QF605 Fixed Income Securities

Project Report

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Part I: Bootstrapping Swap Curves

Q1: Bootstrap the OIS discount factor $D_o(0,T)$ and plot the discount curve for $T \in [0,30]$:

Given the par-swap-rate formula, $PV_{fix}^{1y\ OIS} = PV_{flt}^{1y\ OIS}$, we obtain:

$$D_o(0,1y) * 0.3\% = D_o(0,1y) * \left[\left(1+\frac{f_0}{360}\right)^{180}\left(1+\frac{f_1}{360}\right)^{180} - 1\right]$$

$$f_{1(6m,1y)} = 360 * \left[\left(\frac{1.003}{\left(1+\frac{f_0}{360}\right)^{180}}\right)^{\frac{1}{180}} - 1\right] \qquad D_o(0,1y) = \frac{1}{\left(1+\frac{f_0}{360}\right)^{180}\left(1+\frac{f_{1(6m,1y)}}{360}\right)^{180}}$$

With f_i denote the SOFR rate and D_o as the OIS discount factor. We note that the $f_{1(6m,1y)}$ above is the forward(6m,1y) SOFR rate, hence we converted it to the 1-year SOFR in our code for ease of latter calculations.

Knowing the relationship between $D_o$and f_i, we can write $D_o$in terms of f_i and substitute into the par-swap-rate formula to get only one unknow variable f_{i+1} for every next iteration. This yields the following results (using 2y OIS as example):

$$D_o(0,T_{i+1}) = D_o(0,T_i) * \frac{1}{\left(1+\frac{f_{i+1}}{360}\right)^{360}}$$

$$OIS_{2y} * \frac{1}{\left(1+\frac{0.002995}{360}\right)^{360}} * \left[1+\frac{1}{\left(1+\frac{f_2}{360}\right)^{360}}\right] = \left[1-\frac{1}{\left(1+\frac{0.002995}{360}\right)^{360}}\right] + \left[1-\frac{1}{\left(1+\frac{f_2}{360}\right)^{360}}\right]$$

Capitalizing on the linear interpolation of discount factor formula, we can then solve this unknown discount factor using Brentq root searcher and obtain the bootstrapped discount factors, linear interpolation formula:

$$D_o(0,T_n) = D_o(0,T_{n+i}) + \frac{i}{i+j} * \left[D_o(0,T_{n-j}) - D_o(0,T_{n+i})\right]$$

Using the bootstrapping algorithm, we obtain the OIS discount factors as shown below:

Tenor	DF	Tenor	DF	Tenor	DF	Tenor	DF	Tenor	DF	Tenor	DF
0.5	0.99875	5.5	0.97974	10.5	0.95314	15.5	0.92486	20.5	0.89744	25.5	0.87111
1.0	0.99701	6.0	0.97729	11.0	0.95030	16.0	0.92210	21.0	0.89481	26.0	0.86847
1.5	0.99527	6.5	0.97485	11.5	0.94747	16.5	0.91935	21.5	0.89218	26.5	0.86584
2.0	0.99353	7.0	0.97241	12.0	0.94463	17.0	0.91660	22.0	0.88954	27.0	0.86321
2.5	0.99177	7.5	0.96967	12.5	0.94179	17.5	0.91384	22.5	0.88691	27.5	0.86057
3.0	0.99002	8.0	0.96693	13.0	0.93896	18.0	0.91109	23.0	0.88428	28.0	0.85794
3.5	0.98807	8.5	0.96419	13.5	0.93612	18.5	0.90834	23.5	0.88164	28.5	0.85531
4.0	0.98612	9.0	0.96145	14.0	0.93328	19.0	0.90558	24.0	0.87901	29.0	0.85267
4.5	0.98415	9.5	0.95872	14.5	0.93045	19.5	0.90283	24.5	0.87637	29.5	0.85004
5.0	0.98218	10.0	0.95598	15.0	0.92761	20.0	0.90008	25.0	0.87374	30.0	0.84741

