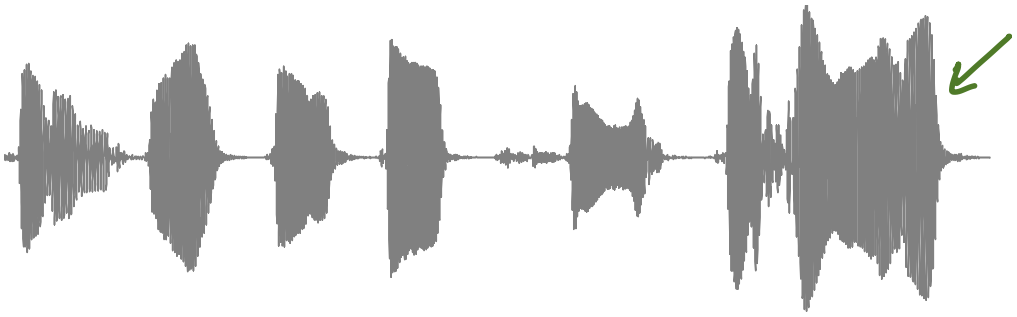
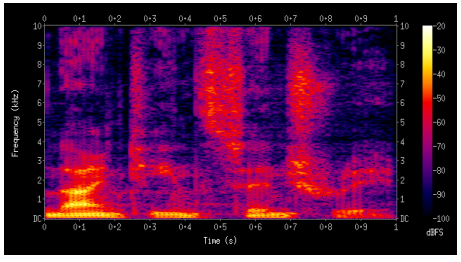
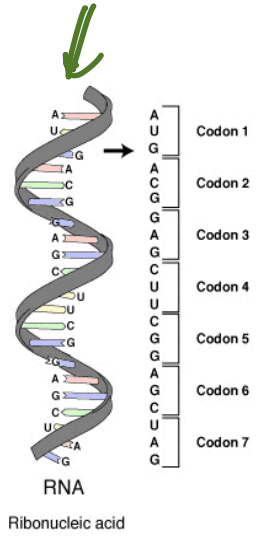
Sequence Modelling

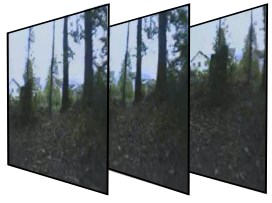
Rich Turner

Sequence data

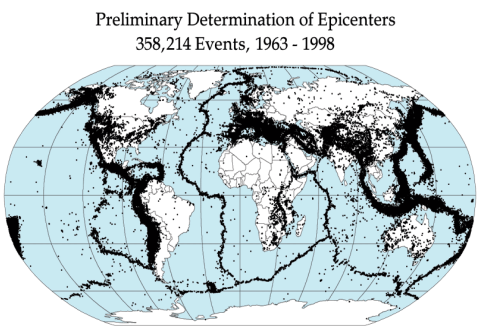


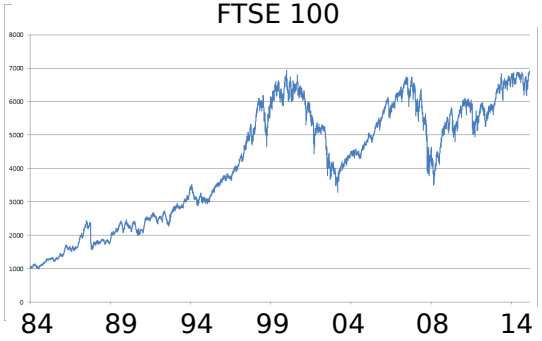


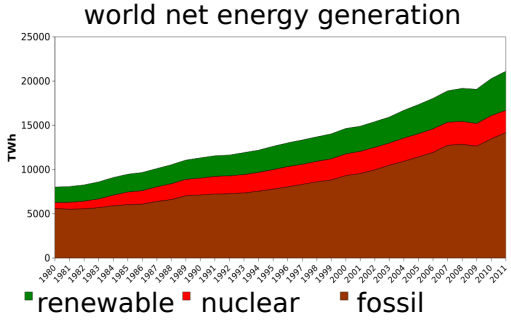




Good King Wenceslas looked out,
On the Feast of Stephen,
When the snow lay round about,
Deep and crisp and even;
Brightly shone the moon that night,
Though the frost was cruel,
When a poor man came in sight,
Gathering winter fuel.







I believe that at the end of the century the use of words and general educated opinion will have altered so much that one will be able to speak of machines thinking without expecting to be contradicted. A. Turing

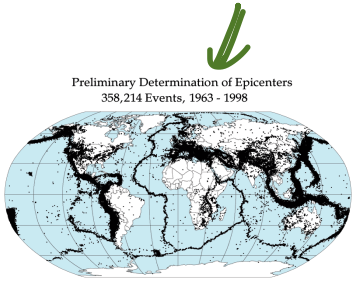
Some images taken from wikipedia

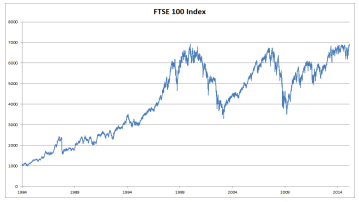
Goals of sequence modelling

observations

Predict future items in sequence

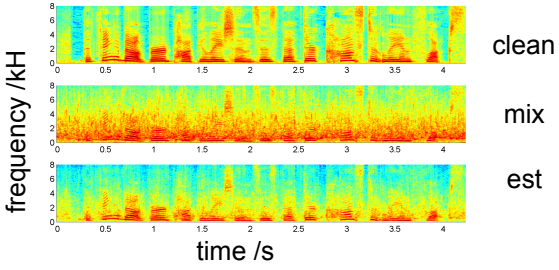
$$\Rightarrow p(y_t | y_1, \ldots, y_{t-1})$$





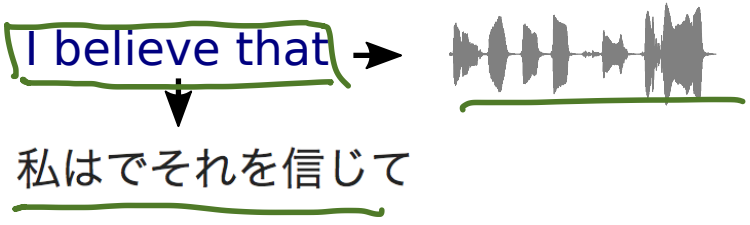
Remove noise from a sequence

$$p(y'_1, \ldots, y'_t | y_1, \ldots, y_t)$$



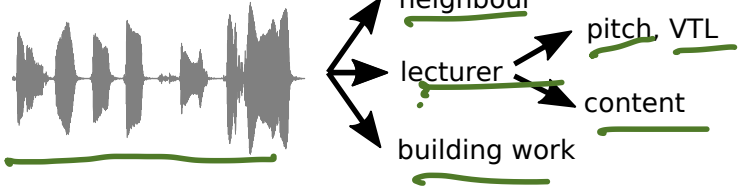
Predict one sequence from another

$$p(y'_1, \ldots, y'_t | y_1, \ldots, y_t)$$



Discover underlying latent variables

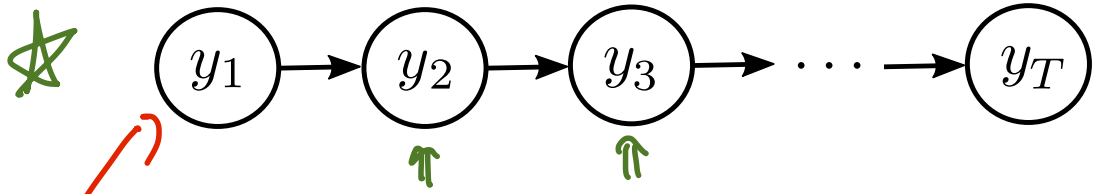
$$p(x_1, \ldots, x_t | y_1, \ldots, y_t)$$



Markov models

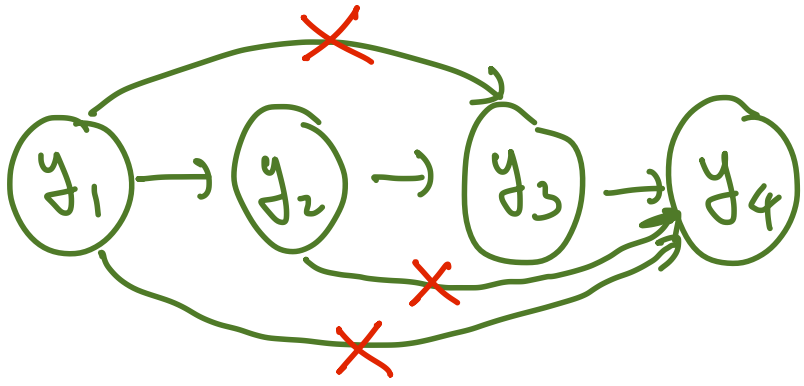
First order Markov

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



$T=4$

$$p(y_1, y_2, y_3, y_4) = p(y_1)\,p(y_2|y_1)\,p(y_3|y_2, \cancel{y_1})\,p(y_4|y_3, \cancel{y_2}, \cancel{y_1})$$



1st order Markov

$$y_t \perp y_{t-2} \mid y_{t-1}$$

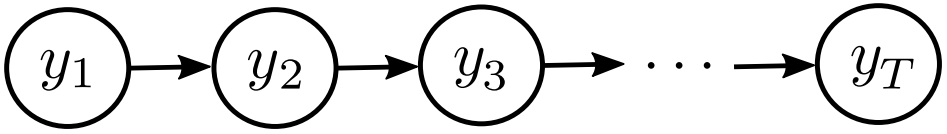
Markov models

parameters tied

∞ number of variables
finite number of parameters

First order Markov

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



$$X \sim \mathcal{G}\left(\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}; \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_T \end{bmatrix}, \Sigma_{1:T,1:T}\right)$$

T

TxT

Gaussian

N

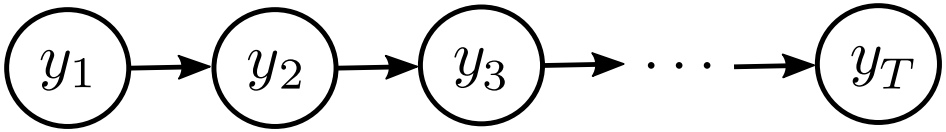
Markov models

parameters tied

∞ number of variables
finite number of parameters

First order Markov

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



Markov model = conditional independence relationship + product rule

future

independent of past

$$y_{t+1} \perp y_{1:t-1} | y_t$$

given present

$$p(y_{1:T}) = \prod_{t=1}^{T} p(y_t|y_{1:t-1})$$

Markov models

parameters tied

∞ number of variables
finite number of parameters

First order Markov

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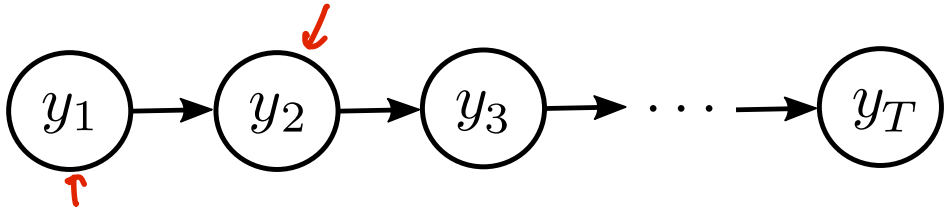

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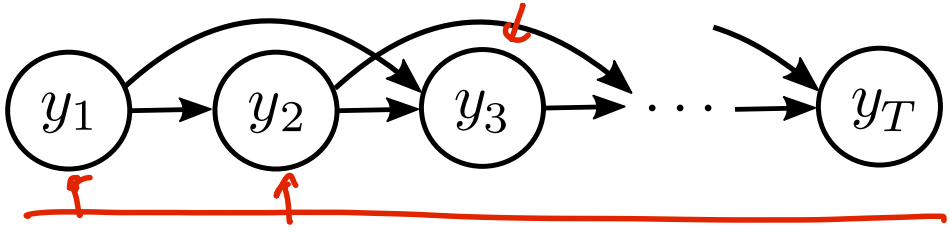
Second order Markov

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2, y_1)\ldots p(y_T|y_{T-1}, y_{T-2})$$

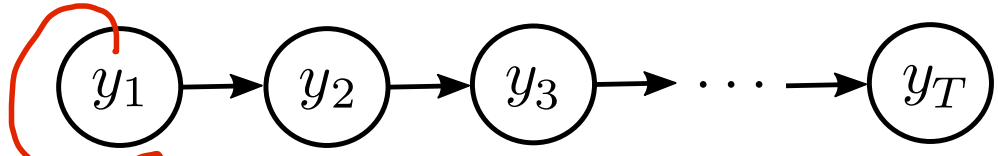


Markov models for discrete data: n-gram models

NLP

First order Markov (bi-gram)

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



$y_t \in \{1, ..., K\}$ discrete states

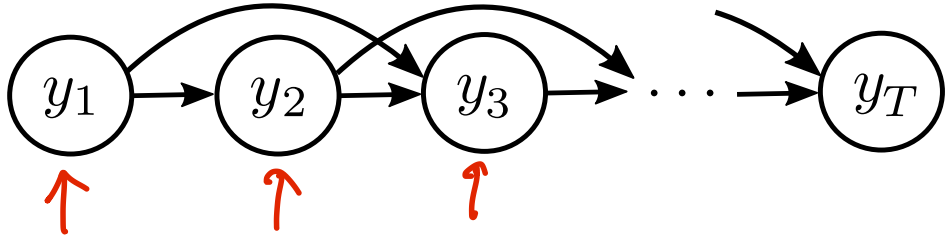
$p(y_1 = k) = \pi_k^0$ initial state probabilities

$p(y_t = k|y_{t-1} = l) = T_{k,l}$ transition probabilities (stochastic matrix)

$$\sum_{k=1}^{K} T_{k,l} = 1$$

Second order Markov (tri-gram)

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2, y_1)\ldots p(y_T|y_{T-1}, y_{T-2})$$



$p(y_t = k|y_{t-1} = l, y_{t-2} = m) = T_{k,l,m}$

n-grams require large multidimensional arrays

Some questions about n-gram models

First order Markov (bi-gram)

$$y_t \in \{1, ..., K\} \quad p(y_1 = k) = \pi_k^0 \qquad p(y_t = k | y_{t-1} = l) = T_{k,l}$$

Q1. How can we compute the marginal distribution over the second state?

$$p(y_2 = k) = \sum_{l=1}^{K} p(y_2 = k | y_1 = l) p(y_1 = l)$$

$$p(y_2 = k) = \sum_{l=1}^{k} T_{kl} \pi_l^0$$

sum

$$p(y_2 = k) = \sum_{l=1}^{K} p(y_2 = k, y_1 = l)$$

product

$$p(y_2 = k) = \left[\underline{\underline{T}} \, \underline{\pi}^0 \right]_k \qquad p(y_2 = k) = \left[\underline{\rho}^0 \, \underline{\underline{S}} \right]_k$$

3F3 $\underline{\underline{S}} = \underline{\underline{T}}^T \qquad \underline{\pi}^0 = \underline{\rho}_0^T$

Some questions about n-gram models

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Q2. How can we compute the stationary distribution for the Markov chain?

$$p(y_\infty = k) = (\pi_\infty)_k$$

$$T \pi_\infty = \pi_\infty \lambda \qquad \lambda = 1$$

$$p(y_t = k) = \sum_{l=1}^{K} p(y_t = k | y_{t-1} = l) \, p(y_{t-1} = l)$$

$$\pi_k^\infty = \sum_{l=1}^{K} T_{k,l} \, \pi_l^\infty$$

Some questions about n-gram models

First order Markov (bi-gram)

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Some questions about n-gram models

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$$\pi_k^\infty = \sum_{l=1}^{K} T_{k,l} \pi_l^\infty$$

eigenvectors of transition matrix with eigenvalue = 1

Some questions about n-gram models

counts

from state

to state	A	B	C
A	2	2	0
B	3	1	1
C	0	2	2
	5	5	3

First order Markov (bi-gram)

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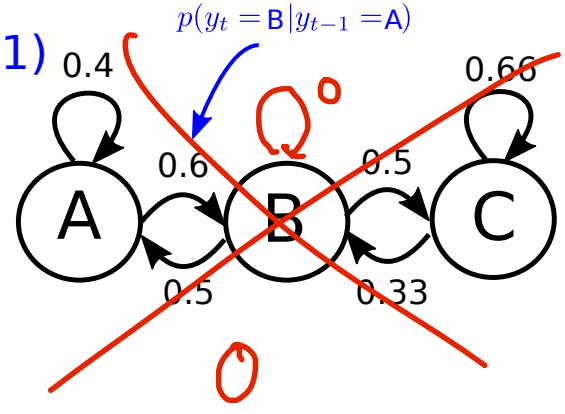
eigenvectors of transition matrix with eigenvalue = 1

Q3. Which transition matrix is most compatible with the following sequence?

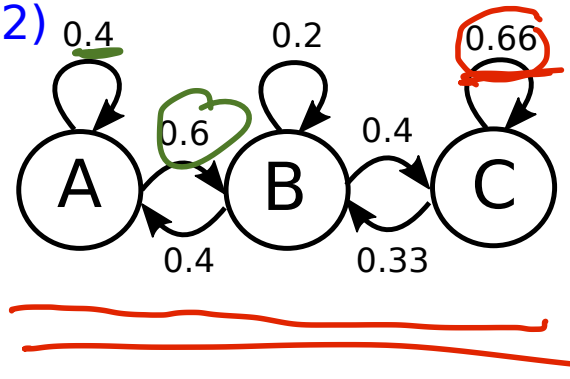
ABAAABBABCCCBC

'State Transition Diagrams'

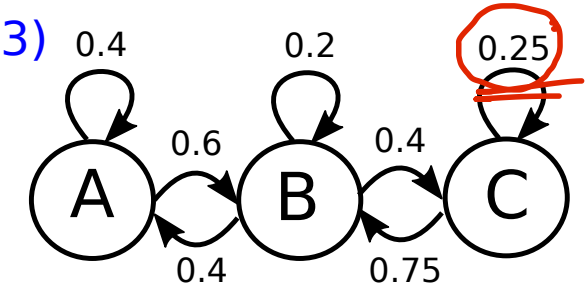
1)



2)



3)



Some questions about n-gram models

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eigenvectors of transition matrix with eigenvalue = 1

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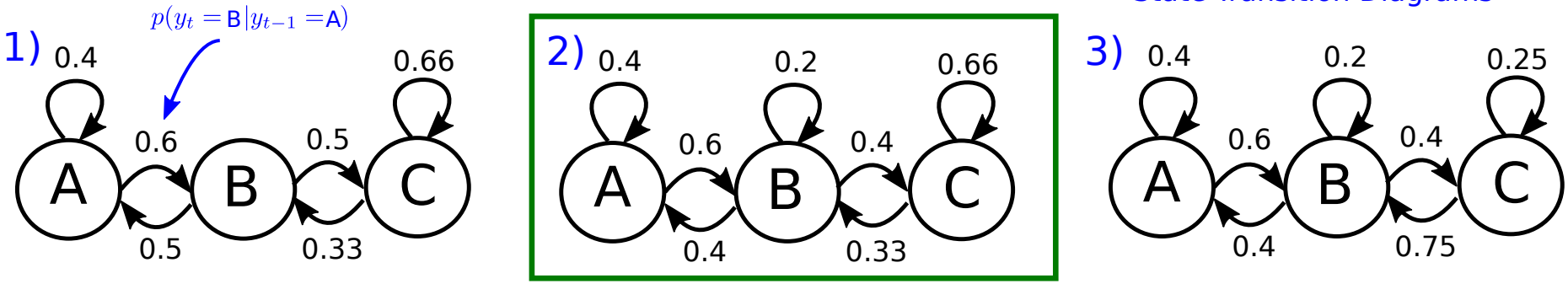
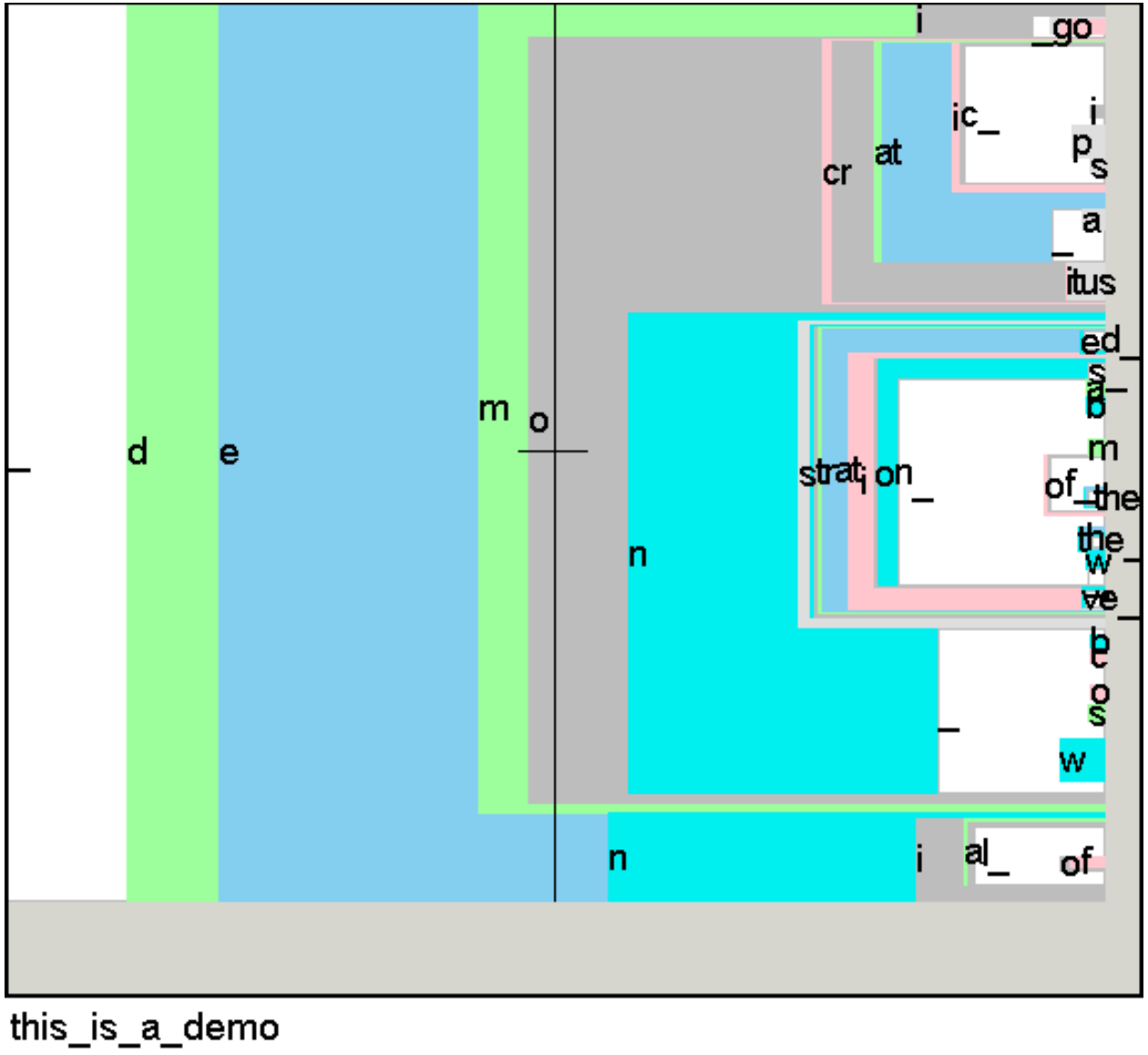


Example application of n-grams: text modelling for dasher



http://www.inference.phy.cam.ac.uk/dasher/ https://www.youtube.com/watch?v=nr3s4613DX8

Markov models for discrete data: n-gram models

First order Markov (bi-gram)

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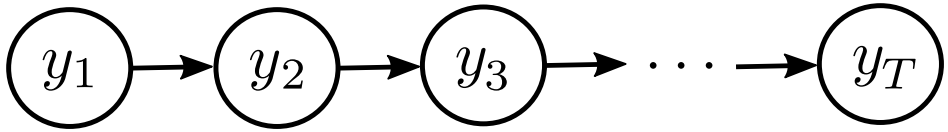

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Second order Markov (tri-gram)

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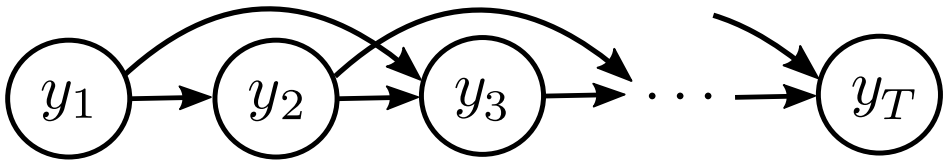


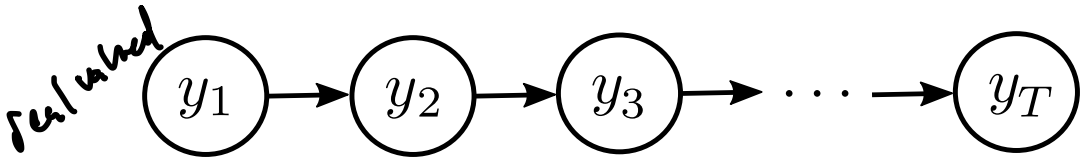
$p(y_t = k|y_{t-1} = l, y_{t-2} = m) = T_{k,l,m}$

n-grams require large multidimensional arrays

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

First order Markov (AR(1))

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



$y_t \in \mathbb{R}^D$ — continuous vector states

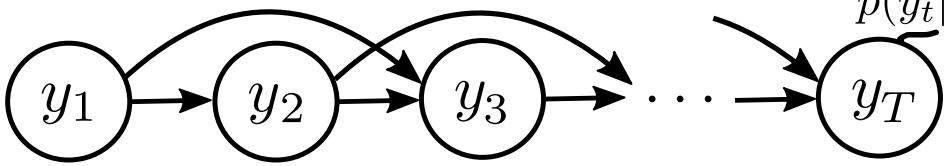
$p(y_1) = \mathcal{G}(y_1; \mu_0, \Sigma_0)$ — initial state density

$p(y_t|y_{t-1}) = \mathcal{G}(y_t; \Lambda y_{t-1}, \Sigma)$ — transition density

$$\mathcal{G}(y; \mu, \Sigma) = \frac{1}{(2\pi)^{D/2}\det(\Sigma)^{1/2}} \exp\left\{-\frac{1}{2}(y-\mu)^\top \Sigma^{-1}(y-\mu)\right\}$$

Second order Markov (AR(2))

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2, y_1)\ldots p(y_T|y_{T-1}, y_{T-2})$$



$p(y_t|y_{t-1}, y_{t-2}) = \mathcal{G}(y_t; \Lambda_1 y_{t-1} + \Lambda_2 y_{t-2}, \Sigma)$

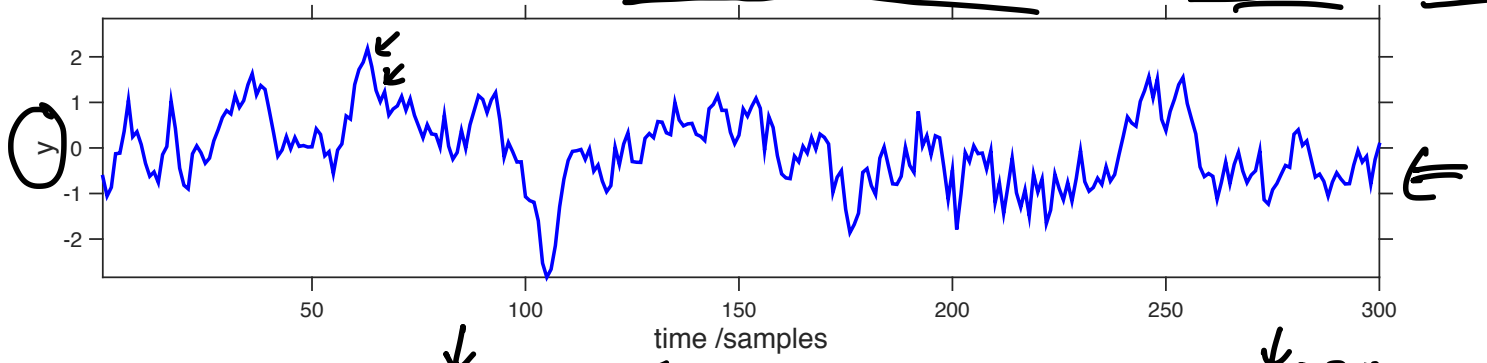
joint distribution over all variables is always multivariate Gaussian

$$p(y_{1:T}) = \mathcal{G}(y_{1:T}; \mu_{1:T}, \Sigma_{1:T,1:T})$$

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

First order Markov (AR(1))

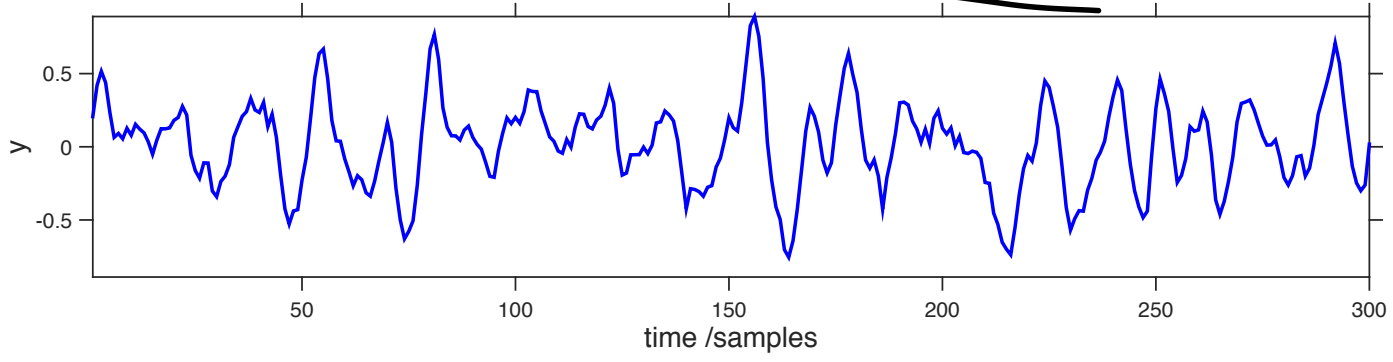
$$y_t \in \mathbb{R}^1 \qquad p(y_t|y_{t-1}) = \mathcal{G}(y_t; \lambda y_{t-1}, \sigma^2) \qquad \lambda = 0.9 \quad \sigma^2 = 0.01$$



Second order Markov (AR(2))

$$y_t \in \mathbb{R}^1 \qquad p(y_t|y_{t-1}, y_{t-2}) = \mathcal{G}(y_t; \lambda_1 y_{t-1} + \lambda_2 y_{t-2}, \sigma^2)$$

$$[\lambda_1, \lambda_2] = [1.57, -0.78] \qquad \sigma^2 = 0.01$$



Markov models for continuous data: Auto-Regressive (AR) Gaussian models

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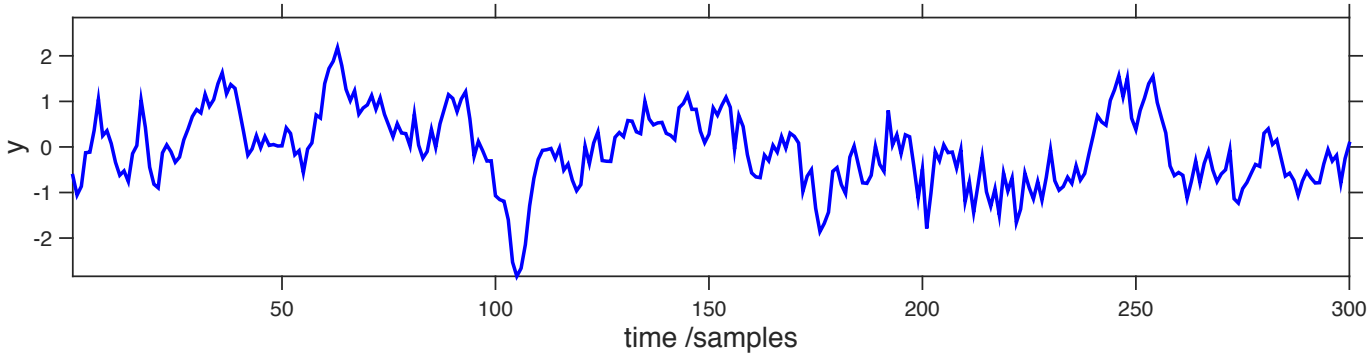

What is the stationary distribution of this process? $p(y_\infty) = ?$

$\mathcal{G}(y_\infty; \mu_\infty, \sigma_\infty^2)$

$y_\infty \to 0$

$\mathbb{E}_{p(y_t)}(y_t) = \mathbb{E}(\lambda y_{t-1} + \epsilon_t \sigma)$

$p(y_{t-1}, \epsilon_t)$

$\mu_t = \lambda \mu_{t-1} + 0.\sigma \Rightarrow \mu_t = \lambda \mu_{t-1}$

$\mu_\infty = \lambda \mu_\infty \Rightarrow \mu_\infty = 0$

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

$y_t = \lambda y_{t-1} + \epsilon_t \sigma$

First order Markov (AR(1))

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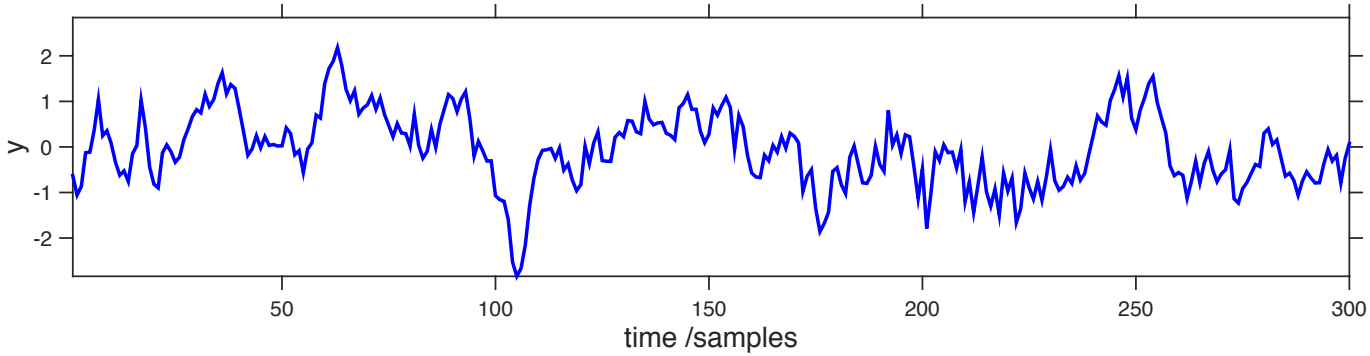

$\mathbb{E}(y_{t-1})\mathbb{E}(\epsilon_t)$ → 0

What is the stationary distribution of this process? $p(y_\infty) = ?$

Everything is linear Gaussian => must be Gaussian $p(y_\infty) = \mathcal{G}(y_\infty; \mu_\infty, \sigma^2_\infty)$

$\mathbb{E}(y_t^2) = \mathbb{E}\left((\lambda y_{t-1} + \epsilon_t \sigma)^2\right) = \lambda^2 \mathbb{E}(y_{t-1}^2) + 2\mathbb{E}(y_{t-1}\epsilon_t)\sigma\lambda + \mathbb{E}(\epsilon_t^2)\sigma^2$ (cross term → 0)

$= \lambda^2 \mathbb{E}(y_{t-1}^2) + \sigma^2$

$\sigma_\infty^2 = \lambda^2 \sigma_\infty^2 + \sigma^2 \qquad \sigma_\infty^2 = \frac{\sigma^2}{1-\lambda^2} = \frac{0.01}{1-0.9^2}$

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

First order Markov (AR(1))

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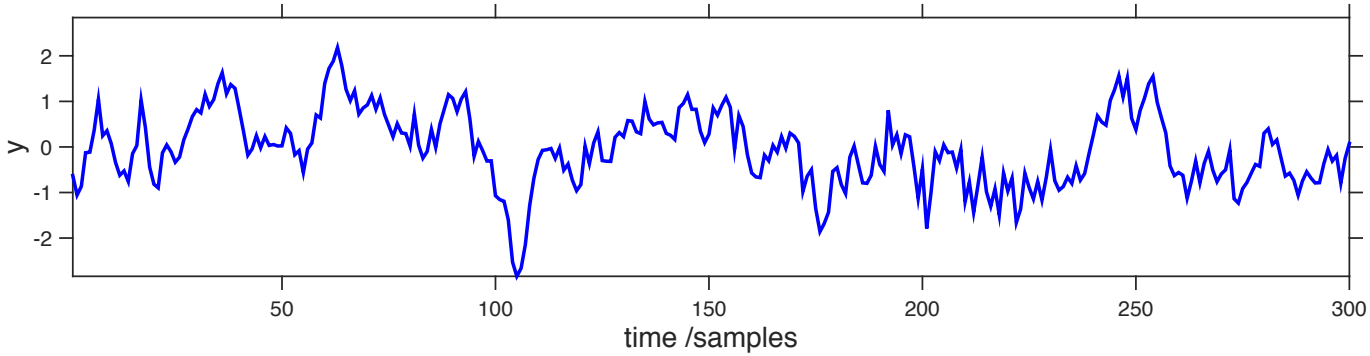

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Markov models for continuous data: Auto-Regressive (AR) Gaussian models

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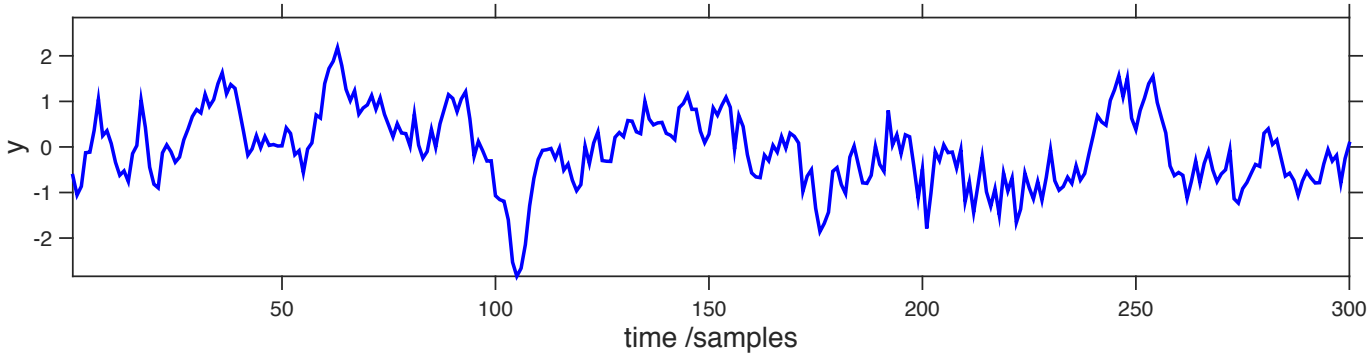

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Mean: $\langle y_t \rangle$

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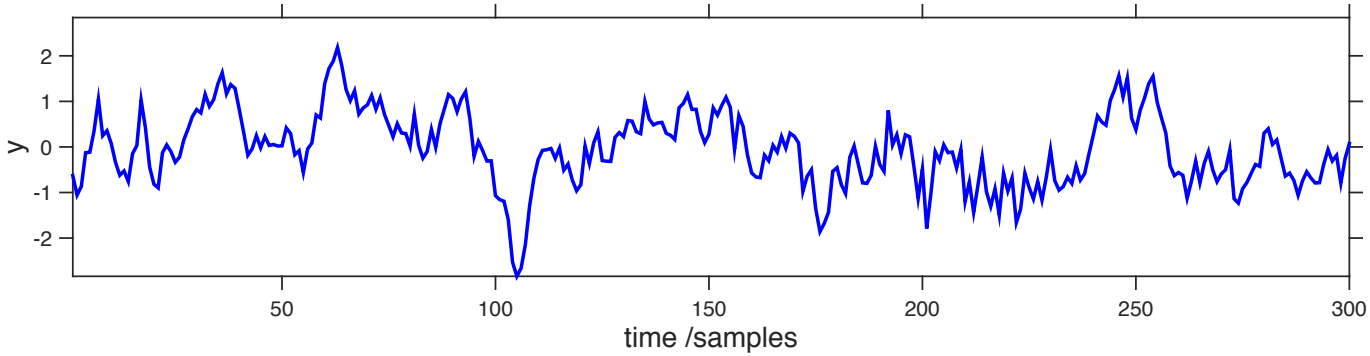

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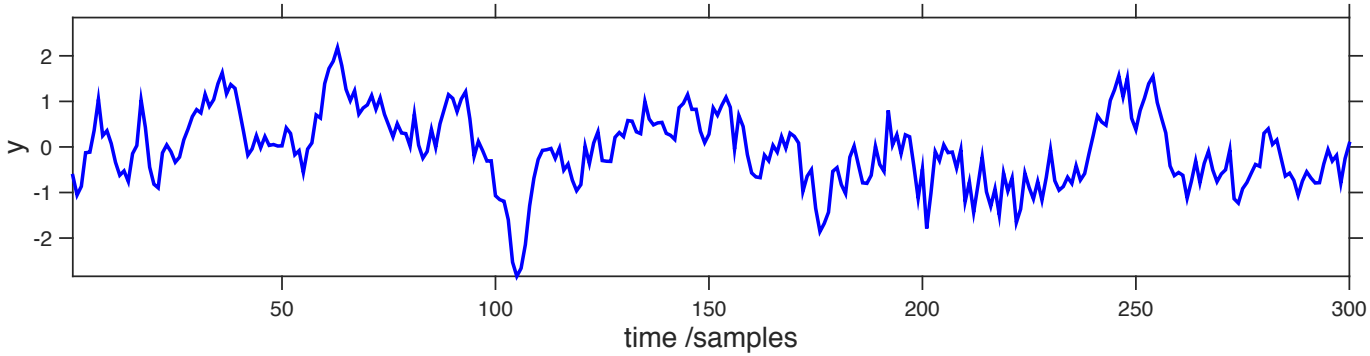

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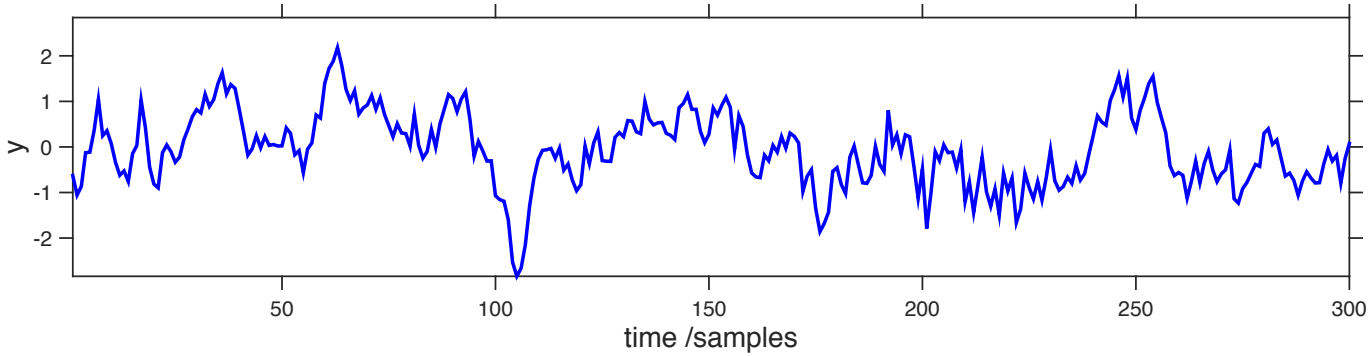

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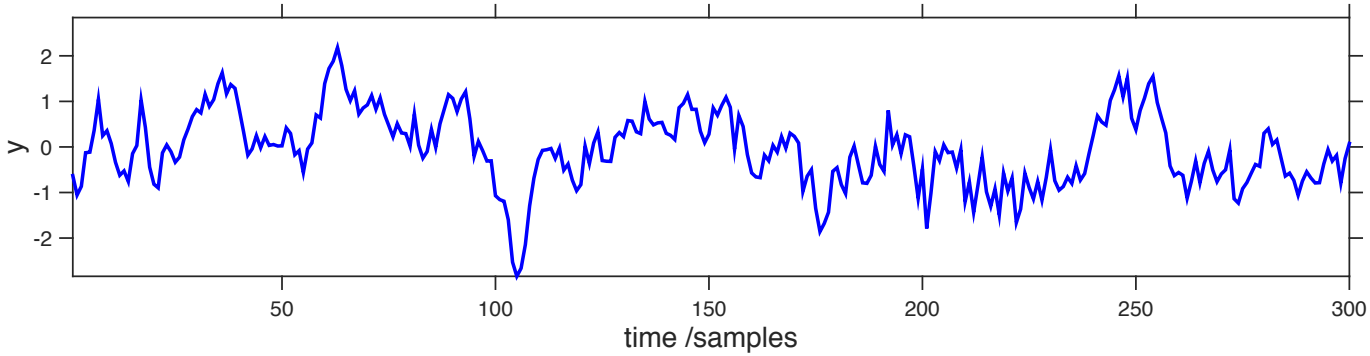

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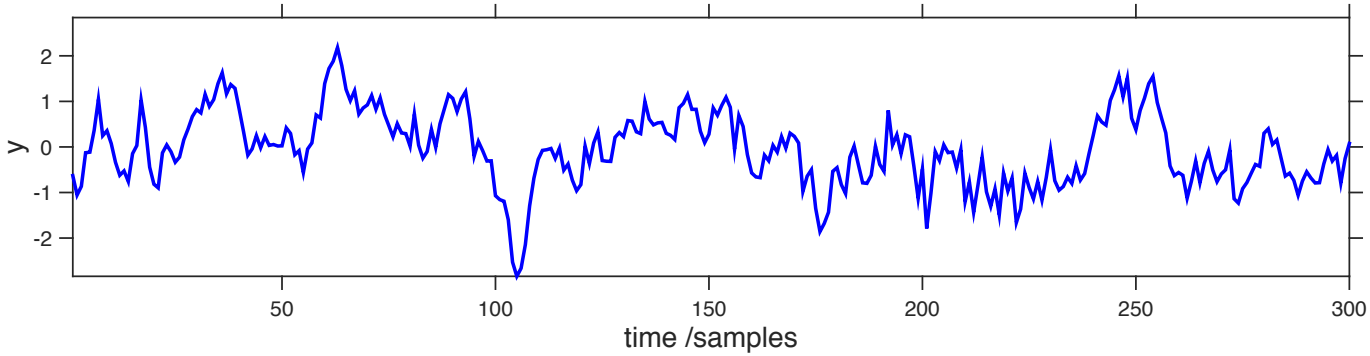

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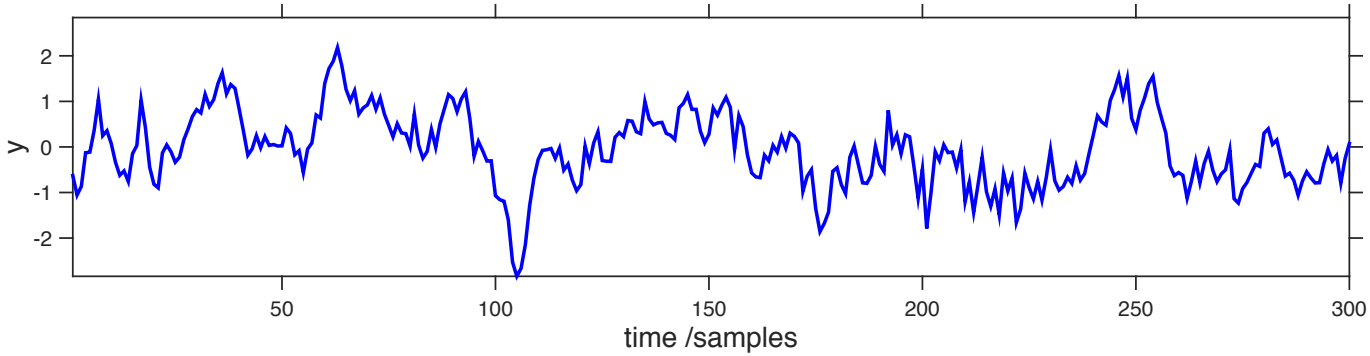

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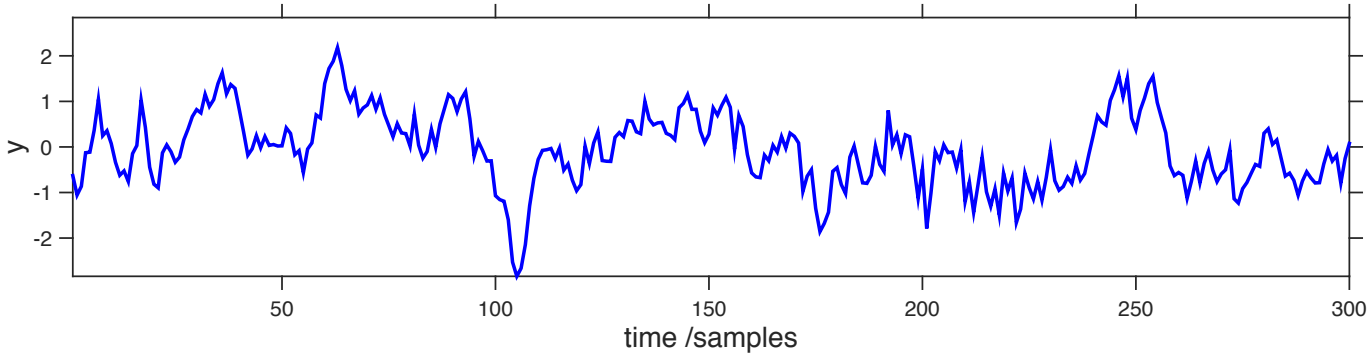

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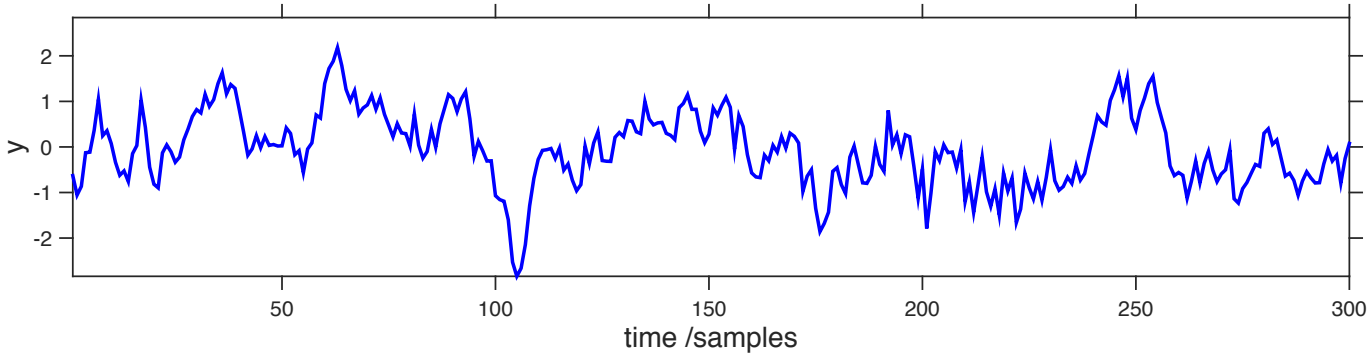

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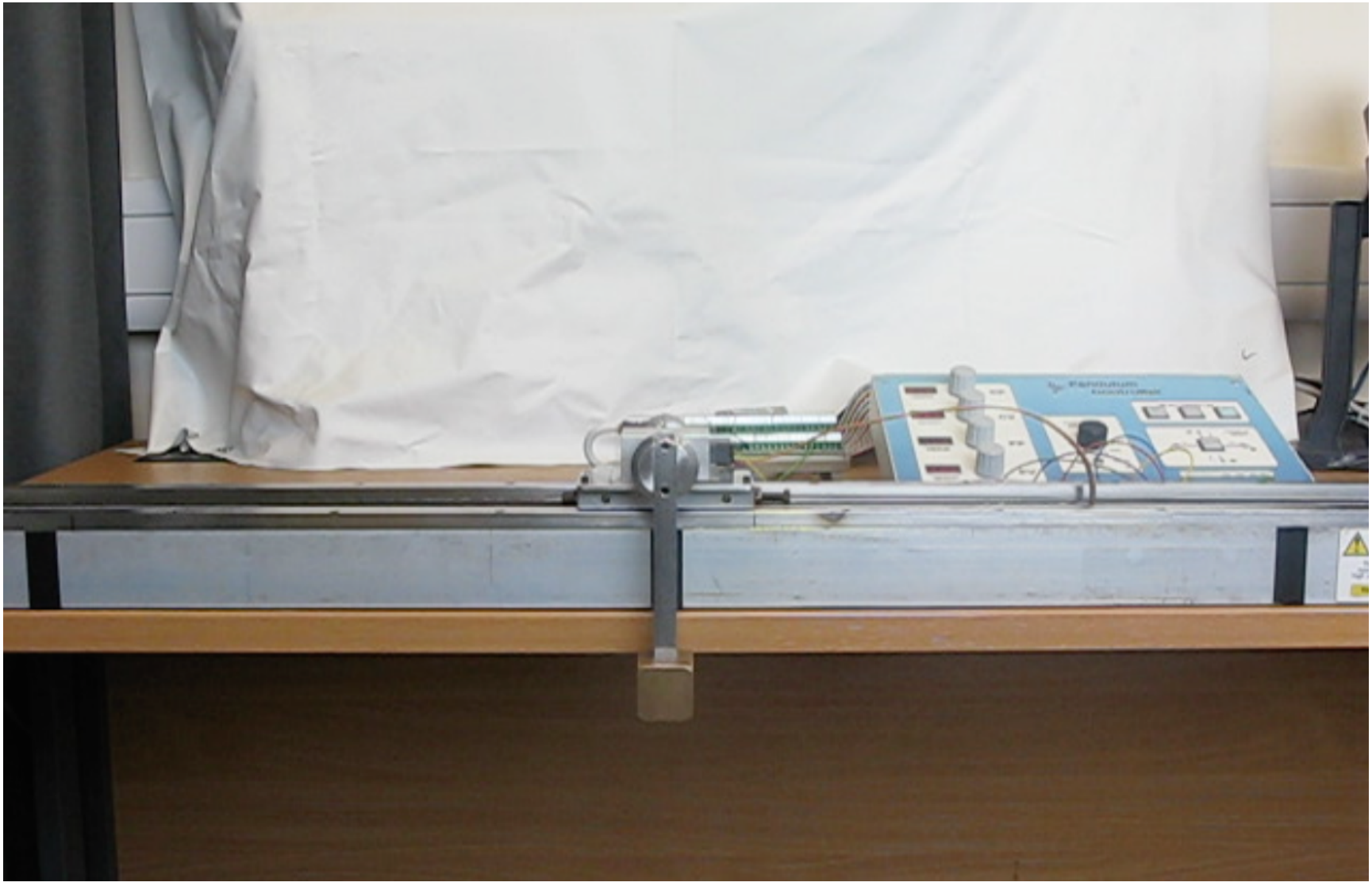
$$\langle y_t^2 \rangle = \lambda^2 \langle y_{t-1}^2 \rangle + \sigma^2 \quad \sigma^2_\infty = \lambda^2 \sigma^2_\infty + \sigma^2 \qquad \sigma^2_\infty = \frac{\sigma^2}{1-\lambda^2}$$

Example application of Markov Models: pendulum swing up control problem



Summary of Markov Models

1st order $p(y_{1:T}) = p(y_1)\,p(y_2|y_1)\,p(y_3|y_2)\,p(y_4|y_3)\cdots p(y_T|y_{T-1})$

Discrete y $\Rightarrow$ Bigram Models

$p(y_1 = k) = \pi^0_k$ ★

$p(y_t = k \mid y_{t-1} = l) = T_{kl}$ ★

$\Downarrow$ stationary distribution

$p(y_\infty = k) = \pi^\infty_k = \sum_l T_{kl}\,\pi^\infty_l$

Continuous y $\Rightarrow$ Autoregressive Models

$p(\underline{y}_1) = \mathcal{G}(\underline{y}_1;\, \underline{\mu}_0, \underline{\underline{\Sigma}}_0)$ ★

$p(\underline{y}_t \mid \underline{y}_{t-1}) = \mathcal{G}(\underline{y}_t;\, \underline{\underline{\Lambda}}\,\underline{y}_{t-1}, \underline{\underline{\Sigma}})$ ★

$\dim(\underline{y}_t) = 1$

$\Downarrow$ stationary distribution

$p(y_\infty) = \mathcal{G}\left(y_\infty;\, 0, \frac{\sigma^2}{1-\lambda^2}\right)$

$\lambda \to 1$
$\mathrm{Var}(y_\infty) \to \infty$

Hidden Markov models

Real data depend on latent variables

ASR

x phonemes/words

y waveform/feature

Computer Vision

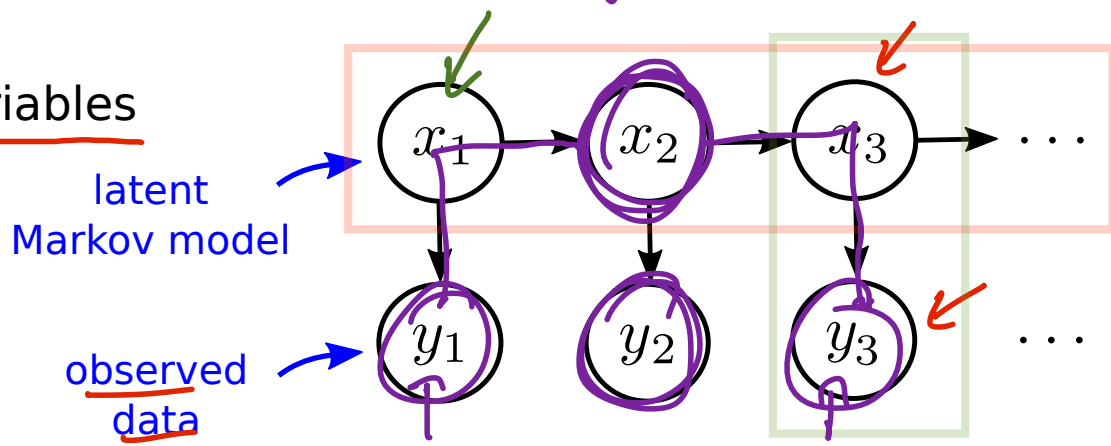
x objects, pose, lighting

y image pixel intensities

Natural Language Processing

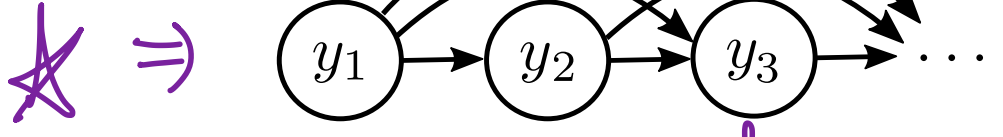
x topics

y words



$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

marginalise latents



$$p(y_{1:T}) = \int p(y_{1:T}, x_{1:T})\, dx_{1:T}$$

fully connected

Two prevelant Examples:

- Hidden Markov Models (discrete x)
- Linear Gaussian State Space Models (Gaussian x and y)

Hidden Markov models: discrete hidden state

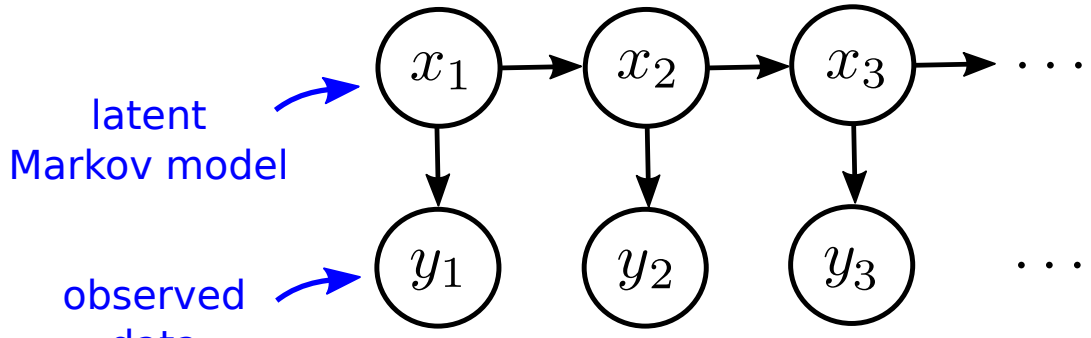
Discrete Hidden State

$$x_t \in \{1, ..., K\}$$

$$p(x_t = k | x_{t-1} = l) = T_{k,l}$$

E.g. in examples below $K = 2$

$$T = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$

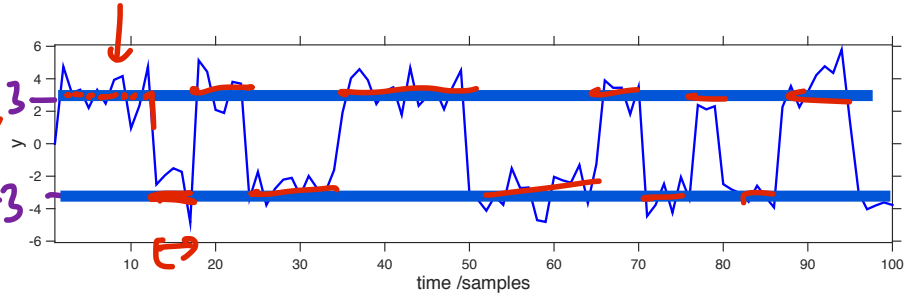


$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t | x_{t-1}) p(y_t | x_t)$$

