MCEN 3030

8 Feb 2024

HW & Quiz #2 Due Sun

Last time: Root Finding

Today: Systems of linear equations

Last time: Root - Finding

Where does $x \cdot \exp(-x) = 0.1$?

$$\Longrightarrow \quad f(x) = \underbrace{x \cdot \exp(-x)}_{\text{nonlinear}} - 0.1$$

Newton-Raphson) Need $f'(x) = e^{-x} - xe^{-x}$

iterative scheme: $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Bisection) No problem in evaluating $f(x)$

iterative scheme: $x_m = \dfrac{x_L + x_u}{2}$

if $f(x_L) \cdot f(x_m) < 0$

there is a root between x_L & x_m

$\rightarrow$ set $x_u = x_m$ & repeat

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Today we will talk about solving <u>systems</u> of <u>linear</u> equations

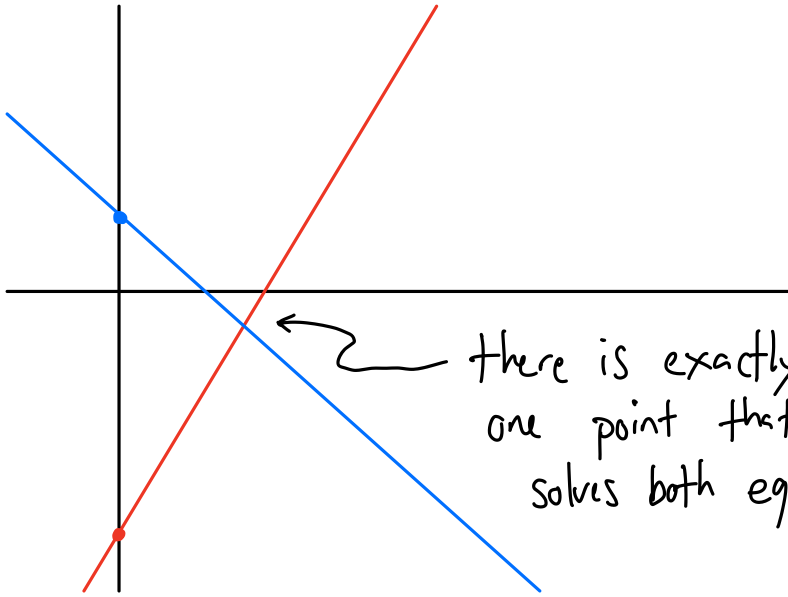
Systems of linear equations

Ex w/ 2 variables:

(1)

$y = -3x + 1$

$y = 4x - 3$

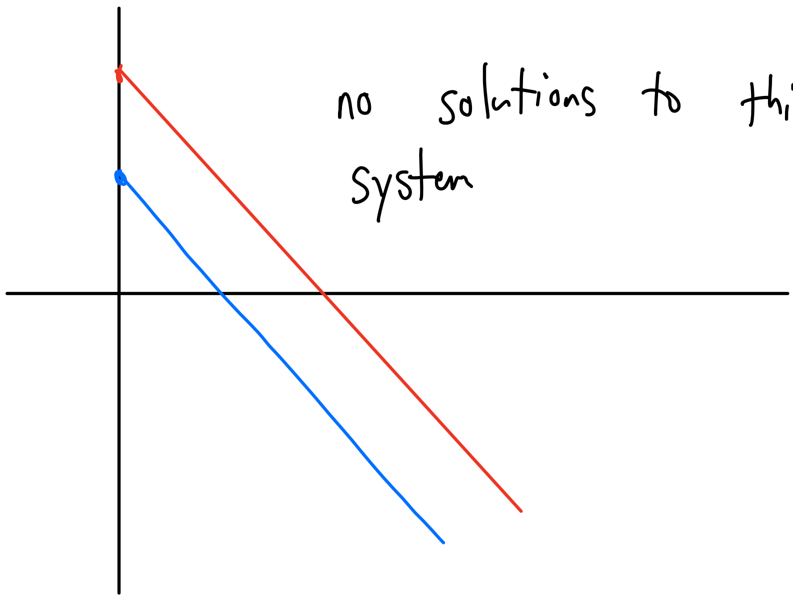


there is exactly one point that solves both equations

(2)

$y = 3 - x$

$y = 4 - x$

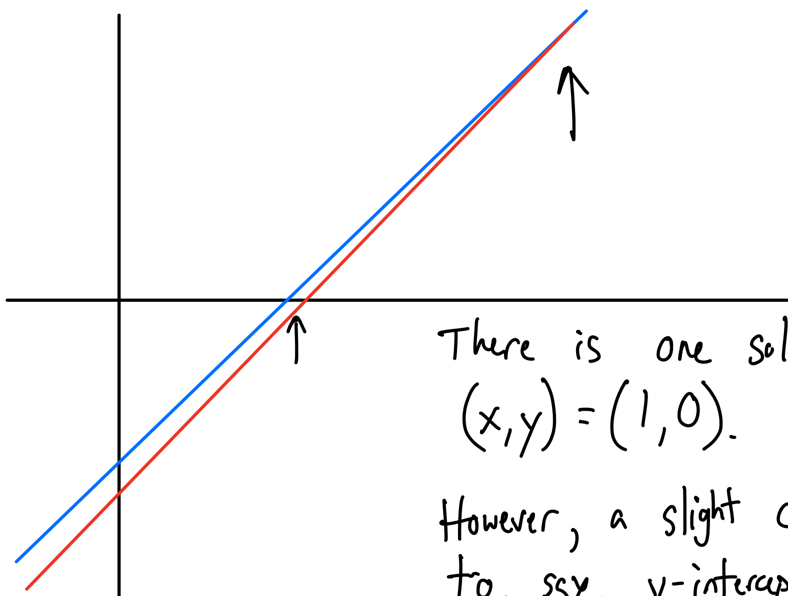


no solutions to this system

(3)

$y = x - 1$

$y = 1.01x - 1.01$



There is one solution, $(x, y) = (1, 0)$.

However, a slight change to, say, y-intercept of 1.05 drastically changes this

Linear Algebra Basics (& we will look at MATLAB)

$y = -3x + 1 \longrightarrow 3x + y = 1$
$y = 4x - 3 \longrightarrow -4x + y = -3$

$$\left.\begin{aligned} 3x + y &= 1 \\ -4x + y &= -3 \end{aligned}\right\} \quad \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

With this matrix representation, we can solve the system a number of ways

1) Gaussian Elimination (next week, w/ LU decomposition)

2) Use the matrix inverse

If we have a system $\underline{\underline{A}}\,\underline{x} = \underline{b}$

its solution is given by $x = A^{-1} b$

At least for 2x2 matrices, the inverse calculation is fairly tractable

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Then $A^{-1} = \dfrac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Ex: $\begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{-1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix}$

But what about $\begin{bmatrix} 1 & -1 \\ 1.01 & -1 \end{bmatrix}$?

$$A^{-1} = \underbrace{\frac{1}{(1)(-1)-(-1)(1.01)}}_{\text{very nearly } \infty} \cdot \begin{bmatrix} \circ & \circ \\ \circ & \circ \end{bmatrix}$$

This is a case of an ill-conditioned system $\Rightarrow$ small changes to values in the problem (e.g. due to cut-off or experimental uncertainty) blow up!