REU(G): BACKGROUND PROBLEMS

ALONSO DELFÍN

The following problems are meant to provide a general background on operator norms and operator algebras for our summer REU(G) project on "Numerical computations of p-operator norms".

1. Operator Norms on $M_d(\mathbb{C})$

In what follows we assume $d \in \mathbb{Z}_{\geq 1}$ and $a = (a_{j,k})_{j,k=1}^d \in M_d(\mathbb{C})$ (that is a is a $d \times d$ matrix with entries $a_{j,k} \in \mathbb{C}$). Recall that for any $p \in [1, \infty)$ we defined the p-operator norm of a as

$$\|a\|_p := \max_{\|\boldsymbol{z}\|_p = 1} \|a(\boldsymbol{z})\|_p$$

Problem 1.1. Show that $\|a\|_1$ is maximum absolute column sum, that is

$$\|a\|_1 = \max_{k \in \{1,\dots,d\}} \|(a_{1,k}, \dots, a_{d,k})\|_1$$

Problem 1.2. Write an algorithm that computes $\|a\|_1$ given $a \in M_d(\mathbb{C})$.

Problem 1.3. Show that $\|a\|_2$ is the largest singular value of A. That is, show that if $\sigma(\overline{a}^T a) = \{\text{eigenvalues of } \overline{a}^T a\}$, then

$$\|a\|_2 = \max_{\lambda \in \sigma(\overline{a}^T a)} \sqrt{\lambda}$$

Problem 1.4. Write an algorithm that computes $\|a\|_2$ given $a \in M_d(\mathbb{C})$. (*Hint:* Optimize using the method of Lagrange Multipliers)

Problem 1.5. Show that $\|a\|_\infty$ is maximum absolute row sum, that is

$$\|a\|_\infty = \max_{j \in \{1,\dots,d\}} \|(a_{j,1}, \dots, a_{j,d})\|_1$$

Problem 1.6. Write an algorithm that computes $\|a\|_\infty$ given $a \in M_d(\mathbb{C})$.

2. General Operator Norms

Let V and W be normed vector spaces with respective norms $\|-\|_V$ and $\|-\|_W$. If $a\colon V \to W$ is a linear transformation we define its operator norm as

$$\|a\|_{V \to W} := \sup\{\|a(\boldsymbol{v})\|_W \colon \|\boldsymbol{v}\|_V = 1\}$$

The set $\mathcal{L}(V, W) = \{a\colon V \to W \mid \|a\|_{V \to W} < \infty\}$ is the space of bouded linear transformations. When $V = W$ we write $\mathcal{L}(V) = \mathcal{L}(V, V)$.

Problem 2.1. Show that

$$\|a\|_{V \to W} = \sup\{\|a(\boldsymbol{v})\|_W \colon \|\boldsymbol{v}\|_V \leq 1\}$$

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Problem 2.2. Show that
$$\|a\|_{V\to W} = \sup\left\{\frac{\|a(\boldsymbol{v})\|_W}{\|\boldsymbol{v}\|_V} : \boldsymbol{v}\neq 0\right\}$$

Problem 2.3. Show that
$$\|a\|_{V\to W} = \inf\{M>0\colon \|a(\boldsymbol{v})\|_W \leq M\|\boldsymbol{v}\|_V\}.$$
Conclude that $\|a(\boldsymbol{v})\|_W \leq \|a\|\|\boldsymbol{v}\|_V$ for all $\boldsymbol{v}\in V$.

3. Hilbert Spaces

Problem 3.1. Let $\mathcal{H}$ be a Hilbert space and let $a\colon \mathcal{H}\to\mathcal{H}$, $b\colon \mathcal{H}\to\mathcal{H}$ be two functions satisfying $\langle a(\xi),\eta\rangle = \langle \xi, b(\eta)\rangle$ for any $\xi,\eta\in\mathcal{H}$. Show that both a and b are linear transformations. For those of you with functional analysis background, use the Closed Graph theorem to show also that a,b are bounded with $\|a\|=\|b\|$.

Problem 3.2. For $d\in\mathbb{Z}_{\geq 1}$, consider the Hilbert space ℓ^2_d so that $\mathcal{L}(\ell^2_d) = M^2_d(\mathbb{C})$. Show that for any $a\in M^2_d(\mathbb{C})$ we have $a^* = \overline{a}^T$.

