**QF 620 - Stochastic Modelling in Finance**

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**Part I (Analytical Option Formulae)**
The first four pricing models we will be going over are as follows; Black-Scholes model, Bachelier Model, Black Model (1976) and Displaced Diffusion (alternatively, Displaced Black Scholes).

As we know, vanilla call and put options are not exactly vanilla; they can be decomposed into digital asset or nothing and cash or nothing components.

$$C(K) = S_{t\,S_t>K} - K \cdot 1_{S_t>K} \quad P(K) = K \cdot 1_{S_t<K} - S_{t\,S_t<K}$$

Where $K$ is the strike price, and $S_t$ is the underlying stock price at time $t$ . $S_{t\,S_t>K}$, $S_{t\,S_t<K}$, $1_{S_t>K}$, $1_{S_t<K}$ denotes asset-or-nothing call and put, and cash-or-nothing call and put digital options respectively.

Therefore, by constructing just 4 option pricing for each model, Call and Put for each of Cash-or-nothing or Asset-or-nothing option, we will also have the building block ready to implement vanilla call and put. This lent itself to object-oriented-programming style very well, and our implemented class structure are as follows:

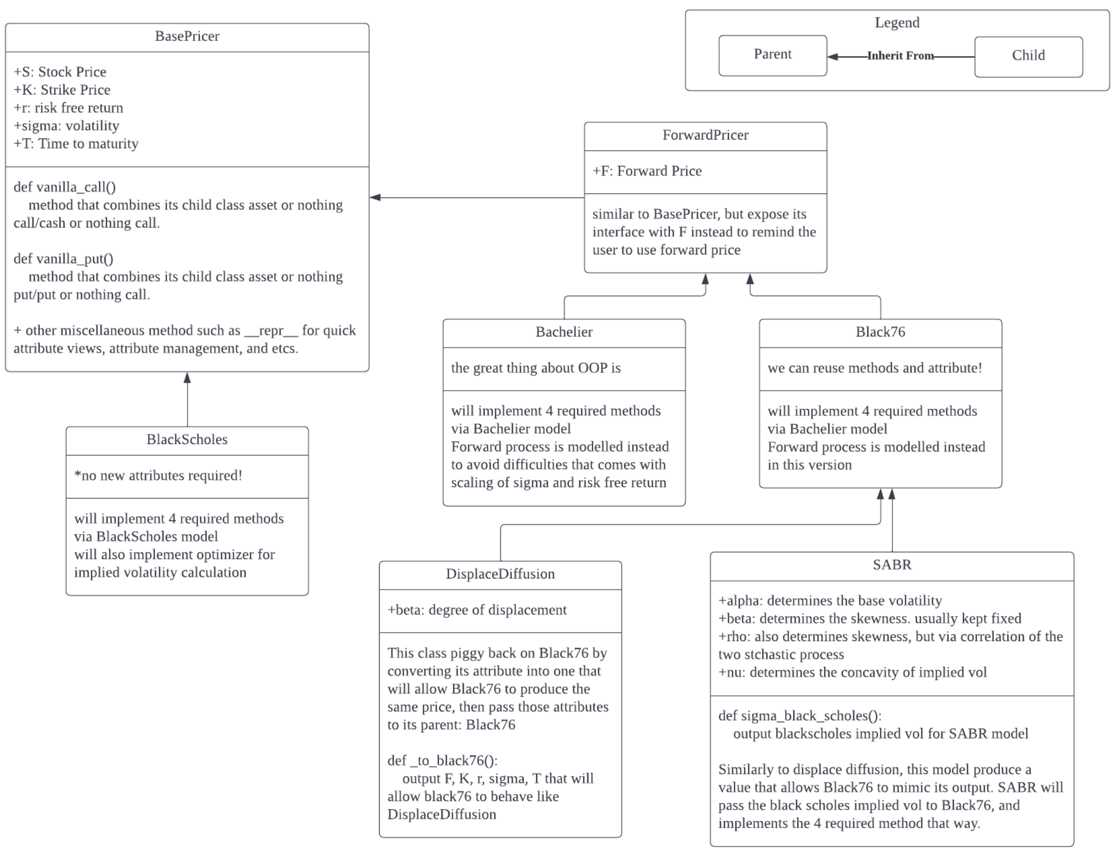



Figure 1. Class structure for derivation of option formulas

For each class implementation, we proceed to derive the valuation formula for the 4 digital options. By inheriting from BasePricer/ForwardPricer, the method that combines the 4 digital options into vanilla call and put will be added automatically.