Continuous Observed State

$$p(y_t | x_t = k) = \mathcal{G}(y_t; \mu_k, \Sigma_k)$$

$\mu_1 = 3$ $\mu_2 = -3$ $\sigma_1^2 = \sigma_2^2 = 1$



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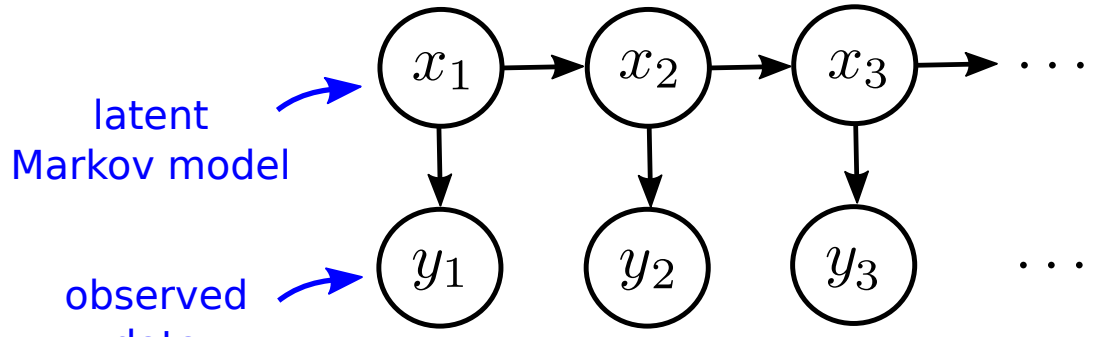

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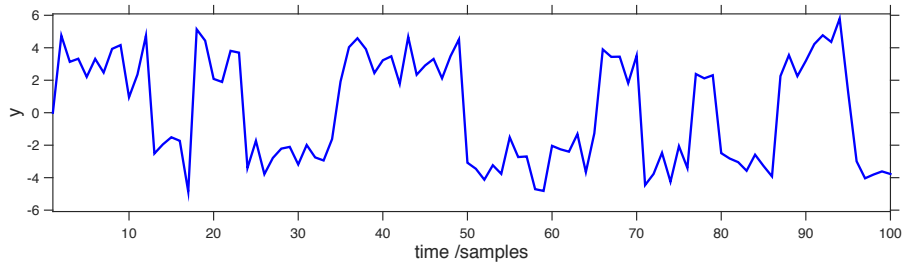
$$p(y_t | x_t = k) = \mathcal{G}(y_t; \mu_k, \Sigma_k)$$

$\mu_1 = 3 \quad \mu_2 = -3 \quad \sigma_1^2 = \sigma_2^2 = 1$



Discrete Observed State

$$p(y_t = l | x_t = k) = S_{l,k}$$

$$S = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

emissions distribution

$x_t = 1$ $x_t = 2$

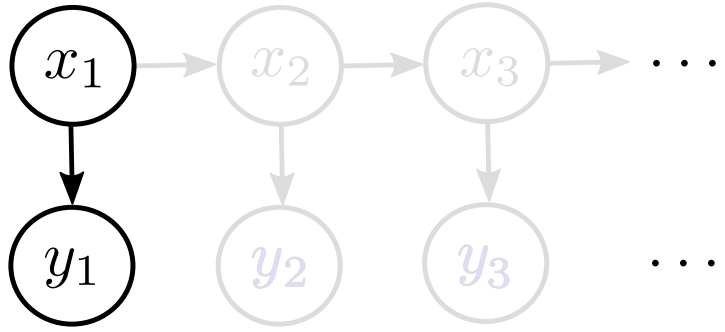
$y_t \; t \in \{A, B, C\}$

ABBBBAAABAAACCCCCBBBBBBCCCCCCCCCCCC
AAABBBBAABAAABBCCCCCCCCCCCCCCCCBBA
AACCCCCCBABCCCCCCCAABBAABABCCCCC

Hidden Markov models: discrete hidden state

Discrete Hidden State, Continuous Observed State

$$x_t \in \{1, ..., K\}$$
$$p(x_1 = k) = \pi_k^0$$
$$p(x_t = k | x_{t-1} = l) = T_{k,l}$$
$$p(y_t | x_t = k) = \mathcal{G}(y_t; \mu_k, \Sigma_k)$$



Consider T = 1

Q1: What type of distribution is $p(y_1)$?

$$p(y_1) = \sum_{k=1}^{K} p(y_1 | x_1 = k)\, p(x_1 = k)$$
$$= \sum_{k=1}^{K} \mathcal{G}(y_1; \mu_k, \Sigma_k)\, \pi_k^0$$

Hidden Markov models: discrete hidden state

Discrete Hidden State, Continuous Observed State

$$x_t \in \{1, ..., K\}$$

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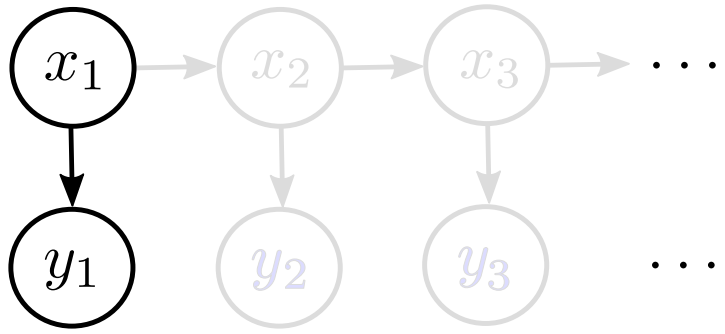

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Discrete Hidden State, Continuous Observed State

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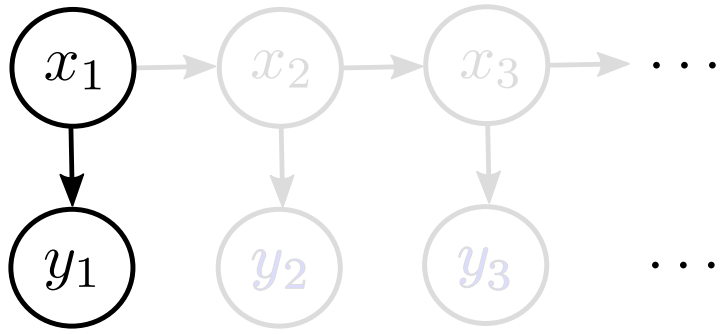

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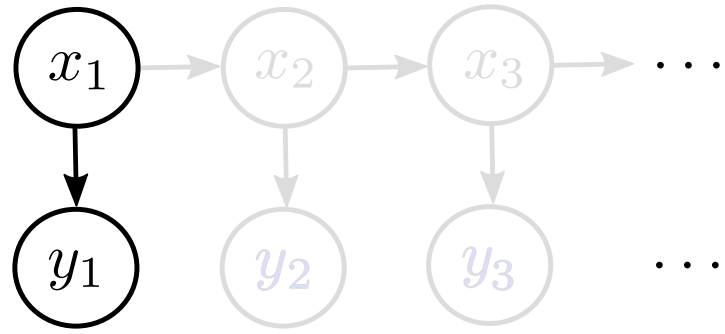

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Q2: What distribution does $p(y_t)$ converge to after a long time?

$$\| \quad p(x_\infty = k) = \pi_k^\infty$$

$$p(y_\infty) = \sum_k p(y_\infty | x_\infty = k) \, p(x_\infty = k) = \sum_k \pi_k^\infty \mathcal{G}(y_\infty; \mu_k, \Sigma_k)$$

Hidden Markov models: discrete hidden state

Discrete Hidden State, Continuous Observed State

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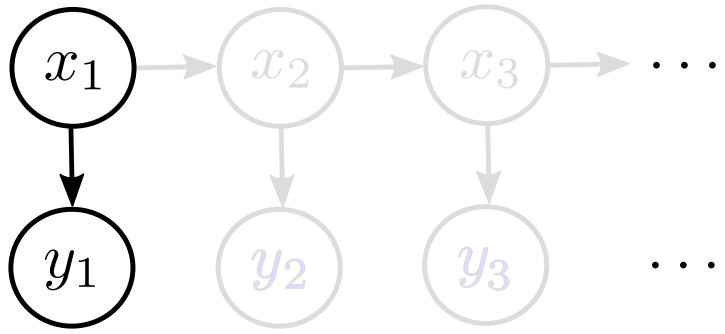

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Hidden Markov models: discrete hidden state

Discrete Hidden State, Continuous Observed State

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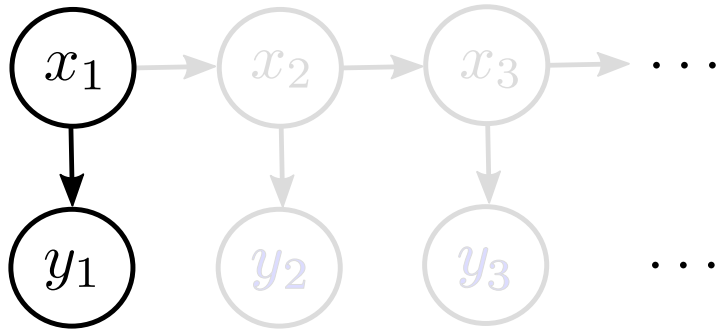

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Hidden Markov models: discrete hidden state

Discrete Hidden State, Continuous Observed State

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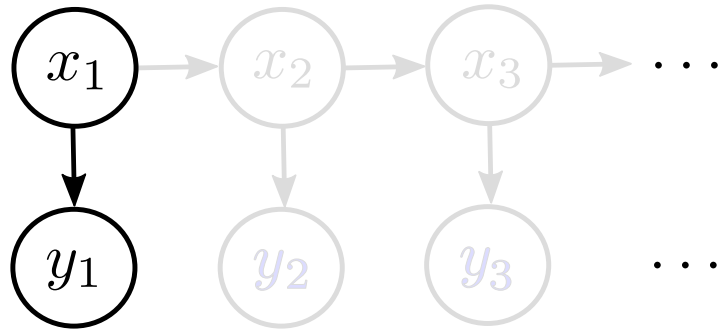

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$$p(y_t) = \sum_k p(y_t | x_t = k) p(x_t = k) \rightarrow \sum_k \pi_k^\infty \mathcal{G}(y_t; \mu_k, \Sigma_k)$$

Hidden Markov models: discrete hidden state

Discrete Hidden State, Continuous Observed State

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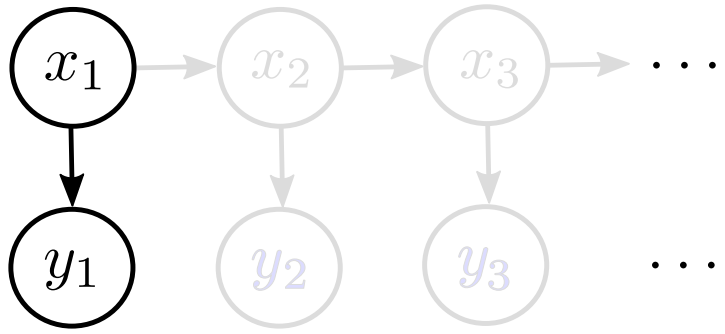

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this HMM = Mixture of Gaussian Models with dynamic cluster assignments

Hidden Markov models: continuous hidden state (LGSSMs)

Linear Gaussian State Space Models

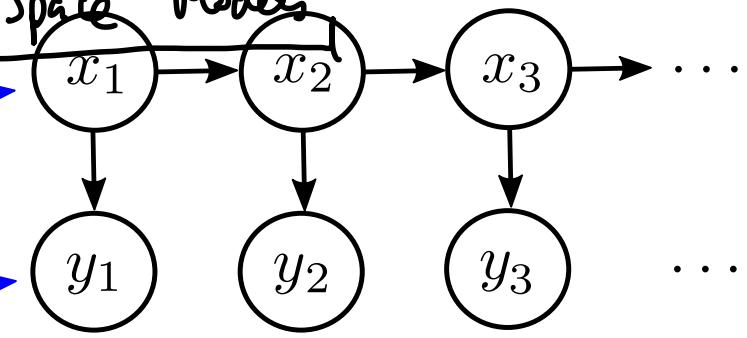
Continuous Hidden State

$x_t \in \mathbb{R}^K$

$p(x_t|x_{t-1}) = \mathcal{G}(x_t; Ax_{t-1}, Q)$

latent Markov model

observed data



Continuous Observed State

$y_t \in \mathbb{R}^D$

$p(y_t|x_t) = \mathcal{G}(y_t; Cx_t, R)$

DxD

DxK

$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

E.g. simple example $K = 2$ $D = 1$

$$A = \lambda \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

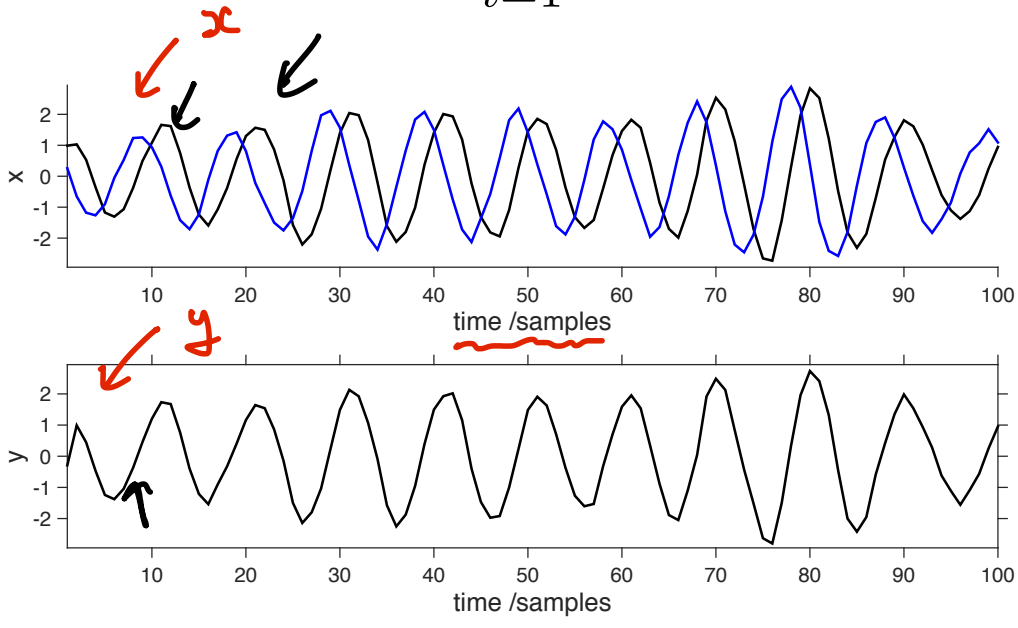
$\lambda = 0.99$ $\theta = 2\pi/10$

$$Q = (1 - \lambda^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$C = [1, 0]$ $R = 0.01$

dynamics model

obs. model

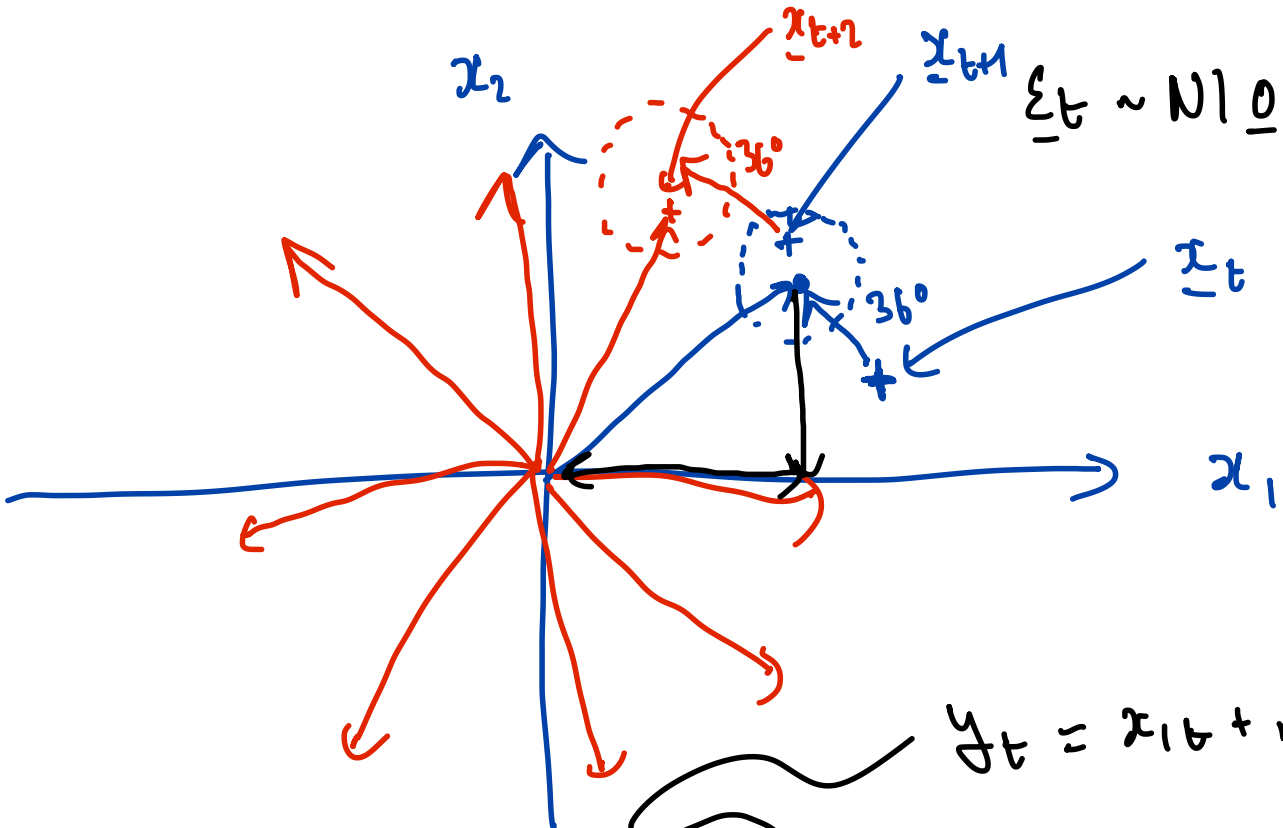


$\lambda = 0.99$

A

$\sigma \in Q$

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \underbrace{\lambda \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{A} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \underbrace{(1-\lambda^2)^{1/2}}_{\sigma} \varepsilon_t$$



$\varepsilon_t \sim N(0, I)$

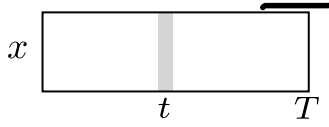
$y_t = x_{1t} + \text{noise}$

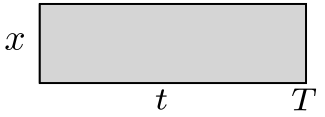
$$y_t = \underbrace{[1,0] \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}}_{x_{1t}} + \sqrt{0.01}\, n_t \qquad n_t \sim N(0,1)$$

Varieties of Inference

Distributional estimates

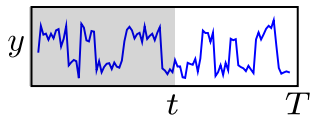
infer single state or sequence?

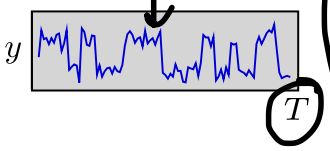




online

future data available?





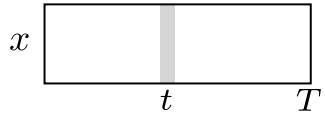
	marginal	joint
filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

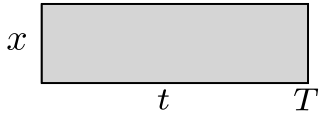
off line

Varieties of Inference

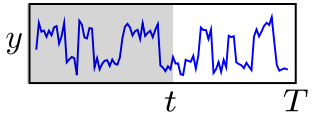
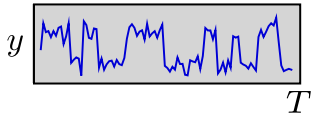
Distributional estimates

infer single state or sequence?





future data available?

	marginal	joint
 filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
 smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

Point estimates

most probable state @ t

$$x_t^* = \arg\max_{x_t} p(x_t \mid y_{1:T})$$

most probable sequence

$$x'_{1:T} = \arg\max_{x_{1:T}} p(x_{1:T} \mid y_{1:T})$$

Summary Sequence Modelling Lecture III

$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t \mid x_{t-1})\, p(y_t \mid x_t) \quad \Leftarrow$$

($y_{1:T}$: observed, $x_{1:T}$: latent; $p(x_t \mid x_{t-1})$: Markov; $p(y_t \mid x_t)$: $\uparrow \uparrow$)

★ **Discrete Hidden State** $x_t \in \{1 \dots K\}$ (also called HMMs!)

$\Rightarrow p(x_t = k \mid x_{t-1} = l) = T_{kl}$ ⇜

$$\begin{cases} p(y_t \mid x_t = k) = G(y_t; \mu_k, \Sigma_k) & \swarrow \quad y_t \in \mathbb{R}^D \\ p(y_t = l \mid x_t = k) = S_{lk} \leftarrow & y_t \in \{1 \dots D\} \end{cases}$$

★ **Continuous Hidden State** $x_t \in \mathbb{R}^K$ (linear Gaussian state space models)

$\Rightarrow p(\underline{x}_t \mid \underline{x}_{t-1}) = G(\underline{x}_t; \underline{\underline{A}}\, \underline{x}_{t-1}, \underline{\underline{Q}})$ $\Leftrightarrow$ $\underline{x}_t = \underline{\underline{A}}\, \underline{x}_{t-1} + \underline{\underline{Q}}^{1/2} \underline{\varepsilon}_t$

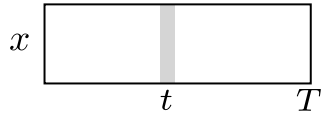
$\Rightarrow p(\underline{y}_t \mid \underline{x}_t) = G(\underline{y}_t; \underline{\underline{C}}\, \underline{x}_t, \underline{\underline{R}})$ $\quad \underline{y}_t \in \mathbb{R}^D$

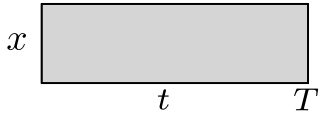
Today: Inference & Learning $p(x \mid y)$

Varieties of Inference

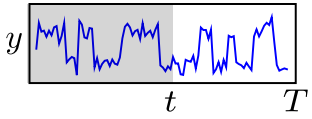
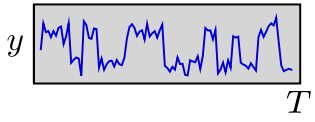
Distributional estimates

infer single state or sequence?





future data available?

	marginal	joint
 filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
 smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

Point estimates

most probable state @ t

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most probable sequence

$$x'_{1:T} = \arg\max_{x_{1:T}} p(x_{1:T} \mid y_{1:T})$$

Question: are these estimates the same $x^*_{1:T} \stackrel{?}{=} x'_{1:T}$ for

1. Linear Gaussian State Spate Models?
2. Discrete Hidden State HMMs?

① LGSSM

$$p(x_{1:T} \mid y_{1:T}) = G(x_{1:T};\ \mu_{1:T},\ \Sigma_{1:T,1:T})$$

$$x^*_{1:T} = \underset{x_{1:T}}{\operatorname{argmax}}\ p(x_{1:T} \mid y_{1:T}) = \mu_{1:T}$$

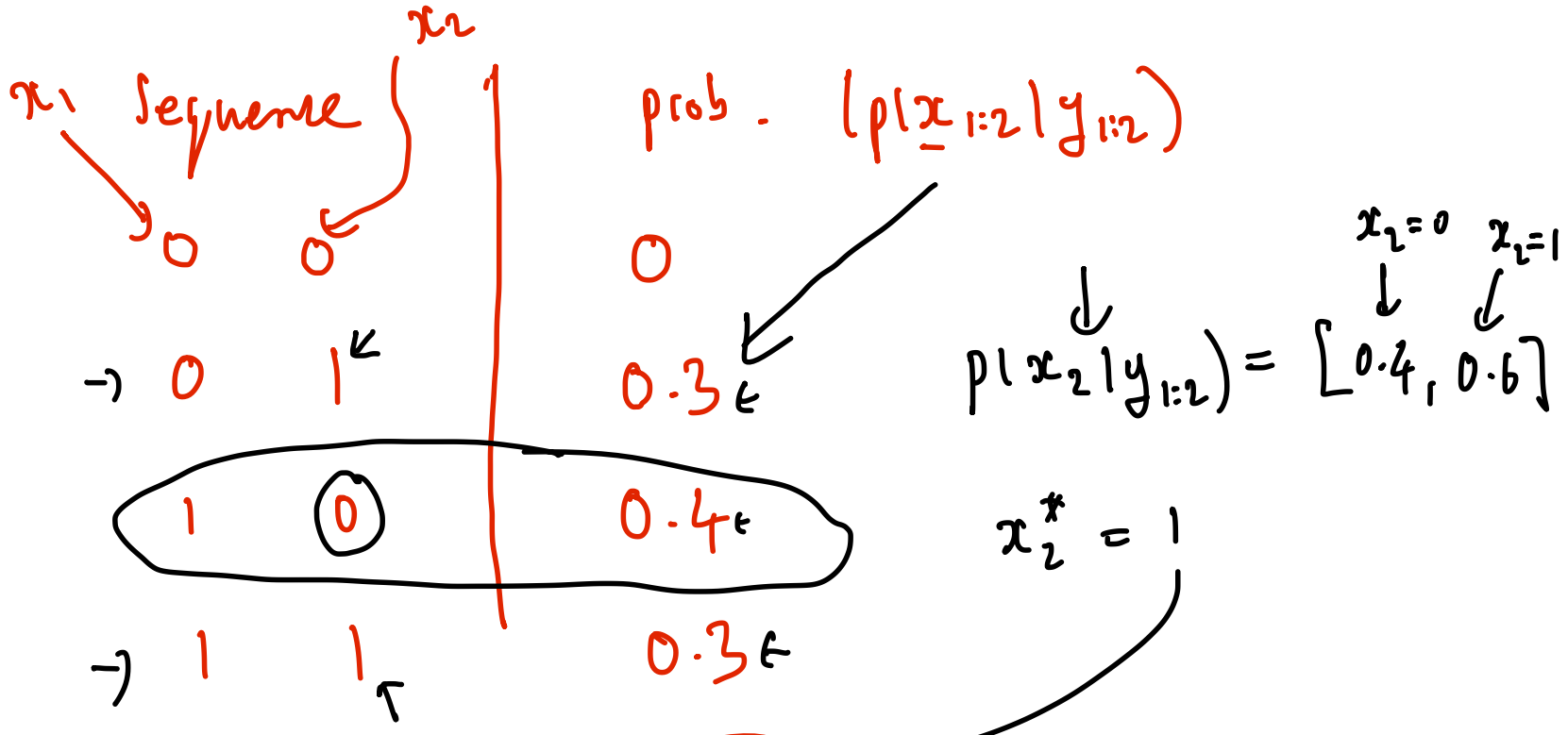
$$p(x_t \mid y_{1:T}) = G(x_t;\ \mu_t,\ \Sigma_{t,t})$$

$$x_t^* = \underset{x_t}{\operatorname{argmax}}\ p(x_t \mid y_{1:T}) = \mu_t$$

$$x^*_{1:T} = \mu_{1:T}$$

Discrete Hidden State HMMs

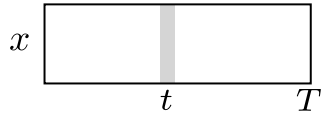
Counter example $T = 2, K = 2$

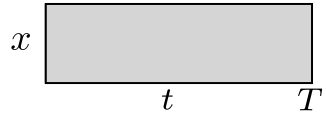


Varieties of Inference

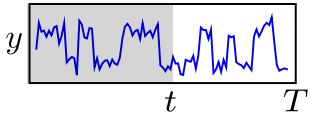
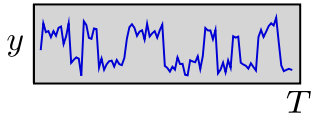
Distributional estimates

infer single state or sequence?





future data available?

	marginal	joint
 filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
 smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

Point estimates

most probable state @ t

$$x_t^* = \arg\max_{x_t} p(x_t \mid y_{1:T})$$

most probable sequence

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1. Linear Gaussian State Spate Models? $x^*_{1:T} = x'_{1:T}$ (Gaussian)

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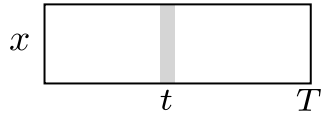
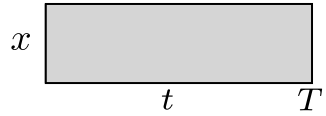
Varieties of Inference

Distributional estimates

infer single state or sequence?





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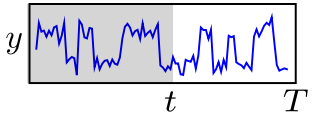
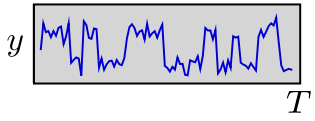
 filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
 smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

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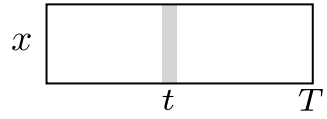
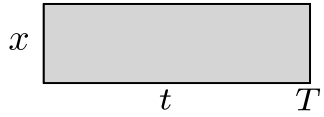
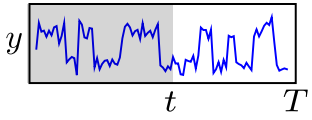


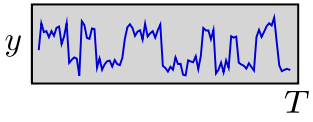


look at this next

future data available?

	marginal	joint
filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$





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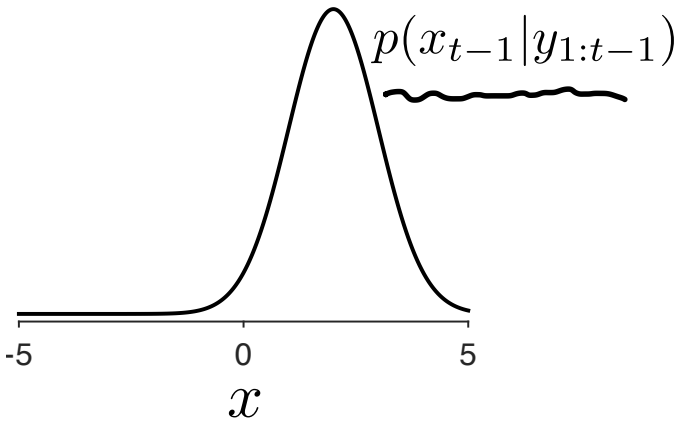
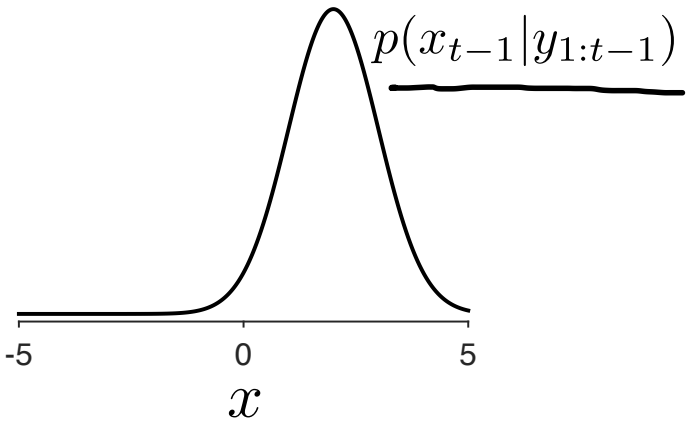
Inference: Kalman Filter

$p(x_t|y_{1:t}) \in$



$p(x_{t-1}|y_{1:t-1})$

Inference: Kalman Filter



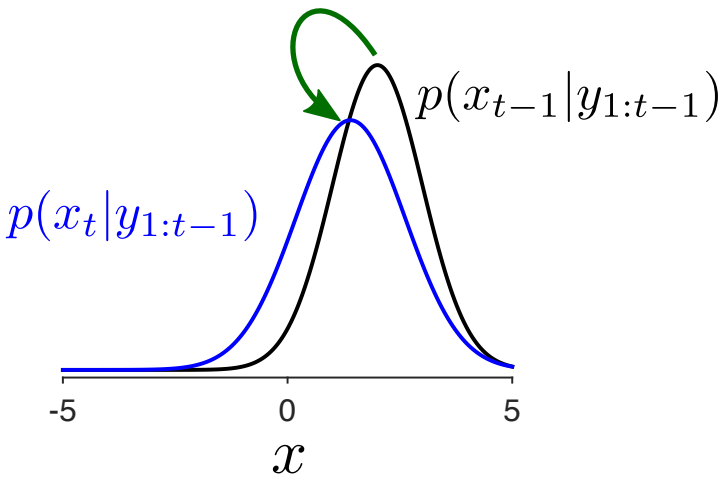
diffuse via dynamics

$$p(x_{t-1}|y_{1:t-1})$$

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

sum for discrete hidden state

Inference: Kalman Filter



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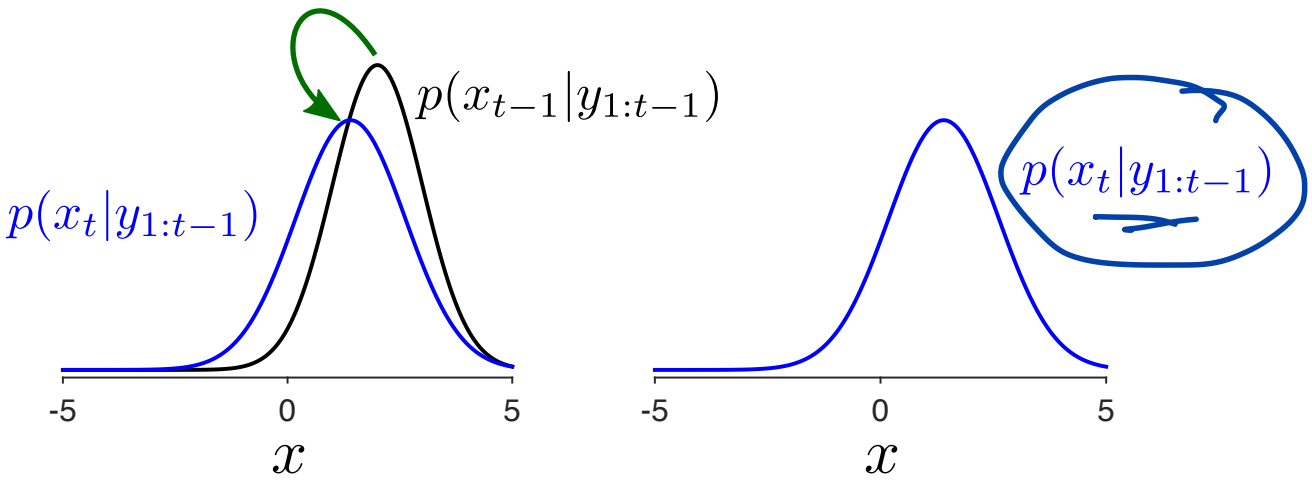

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combine with likelihood

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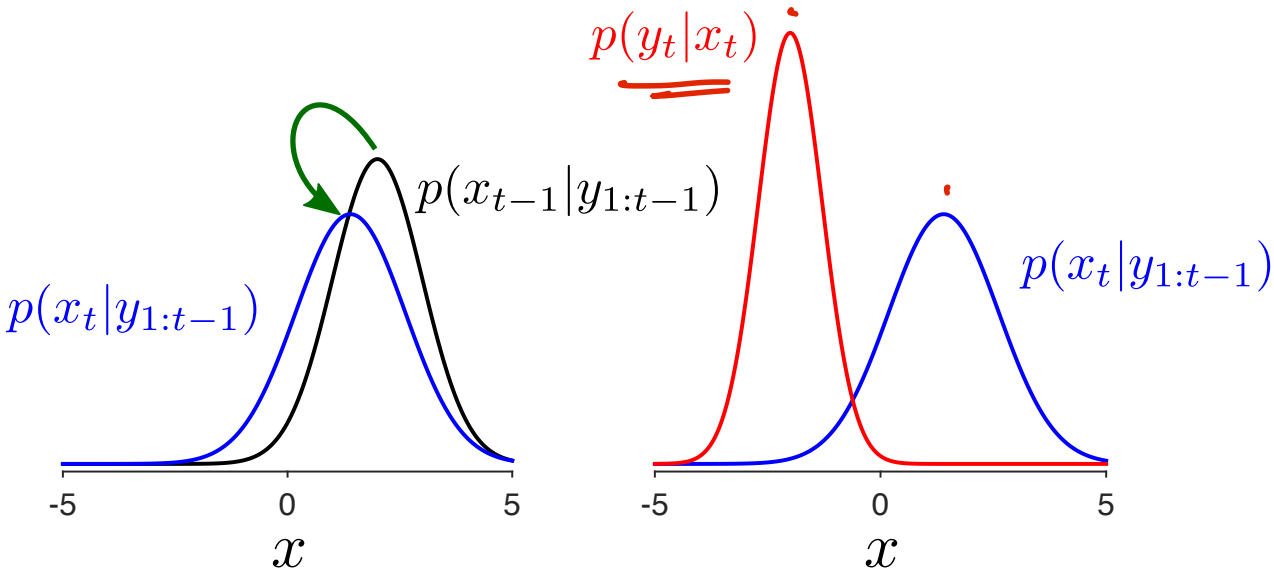
prior

likelihood

Bayes' Rule

Inference: Kalman Filter



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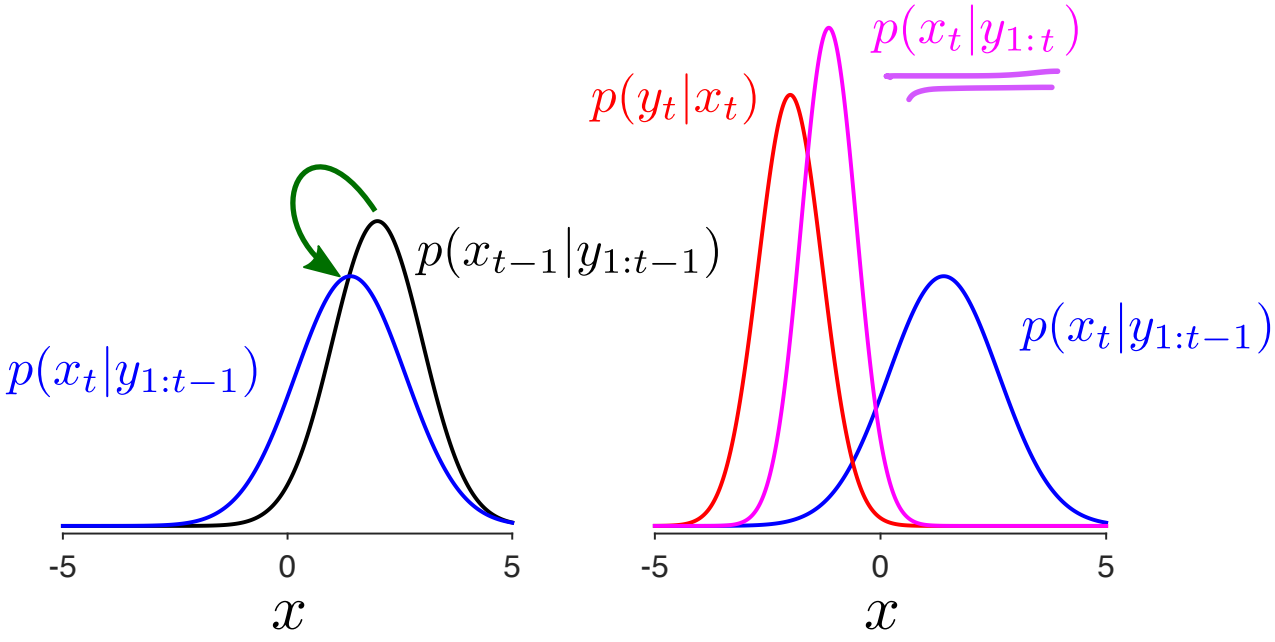

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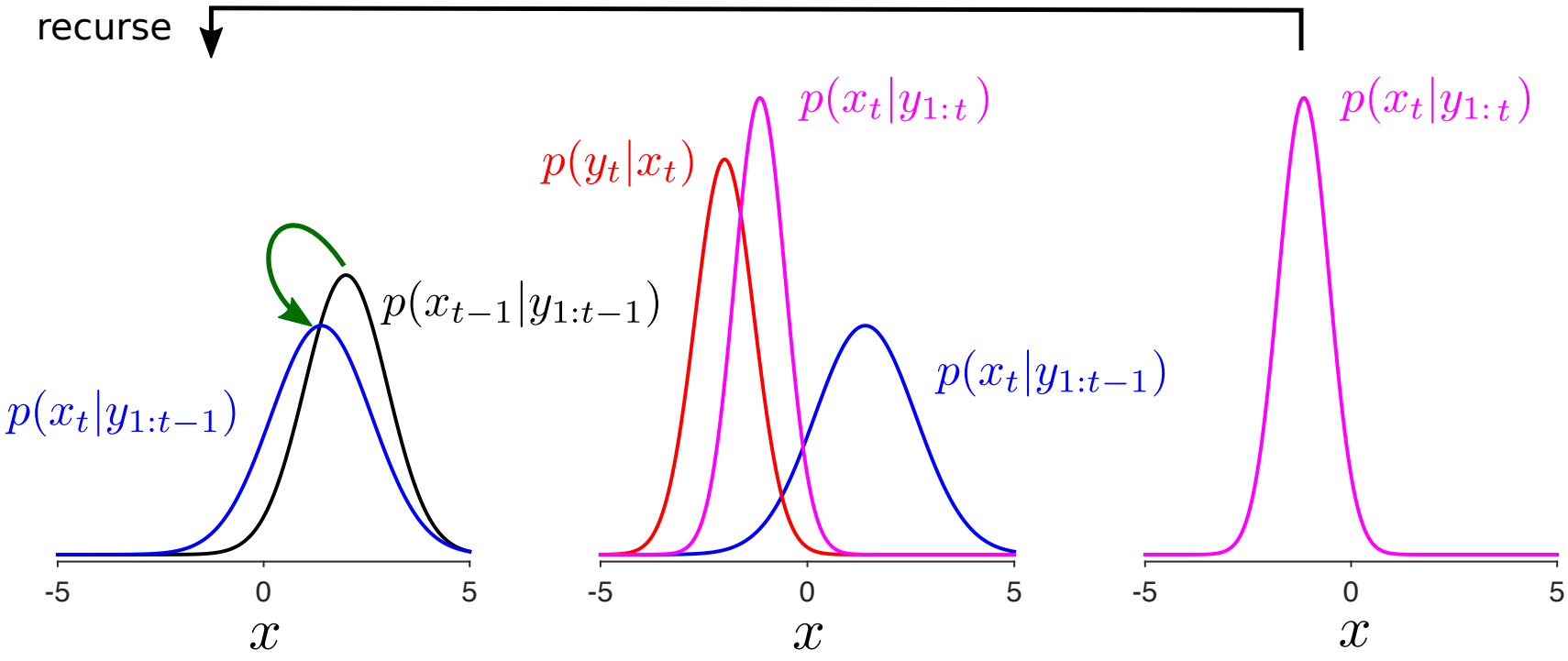

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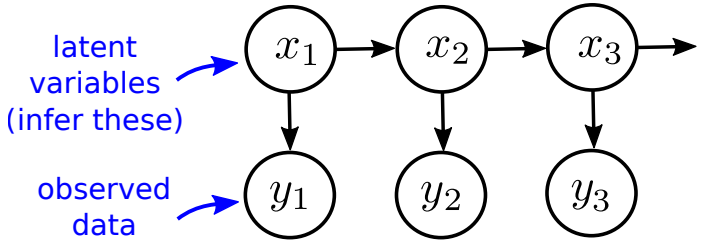
Bayes' Rule

prior likelihood

Inference: Derivation of General Filtering Equations

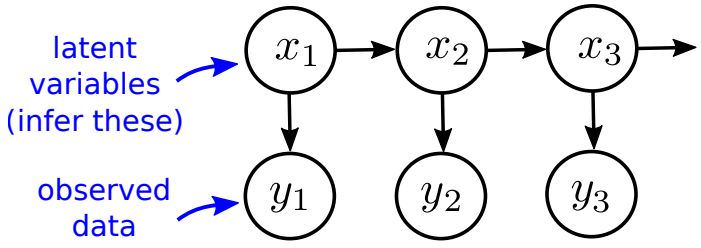
Model



$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

Inference: Derivation of General Filtering Equations

Model



+

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Rules of probability

product rule

$$p(A|B,C) = \frac{1}{p(B|C)}\, p(B|A,C)p(A|C)$$

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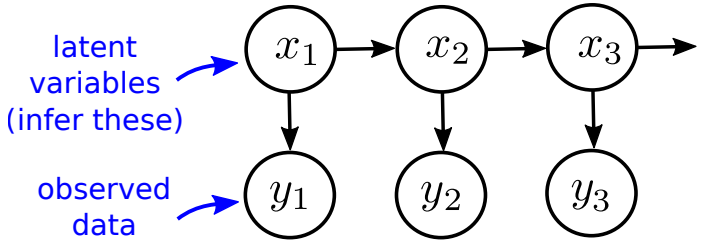
$$p(A|C) = \sum_B p(A,B|C)$$

Inference: Derivation of General Filtering Equations

Model



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Inference

$$= \ ?$$

Inference: Derivation of General Filtering Equations

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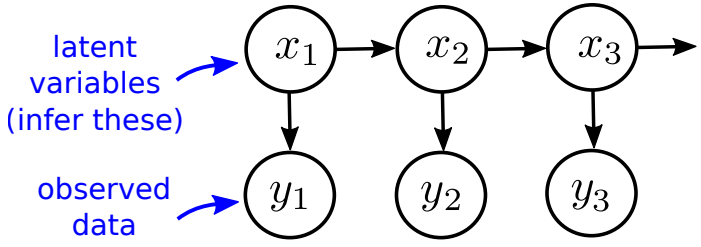

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Inference: Derivation of General Filtering Equations

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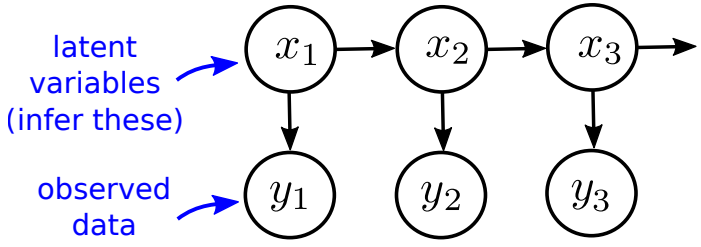

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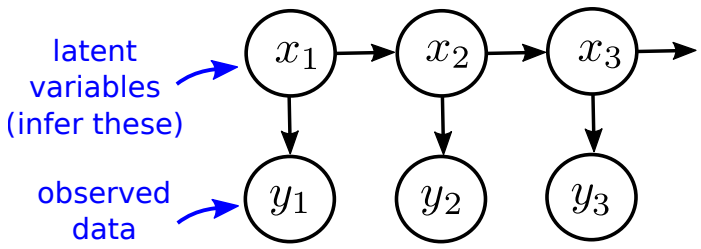
Inference: Derivation of General Filtering Equations

Model



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Inference: Derivation of General Filtering Equations

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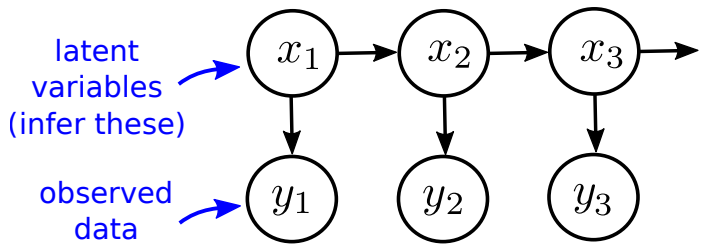

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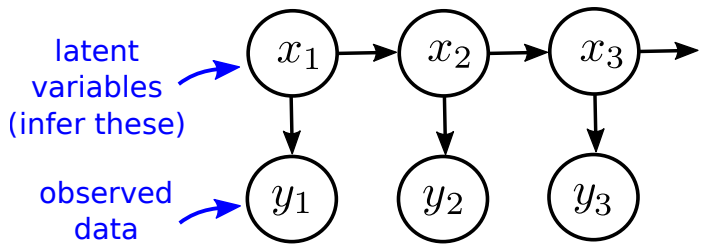
conditional independence from model

$y_t \perp y_{1:t-1}|x_t$

Inference: Derivation of General Filtering Equations

Model



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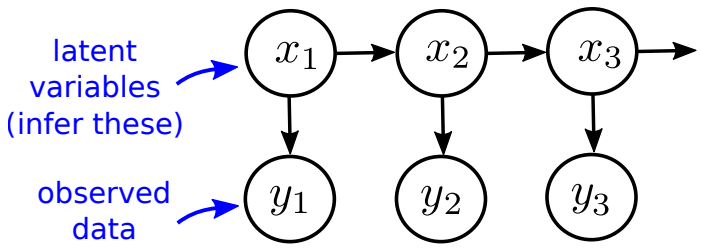
conditional independence from model $y_t \perp y_{1:t-1}|x_t$

constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

Inference: Derivation of General Filtering Equations

Model



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Inference: Derivation of General Filtering Equations

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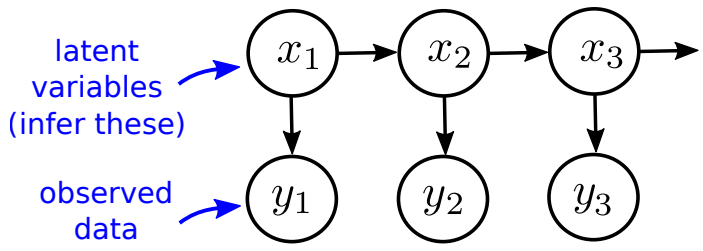

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product rule

$$p(A|B,C) = \frac{1}{p(B|C)}\, p(B|A,C)p(A|C)$$

sum rule

$$p(A|C) = \sum_B p(A,B|C)$$

Inference

$= ?$

$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

$$\begin{aligned}
p(x_t|y_{1:t}) &= p(x_t|y_t, y_{1:t-1}) \\
&= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t, y_{1:t-1})p(x_t|y_{1:t-1}) \\
&= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t)p(x_t|y_{1:t-1}) \\
&\propto p(y_t|x_t)p(x_t|y_{1:t-1})
\end{aligned}$$

product rule
$A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

conditional independence from model
$y_t \perp y_{1:t-1}|x_t$

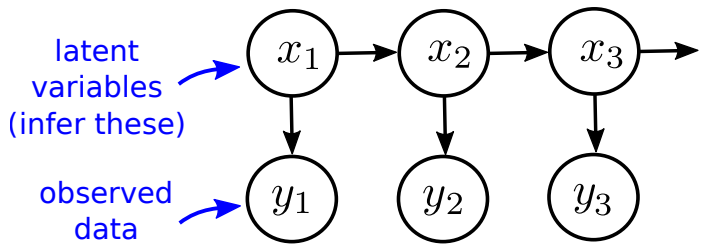
constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

$$p(x_t|y_{1:t-1}) = \int p(x_t, x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

sum rule
$A = x_t \;\; B = x_{t-1} \;\; C = y_{1:t-1}$

Inference: Derivation of General Filtering Equations

Model



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Rules of probability

product rule

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\end{aligned}$$

product rule $A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

conditional independence from model $y_t \perp y_{1:t-1}|x_t$

constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

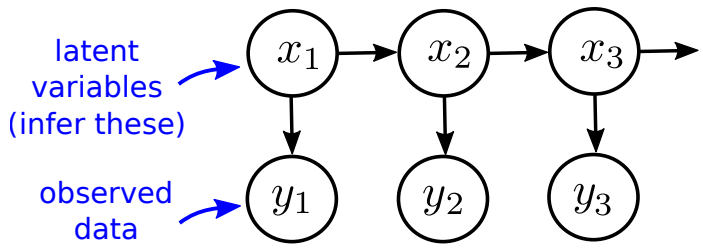
$$\begin{aligned}
p(x_t|y_{1:t-1}) &= \int p(x_t, x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1} \\
&= \int p(x_t|x_{t-1}, y_{1:t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}
\end{aligned}$$

sum rule $A = x_t \;\; B = x_{t-1} \;\; C = y_{1:t-1}$

product rule

Inference: Derivation of General Filtering Equations

Model



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Rules of probability

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Inference

= ?

