03 - tensor calculus - tensor analysis

tensor algebra - invariants

- (principal) invariants of second order tensor

$$
\begin{aligned}
I_A &= \mathrm{tr}\,(\boldsymbol{A}) \\
II_A &= \tfrac{1}{2}\,[\mathrm{tr}^2\,(\boldsymbol{A}) - \mathrm{tr}\,(\boldsymbol{A}^2)] \\
III_A &= \det\,(\boldsymbol{A})
\end{aligned}
$$

- derivatives of invariants wrt second order tensor

$$
\begin{aligned}
\partial_{\boldsymbol{A}}\; I_A &= \boldsymbol{I} \\
\partial_{\boldsymbol{A}}\; II_A &= I_A\,\boldsymbol{I} - \boldsymbol{A} \\
\partial_{\boldsymbol{A}}\; III_A &= III_A\,\boldsymbol{A}^{-\mathrm{t}}
\end{aligned}
$$

tensor algebra - trace

- trace of second order tensor $\qquad \mathrm{tr}\,(\boldsymbol{u}\otimes\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{v}$

$$
\begin{aligned}
\mathrm{tr}(\boldsymbol{A}) &= \mathrm{tr}(A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j) \\
&= A_{ij}\,\mathrm{tr}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j) = A_{ij}\,\boldsymbol{e}_i\cdot\boldsymbol{e}_j \\
&= A_{ij}\,\delta_{ij} = A_{ii} = A_{11} + A_{22} + A_{33}
\end{aligned}
$$

- properties of traces of second order tensors

$$
\begin{aligned}
&\mathrm{tr}\,(\boldsymbol{I}) = 3 \\
&\mathrm{tr}\,(\boldsymbol{A}^{\mathrm{t}}) = \mathrm{tr}\,(\boldsymbol{A}) \\
&\mathrm{tr}\,(\boldsymbol{A}\cdot\boldsymbol{B}) = \mathrm{tr}\,(\boldsymbol{B}\cdot\boldsymbol{A}) \\
&\mathrm{tr}\,(\alpha\,\boldsymbol{A} + \beta\,\boldsymbol{B}) = \alpha\,\mathrm{tr}(\boldsymbol{A}) + \beta\,\mathrm{tr}(\boldsymbol{B}) \\
&\mathrm{tr}\,(\boldsymbol{A}\cdot\boldsymbol{B}^{\mathrm{t}}) = \boldsymbol{A}:\boldsymbol{B} \\
&\mathrm{tr}\,(\boldsymbol{A}) = \mathrm{tr}\,(\boldsymbol{A}\cdot\boldsymbol{I}) = \boldsymbol{A}:\boldsymbol{I}
\end{aligned}
$$

tensor algebra - determinant

- determinant of second order tensor $\qquad III_A = \det(\boldsymbol{A})$

$$
\begin{aligned}
\det(\boldsymbol{A}) &= \det(A_{ij}) = \tfrac{1}{6}\,e_{ijk}\,e_{abc}\,A_{ia}\,A_{jb}\,A_{kc} \\
&= A_{11}A_{22}A_{33} + A_{21}A_{32}A_{13} + A_{31}A_{12}A_{23} \\
&- A_{11}A_{23}A_{32} - A_{22}A_{31}A_{13} - A_{33}A_{12}A_{21}
\end{aligned}
$$

- properties of determinants of second order tensors

$$
\begin{aligned}
&\det\,(\boldsymbol{I}) = 1 \\
&\det\,(\boldsymbol{A}^{\mathrm{t}}) = \det\,(\boldsymbol{A}) \\
&\det\,(\alpha\,\boldsymbol{A}) = \alpha^3\det(\boldsymbol{A}) \\
&\det\,(\boldsymbol{A}\cdot\boldsymbol{B}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) \\
&\det(\boldsymbol{u}\otimes\boldsymbol{v}) = 0
\end{aligned}
$$

tensor algebra - determinant

- determinant defining vector product

$$\boldsymbol{u} \times \boldsymbol{v} = \det \begin{bmatrix} u_1 & v_1 & \boldsymbol{e}_1 \\ u_2 & v_2 & \boldsymbol{e}_2 \\ u_3 & v_3 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 & - & u_3 v_2 \\ u_3 v_1 & - & u_1 v_3 \\ u_1 v_2 & - & u_2 v_1 \end{bmatrix}$$