Next is the actual derivation of the 4 digital options for each of the classes. $r$ denotes the risk-free rate, $\sigma$ is the standard deviation (volatility), $T$ is the time at maturity, $\Phi$ and $\phi$ are the CDF and PDF of normal distributions, respectively. We would not state any formula for vanilla put and call since the general formula is the same for every model.

$$C(K) = S_{t\,S_t>K} - K \cdot 1_{S_t>K} \quad P(K) = K \cdot 1_{S_t<K} - S_{t\,S_t<K}$$

**BlackScholes:** $dS_t = rS_t dt + \sigma S_t dW_t^*$

| **BlackScholes** | Call | Put |
|---|---|---|
| Cash or Nothing ($1_{S_t>K}$) | $e^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$ | $e^{-rT}\Phi\left(\frac{\log\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$ |
| Asset or Nothing ($S_{t\,S_t>K}$) | $S_0\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$ | $S_0\Phi\left(\frac{\log\left(\frac{K}{S_0}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$ |

Table 1. Black-Scholes option valuations

**Bachelier:** $dF_t = \sigma_{ba} W_t^*$

Note that we choose to model forward process here instead so we can simplify the risk-free rate scaling issue with Bachelier. $\sigma_{ba}$ for Bachelier will always be scaled up with $\sigma_{ba} = \sigma F_0$. $F_t$ is equal to forward price calculated at time $t$ where $F_t = S_t e^{r(T-t)}$ and $F_0 = S_0 e^{rT}$

| **Bachelier** | Call | Put |
|---|---|---|
| Cash or Nothing ($1_{S_t>K}$) | $e^{-rT}\Phi\left(\frac{F_0 - K}{\sigma_{ba}\sqrt{T}}\right)$ | $e^{-rT}\Phi\left(\frac{F_0 - K}{\sigma_{ba}\sqrt{T}}\right)$ |
| Asset or Nothing ($S_{t\,S_t>K}$) | $e^{-rT}\left[F_0\Phi\left(\frac{F_0 - K}{\sigma_{ba}\sqrt{T}}\right) + \sigma_{ba}\sqrt{T}\phi\left(\frac{F_0 - K}{\sigma_{ba}\sqrt{T}}\right)\right]$ | $e^{-rT}\left[F_0\Phi\left(\frac{K - F_0}{\sigma_{ba}\sqrt{T}}\right) - \sigma_{ba}\sqrt{T}\phi\left(\frac{K - F_0}{\sigma_{ba}\sqrt{T}}\right)\right]$ |

Table 2. Bachelier option valuations

**Black76:** $dF_t = \sigma F_t W_t^*$

| **Black76** | Call | Put |
|---|---|---|
| Cash or Nothing ($1_{S_t>K}$) | $e^{-rT}\Phi\left(\frac{\log\left(\frac{F_0}{K}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$ | $e^{-rT}\Phi\left(\frac{\log\left(\frac{F_0}{K}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$ |
| Asset or Nothing ($S_{t\,S_t>K}$) | $e^{-rT}F_0\Phi\left(\frac{\log\left(\frac{F_0}{K}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$ | $e^{-rT}F_0\Phi\left(\frac{\log\left(\frac{F_0}{K}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$ |

Table 3. Black76 option valuations

**Displaced-Diffusion:** $dF_t = \sigma[\beta F_t - (1-\beta)F_0]dW_t$

Here we convert our parameter as follows:

$$F_0^{dd} = \frac{F_0}{\beta}, \qquad K^{dd} = K + \left(\frac{1-\beta}{\beta}\right) F_0, \quad \sigma_{dd} = \sigma\beta$$

