MCEN 3030

6 Feb 2024

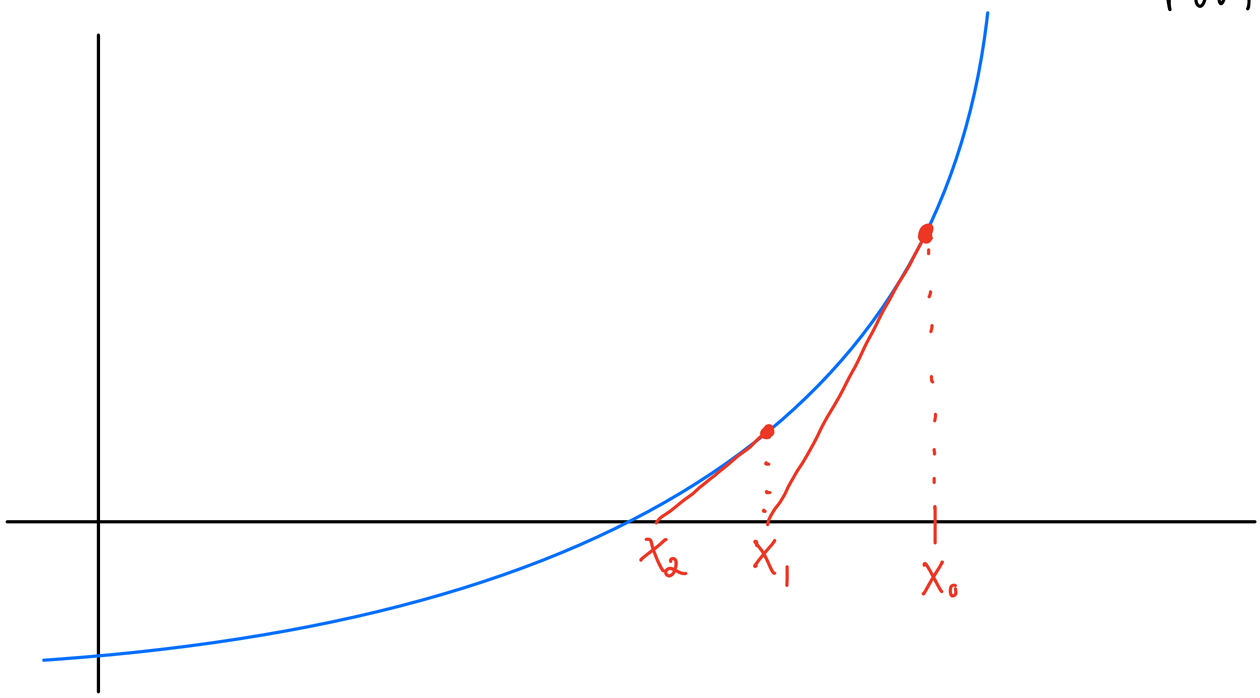
HW & Quiz #2 Due Sunday

Last week: Root finding

- open and bracketing methods

Today: Another open method

Last week: Newton's Method – an "open" method for root finding



Algorithm:

1) Start with a guess (the "seed") for the root, x_0

2) Update our guess: $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

3) Iterate until we are happy with the convergence/it diverges (e.g. could set a max number of iterations).

- This algorithm converges quickly

- However, it may not converge at all or for certain seed values.

Fixed-Point Iteration Method

* We will program this one in class

Idea: Suppose we are interested in where $f(x) = 0$.

Find a way * to rearrange to $x = g(x)$

a quick example: $g(x) = x + f(x) \rightarrow$ seeking $x = g(x)$

If we find where $x = g(x) \Rightarrow f(x) = 0$.

Other examples below *

Algorithm:

1) Start with an initial guess (the "seed") x_0
2) Iterative procedure: $x_{i+1} = g(x_i)$
3) Proceed until a satisfactory level of convergence is achieved