Q2: Bootstrap the LIBOR discount factor $D_L(0,T)$, and plot it for $T \in [0,30]$:

The LIBOR discount factor and its relationship with LIBOR is described as follows:

$$D_L\left(0,T_{\frac{i}{m}}\right) = \frac{1}{1+0.5*L\left(0,T_{\frac{i}{m}}\right)} \qquad L\left(T_{\frac{i-1}{m}},T_{\frac{i}{m}}\right) = \frac{D_L\left(0,T_{\frac{i-1}{m}}\right)-D_L\left(0,T_{\frac{i}{m}}\right)}{\frac{1}{m}*D_L\left(0,T_{\frac{i}{m}}\right)}$$

The par-swap-rate formula, $PV_{fix}^{IRS} = PV_{flt}^{IRS}$ is given as:

$$0.5 * IRS_N * \sum_{i=1}^{N*m} D_o\left(0,T_{\frac{i}{m}}\right) = 0.5 * \sum_{i=1}^{N*m} D_o\left(0,T_{\frac{i}{m}}\right) * L\left(T_{\frac{i-1}{m}},T_{\frac{i}{m}}\right)$$

With the above formulae, we can obtain the below by substituting the $L\left(0,T_{\frac{i}{m}}\right)$ into the par-swap-rate formula, $PV_{fix}^{IRS} = PV_{flt}^{IRS}$ (using 1-year IRS as example):

$$0.5 * IRS_1 * \left[D_o\left(0,T_{\frac{1}{2}}\right) + D_o(0,T_1)\right] = 0.5 * \left[D_o\left(0,T_{\frac{1}{2}}\right) * \frac{D_L(0,T_0) - D_L\left(0,T_{\frac{1}{2}}\right)}{\frac{1}{2} * D_L\left(0,T_{\frac{1}{2}}\right)} + D_o(0,T_1) * \frac{D_L\left(0,T_{\frac{1}{2}}\right) - D_L(0,T_1)}{\frac{1}{2} * D_L(0,T_1)}\right]$$

With the above formula, we only have only one unknown $D_L(0,T_1)$, and we can adopt similar approach in Q1 to bootstrap the LIBOR discount factors, namely the linear interpolation root searching approach:

$$D_L(0,T_n) = D_L(0,T_{n+i}) + \frac{i}{i+j} * \left[D_L\left(0,T_{n-j}\right) - D_L(0,T_{n+i})\right]$$

Therefore, the LIBOR discount factors are as follows:

Tenor	**DF**	**Tenor**	**DF**	**Tenor**	**DF**	**Tenor**	**DF**	**Tenor**	**DF**	**Tenor**	**DF**
0.5	0.98765	5.5	0.83280	10.5	0.67855	15.5	0.53679	20.5	0.39899	25.5	0.30671
1.0	0.97258	6.0	0.81660	11.0	0.66438	16.0	0.52251	21.0	0.38976	26.0	0.29748
1.5	0.95738	6.5	0.80041	11.5	0.65022	16.5	0.50822	21.5	0.38053	26.5	0.28826
2.0	0.94218	7.0	0.78422	12.0	0.63606	17.0	0.49394	22.0	0.37131	27.0	0.27903
2.5	0.92633	7.5	0.76897	12.5	0.62190	17.5	0.47965	22.5	0.36208	27.5	0.26980
3.0	0.91048	8.0	0.75371	13.0	0.60773	18.0	0.46536	23.0	0.35285	28.0	0.26057
3.5	0.89473	8.5	0.73846	13.5	0.59357	18.5	0.45108	23.5	0.34362	28.5	0.25135
4.0	0.87898	9.0	0.72321	14.0	0.57941	19.0	0.43679	24.0	0.33440	29.0	0.24212
4.5	0.86398	9.5	0.70796	14.5	0.56524	19.5	0.42250	24.5	0.32517	29.5	0.23289
5.0	0.84899	10.0	0.69271	15.0	0.55108	20.0	0.40822	25.0	0.31594	30.0	0.22366

