2, \ldots, x_N$

$$x_i \in \{\underset{H}{\underset{\parallel}{0}}, \underset{T}{\underset{\parallel}{1}}\}$$

- For notational convenience we will be writing:

$$x_{1:N} = (x_1, \ldots, x_N)$$

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Example: Coin toss

- The probability of success of the coin toss: θ

- How can we describe our uncertainty about it?

$$\theta \in [0,1] \xrightarrow[\text{ent}]{\text{max}} \theta \sim U([0,1])$$ prior

state of knowledge before we see any data

- Each coin toss experiment corresponds is:

$$X_i | \theta \sim \text{Bernoulli}(\theta)$$ likelihood

a model of the measurement process

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The likelihood of the data

$$x_i | \theta \sim \text{Bernoulli}(\theta)$$

$$p(x_i|\theta) = \theta^{x_i}(1-\theta)^{1-x_i} = \begin{cases} \theta, & x_i = 1 \\ 1-\theta, & x_i = 0 \end{cases}$$

likelihood of single data point

x_i ind.

$$p(x_{1:N}|\theta) = p(x_1|\theta)\cdot p(x_2|\theta)\cdots p(x_N|\theta)$$

likelihood of entire data set

$$= \prod_{i=1}^{N} p(x_i|\theta)$$

$$= \prod_{i=1}^{N} \theta^{x_i}(1-\theta)^{1-x_i}$$

$$= \theta^{\sum_{i=1}^{N} x_i}\cdot(1-\theta)^{N-\sum_{i=1}^{N} x_i}$$

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Bayes' rule applied

$$p(A|B) = \frac{p(AB)}{p(B)}.$$

$$A = \textbf{the model parameters} = \theta$$

$$B = \textbf{the data} = x_{1:N}$$

The joint probability density

Posterior state of knowledge - posterior

$$p(\textbf{the model parameters}|\textbf{the data}) = \frac{p(\textbf{the parameters and the data})}{p(\textbf{the data})}$$

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Simplifying the joint

$$p(AB) = p(B|A)p(A)$$

$$A = \textbf{the model parameters} = \theta$$

$$B = \textbf{the data} = x_{1:N}$$

Joint | Likelihood | Prior

$$\boxed{p(\textbf{the parameters and the data})} = \boxed{p(\textbf{the data}|\textbf{the parameters})}\boxed{p(\textbf{the parameters})}$$

$$\boxed{\textbf{posterior} \propto \textbf{likelihood} \times \textbf{prior}}$$

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The posterior for the coin toss example

posterior $\propto$ likelihood $\times$ prior

$$p(\theta \mid x_{1:N}) \propto p(x_{1:N} \mid \theta) p(\theta)$$

$$= \theta^{\sum_{i=1}^{N} x_i} (1-\theta)^{N - \sum_{i=1}^{N} x_i} \mathbb{1}_{[0,1]}(\theta)$$

$$\theta \mid x_{1:N} \sim \underline{\mathrm{Beta}\left(\alpha = \sum_{i=1}^{N} x_i + 1,\ \beta = N - \sum_{i=1}^{N} x_i + 1\right)}$$

