MCEN 3030

11 Apr 2024

HW #7 Due <u>Tues</u>

Last time:
- Implicit Euler
- Runge - Kutta

Today: Extensions of RK:
- Coupled ODEs
- Higher-order ODEs
- Shooting Method

Last time $\quad \frac{dx}{dt} = f(x,t) \quad$ w/ $\quad x(0) = x_0$

Explicit Euler:

$$x_{i+1} = x_i + \Delta t \cdot f(x_i, t_i)$$

Implicit Euler:

$$x_{i+1} = x_i + \Delta t \cdot f(x_{i+1}, t_{i+1})$$

(stable)

Runge-Kutta 4:

$$x_{i+1} = x_i + \Delta t \cdot \phi(x,t)$$

$$\phi(x,t) = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

where
$$k_1 = f(x_i, t_i)$$
$$k_2 = f\left(x_i + \tfrac{1}{2} k_1 \cdot \Delta t,\ t_i + \tfrac{1}{2}\Delta t\right)$$
$$k_3 = f\left(x_i + \tfrac{1}{2} k_2 \cdot \Delta t,\ t_i + \tfrac{1}{2}\Delta t\right)$$
$$k_4 = f\left(x_i + k_3 \cdot \Delta t,\ t_i + \Delta t\right)$$

Reminder: (Euler)

$$\frac{dx}{dt} = f(x,t)$$

$$\hookrightarrow \frac{x_{i+1} - x_i}{\underbrace{t_{i+1} - t_i}_{\Delta t}} = f(x_i, t_i) \longrightarrow x_{i+1} = x_i + \Delta t \cdot f(x_i, t_i)$$

Coupled ODEs

$$\begin{cases} \dfrac{dx}{dt} = f(x, y, t) & \text{w/ } x(0) = x_0 \\ \dfrac{dy}{dt} = g(x, y, t) & y(0) = y_0 \end{cases}$$

$\rightarrow$ $x(t)$ & $y(t)$, but maybe can't solve analytically: the dynamics depend on each other

Ex: $$\left.\begin{aligned} \dot{x} &= ax - bxy \\ \dot{y} &= cxy - dy \end{aligned}\right\} \text{ Predator-prey model}$$

Idea is the same as one variable RK4, but with 2×1 vectors:

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \frac{\Delta t}{6} \begin{bmatrix} k_{1x} + 2k_{2x} + 2k_{3x} + k_{4x} \\ k_{1y} + 2k_{2y} + 2k_{3y} + k_{4y} \end{bmatrix}$$

where
$$k_{1x} = f(x_i, y_i, t_i)$$
$$k_{1y} = g(x_i, y_i, t_i)$$
$$k_{2x} = f\left(x_i + \tfrac{1}{2}k_{1x} \cdot \Delta t,\ y_i + \tfrac{1}{2}k_{1y} \cdot \Delta t,\ t_i + \tfrac{1}{2}\Delta t\right)$$
$$k_{2y} = g\left(x_i + \tfrac{1}{2}k_{1x} \cdot \Delta t,\ y_i + \tfrac{1}{2}k_{1y} \cdot \Delta t,\ t_i + \tfrac{1}{2}\Delta t\right)$$
$$k_{3x} = f\left(x_i + \tfrac{1}{2}k_{2x} \cdot \Delta t,\ y_i + \tfrac{1}{2}k_{2y}\Delta t,\ t_i + \tfrac{1}{2}\Delta t\right)$$
$$k_{3y} = g\left(x_i + \tfrac{1}{2}k_{2x} \cdot \Delta t,\ y_i + \tfrac{1}{2}k_{2y}\Delta t,\ t_i + \tfrac{1}{2}\Delta t\right)$$
$$k_{4x} = f\left(x_i + k_{3x} \cdot \Delta t,\ y_i + k_{3y}\Delta t,\ t_i + \Delta t\right)$$
$$k_{4y} = g\left(x_i + k_{3x} \cdot \Delta t,\ y_i + k_{3y}\Delta t,\ t_i + \Delta t\right)$$

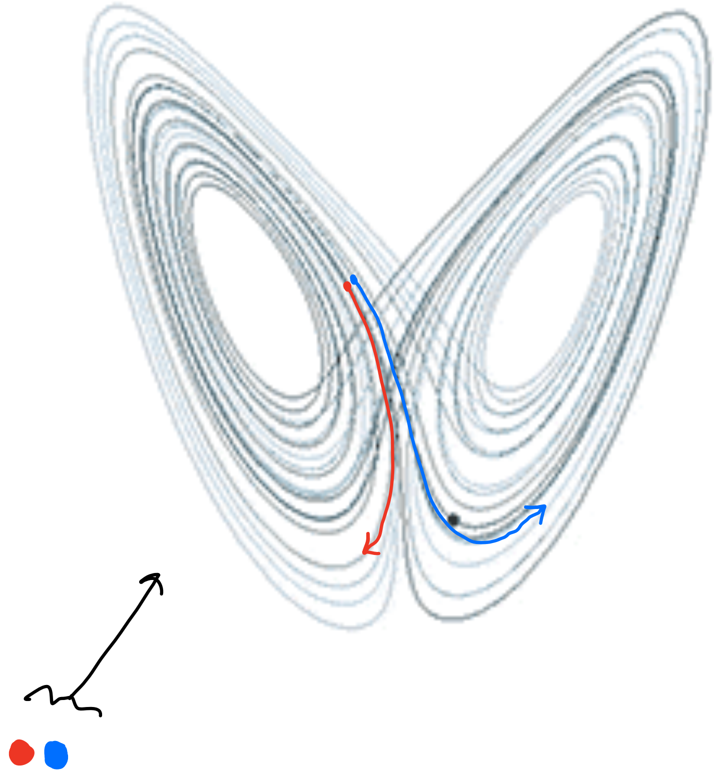
Famous example: Lorenz Equations "Butterfly Effect"

$x(t) \quad y(t) \quad z(t)$

$$\frac{dx}{dt} = \sigma(y-x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$



- No analytical solution
- Mathematical chaos
  - small changes in initial conditions will become large differences after a period of time.