Q3: Calculate the following forward swap rates:

The Forward Swap Rate formula is given by:

$$S(T_i, T_{i+t}) = \frac{0.5 * \sum_{n=i+1}^{i+t} D_o(0, T_n) * L(T_{n-1}, T_n)}{0.5 * \sum_{n=i+1}^{i+t} D_o(0, T_n)}$$

And the Forward Swap Rates are as follows:

Forward Swap Rates					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.032007	0.033259	0.034011	0.035255	0.038428
5Y	0.039274	0.040075	0.040072	0.041093	0.043634
10Y	0.042189	0.043116	0.044097	0.046249	0.053458

<u>OIS discount curve and LIBOR discount curve are shown as follow:</u>

Part II: B Swaption Calibration

Q1: Displaced-Diffusion calibration

In terms of model calibration, the primary objective is to achieve a match between the calibrated model implied volatilities and the implied volatilities observed in the market. An optimized function can be created to determine the optimal parameters which minimize the sum of squared error between the market implied volatilities and the implied volatilities calibrated from the Displaced-Diffusion model.

First, obtain the Swaption price and convert it to implied volatilities by using the DD model, which in essential the same as Black 76 model, except that it models the movement of shift. The formula of that is shown as a "weighted average" between a normal and a lognormal model, with β weight given to the normal model.

Formula for pricing the Swaption is given by:

$$V_{n,N}(0) = P_{n+1,N}(0) Black76\left(\frac{S_{n,N}(0)}{\beta}, K + \frac{1-\beta}{\beta} S_{n,N}(0), \sigma\beta, T\right)$$

Where PVBP is computed as:

$$P_{n+1,N}(t) = \sum_{i=n+1}^{N} \Delta_{i-1}\, D_i(t)$$

Also, in order to make the optimization result more stable, we choose to weight on ATM swaption more.

Calibrated DD Parameters:

Sigma					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.2250	0.2872	0.2978	0.2607	0.2447
5Y	0.2726	0.2983	0.2998	0.2660	0.2451
10Y	0.2854	0.2928	0.2940	0.2674	0.2437

Beta					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	5.025672e-08	1.848757e-07	1.079396e-11	0.000004	0.000007
5Y	1.425032e-12	5.500512e-08	2.277205e-06	0.000143	0.055462
10Y	1.395571e-07	7.489643e-06	8.154945e-05	0.000001	0.001745

Please refer to **Appendix for plotted DD presentation

Q2: SABR calibration

Using the similar method as Q1, first we get σ_{SABR}using the SABR pricing function, and then use the calibration function to minimize the sum of square error between $\sigma_{SABR}$and σ_{Market}, after which we could find out the parameters that fit the market best using least squares methods.

Calibrated SABR Parameters:

Alpha					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	0.139070	0.184651	0.196851	0.178052	0.170749
5Y	0.166527	0.199497	0.210348	0.191091	0.177421
10Y	0.177383	0.195282	0.207103	0.201575	0.181396

Rho					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	-0.633224	-0.525113	-0.482846	-0.414426	-0.256539
5Y	-0.585127	-0.546873	-0.549775	-0.511469	-0.440661
10Y	-0.545252	-0.544508	-0.550637	-0.562735	-0.513125

Rho					
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**5Y**	**10Y**
1Y	2.049486	1.677386	1.438137	1.064877	0.781464
5Y	1.339590	1.061907	0.936714	0.671422	0.494049
10Y	1.006550	0.925535	0.868939	0.720157	0.577509

Please refer to **Appendix for plotted SABR presentation

Q3: Price the swaptions using calibrated Displaced-diffusion and SABR model.

We set an interpolation function using cubic spline method to acquire calibrated sigma, alpha, beta, rho and nu for 2y x 10y and 8y x10y respectively, using the parameters we obtained from the two parts above.