- error $\equiv \left| \dfrac{x_{i+1} - x_i}{x_{i+1}} \right|$

- reach a defined number of iterations

* Ex: $f(x) = x^2 - x + 1$

Suppose $f(x) = x^2 - x + 1 = 0$

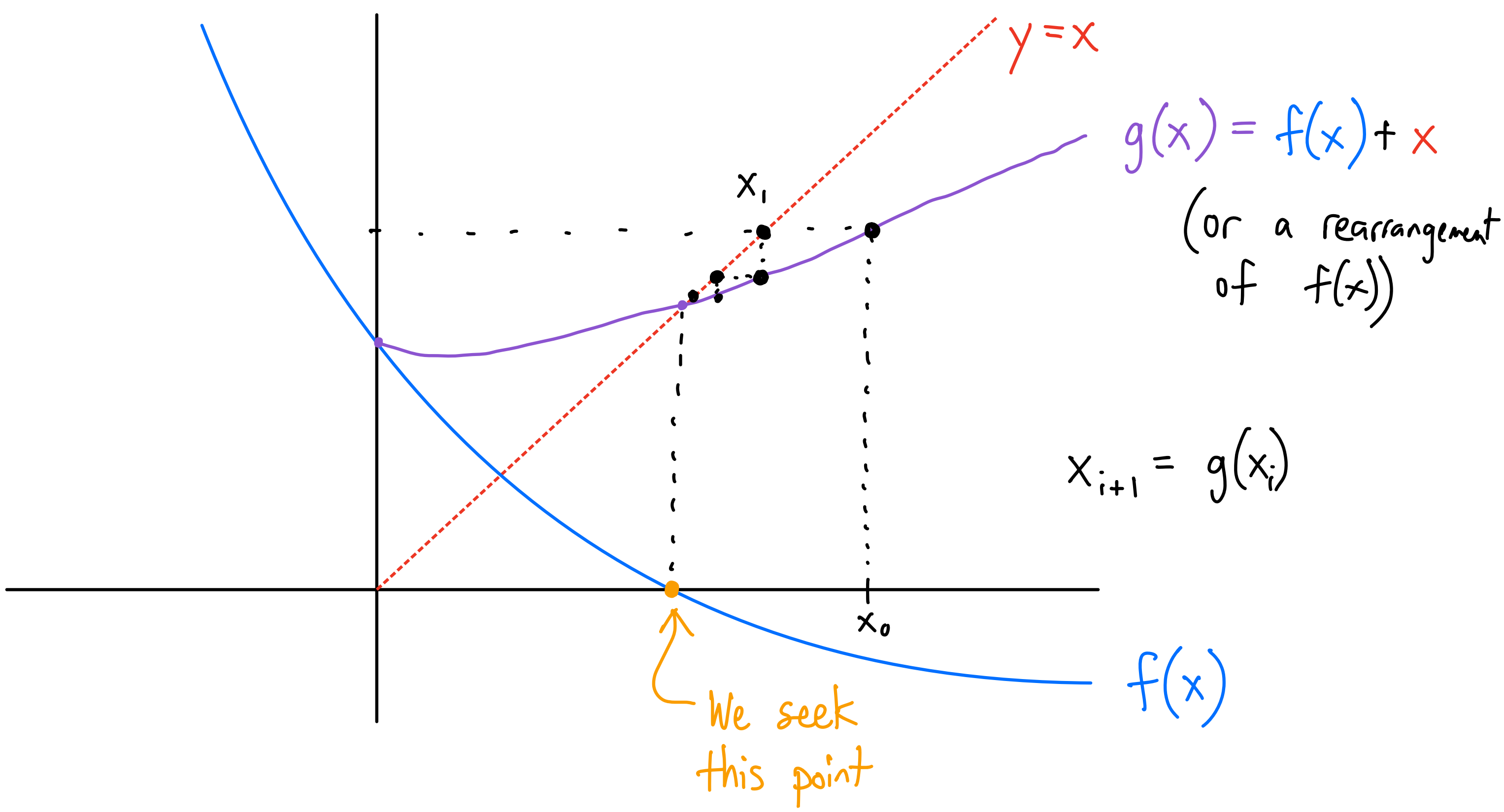
$$\Rightarrow \underbrace{x^2 + 1}_{g(x)} = x$$

