MCEN 3030

9 Apr 2024

HW #7 Due Monday

Last time: ("Explicit") Euler's Method (see added notes)

Today: Improvements on Euler's Method

Last time: Euler's Method for finding $x(t)$

For $\frac{dx}{dt} = f(x,t)$, can turn into a

numerical scheme:

$$\frac{dx}{dt} \rightarrow \frac{x_{i+1} - x_i}{\underbrace{\Delta t}_{t_{i+1} - t_i}} = f(x_i, t_i)$$

$$\rightarrow \boxed{x_{i+1} = x_i + \Delta t \cdot f(x_i, t_i)}$$ "Explicit" Euler's Method

$$x_i + \Delta t \cdot \frac{\Delta x}{\Delta t}$$

- Basic numerical scheme, easy-to-describe (Taylor Series)

- But – unstable, meaning we can get ridiculous results if we naively implement this to any equation

- Fairly generally: we can get a decent approximation for $x(t)$ if we choose a small enough step size Δt.

"Implicit" Euler's Method

Idea: Evaluate $f(x,t)$ at next position?!

$$x_{i+1} = x_i + \Delta t \cdot f(x_{i+1}, t_{i+1})$$

x_{i+1} appears on both sides of the equation

→ use Newton-Raphson or Fixed-Point Iteration method to solve for x_{i+1}

e.g. $f(x,t) = t \sin x$

$$0 = -x_{i+1} + x_i + \Delta t \cdot \left(t_{i+1} \sin(x_{i+1})\right)$$

← function to find root for, call it $h(x_{i+1})$

If using Newton-Raphson:

$$\frac{dh}{dx_{i+1}} = -1 + \Delta t \cdot t_{i+1} \cdot \cos(x_{i+1})$$

and $$x_{i+1,new} = x_{i+1,old} - \frac{h(x_{i+1,old})}{h'(x_{i+1,old})}$$

Huen's Method and Other "Higher-Order" Methods

Recall that we "derived" Euler's Method from Taylor Series

$$x(t_{i+1}) = x(t_i) + \left.\frac{dx}{dt}\right|_{t_i} \cdot \Delta t + \underbrace{\frac{1}{2} \left.\frac{d^2x}{dt^2}\right|_{t_i} \cdot (\Delta t)^2 + \ldots}_{}$$

Error in the description of the linearized function

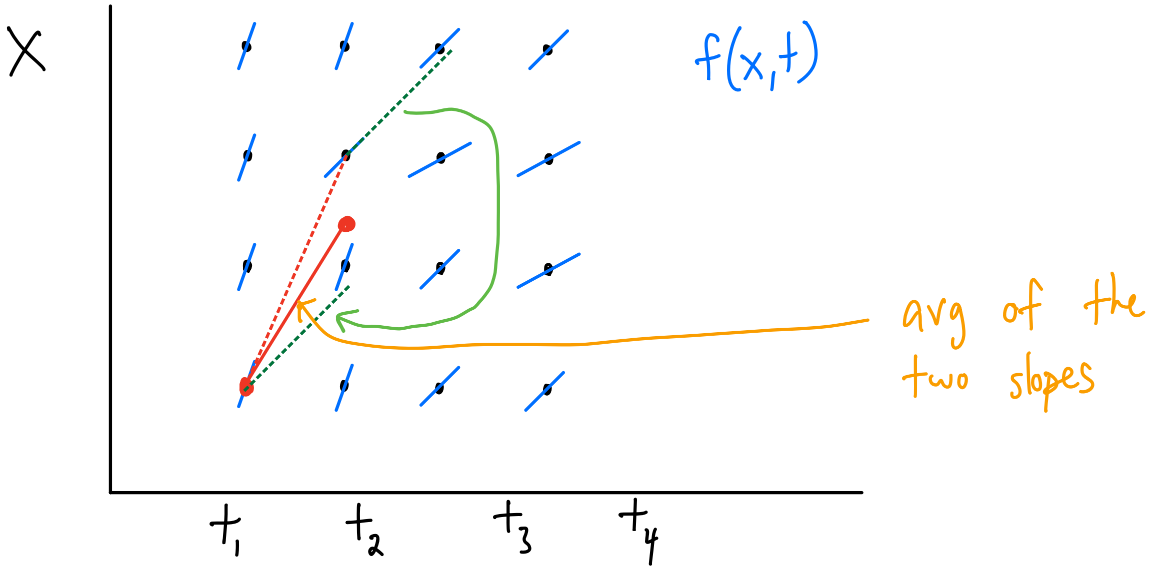
Huen's Method, allows us to look forward to the next time step and assess how the slope has changed. In a sense, we think a bit about $\left.\frac{d^2x}{dt^2}\right|_{t_i}$

(Explicit) Euler's Method is the "predictor"

$$x_{i+1}^E = x_i + \Delta t \cdot f(x_i, t_i)$$

$$\boxed{x_{i+1} = x_i + \Delta t \cdot \left(\frac{f(x_i, t_i) + f(x_{i+1}^E, t_{i+1})}{2} \right)}$$

Huen's Method



<u>Runge-Kutta Method</u> (actually "Runge-Kutta" refers to a family of methods, this one is sometimes called RK4)

Similar in spirit to Huen, but "looks to the future" in more ways.

$$x_{i+1} = x_i + \Delta t \cdot \left(\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \right)$$

where

$k_1 = f(x_i, t_i)$ ⟶ Euler's Method - like

$k_2 = f(x_i + \frac{1}{2} k_1 \cdot \Delta t,\ t_i + \frac{1}{2} \Delta t)$ ⟶ Halfway Huen's

$k_3 = f(x_i + \frac{1}{2} k_2 \cdot \Delta t,\ t_i + \frac{1}{2} \Delta t)$ ⟶ Halfway Huen's, w/ correction

$k_4 = f(x_i + k_3 \cdot \Delta t,\ t_i + \Delta t)$ ⟶ Huen's w/ correction