$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

$p(y_t|y_{1:t-1})$

$$p(x_t|y_{1:t}) = p(x_t|y_t, y_{1:t-1})$$

$$= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t, y_{1:t-1})p(x_t|y_{1:t-1})$$

product rule $A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

$$= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t)p(x_t|y_{1:t-1})$$

conditional independence from model $y_t \perp y_{1:t-1}|x_t$

$$\propto p(y_t|x_t)p(x_t|y_{1:t-1})$$

constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

$$p(x_t|y_{1:t-1}) = \int p(x_t, x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

sum rule $A = x_t \;\; B = x_{t-1} \;\; C = y_{1:t-1}$

$$= \int p(x_t|x_{t-1}, y_{1:t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

product rule

$$= \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

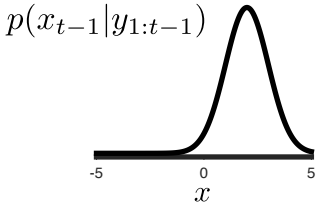
conditional independence from model

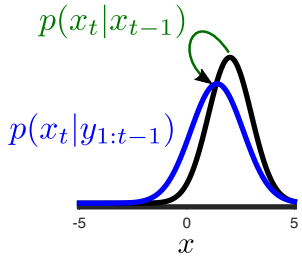
Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1})$$

diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

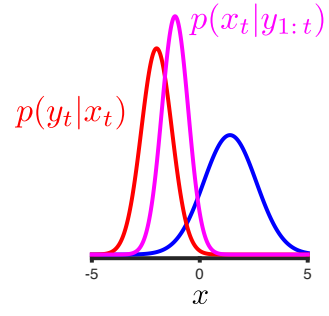




combine with likelihood

$$p(x_t|y_{1:t}) \propto p(x_t|y_{1:t-1})p(y_t|x_t)$$

prior likelihood

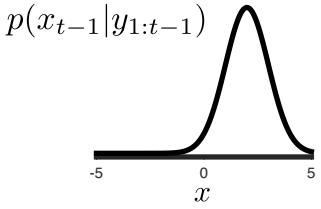


Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

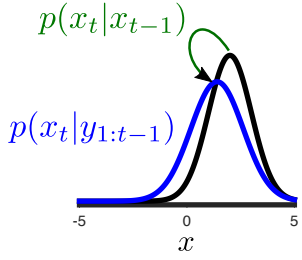
most recent data used in prediction

variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$

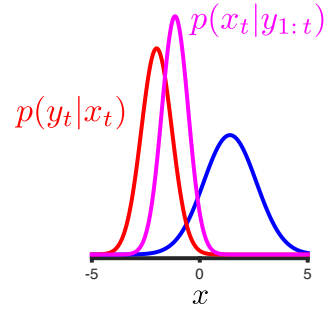


combine with likelihood

$$p(x_t|y_{1:t}) \propto p(x_t|y_{1:t-1}) p(y_t|x_t)$$

prior

likelihood

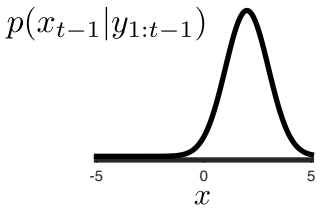


Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

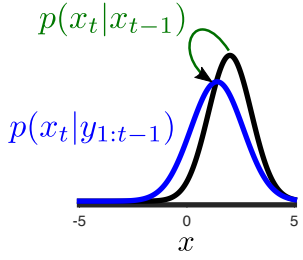
most recent data used in prediction

variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

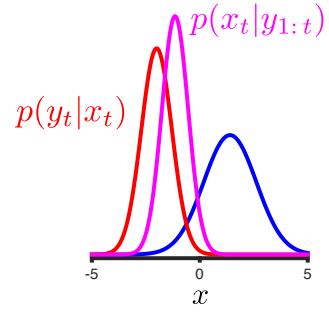


$$p(x_t|y_{1:t-1}) = \mathcal{G}(x_t; \mu_t^{t-1}, V_t^{t-1})$$

combine with likelihood

$$p(x_t|y_{1:t}) \propto p(x_t|y_{1:t-1})p(y_t|x_t)$$

prior likelihood



Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

most recent data used in prediction

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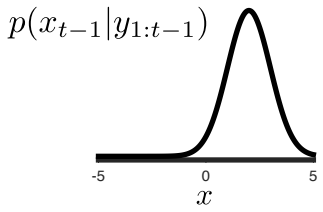


diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$

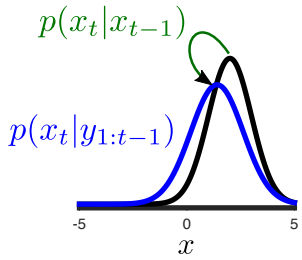
diffuses toward 0

$$\Rightarrow \quad p(x_t|y_{1:t-1}) = \mathcal{G}(x_t; \mu_t^{t-1}, V_t^{t-1})$$

$$\mu_t^{t-1} = A\mu_{t-1}^{t-1}$$

$$V_t^{t-1} = AV_{t-1}^{t-1}A^\top + Q$$

variance inflates



combine with likelihood

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prior

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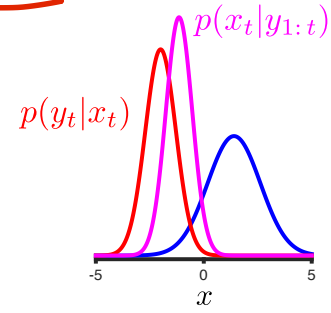


$$p(x_t|x_{t-1}) = G(x_t; \underset{A}{a}x_t, \underset{Q}{q})$$

$$p(x_{t-1}) = G(x_{t-1}; \underset{=}{\mu_{t-1}^{t-1}}, V_{t-1}^{t-1})$$

$$E(x_t) = E(a x_t + \varepsilon_t q^{1/2}) = a\mu_{t-1}^{t-1} + 0$$

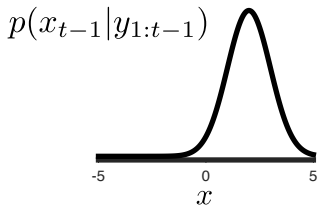
$$E(x_t^2) = E[(a x_t + \varepsilon_t q^{1/2})^2]$$

Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

most recent data used in prediction

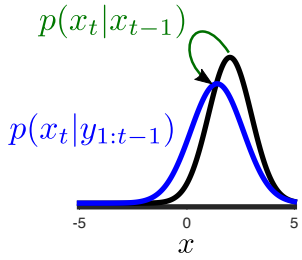
variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$

diffuses toward 0



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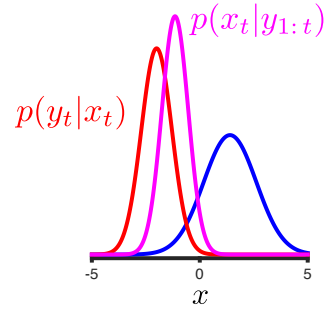

$$p(x_t|y_{1:t}) = \mathcal{G}(x_t; \mu_t^t, V_t^t)$$

Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

most recent data used in prediction

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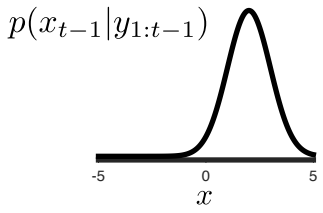

diffuse via dynamics

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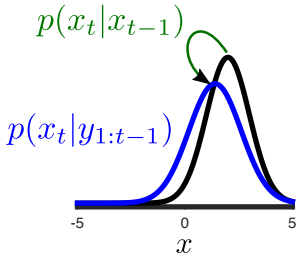

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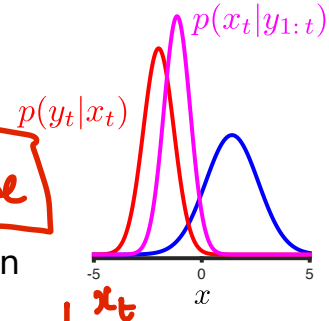


prediction correction

$$p(x_t|y_{1:t}) = \mathcal{G}(x_t; \mu_t^t, V_t^t)$$

$$\mu_t^t = \mu_t^{t-1} + K_t(y_t - C\mu_t^{t-1})$$

$$V_t^t = V_t^{t-1} - K_t C V_t^{t-1}$$

Kalman gain

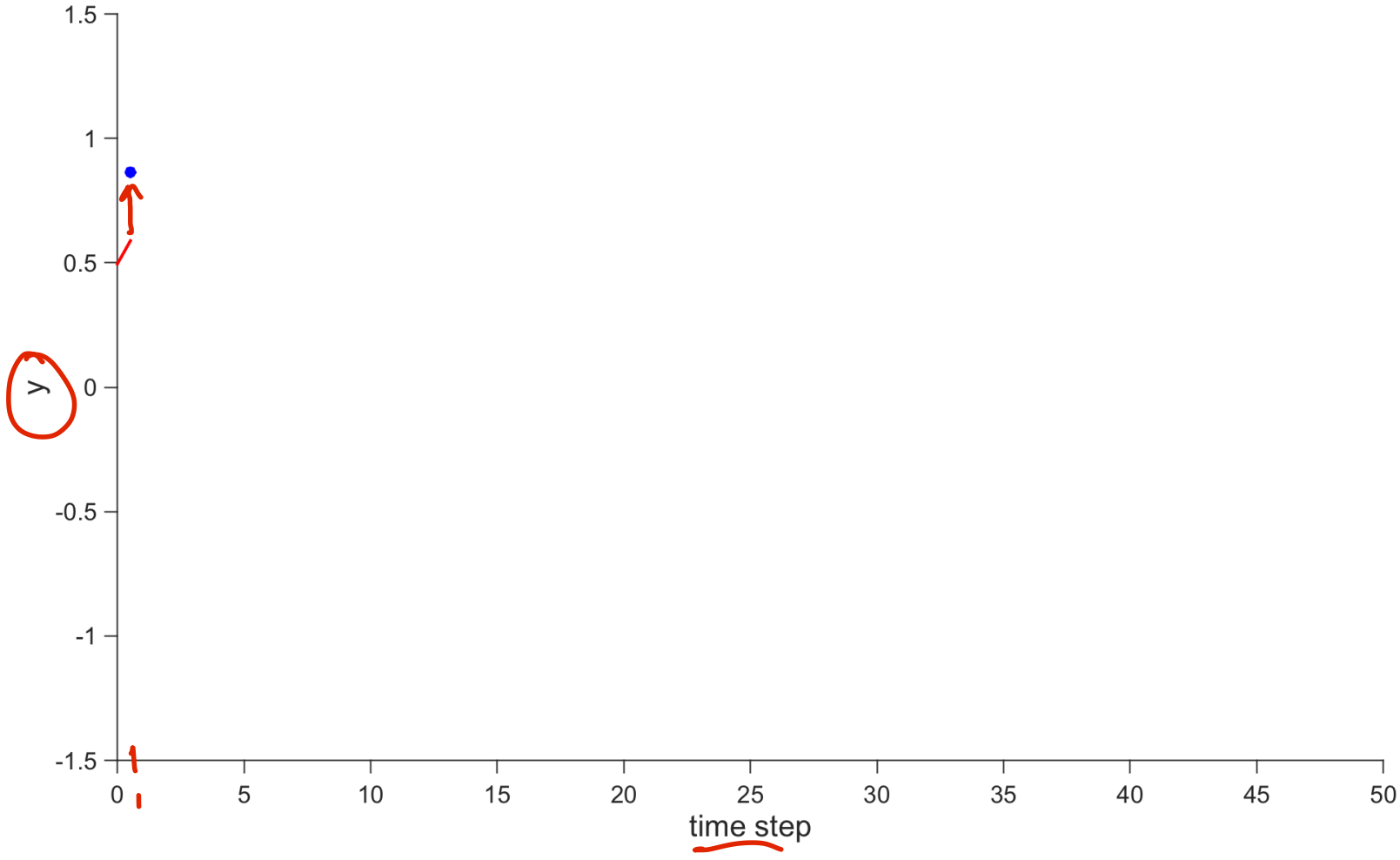
$$K_t = V_t^{t-1} C^\top (CV_t^{t-1} C^\top + R)^{-1}$$

Kalman Filter Demo

- data: $y_t = \sin(\omega t) + \sigma_y \epsilon_t$ where $\sigma_y^2 = 0.1$
- model: $x_t = \lambda x_{t-1} + \sigma\eta$ and $y_t = x_t + \sigma_y \eta'_t$
  where $\lambda = 0.99$ and $\sigma^2 = 1 - \lambda^2$
- demo shows how the Kalman filter processes the data to form estimates of the hidden state at each time point $p(x_t | y_{1:t})$

Kalman Filter Demo

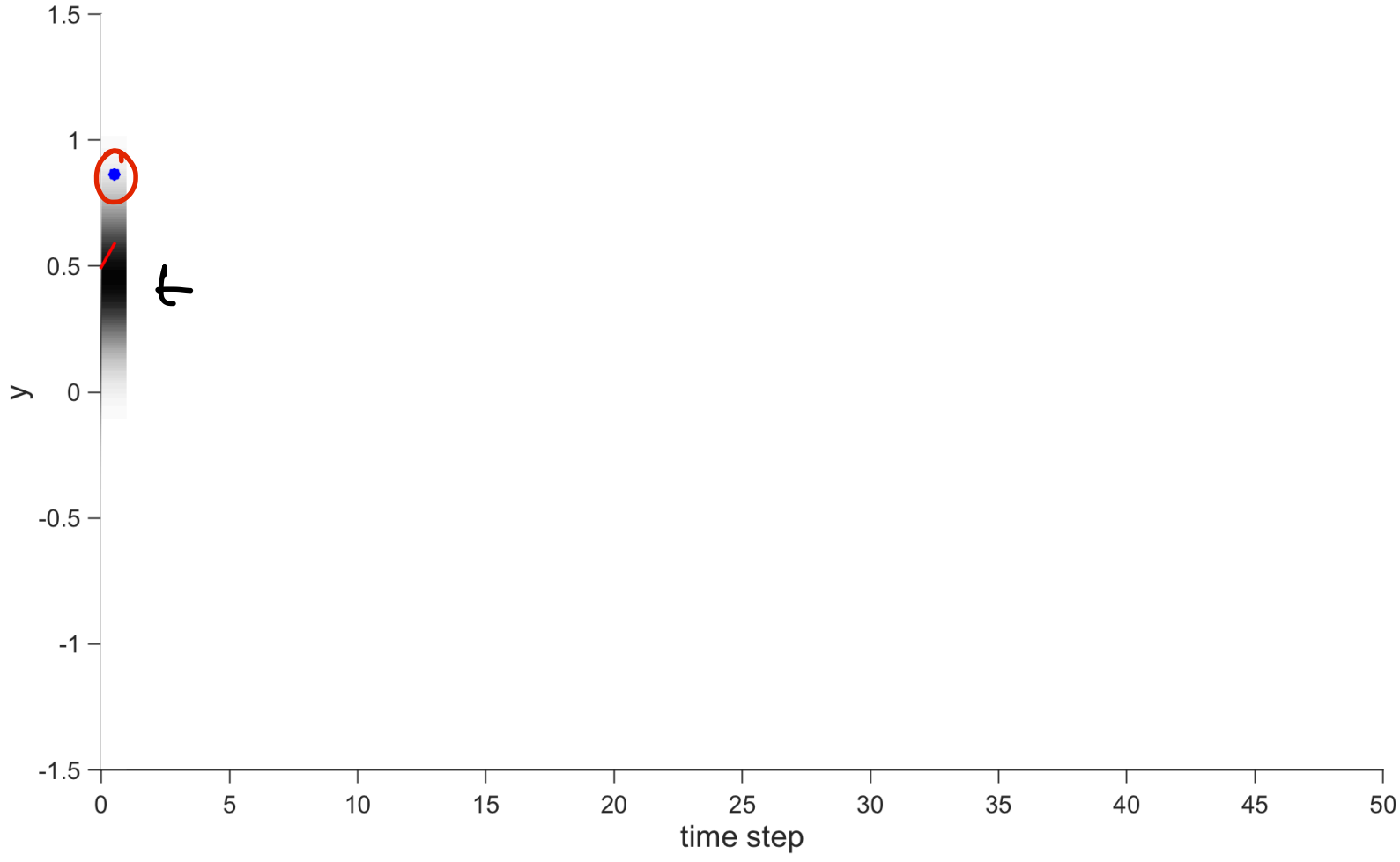
observed noisy data y_t, ground truth sinusoid



observe first data point y_1

Kalman Filter Demo

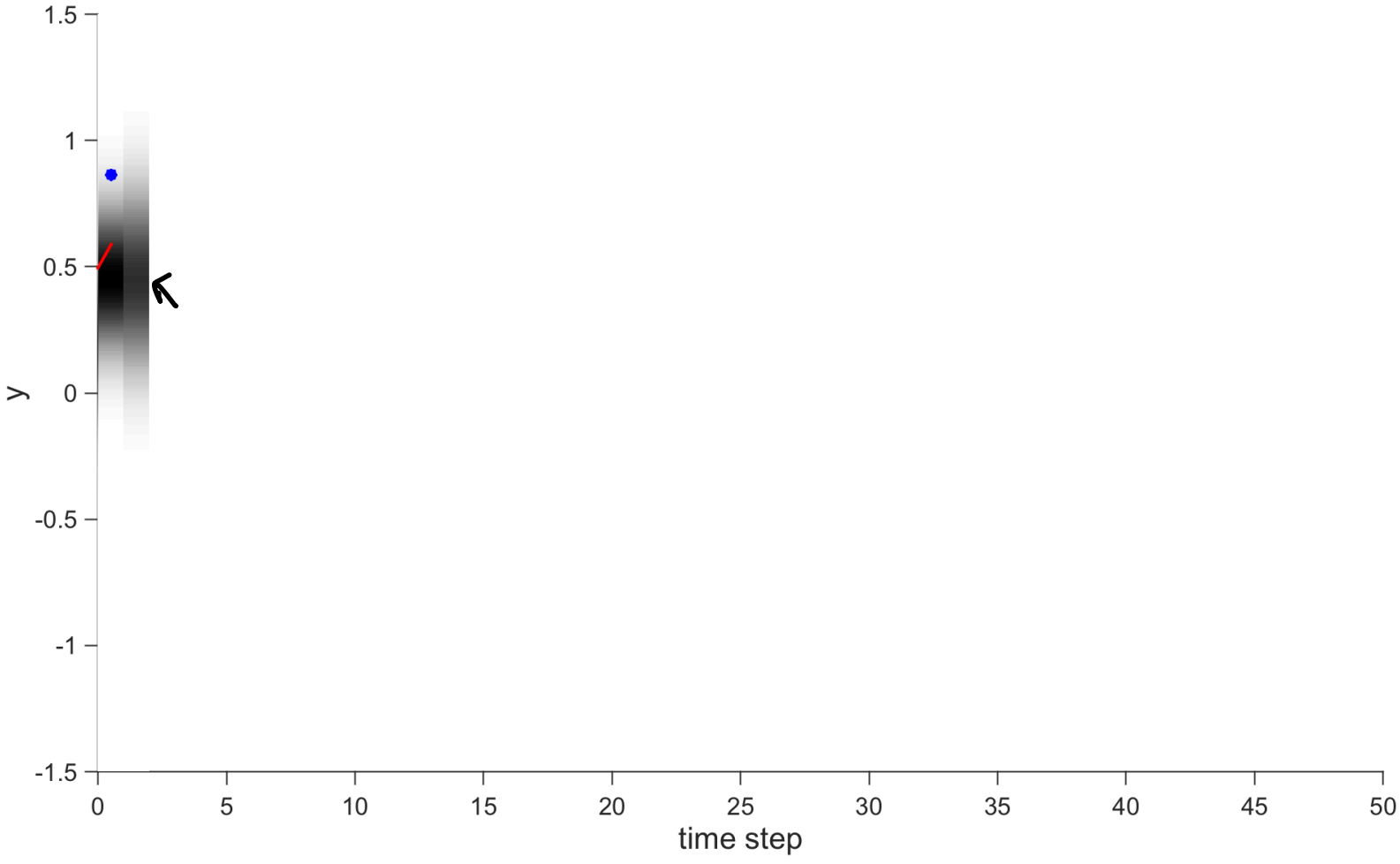
observed noisy data y_t, ground truth sinusoid



posterior over first latent variable $p(x_1|y_1)$

Kalman Filter Demo

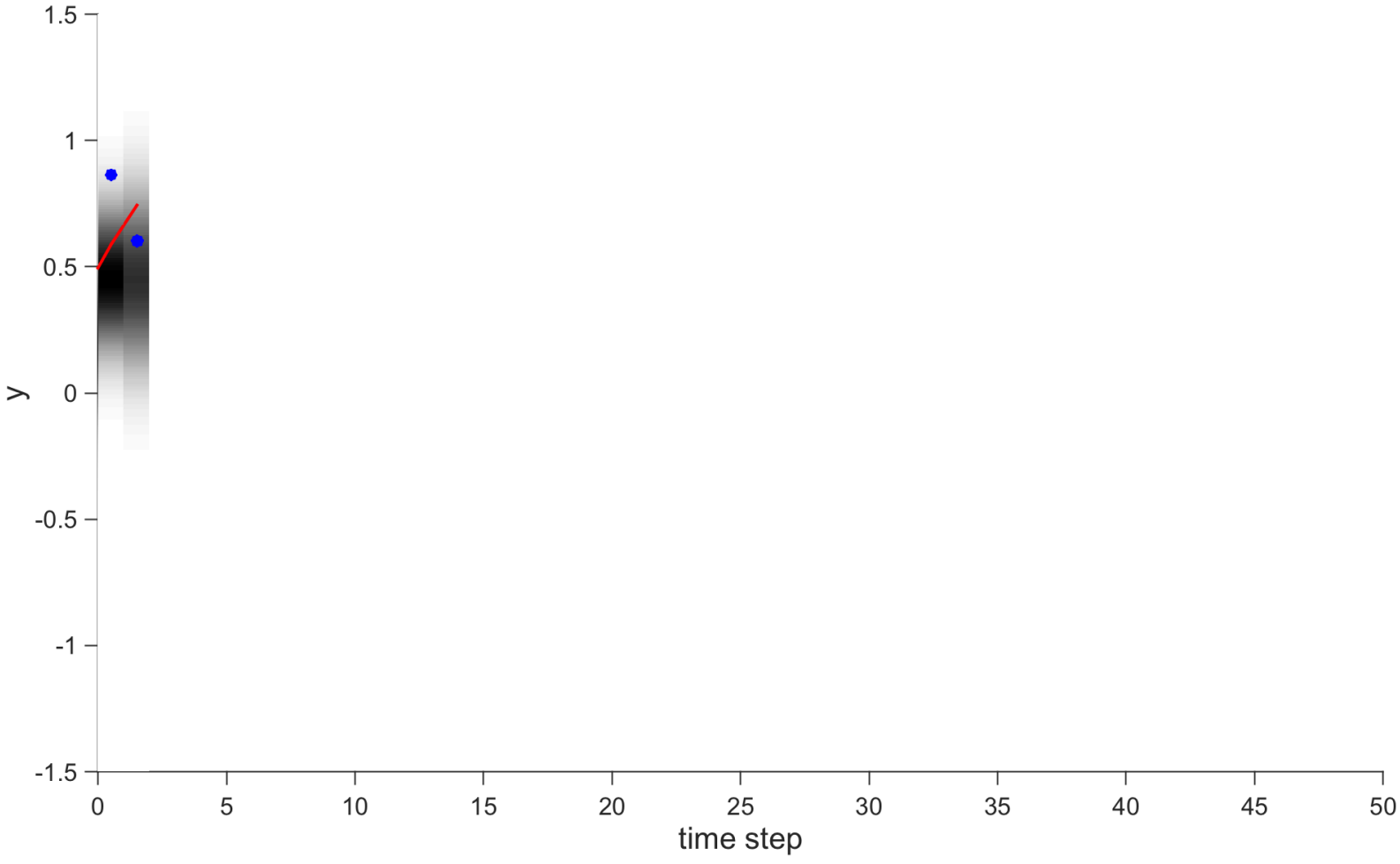
observed noisy data y_t, ground truth sinusoid



prediction for second latent variable $p(x_2|y_1)$

Kalman Filter Demo

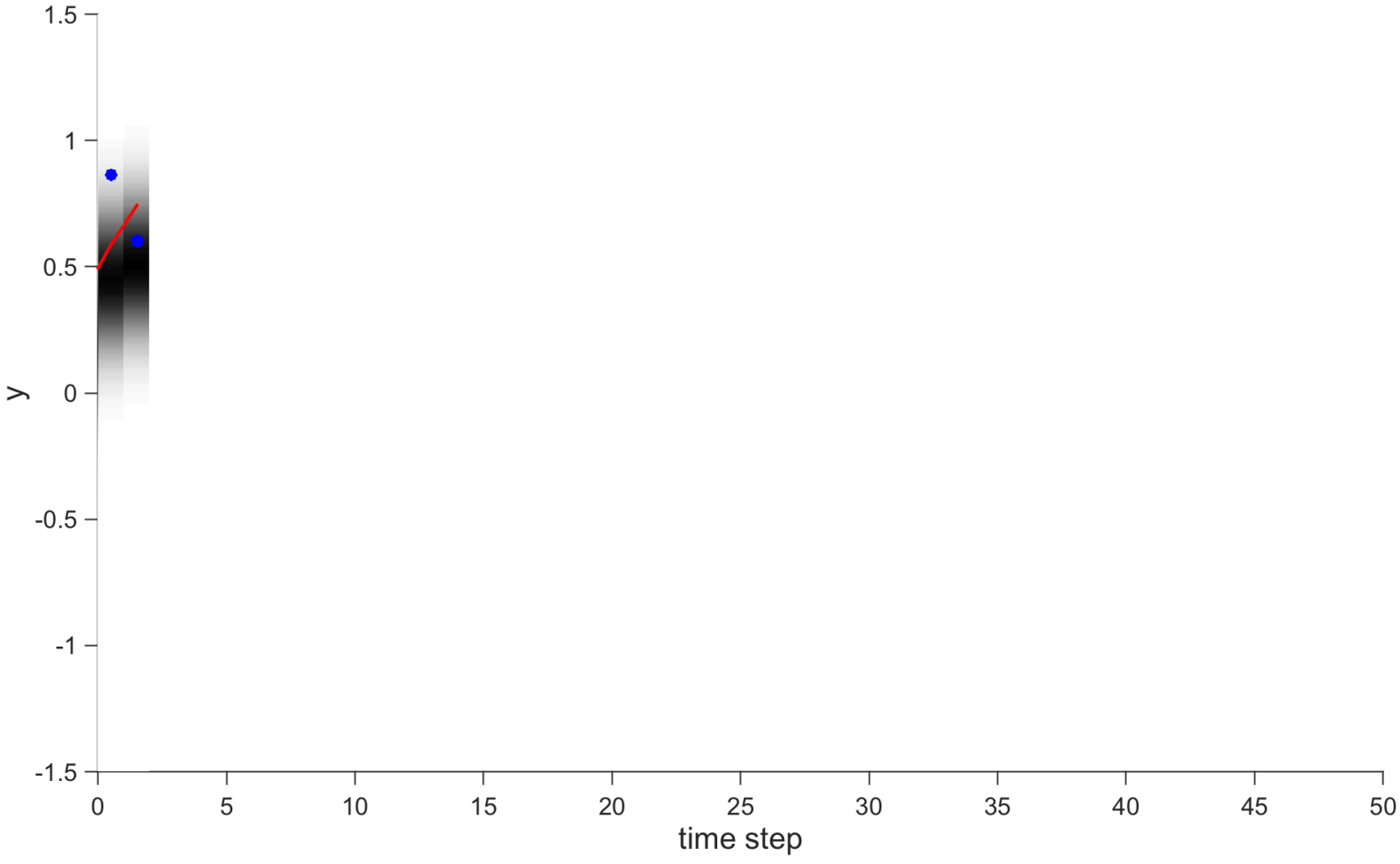
observed noisy data y_t, ground truth sinusoid



observe next data point y_2

Kalman Filter Demo

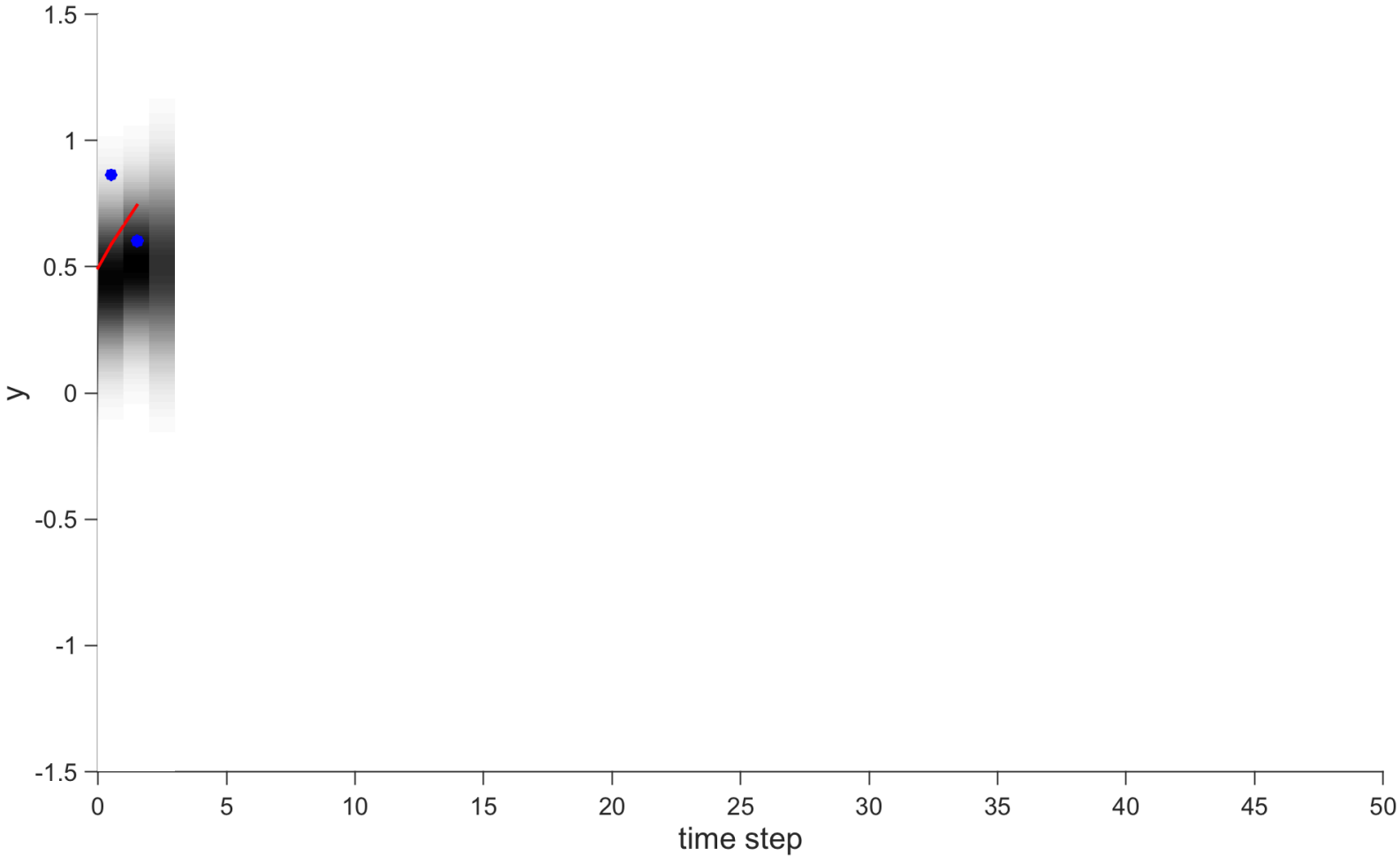
observed noisy data y_t, ground truth sinusoid



form posterior over second latent variable $p(x_2|y_1, y_2)$

Kalman Filter Demo

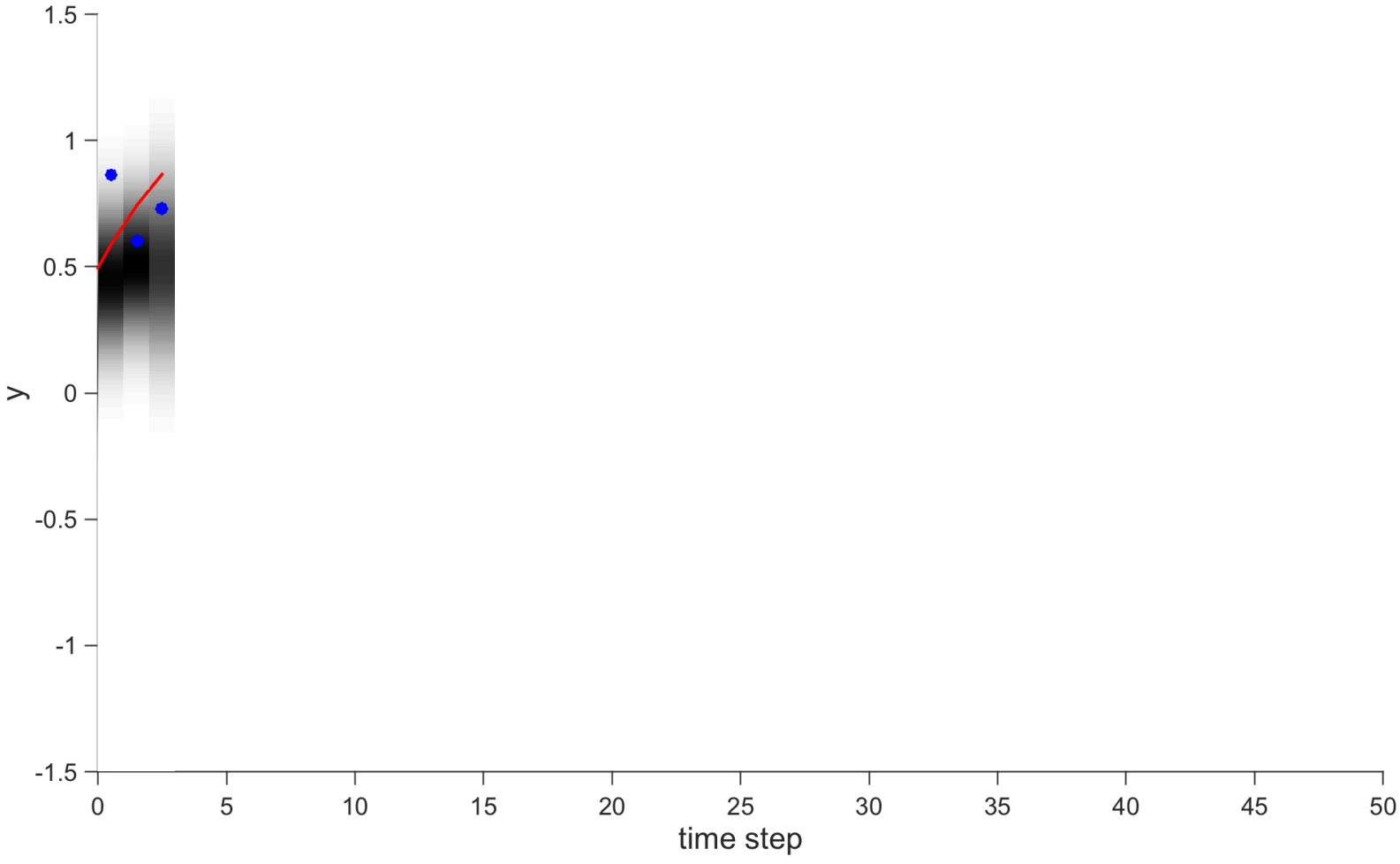
observed noisy data y_t, ground truth sinusoid



prediction for third latent variable $p(x_3|y_1, y_2)$

Kalman Filter Demo

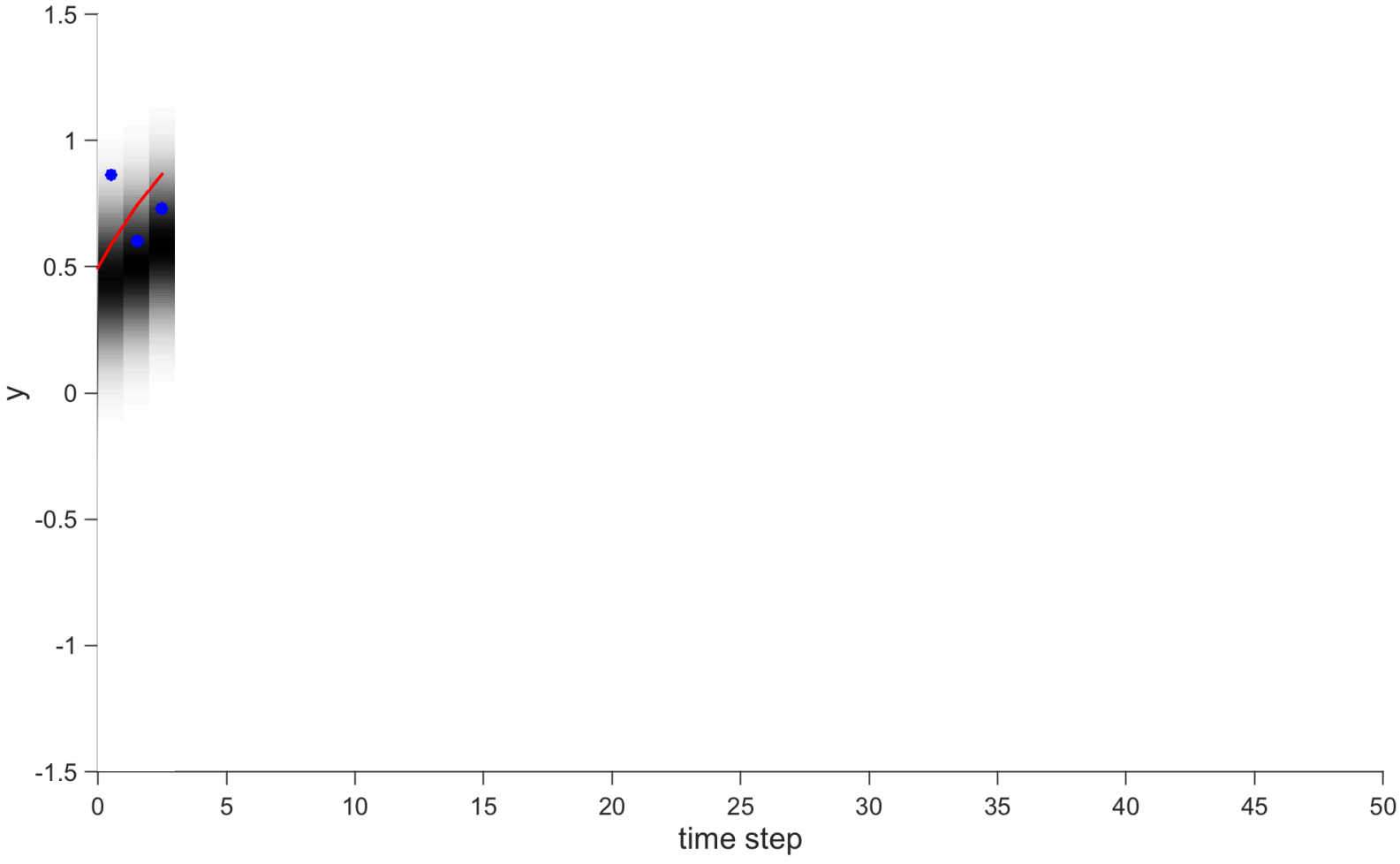
observed noisy data y_t, ground truth sinusoid



observe next data point y_3

Kalman Filter Demo

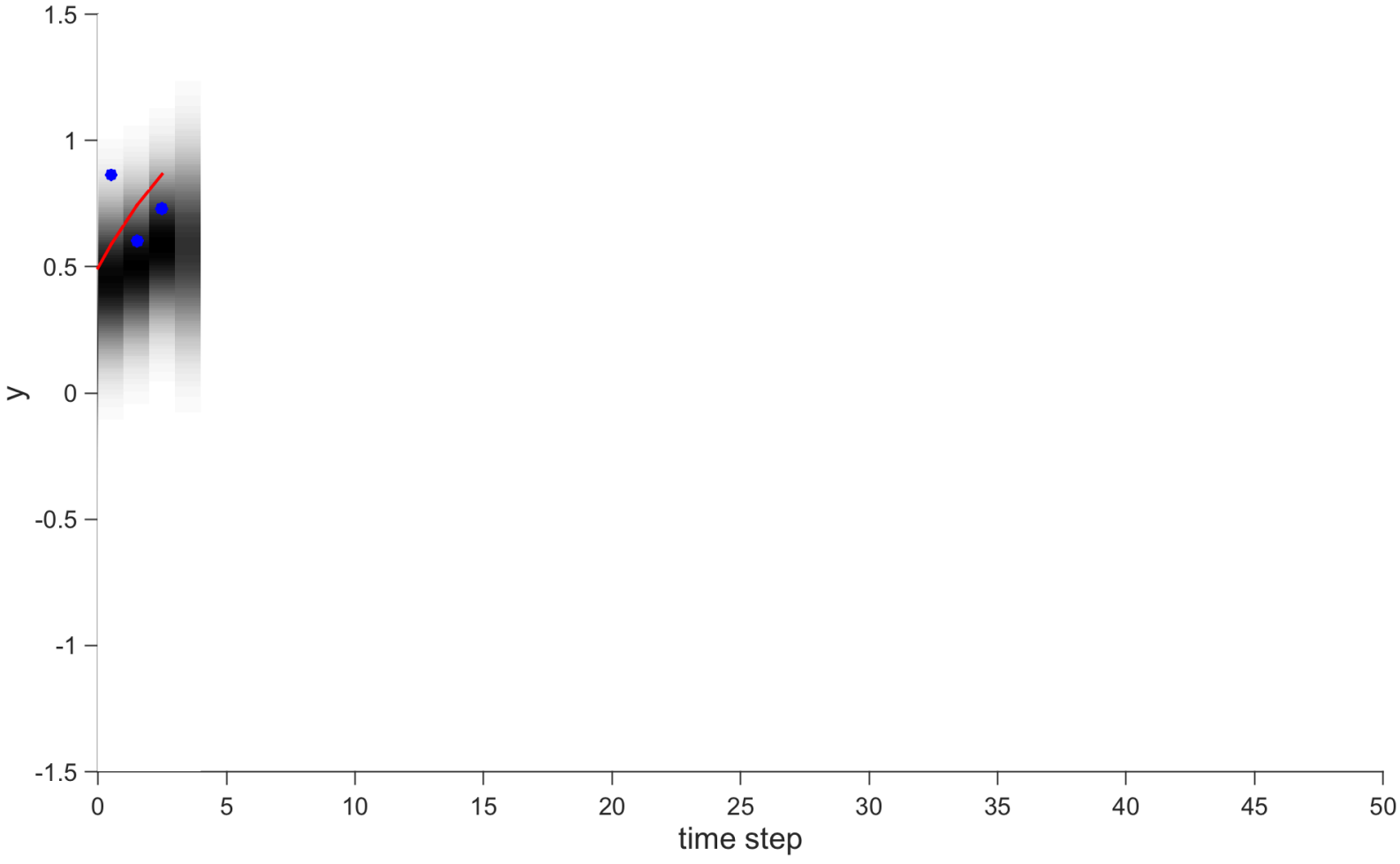
observed noisy data y_t, ground truth sinusoid



form posterior over third latent variable $p(x_3|y_1, y_2, y_3)$

Kalman Filter Demo

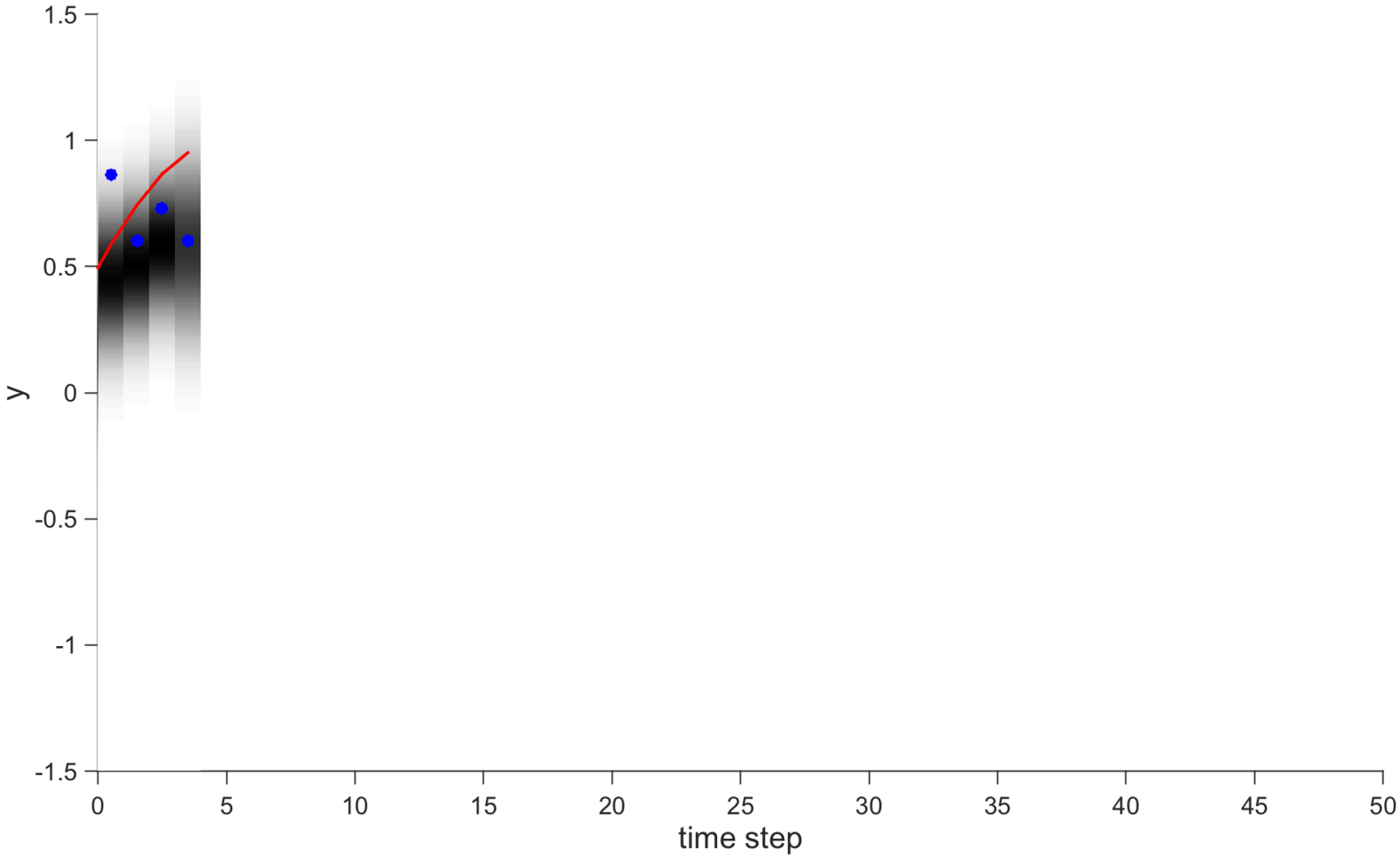
observed noisy data y_t, ground truth sinusoid



prediction for fourth latent variable $p(x_4|y_{1:3})$

Kalman Filter Demo

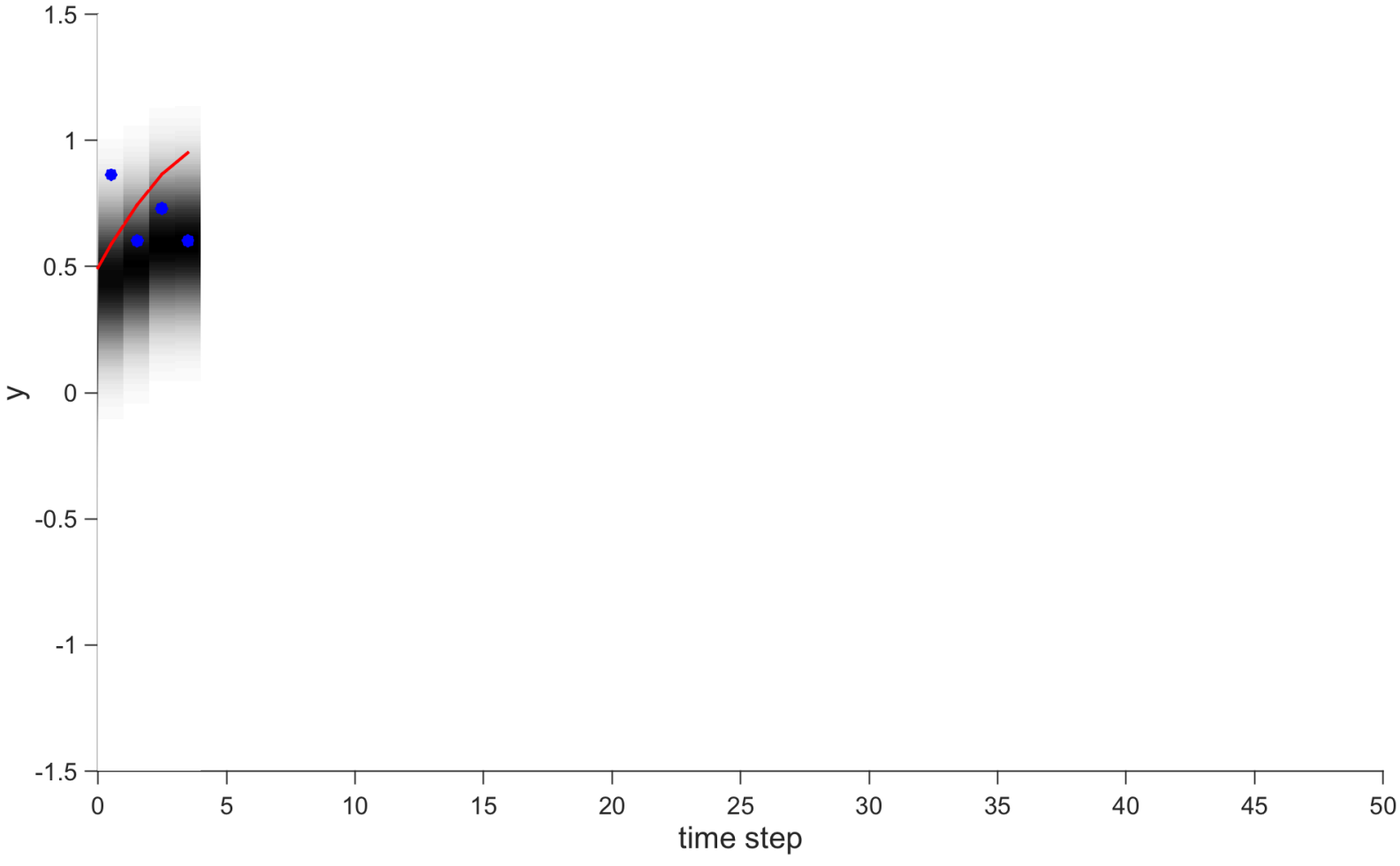
observed noisy data y_t, ground truth sinusoid



observe next data point y_4

Kalman Filter Demo

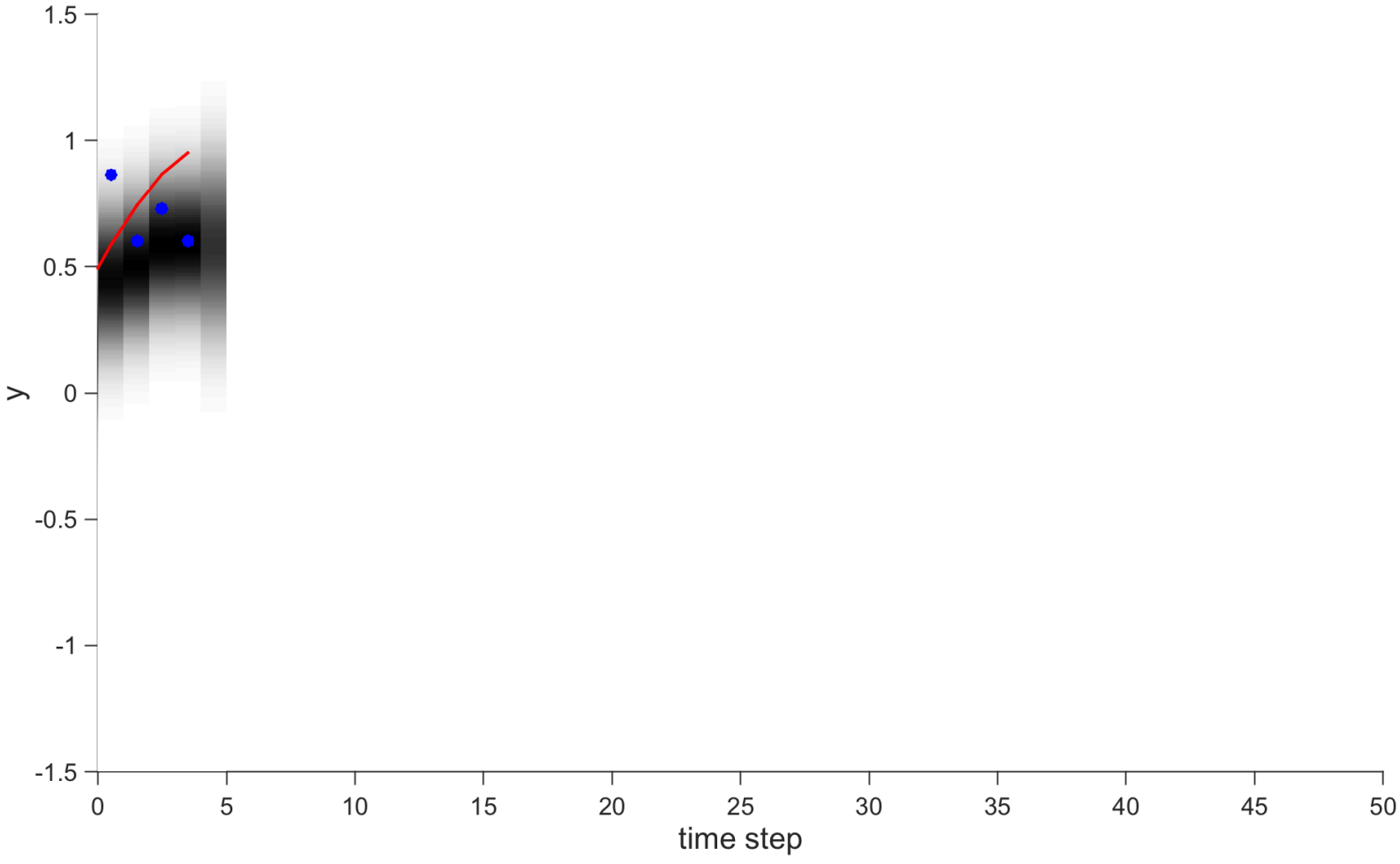
observed noisy data y_t, ground truth sinusoid



form posterior over fourth latent variable $p(x_4|y_{1:4})$

Kalman Filter Demo

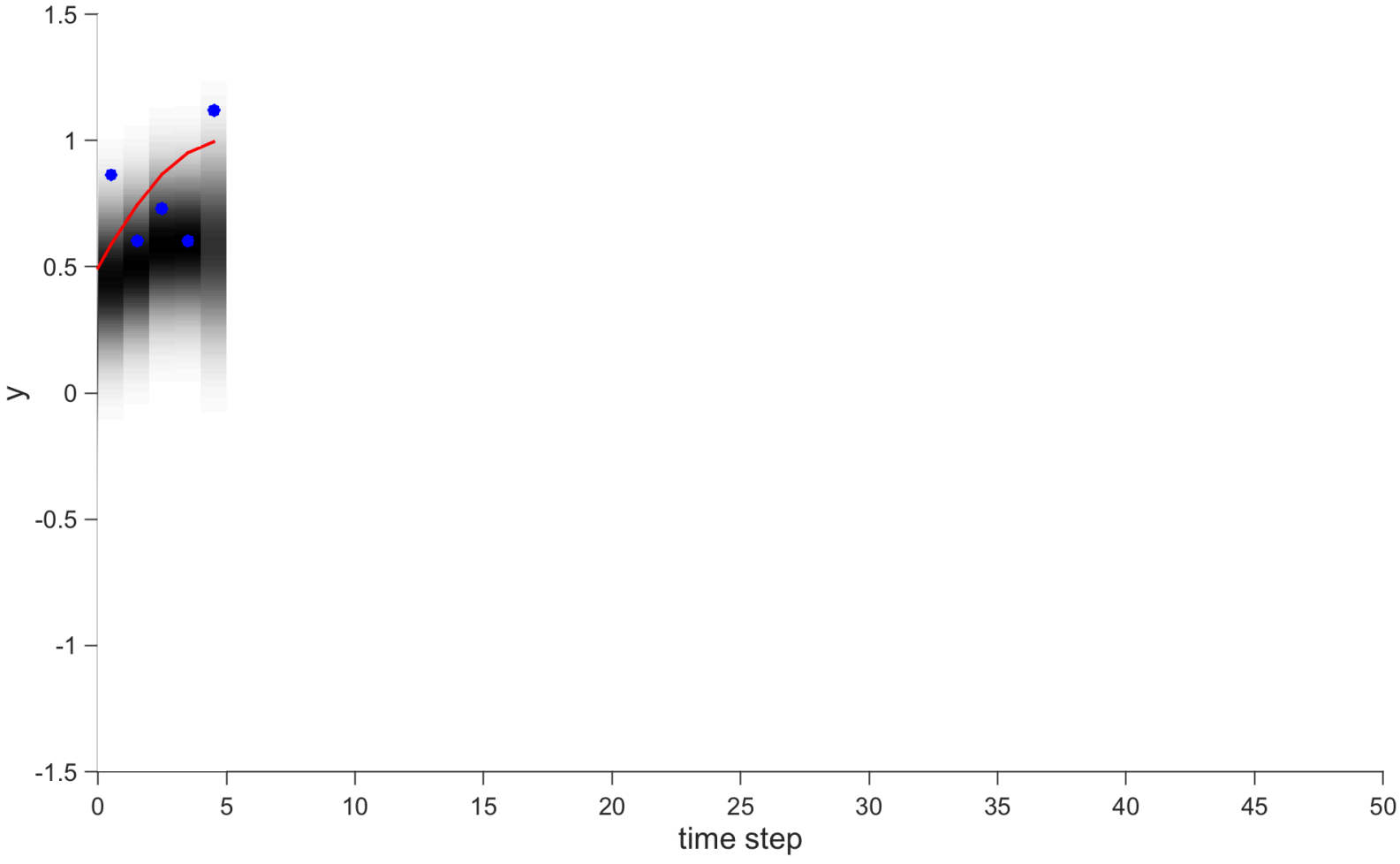
observed noisy data y_t, ground truth sinusoid



prediction for fifth latent variable $p(x_5|y_{1:4})$

Kalman Filter Demo

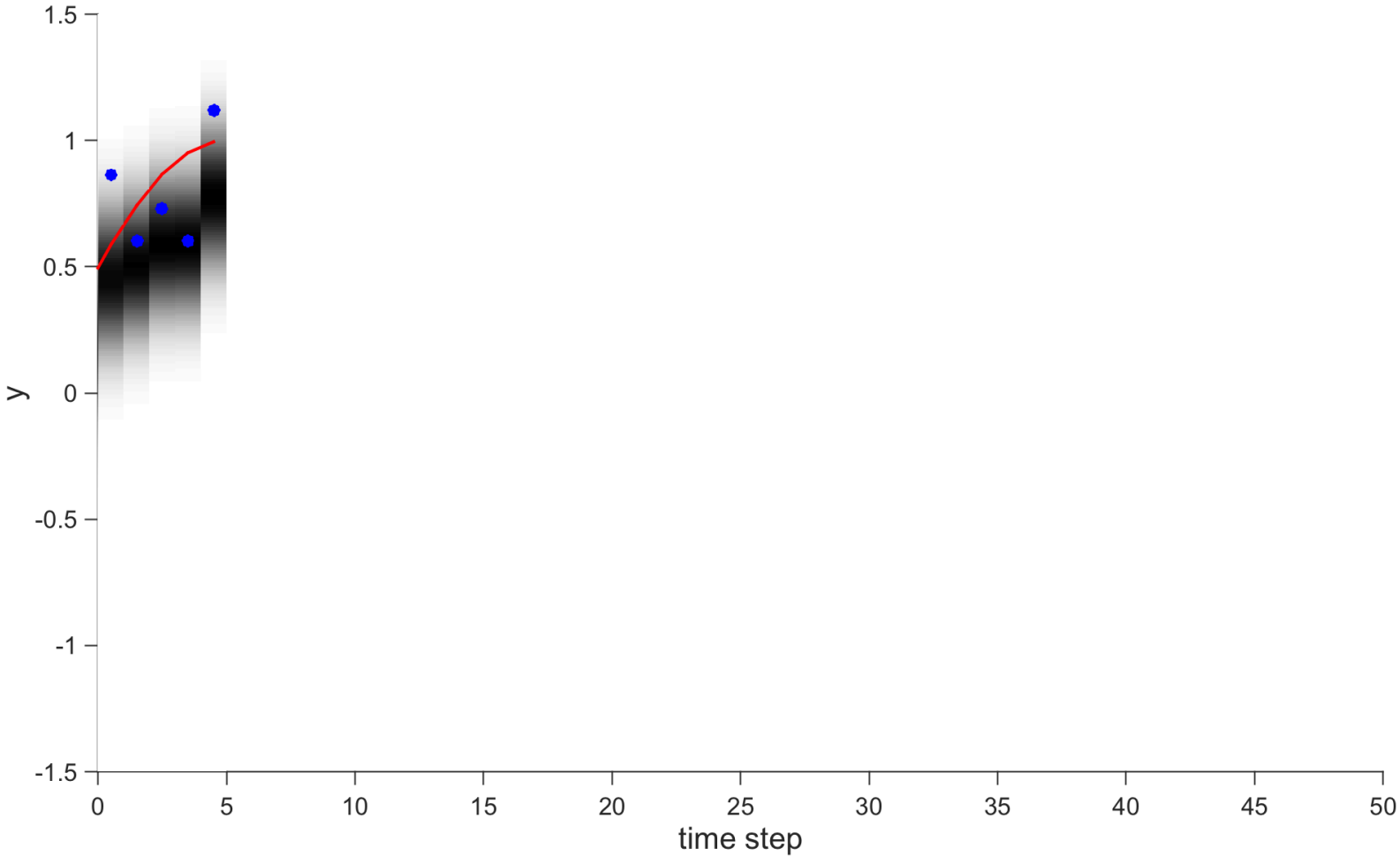
observed noisy data y_t, ground truth sinusoid



observe next data point y_5

Kalman Filter Demo

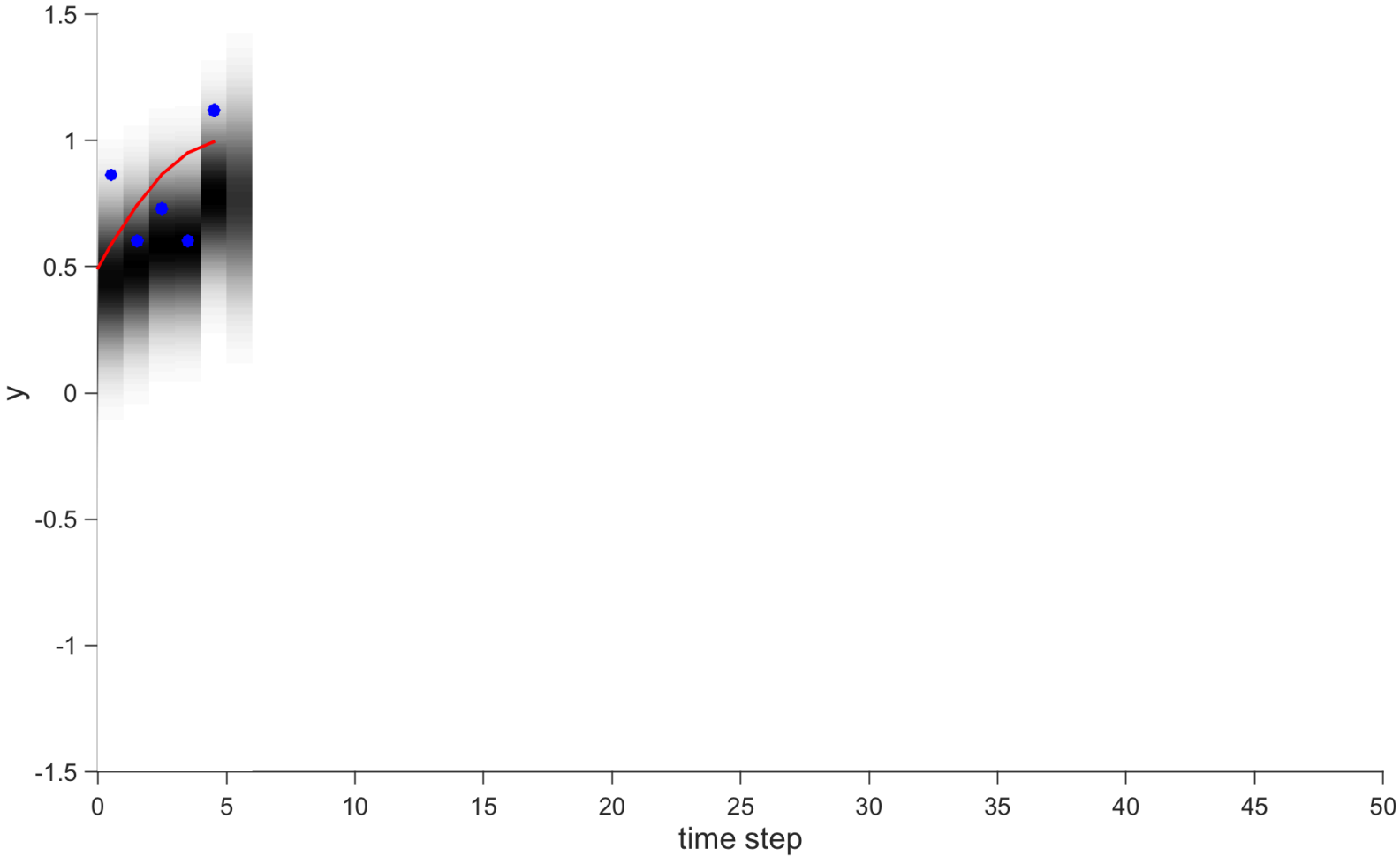
observed noisy data y_t, ground truth sinusoid