Using the new parameter, we reuse the Black76 model, with some slight modifications

| **Displaced-Diffusion** | Call | Put |
|---|---|---|
| Cash or Nothing ($1_{S_t>K}$) | $e^{-rT}\Phi\left(\frac{\log\left(\frac{F_0^{dd}}{K^{dd}}\right) - \frac{\sigma_{dd}^2}{2}T}{\sigma_{dd\sqrt{T}}}\right)$ | $e^{-rT}\Phi\left(\frac{\log\left(\frac{F_0^{dd}}{K^{dd}}\right) + \frac{\sigma_{dd}^2}{2}T}{\sigma_{dd\sqrt{T}}}\right)$ |
| Asset or Nothing ($S_{t\, S_t>K}$) | $e^{-rT}\left[F_0^{dd}\Phi\left(\frac{\log\left(\frac{F_0^{dd}}{K^{dd}}\right) + \frac{\sigma_{dd}^2}{2}T}{\sigma_{dd\sqrt{T}}}\right) - \left(\frac{1-\beta}{\beta}\right)F_0 \cdot 1_{S_t>K}\right]$ | $e^{-rT}\left[F_0^{dd}\Phi\left(\frac{\log\left(\frac{F_0^{dd}}{K^{dd}}\right) - \frac{\sigma_{dd}^2}{2}T}{\sigma_{dd\sqrt{T}}}\right) - \left(\frac{1-\beta}{\beta}\right)F_0 \cdot 1_{S_t<K}\right]$ |

Table 4. Displaced-Diffusion option valuations

One feature to note is that the asset-or-nothing options for Displaced-Diffusion (DD) model also has a valuation of a cash-or-nothing. This is attributed to the fact that the subtraction of $\left((1-\beta)/\beta\right)F_0$ to the strike price does not actually affect the strike price of the option, and it would also be priced into the asset-or-nothing digital option payoff. The general formula of $C(K) = S_{t\, S_t>K} - K \cdot 1_{S_t>K}$ for call options will still be valid here, as well as for put options.

The above implementation can be easily done via inheriting the Black76 class, and utilizing the super() method of python to let DD call Black76's method with the modified attributes $F_0^{dd}$, $K^{dd}$, $\sigma_{dd}$.

The black implied vol approximator, which we will cover in part 2, is also implemented within this class.

**SABR:** $dF_t = \alpha_t F_t^{\beta} dW_t^F,\ d\alpha_t = \nu\alpha_t dW_t^{\alpha},\ dW_t^F dW_t^{\alpha} = \rho dt$

Although not required in part 1, later parts will require us to implement the SABR model and so it would be more coherent if we briefly cover how we have implemented SABR within our framework in this part.

The SABR model can be easily implemented if we already have a function that reports the SABR's implied vol in lognormal model (Black Scholes). We can use it to calculate $\sigma_{\text{SABR}}$, which, given the same set of $F_0$, $K$, $r$, $T$ will allow the black76 model to output the price that would be output by SABR model. The equation of this approximation will be covered in part 2.

In a similar fashion to the DD model, we have implemented SABR by inheriting from Black76.

The python codes of the above implementation will be in the file option_pricer.py which also contains unit tests for put-call parity, amongst other.

# Part II (Model Calibration)

In order to calibrate the DD and SABR model effectively, we need to be able to invert the volatility from each respective model to Black-Scholes implied volatility. An approximation functions we will use are:

**SABR model** (Hagan et al., 2014, Eq. (14))

$$\sigma_{\text{BS}-\text{SABR}} = \frac{\alpha}{(F_0K)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{F_0}{K}\right)+\frac{(1-\beta)^4}{1920}\log^4\left(\frac{F_0}{K}\right)+\ \dots\right\}}$$

$$\times\frac{z}{x(z)}\times\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(F_0K)^{(1-\beta)/2}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(F_0K)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]T+\cdots\right\}$$

Where:

$$z=\frac{\nu}{\alpha}(F_0K)^{(1-\beta)/2}\log\left(\frac{F_0}{K}\right),\ x(z)=\log\left[\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right]$$

**Displaced-Diffusion model** (Choi et al., 2022, Eq. (20)):

$$\sigma_{\text{BS-DD}}=\sigma\frac{D(F_0)}{F_0}\sqrt{\left(\frac{k_D}{k}\right)\frac{1+(\log^2 k_D)/24}{1+(\log^2 k)/24}\frac{1+\sigma^2(D(F_0)/F_0)^2(k_D/k)T/24}{1+\beta^2\sigma_D T/24}}$$

Where:

$$D(x)\ =\ \beta x\ -\ (1-\beta)A,\ k\ =\ \frac{K}{F_0},\ k_D=\frac{D(K)}{D(F_0)},\text{ and without loss of generality, }A\ =\ F_0$$

