MCEN 3030

29 Feb 2024

No HW due Monday!

Last time: Exam

OK, before that: Fitting nonlinear models

Today: Nonlinear system<u>s</u>

w/ Newton-Raphson

Quick Linear Algebra Aside: Cramer's Rule

If we are trying to solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$ (and if a solution exists)

$$x_j = \frac{\det(\tilde{A}_j)}{\det(A)}$$

$\tilde{A}_j$ = the matrix w/ j^{th} column replaced by b

Ex: $\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$ $\qquad \det(A) = 20$

$$x_1 = \frac{1}{|\underline{\underline{A}}|} \left| \begin{bmatrix} 5 & 3 & -1 \\ 3 & 4 & -3 \\ 1 & 3 & -1 \end{bmatrix} \right| = \frac{20}{20} = 1$$

$$x_2 = \frac{1}{|\underline{\underline{A}}|} \left| \begin{bmatrix} 2 & 5 & -1 \\ 4 & 3 & -3 \\ -2 & 1 & -1 \end{bmatrix} \right| = \frac{40}{20} = 2$$

$$x_3 = \frac{1}{|\underline{\underline{A}}|} \left| \begin{bmatrix} 2 & 3 & 5 \\ 4 & 4 & 3 \\ -2 & 3 & 1 \end{bmatrix} \right| = \frac{60}{20} = 3$$

Previously: We considered linear systems

$$\left.\begin{aligned} x + 2y &= 5 \\ -x + 2y &= 3 \end{aligned}\right\} \longrightarrow \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$\Longrightarrow$ Only for linear systems,

can use LU decomposition, etc.

But: $$y^2 e^{-x/15} = -3$$
$$(y+1)\arctan\left(\frac{x}{9}\right) = 14$$

We <u>are</u> going to be able to solve this type of system using an iterative method

– Higher-dimensional Newton-Raphson

Newton's Method for 2+D nonlinear systems

Now that we introduced 2D Taylor Series, I want to circle back and talk about solving 2D nonlinear systems.

Our system:
$$f(x,y) = 0$$
$$g(x,y) = 0$$

& we guess the solution (a seed): (x_0, y_0).

Almost certainly $f(x_0, y_0) \neq 0$, $g(x_0, y_0) \neq 0$

$\rightarrow$ iterate on our guess to hopefully converge upon the true solution

Idea: Use a 2D Newton-Raphson Method to iterate on our guess and hopefully converge

Write our guess adjustments (in a Newton-Raphson sense) via a 2D Taylor Series

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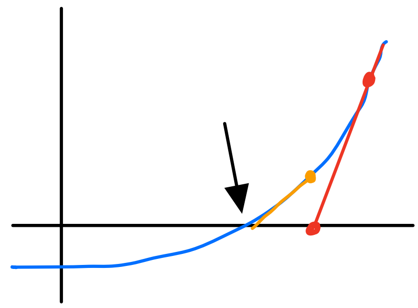
In 1D it was: (for $f(x)=0$)

$$f(x) \approx f(x_i) + \left.\frac{df}{dx}\right|_{x_i} \cdot (x - x_i)$$



$$\xrightarrow[\text{solve for } x = x_{i+1}]{f(x)=0} \qquad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

For $f(x,y)=0$ & $g(x,y)=0$ (linear approx)

$$f(x,y) \approx f(x_i,y_i) + \left.\frac{\partial f}{\partial x}\right|_{(x_i,y_i)} \cdot \overset{\Delta x}{(x - x_i)} + \left.\frac{\partial f}{\partial y}\right|_{(x_i,y_i)} \cdot (y - y_i)$$

$$g(x,y) \approx g(x_i,y_i) + \left.\frac{\partial g}{\partial x}\right|_{(x_i,y_i)} \cdot (x - x_i) + \left.\frac{\partial g}{\partial y}\right|_{(x_i,y_i)} \cdot (y - y_i)$$

$$\Longrightarrow \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -f(x_i,y_i) \\ -g(x_i,y_i) \end{bmatrix}$$

Jacobian Matrix

eval (x_i,y_i)

$\rightarrow$ Δx & Δy are how much to adjust x_i & y_i to get $f(x_i,y_i)$ & $g(x_i,y_i)$ closer to zero.

$$x_{i+1} = x_i + \Delta x \quad \& \quad y_{i+1} = y_i + \Delta y$$

Can solve for $\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}^{-1} \begin{bmatrix} -f \\ -g \end{bmatrix}$

eval at (x_i, y_i)

or use Cramer's Rule

$$\Delta x = \frac{\left| \begin{bmatrix} -f & \frac{\partial f}{\partial y} \\ -g & \frac{\partial g}{\partial y} \end{bmatrix} \right|}{\det(\text{Jacobian})} \quad \& \quad \Delta y = \frac{\left| \begin{bmatrix} \frac{\partial f}{\partial x} & -f \\ \frac{\partial g}{\partial x} & -g \end{bmatrix} \right|}{\det(\text{Jacobian})}$$

$$\Longrightarrow \quad \Delta x = \frac{-f \cdot \frac{\partial g}{\partial y} + \frac{\partial f}{\partial y} \cdot g}{\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \cdot \frac{\partial f}{\partial y}}$$

$$\Delta y = \frac{-\frac{\partial f}{\partial x} \cdot g + f \frac{\partial g}{\partial x}}{\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \cdot \frac{\partial f}{\partial y}}$$

evaluated @ current guess (x_i, y_i)