rare situation where the posterior
is analytically available

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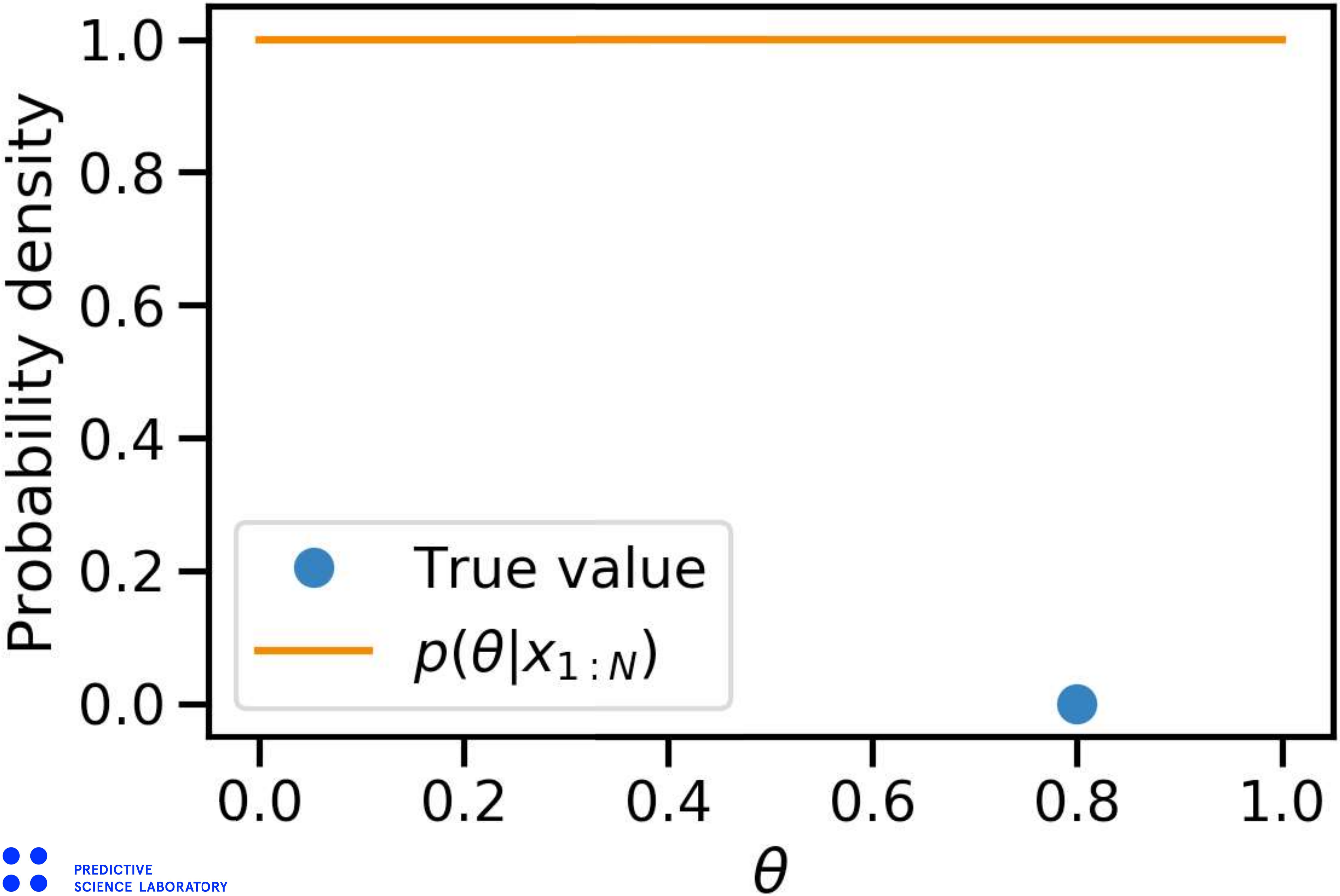
$N = 0$ - no data/observations still uniform



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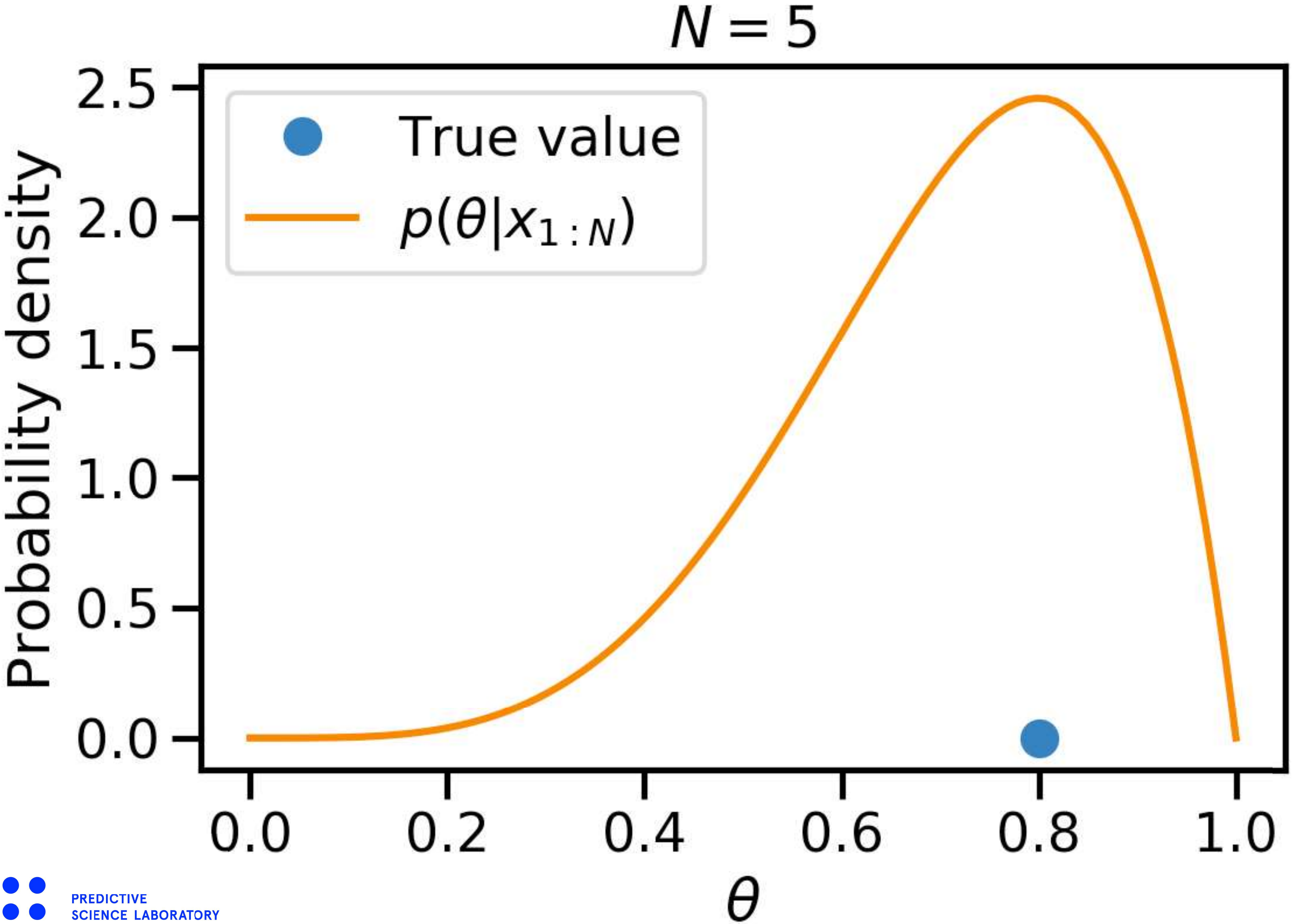