We can now numerically optimize for the parameters to approach observed implied volatility as much as possible. The implied volatility target is those reported by Black-Scholes model. We also assume the early exercise premium should be negligible for model calibrations. The results are shown in Figure 2 and 3, and Table 5:

Unfortunately, Displaced-Diffusion model approaches Bachelier as it tries to fit to the market volatility skew as seen in how its beta value are ≈ 0 which will cause DD to approach Bachelier (this is also one of the unit tests in option_pricer.py!). Conversely, $\beta\approx 1$ makes DD approaches Black-Scholes.

| | Underlying | SPX | | | SPY | | |
|---|---|---|---|---|---|---|---|
| Model | Param\T | 17 | 45 | 80 | 17 | 45 | 80 |
| Displace Diffusion | Beta ($\beta$) | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| | Sigma ($\sigma$) | 0.5345 | 0.3786 | 0.3345 | 0.5433 | 0.4218 | 0.2687 |
| SABR | Alpha ($\alpha$) | 1.212 | 1.8166 | 2.1398 | 0.6644 | 0.9085 | 1.1209 |
| | Rho ($\rho$) | -0.3009 | -0.4044 | -0.5748 | 0.4118 | 0.4889 | 0.633 |
| | Nu ($\nu$) | 5.4602 | 2.7902 | 1.8421 | 5.2524 | 2.728 | -1.7422 |

Table 5. Fitted Parameters

Contrastingly, SABR model has a much more robust degree of freedom, and quite accurately captures the volatility skew.

With positive $\rho$, volatility will be more volatile with positive returns, something which is not as commonly observed as the opposite. With negative $\rho$, we end up simulating a negative skewness where highly volatile periods are associated with losses. This exhibits itself as the implied volatility skewness where out-the-money put are more valued than out-the-money call.

With high $\nu$, we observe increased concavity as volatility is allowed to fluctuate more. With more uncertainty in the magnitude of volatility, the implied volatility of the options far out of the money can be increased higher and reflect the use of option as an insurance for unexpected volatility spike.