So: when $g(x) = x^2 + 1 = x$

$$f(x) = (x^2 + 1) - x = 0 \quad \checkmark$$

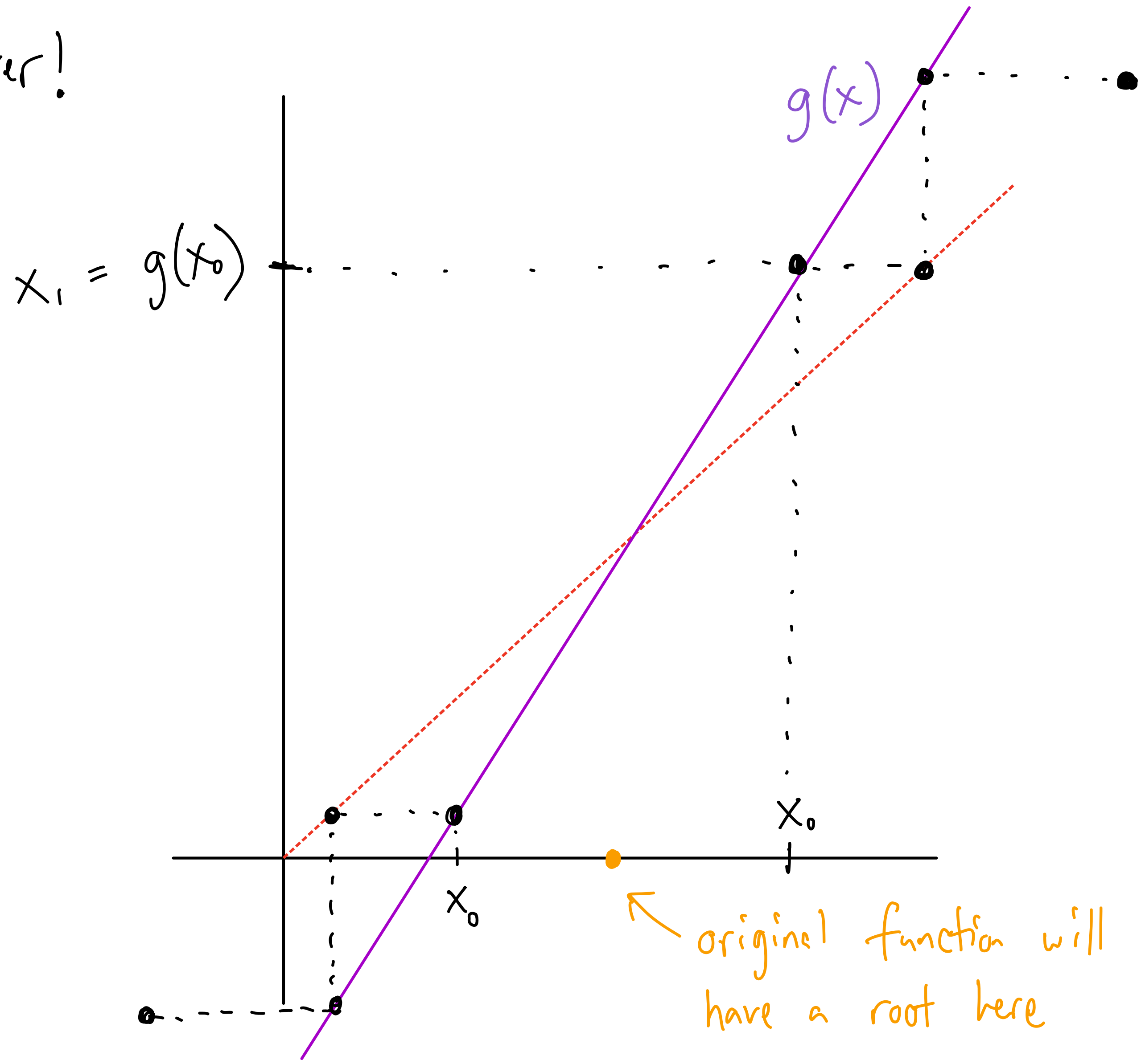
(More examples below)

What happens graphically



$$x_{i+1} = g(x_i)$$

However!



Does not converge!

A condition for convergence: $|g'(x)| < 1$ in the neighborhood of the root of $f(x)$

With the limitation $|g'(x) < 1|$, there may be some options for $g(x)$ that work, others that don't

$$f(x) = xe^{0.5x} + 1.2x - 5 \overset{?}{=} 0$$

Solution: $x \approx 1.50$

$$x = \underbrace{g(x)}$$

one option: $x = \dfrac{5 - 1.2x}{e^{0.5x}}$

$g'(1) = -1.88$
$g'(2) = -0.92$

Rearrangements to get $x =$ something

another: $x = \dfrac{5 - xe^{0.5x}}{1.2}$

$g'(1) = -2.06$
$g'(2) = -4.53$

$$\boxed{\text{another: } x = \frac{5}{e^{0.5x} + 1.2} \qquad g'(1) = -0.50 \quad g'(2) = -0.44}$$

another: $x = xe^{0.5x} + 2.2x - 5$

$f(x) + x$

$g'(1) = 4.67$
$g'(2) = 7.63$

Only this one is "robust" to seed choice.

Let's check the algebra with, e.g.

$$x = \frac{5 - xe^{0.5x}}{1.2} \equiv g(x)$$

$g(x)$ is defined as $\frac{5-xe^{0.5x}}{1.2}$

If this is true, $f(x) = 0$.

$$1.2x = 5 - xe^{0.5x} \quad \text{or} \quad xe^{0.5x} - 5 = -1.2x$$

Then
$$f(x) = xe^{0.5x} + 1.2x - 5$$
$$= \left(xe^{0.5x} - 5\right) + 1.2x$$
$$= -1.2x + 1.2x = 0$$

Yep!