REU(G): Numerical computations of p-operator norms

Alonso Delfín

CU Boulder

May 13, 2024

Outline

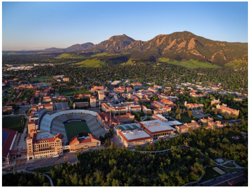
Outline

About me

ITAM

Cinvestav

UNIVERSITY OF OREGON



University of Colorado **Boulder**

About me

Introductions

- Preferred name and pronouns.
- Year, major, etc.
- Math interests.
- 1 Career Goal.
- Anything extra you'd like to add.

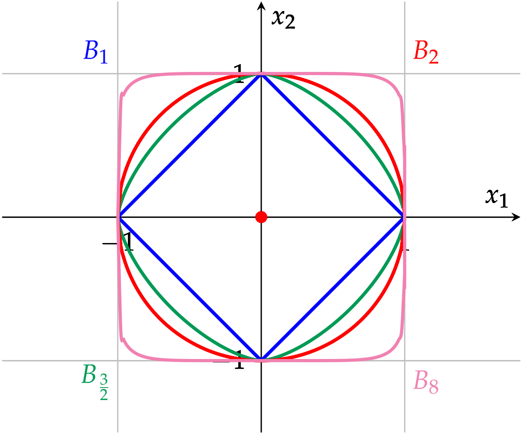
1. Alessandra
2. Ian
3. Luke
4. CJ
5. Jack
6. Anoushka
7. Wilson

Outline

Unit p-circles in $\mathbb{R}^2$

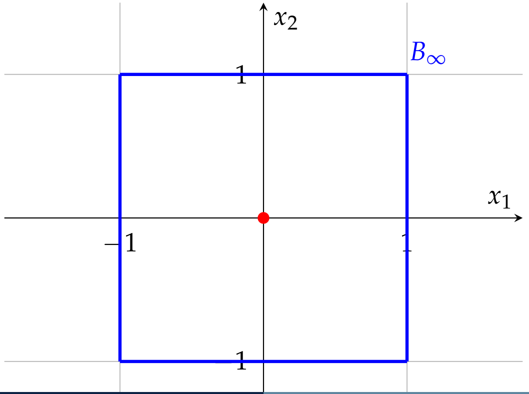
For $p \in [1, \infty)$ let

$$B_p := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1|^p + |x_2|^p = 1\}.$$



Letting $p \to \infty$

$$B_\infty := \{(x_1, x_2) \in \mathbb{R}^2 \colon \lim_{p\to\infty} |x_1|^p + |x_2|^p = 1\}$$
$$= \{(x_1, x_2) \in \mathbb{R}^2 \colon \max\{|x_1|, |x_2|\} = 1\}.$$



p-operator norms in $\mathbb{C}^d$

Let $d \in \mathbb{Z}_{\geq 1}$ and let $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$. We define

$$\|\boldsymbol{z}\|_p := \begin{cases} \left(\sum_{j=1}^{d} |z_j|^p \right)^{1/p} & p \in [1, \infty) \\ \max_{j \in \{1,\ldots,d\}} |z_j| & p = \infty \end{cases}.$$

The normed space $(\mathbb{C}^d, \| - \|_p)$ will be denoted by ℓ_d^p. Any linear map $a \colon \mathbb{C}^d \to \mathbb{C}^d$ is a $d \times d$ matrix with complex entries. For $p_1, p_2 \in [1, \infty]$ we define the $(p_1 \to p_2)$-operator norm of a by

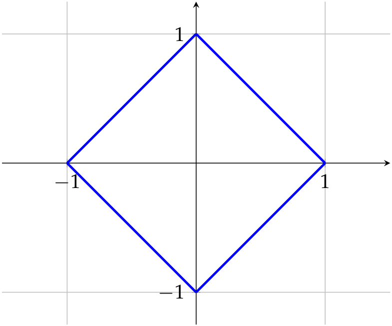
$$\|a\|_{p_1 \to p_2} := \max_{\|\boldsymbol{z}\|_{p_1} = 1} \|a(\boldsymbol{z})\|_{p_2}.$$

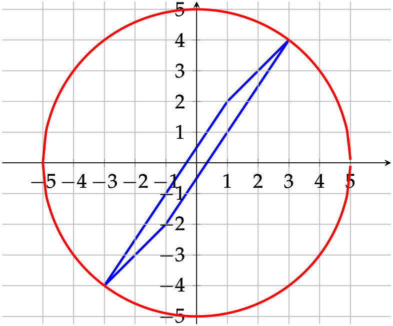
$\|a\|_{p_1 \to p_2}$ is the radius of the smallest p_2-circle that contains $a(B_{p_1})$.

When $p_1 = p_2$, we put $\|a\|_p := \|a\|_{p \to p}$.

$(1 \to 2)$-Operator Norm: Example

Let $a : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $a = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. How to find $\|a\|_{1\to 2}$?

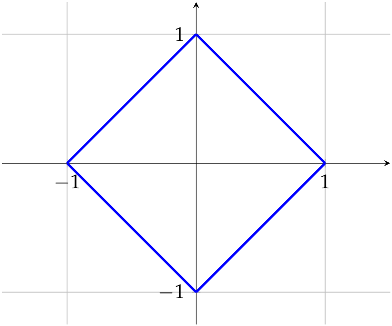


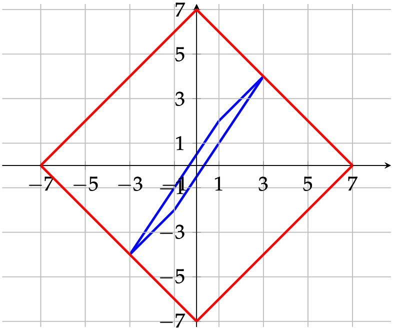


$$\|a\|_{1\to 2} = 5$$

1-Operator Norm: Example

Let $a : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $a = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. How to find $\|a\|_1$?





$$\|a\|_1 = 7$$

p-operator norms in $\mathbb{C}^d$: Known Cases

$M_d^p(\mathbb{C})$ is the algebra of $d \times d$ complex valued matrices equipped with the p-operator norm:

$$M_d^p(\mathbb{C}) = \mathcal{L}(\ell_d^p)$$

For $a \in M_d^p(\mathbb{C})$ we defined $\|a\|_p := \max_{\|z\|_p=1} \|a(z)\|_p$.

If $a = (a_{j,k})_{j,k=1}^d$, then

$$\|a\|_1 = \max_{k\in\{1,\dots,d\}} \sum_{j=1}^{d} |a_{j,k}| = \max_k \|(a_{1,k}, \dots, a_{d,k})\|_1,$$

$$\|a\|_2 = \max_{\lambda\in\sigma(\overline{a}^T a)} \sqrt{|\lambda|},$$

$$\|a\|_\infty = \max_{j\in\{1,\dots,d\}} \sum_{k=1}^{d} |a_{j,k}| = \max_j \|(a_{j,1}, \dots, a_{j,d})\|_1.$$

Otherwise, for a general matrix a, the value $\|a\|_p$ is NP-hard to compute.

$(p_1 \to p_2)$-operator norms in $\mathbb{C}^d$: Known cases

$M_d^{p_1 \to p_2}(\mathbb{C})$ is the algebra of $d \times d$ complex valued matrices equipped with the $(p_1 \to p_2)$-operator norm:

$$M_d^{p_1 \to p_2}(\mathbb{C}) = \mathcal{L}(\ell_d^{p_1}, \ell_d^{p_2})$$

For $a \in M_d^{p_1 \to p_2}(\mathbb{C})$ we defined $\|a\|_{p_1 \to p_2} := \max\limits_{\|z\|_{p_1}=1} \|a(z)\|_{p_2}$.

For $a = (a_{j,k})_{j,k=1}^d$:

$$\|a\|_{1\to 2} = \max_k \|(a_{1,k}, \ldots, a_{d,k})\|_2,$$

$$\|a\|_{1\to \infty} = \max_k \|(a_{1,k}, \ldots, a_{d,k})\|_\infty,$$

$$\|a\|_{2\to \infty} = \max_j \|(a_{j,1}, \ldots, a_{j,d})\|_\infty.$$

However, the computability of $\|a\|_{2\to 1}$, $\|a\|_{\infty\to 1}$, and $\|a\|_{\infty\to 2}$ is NP-hard.

Approximating $\|a\|_{p_1 \to p_2}$

Let $p_1, p_2 \in [1, \infty]$ and find $q_2 \in [1, \infty]$ such that

$$\frac{1}{p_2} + \frac{1}{q_2} = 1$$

Then, Hölder's inequality gives

$$\|a\|_{p_1 \to p_2} = \max_{\|\mathbf{x}\|_{p_1}=1, \|\mathbf{y}\|_{q_2}=1} \left|\mathbf{y}^T a \mathbf{x}\right|$$

Thus, $\|a\|_{p_1 \to p_2}$ can be approximated by maximizing over $\mathbf{x}$ and $\mathbf{y}$ one at a time, alternately.

Outline

Hilbert Spaces

A Hilbert space is a complex vector space $\mathcal{H}$ with a complex valued inner product

$$\mathcal{H} \times \mathcal{H} \ni (\xi, \eta) \mapsto \langle \xi, \eta \rangle \in \mathbb{C},$$

such that $\mathcal{H}$ is complete with the norm $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$.

Example

Let I be any set and put

$$\ell^2(I) := \left\{ \xi \colon I \to \mathbb{C} \;\middle|\; \sum_{j \in I} |\xi(j)|^2 < \infty \right\}.$$

Then $\ell^2(I)$ is a Hilbert space with

$$\langle \xi, \eta \rangle := \sum_{j \in I} \xi(j) \overline{\eta(j)}.$$

Notice that $\ell_d^2 = \ell^2(\{1, \dots, d\})$, which is simply $\mathbb{C}^d$ with the 2-norm.

Operators on Hilbert spaces

Let $\mathcal{H}$ be a Hilbert space. A linear map $a\colon \mathcal{H} \to \mathcal{H}$ is said to be bounded if

$$\|a\| := \sup_{\|\xi\|=1} \|a(\xi)\| < \infty$$

We denote by $\mathcal{L}(\mathcal{H})$ the space of bounded linear maps $\mathcal{H} \to \mathcal{H}$.

Remark.

A linear map $a\colon \mathcal{H} \to \mathcal{H}$ is bounded if and only if it's continuous.

We say function $a\colon \mathcal{H} \to \mathcal{H}$ is adjointable if there is another function $b\colon \mathcal{H} \to \mathcal{H}$ satisfying

$$\langle a\xi, \eta \rangle = \langle \xi, b\eta \rangle$$

for all $\xi, \eta \in \mathcal{H}$. Then, $a, b \in \mathcal{L}(\mathcal{H})$ and moreover $\|a\| = \|b\|$. In fact, b is uniquely determined by a, denoted by $b = a^*$, and it's known as the adjoint of a.

Theorem

Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{L}(\mathcal{H}) =$adjointable maps.

Subspaces of Operators

Let $\mathcal{H}$ be a Hilbert space. Broadly speaking, operator algebraist study closed subspaces of $\mathcal{L}(\mathcal{H})$. In particular

- A closed subspace X $\subseteq \mathcal{L}(\mathcal{H})$ it's known as an operator space.
- A closed subalgebra $A \subseteq \mathcal{L}(\mathcal{H})$ it's known as an operator algebra.
- A closed and selfadjoint subalgebra $A \subseteq \mathcal{L}(\mathcal{H})$ it's known as a C*-algebra.

A **Banach algebra** is a complete normed complex algebra A such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.

Theorem (Gelfand-Naimark-Segal (1943))

Let A be a Banach algebra with an involution $A \ni a \mapsto a^* \in A$, satisfying

$$\|a^*a\| = \|a\|^2 \quad \textit{for all} \quad a \in A$$

Then there is a Hilbert space $\mathcal{H}$ and an isometric $*$-homomorphism $\varphi \colon A \to \mathcal{L}(\mathcal{H})$. Thus, A can be isometrically identified with the C-algebra $\varphi(A) \subseteq \mathcal{L}(\mathcal{H})$.*

C*-algebras: Examples

The following are basic examples of C*-algebra.

1. $\mathcal{L}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$,
2. $M_n^2(\mathbb{C}) = \mathcal{L}(\ell_n^2)$,
3. $\ell^\infty(I, A)$, for any set I and any C*-algebra A, with pointwise multiplication and pointwise involution,
4. $(C(\Omega), \| - \|_{\text{sup}})$ for a compact Hausdorff space Ω,
5. $(C_0(\Omega), \| - \|_{\text{sup}})$ for a locally compact Hausdorff space Ω.

Theorem (Gelfand-Naimark (1943))

Let A be a nonzero commutative C-algebra. Then A is *-isomorphic to $C_0(\Omega)$ for a locally compact Hausdorff space Ω.*

Isometries in $\mathcal{L}(\mathcal{H})$

Let $\mathcal{H} = \ell^2(\mathbb{Z}_{\geq 1})$ and $d \in \mathbb{Z}_{\geq 2}$. Then there are elements $s_1, \ldots, s_d \in \mathcal{L}(\mathcal{H})$ such that

$$s_j^* s_j = \mathrm{id}_{\mathcal{H}}, \quad \text{and} \quad \sum_{j=1}^{d} s_j s_j^* = \mathrm{id}_{\mathcal{H}}$$

Indeed, let $(\delta_j)_{j=1}^{\infty}$ the canonical orthonormal basis for $\ell^2(\mathbb{Z}_{\geq 1})$, that is, $\delta_j(k) = \delta_{j,k}$. For $d = 2$, we put $s_1(\delta_j) = \delta_{2j}$, and $s_2(\delta_j) = \delta_{2j-1}$. Then,

$$s_1^*(\delta_j) = \begin{cases} \delta_{j/2} & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases} \quad \text{and} \quad s_2^*(\delta_j) = \begin{cases} \delta_{(j+1)/2} & \text{if } j \text{ is odd} \\ 0 & \text{if } j \text{ is even} \end{cases}$$

$s_1^* s_1 = \mathrm{id}_{\mathcal{H}} = s_2^* s_2$

Recall $s_1(\delta_j) = \delta_{2j}$, $s_2(\delta_j) = \delta_{2j-1}$,

$$s_1^*(\delta_j) = \begin{cases} \delta_{j/2} & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases} \quad \text{and} \quad s_2^*(\delta_j) = \begin{cases} \delta_{(j+1)/2} & \text{if } j \text{ is odd} \\ 0 & \text{if } j \text{ is even} \end{cases}$$

$$\begin{array}{ccccc} & s_1 & & s_1^* & \\ \delta_1 & \mapsto & \delta_2 & \mapsto & \delta_1 \\ \delta_2 & \mapsto & \delta_4 & \mapsto & \delta_2 \\ \delta_3 & \mapsto & \delta_6 & \mapsto & \delta_3 \\ \delta_4 & \mapsto & \delta_8 & \mapsto & \delta_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \quad \text{and} \quad \begin{array}{ccccc} & s_2 & & s_2^* & \\ \delta_1 & \mapsto & \delta_1 & \mapsto & \delta_1 \\ \delta_2 & \mapsto & \delta_3 & \mapsto & \delta_2 \\ \delta_3 & \mapsto & \delta_5 & \mapsto & \delta_3 \\ \delta_4 & \mapsto & \delta_7 & \mapsto & \delta_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Therefore, $s_1^* s_1 = \mathrm{id}_{\mathcal{H}} = s_2^* s_2$.

$s_1 s_1^* + s_2 s_2^* = \mathrm{id}_{\mathcal{H}}$

Recall $s_1(\delta_j) = \delta_{2j}$, $s_2(\delta_j) = \delta_{2j-1}$,

$$s_1^*(\delta_j) = \begin{cases} \delta_{j/2} & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases} \quad \text{and} \quad s_2^*(\delta_j) = \begin{cases} \delta_{(j+1)/2} & \text{if } j \text{ is odd} \\ 0 & \text{if } j \text{ is even} \end{cases}$$

$$\begin{array}{ccccc} & s_1^* & & s_1 & \\ \delta_1 & \mapsto & 0 & \mapsto & 0 \\ \delta_2 & \mapsto & \delta_1 & \mapsto & \delta_2 \\ \delta_3 & \mapsto & 0 & \mapsto & 0 \\ \delta_4 & \mapsto & \delta_2 & \mapsto & \delta_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \quad \text{and} \quad \begin{array}{ccccc} & s_2^* & & s_2 & \\ \delta_1 & \mapsto & \delta_1 & \mapsto & \delta_1 \\ \delta_2 & \mapsto & 0 & \mapsto & 0 \\ \delta_3 & \mapsto & \delta_2 & \mapsto & \delta_3 \\ \delta_4 & \mapsto & 0 & \mapsto & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Hence, $s_1 s_1^* + s_2 s_2^* = \mathrm{id}_{\mathcal{H}}$.

The Cuntz Algebra $\mathcal{O}_d$

Definition

We define $\mathcal{O}_d$, the Cuntz algebra of order $d \in \mathbb{Z}_{\geq 2}$, as the C* algebra in $\mathcal{L}(\mathcal{H})$ generated by $s_1, \dots, s_d$.

Some interesting facts about $\mathcal{O}_d$:

1. $\mathcal{O}_d$ is a simple, unital C*-algebra,
2. $\mathcal{O}_d$ has the following universal property: If A is a unital C*-algebra containing elements $a_1, \dots, a_d$ such that

$$a_j^* a_j = 1_A \quad \text{and} \quad \sum_{j=1}^{d} a_j a_j^* = 1_A,$$

   then there is a unique $*$-homomorphism $\varphi : \mathcal{O}_d \to A$ such that $\varphi(s_j) = a_j$.
3. If $d_1 \neq d_2$, then $\mathcal{O}_{d_1} \not\simeq \mathcal{O}_{d_2}$. This follows from applying the functor $K_0 \colon \mathbf{C^*Alg} \to \mathbf{Ab}$ which satisfies

$$K_0(\mathcal{O}_d) \simeq \mathbb{Z}/(d-1)\mathbb{Z}$$

Outline

Hilbert Spaces vs L^p-spaces

Any Hilbert space $\mathcal{H}$ is unitarily isomorphic to

$$\ell^2(I) := \left\{ \xi \colon I \to \mathbb{C} \;\middle|\; \sum_{j \in I} |\xi(j)|^2 < \infty \right\}.$$

for a set I.

Thus, if $p \in [1, \infty)$ and (Ω, μ) is a measure space, the Banach space

$$L^p(\Omega, \mu) := \left\{ [f \colon \Omega \to \mathbb{C}]_\mu \colon \int_\Omega |f|^p d\mu < \infty \right\}$$

generalizes the concept of a Hilbert space.

Definition

For a fixed $p \in [1, \infty)$, we say a Banach Algebra A is an L^p-operator algebra if there is a (Ω, μ) is a measure space and an isometric homomorphism $\varphi \colon A \to \mathcal{L}(L^p(\Omega, \mu))$.

Differences between C*-algebras and L^p-operator algebras

If A is a C*-algebra, the GNS construction yields a Hilbert space and a nondegenerate isometric *-homomorphism $\varphi\colon A \to \mathcal{L}(\mathcal{H})$:

$$\overline{\operatorname{span}\{\varphi(a)\xi \mid a \in A, \xi \in \mathcal{H}\}} = \mathcal{H}$$

Furthermore, A always has a contractive approximate unit, that is a net $(e_\lambda)_\lambda$ with $\|e_\lambda\| \leq 1$ satisfying

$$\lim_{\lambda} \|a - e_\lambda a\| = \lim_{\lambda} \|e_\lambda a - a\| = 0.$$

Finally, C*-norms are unique.

1. L^p-operator algebras lack involution,
2. Some L^p-operator algebras can't be nondegenerately represented,
3. Some L^p-operator algebras don't have contractive approximate units,
4. An abstract characterization of L^p-operator algebras, among all Banach algebras, is not known,
5. L^p-operator norms are generally hard to compute,
6. L^p-operator norms are not unique.

Outline

Two norms that make $\mathbb{C}^2$ an L^1-Operator Algebra

Equip $\mathbb{C}^2$ with the usual max norm $\|(z_1, z_2)\|_\infty = \max\{|z_1|, |z_2|\}$.
Then, $\varphi\colon \mathbb{C}^2 \to \mathcal{L}(\ell_2^1)$ given as

$$[\varphi(z_1, z_2)]\xi = (z_1\xi(1), z_2\xi(2))$$

is an isometric homomorphism. Thus $(\mathbb{C}^2, \| - \|_\infty)$ is an L^1-operator algebra.
Now consider $\ell^1(\mathbb{Z}/2\mathbb{Z})$ with multiplication given via convolution:

$$(\xi * \eta)(n) = \xi(0)\eta(n) + \xi(1)\eta(n-1)$$

Then $\psi\colon \ell^1(\mathbb{Z}/2\mathbb{Z}) \to \mathcal{L}(\ell^1(\mathbb{Z}/2\mathbb{Z}))$ given by $\psi(\xi)\eta = \xi * \eta$ is an isometric homomorphism, making $\ell^1(\mathbb{Z}/2\mathbb{Z})$ an L^1-operator algebra.
The Fourier transform $\mathcal{F}\colon \ell^1(\mathbb{Z}/2\mathbb{Z}) \to C(\mathbb{Z}/2\mathbb{Z})$ given by

$$(\mathcal{F}\xi)(n) = \xi(0) + (-1)^n\xi(1)$$

is an algebra isomorphism. Hence, $\ell^1(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{C}^2$ as algebras, which endows $\mathbb{C}^2$ with another L^1-operator norm.

A norm in $\mathbb{C}^2$ that does not make it an L^1-operator algebra

Theorem (Bernau-Lacey (1977))

Let $p \in [1, \infty)$ and let $e \in \mathcal{L}(L^p(\mu))$ be a bicontractive idempotent (i.e. $e^2 = e$, $\|e\| \leq 1$, and $\|1 - e\| \leq 1$). Then $\|1 - 2e\| = 1$.

In $\ell^1(\mathbb{Z}/3\mathbb{Z})$ we consider the ideal $\ell^1_0(\mathbb{Z}/3\mathbb{Z})$ generated by $\Delta_1 := \delta_1 - \delta_0$ and $\Delta_2 := \delta_2 - \delta_0$. Equip $\ell^1_0(\mathbb{Z}/3\mathbb{Z})$ with the following norm

$$\|\alpha\Delta_1 + \beta\Delta_2\|_0 = \sup_{\|a\Delta_1 + b\Delta_2\|_1 = 1} \|(\alpha\Delta_1 + \beta\Delta_2) * (a\Delta_1 + b\Delta_2)\|_1$$

Theorem (Blinov-D-Weld (2024))

The ideal $\ell^1_0(\mathbb{Z}/3\mathbb{Z})$ is algebraically isomorphic to $\mathbb{C}^2$ and the element $e = \gamma\Delta_1 + \overline{\gamma}\Delta_2$, where $\gamma := \frac{e^{2\pi i/3}}{3}$, is a bicontractive idempotent with $\|1 - 2e\|_0 = \frac{2}{\sqrt{3}} > 1$. In particular, $(\ell^1_0(\mathbb{Z}/3\mathbb{Z}), \| - \|_0)$ is not an L^p-operator algebra for any $p \in [1, \infty)$.

Outline

Main goal

Let $p \in [1, \infty)$ and consider the vector space $\mathsf{X} = M_d \oplus M_d$ with norm

$$\|(a_1, a_2)\|_{\mathsf{X}} := \max\{(\|a_1 z\|_p^p + \|a_2 z\|_p^p)^{1/p} \colon \|z\|_p = 1\}.$$

Similarly, let $\mathsf{Y} := M_d \oplus M_d$, but now equipped with the norm

$$\|(b_1, b_2)\|_{\mathsf{Y}} := \max\{(\|b_1 z_1 + b_2 z_2\|_p \colon \|z_1\|_p^p + \|z_2\|_p^p = 1\}.$$

Conjecture 1:

$$\|(a_1, a_2)\|_{\mathsf{X}} = \sup\{\|b_1 a_1 + b_2 a_2\|_{p \to p} \colon \|(b_1, b_2)\|_{\mathsf{Y}} = 1\}$$

Conjecture 2:

$$\|(b_1, b_2)\|_{\mathsf{Y}} = \sup\{\|b_1 a_1 + b_2 a_2\|_{p \to p} \colon \|(a_1, a_2)\|_{\mathsf{X}} = 1\}$$