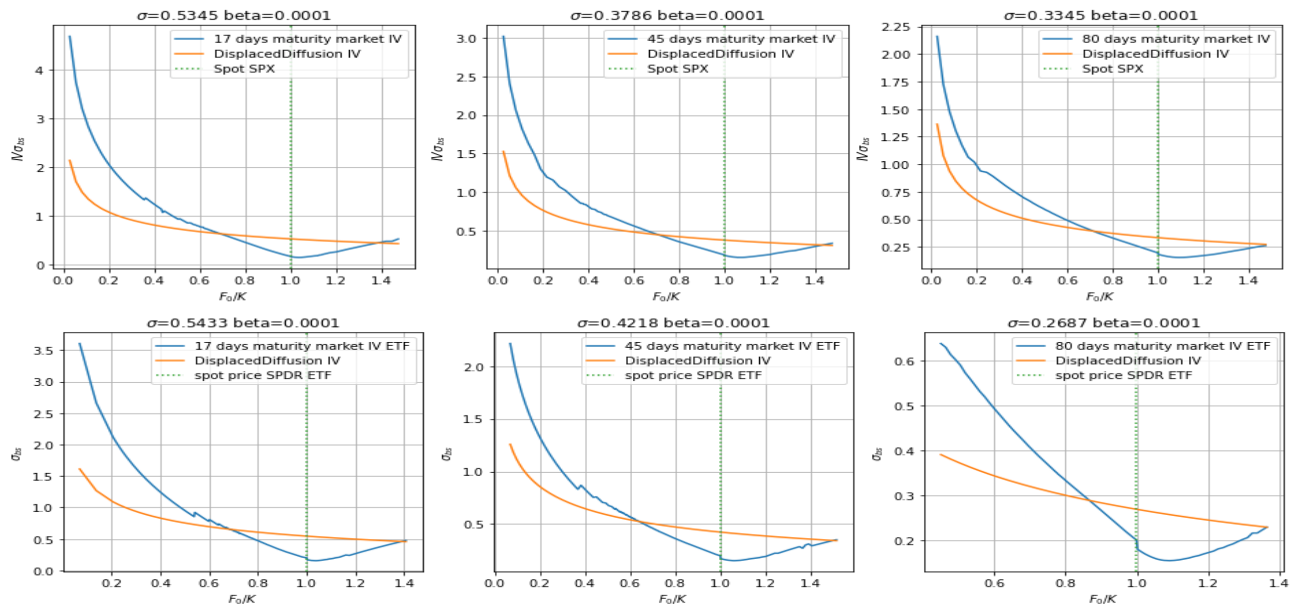



Figure 2. Displace Diffusion implied volatility curve

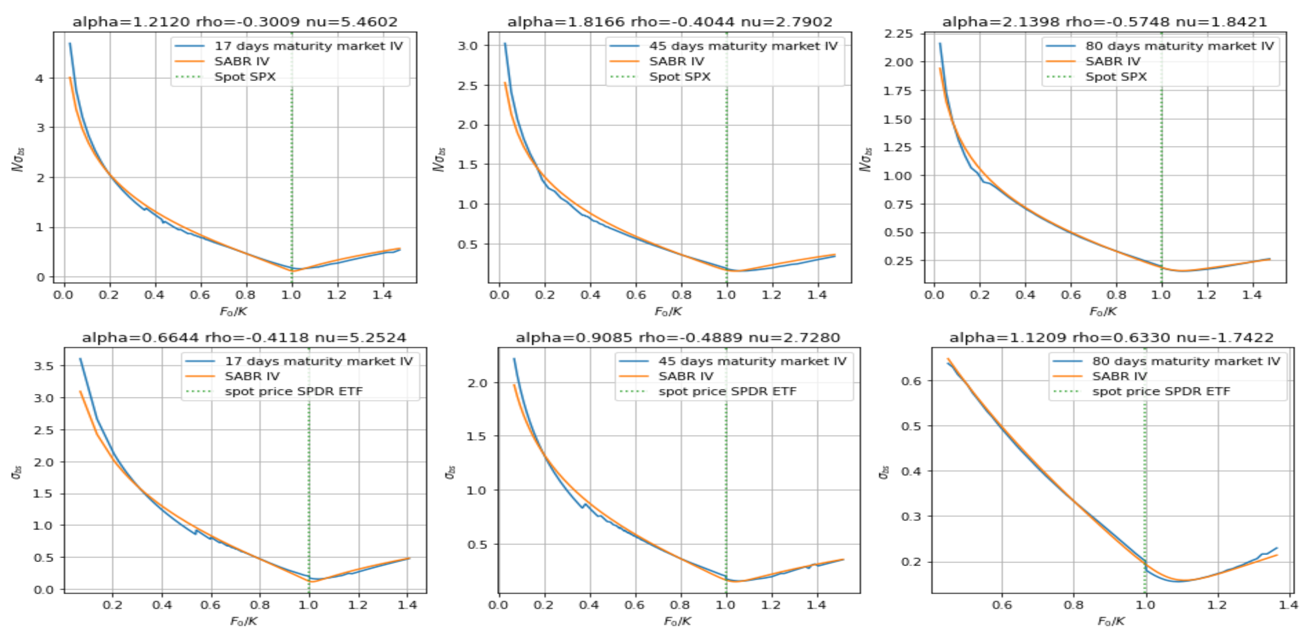



Figure 3. SABR implied volatility curve

## Part III (Static Replication)

We first derive the implied volatility using Black Scholes call and put options pricing formula following the SPY index price. Following the derivative expected payoff, $S_T^{\frac{1}{3}} + 1.5 \times \log(S_T) + 10.0$ and model free integrated variance $\sigma_{MF}^2 T = E\left[\int_0^T \sigma^2\, t\, dt\right]$, we first derive the expectation of the model free integrated variance.

$$E\left[\int_0^T \sigma_t^2\, dt\right] = 2e(rT)\int_0^F \frac{P(K)}{k^2} dK + 2e(rT)\int_0^F \frac{C(K)}{k^2} dK$$

Thereafter we derive the sigma from "At-the-Money" options volatility and apply it to the Black Scholes option pricing model. Using the models that we have developed in part 1, we can use it as the integrand of the above integral to calculate the integrated variance of **0.004236501** and integrated volatility of **0.1853718792209719** for Black-Scholes. This is exactly the value of at-the-money implied volatility, which makes sense as we used it to calculate the $P(K)$, $C(K)$ used for the integrand. The Bachelier Model also arrives at a similar integrated variance of **0.004263876** and volatility of **0.1859698234524026**.