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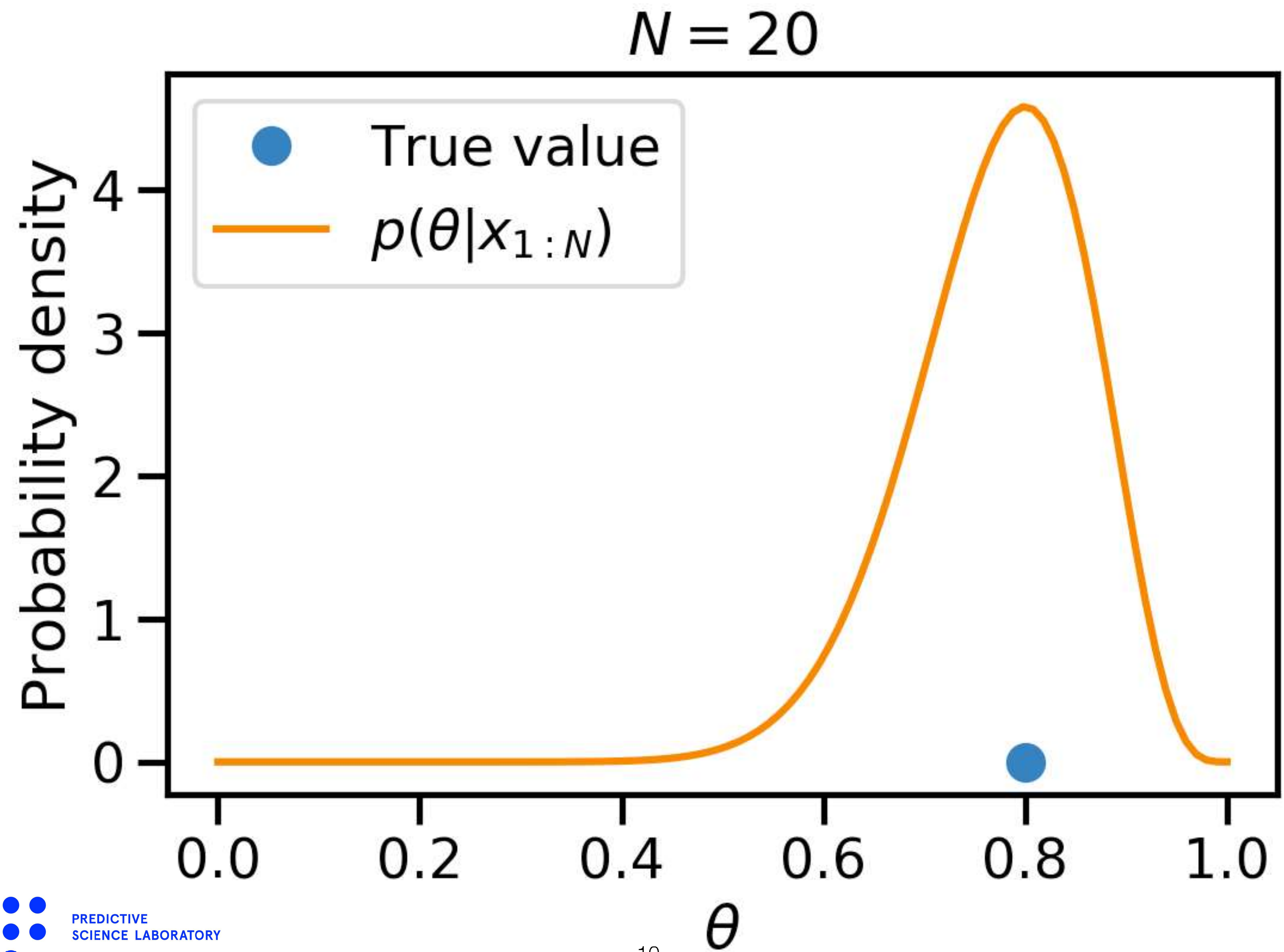
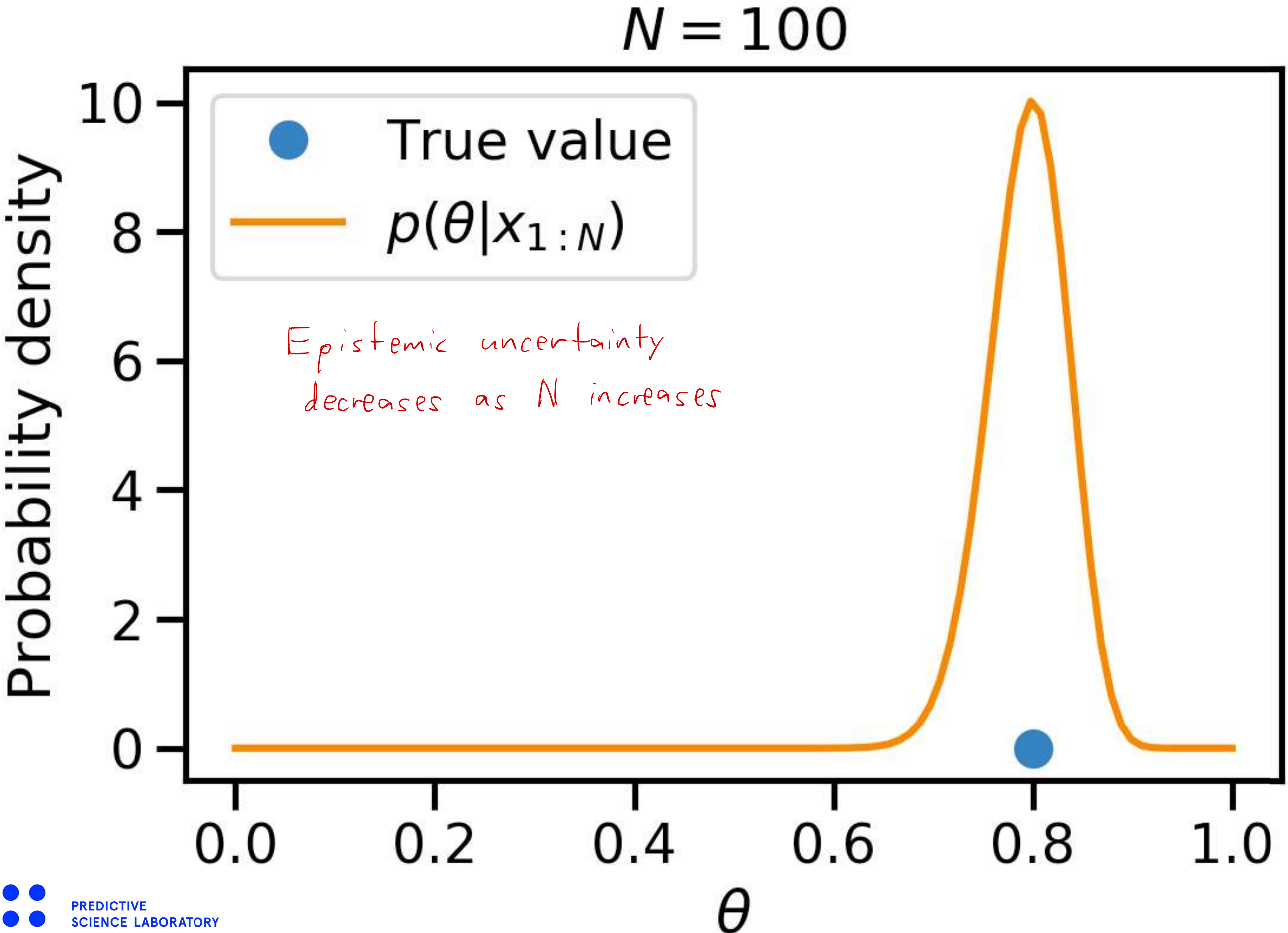


$N = 20$
True value
$p(\theta|x_{1:N})$
Probability density
0
1
2
3
4
0.0
0.2
0.4
0.6
0.8
1.0
θ



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