- determinant defining scalar triple product

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

tensor algebra - inverse

- inverse of second order tensor

$\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}$ in particular $\boldsymbol{v} = \boldsymbol{A} \cdot \boldsymbol{u} \quad \boldsymbol{A}^{-1} \cdot \boldsymbol{v} = \boldsymbol{u}$

- adjoint and cofactor

$$\boldsymbol{A}^{\text{adj}} = \det(\boldsymbol{A})\, \boldsymbol{A}^{-1} \qquad \boldsymbol{A}^{\text{cof}} = \det(\boldsymbol{A})\, \boldsymbol{A}^{-\text{t}} = (\boldsymbol{A}^{\text{adj}})^{\text{t}}$$

$$\partial_{\boldsymbol{A}} \det(\boldsymbol{A}) = \det(\boldsymbol{A})\, \boldsymbol{A}^{-\text{t}} = III_A\, \boldsymbol{A}^{-\text{t}} = \boldsymbol{A}^{\text{cof}}$$

- properties of inverse $\qquad \det(\boldsymbol{A}^{-1}) = 1/\det(\boldsymbol{A})$

$$\begin{aligned} (\boldsymbol{A}^{-1})^{-1} &= \boldsymbol{A} \\ (\alpha\, \boldsymbol{A}^{-1})^{-1} &= \alpha^{-1}\, \boldsymbol{A} \\ (\boldsymbol{A} \cdot \boldsymbol{B})^{-1} &= \boldsymbol{B}^{-1} \cdot \boldsymbol{A}^{-1} \end{aligned}$$

tensor algebra - spectral decomposition

- eigenvalue problem of second order tensor

$$\boldsymbol{A} \cdot \boldsymbol{n}_A = \lambda_A\, \boldsymbol{n}_A \qquad [\, \boldsymbol{A} - \lambda_A\, \boldsymbol{I}\,] \cdot \boldsymbol{n}_A = \boldsymbol{0}$$

- solution $\det(\boldsymbol{A} - \lambda_A\, \boldsymbol{I}) = 0$ in terms of scalar triple product

$$[\boldsymbol{A} \cdot \boldsymbol{u} - \lambda_A \boldsymbol{u}, \boldsymbol{A} \cdot \boldsymbol{v} - \lambda_A \boldsymbol{v}, \boldsymbol{A} \cdot \boldsymbol{w} - \lambda_A \boldsymbol{w}] = 0$$

- characteristic equation

$$\lambda_A^3 - I_A\, \lambda_A^2 + II_A\, \lambda_A - III_A = 0$$

$$\begin{aligned} I_A &= \operatorname{tr}(\boldsymbol{A}) \\ II_A &= \tfrac{1}{2}\, [\operatorname{tr}^2(\boldsymbol{A}) - \operatorname{tr}(\boldsymbol{A}^2)] \\ III_A &= \det(\boldsymbol{A}) \end{aligned}$$

- spectral decomposition

$$\boldsymbol{A} = \textstyle\sum_{i=1}^{3} \lambda_{Ai}\, \boldsymbol{n}_{Ai} \otimes \boldsymbol{n}_{Ai}$$

- cayleigh hamilton theorem

$$\boldsymbol{A}^3 - I_A\, \boldsymbol{A}^2 + II_A\, \boldsymbol{A} - III_A\, \boldsymbol{I} = \boldsymbol{0}$$

tensor algebra - sym/skw decomposition

- symmetric - skew-symmetric decomposition

$$\boldsymbol{A} = \tfrac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\text{t}}] + \tfrac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\text{t}}] = \boldsymbol{A}^{\text{sym}} + \boldsymbol{A}^{\text{skw}}$$

- symmetric and skew-symmetric tensor

$$\boldsymbol{A}^{\text{sym}} = (\boldsymbol{A}^{\text{sym}})^{\text{t}} \qquad \boldsymbol{A}^{\text{skw}} = -(\boldsymbol{A}^{\text{skw}})^{\text{t}}$$

- symmetric tensor

$$\boldsymbol{A}^{\text{sym}} = \tfrac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\text{t}}] = \mathsf{I}^{\text{sym}} : \boldsymbol{A}$$