Pricing results with different strikes

Strikes	**1%**	**2%**	**3%**	**4%**	**5%**	**6%**	**7%**	**8%**
PayerDD_2x10	0.288142	0.194936	0.112326	0.051345	0.017366	0.004106	0.000651	0.000067
PayerSABR_2x10	0.289615	0.198313	0.115173	0.052170	0.021481	0.010869	0.006740	0.004745
ReceiverDD_8x10	0.018985	0.033904	0.056649	0.088980	0.132050	0.186136	0.250582	0.323971
ReceiverSABR_8x10	0.019194	0.038398	0.061147	0.090392	0.130409	0.186129	0.257247	0.338370

The data above implies that as the strikes become higher, the payer swaption would be cheaper, while it is the opposite for the receiver one. Furthermore, when k increases, receiver swaption priced with DD model would be closer to that with SABR model.

Part III: Convexity Correction

Q1: Calculate PV of a leg receiving CMS10y semi-annually over the next 5 years.
Calculate PV of a leg receiving CMS2y quarterly over the next 10 years.

A CMS leg is a collection of CMS rates paid over a period. To calculate the PV, we need the following formula:

$$PV_{CMS10y} = D\,(0{,}6m) \times 0.5 \times CMS\left(S_{6m,10y6m}(6m)\right) + D\,(0{,}1y) \times 0.5 \times$$
$$CMS\left(S_{1y,11y}(1y)\right) + \cdots + D\,(0{,}5y) \times 0.5 \times CMS\left(S_{5y,15y}(15y)\right)$$
$$PV_{CMS2y} = D\,(0{,}3m) \times 0.25 \times CMS\left(S_{3m,2y3m}(3m)\right) + D\,(0{,}6m) \times 0.5 \times$$
$$CMS\left(S_{6m,2y6m}(6m)\right) + \cdots + D\,(0{,}10y) \times 0.25 \times CMS\left(S_{10y,12y}(10y)\right)$$

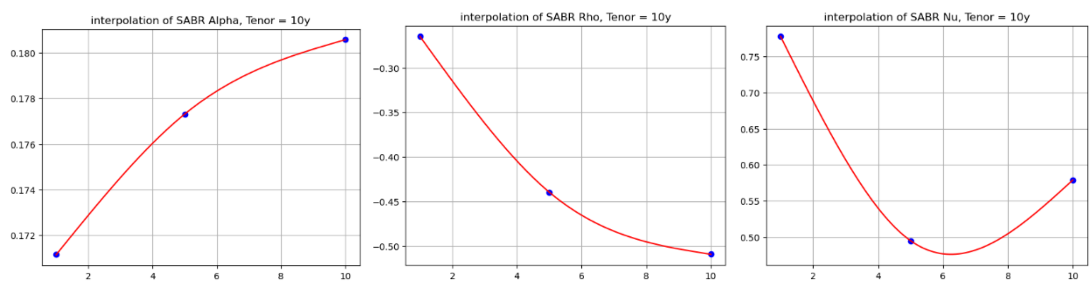
And the $CMS\left(S_{n,N}(T_i)\right)$ is given by:

$$E^T[S_{n,N}\,(T)] = g(F) + \frac{1}{D_0(0,T)}\left[\int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK\right]$$

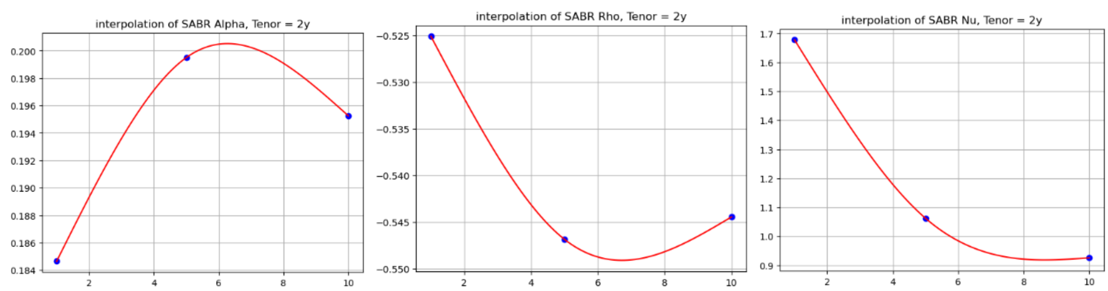
IRR-settled option price (V^{rec} or V^{pay}) is given by:

$$V_{n,N}(0) = D_0(0,T_n) \times IRR(S_{n,N}(0)) \times Black76(S_{n,N}(0), K, \sigma_{SABR}, T)$$