As for the exotic derivatives, we get the following payoff under Black-Scholes:

$$S_0^{\frac{1}{3}} \exp\left\{\left(r - \frac{\sigma^2}{3}\right)\frac{T}{3}\right\} + \left(\frac{3}{2}\right)\log(S_0) + \left(\frac{3}{2}\right)\left(r - \frac{\sigma^2}{2}\right)T + 10$$

We get the option price of **37.714081753** using Black Scholes model.

For Bachelier model:

$$V_0 = e^{-rT} E\left[\left(F_0 + \sigma\sqrt{T}x\right)^{\frac{1}{3}} + 1.5\log\left(F_0 + \sigma\sqrt{T}x\right) + 10\right]$$

Using the formula for raw moments (Winkelbauer, 2012):

$$E\{X^\nu\} = (j\sigma)^\nu 2^{\nu/2}\left[\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)_1} F_1\left(-\frac{\nu}{2}, \frac{1}{2}; -\frac{\mu^2}{2\sigma^2}\right) + j\frac{\mu}{\sigma}\frac{\sqrt{2\pi}}{\Gamma\left(-\frac{\nu}{2}\right)_1} F_1\left(\frac{1-\nu}{2}, \frac{3}{2}; -\frac{\mu^2}{2\sigma^2}\right)\right]$$

Where $X \sim N(\mu, \sigma^2)$ and $F_1(\alpha, \gamma; z)$ being Kummer's confluent hypergeometric function. Going through the calculations, Bachelier model gives us option valuation at **37.704720822**

After calibrating the implied volatility to the market volatility using the SABR model, proceed to use it to calculate put and call price for static replication. The integral to be calculated are derived as follows:

$$V_0 = e^{-rT}F_0 + \int_0^F \left(-\frac{2}{9}\frac{P(K)}{K^{\left(\frac{5}{3}\right)}} - \frac{1.5}{K^2}\right) dK + \int_F^\infty \left(-\frac{2}{9}\frac{C(K)}{K^{\left(\frac{5}{3}\right)}} - \frac{1.5}{K^2}\right) dK$$

After integrating the above, the exotic option price is **37.71098680265409.** We also found $\sigma$ to be **0.22669860136104247** from the model free integrated variance equation earlier. The expected value of integrated variance is much higher in this case as SABR model can capture the market behavior more accurately, especially when it comes to the skewness of implied volatility.

# Part IV (Dynamic Hedging)

The Black-Scholes assumes that the continuous hedging is possible. As long as this assumption hold, the final profit and loss will be exactly zero or there is no replication error regardless of the movement of stock price.

Dropping the Black-Scholes assumption about the possibility of continuous hedging, we investigate the error from replicating a single option throughout its lifetime according to the Black-Scholes replication strategy, with the constraint that only a discrete number of rebalancing trades at regular intervals are allowed.

Suppose $S_0 = \$100, \sigma = 0.2, r = 5\%, T = \frac{1}{12} year, and\ K = \$100$. We simulated 50,000 paths of possible future stock price with the help of Monte Carlo simulations to understand effect of the discrete hedging strategy on replication value. In addition, we assess 2 scenarios where we hedge 21 times and 84 times with the option time to expiration is 1 month which consists of 21 trading days. Black-Scholes assumes stock price evolves lognormally and follow:

$$S_T = S_0\, e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}$$

The dynamic hedging strategy for an option is:

$$C_t = \phi_t S_t - \psi_t B_t,$$

The hedging error is calculated from:

$$\text{Hedging error} = \mathrm{C_t}(\text{replicated position}) + \mathrm{C_0}(\text{call option price at time } 0) - \text{final call option payoff}$$