- skew-symmetric tensor

$$\boldsymbol{A}^{\text{skw}} = \tfrac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\text{t}}] = \mathsf{I}^{\text{skw}} : \boldsymbol{A}$$

tensor algebra - symmetric tensor

- symmetric second order tensor

$$\boldsymbol{A}^{\mathrm{sym}} = \tfrac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\mathrm{t}}] \qquad \boldsymbol{A}^{\mathrm{sym}} = (\boldsymbol{A}^{\mathrm{sym}})^{\mathrm{t}} \qquad \boldsymbol{A}^{\mathrm{sym}} = \boldsymbol{S}$$

- processes three real eigenvalues and corresp.eigenvectors

$$\boldsymbol{S} = \textstyle\sum_{i=1}^{3} \lambda_{Si}\,(\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \qquad \begin{aligned} I_S &= \lambda_{S1} + \lambda_{S2} + \lambda_{S3} \\ II_S &= \lambda_{S2}\,\lambda_{S3} + \lambda_{S3}\,\lambda_{S1} + \lambda_{S1}\,\lambda_{S2} \\ III_S &= \lambda_{S1}\,\lambda_{S2}\,\lambda_{S3} \end{aligned}$$

- square root, inverse, exponent and log

$$\begin{aligned} \sqrt{\boldsymbol{S}} &= \textstyle\sum_{i=1}^{3} \sqrt{\lambda_{Si}}\;(\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \boldsymbol{S}^{-1} &= \textstyle\sum_{i=1}^{3} \lambda_{Si}^{-1}\;(\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \exp(\boldsymbol{S}) &= \textstyle\sum_{i=1}^{3} \exp(\lambda_{Si})\;(\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \ln(\boldsymbol{S}) &= \textstyle\sum_{i=1}^{3} \ln(\lambda_{Si})\;(\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \end{aligned}$$

tensor algebra - skew-symmetric tensor

- skew-symmetric second order tensor

$$\boldsymbol{A}^{\mathrm{skw}} = \tfrac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\mathrm{t}}] \qquad \boldsymbol{A}^{\mathrm{skw}} = -(\boldsymbol{A}^{\mathrm{skw}})^{\mathrm{t}} \qquad \boldsymbol{A}^{\mathrm{skw}} = \boldsymbol{W}$$

- processes three independent entries defining axial vector

$$\boldsymbol{w} = -\tfrac{1}{2}\,\overset{3}{\boldsymbol{e}} : \boldsymbol{W} \qquad \boldsymbol{w} = -\,\overset{3}{\boldsymbol{e}} \cdot \boldsymbol{w} \quad \text{such that} \quad \boldsymbol{W} \cdot \boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{v}$$

- invariants of skew-symmetric tensor

$$\begin{aligned} I_W &= \mathrm{tr}(\boldsymbol{W}) &&= 0 \\ II_W &= \boldsymbol{w} \cdot \boldsymbol{w} && \\ III_W &= \det(\boldsymbol{W}) &&= 0 \end{aligned}$$

tensor algebra - vol/dev decomposition

- volumetric - deviatoric decomposition

$$\boldsymbol{A} = \boldsymbol{A}^{\mathrm{vol}} + \boldsymbol{A}^{\mathrm{dev}}$$

- volumetric and deviatoric tensor

$$\mathrm{tr}(\boldsymbol{A}^{\mathrm{vol}}) = \mathrm{tr}(\boldsymbol{A}) \qquad \mathrm{tr}(\boldsymbol{A}^{\mathrm{dev}}) = 0$$

- volumetric tensor

$$\boldsymbol{A}^{\mathrm{vol}} = \tfrac{1}{3}[\boldsymbol{A} : \boldsymbol{I}]\,\boldsymbol{I} = \mathsf{I}^{\mathrm{vol}} : \boldsymbol{A}$$

- deviatoric tensor

$$\boldsymbol{A}^{\mathrm{dev}} = \boldsymbol{A} - \tfrac{1}{3}[\boldsymbol{A} : \boldsymbol{I}]\,\boldsymbol{I} = \mathsf{I}^{\mathrm{dev}} : \boldsymbol{A}$$

tensor algebra - orthogonal tensor

- orthogonal second order tensor $\boldsymbol{Q} \in \mathrm{S0}(3)$

$$\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{\mathrm{t}} \quad \Leftrightarrow \quad \boldsymbol{Q}^{\mathrm{t}} \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathrm{t}} = \boldsymbol{I}$$

- decomposition of second order tensor

$$\boldsymbol{A} = \boldsymbol{Q} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{Q}$$

such that $\boldsymbol{a} \cdot \boldsymbol{U} \cdot \boldsymbol{a} \geq 0$ and $\boldsymbol{a} \cdot \boldsymbol{V} \cdot \boldsymbol{a} \geq 0$

- proper orthogonal tensor $\boldsymbol{Q} \in \mathrm{S0}(3)$ has eigenvalue $\lambda_Q = 1$

$$\boldsymbol{Q} \cdot \boldsymbol{n}_Q = \boldsymbol{n}_Q \quad \text{with} \quad [Q_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +\cos\varphi & +\sin\varphi \\ 0 & -\sin\varphi & +\cos\varphi \end{bmatrix}$$

interpretation: finite rotation around axis $\boldsymbol{n}_Q$

tensor analysis - frechet derivative

- consider smooth differentiable scalar field Φ with

$$\begin{array}{llll} \text{scalar argument} & \Phi: & \mathcal{R} \to \mathcal{R}; & \Phi(x) = \alpha \\ \text{vector argument} & \Phi: & \mathcal{R}^3 \to \mathcal{R}; & \Phi(\boldsymbol{x}) = \alpha \\ \text{tensor argument} & \Phi: & \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}; & \Phi(\boldsymbol{X}) = \alpha \end{array}$$

- frechet derivative (tensor notation)

$$\begin{array}{ll} \text{scalar argument} & \mathrm{D}\Phi(x) = \dfrac{\partial \Phi(x)}{\partial x} = \partial_x \Phi(x) \\ \text{vector argument} & \mathrm{D}\Phi(\boldsymbol{x}) = \dfrac{\partial \Phi(\boldsymbol{x})}{\partial \boldsymbol{x}} = \partial_{\boldsymbol{x}} \Phi(\boldsymbol{x}) \\ \text{tensor argument} & \mathrm{D}\Phi(\boldsymbol{X}) = \dfrac{\partial \Phi(\boldsymbol{X})}{\partial \boldsymbol{X}} = \partial_{\boldsymbol{X}} \Phi(\boldsymbol{X}) \end{array}$$

tensor analysis - gateaux derivative

- consider smooth differentiable scalar field Φ with

$$\begin{array}{llll} \text{scalar argument} & \Phi: & \mathcal{R} \to \mathcal{R}; & \Phi(x) = \alpha \\ \text{vector argument} & \Phi: & \mathcal{R}^3 \to \mathcal{R}; & \Phi(\boldsymbol{x}) = \alpha \\ \text{tensor argument} & \Phi: & \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}; & \Phi(\boldsymbol{X}) = \alpha \end{array}$$

- gateaux derivative,i.e.,frechet wrt direction (tensor notation)

$$\begin{array}{ll} \text{scalar argument} & \mathrm{D}\Phi(x)\ u = \dfrac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi(x + \epsilon u)\,|_{\epsilon=0} \quad \forall\, u \in \mathcal{R} \\ \text{vector argument} & \mathrm{D}\Phi(\boldsymbol{x}) \cdot \boldsymbol{u} = \dfrac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi(\boldsymbol{x} + \epsilon \boldsymbol{u})\,|_{\epsilon=0} \quad \forall\, \boldsymbol{u} \in \mathcal{R}^3 \\ \text{tensor argument} & \mathrm{D}\Phi(\boldsymbol{X}) : \boldsymbol{U} = \dfrac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi(\boldsymbol{X} + \epsilon \boldsymbol{U})|_{\epsilon=0} \quad \forall\, \boldsymbol{U} \in \mathcal{R}^3 \otimes \mathcal{R}^3 \end{array}$$

tensor analysis - gradient

- consider scalar- and vector field in domain $\mathcal{B} \in \mathcal{R}^3$

$$\begin{array}{ll} f: \mathcal{B} \to \mathcal{R} & f: \boldsymbol{x} \to f(\boldsymbol{x}) \\ \boldsymbol{f}: \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{f}: \boldsymbol{x} \to \boldsymbol{f}(\boldsymbol{x}) \end{array}$$

- gradient of scalar- and vector field

$$\nabla f(\boldsymbol{x}) = \frac{\partial f(\boldsymbol{x})}{\partial x_i} = f_{,i}(\boldsymbol{x})\, \boldsymbol{e}_i \qquad \nabla f(\boldsymbol{x}) = \begin{bmatrix} f_{,1} \\ f_{,2} \\ f_{,3} \end{bmatrix}$$

$$\nabla \boldsymbol{f}(\boldsymbol{x}) = \frac{\partial f_i(\boldsymbol{x})}{\partial x_j} = f_{i,j}(\boldsymbol{x})\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \nabla \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{bmatrix}$$

renders vector- and 2nd order tensor field

tensor analysis - divergence

- consider vector- and 2nd order tensor field in domain $\mathcal{B}$

$$\begin{array}{ll} \boldsymbol{f}: \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{f}: \boldsymbol{x} \to \boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{F}: \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & \boldsymbol{F}: \boldsymbol{x} \to \boldsymbol{F}(\boldsymbol{x}) \end{array}$$

- divergence of vector- and 2nd order tensor field

$$\mathrm{div}(\boldsymbol{f}(\boldsymbol{x})) = \mathrm{tr}(\nabla \boldsymbol{f}(\boldsymbol{x})) = \nabla \boldsymbol{f}(\boldsymbol{x}) : \boldsymbol{I}$$

$$\mathrm{div}(\boldsymbol{f}(\boldsymbol{x})) = f_{i,i}(\boldsymbol{x}) = f_{1,1} + f_{2,2} + f_{3,3}$$

$$\mathrm{div}(\boldsymbol{F}(\boldsymbol{x})) = \mathrm{tr}(\nabla \boldsymbol{F}(\boldsymbol{x})) = \nabla \boldsymbol{F}(\boldsymbol{x}) : \boldsymbol{I}$$

$$\mathrm{div}(\boldsymbol{F}(\boldsymbol{x})) = F_{ij,j}(\boldsymbol{x}) = \begin{bmatrix} F_{11,1} + F_{12,2} + F_{13,3} \\ F_{21,1} + F_{22,2} + F_{23,3} \\ F_{31,1} + F_{32,2} + F_{33,3} \end{bmatrix}$$

renders scalar- and vector field

tensor analysis - laplace operator

- consider scalar- and vector field in domain $\mathcal{B} \in \mathcal{R}^3$

$$\begin{array}{ll} f : \mathcal{B} \to \mathcal{R} & f : \boldsymbol{x} \to f \ (\boldsymbol{x}) \\ \boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f} \ (\boldsymbol{x}) \end{array}$$

- laplace operator acting on scalar- and vector field

$$\Delta f(\boldsymbol{x}) = \mathrm{div}(\nabla(f(\boldsymbol{x}))) \quad \Delta f(\boldsymbol{x}) = f_{,ii} = f_{,11} + f_{,22} + f_{,33}$$

$$\Delta \boldsymbol{f}(\boldsymbol{x}) = \mathrm{div}(\nabla(\boldsymbol{f}(\boldsymbol{x}))) \quad \Delta \boldsymbol{f}(\boldsymbol{x}) = f_{i,jj} = \begin{bmatrix} f_{1,11} + f_{1,22} + f_{1,33} \\ f_{2,11} + f_{2,22} + f_{2,33} \\ f_{3,11} + f_{3,22} + f_{3,33} \end{bmatrix}$$

renders scalar- and vector field

tensor analysis - transformation formulae

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$\begin{array}{ll} \alpha : \mathcal{B} \to \mathcal{R} & \alpha : \boldsymbol{x} \to \alpha \ (\boldsymbol{x}) \\ \boldsymbol{u} : \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{u} : \boldsymbol{x} \to \boldsymbol{u} \ (\boldsymbol{x}) \\ \boldsymbol{v} : \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{v} : \boldsymbol{x} \to \boldsymbol{v} \ (\boldsymbol{x}) \\ \boldsymbol{A} : \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & \boldsymbol{A} : \boldsymbol{x} \to \boldsymbol{A} \ (\boldsymbol{x}) \end{array}$$

- useful transformation formulae (tensor notation)

$$\begin{array}{rcl} \nabla \ (\alpha \, \boldsymbol{u}) &=& \boldsymbol{u} \otimes \nabla\alpha \ + \ \alpha \, \nabla \boldsymbol{u} \\ \nabla \ (\boldsymbol{u} \cdot \boldsymbol{v}) &=& \boldsymbol{u} \cdot \nabla \boldsymbol{v} \ + \ \boldsymbol{v} \cdot \nabla \boldsymbol{u} \\ \mathrm{div} \ (\alpha \, \boldsymbol{u}) &=& \alpha \, \mathrm{div}(\boldsymbol{u}) \ + \ \boldsymbol{u} \cdot \nabla\alpha \\ \mathrm{div} \ (\alpha \, \boldsymbol{A}) &=& \alpha \, \mathrm{div}(\boldsymbol{A}) \ + \ \boldsymbol{A} \cdot \nabla\alpha \\ \mathrm{div} \ (\boldsymbol{u} \cdot \boldsymbol{A}) &=& \boldsymbol{u} \cdot \mathrm{div}(\boldsymbol{A}) \ + \ \boldsymbol{A} : \nabla \boldsymbol{u} \\ \mathrm{div} \ (\boldsymbol{u} \otimes \boldsymbol{v}) &=& \boldsymbol{u} \, \mathrm{div}(\boldsymbol{v}) \ + \ \boldsymbol{v} \cdot \nabla \boldsymbol{u}^{\mathrm{t}} \end{array}$$

tensor analysis - transformation formulae

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$\begin{array}{llll} \alpha : & \mathcal{B} \to \mathcal{R} & \alpha : & x_k \to \alpha \quad (x_k) \\ u_i : & \mathcal{B} \to \mathcal{R}^3 & u_i : & x_k \to u_i \quad (x_k) \\ v_i : & \mathcal{B} \to \mathcal{R}^3 & v_i : & x_k \to v_i \quad (x_k) \\ A_{ij} : & \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & A_{ij} : & x_k \to A_{ij} \ (x_k) \end{array}$$

- useful transformation formulae (index notation)

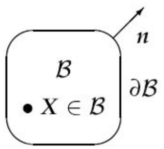
$$\begin{array}{rcl} (\ \alpha \, u_i \ )_{,j} &=& u_i \, \alpha_{,j} \ + \ \alpha \, u_{i,j} \\ (\ u_i \, v_i \ )_{,j} &=& u_i \, v_{i,j} \ + \ v_i \, u_{i,j} \\ (\ \alpha \, u_i \ )_{,i} &=& \alpha \, u_{i,i} \ + \ u_i \, \alpha_{,i} \\ (\ \alpha \, A_{ij} \ )_{,j} &=& \alpha \, A_{ij,j} \ + \ A_{ij} \, \alpha_{,j} \\ (\ u_i \, A_{ij} \ )_{,j} &=& u_i \, A_{ij,j} \ + \ A_{ij} \, u_{i,j} \\ (\ u_i \, v_j \ )_{,j} &=& u_i \, v_{j,j} \ + \ v_j \, u_{i,j} \end{array}$$

tensor analysis - integral theorems

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$\begin{array}{ll} \alpha : \mathcal{B} \to \mathcal{R} & \alpha : \boldsymbol{x} \to \alpha \ (\boldsymbol{x}) \\ \boldsymbol{u} : \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{u} : \boldsymbol{x} \to \boldsymbol{u} \ (\boldsymbol{x}) \\ \boldsymbol{A} : \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & \boldsymbol{A} : \boldsymbol{x} \to \boldsymbol{A} \ (\boldsymbol{x}) \end{array}$$

- integral theorems (tensor notation)



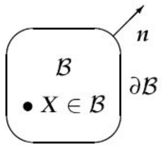
$$\begin{array}{rcll} \int_{\partial\mathcal{B}} \alpha \, \boldsymbol{n} \ \mathrm{d}A &=& \int_{\mathcal{B}} \nabla \alpha \, \mathrm{d}V & \text{green} \\ \int_{\partial\mathcal{B}} \boldsymbol{u} \cdot \boldsymbol{n} \ \mathrm{d}A &=& \int_{\mathcal{B}} \mathrm{div}(\boldsymbol{u}) \, \mathrm{d}V & \text{gauss} \\ \int_{\partial\mathcal{B}} \boldsymbol{A} \cdot \boldsymbol{n} \ \mathrm{d}A &=& \int_{\mathcal{B}} \mathrm{div}(\boldsymbol{A}) \, \mathrm{d}V & \text{gauss} \end{array}$$

tensor analysis - integral theorems

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$\begin{array}{lll} \alpha: & \mathcal{B} \to \mathcal{R} \\ u_i: & \mathcal{B} \to \mathcal{R}^3 \\ A_{ij}: & \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 \end{array} \qquad \begin{array}{lll} \alpha: & x_k \to \alpha & (x_k) \\ u_i: & x_k \to u_i & (x_k) \\ A_{ij}: & x_k \to A_{ij} & (x_k) \end{array}$$

- integral theorems (tensor notation)



$$\begin{array}{llll} \int_{\partial\mathcal{B}} \alpha\, n_i \, \mathrm{d}A = \int_{\mathcal{B}} \alpha_{,i} \mathrm{d}V & \text{green} \\ \int_{\partial\mathcal{B}} u_i\, n_i \, \mathrm{d}A = \int_{\mathcal{B}} u_{i,i} \mathrm{d}V & \text{gauss} \\ \int_{\partial\mathcal{B}} A_{ij}\, n_j \, \mathrm{d}A = \int_{\mathcal{B}} A_{ij,j} \mathrm{d}V & \text{gauss} \end{array}$$

voigt / matrix vector notation

- strain tensors as vectors in voigt notation

$$E_{ij} = \begin{bmatrix} E_{11} & E_{12} & E_{31} \\ E_{12} & E_{22} & E_{23} \\ E_{31} & E_{23} & E_{33} \end{bmatrix}$$

$$E^{\text{voigt}} = [E_{11}, E_{22}, E_{33}, 2\,E_{12}, 2\,E_{23}, 2\,E_{31}]^{\text{t}}$$

- stress tensors as vectors in voigt notation

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{31} \\ S_{12} & S_{22} & S_{23} \\ S_{31} & S_{23} & S_{33} \end{bmatrix}$$

$$S^{\text{voigt}} = [S_{11}, S_{22}, S_{33}, S_{12}, S_{23}, S_{31}]^{\text{t}}$$

- why are strain & stress different? check energy expression!

$$\psi = \tfrac{1}{2}\,\boldsymbol{E} : \boldsymbol{S} \qquad \psi = \tfrac{1}{2}\,\boldsymbol{E}^{\text{voigt}} : \boldsymbol{S}^{\text{voigt}}$$

voigt / matrix vector notation

- fourth order material operators as matrix in voigt notation

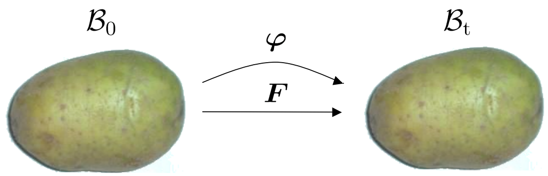
$$C^{\text{voigt}} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3331} \end{bmatrix}$$

- why are strain & stress different? check these expressions!

$$\boldsymbol{S} = \mathbf{C} : \boldsymbol{E} \qquad \boldsymbol{S}^{\text{voigt}} = \boldsymbol{C}^{\text{voigt}} \cdot \boldsymbol{E}^{\text{voigt}}$$

deformation gradient

- given the deformation gradient, play with matlab to become familiar with basic tensor operations!



- uniaxial tension (incompressible), simple shear, rotation

$$F^{\text{uni}}_{ij} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-\frac{1}{2}} & 0 \\ 0 & 0 & \alpha^{-\frac{1}{2}} \end{bmatrix} \quad F^{\text{shr}}_{ij} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F^{\text{rot}}_{ij} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

second order tensors - scalar products

- inverse of second order tensor

$$\boldsymbol{I} = \boldsymbol{F} \cdot \boldsymbol{F}^{-1} \qquad \delta_{ij} = F_{ik}\, F^{-1}_{kj}$$

- right / left cauchy green and green lagrange strain tensor

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{t}} \cdot \boldsymbol{F} \qquad C_{ij} = F_{ki}\, F_{kj}$$

$$\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathrm{t}} \qquad b_{ij} = F_{ik}\, F_{jk}$$

$$\boldsymbol{E} = \tfrac{1}{2}[\boldsymbol{C} - \boldsymbol{I}] \qquad E_{ij} = \tfrac{1}{2}[C_{ij} - \delta_{ij}]$$

- trace of second order tensor

$$\begin{aligned} \mathrm{tr}(\boldsymbol{C}) &= \mathrm{tr}(C_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \\ &= C_{ij}\,\delta_{ij} = C_{ii} = C_{11} + C_{22} + C_{33} \end{aligned}$$