form posterior over fifth latent variable $p(x_5 | y_{1:5})$

Kalman Filter Demo

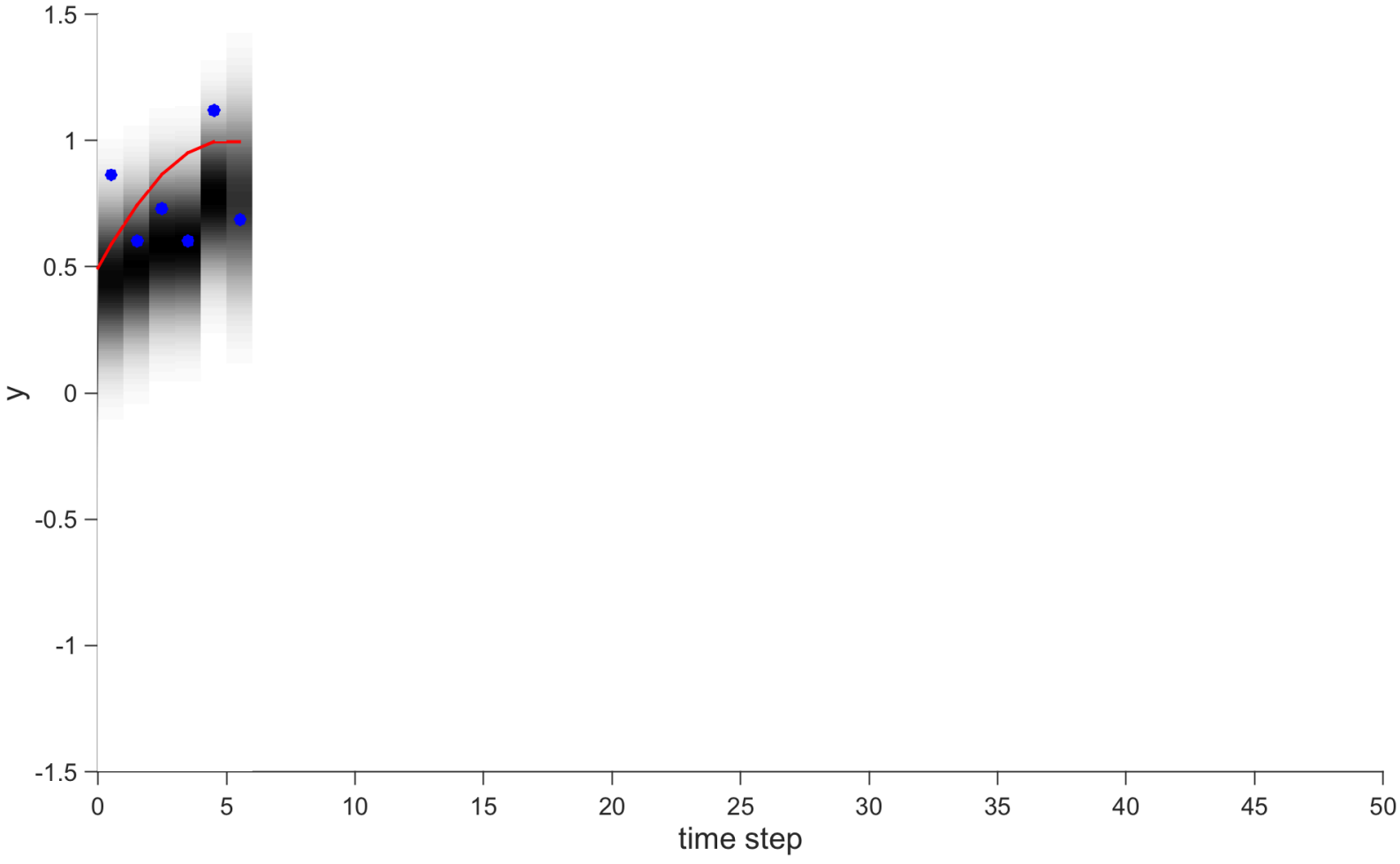
observed noisy data y_t, ground truth sinusoid



prediction for sixth latent variable $p(x_6|y_{1:5})$

Kalman Filter Demo

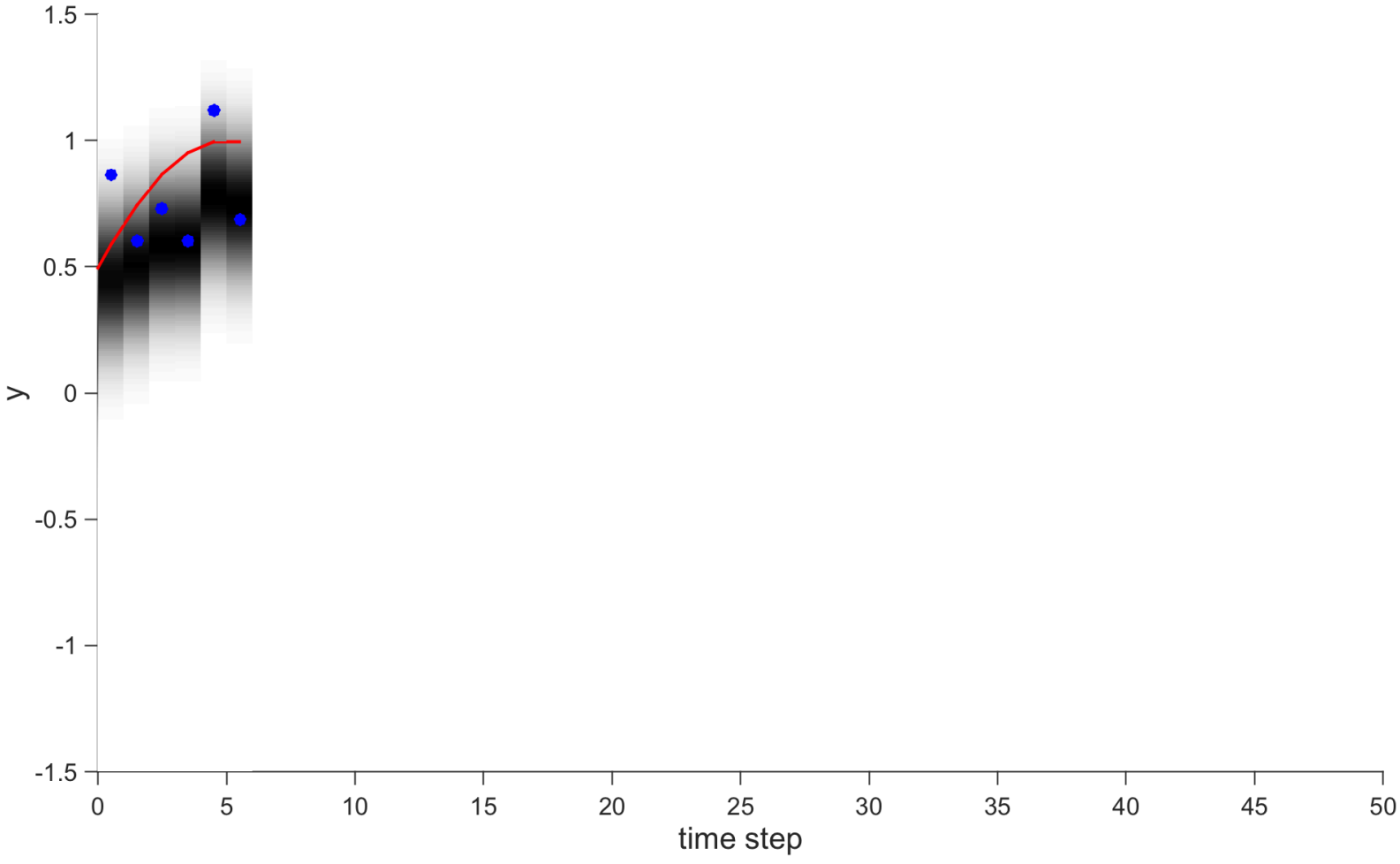
observed noisy data y_t, ground truth sinusoid



observe next data point y_6

Kalman Filter Demo

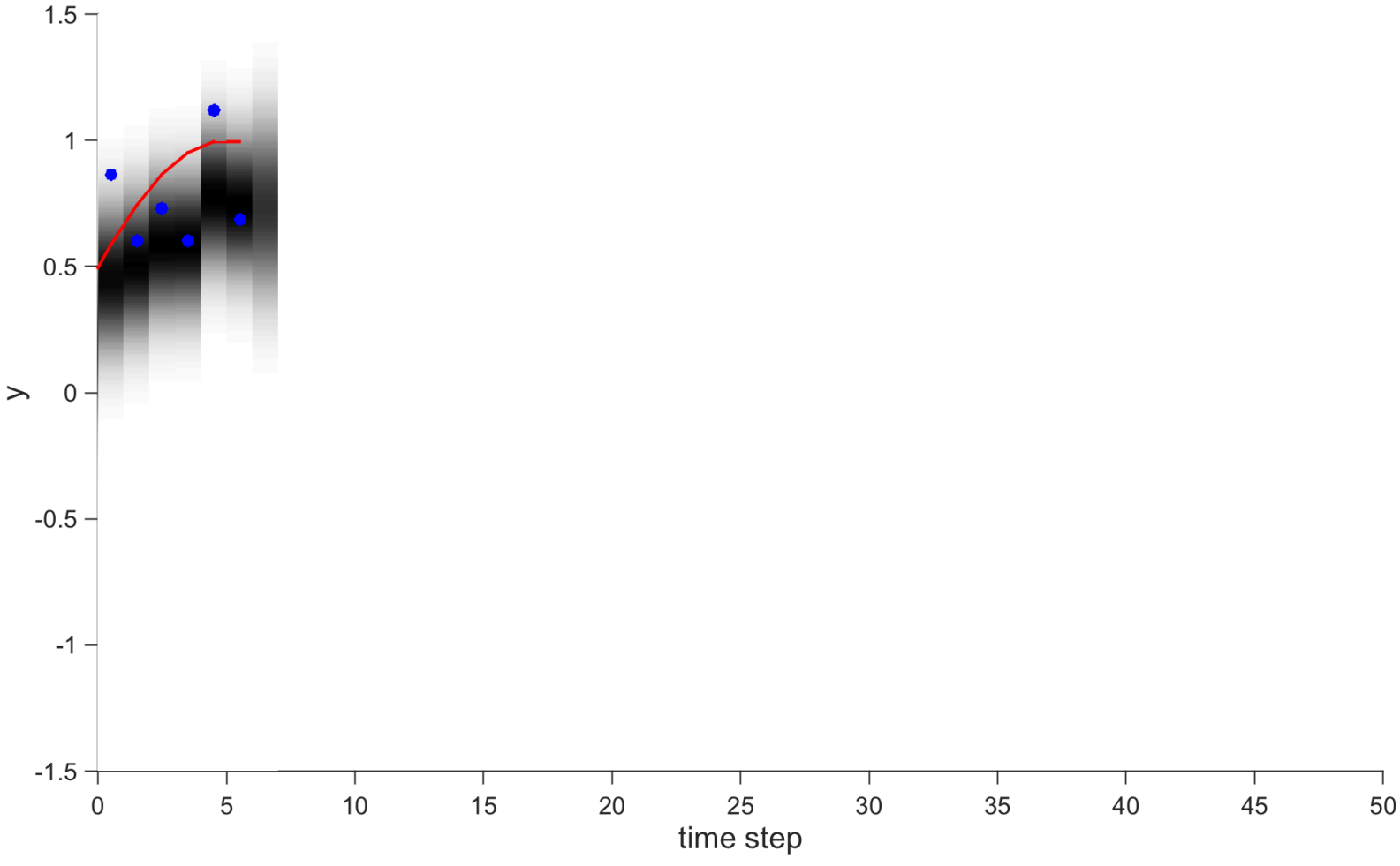
observed noisy data y_t, ground truth sinusoid



form posterior over sixth latent variable $p(x_6|y_{1:6})$

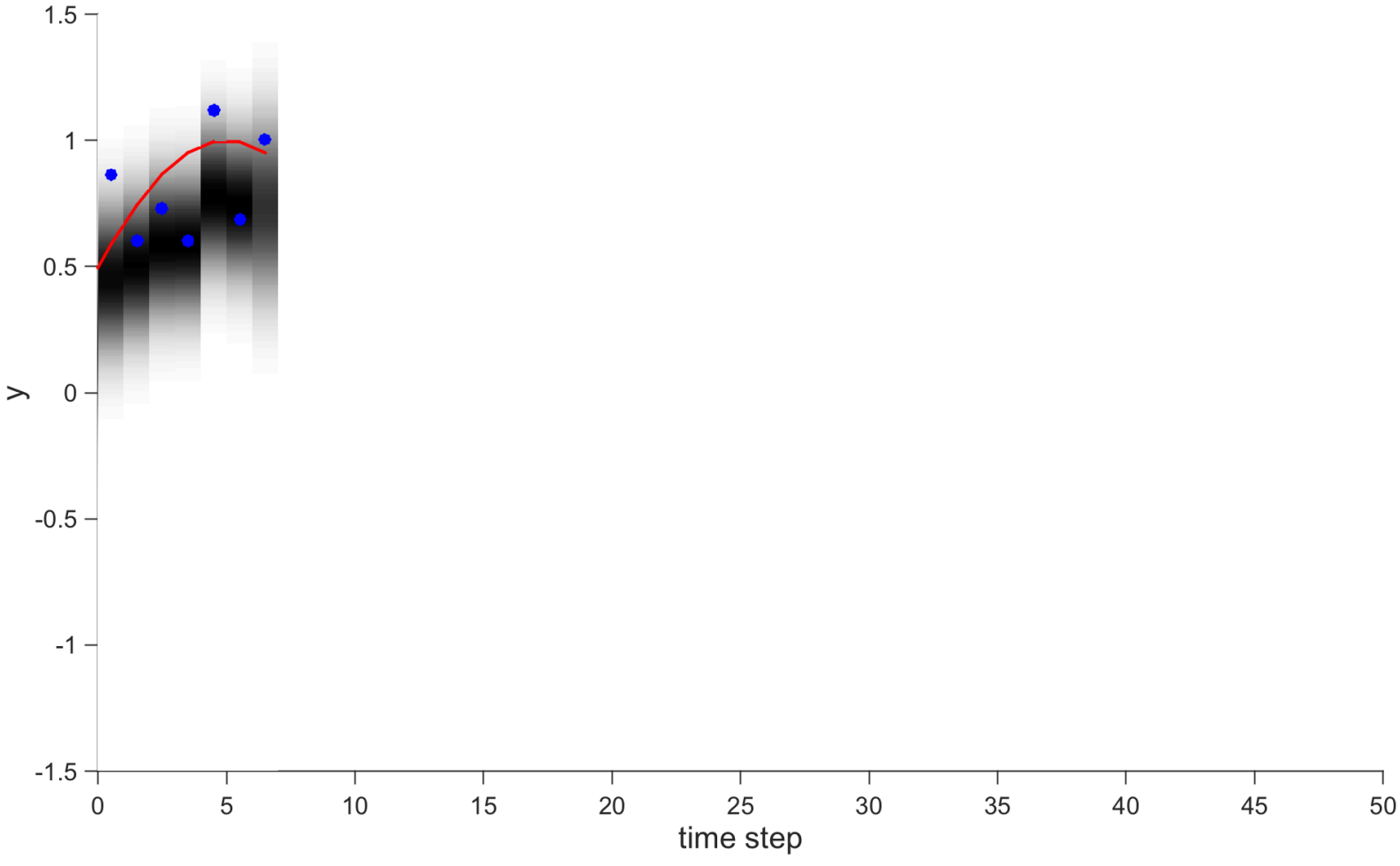
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



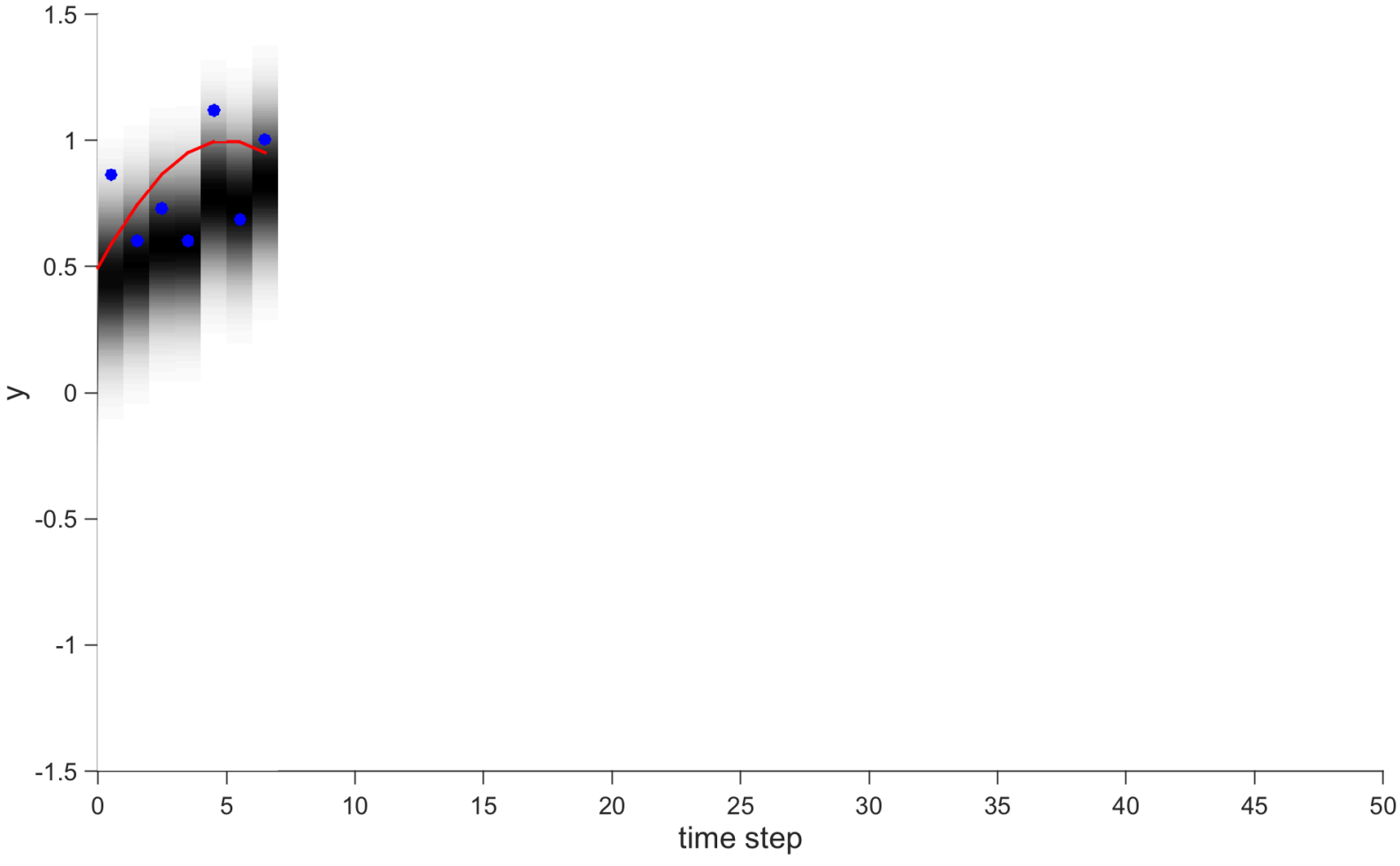
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



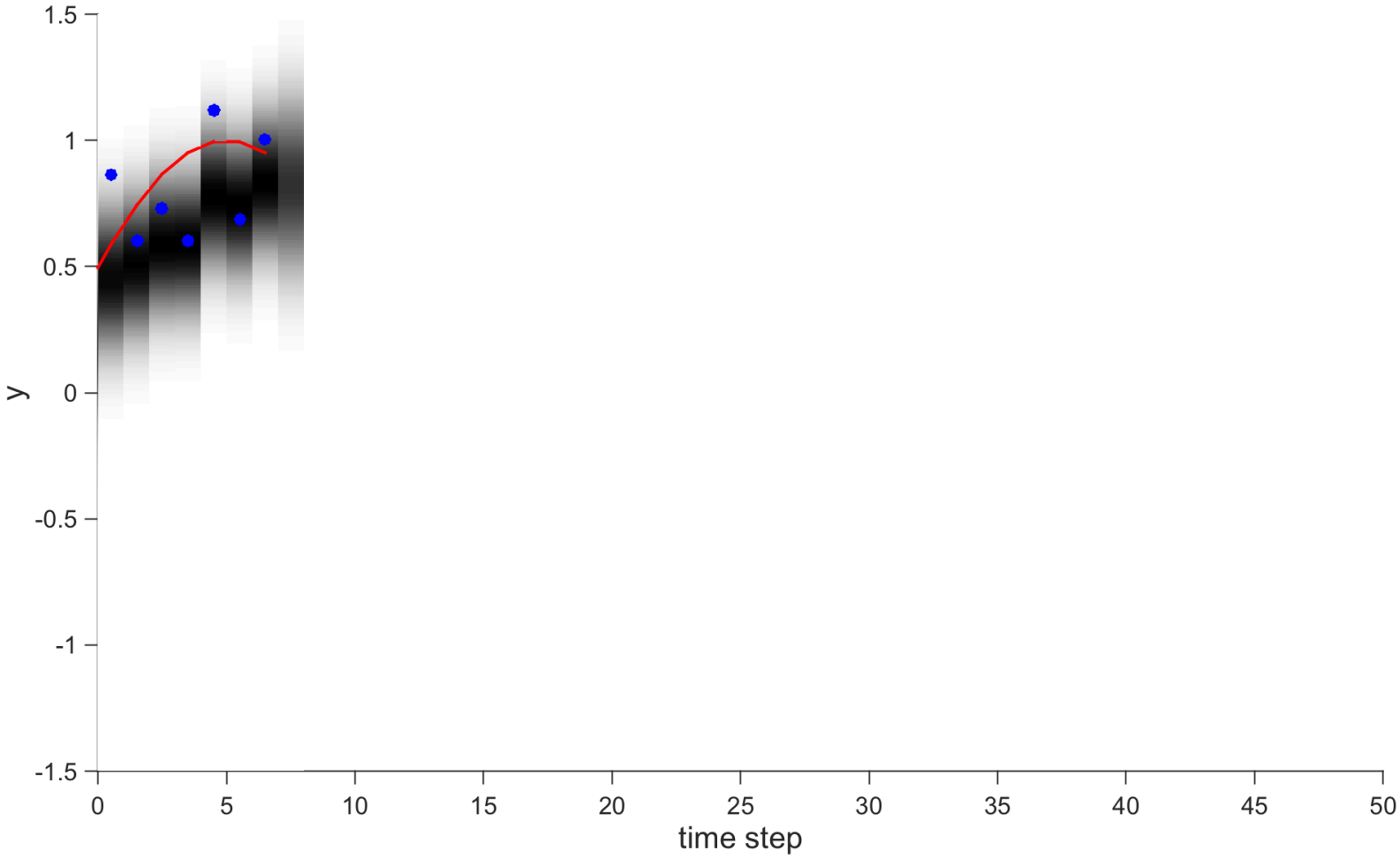
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



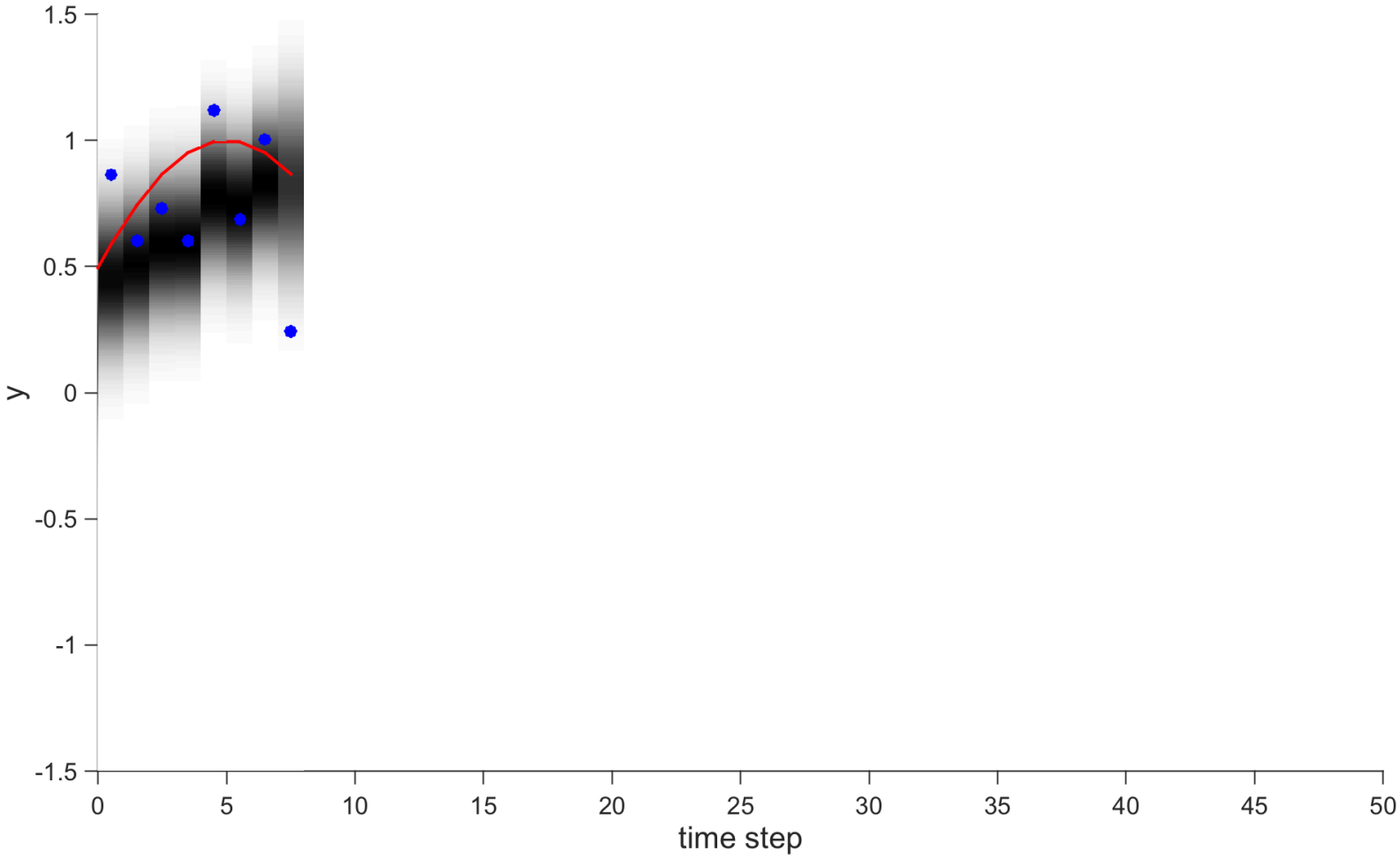
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



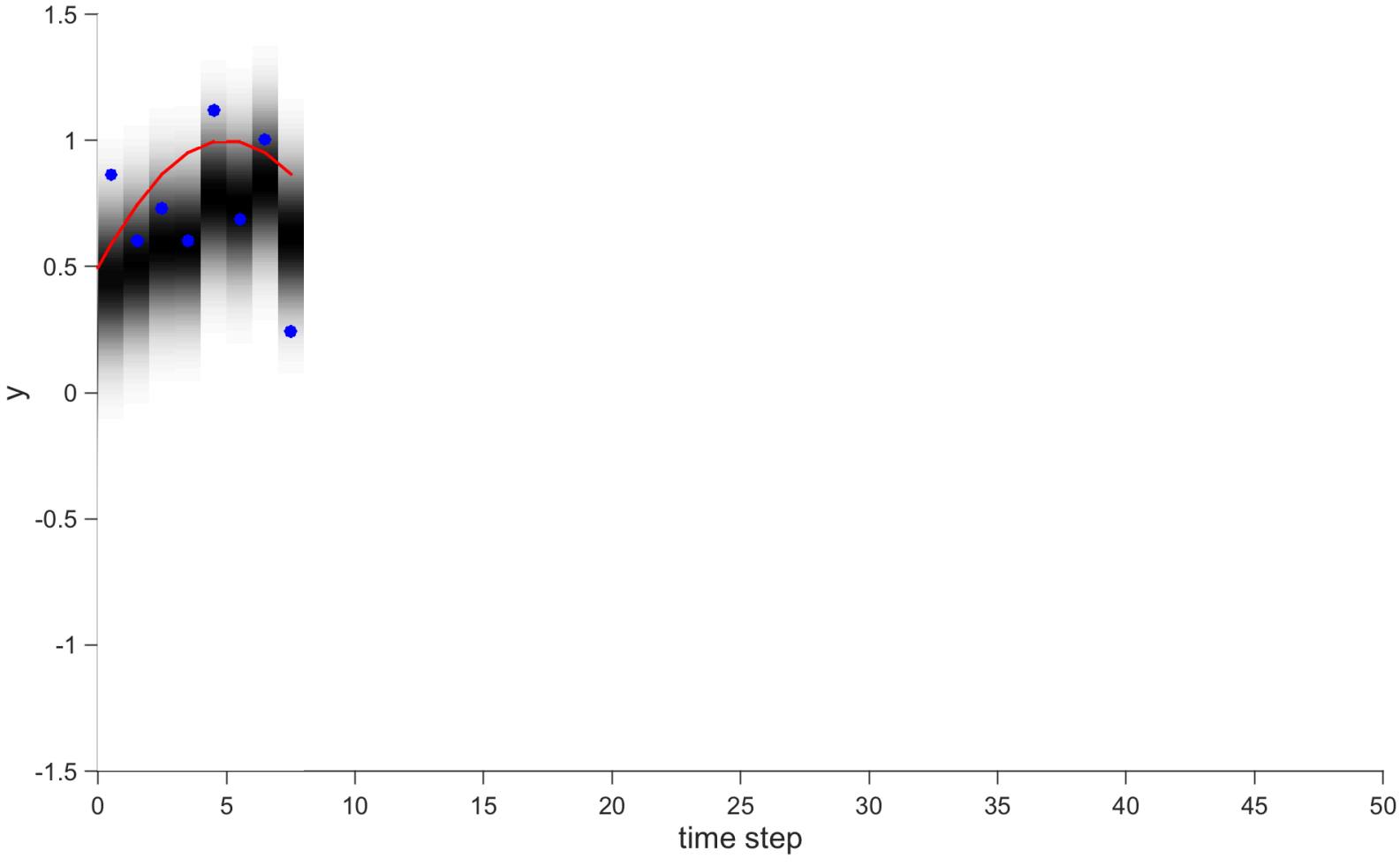
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



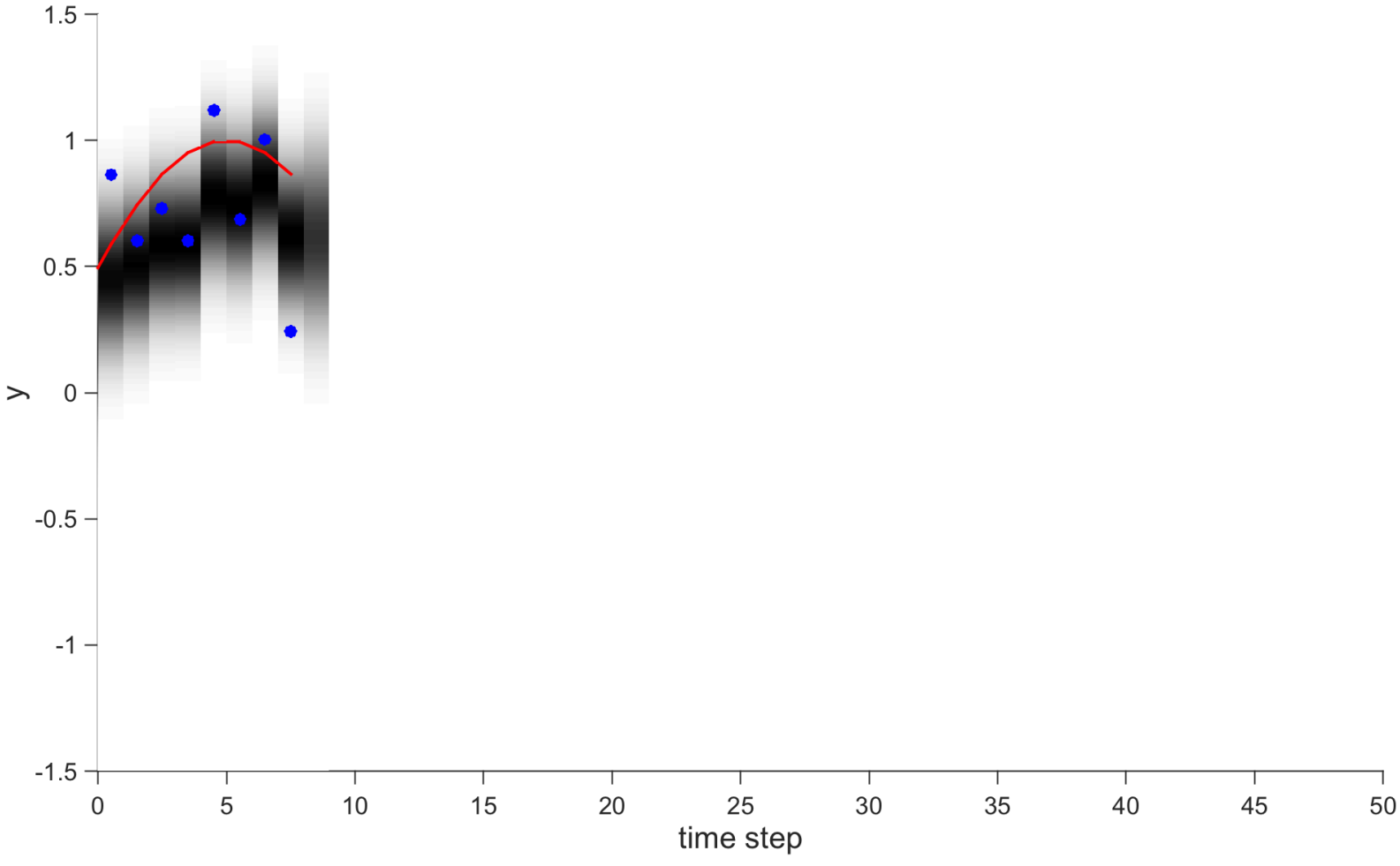
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



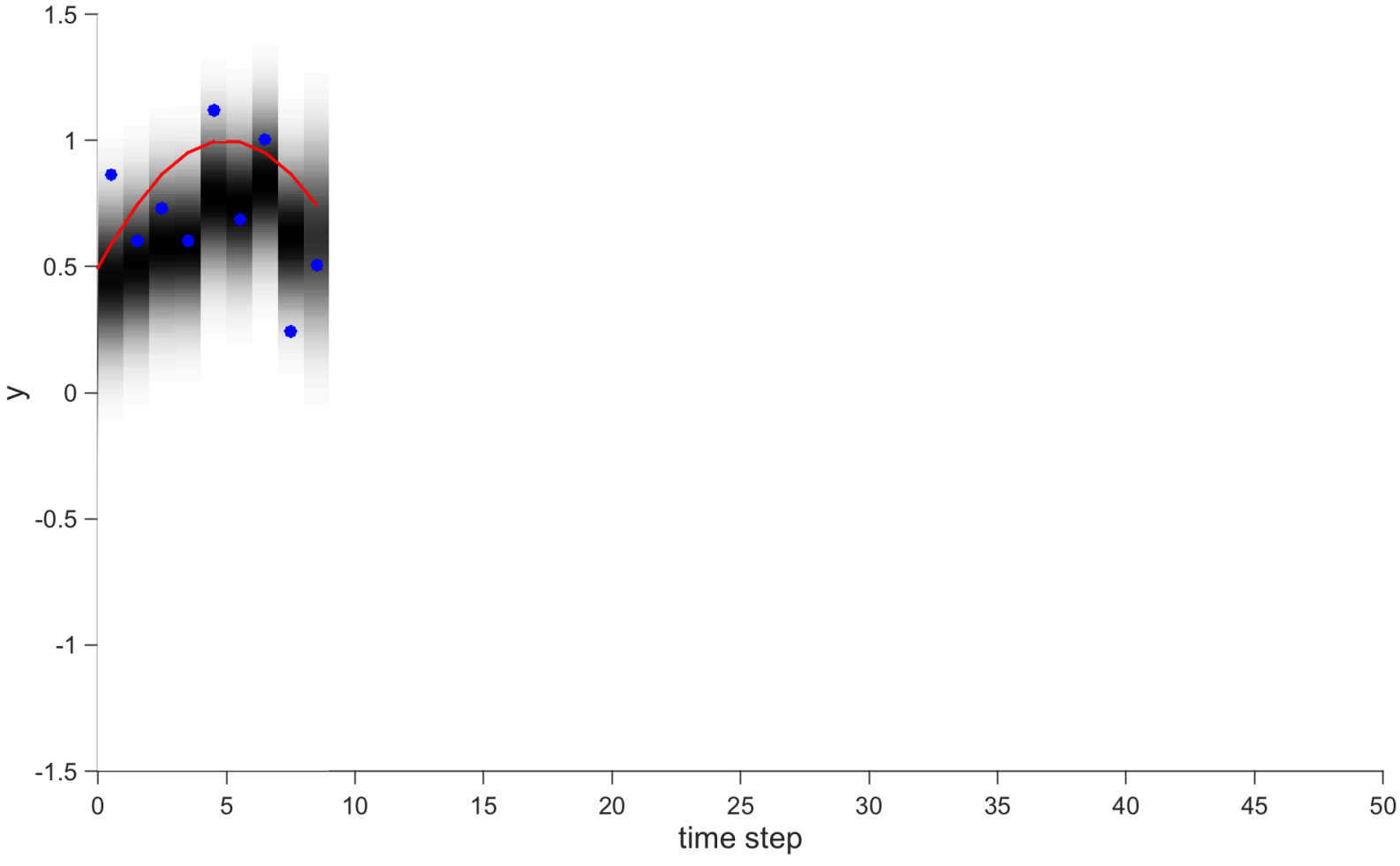
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



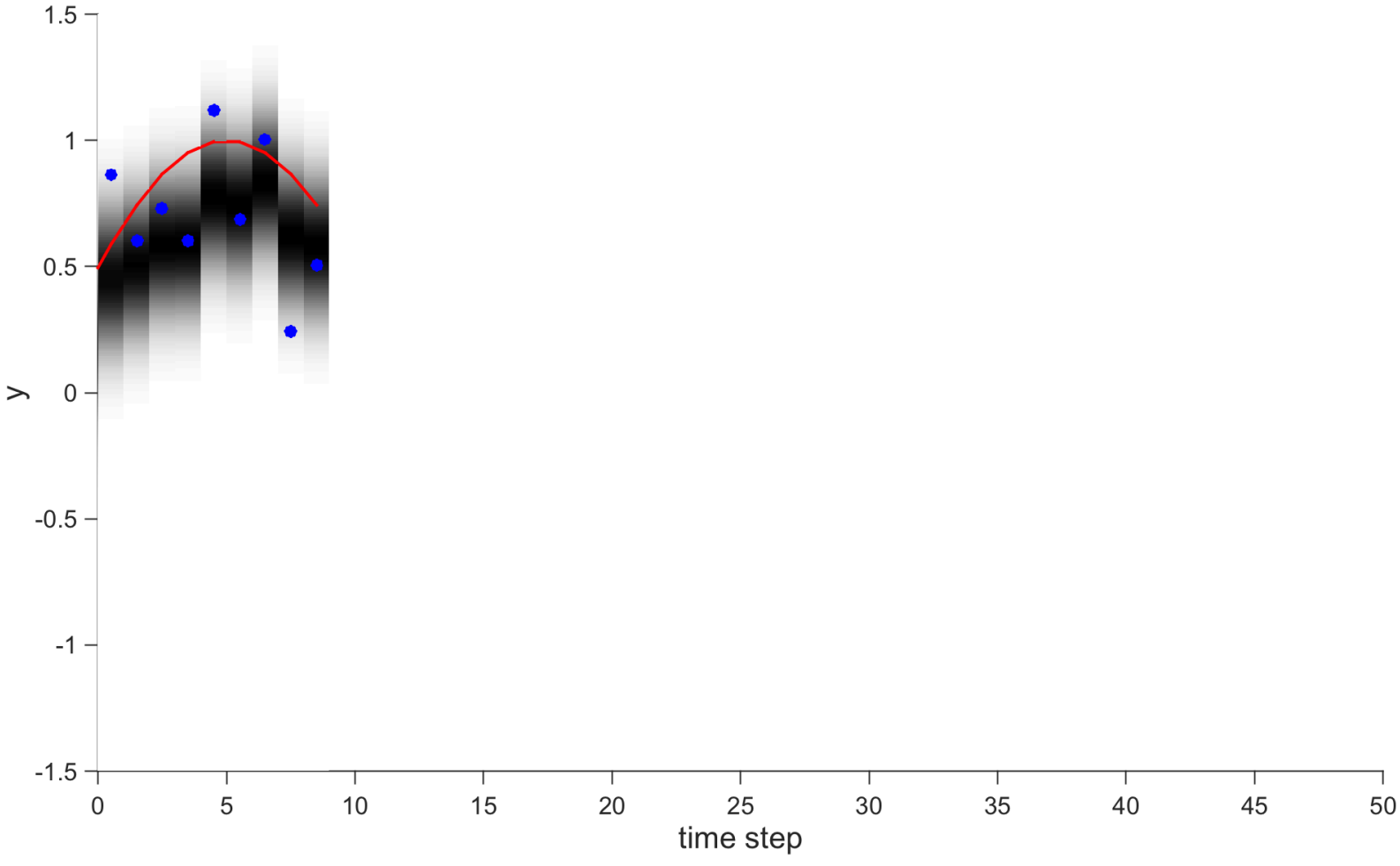
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



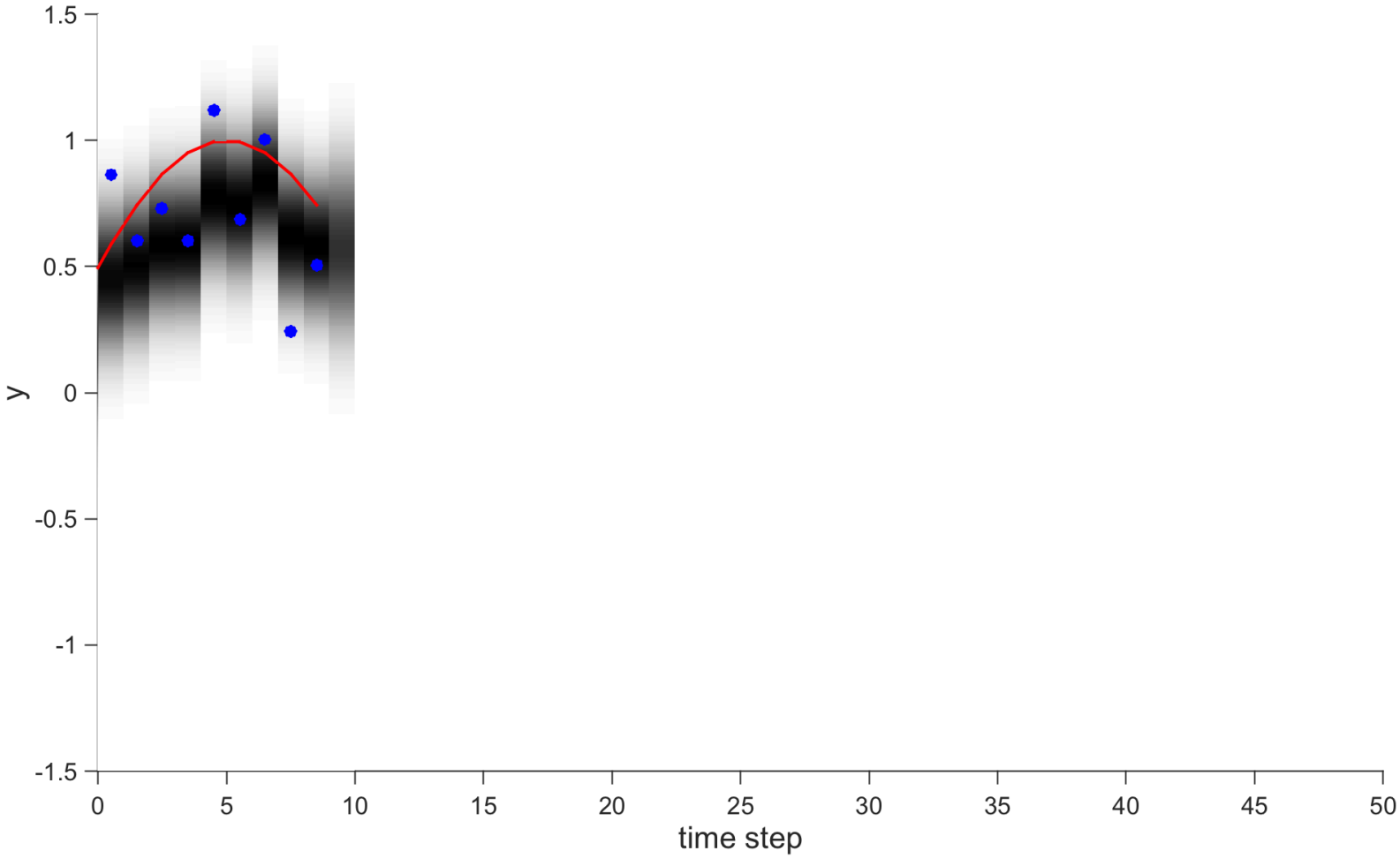
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



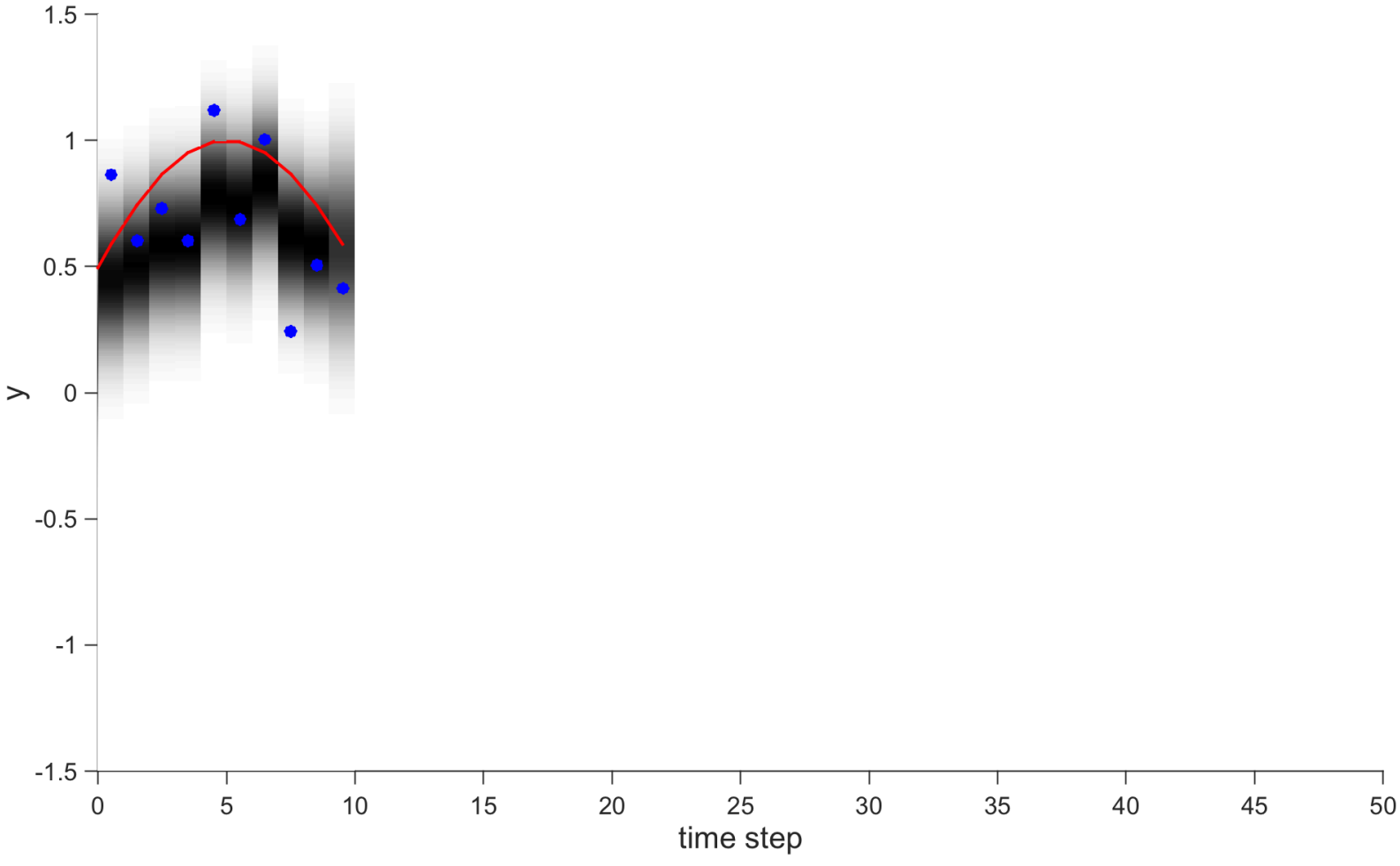
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



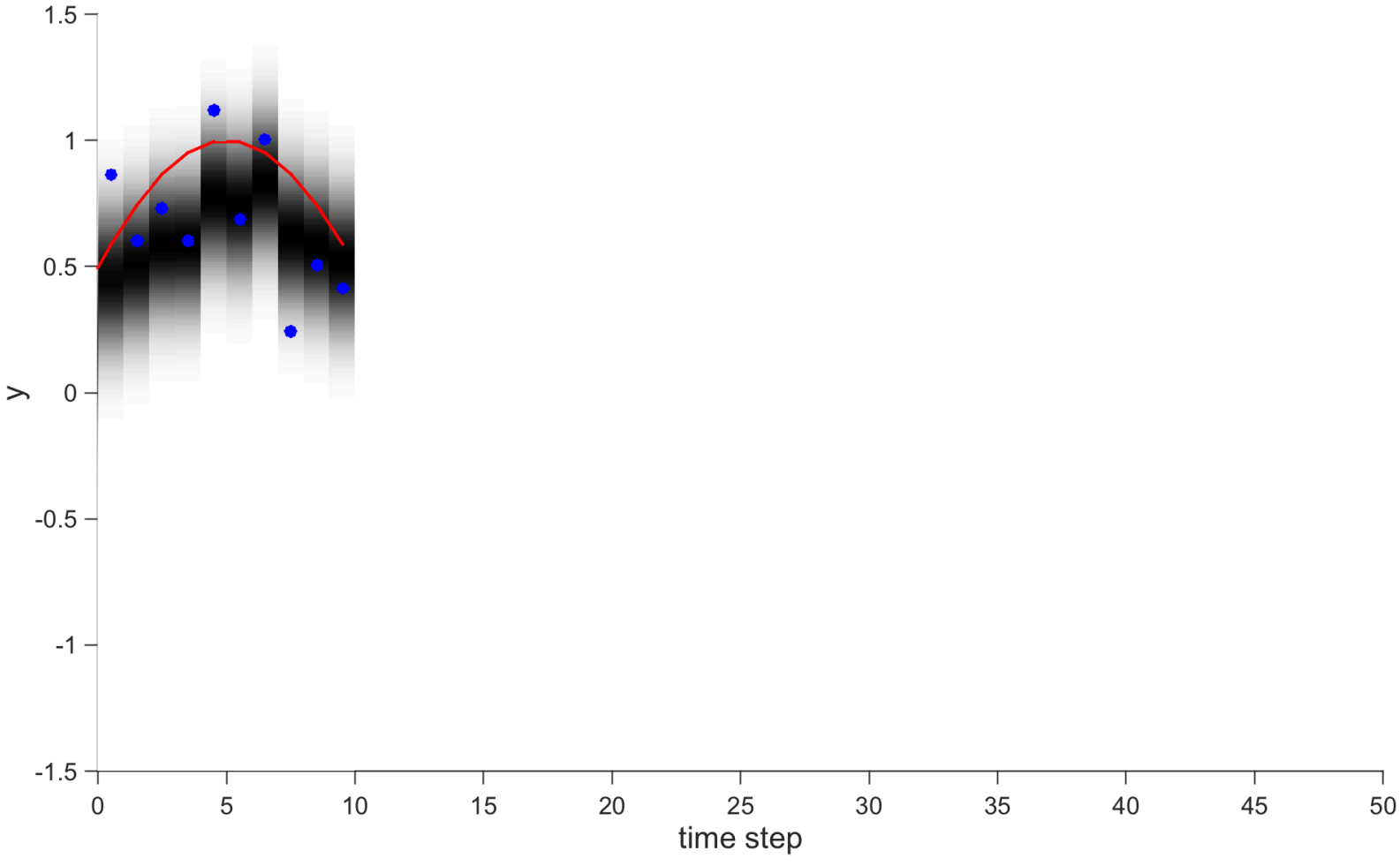
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



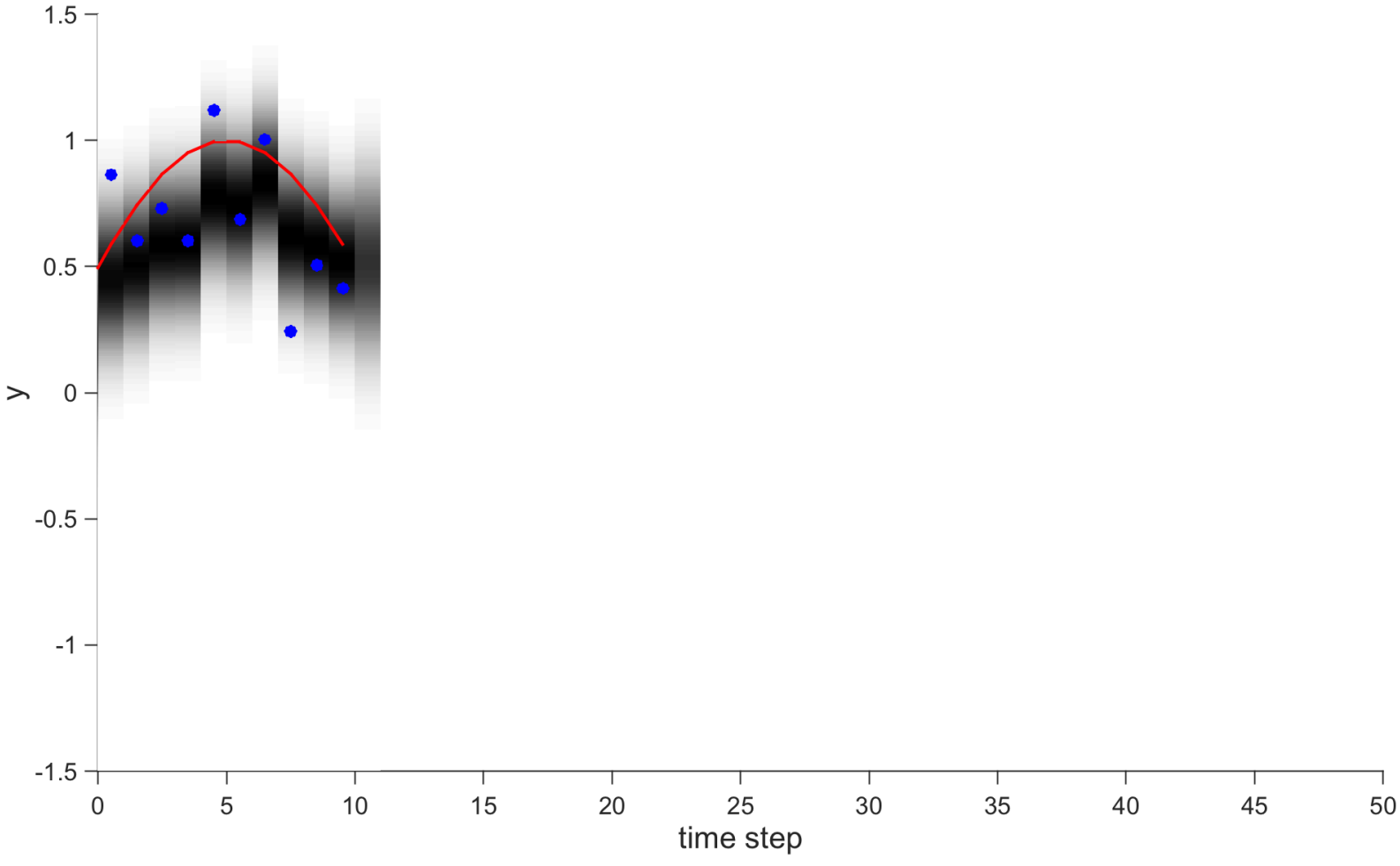
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



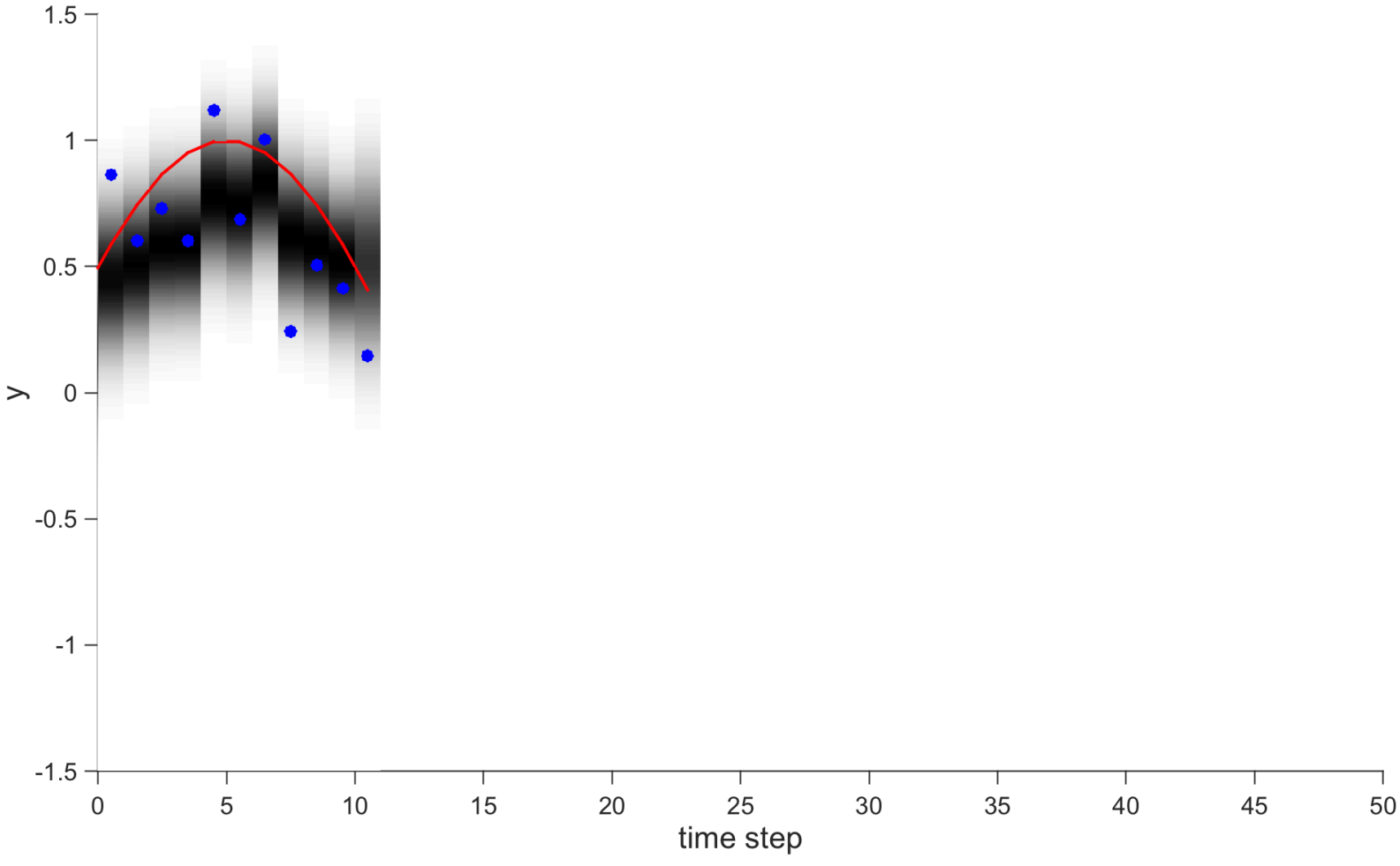
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