Higher-Order ODEs

... can be treated as coupled ODEs, e.g.

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

(mass damper spring)

need two initial conditions:

$x(0) = x_0$

$u(0) = u_0$

Define $u \equiv \dot{x} \implies \ddot{x} = \dot{u}$

$$(\ddot{x} =)\ \dot{u} = -\frac{b}{m}u - \frac{k}{m}x + \frac{F(t)}{m}$$

$$\dot{x} = u$$

everything on the RHS is in terms of u, x, t, not in terms of $\dot{u}, \dot{x}$

→ looks like coupled ODEs above, and solve similarly

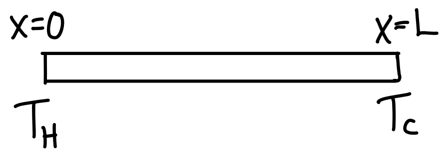
Shooting Method

With Euler-like methods we are always time-stepping forwards: current information $\rightarrow$ info at next step

– always initial-value problems

But what about boundary-value problems?

e.g. $$\frac{d^2T}{dx^2} = m(T - T_\infty)$$



w/ $T(x=0) = T_H$ & $T(x=L) = T_c$

Proceed as we did on previous page:

define $Q \equiv \dot{T} \implies \ddot{T} = \dot{Q}$

$$\dot{Q} = m(T - T_\infty) \qquad \text{w/out } Q(0) = ???$$

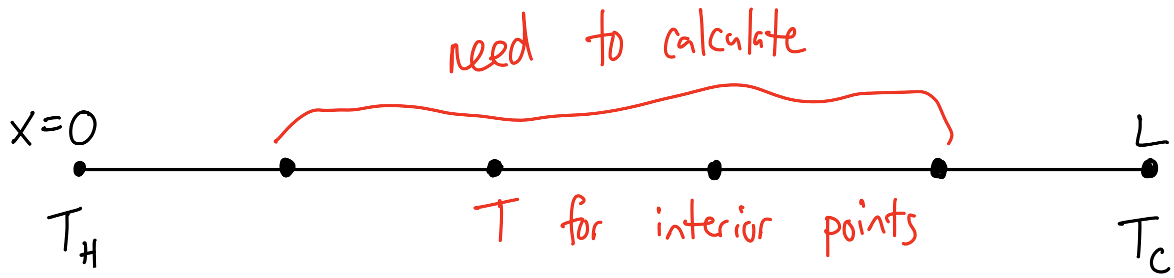
$$\dot{T} = Q \qquad \text{w/ } T(0) = T_H$$

Algorithm:

1) Guess an initial condition $Q(0) = Q_0$
2) Numerically solve for the function T via RK4
3) Compare the solution's $T(L)$ to the BC $T(L) = T_c$
4) Iterate, guessing a new Q_0, until convergence

Another approach to BVPs...

Again, let's consider $\dfrac{d^2T}{dx^2} = m(T - T_\infty)$



N=6 "nodes"

$$\frac{d^2T}{dx^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2} = \frac{\dfrac{T_{i+1} - T_i}{\Delta x} - \dfrac{T_i - T_{i-1}}{\Delta x}}{\Delta x}$$

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2} = m(T_i - T_\infty)$$

$$T_{i+1} - 2T_i + T_{i-1} = (\Delta x)^2 m T_i - (\Delta x)^2 m T_\infty$$

$$\frac{T_{i+1} + T_{i-1} + (\Delta x)^2 m T_\infty}{2 + (\Delta x)^2 m} = T_i$$

Create a system of equations:

$$T_1 = T_H \quad (\text{a BC})$$

$$T_2 = \frac{T_3 + T_1 + (\Delta x)^2 m T_\infty}{2 + (\Delta x)^2 m}$$

Define $c = \dfrac{1}{2 + (\Delta x)^2 m}$

$$\vdots \qquad \vdots$$

$$T_{N-1} = \frac{T_N + T_{N-2} + (\Delta x)^2 m T_\infty}{2 + (\Delta x)^2 m}$$

$$T_N = T_C \quad (\text{other BC})$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -c & 1 & -c & 0 & 0 & 0 \\ 0 & -c & 1 & -c & 0 & 0 \\ 0 & 0 & -c & 1 & -c & 0 \\ 0 & 0 & 0 & -c & 1 & -c \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} T_H \\ c(\Delta x)^2 m T_\infty \\ c(\Delta x)^2 m T_\infty \\ c(\Delta x)^2 m T_\infty \\ c(\Delta x)^2 m T_\infty \\ T_C \end{bmatrix}$$

Solve for unknown vector T using

$Ax = b$ techniques