4. L^1-operator algebras

Problem 4.1. Equip $\mathbb{C}^2$ with point-wise multiplication and max norm so that $\mathbb{C}^2$ becomes a normed algebra. That is, if $\boldsymbol{z}=(z_1,z_2)\in\mathbb{C}^2$ we put as usual $\|\boldsymbol{z}\|_\infty = \max\{|z_1|,|z_2|\}$, and if in addition $\boldsymbol{w}=(w_1,w_2)\in\ell^\infty(\{0,1\})$ we define a new element $\boldsymbol{z}\cdot\boldsymbol{w}\in\mathbb{C}^2$ by
$$\boldsymbol{z}\cdot\boldsymbol{w} = (z_1w_1, z_2w_2).$$
Define a map $\varphi\colon\mathbb{C}^2\to\mathcal{L}(\ell^1(\{0,1\}))$ as follows: For every $\boldsymbol{z}=(z_1,z_2)\in\mathbb{C}^2$, we define $\varphi(\boldsymbol{z})\in\mathcal{L}(\ell^1(\{0,1\}))$ by the formula
$$[\varphi(\boldsymbol{z})](\xi) := (z_1\xi(0), z_2\xi(1)) \ \ \forall \ \ \xi=(\xi(0),\xi(1))\in\ell^1(\{0,1\}).$$
Show that φ is an isometric algebra homomorphism, that is show that for any $\boldsymbol{z},\boldsymbol{w}\in\mathbb{C}^2$, $\alpha\in\mathbb{C}$, we have

(1) $\varphi(\boldsymbol{z}+\alpha\boldsymbol{w}) = \varphi(\boldsymbol{z})+\alpha\varphi(\boldsymbol{w})$.
(2) $\varphi(\boldsymbol{z}\cdot\boldsymbol{w}) = \varphi(\boldsymbol{z})\circ\varphi(\boldsymbol{w})$.
(3) $\|\varphi(\boldsymbol{z})\|_{1\to 1} = \|\boldsymbol{z}\|_\infty$.

Problem 4.2. Equip $\ell^1(\{0,1\})$ with multiplication via convolution modulo 2 so that $\ell^1(\{0,1\})$ becomes a normed algebra. That is, for $\xi,\eta\in\ell^1(\{0,1\})$ define a new element $\xi*\eta\in\ell^1(\{0,1\})$ as follows
$$\xi*\eta = (\xi(0)\eta(0)+\xi(1)\eta(1), \xi(0)\eta(1)+\xi(1)\eta(0))$$
Define a map $\psi\colon\ell^1(\{0,1\})\to\mathcal{L}(\ell^1(\{0,1\}))$ as follows: For every $\xi\in\ell^1(\{0,1\})$, we define $\varphi(\xi)\in\mathcal{L}(\ell^1(\{0,1\}))$ by the formula
$$[\psi(\xi)](\eta) := \xi*\eta \ \ \forall \ \ \eta\in\ell^1(\{0,1\})$$
Show that ψ is an isometric algebra homomorphism, that is show that for any $\xi,\zeta\in\ell^1(\{0,1\})$, $\alpha\in\mathbb{C}$, we have

(1) $\psi(\xi+\alpha\zeta) = \psi(\xi)+\alpha\psi(\zeta)$.
(2) $\psi(\xi*\zeta) = \psi(\xi)\circ\psi(\zeta)$.
(3) $\|\psi(\xi)\|_{1\to 1} = \|\xi\|_1$.

Problem 4.3. Equip $\mathbb{C}^2$ with point-wise multiplication, as in Problem 4.1, and equip $\ell^1(\{0,1\})$ with multiplication via convolution, as in Problem 4.2.
Define a map $\mathcal{F}\colon \ell^1(\{0,1\}) \to \mathbb{C}^2$ as follows: For every $\xi \in \ell^1(\{0,1\})$, we define $\mathcal{F}(\xi) \in \mathbb{C}^2$ by the formula

$$\mathcal{F}(\xi) := (\xi(0) + \xi(1), \xi(0) - \xi(1))$$

Show that $\mathcal{F}$ is an algebra isomorphism, that is show that

1. $\mathcal{F}(\xi + \alpha\zeta) = \mathcal{F}(\xi) + \alpha\mathcal{F}(\zeta)$.
2. $\mathcal{F}(\xi * \zeta) = \mathcal{F}(\xi) \cdot \mathcal{F}(\zeta)$.
3. $\mathcal{F}$ is both one-to-one and onto.

Problem 4.4. Conclude using Problems 4.1, 4.2, and 4.3 above that $\mathbb{C}^2$ can be equipped with **two different norms** that make $\mathbb{C}^2$ an L^1-operator algebra.