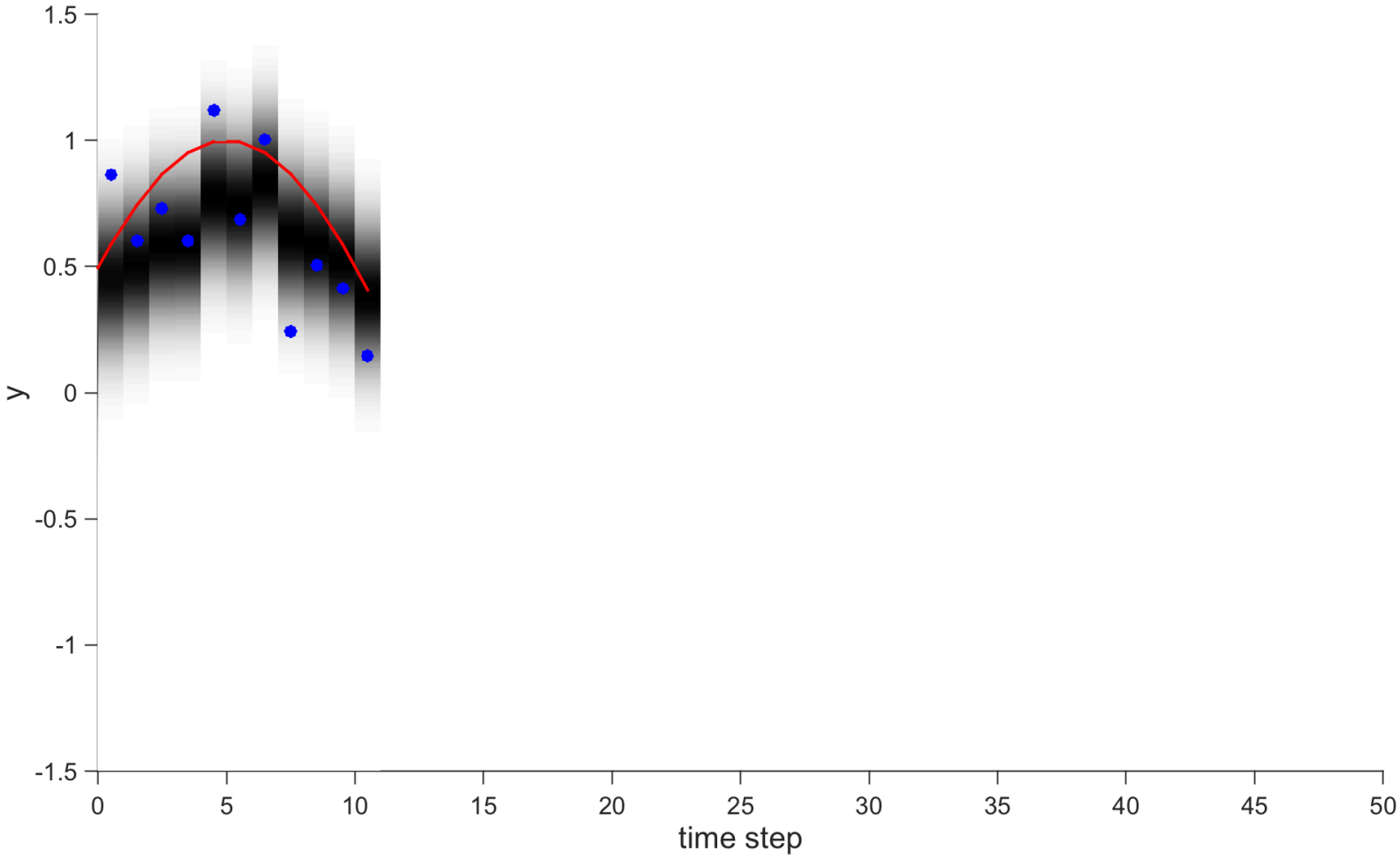
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



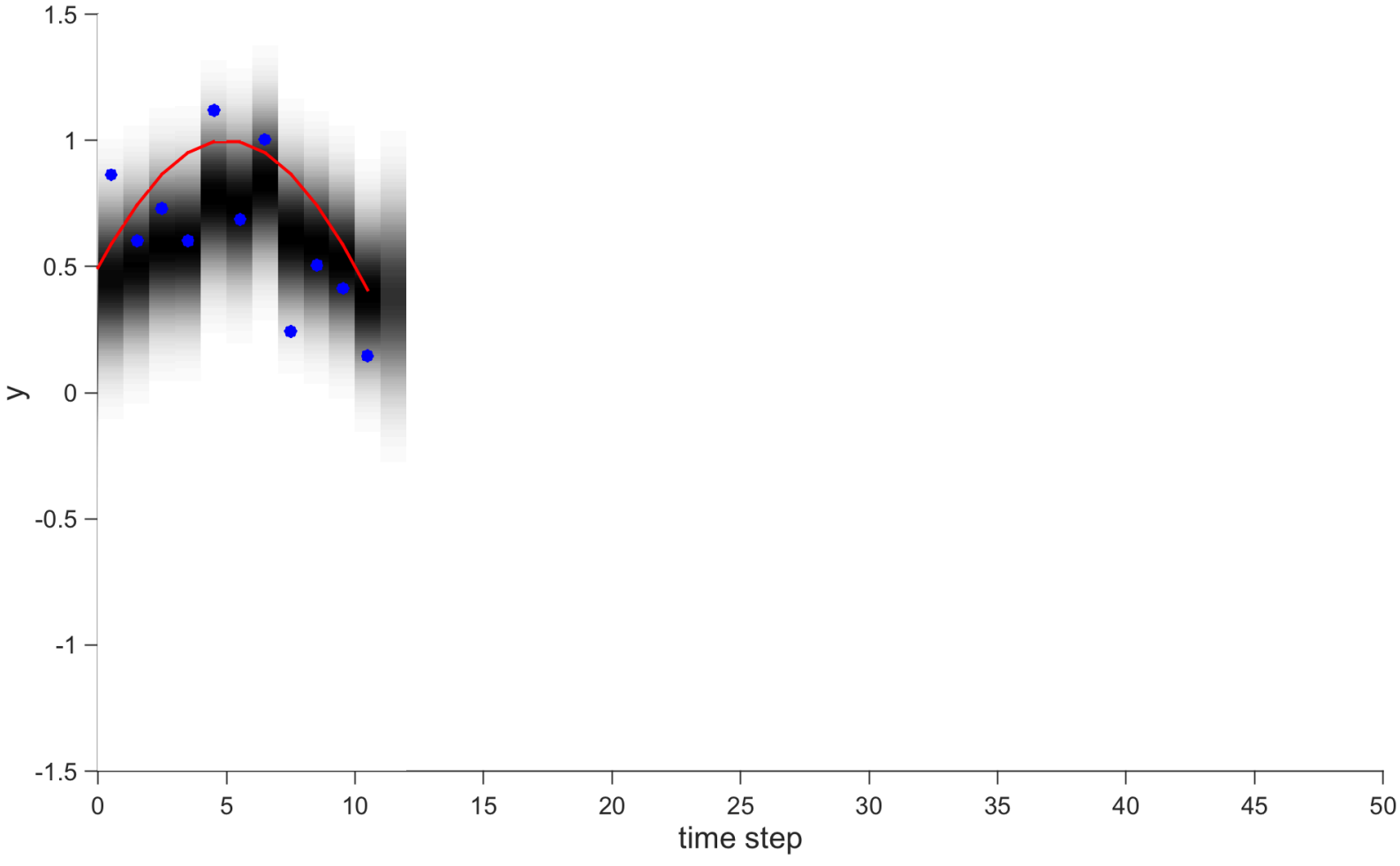
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



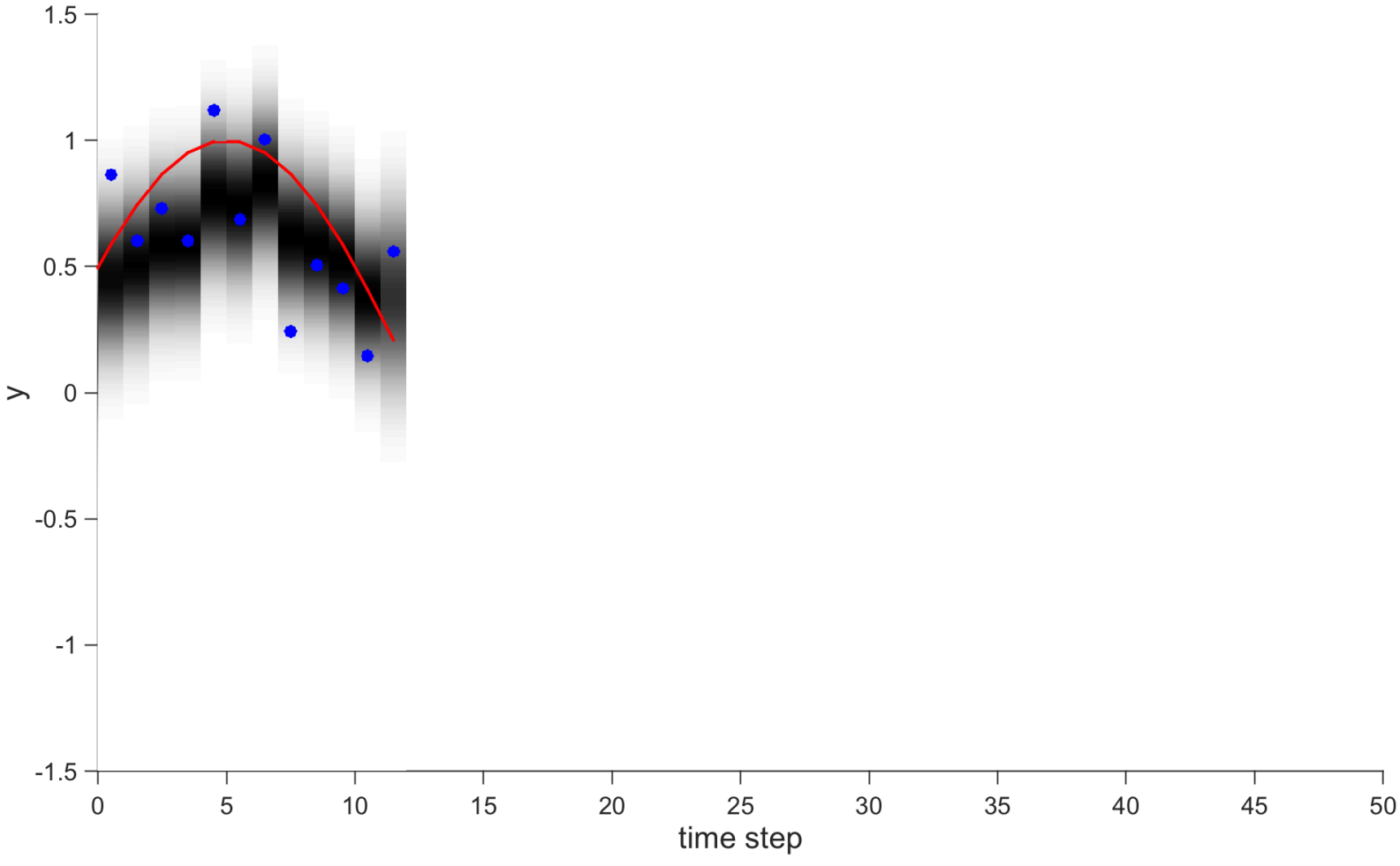
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



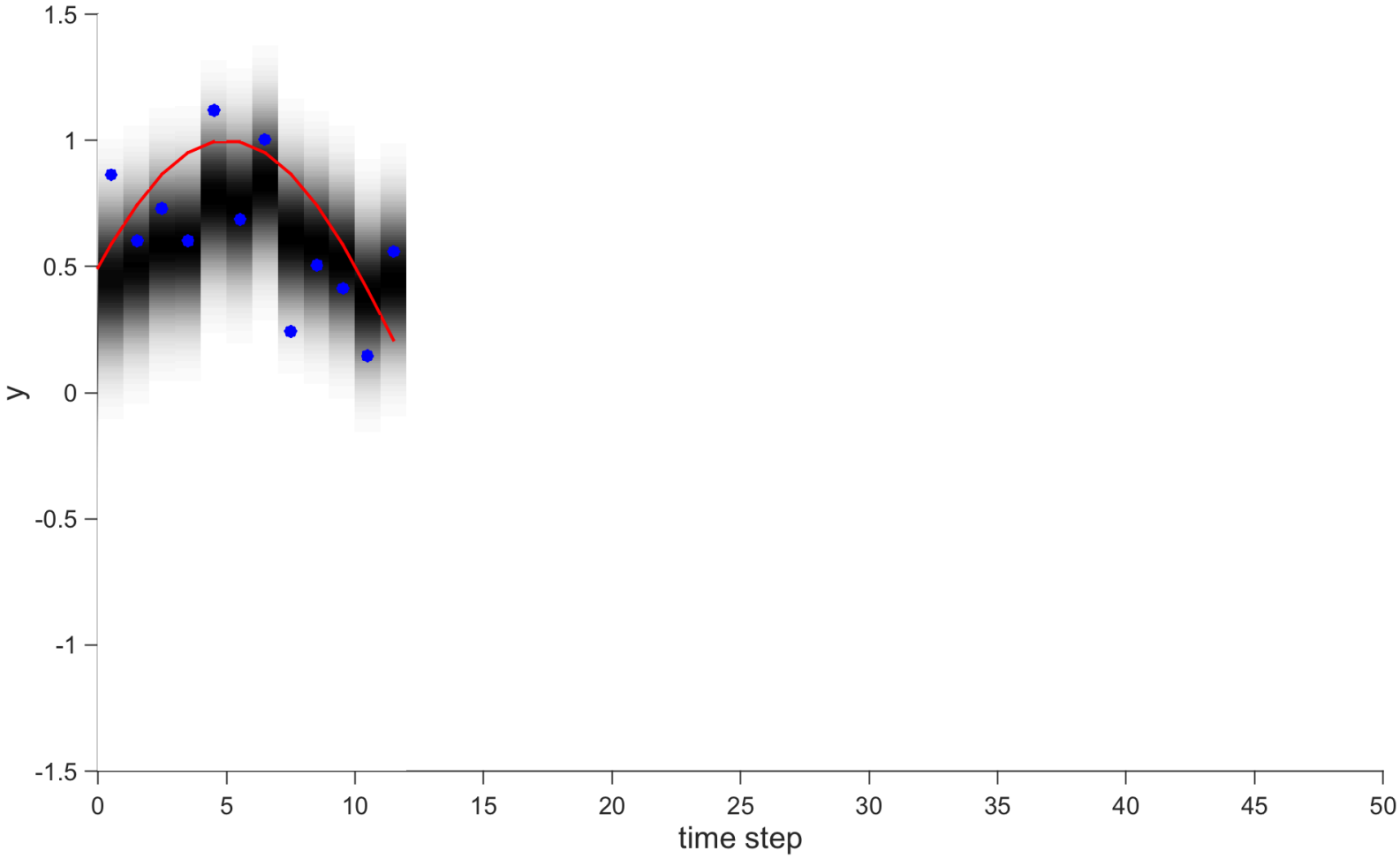
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



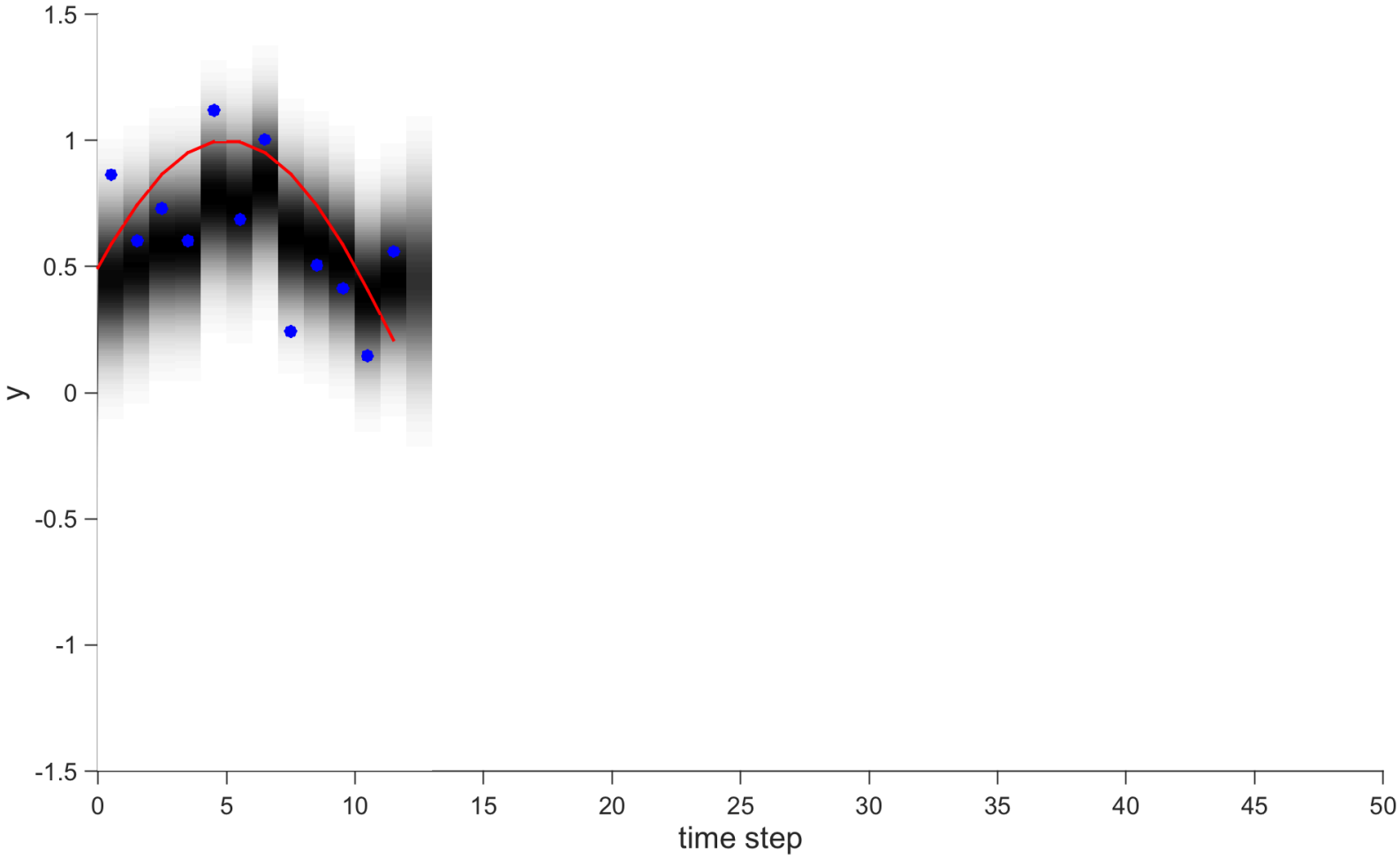
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



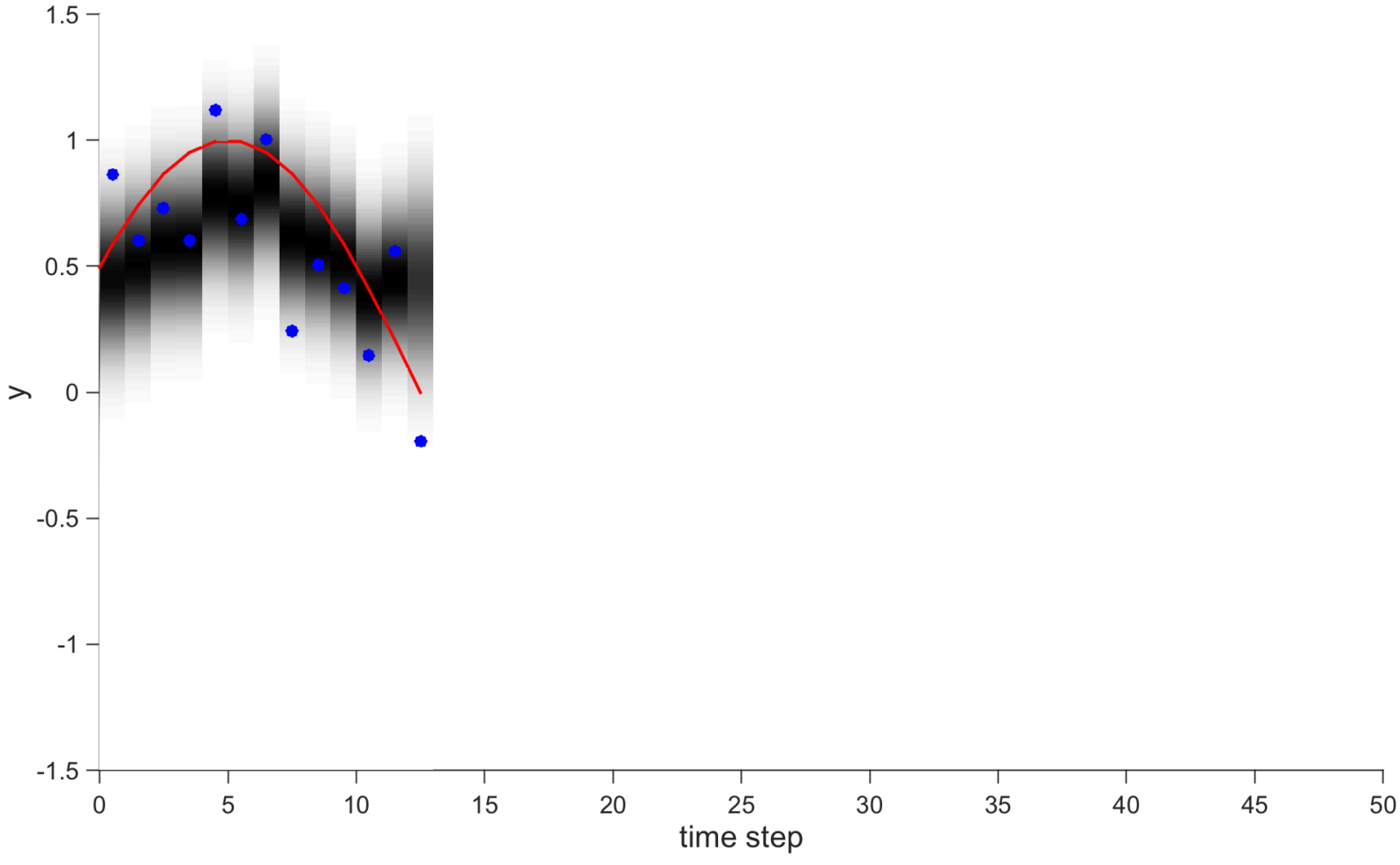
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



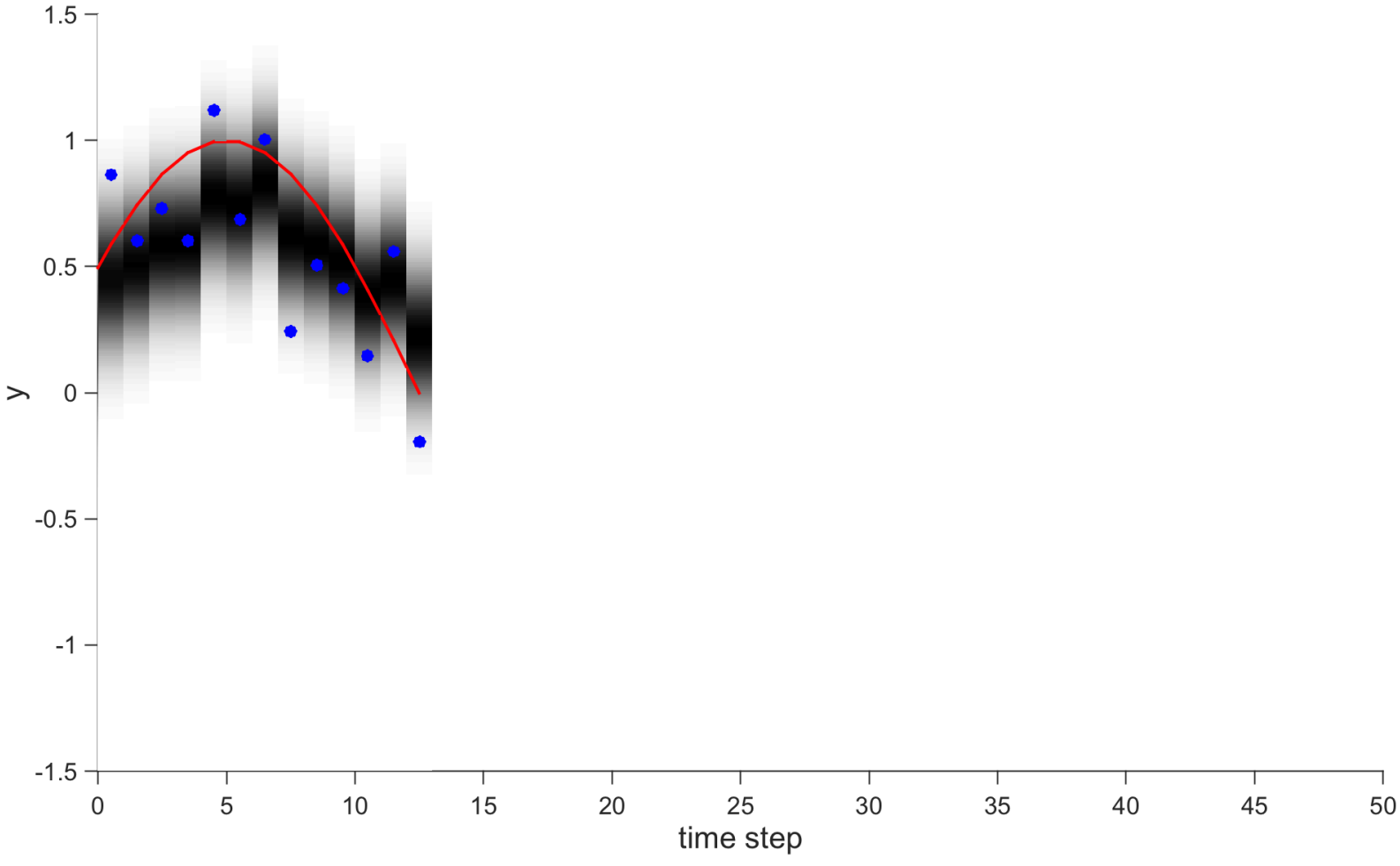
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



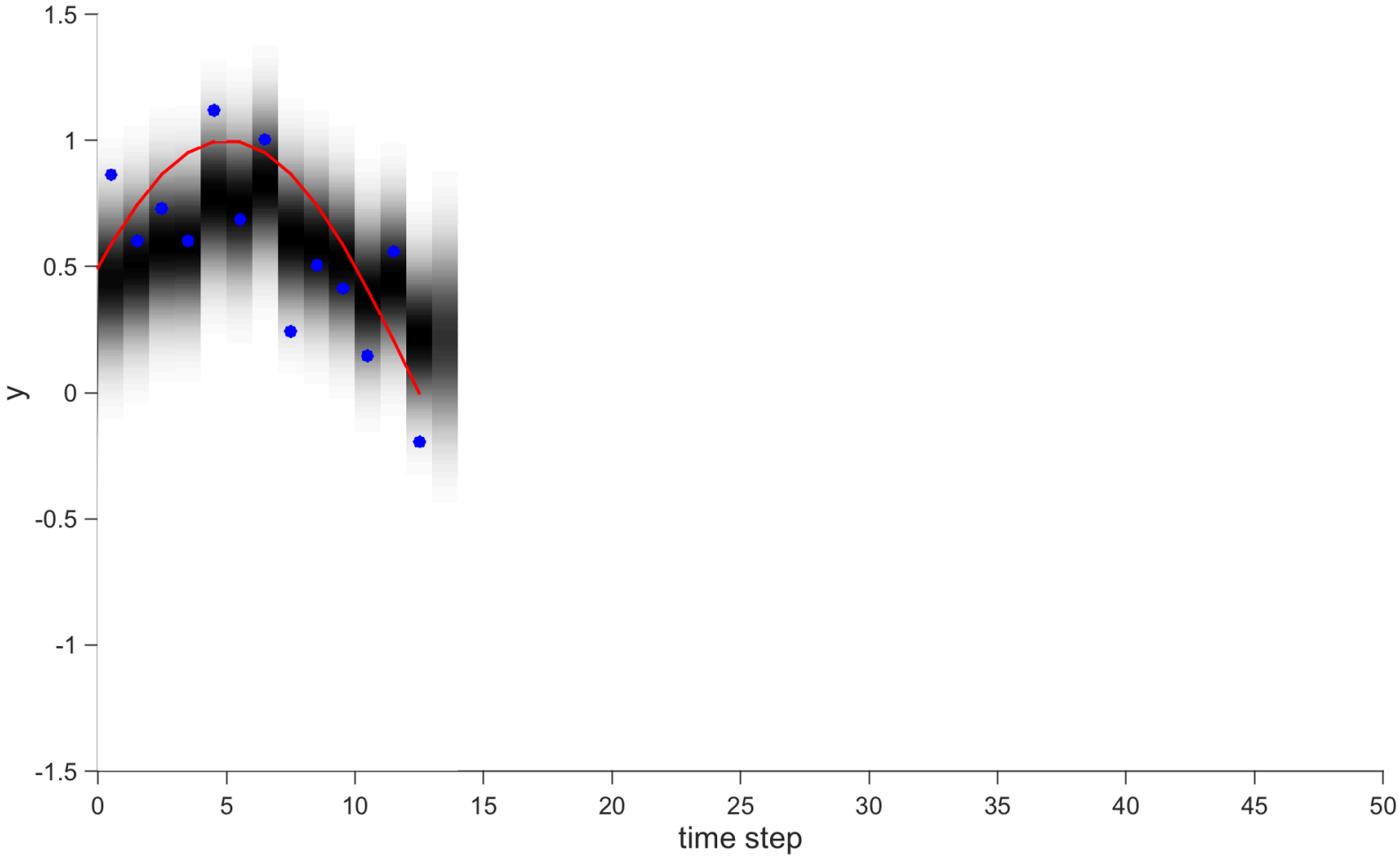
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



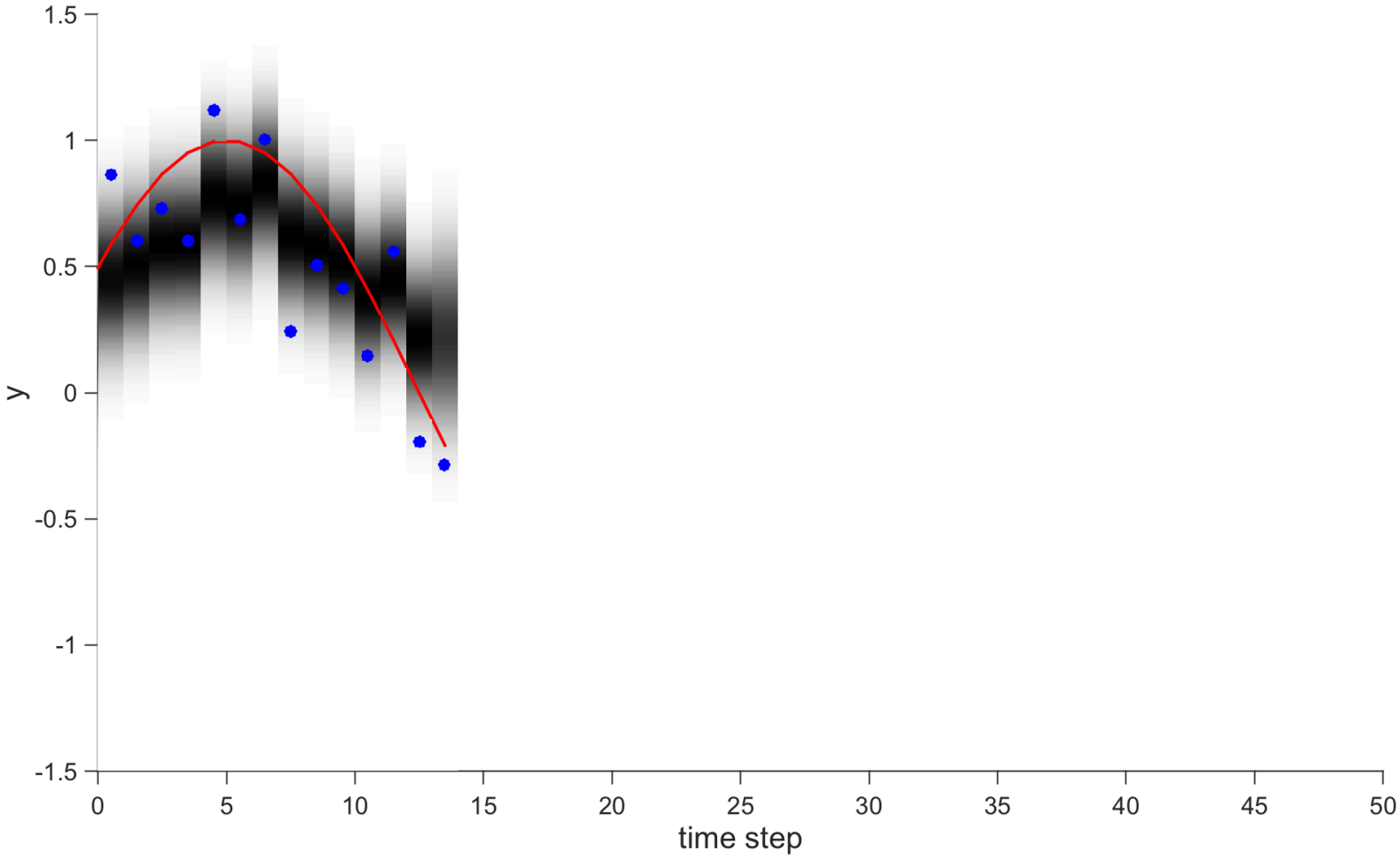
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



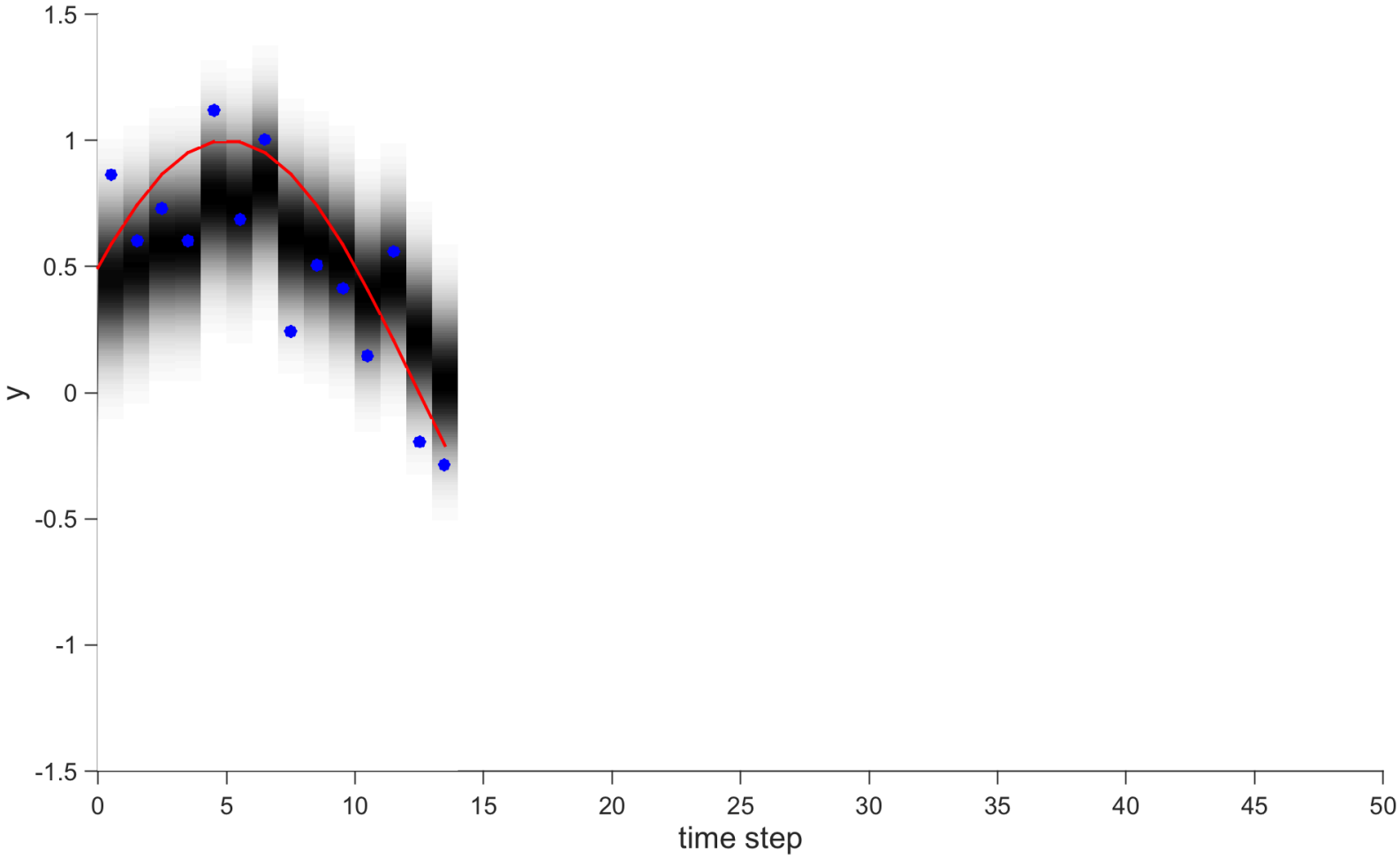
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



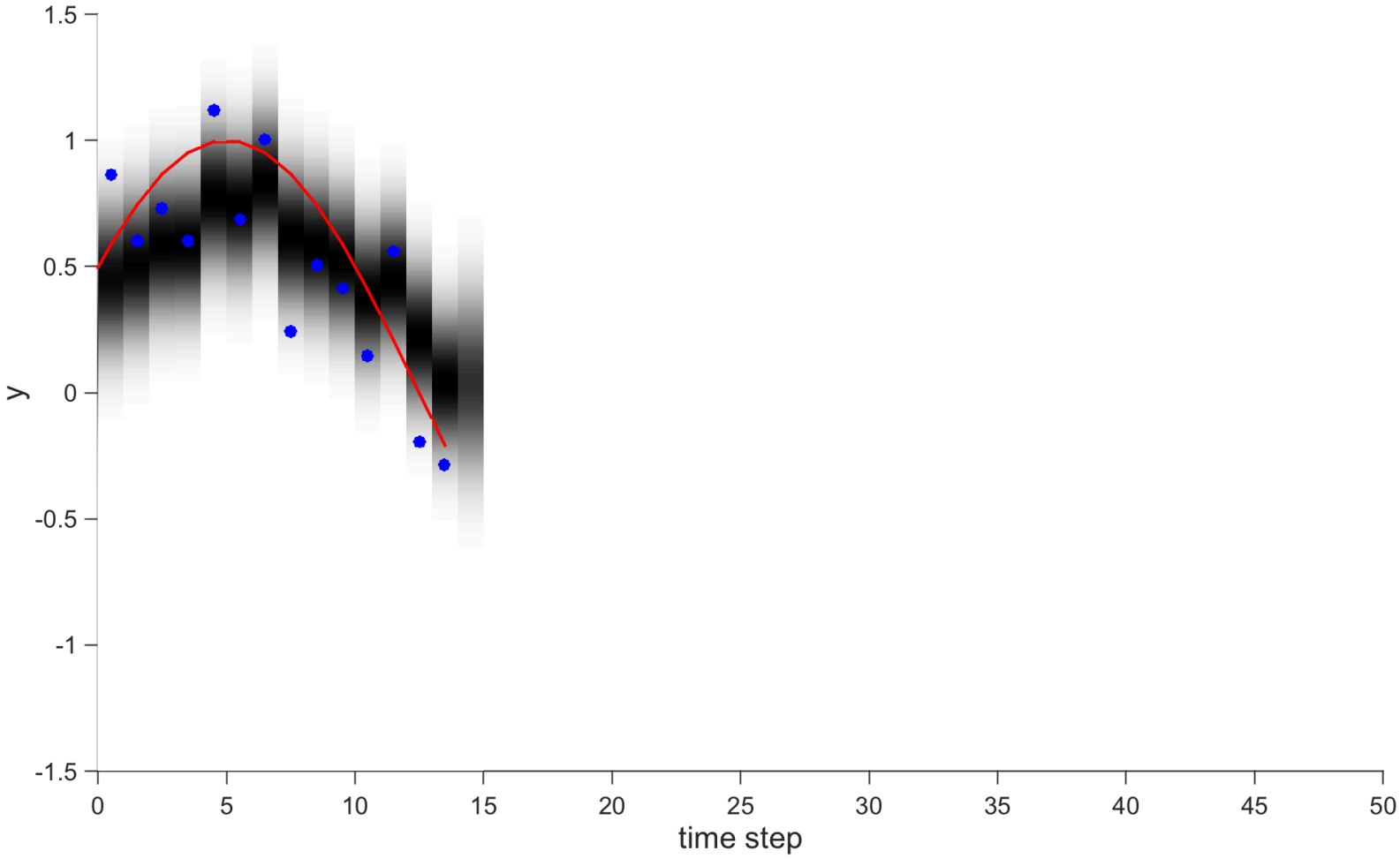
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



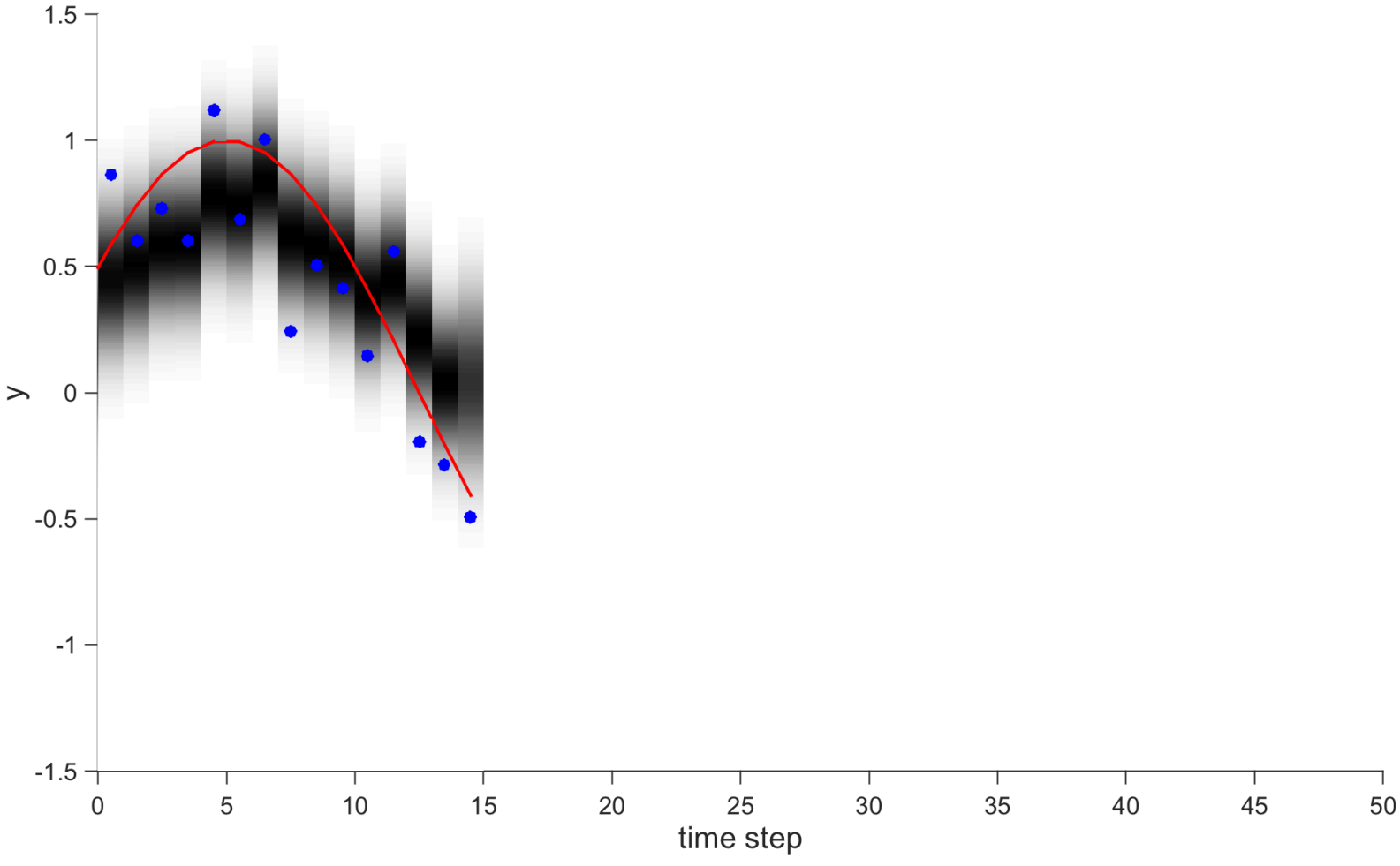
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



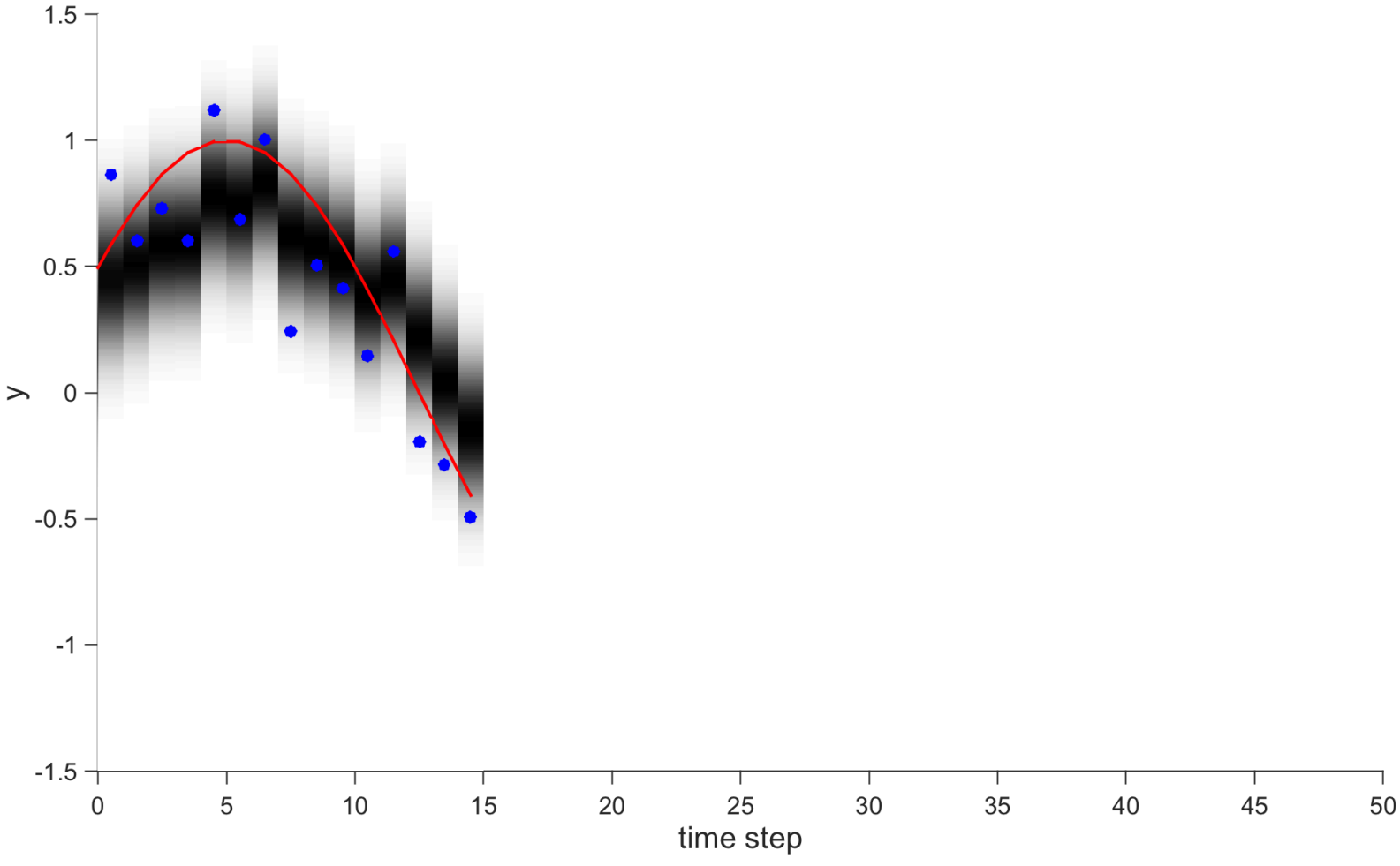
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



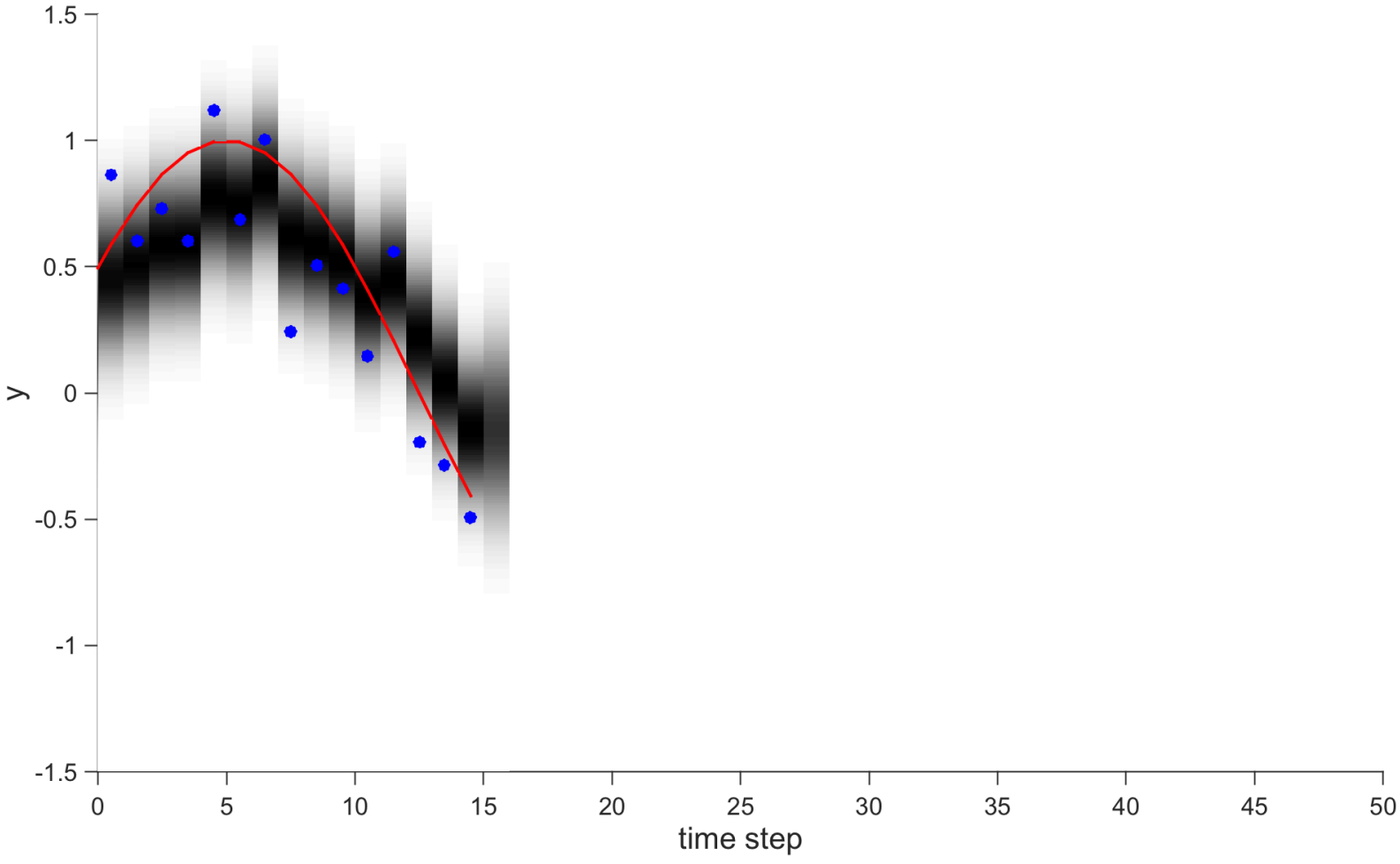
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



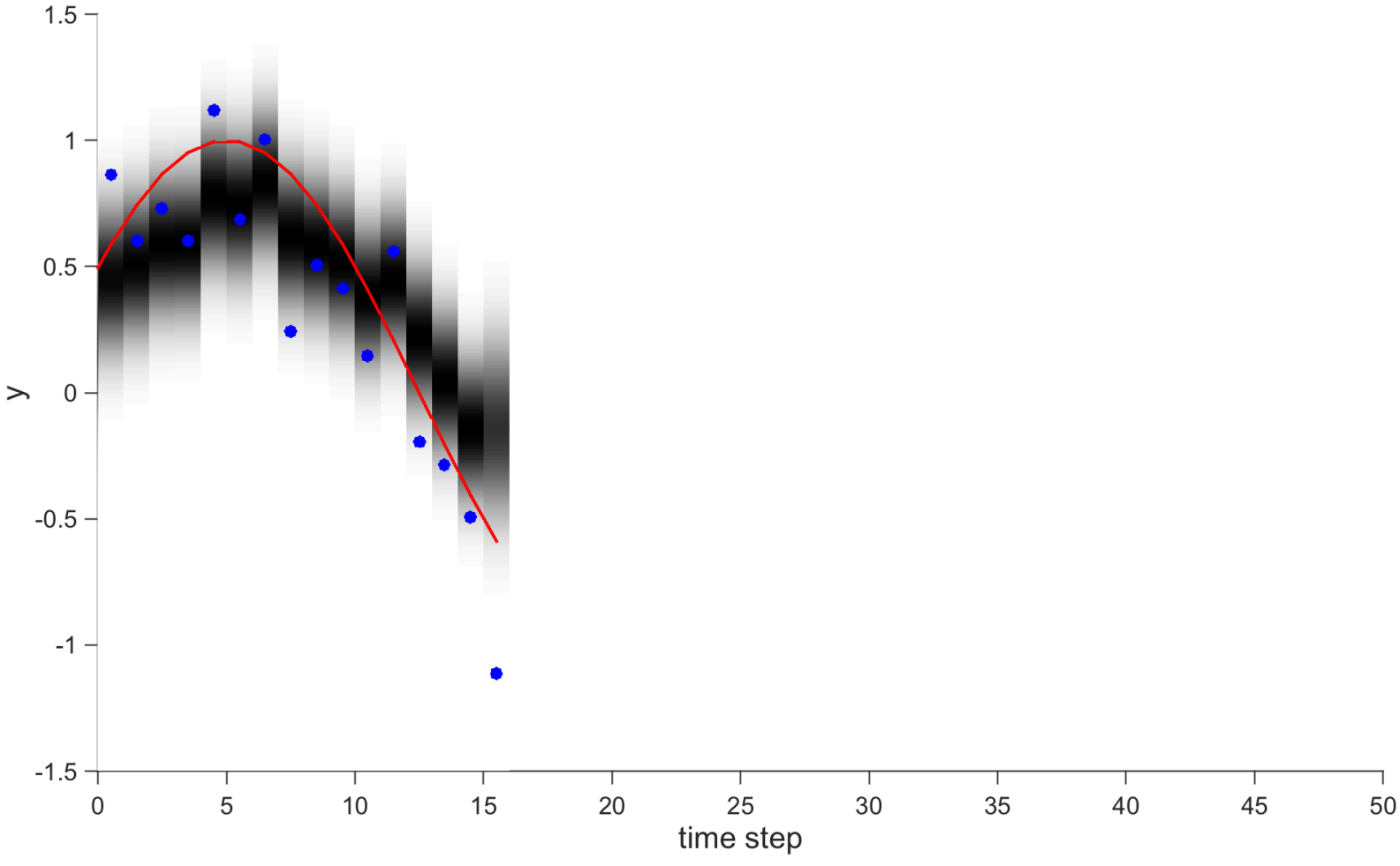
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



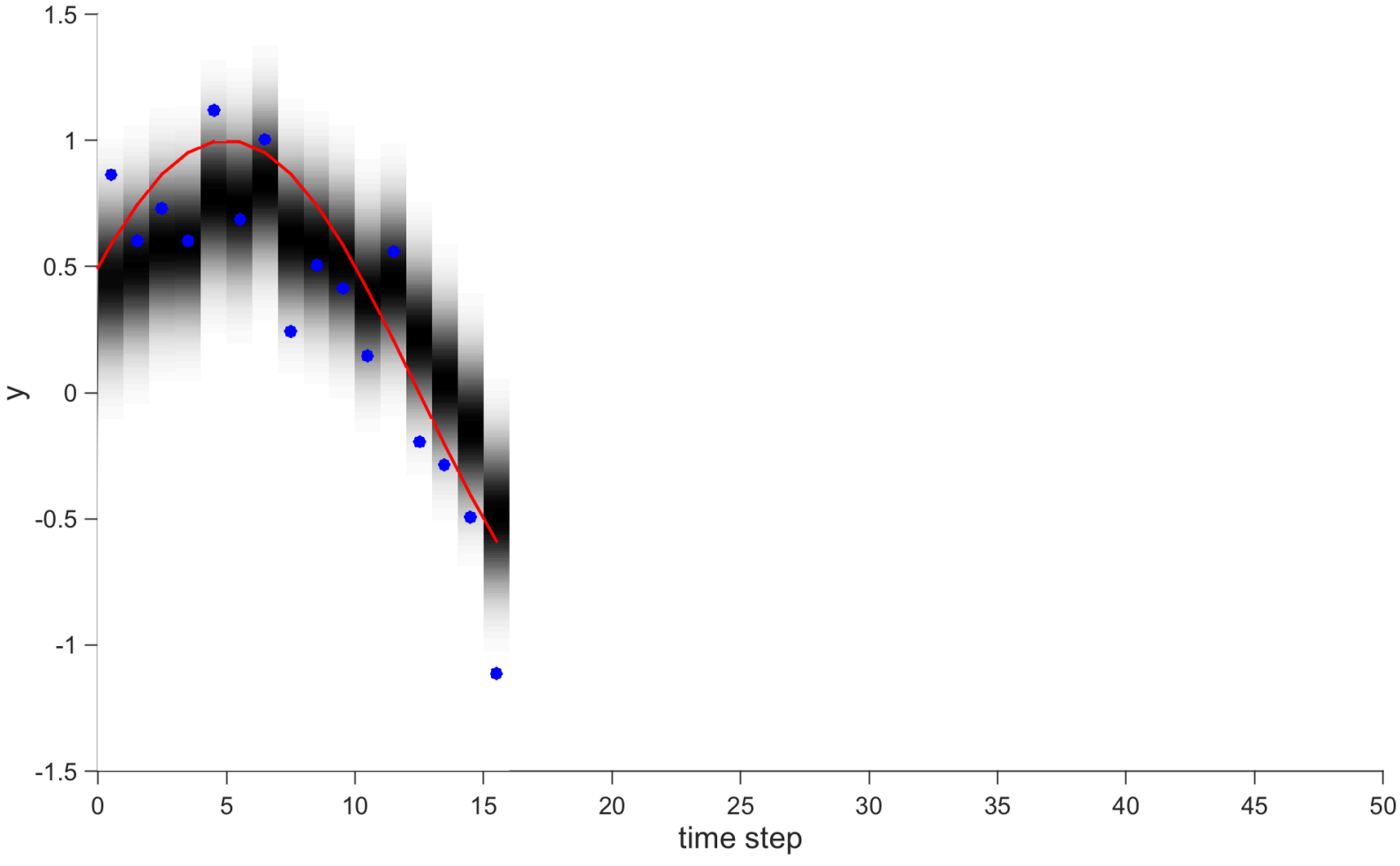
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



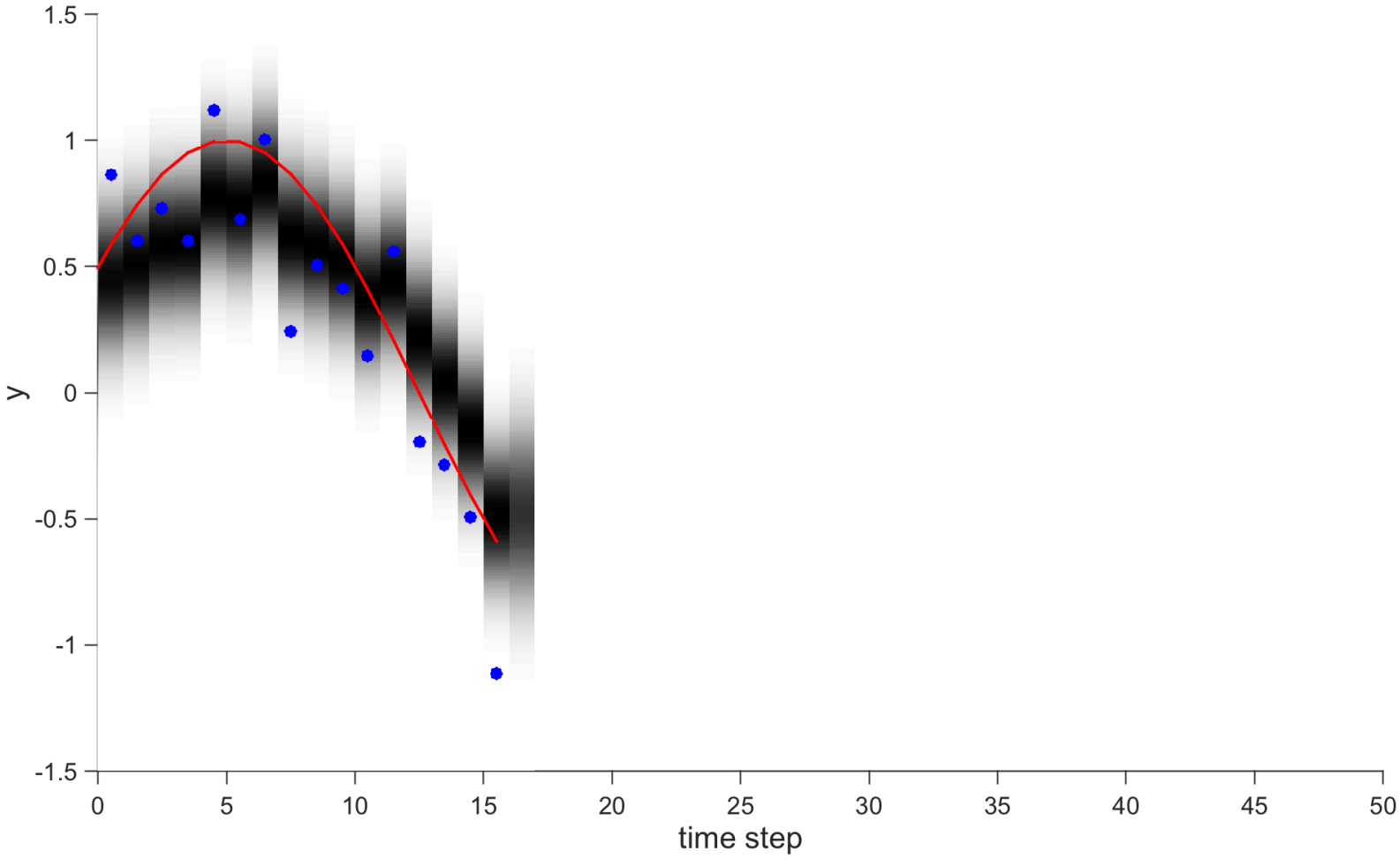
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