To calculate PV of leg receiving CMS10y semi-annually over the next 5 years, we need to find out SABR parameters at different expiries in order to price each CMS rate. Hence, we use cubic spline interpolation between alpha, rho, and nu of 1y × 10y, 5y × 10y and 10y × 10y SABR models. The profiles are demonstrated below:



Similarly, for CMS2y processed quarterly, alpha, rho, and nu can be interpolated between 1y×2y, 5y×2y,10y×2y, whose profiles are demonstrated below:



Based on our calculation, **the results of question 1 are**:

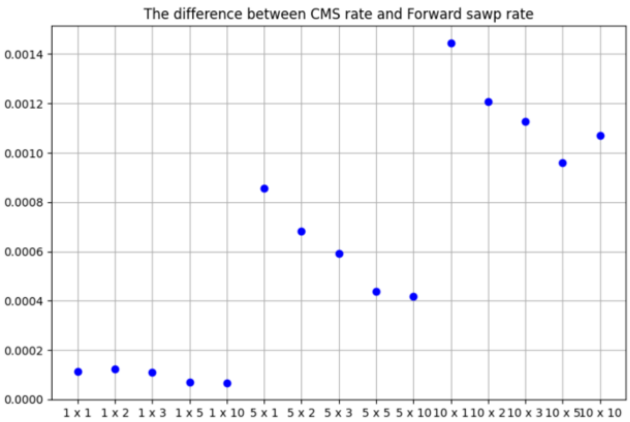
The PV of a leg receiving CMS10y semi-annually over the next 5 years is **0.20209.**

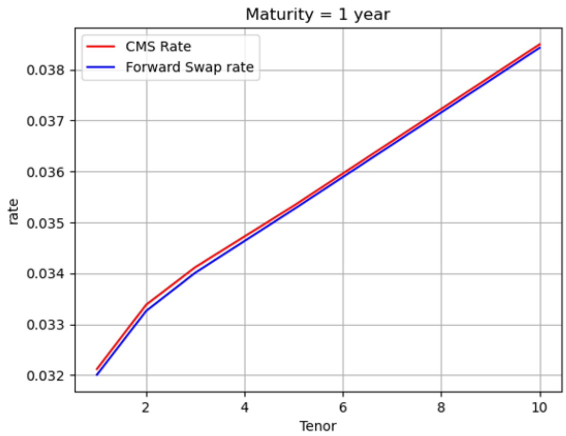
The PV of a leg receiving CMS2y quarterly over the next 10 years is **0.38106.**

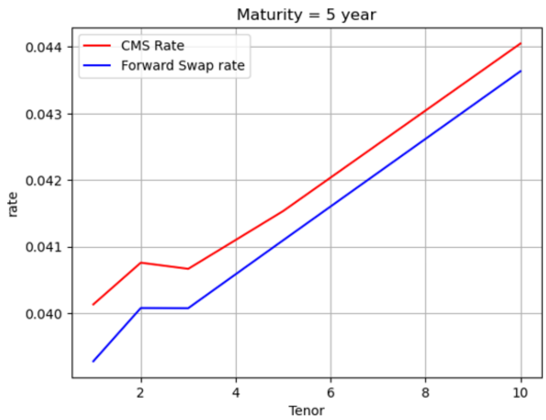
Q2: Compare the forward swap rates with the CMS rates:

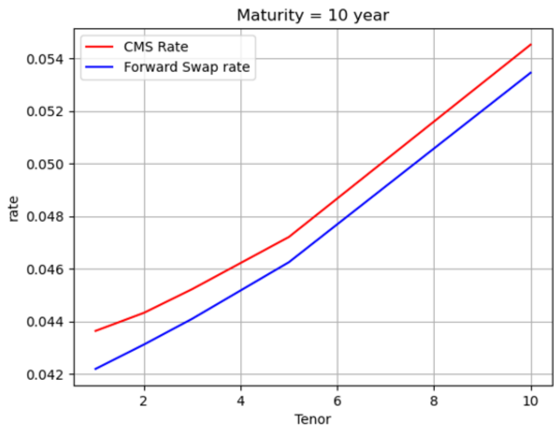
The results are as follows:

Terms	CMS rate	Forward swap rate	Difference	Terms	CMS rate	Forward swap rate	Difference
1 x 1	0.03212	0.032007	0.000113	**5 x 5**	0.041532	0.041093	0.0004391
1 x 2	0.033382	0.033259	0.000122	**5 x 10**	0.044051	0.043634	0.0004169
1 x 3	0.03412	0.034011	0.000109	**10 x 1**	0.043635	0.042189	0.0014455
1 x 5	0.035326	0.035255	0.000070	**10 x 2**	0.044322	0.043116	0.0012061
1 x 10	0.038496	0.038428	0.000068	**10 x 3**	0.045225	0.044097	0.0011280
5 x 1	0.040129	0.039274	0.000855	**10 x 5**	0.047209	0.046249	0.0009604
5 x 2	0.040756	0.040075	0.000681	**10 x 10**	0.054526	0.053458	0.0010687
5 x 3	0.040664	0.040072	0.000591				









Comparing CMS rates with forward swap rates of corresponding expiry and tenor from Part 1, we can observe that, CMS rate is always higher than the forward swap rate, and the difference can be explained by convexity adjustment. Also, we can recognize that the difference between CMS and forward swap rate increases as the expiry lengthens. It means that the longer expiry becomes, the greater the magnitude of convexity correction grows.

On the contrary, Tenor has little impact on convexity correction.

Part IV: Decompounded Options

Question 1: **Use static replication to value the PV of:** $\boldsymbol{CMS\ 10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}}}$ **at time T=5y:**

We start with:

$$D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK,$$

Where, $h(K) = \frac{g(K)}{IRR(K)}$,

$$h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2} ,$$

$$h''(K) = \frac{IRR(K)g''(K) - IRR''(K)g(K) - 2 \cdot IRR'(K)g'(K)}{IRR(K)^2} + \frac{2 \cdot IRR'(K)^2 g(K)}{IRR(K)^3} ,$$

$$g(K) = K^{\frac{1}{4}} - 0.04^{\frac{1}{2}}, \qquad g'(K) = \frac{1}{4} \cdot K^{-\frac{3}{4}}, \qquad g''(K) = -\frac{3}{16} \cdot K^{-\frac{7}{4}}$$

Discount factor is the 5y OIS discount factor from part 1. F is the $5y \times 10y$ forward swap calculated from part 1. V^{rec} and V^{pay} are European put and call that can be calculated by Black76, where their volatility is found by SABR model from part 2, calibrated parameters located at 5y expiry, 10y tenor.

The PV of this payoff is **<u>0.249863</u>**.

Question 2: **Use static replication to value the PV of:** $\boldsymbol{\left(CMS\ 10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}}\right)^+}$ **at time T=5y:**

This is a CMS caplet. We only need to value to part where $F^{\frac{1}{4}} - 0.04^{\frac{1}{2}} > 0$, which is $F > X = 0.0016$. We start with:

$$Caplet = h'(X)V^{pay}(X) + \int_X^\infty h''(K)V^{pay}(K)dK$$

The PV of this payoff is **<u>0.031031</u>**.

Appendix:

Displaced-Diffusion vol vs. SABR vol vs. Market vol: