Random Vectors

Contents

Definition

Take N random variables $X_1, \dots, X_N$ and put them in a vector:

$$\mathbf{X} = (X_1, \dots, X_N).$$

We say that $\mathbf{X}$ is a *random vector*. Random vectors are used to model *uncertain stuff* that require multiple numbers to be described. For example:

- The unobserved state of a multi-body system can be described by the random vector of coordinates and velocities.
- An "uncertain function" could be modeled by the random vector of its function values at N test points.

Probability density function of a random vector

The the PDF of the random vector is the joint PDF of the components. We write:

$$p(\mathbf{x}) = p(x_1, \dots, x_N).$$

Expectation of a random vector

The expectation of a random vector is the vector of expectations of each component:

$$\mathbb{E}[\mathbf{X}] = \begin{pmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_N] \end{pmatrix}$$

This satisfies properties similar to the expectation of scalar random variables. For example, for any real number λ we have that:

$$\mathbb{E}[\lambda\mathbf{X}] = \lambda\mathbb{E}[X].$$

Also, if $\mathbf{Y}$ is another N-dimensional random vector, we have:

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Covariance matrix of two random vectors

Let $\mathbf{X}$ and $\mathbf{Y}$ be N- and M-dimensional random vectors, respectively. The covariance of $\mathbb{X}$ and $\mathbf{Y}$ is the $N \times M$ matrix consisting of all covariances between the components of $\mathbf{X}$ and $\mathbf{Y}$, i.e.,

$$\mathbb{C}[\mathbf{X}, \mathbf{Y}] = (\mathbb{C}[X_i, Y_j]).$$

It can also be rewritten as the expectation of a matrix:

$$\mathbb{C}[\mathbf{X},\mathbf{Y}] = \mathbb{E}\left[(\mathbf{X}-\mathbb{E}[\mathbf{X}])(\mathbf{Y}-\mathbb{E}[\mathbf{Y}])^T\right].$$

Here we assumed that the expectation operator is applied on each one of the matrix components.

It is easy to show that the covariance is a linear function of each argument.

The $N\times N$ matrix $\mathbb{C}[X,X]$ is the *self covariance* matrix (or just covariance matrix) of $\mathbf{X}$. The diagonal of the covariance matrix of $\mathbf{X}$ contains the variances of each of the components of $\mathbf{X}$. If self-covariance is diagonal, then entries of X are uncorrelated (implied from them being independent)

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