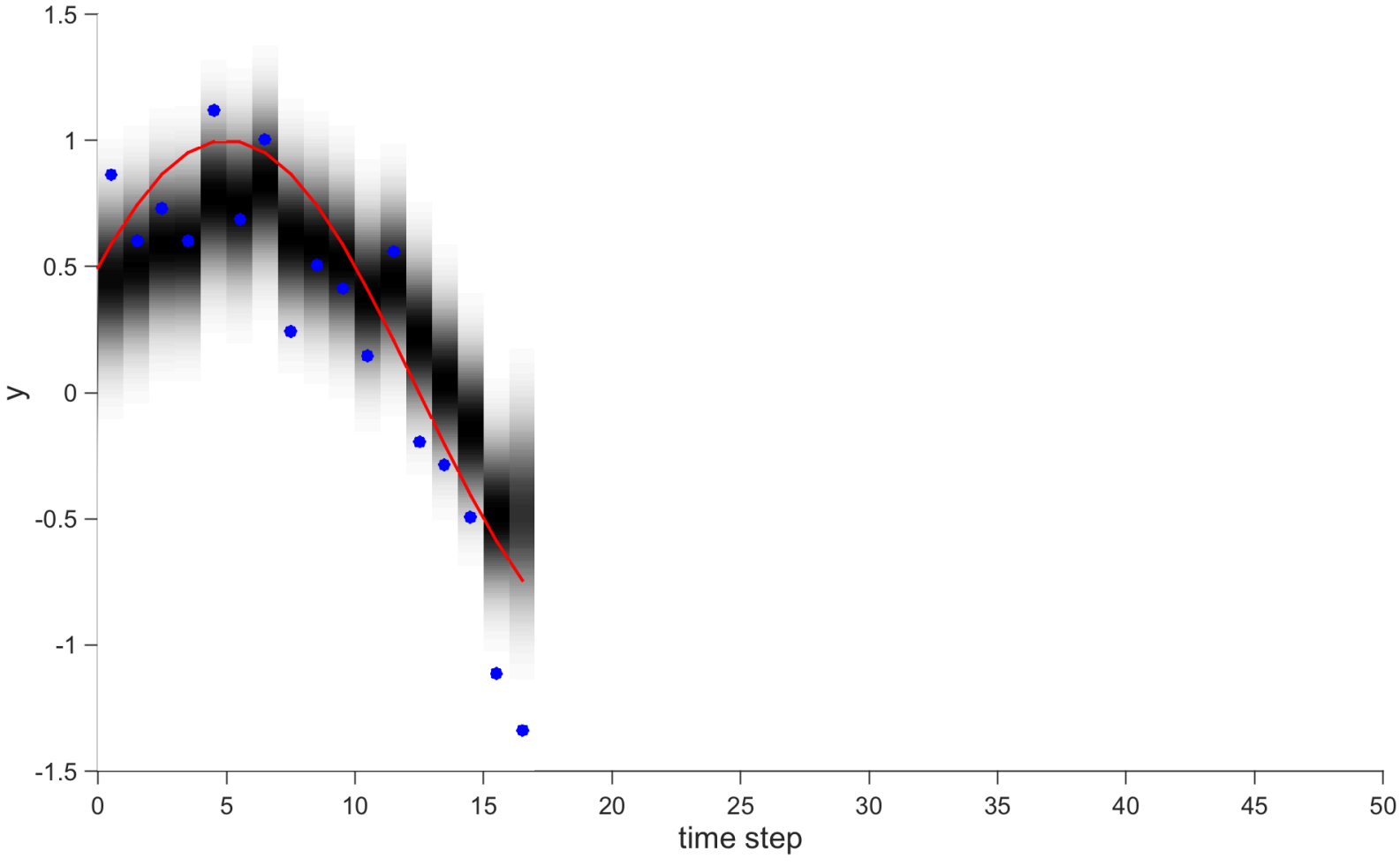
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



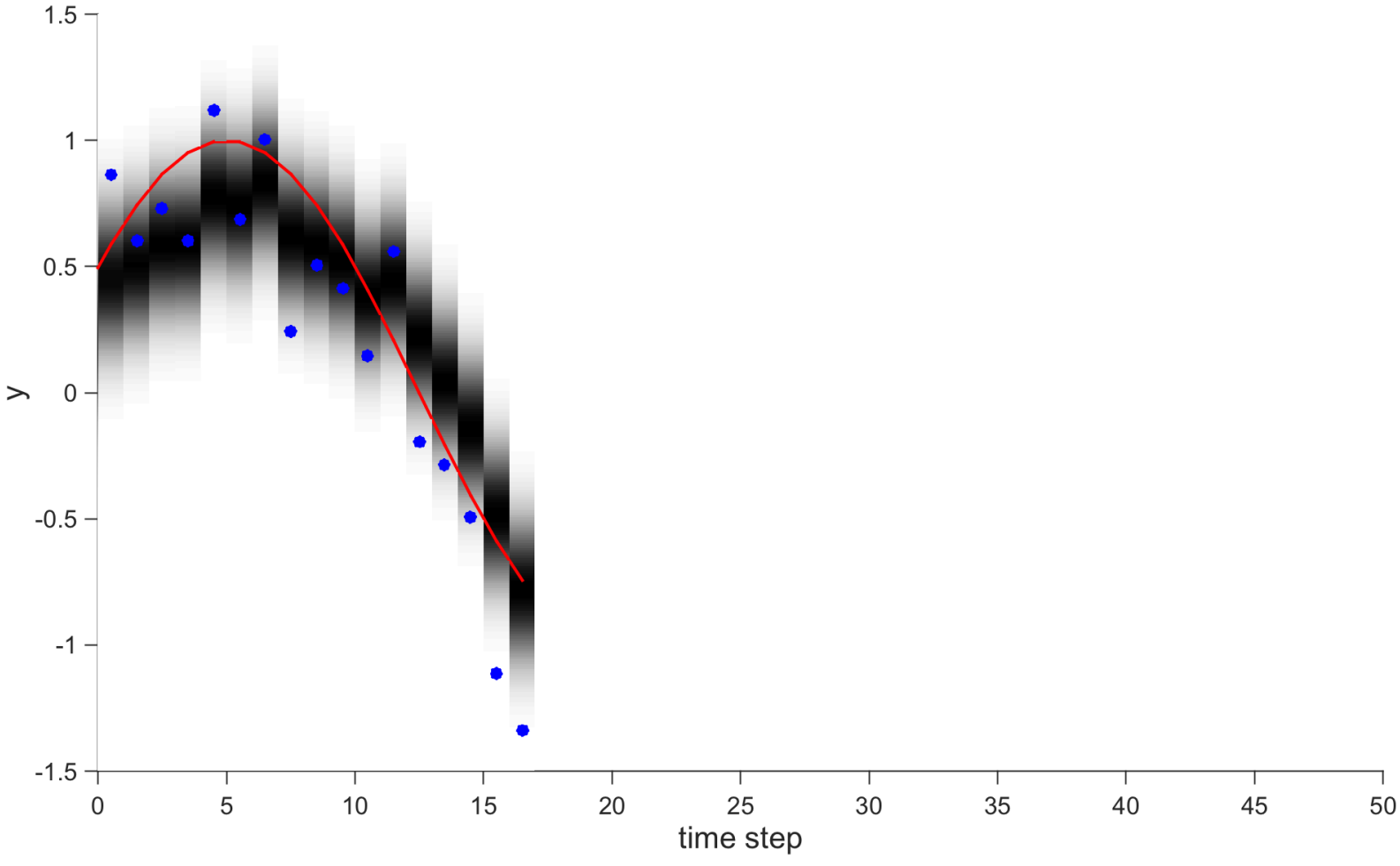
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



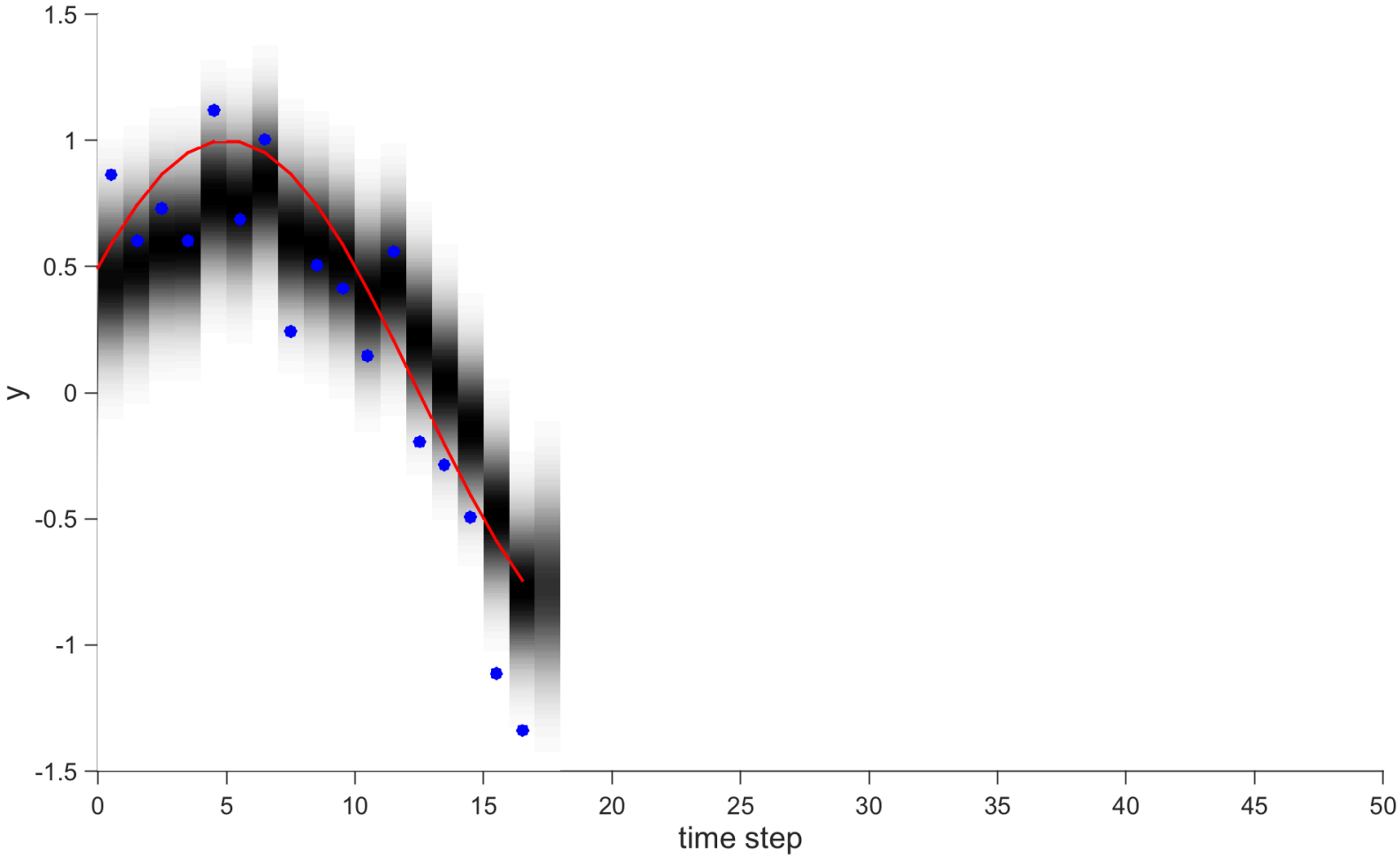
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



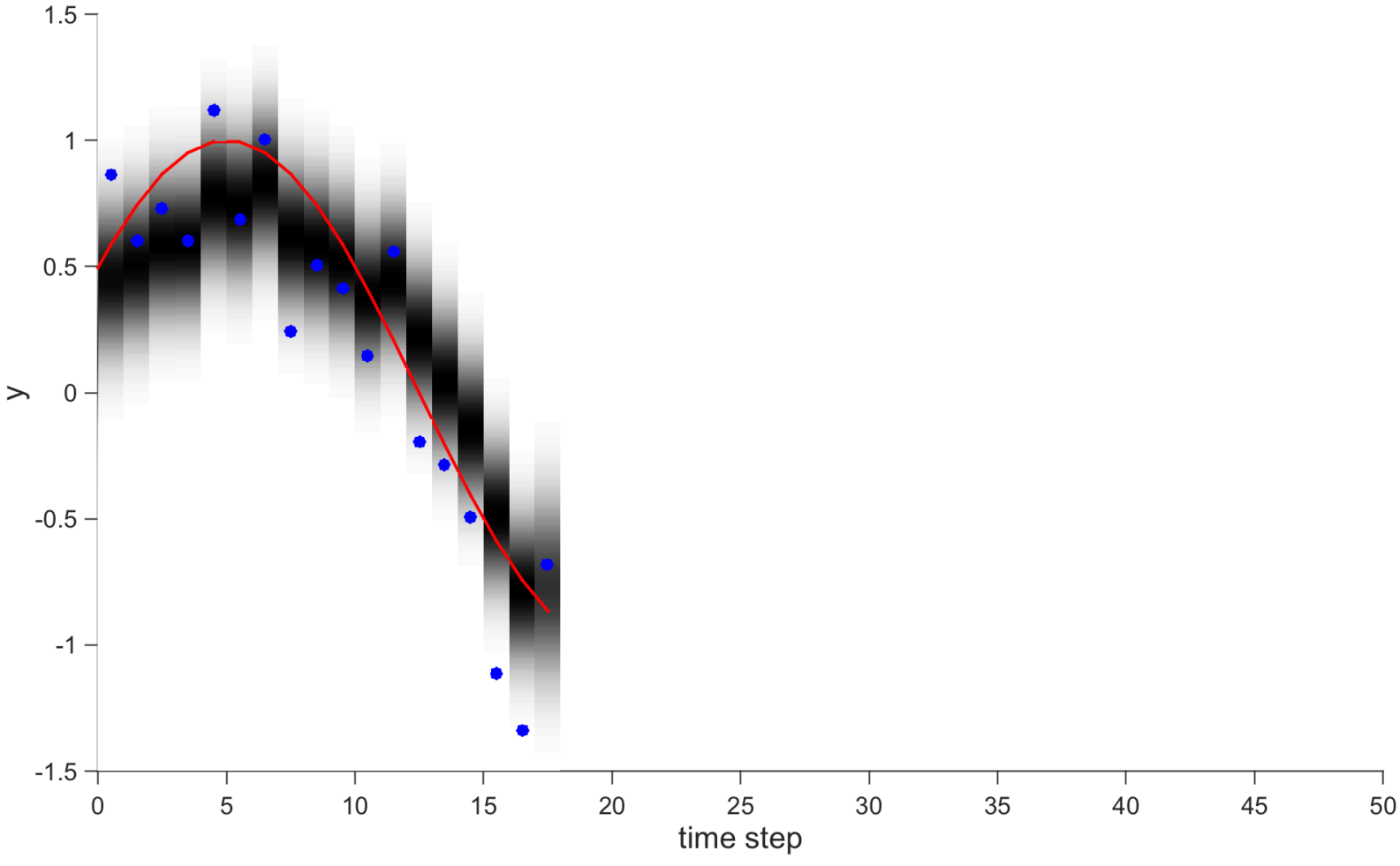
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



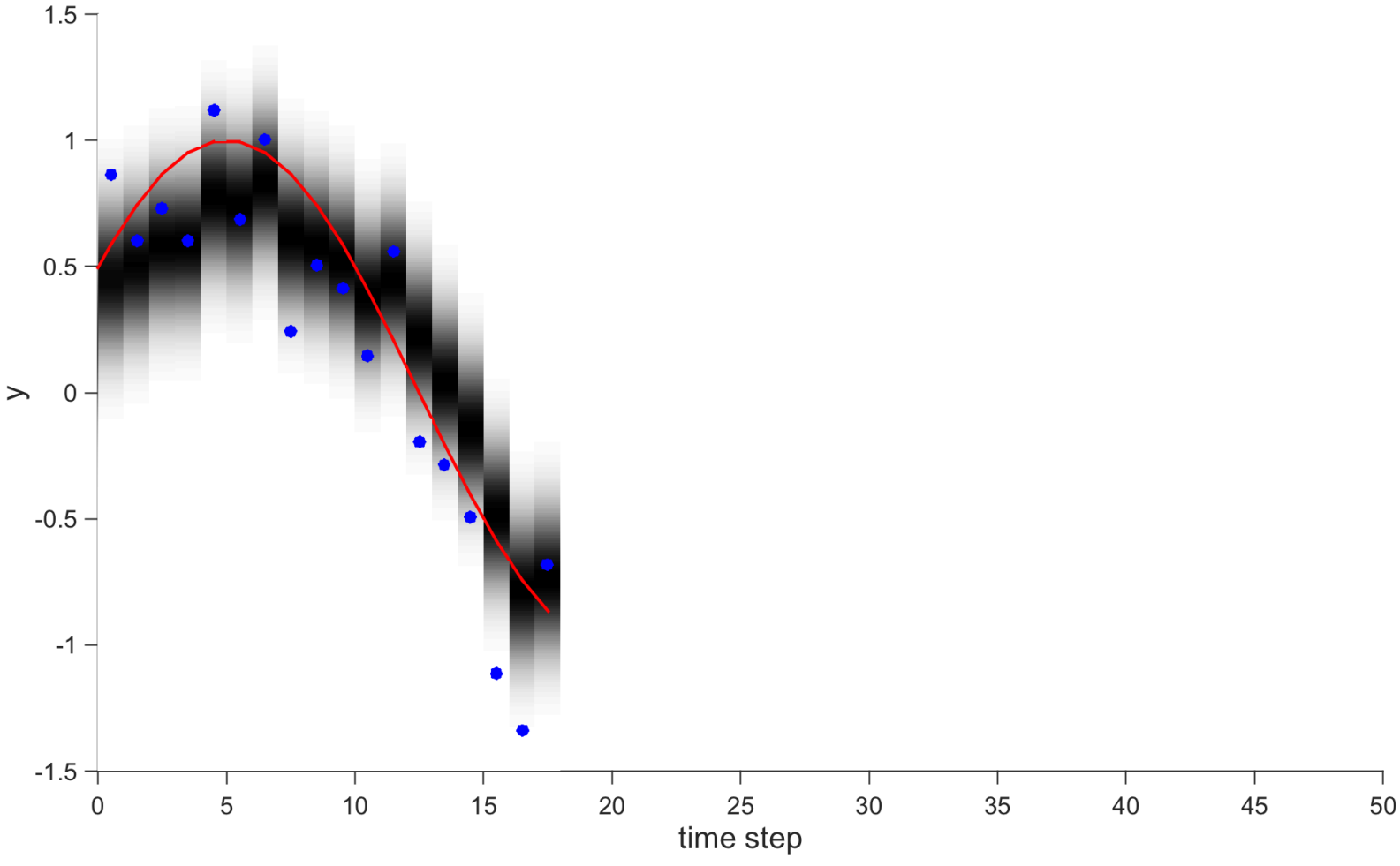
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



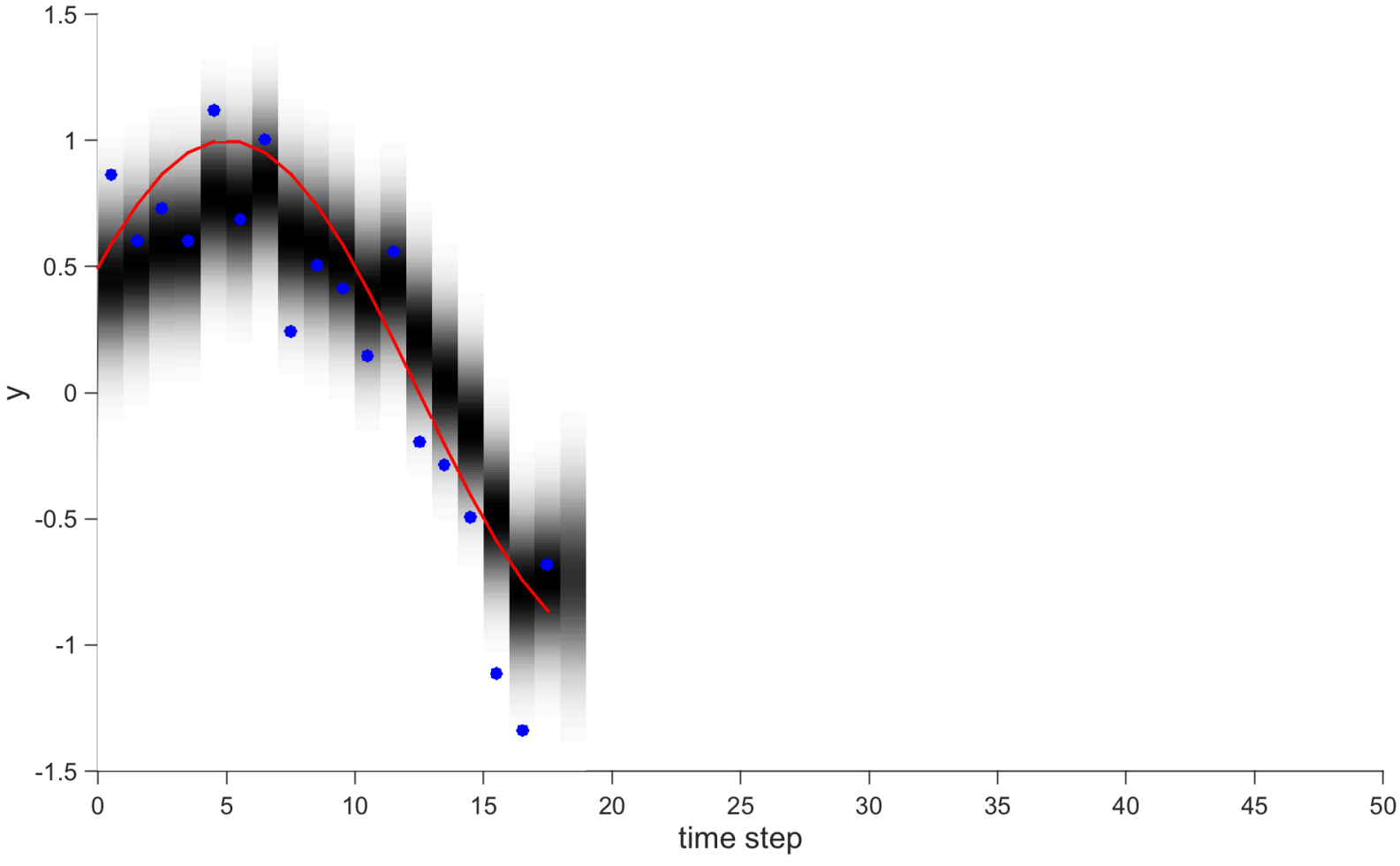
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



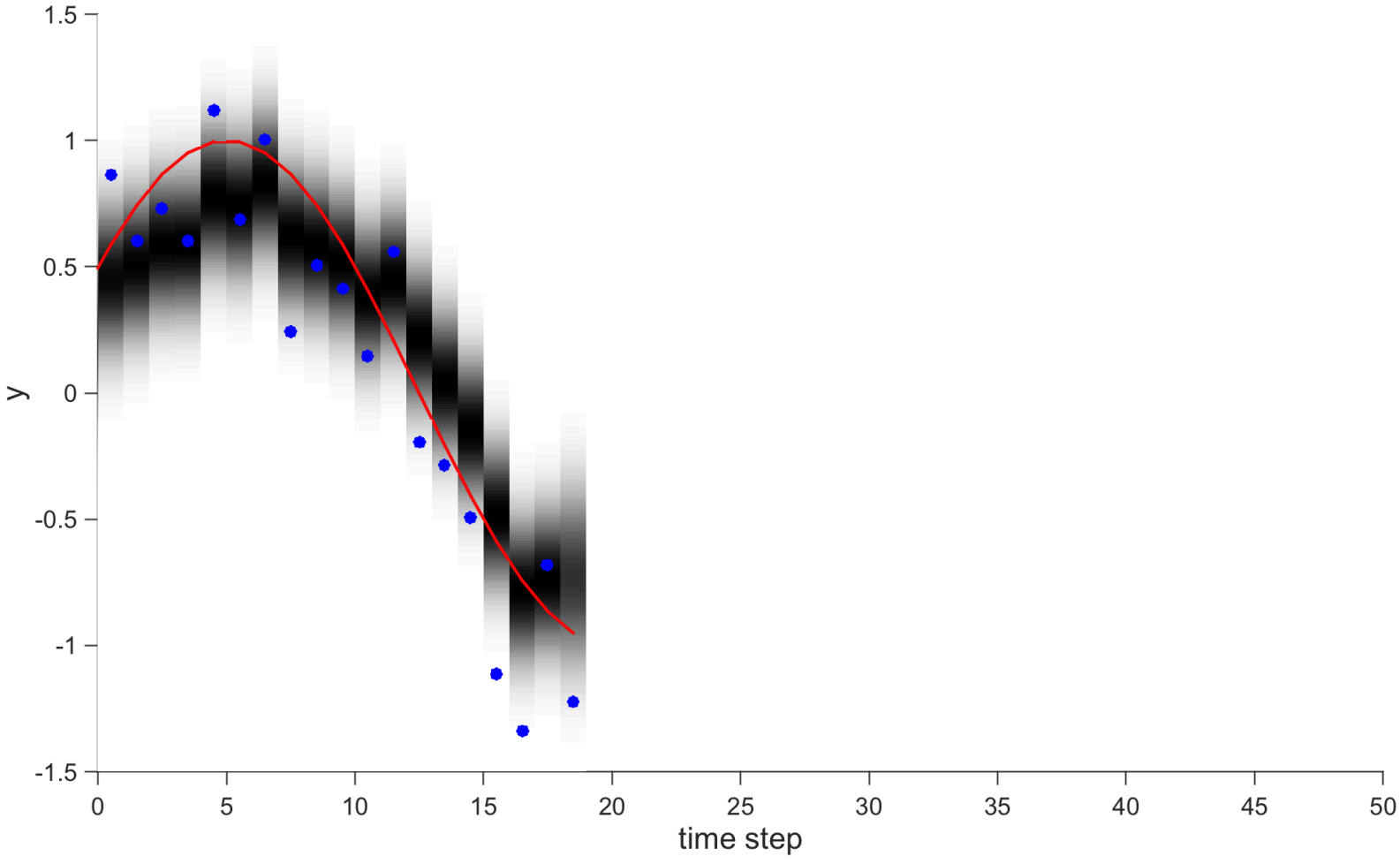
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



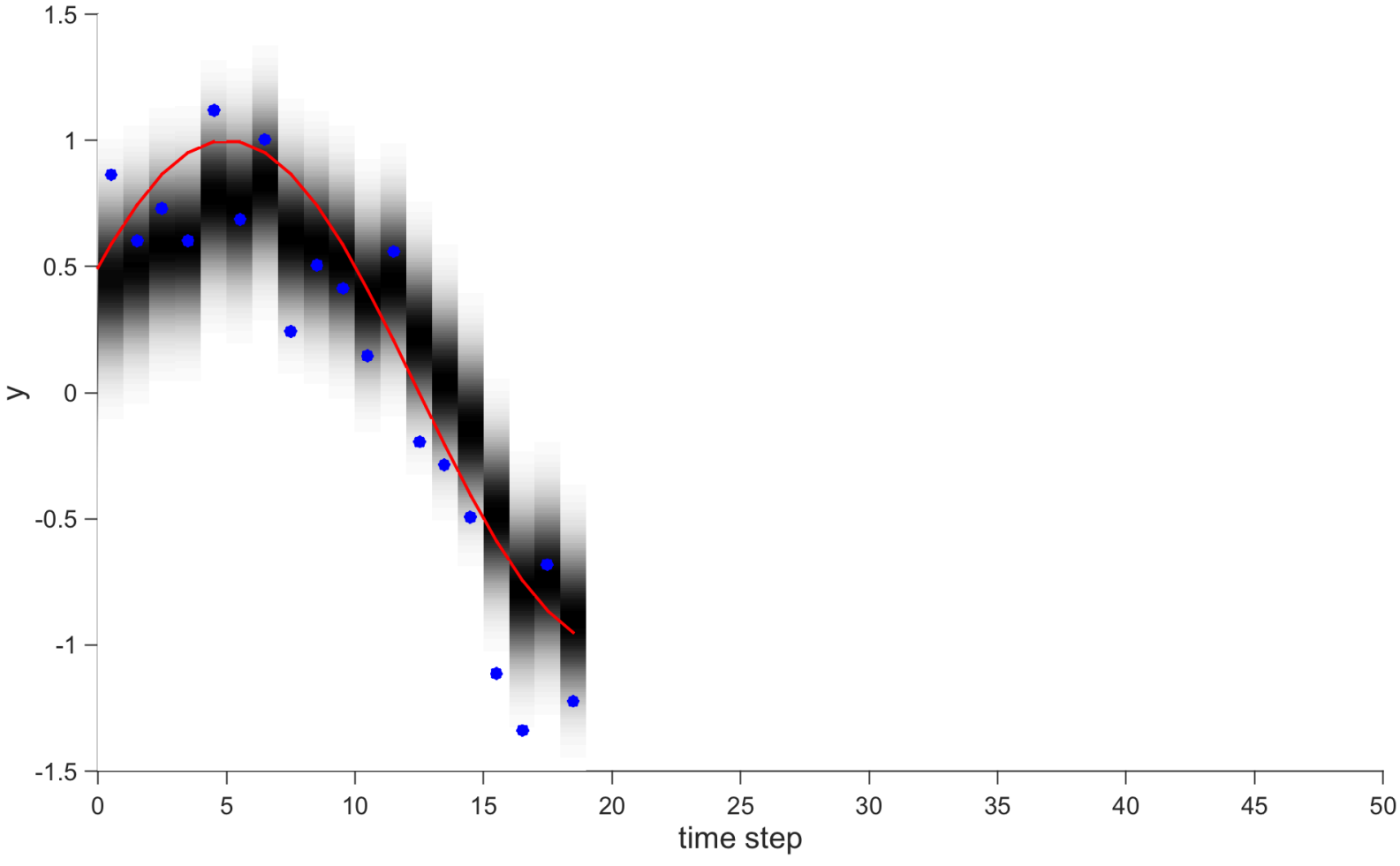
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



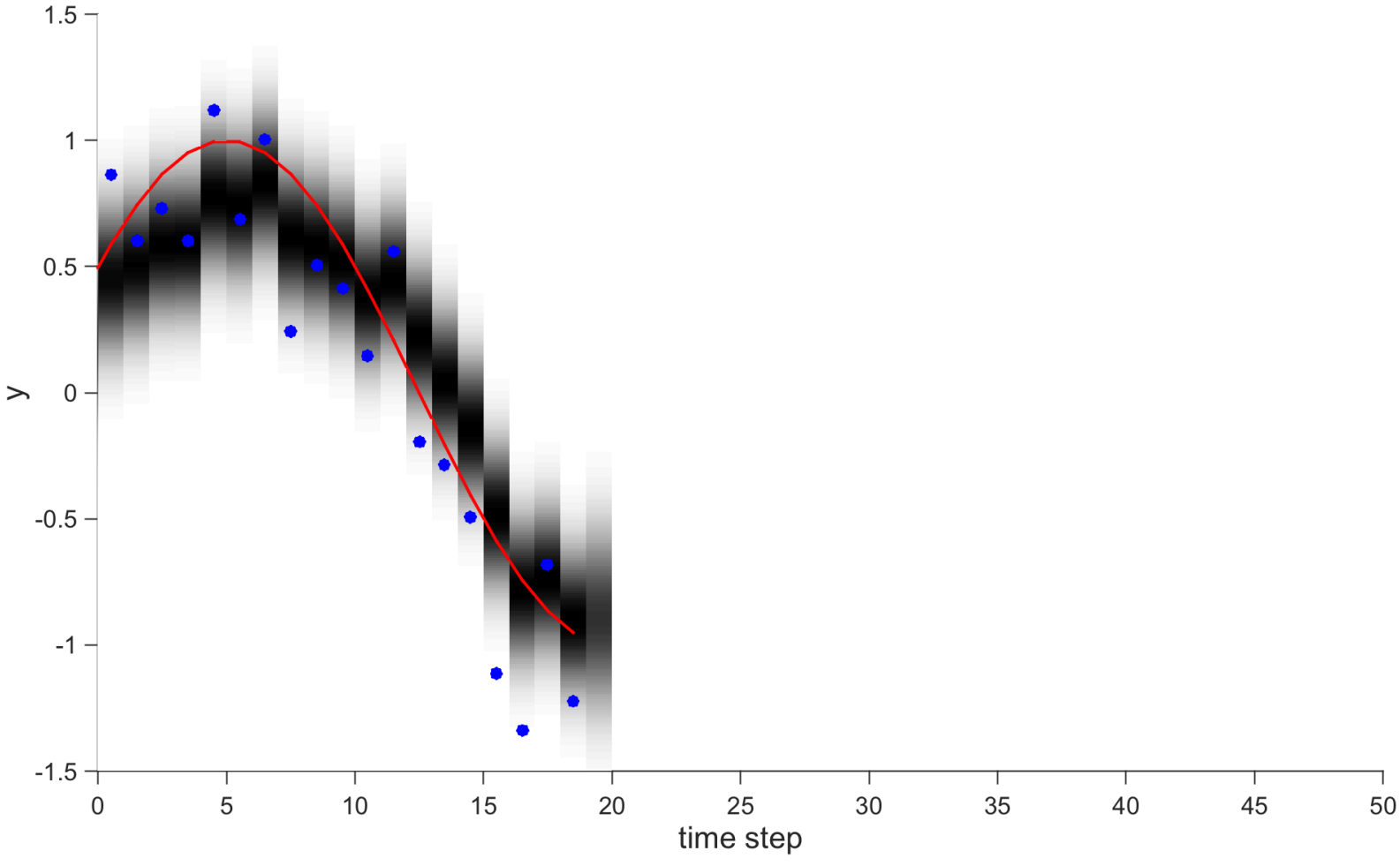
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



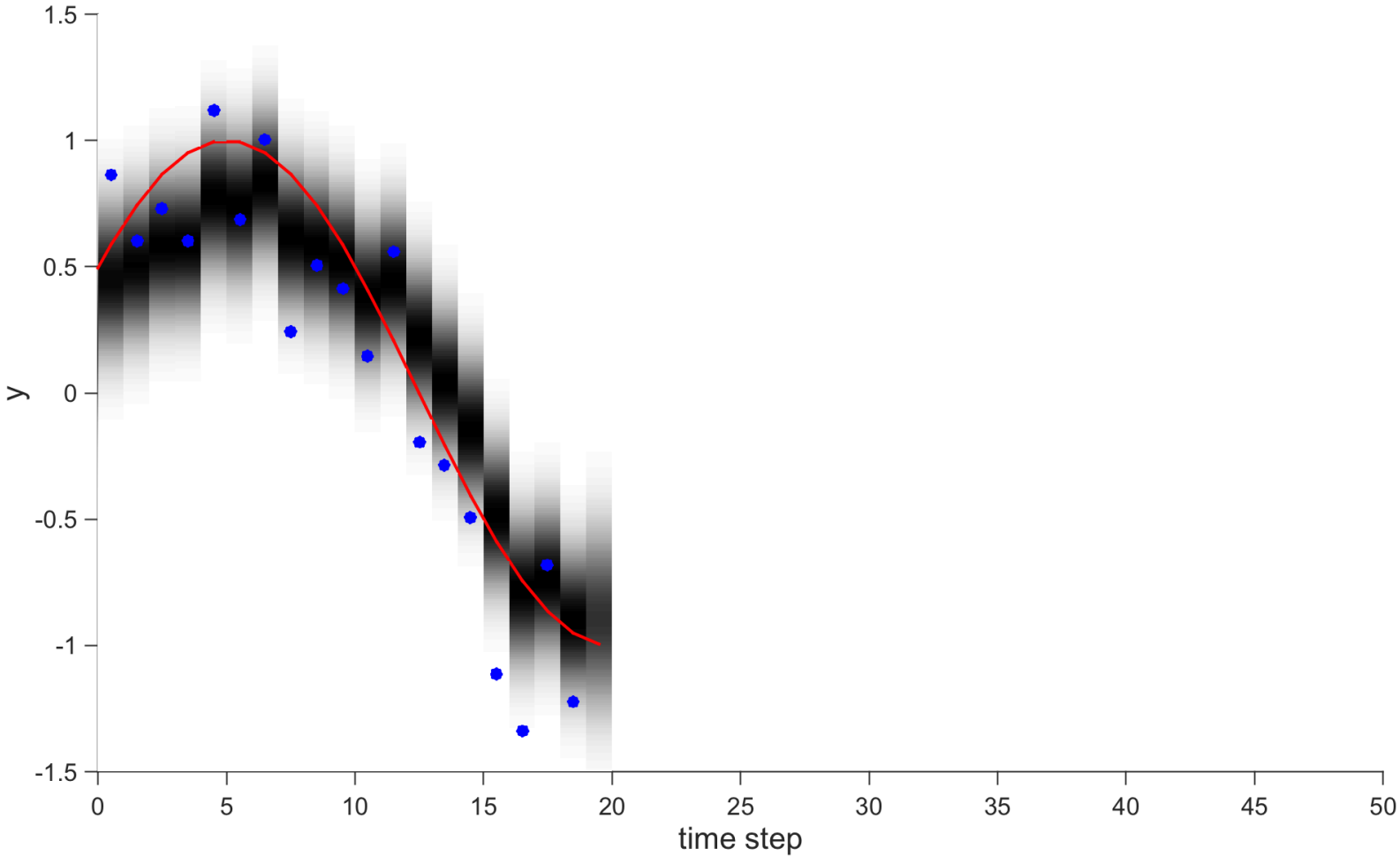
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



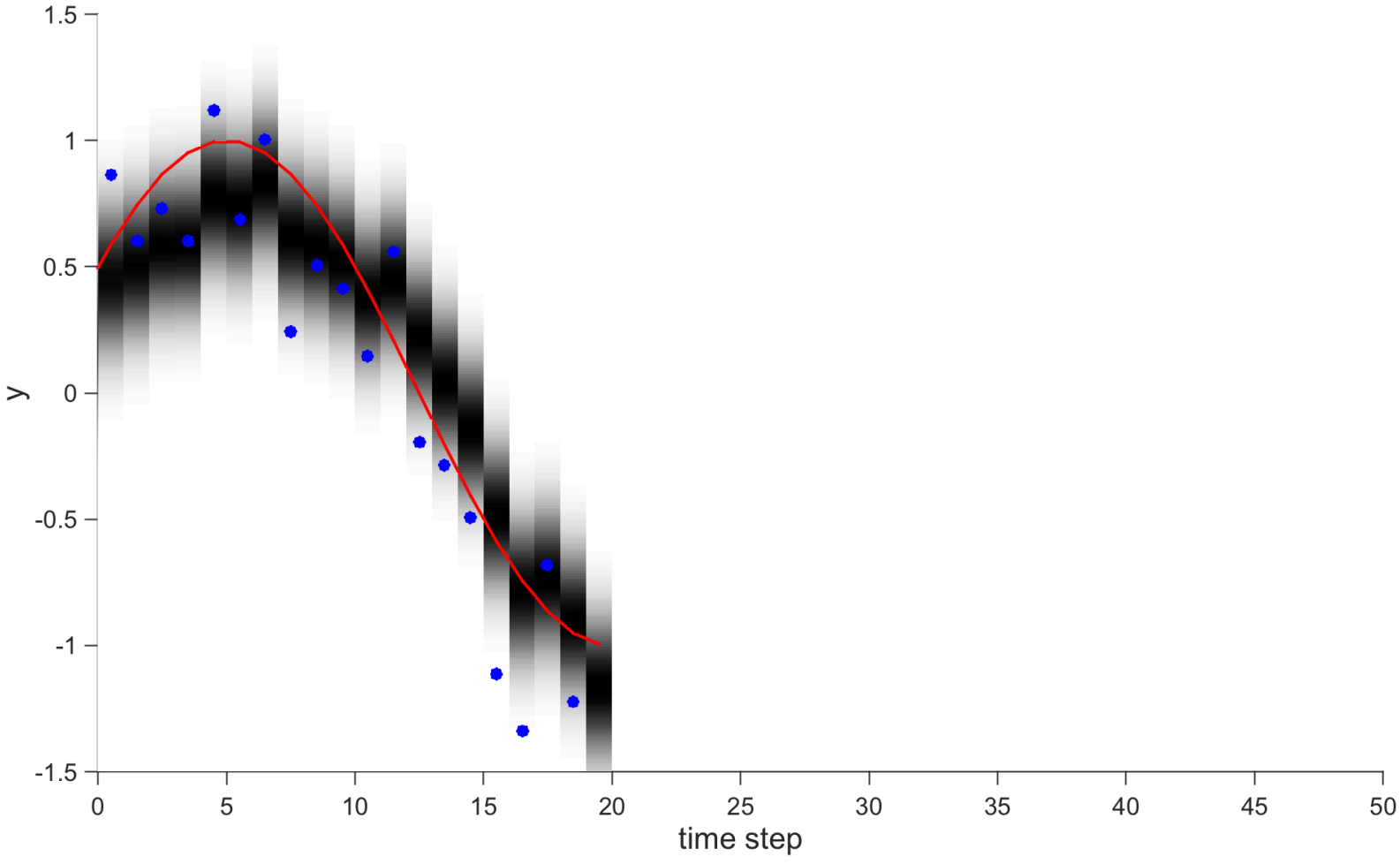
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



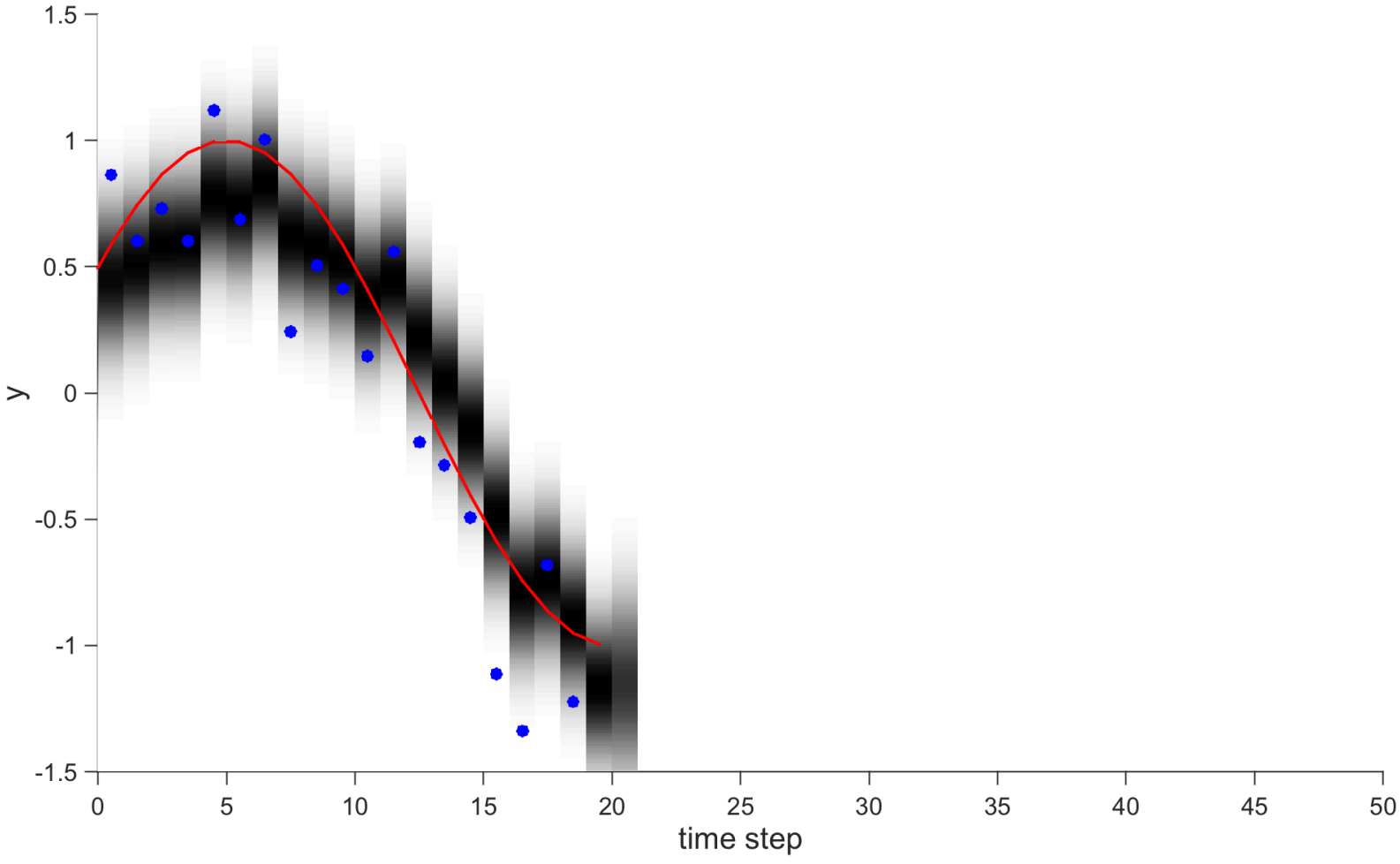
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



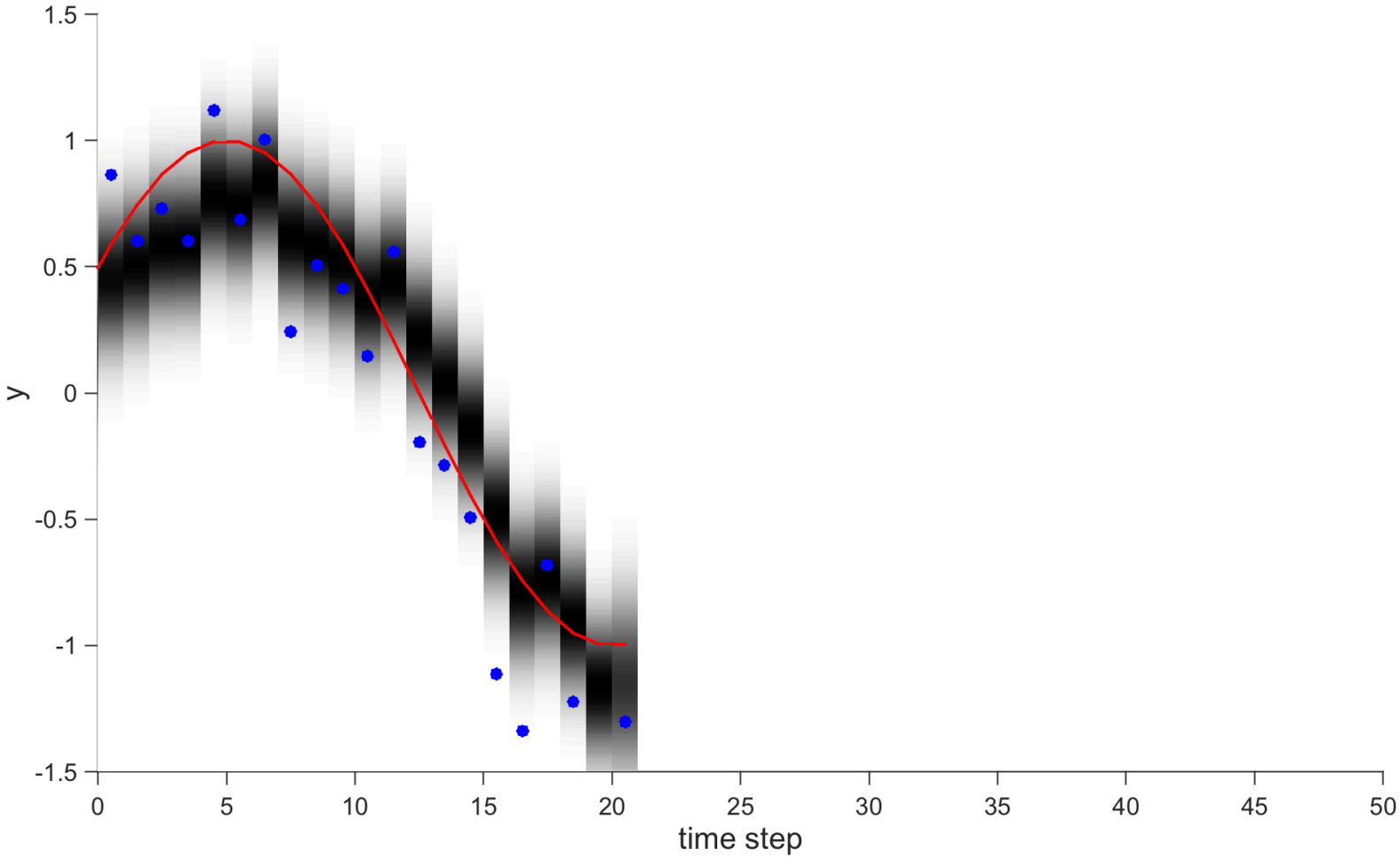
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



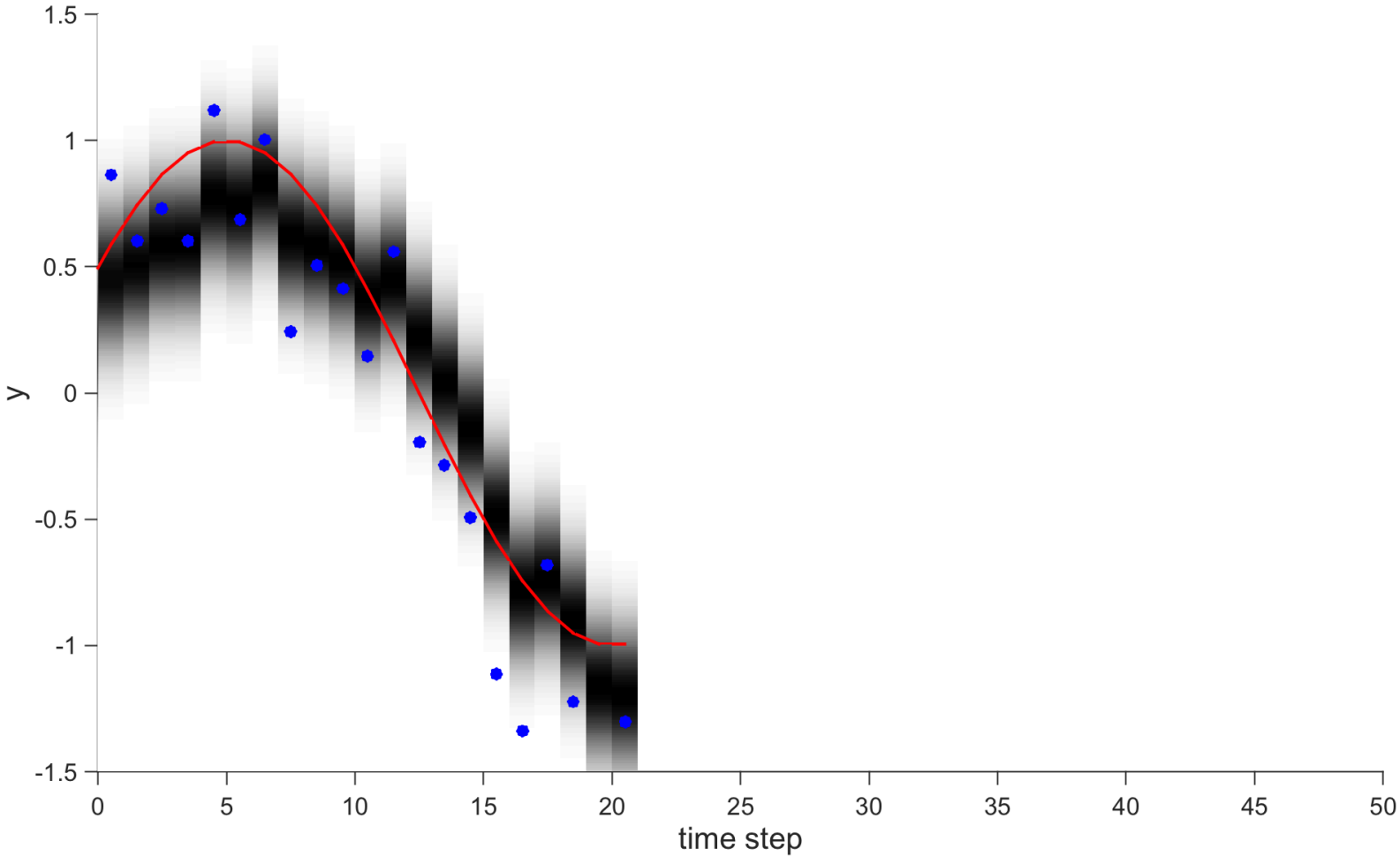
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



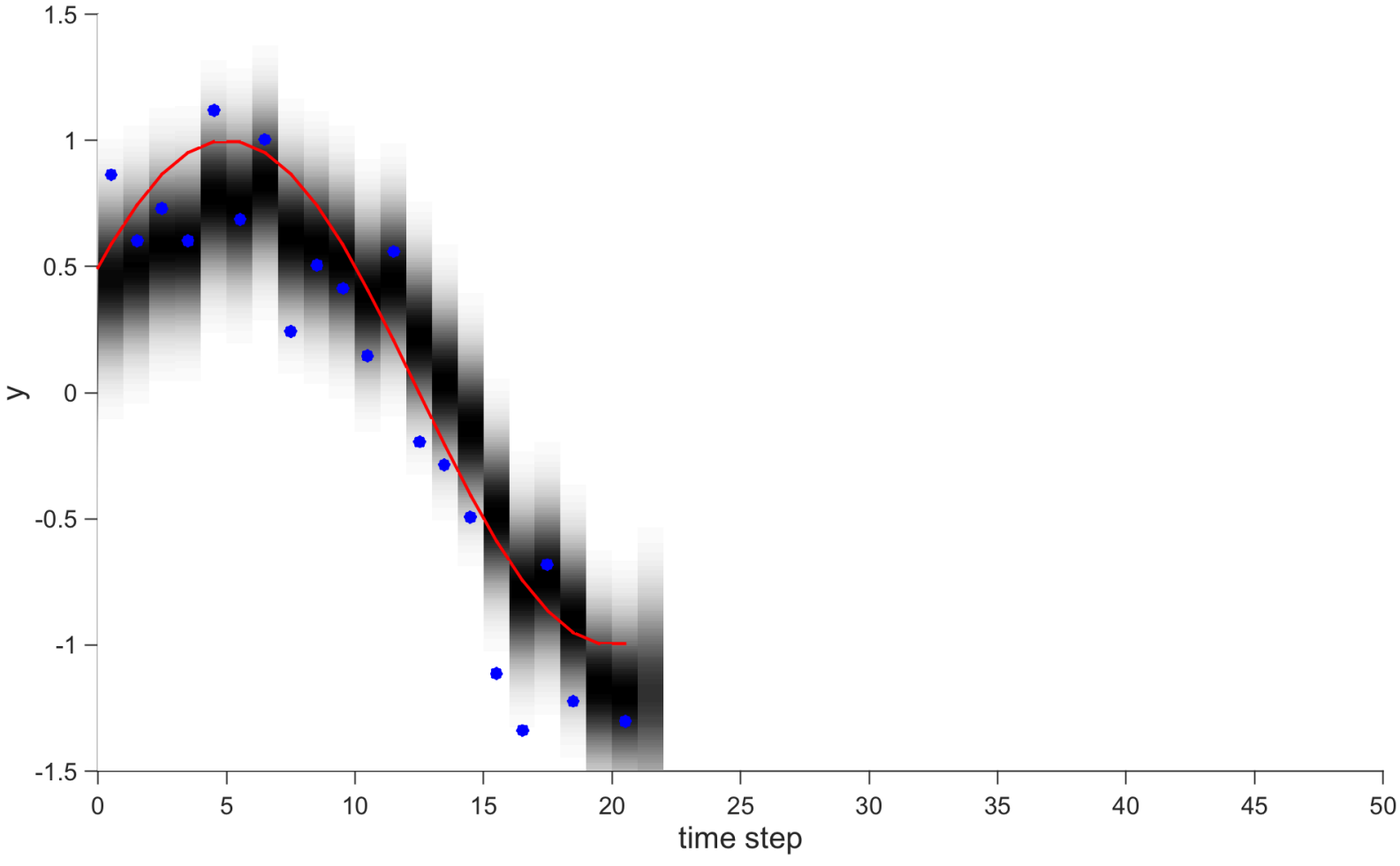
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