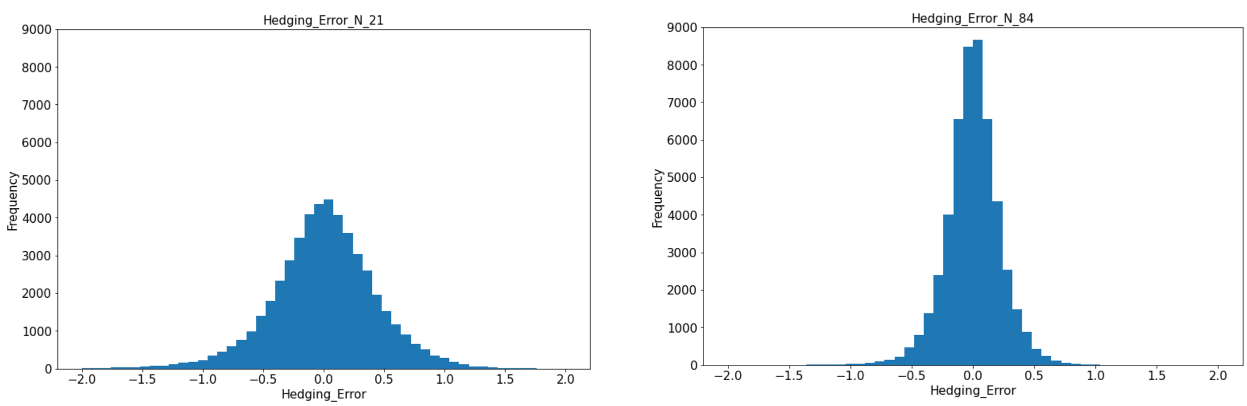



Figure 4. Hedging error for N = 21 (left) and N = 84 (right)

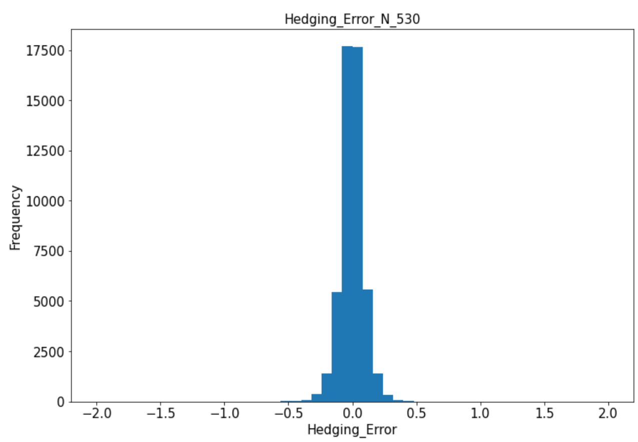
| Number of hedging | N = 21 | N = 84 |
|---|---|---|
| Mean of error | 0.003176580109724182 | 0.0009547969376775895 |
| Standard deviation of error | 0.42890919504675684 | 0.21788990986855641 |
| Standard deviation of error as percentage of option premium | 17.07% | 8.67% |

Table 6. Statistical summary of the 50,000 simulations

From the histogram (Figure 4) we can identify the range of hedging errors for 21 times hedging or once per trading day is approximately between -1.5 to 1.5 compared to -1 and 1 for 84 times hedging or 4 times per trading day.

As the hedging getting more frequent, the mean and standard deviation of hedging error becomes smaller and closer to 0. This also proven by statistical summary from Table 6 which shows the lower mean and standard deviation of error when we do more frequent hedging.

We also try to simulate higher frequency hedging with N = 530 or 25 times hedging in a day. The result of this simulation (Figure 5) is still in-line with our believe that the mean and standard error will get smaller and nearer to 0 as the number of hedging increase.




```
mean_error_N_530 =  0.00013470603967957233
standard_deviation_error_N_530 =  0.08747493993736252
standard_deviation_error_as_percentage of option_premium_N_530 =  3.482189644674707
```

Figure 5 Hedging error for N =530 and statistical summary on the error

As we rebalance and at discrete rather than continuous intervals, the hedge is imperfect, and the replication is inexact. We found that the typical error in the replication value is proportional to the vega of the option (its volatility sensitivity, Kappa) multiplied by the uncertainty in its observed volatility.

$$\sigma_{Error} = \sqrt{\frac{\pi}{4}}(\kappa)\frac{\sigma}{\sqrt{N}}$$

Here κ is the options vega, the standard Black-Scholes sensitivity of the option price with respect to volatility, evaluated at the initial spot and trading date.

$$\kappa = \frac{dC}{d\sigma}(t = 0, S = S_0) = \frac{S_{0\sqrt{T}}}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}$$

We found that κ is 0.1145/vol point and using that to estimate the $\sigma_{Error}$ are 0.4432 and 0.2216 for N=21 and N=84 respectively. These estimates are quite close to the simulation results of Table 6.

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