- (principal) invariants of second order tensor

$$\begin{aligned} I_C &= \mathrm{tr}\,(\boldsymbol{C}) & I_b &= \mathrm{tr}\,(\boldsymbol{b}) \\ II_C &= \tfrac{1}{2}\,[\mathrm{tr}^2\,(\boldsymbol{C}) - \mathrm{tr}\,(\boldsymbol{C}^2)] & II_b &= \tfrac{1}{2}\,[\mathrm{tr}^2\,(\boldsymbol{b}) - \mathrm{tr}\,(\boldsymbol{b}^2)] \\ III_C &= \det\,(\boldsymbol{C}) & III_b &= \det\,(\boldsymbol{b}) \end{aligned}$$

fourth order tensors - scalar products

- symmetric fourth order unit tensor

$$\mathsf{I}^{\mathrm{sym}} = \tfrac{1}{2}\,[\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}]\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\mathrm{sym}} : \boldsymbol{E} = \boldsymbol{E}^{\mathrm{sym}}$$

- screw-symmetric fourth order unit tensor

$$\mathsf{I}^{\mathrm{skw}} = \tfrac{1}{2}\,[\delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}]\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\mathrm{skw}} : \boldsymbol{E} = \boldsymbol{E}^{\mathrm{skw}}$$

- volumetric fourth order unit tensor

$$\mathsf{I}^{\mathrm{vol}} = \tfrac{1}{3}\,\delta_{ij}\,\delta_{kl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\mathrm{vol}} : \boldsymbol{E} = \boldsymbol{E}^{\mathrm{vol}}$$

- deviatoric fourth order unit tensor $\qquad \mathsf{I}^{\mathrm{dev}} : \boldsymbol{E} = \boldsymbol{E}^{\mathrm{dev}}$

$$\mathsf{I}^{\mathrm{dev}} = [-\tfrac{1}{3}\,\delta_{ij}\,\delta_{kl} + \tfrac{1}{2}\,\delta_{ik}\,\delta_{jl} + \tfrac{1}{2}\,\delta_{il}\,\delta_{jk}]\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

neo hooke'ian elasticity

- free energy $\psi_0^{\mathrm{neo}} = \tfrac{1}{2}\,\lambda_0 \ln^2(\det(F_{ij})) + \tfrac{1}{2}\,\mu_0[\,F_{ij}F_{ij} - n^{\mathrm{dim}} - 2\,\ln(\det(F_{ij}))\,]$
- 1st and 2nd piola kirchhoff stress and cauchy stress

$$\begin{aligned} P_{ij}^{\mathrm{neo}} &= \mu_0\,F_{ij} + [\,\lambda_0 \ln(\det(F_{ij})) - \mu_0\,]F^{-1}_{ji} \\ S_{ij}^{\mathrm{neo}} &= \mu_0\,\delta_{ij} + [\,\lambda_0 \ln(\det(F_{ij})) - \mu_0\,]C^{-1}_{ji} \\ \sigma_{ij}^{\mathrm{neo}} &= \tfrac{1}{J}[\,\mu_0\,b_{ij} + \lambda_0 \ln(\det(F_{ij})) - \mu_0\,]\delta_{ij} \end{aligned}$$

- 4th order tangent operators

$$\begin{aligned} \mathsf{A}_{ijkl}^{\mathrm{neo}} &= \lambda_0 F^{-1}_{ji}\,F^{-1}_{lk} + \mu_0 I_{ik}\,I_{jl} \\ &\quad + [\mu_0 - \lambda_0 \ln(\det(F_{ij}))]\,F^{-1}_{li}\,F^{-1}_{jk} \\ \mathsf{C}_{ijkl}^{\mathrm{neo}} &= \lambda_0 C^{-1}_{ji}\,C^{-1}_{lk} \\ &\quad + [\mu_0 - \lambda_0 \ln(\det(F_{ij})]\,[C^{-1}_{ik}\,C^{-1}_{jl} + C^{-1}_{il}\,C^{-1}_{jk}] \end{aligned}$$

matlab

- play with the matlab routine to familiarize yourself with tensor expressions!
- calculate the stresses for different deformation gradients!
- which of the following stress tensors is symmetric and could be represented in voigt notation? P_{ij}^{neo} S_{ij}^{neo} $\sigma_{ij}^{\mathrm{neo}}$
- what would $\boldsymbol{C}^{\mathrm{voigt}}$ look like in the linear limit, for $F_{ij} \approx \delta_{ij}$
- what are the advantages of using the voigt notation?