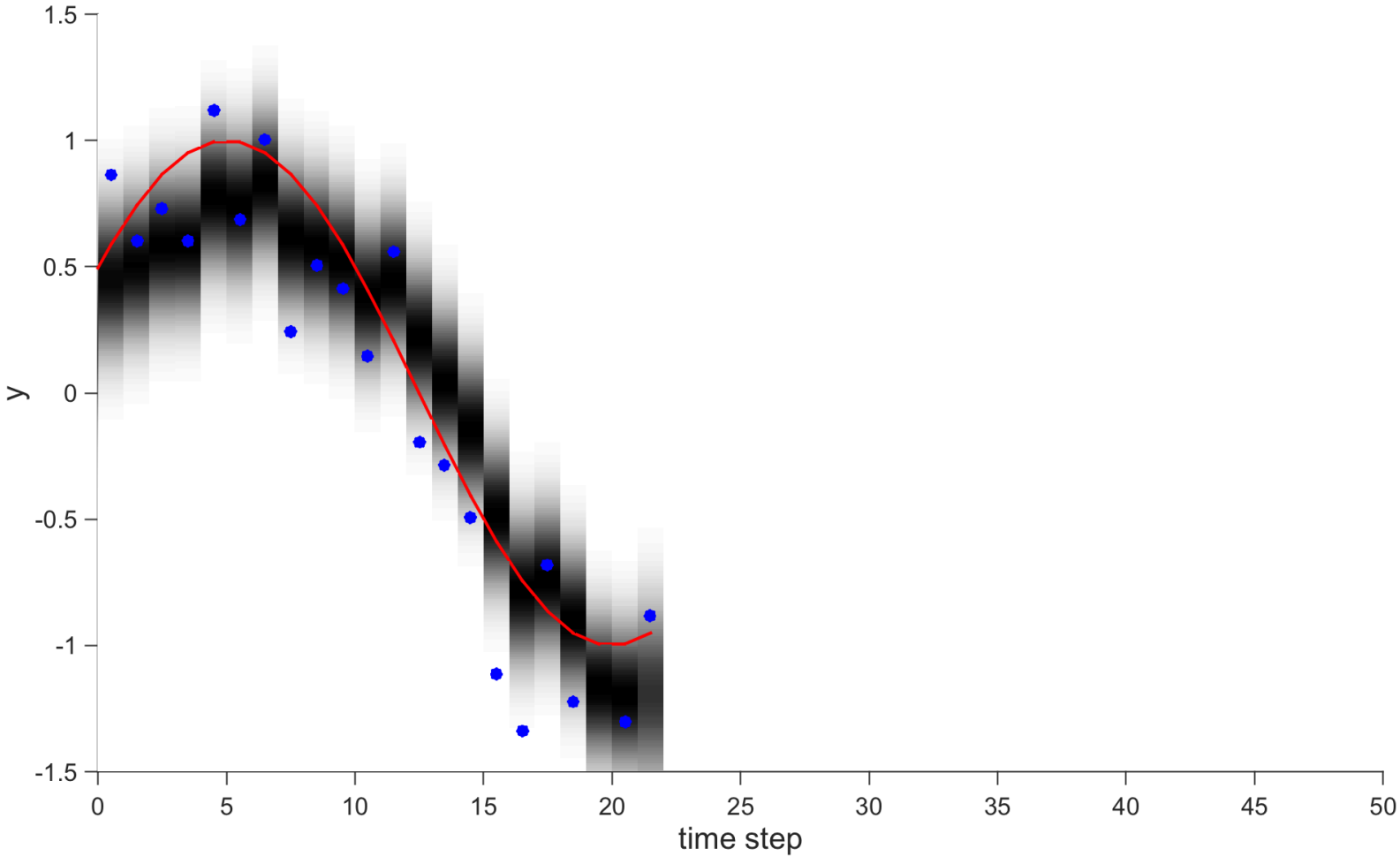
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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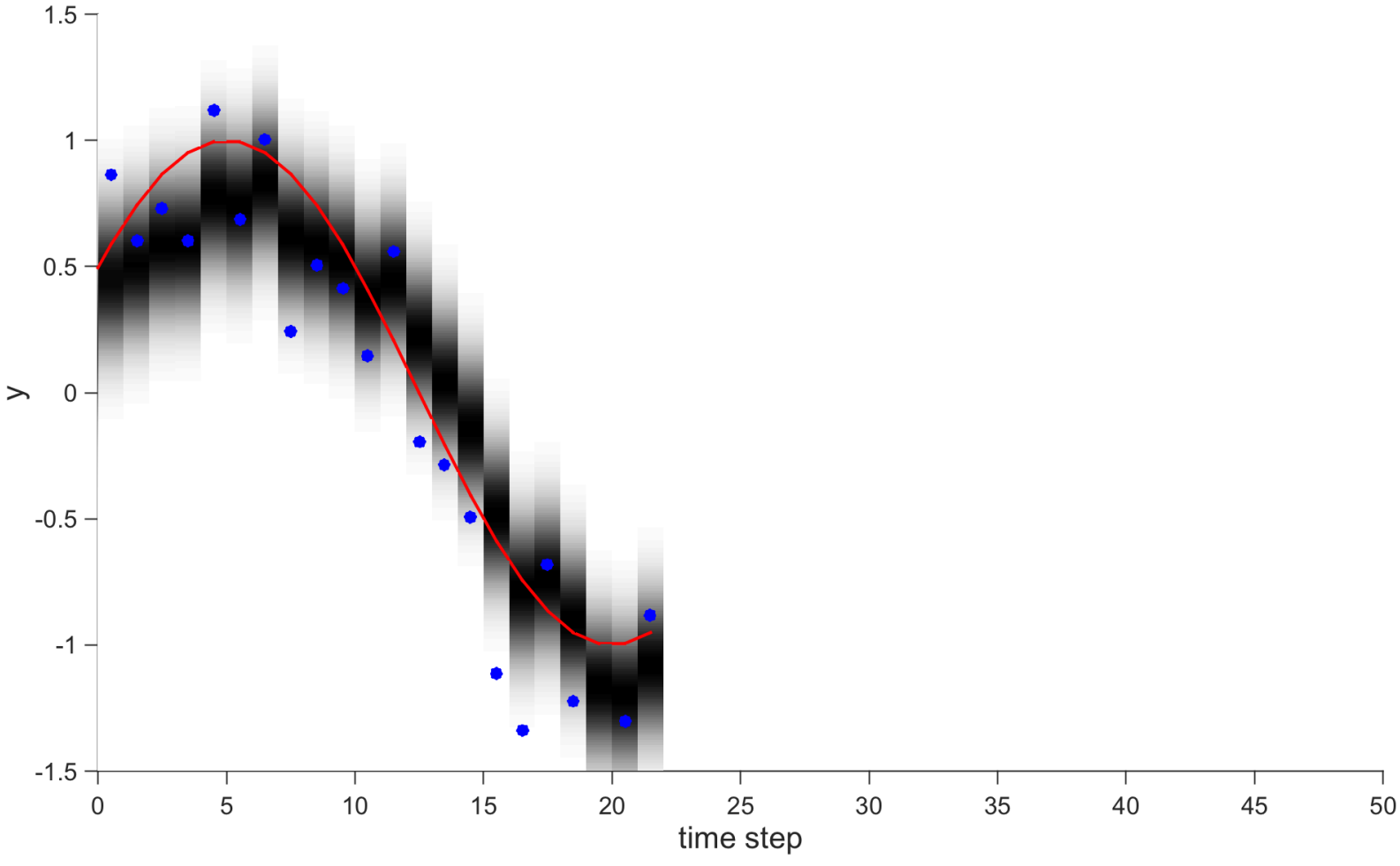
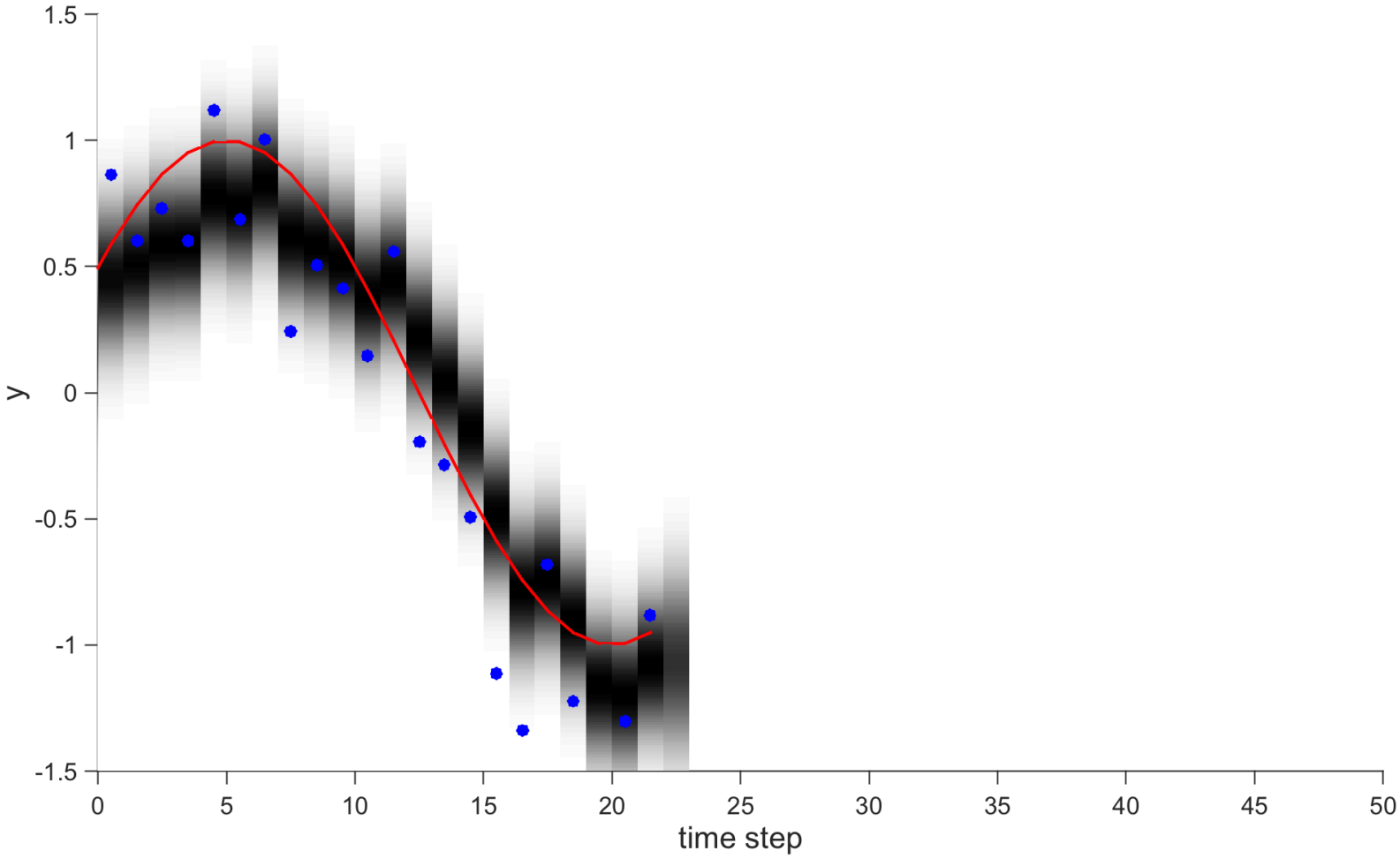


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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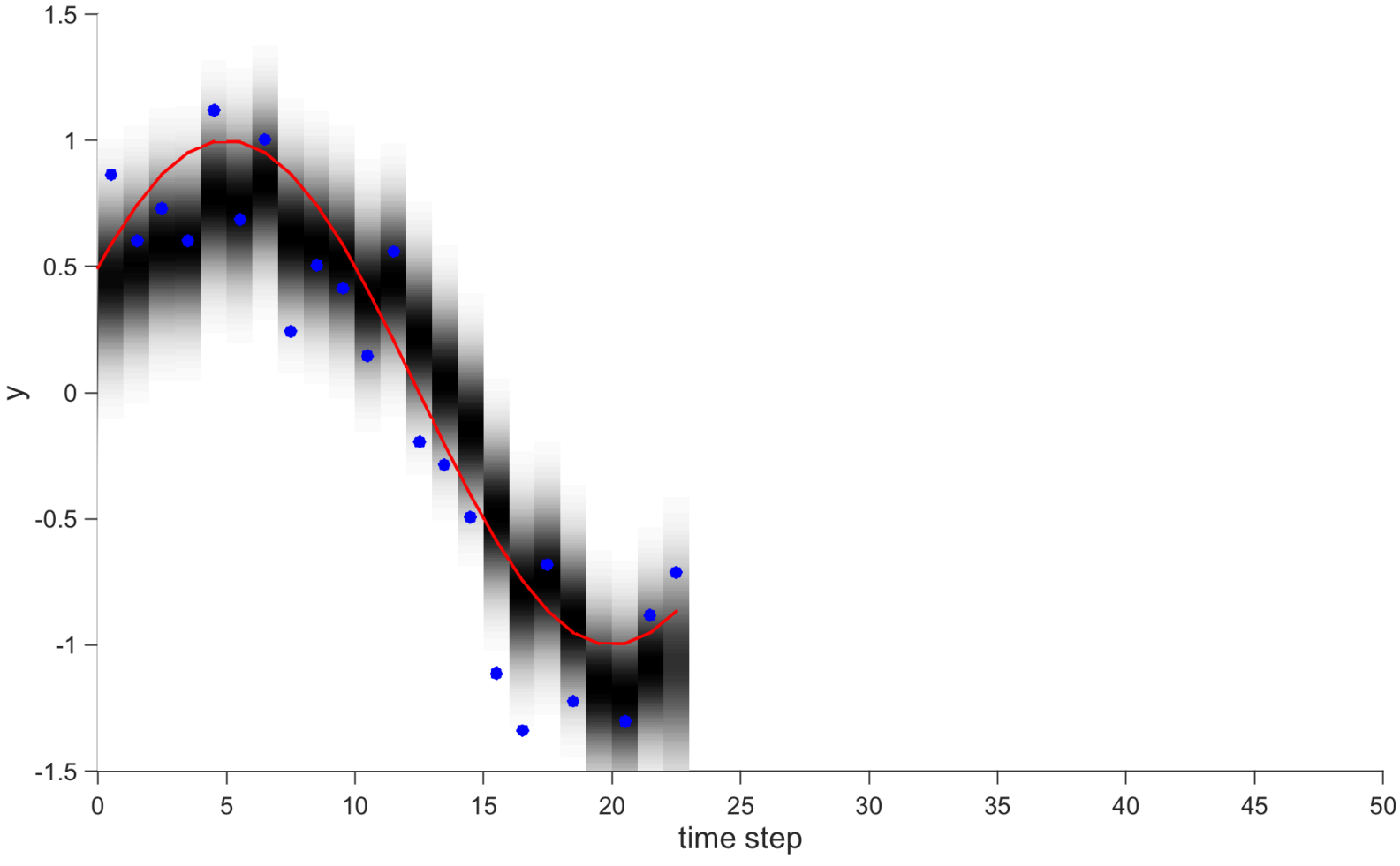

Kalman Filter Demo

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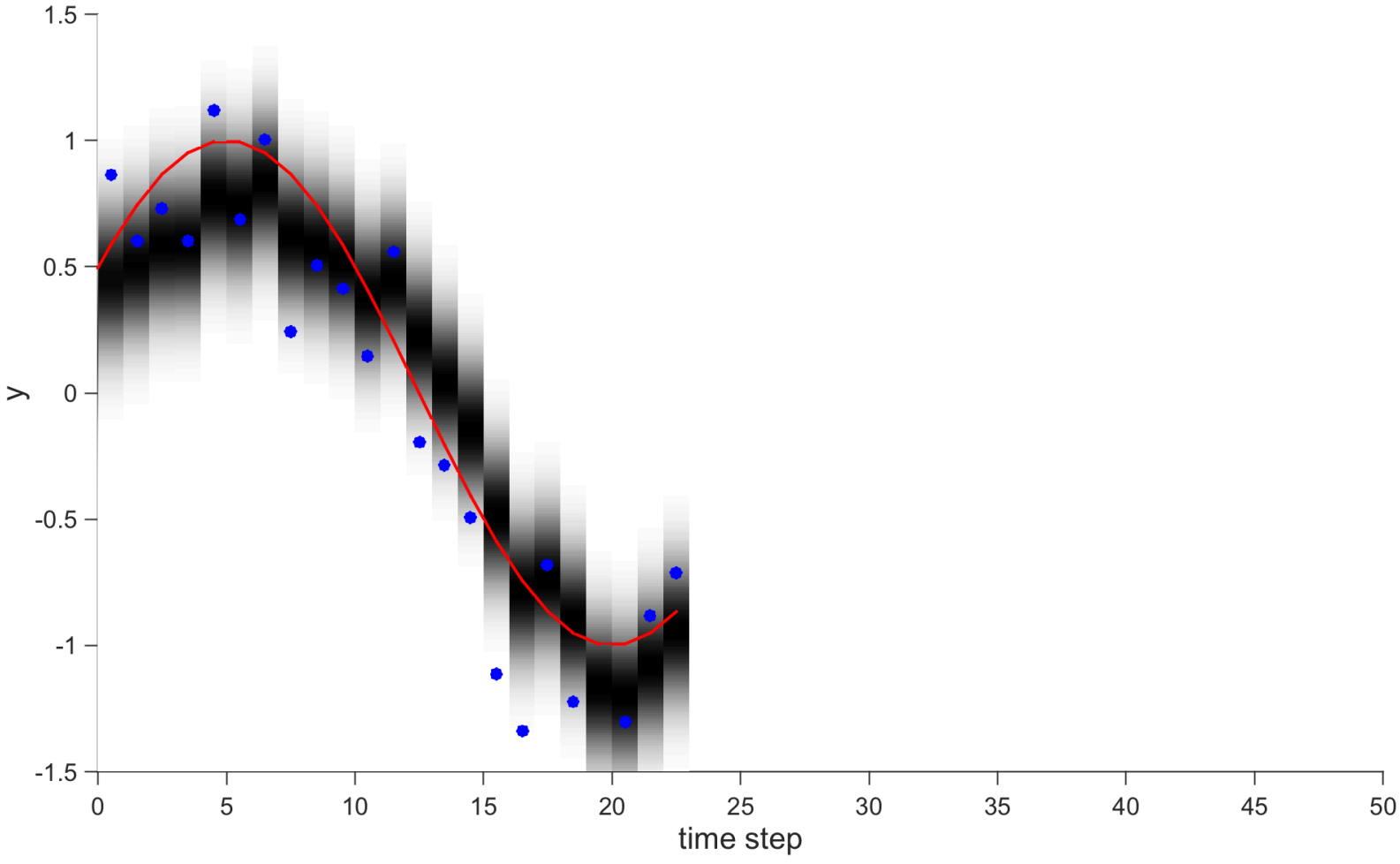

Kalman Filter Demo

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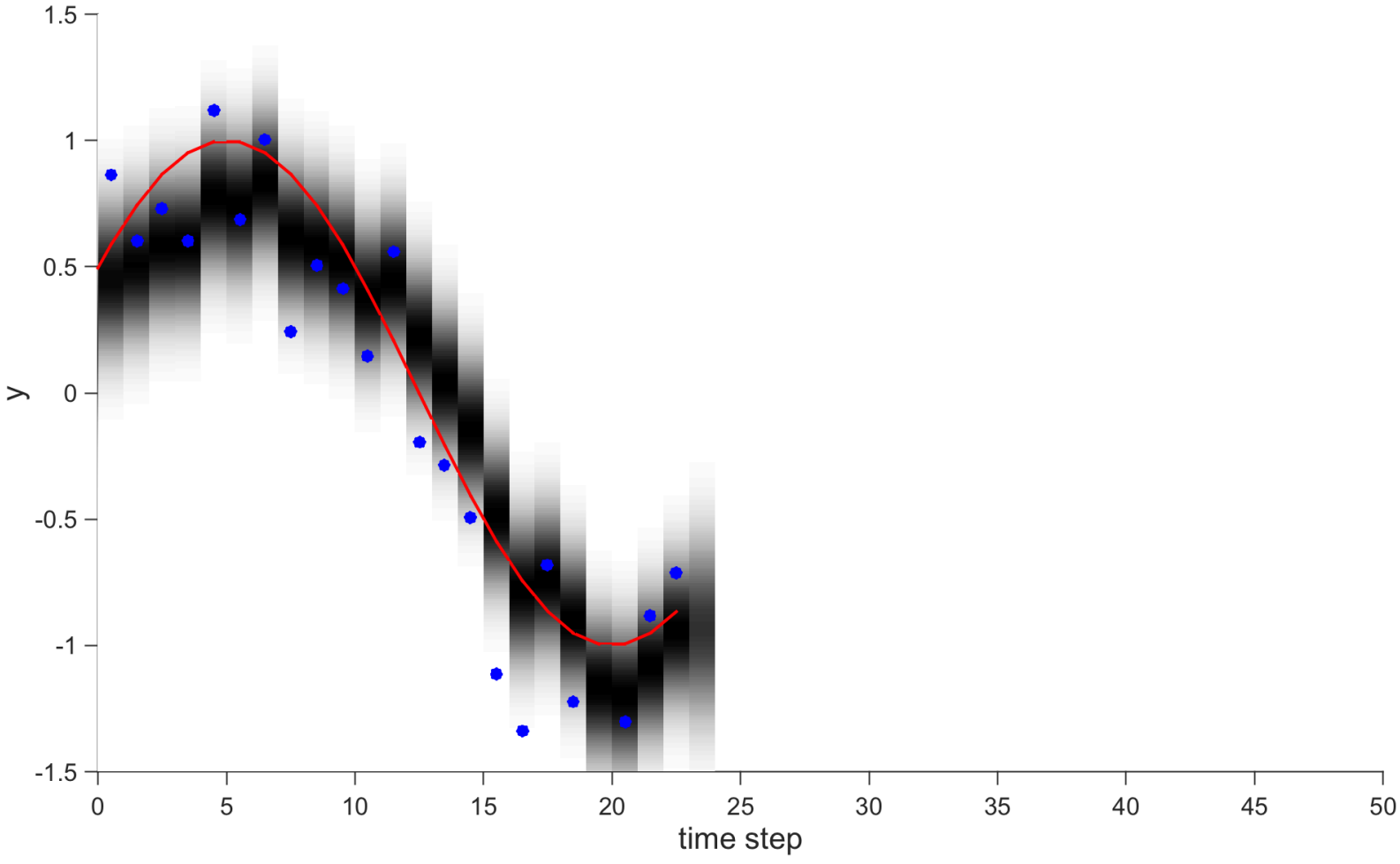
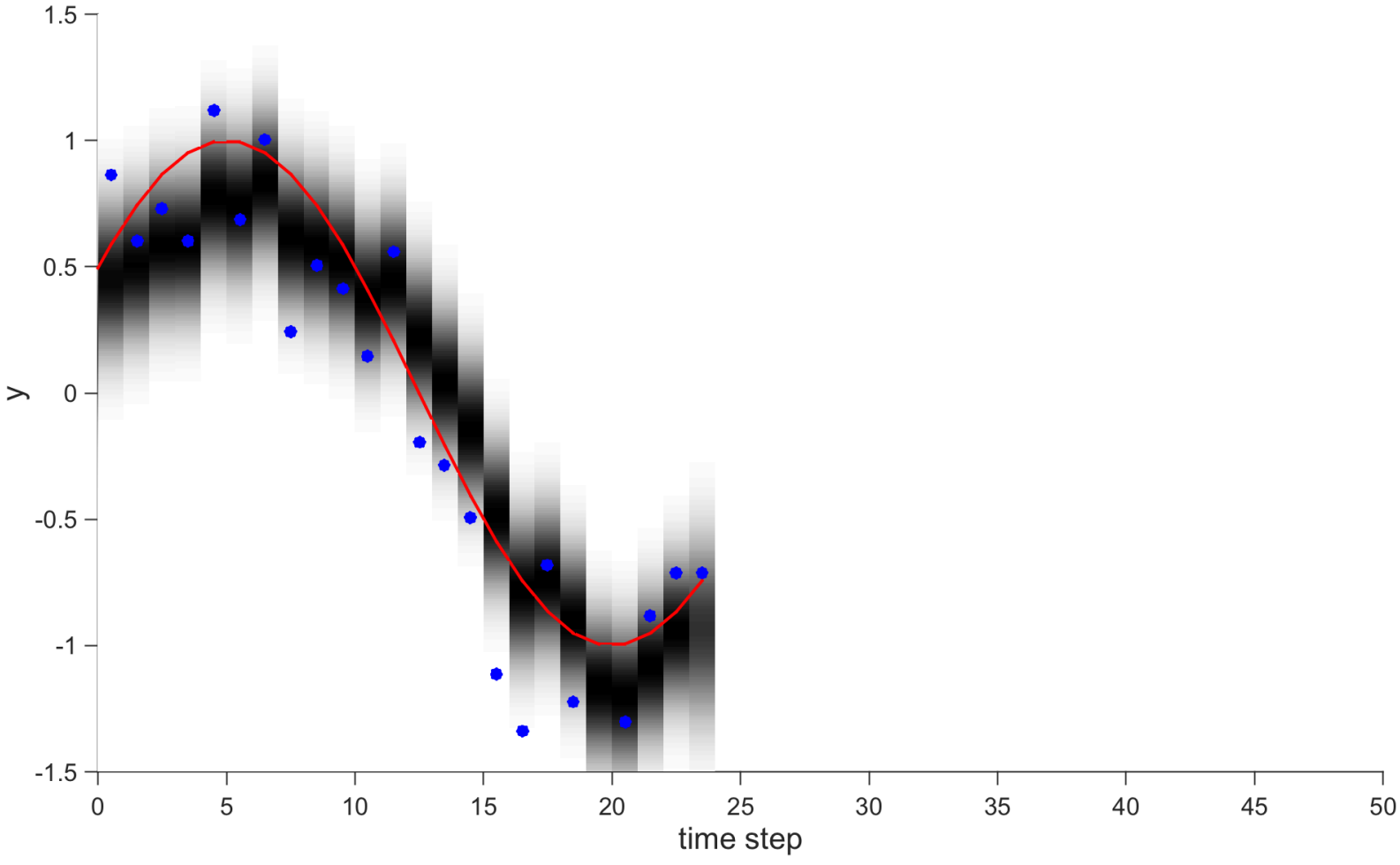


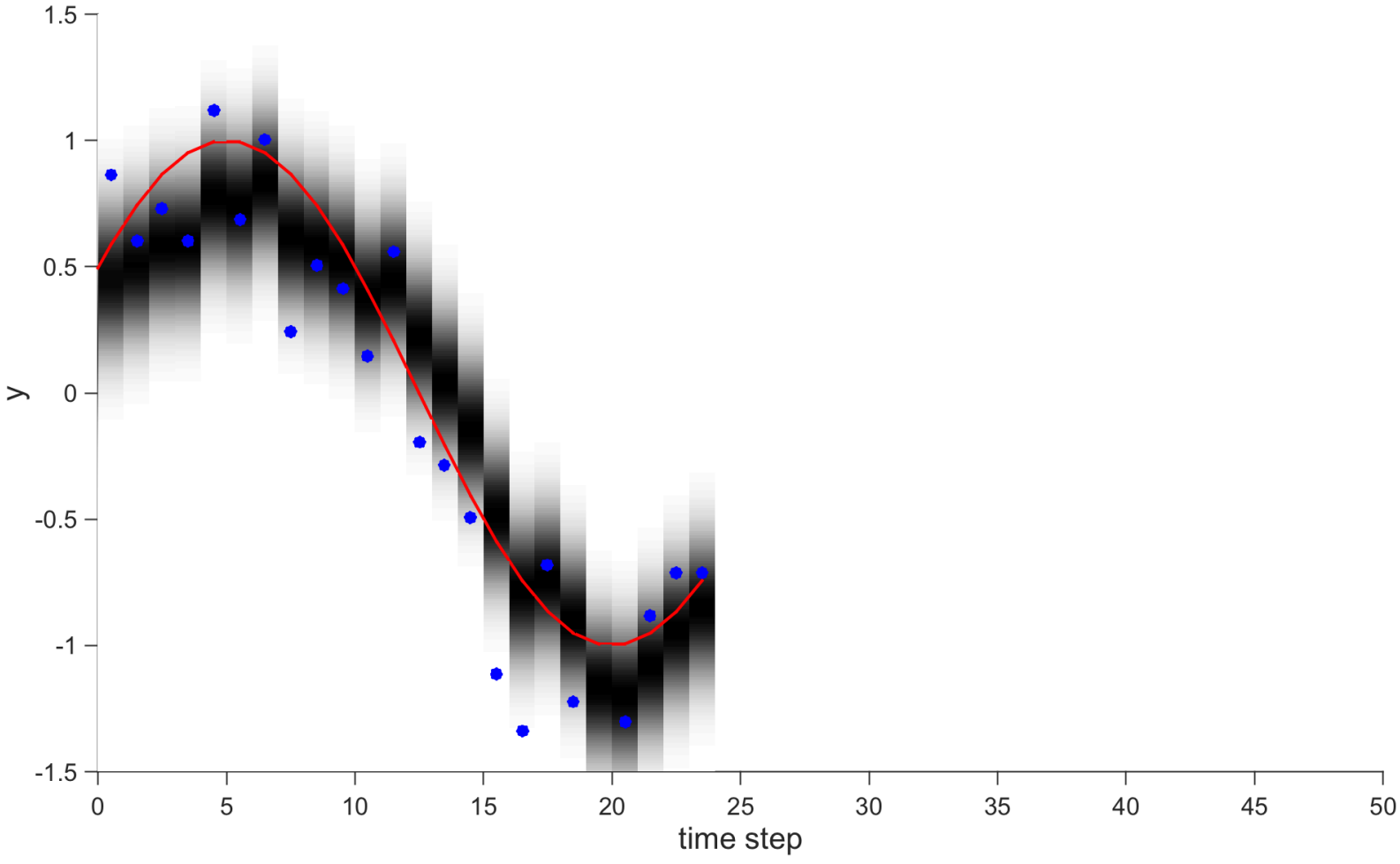
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



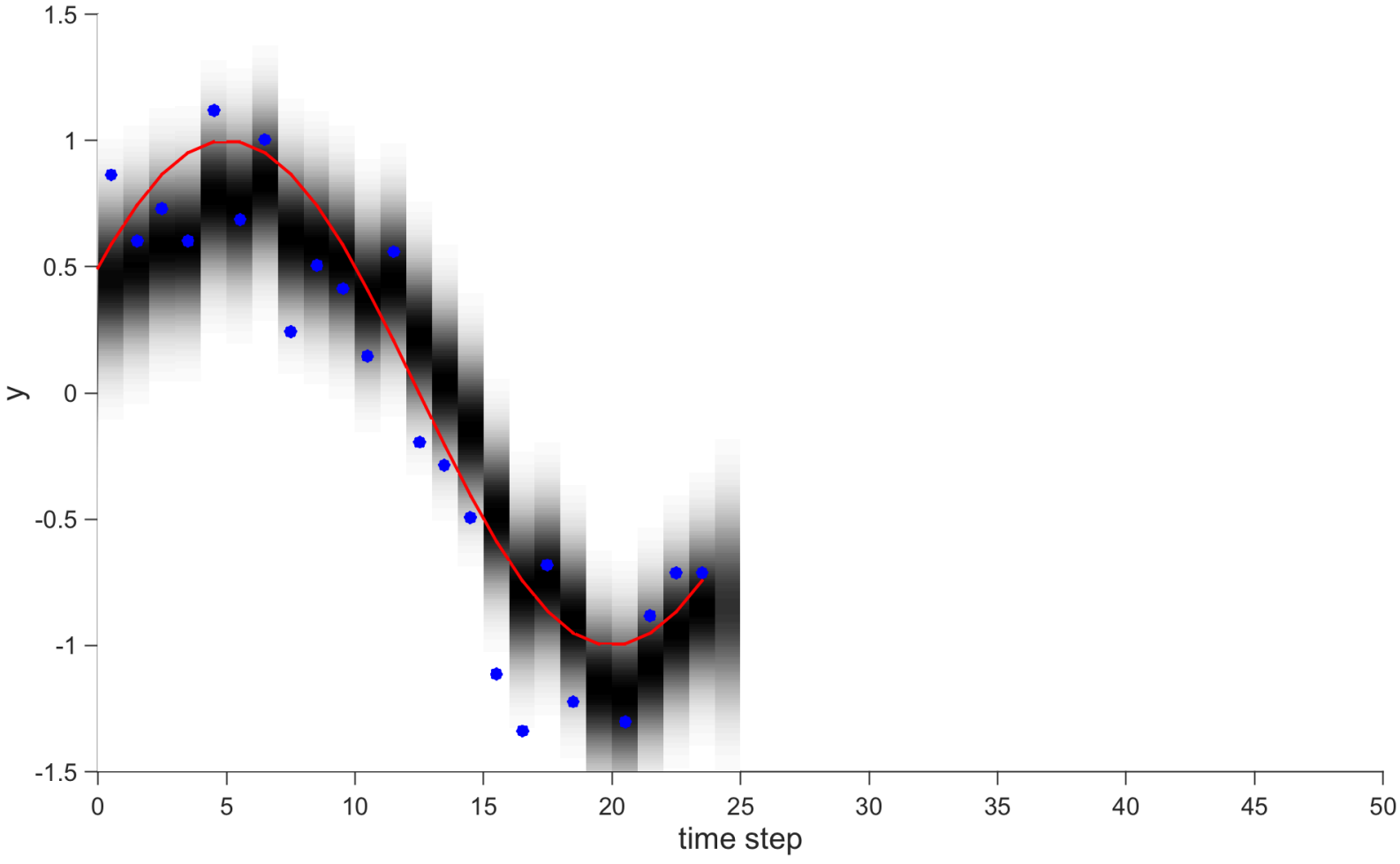
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



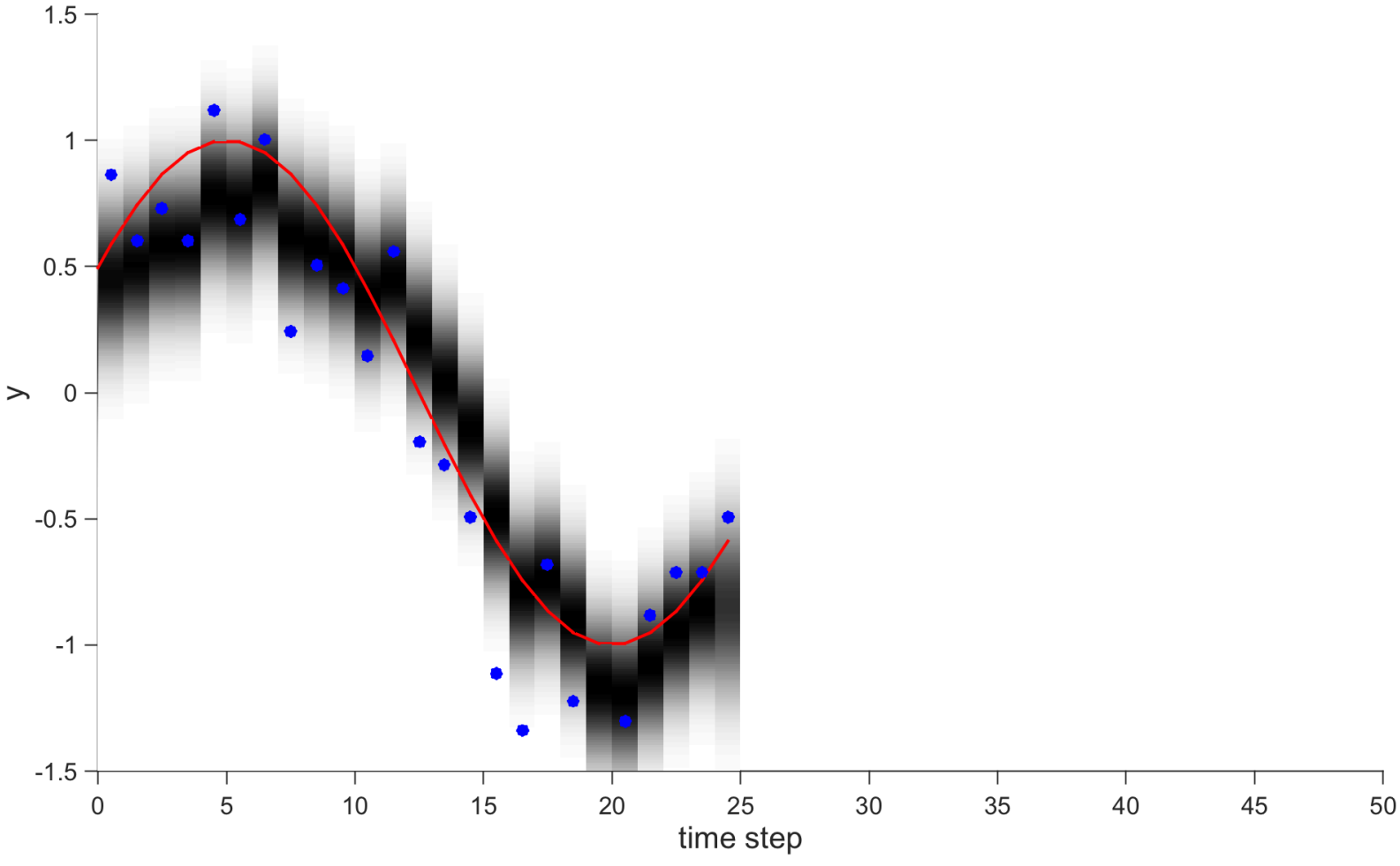
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



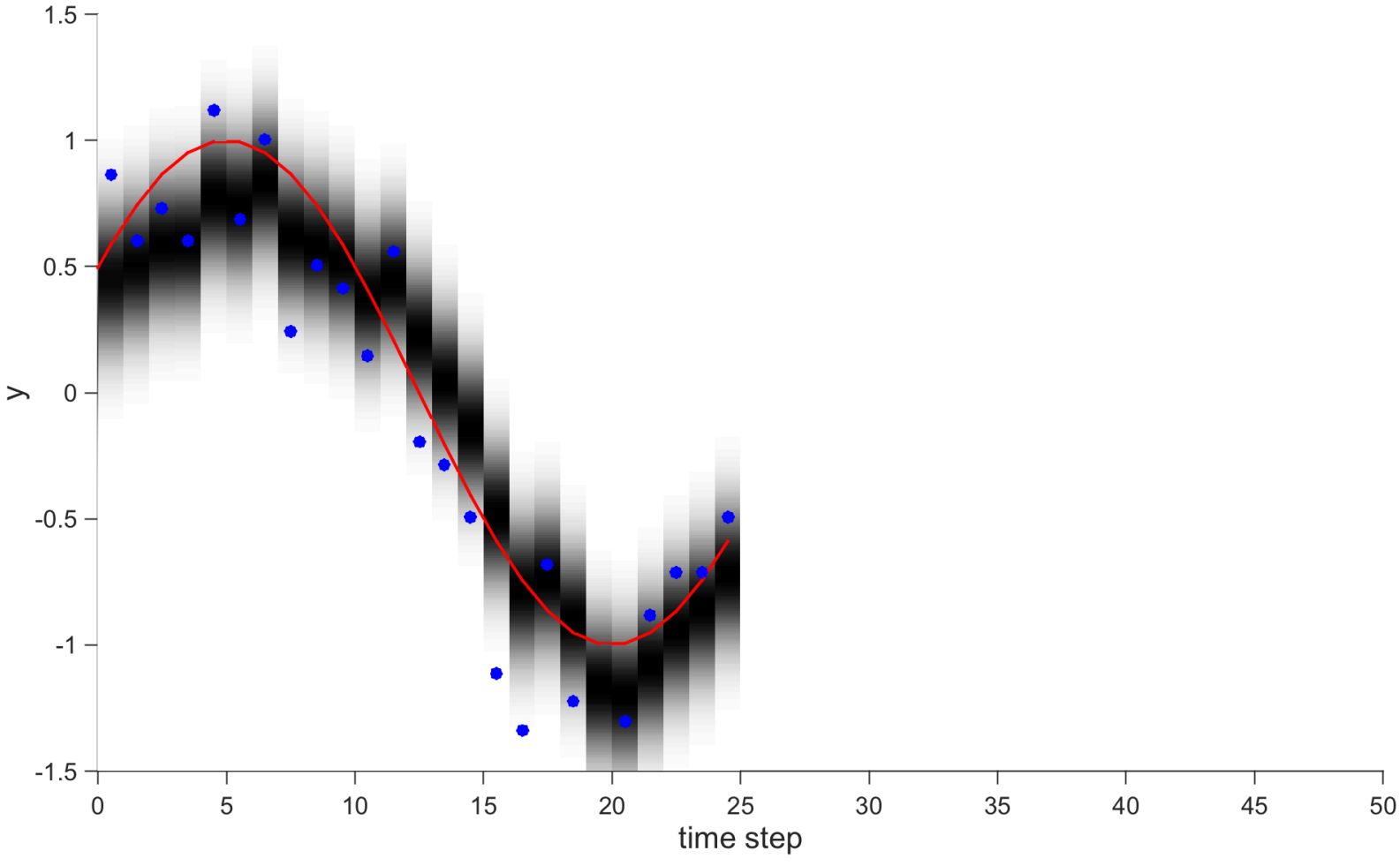
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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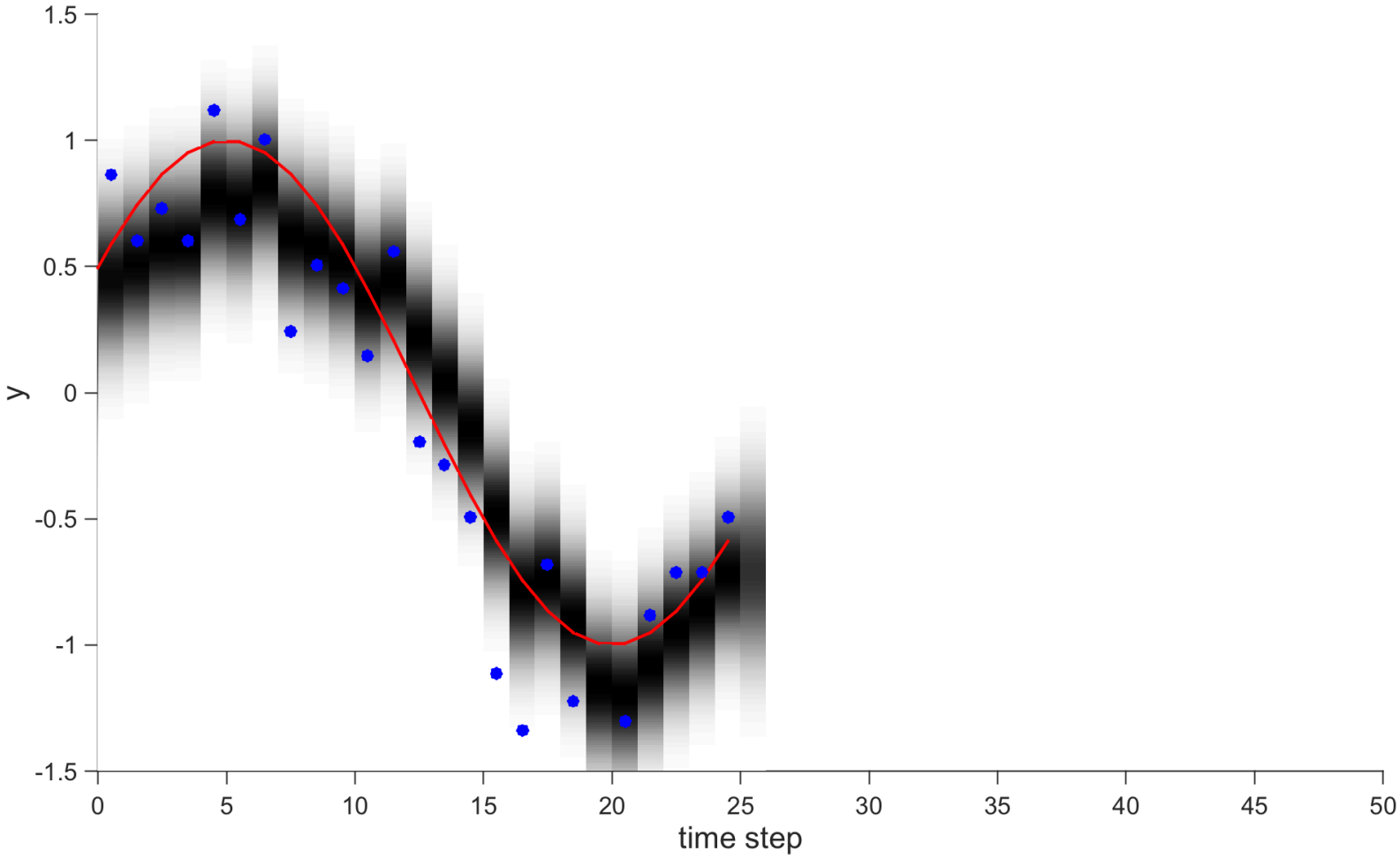

Kalman Filter Demo

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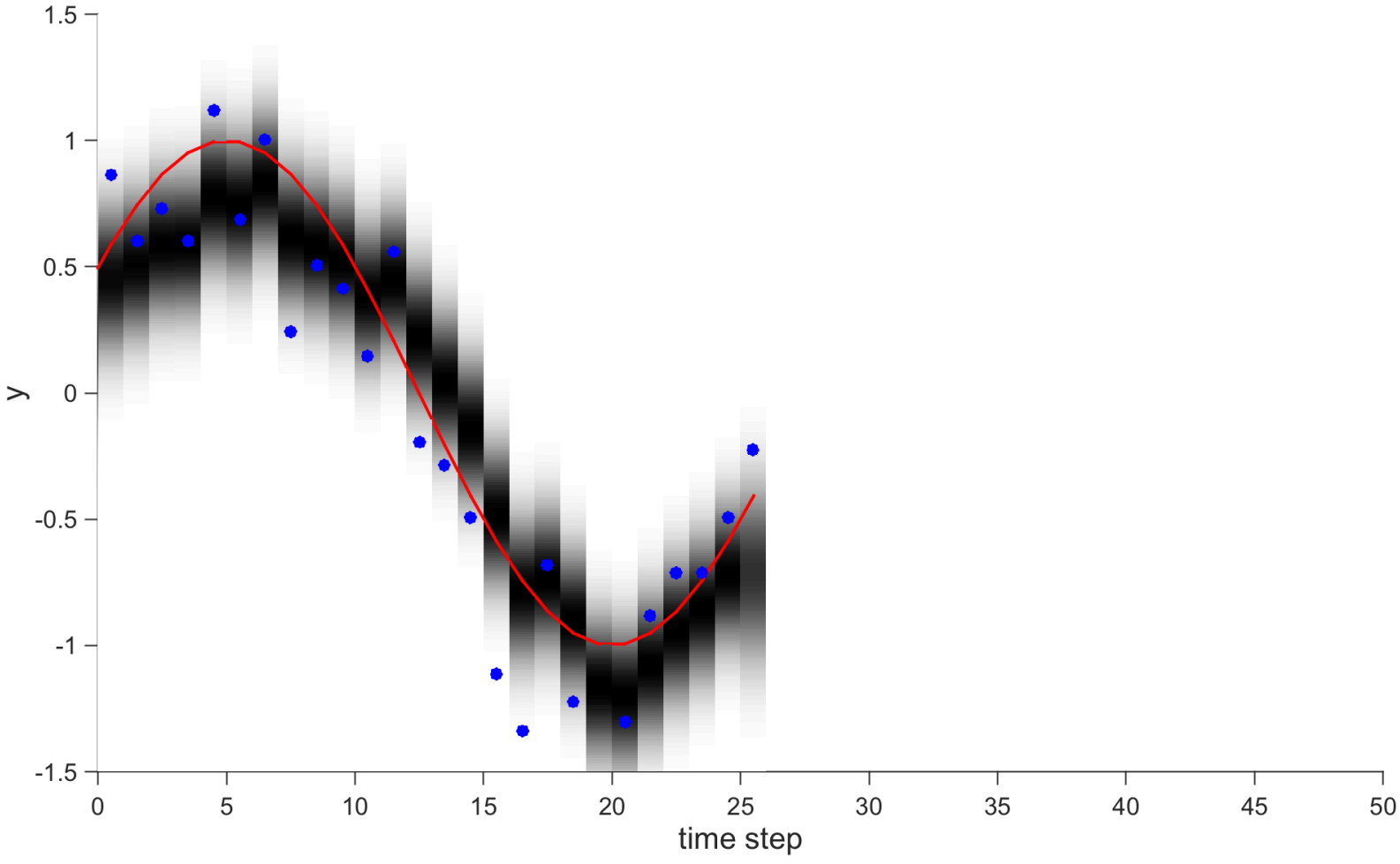
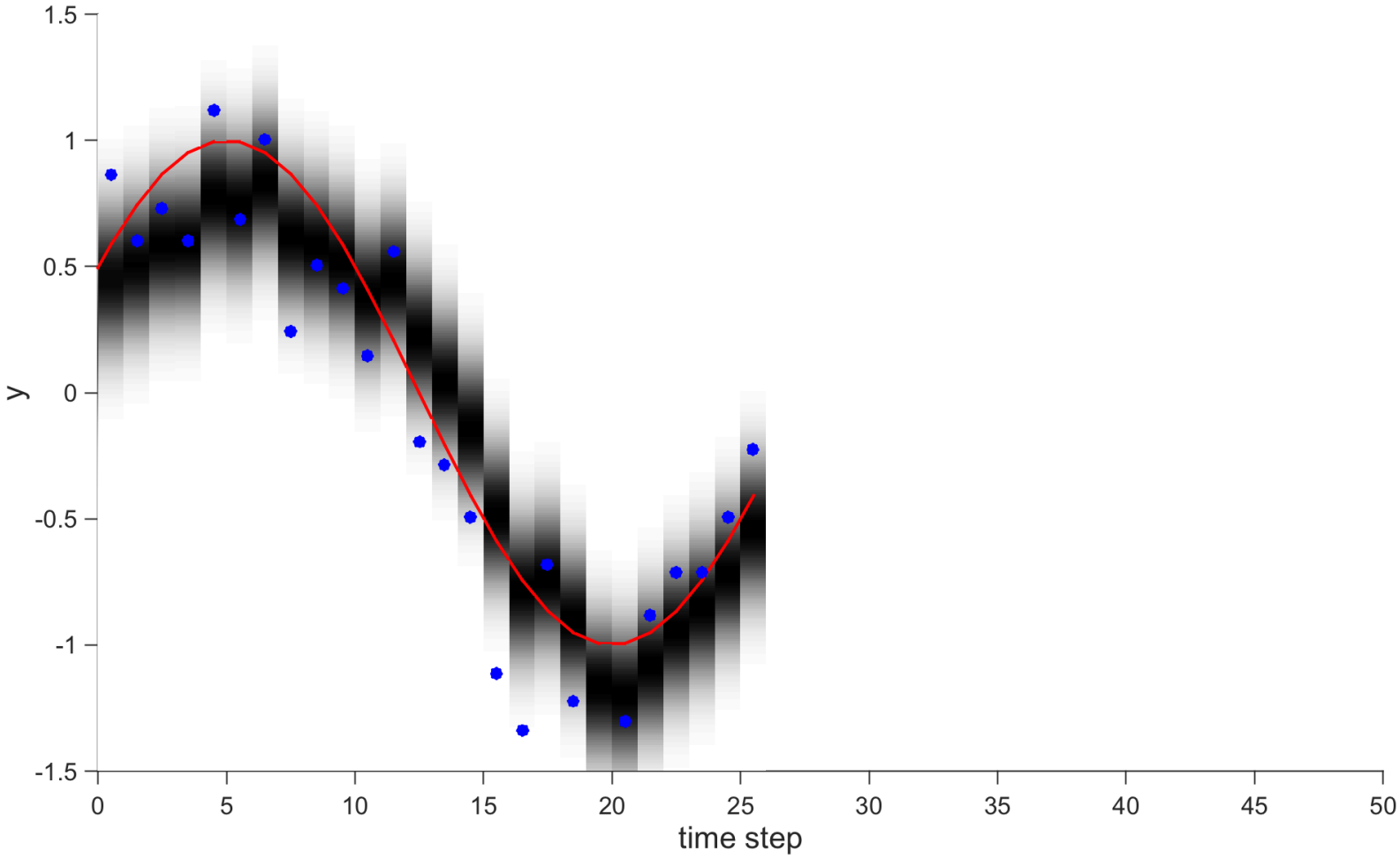


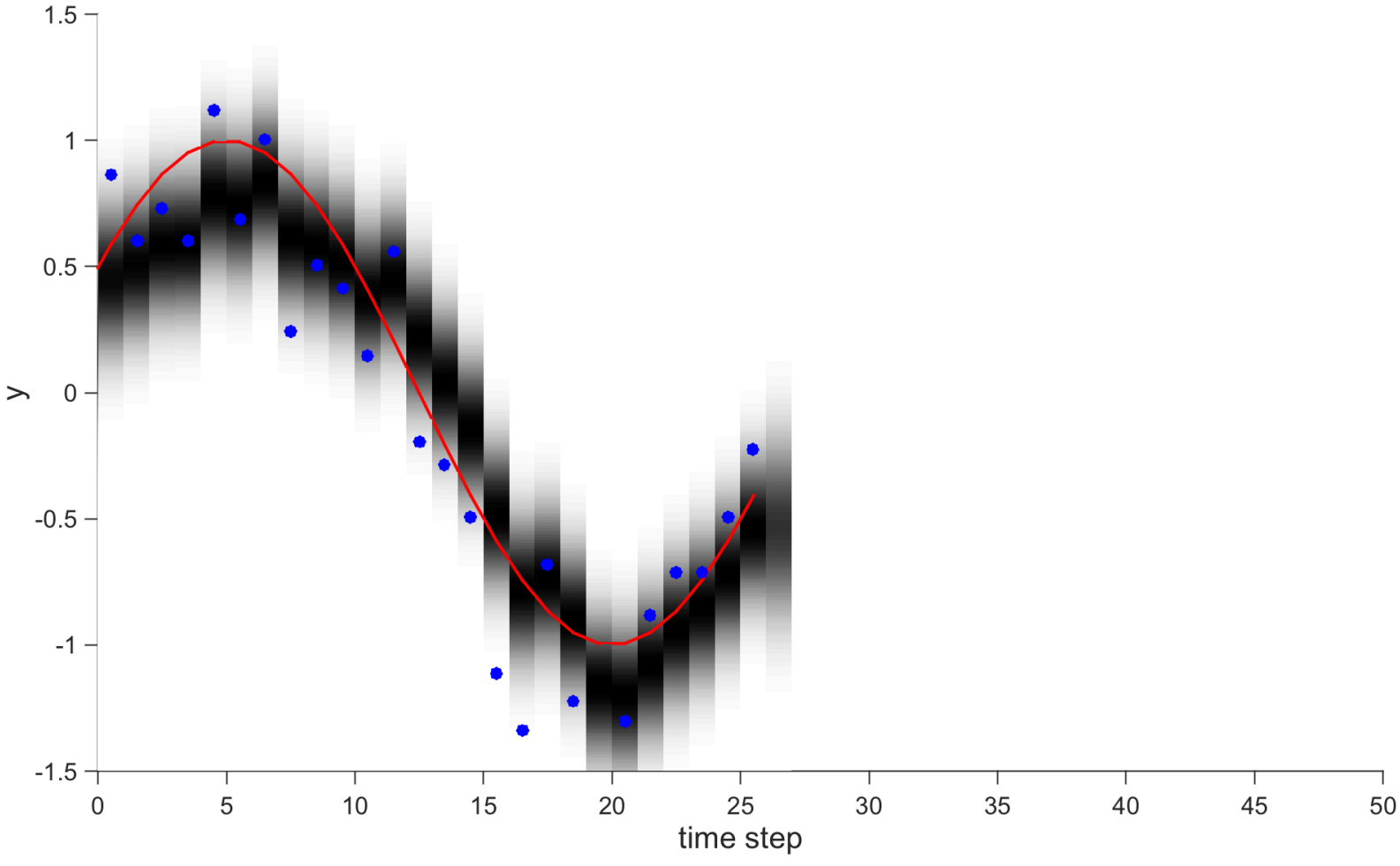
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



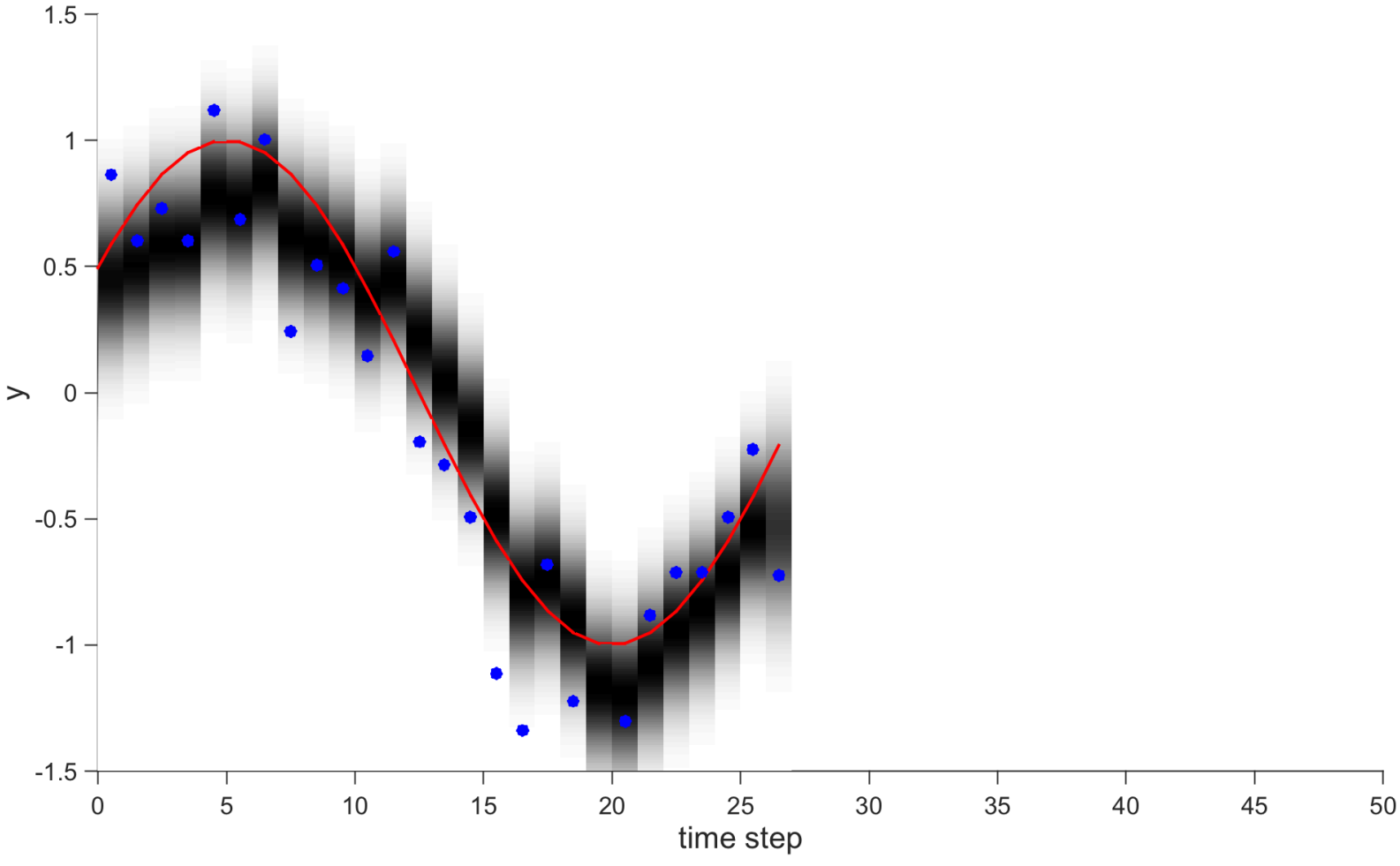
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



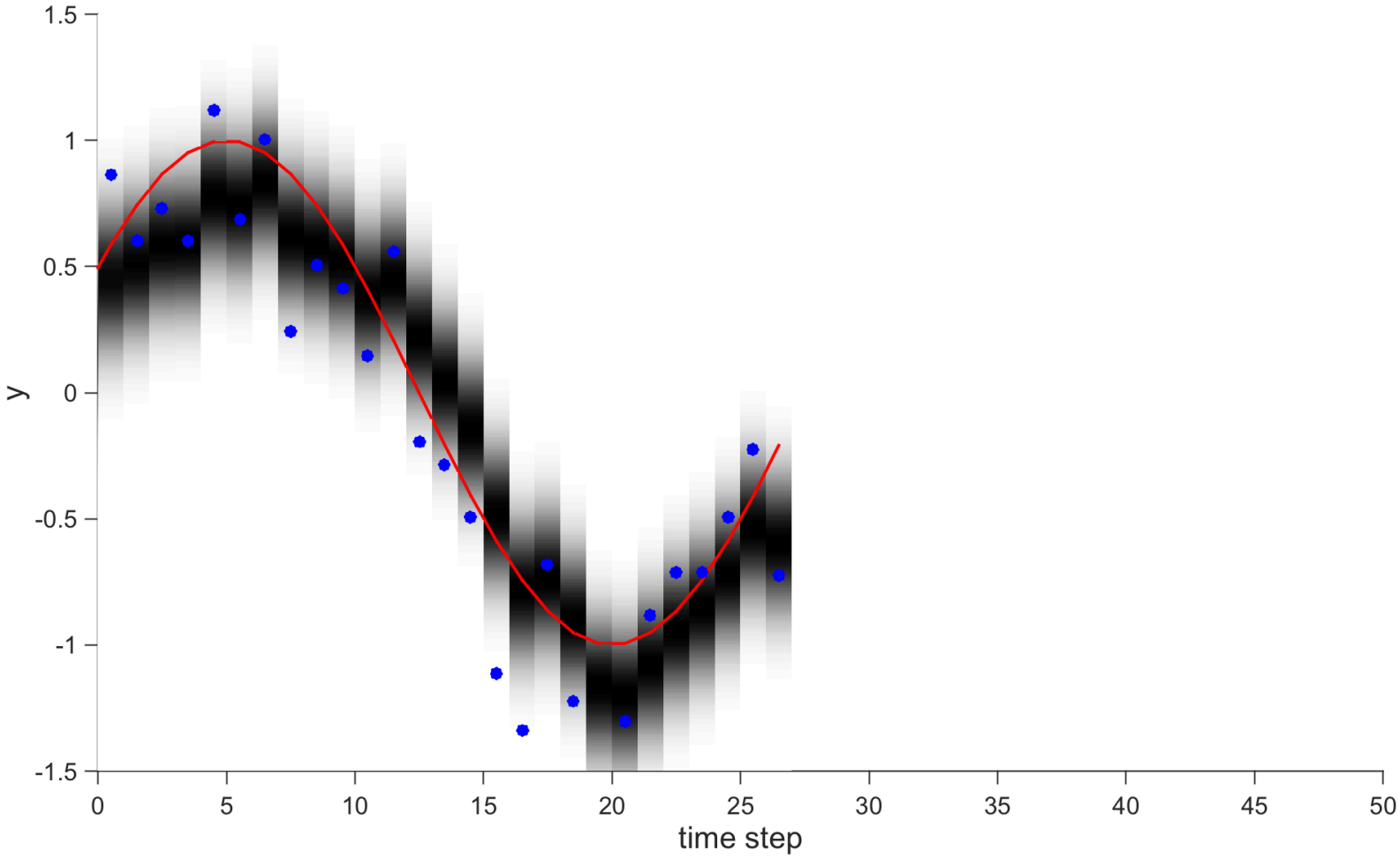
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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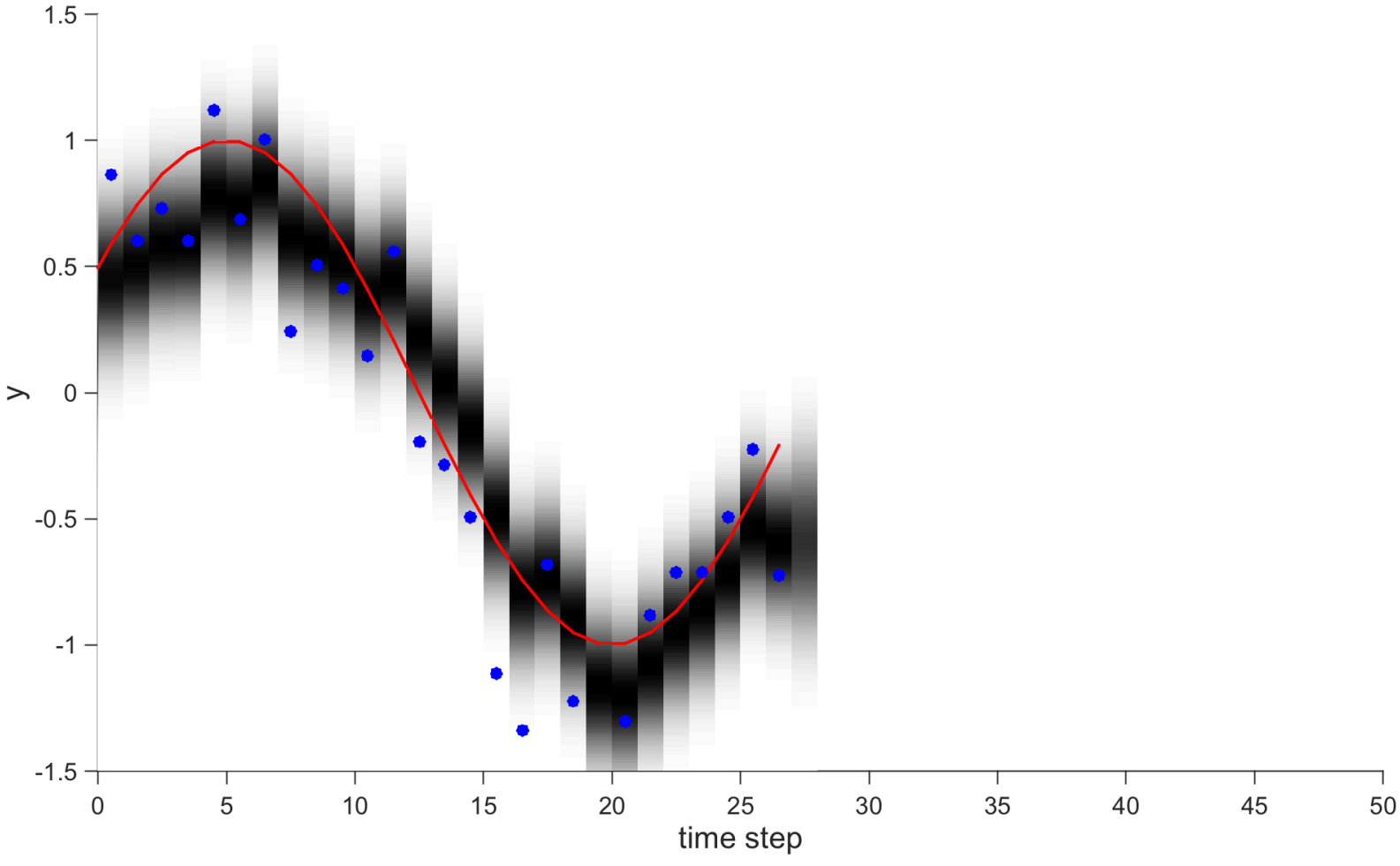

Kalman Filter Demo

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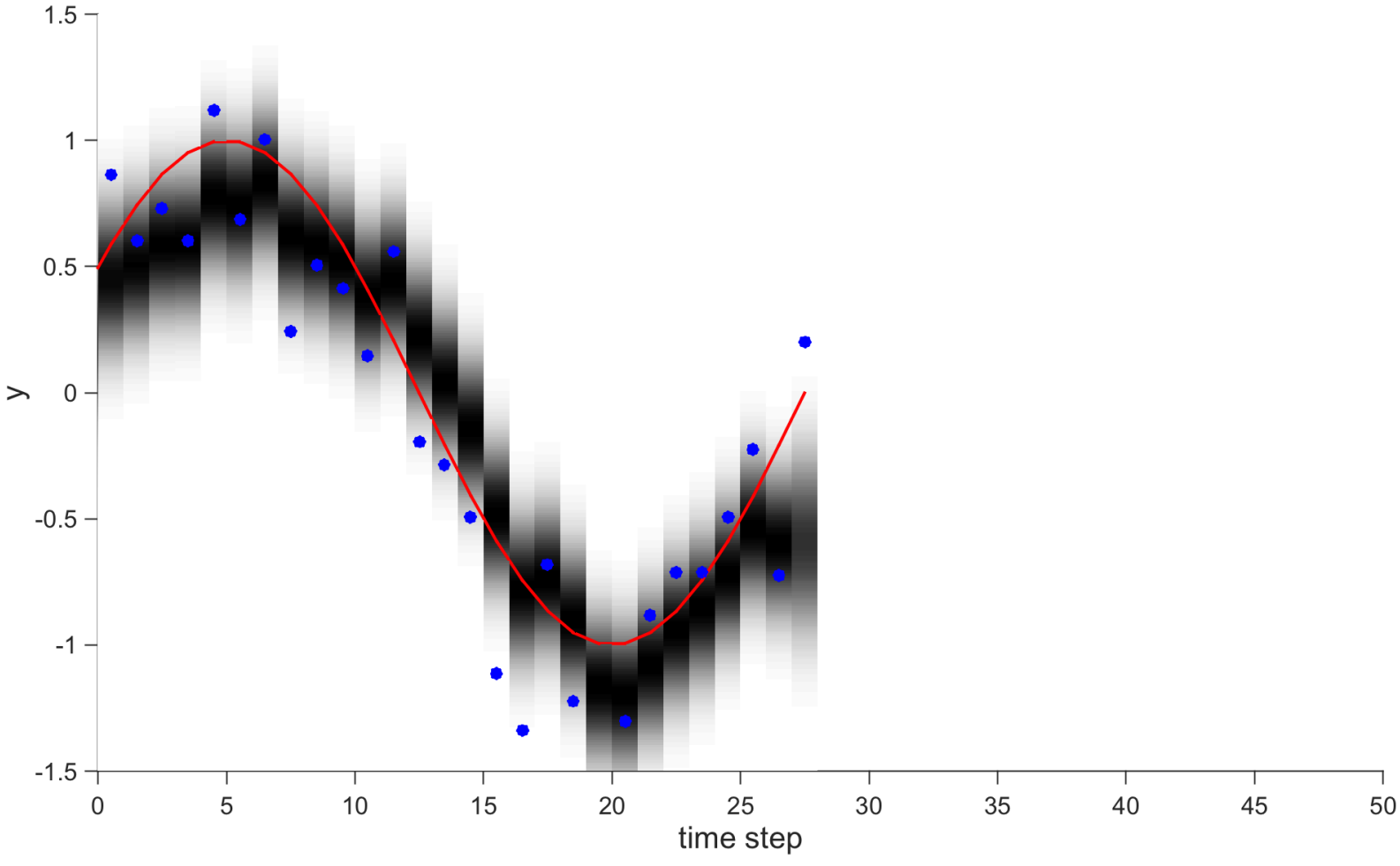

Kalman Filter Demo

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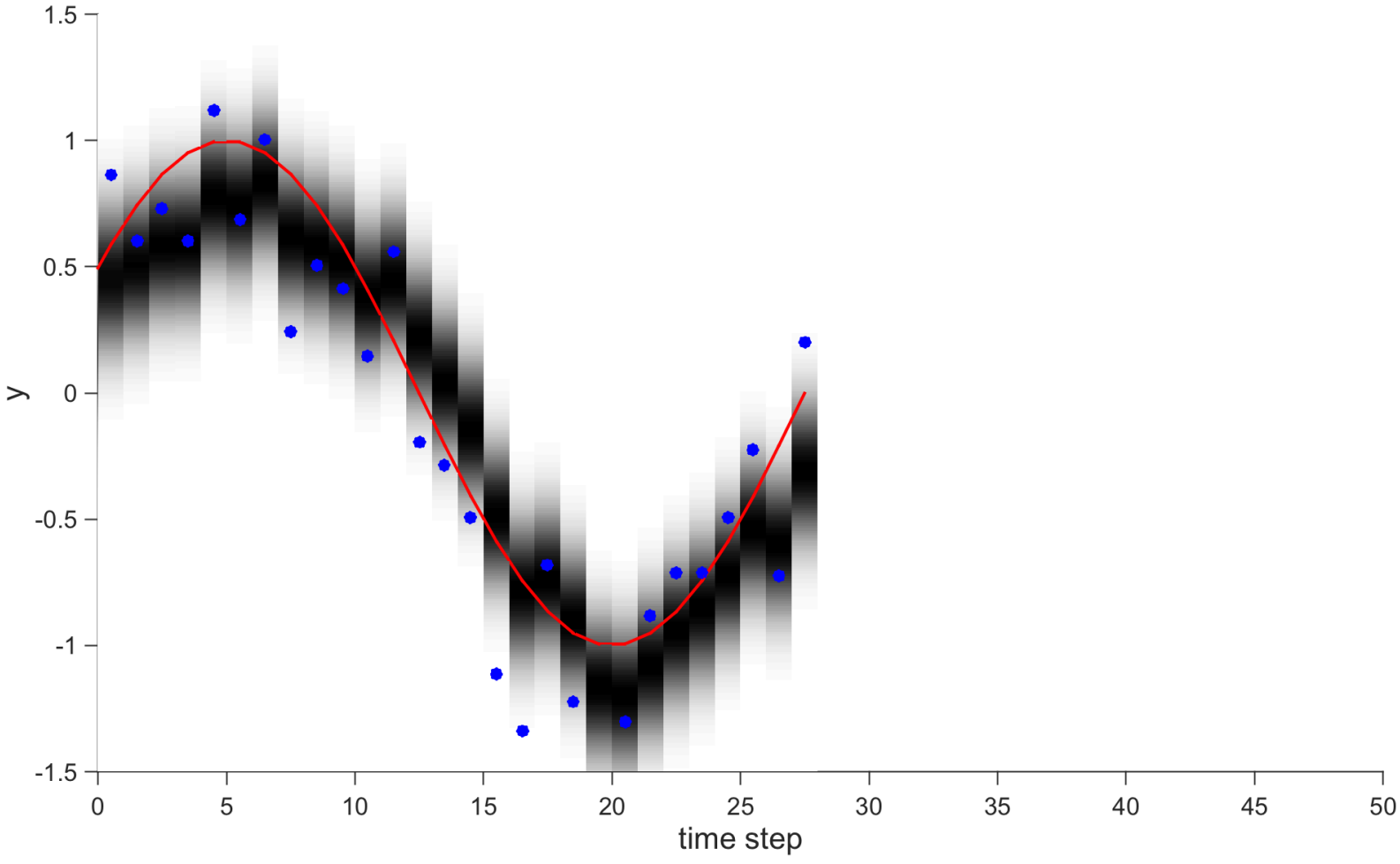

Kalman Filter Demo

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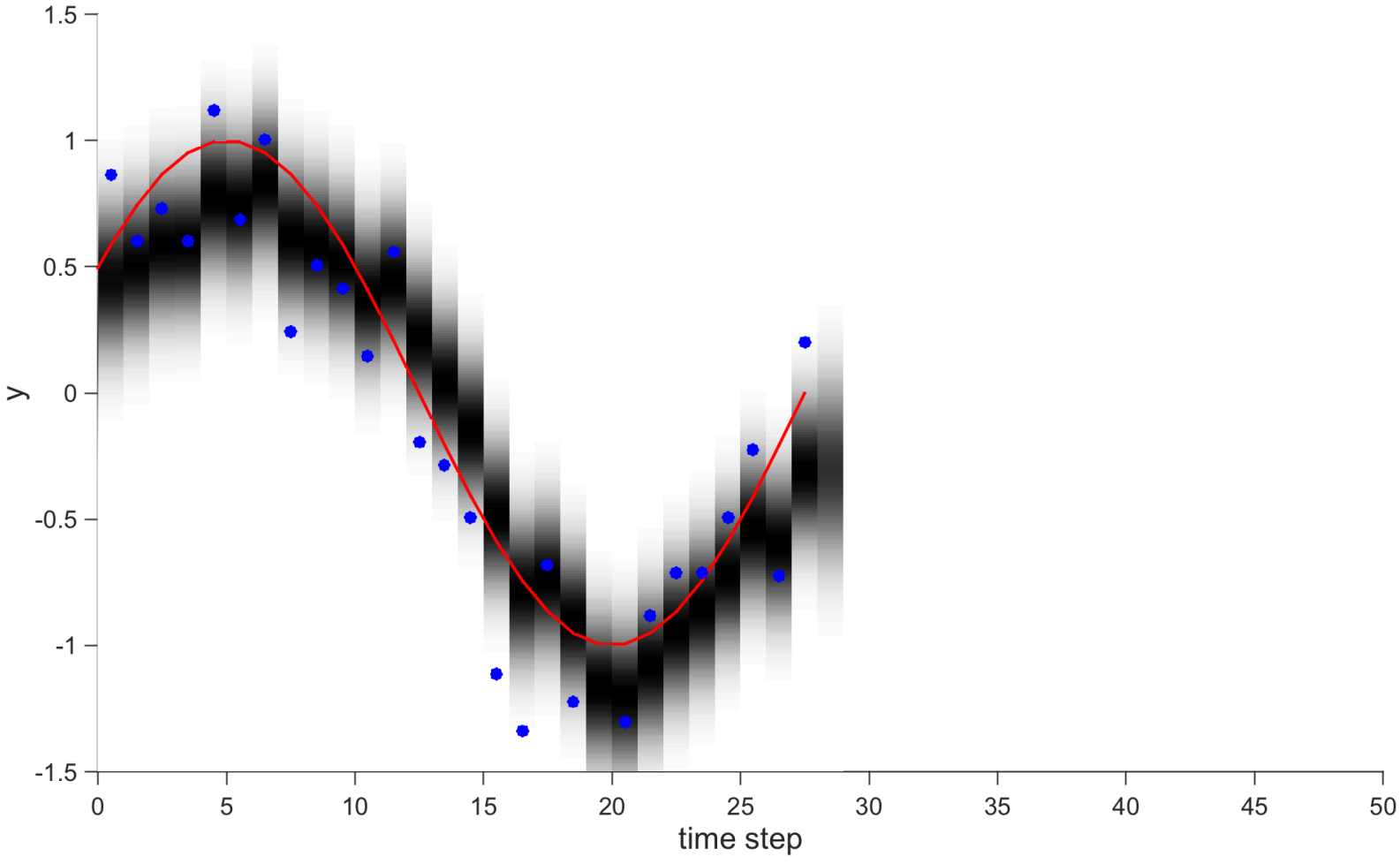

Kalman Filter Demo

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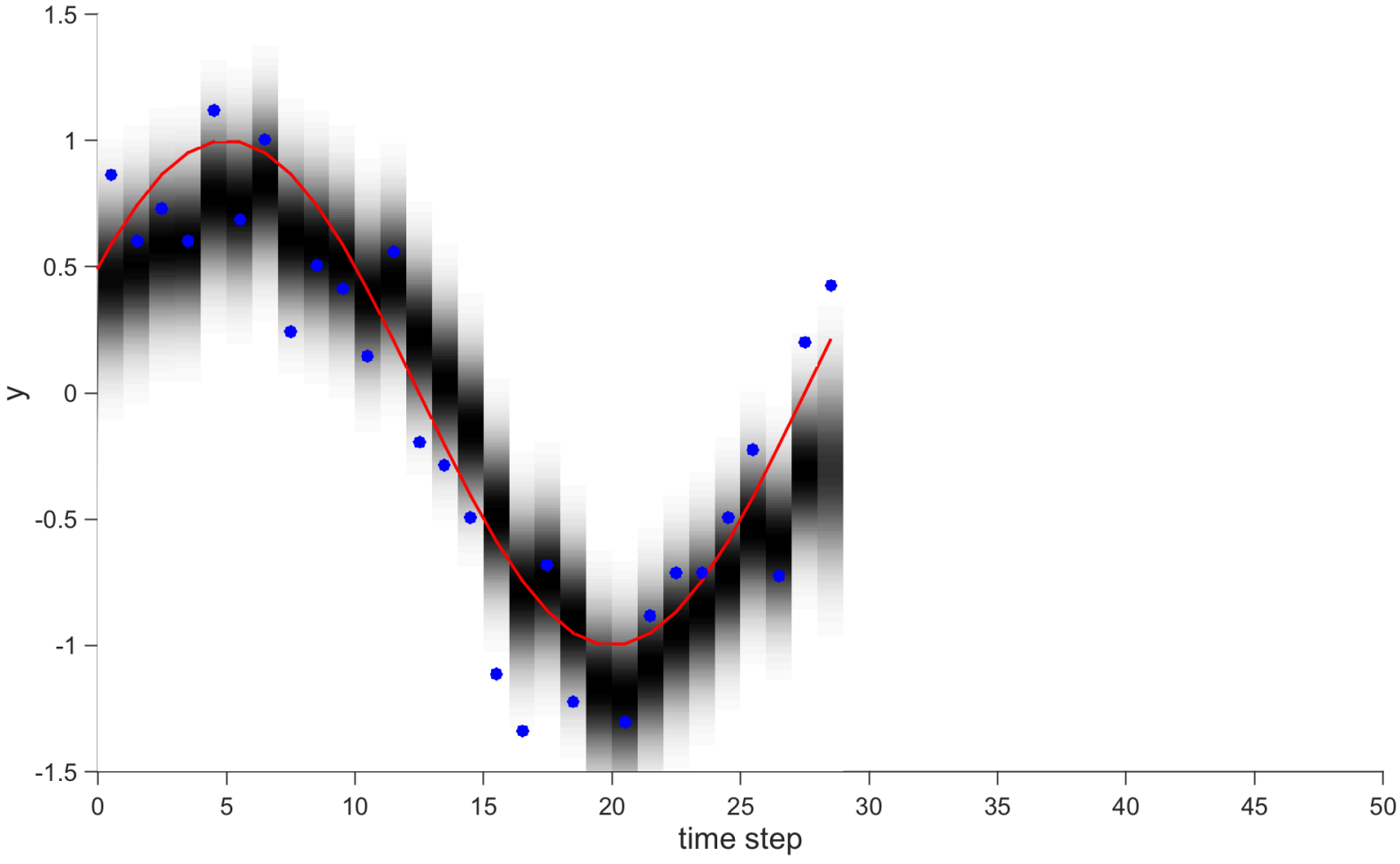

Kalman Filter Demo

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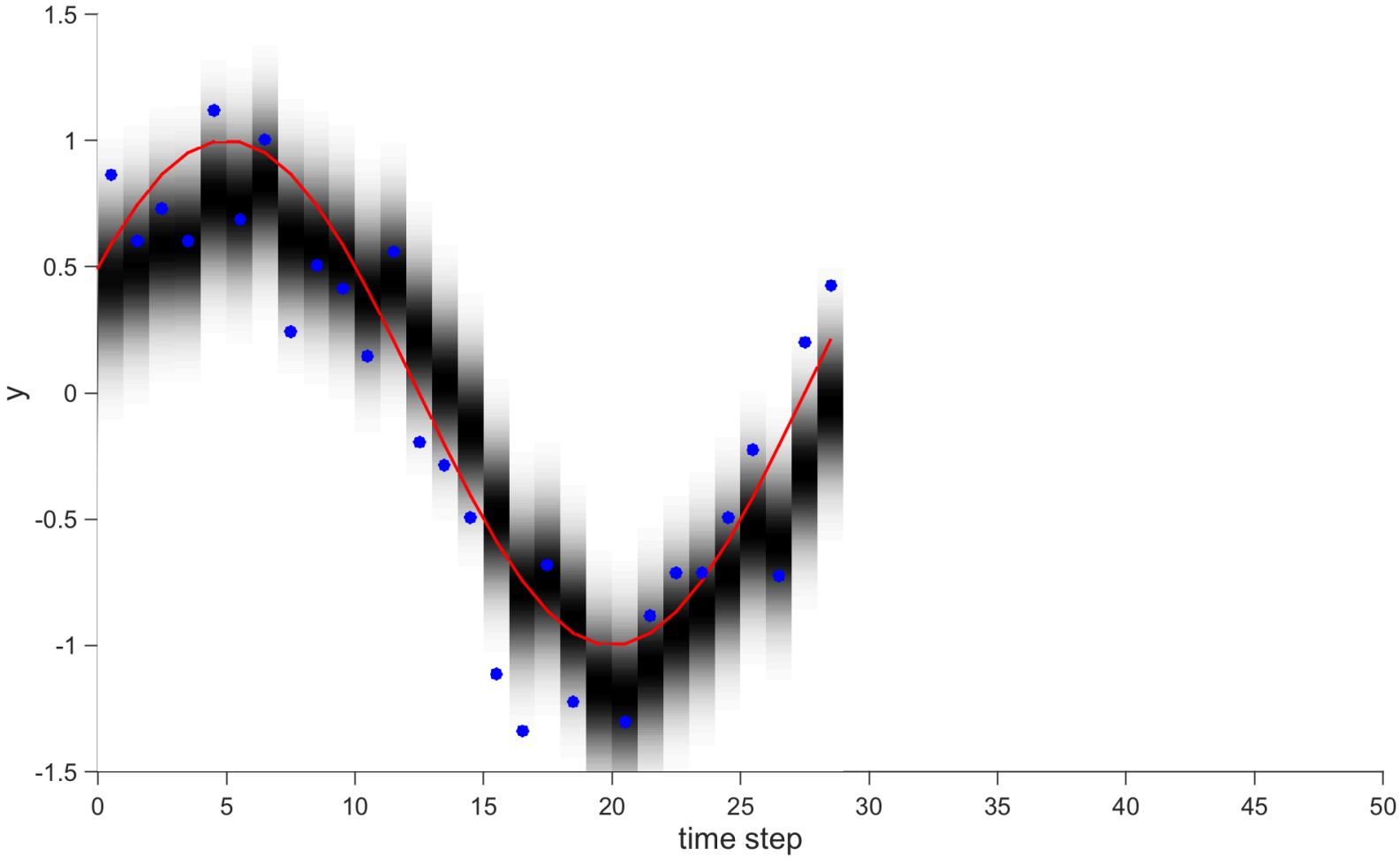

Kalman Filter Demo

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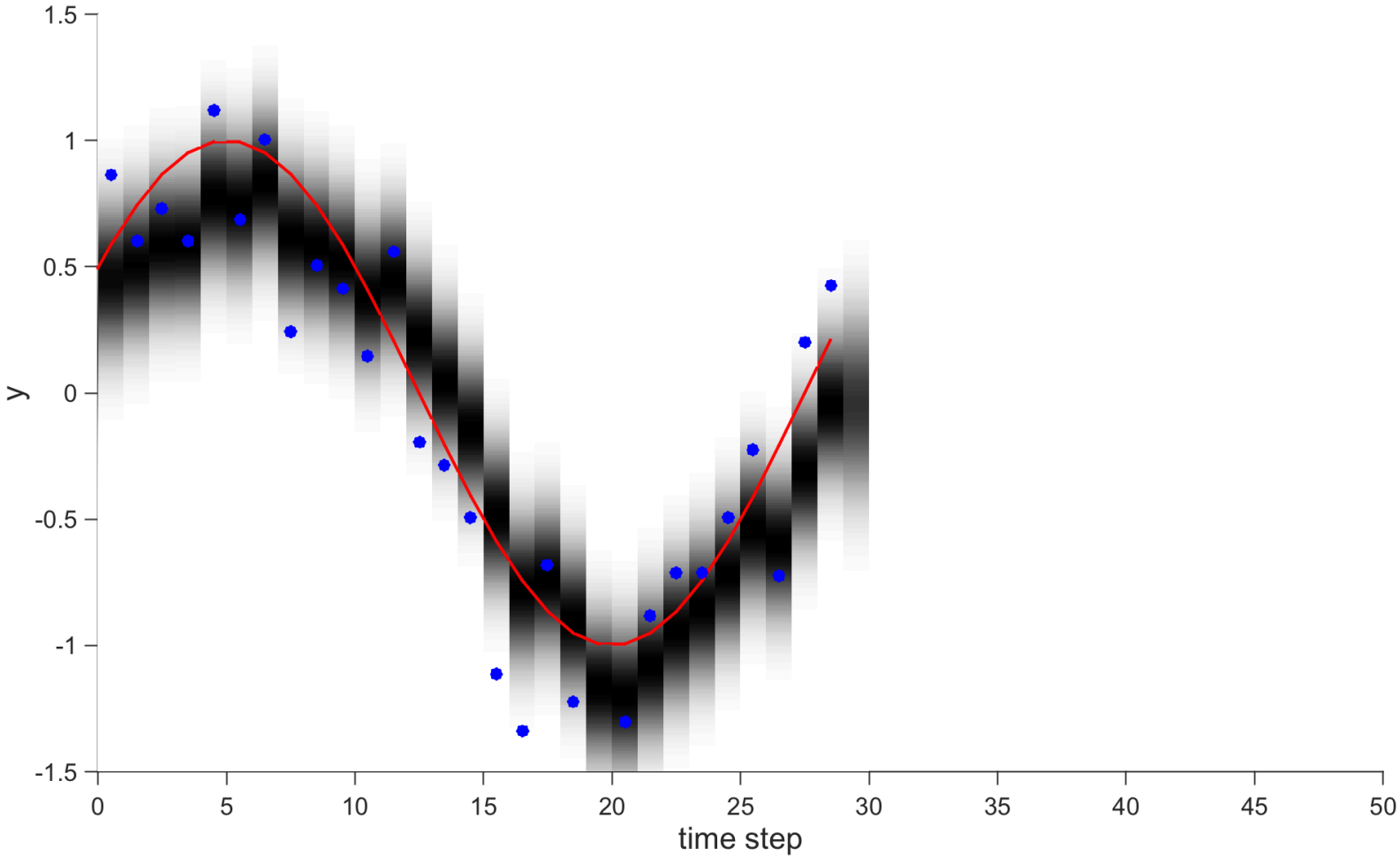

Kalman Filter Demo

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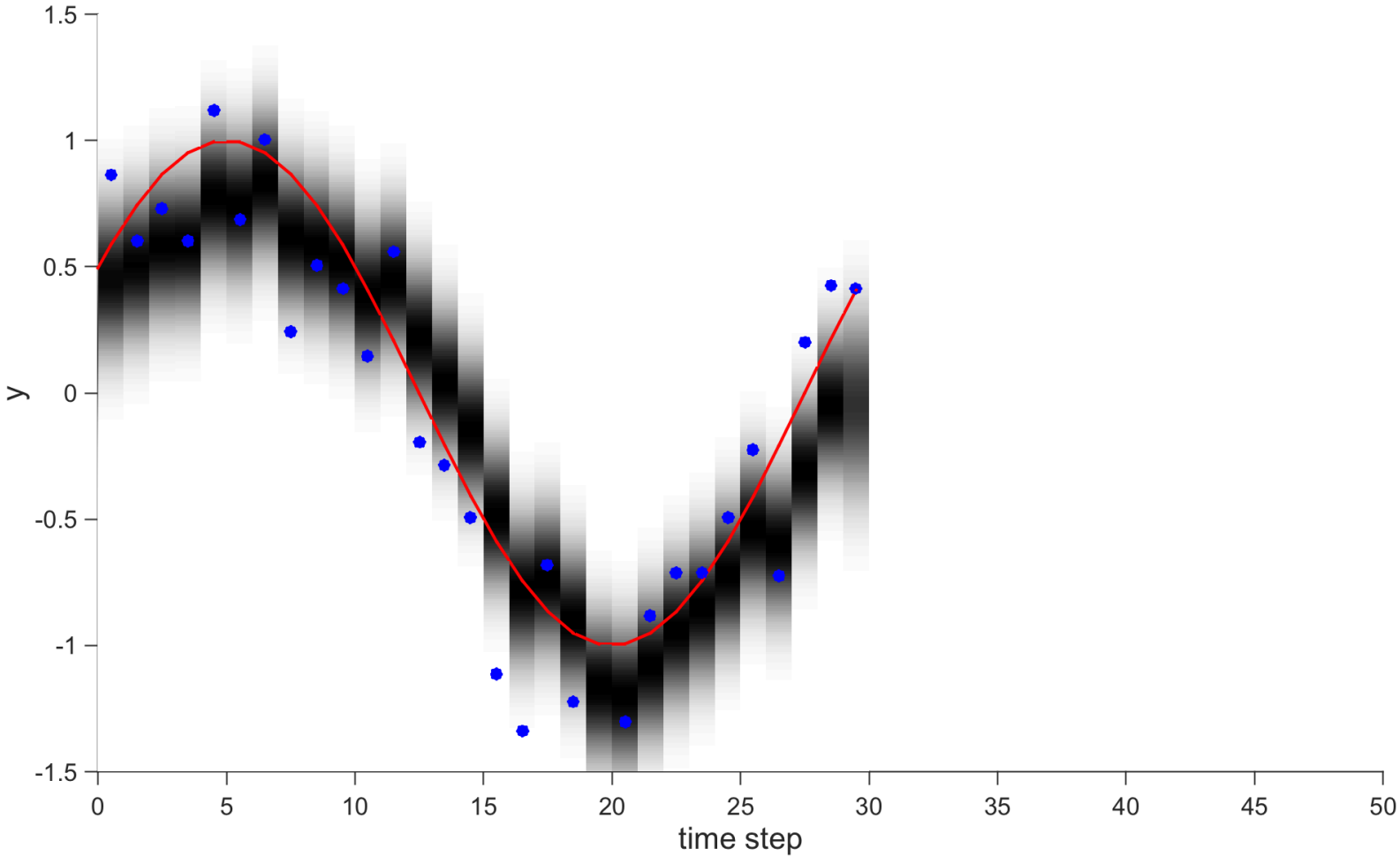

Kalman Filter Demo

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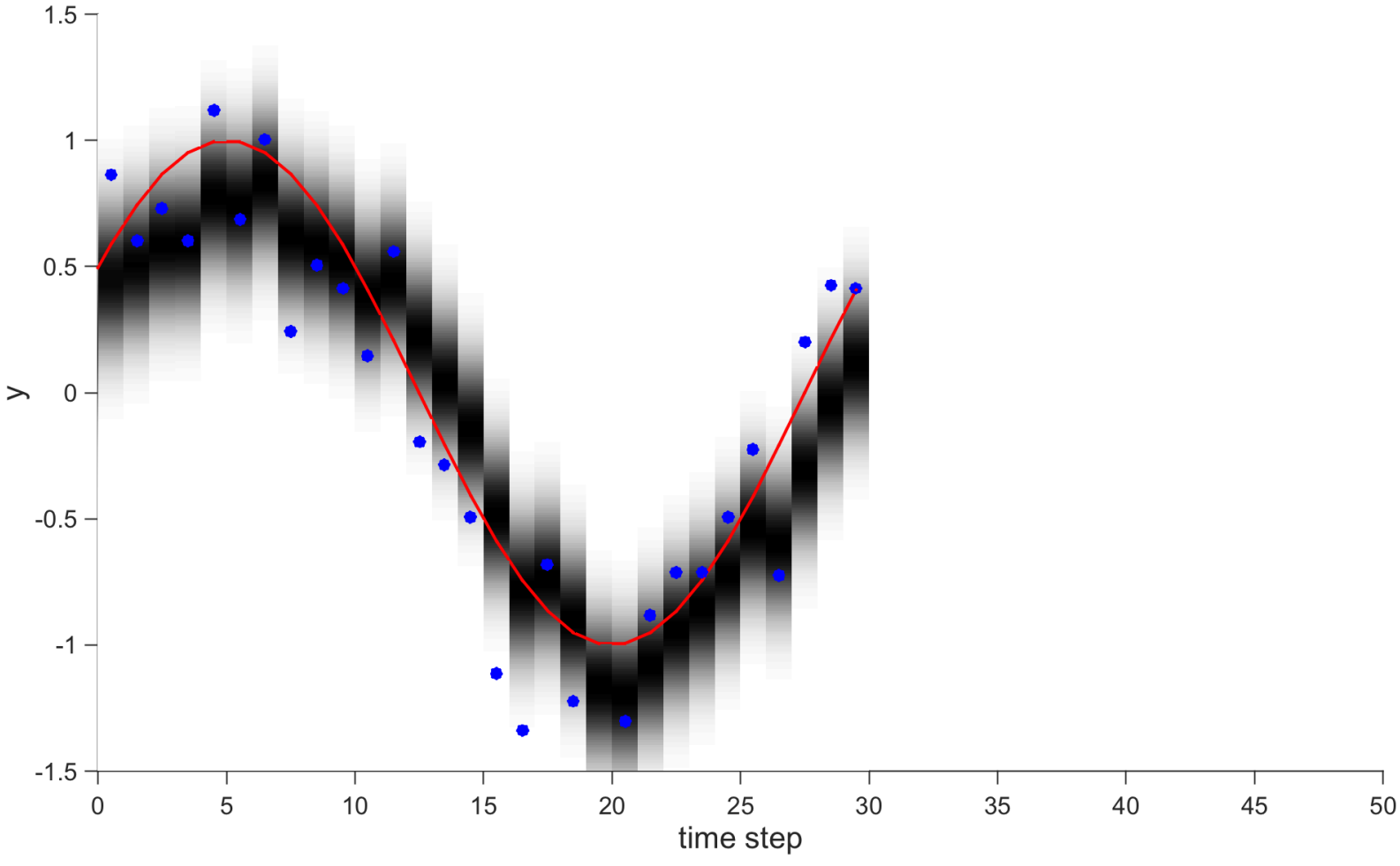

Kalman Filter Demo

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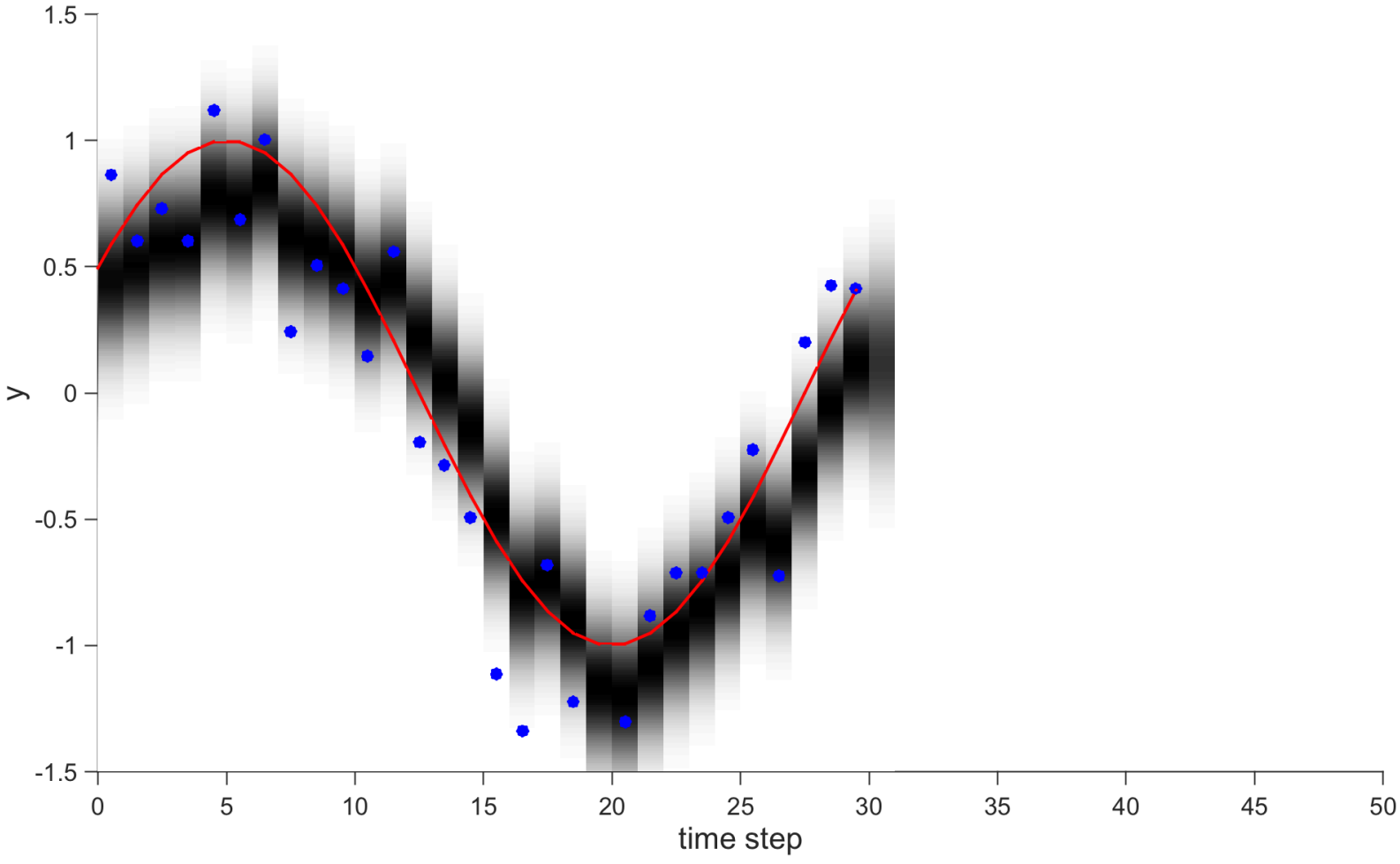

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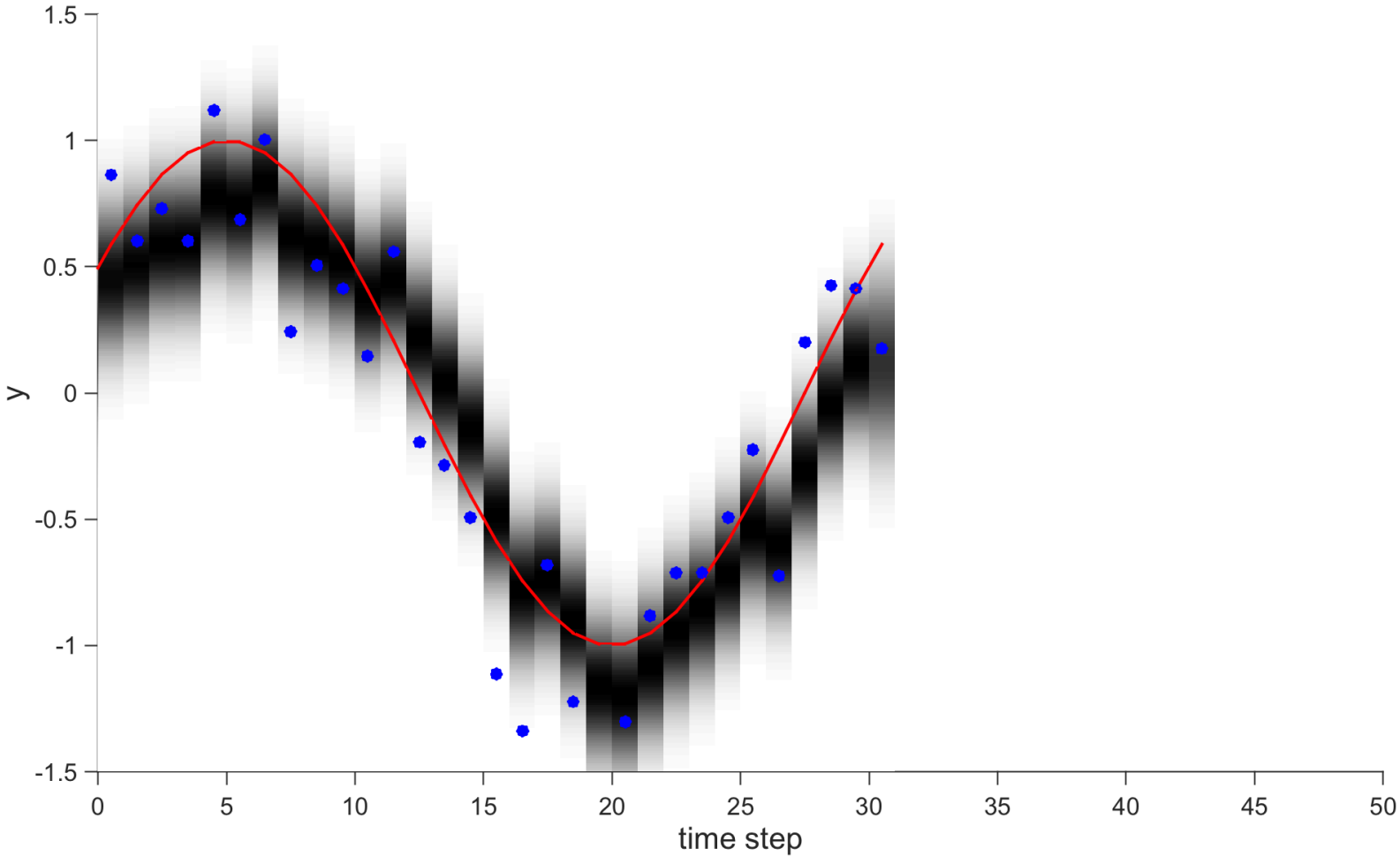

Kalman Filter Demo

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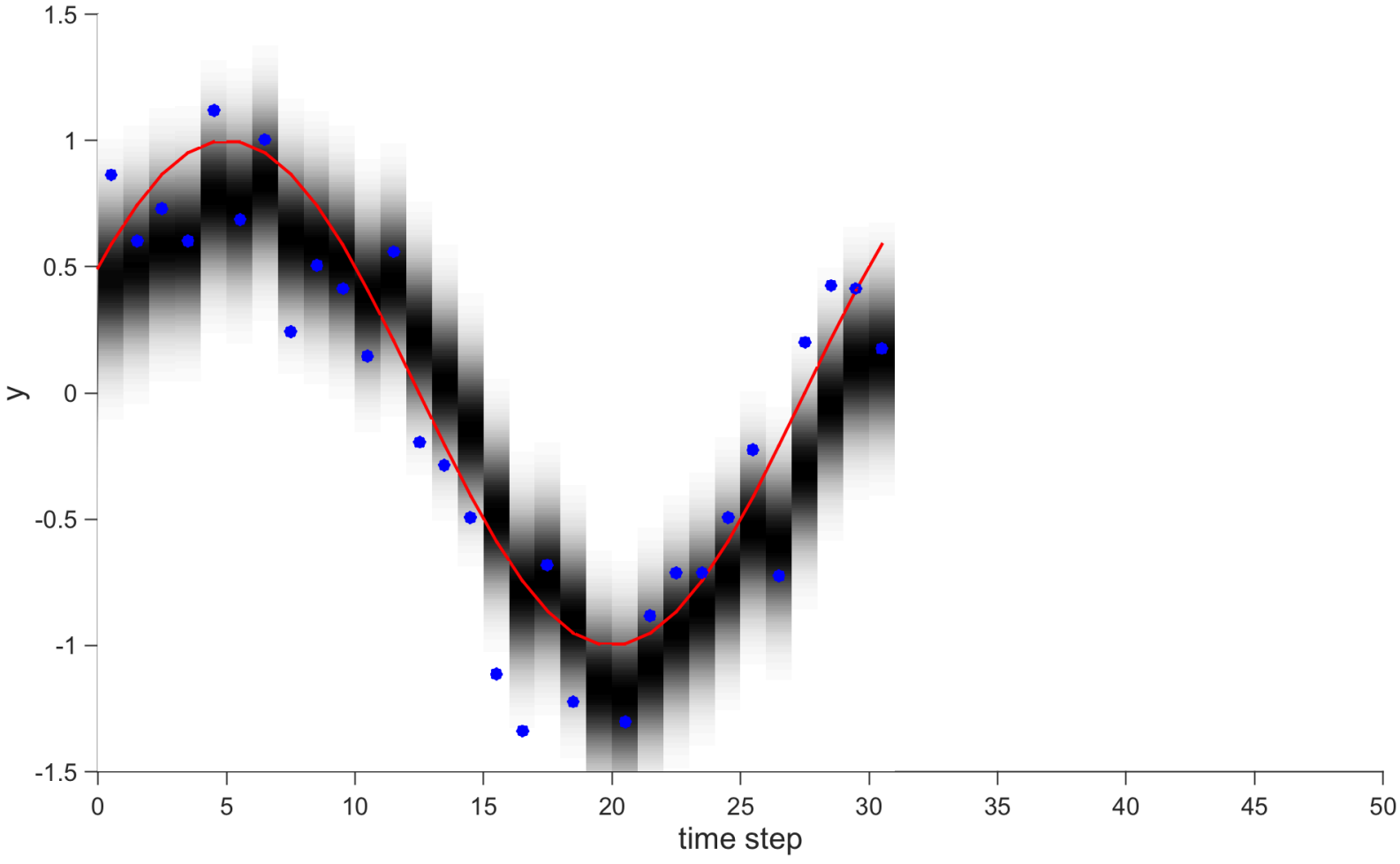

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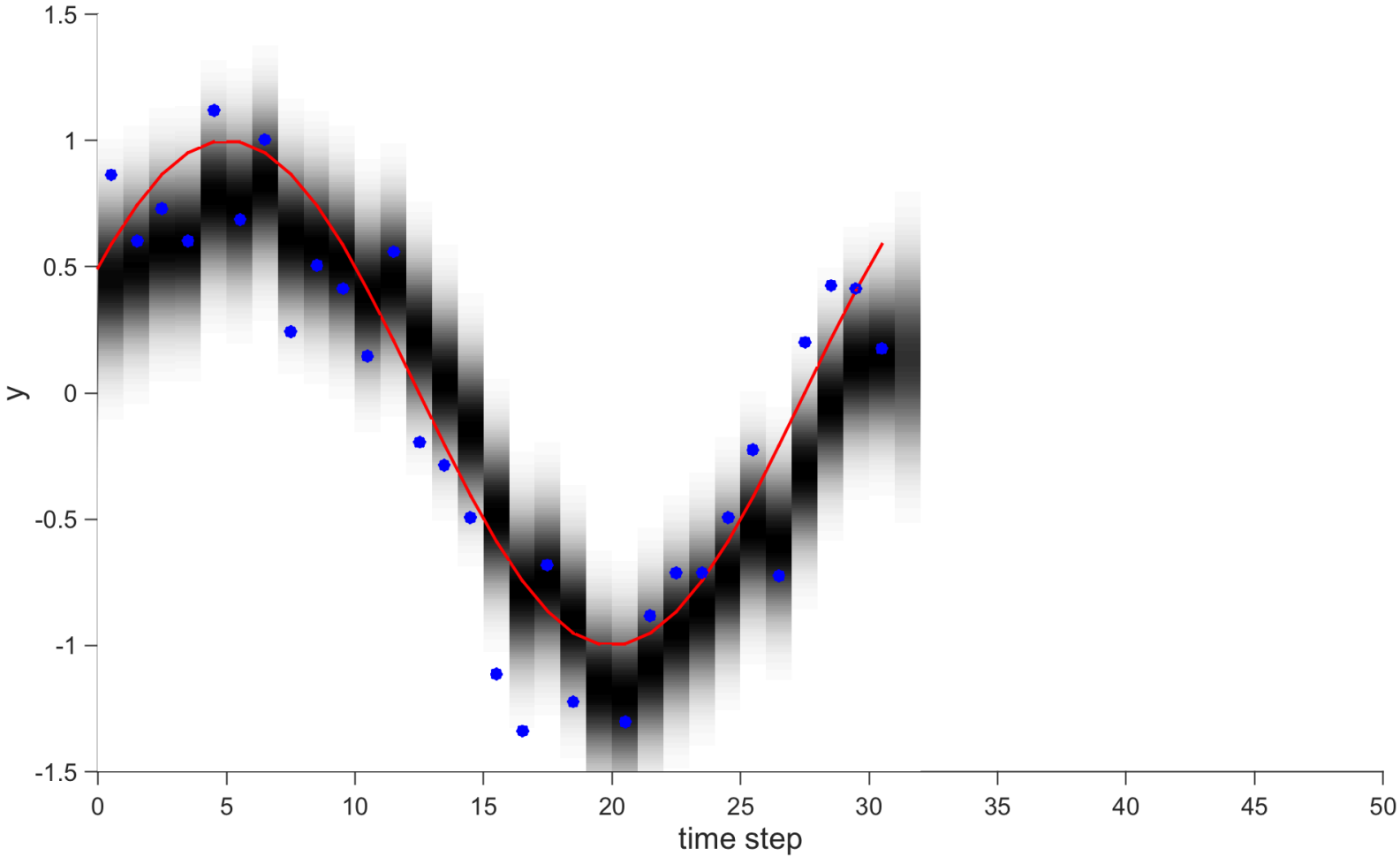

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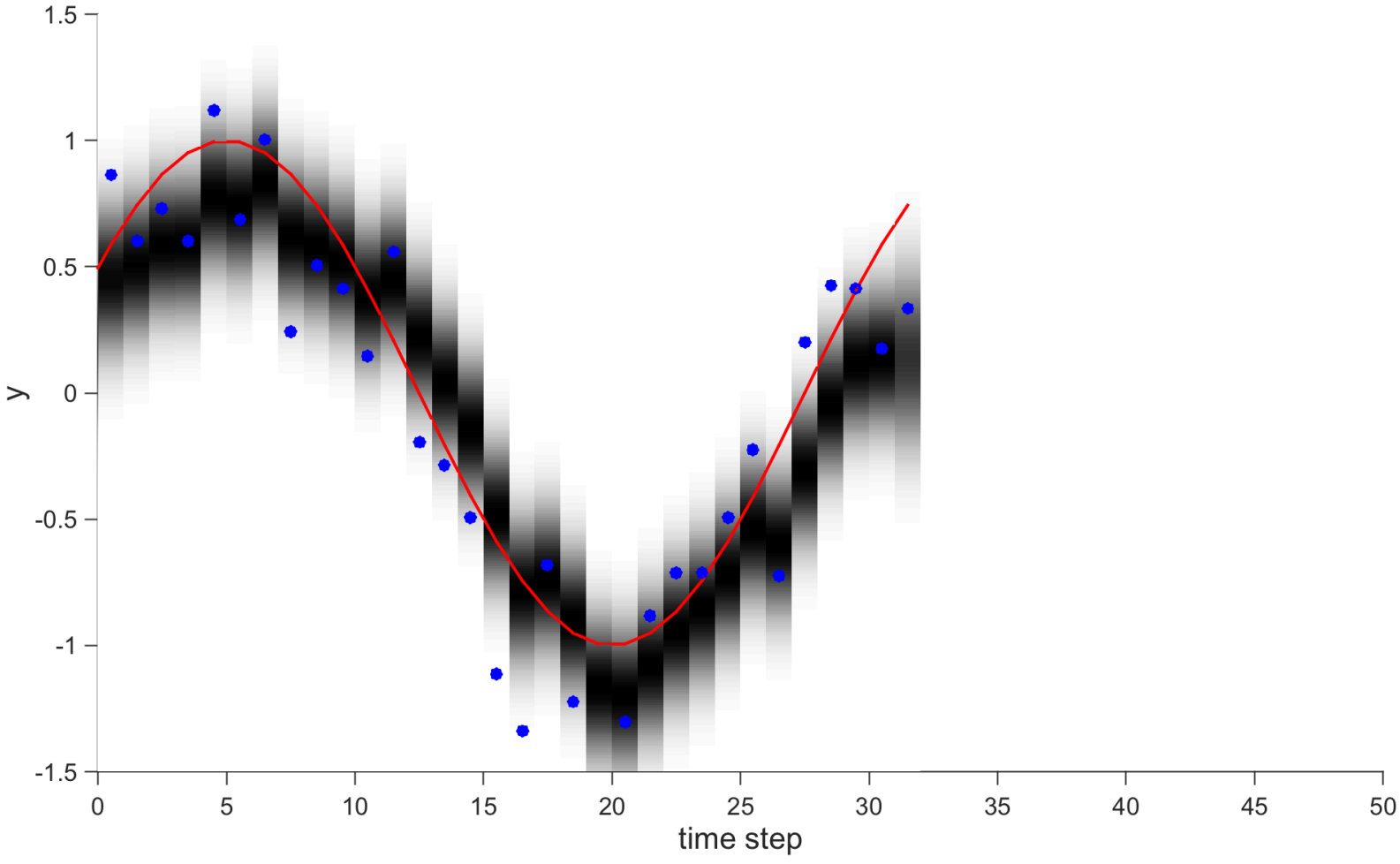

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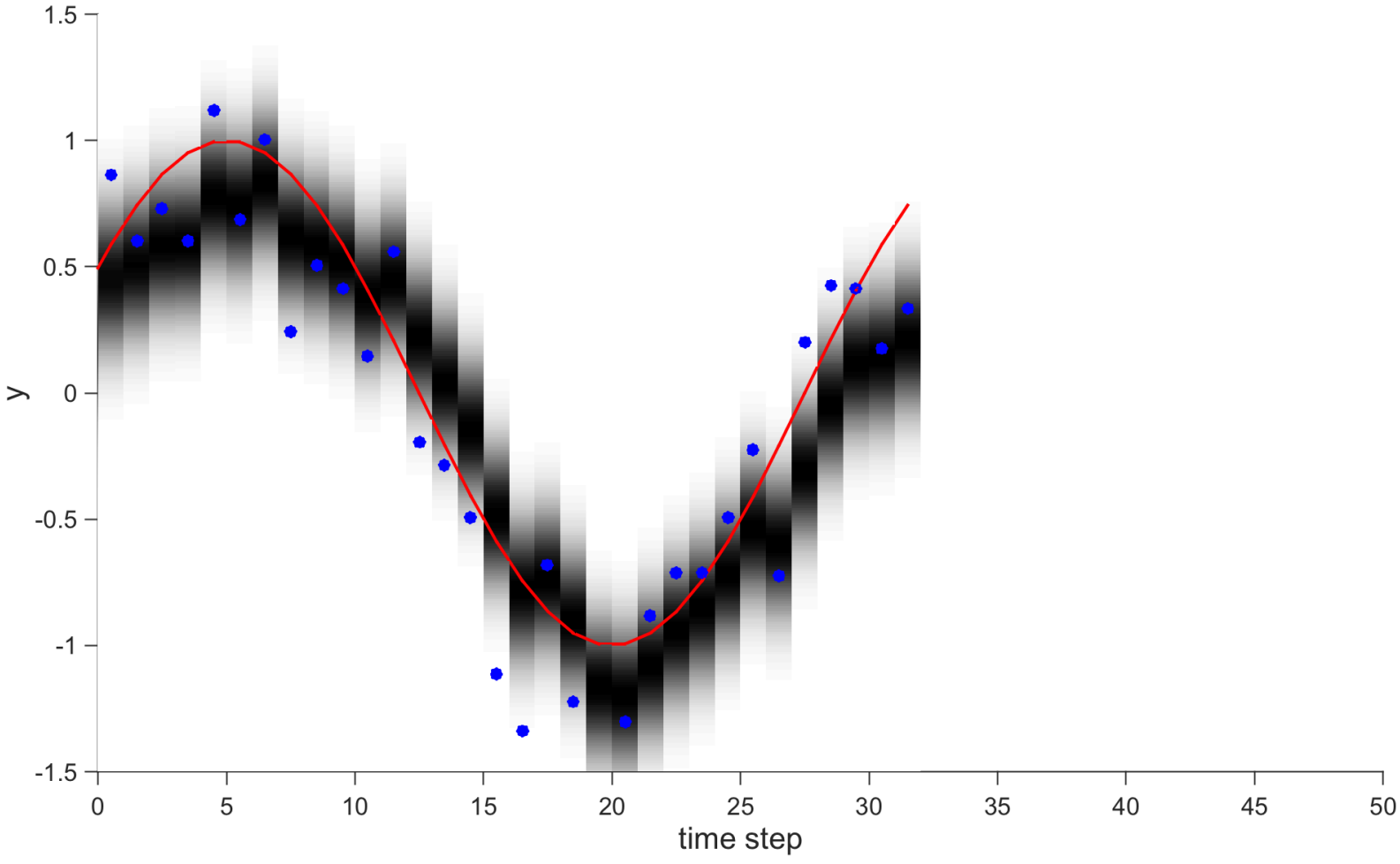

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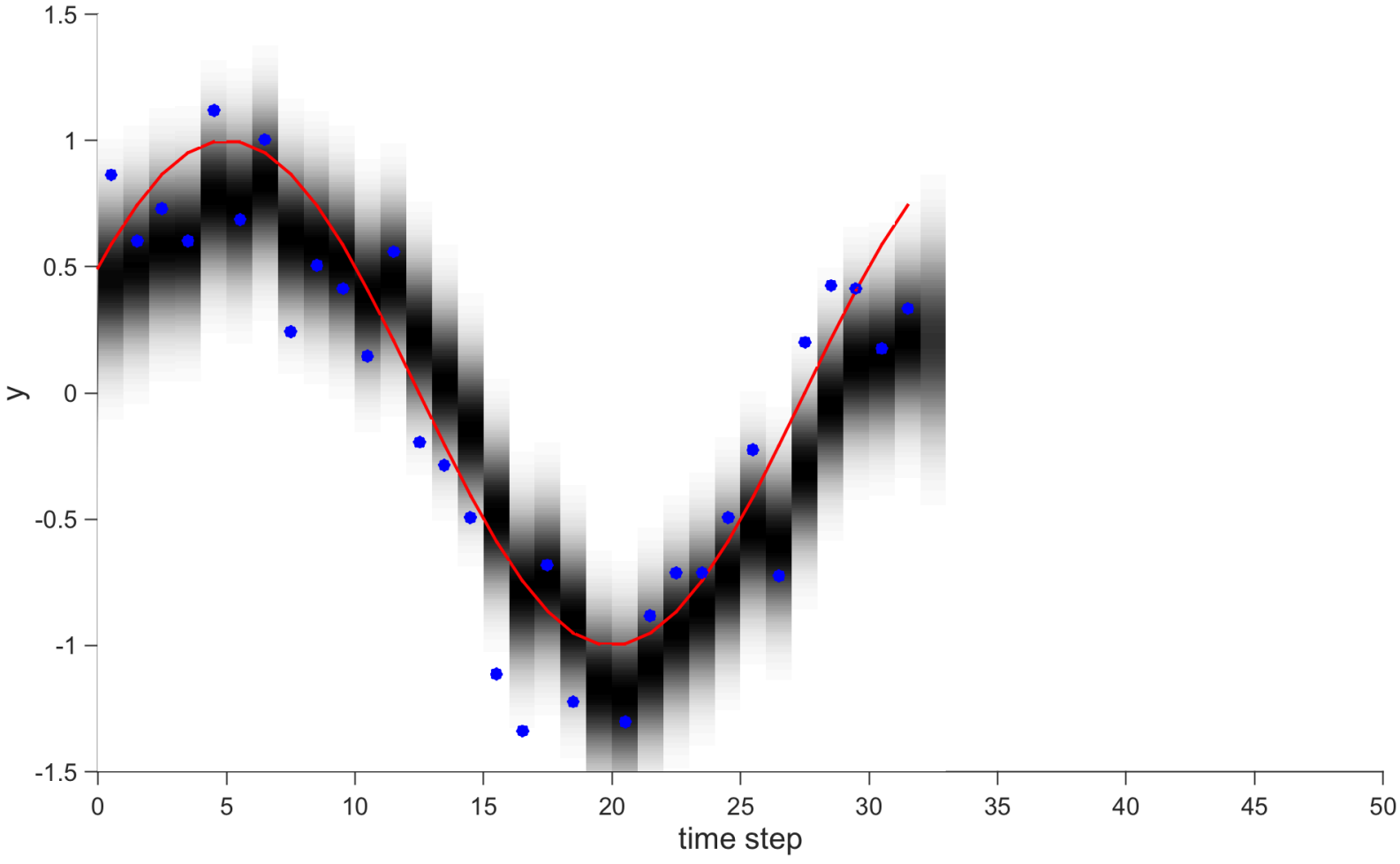

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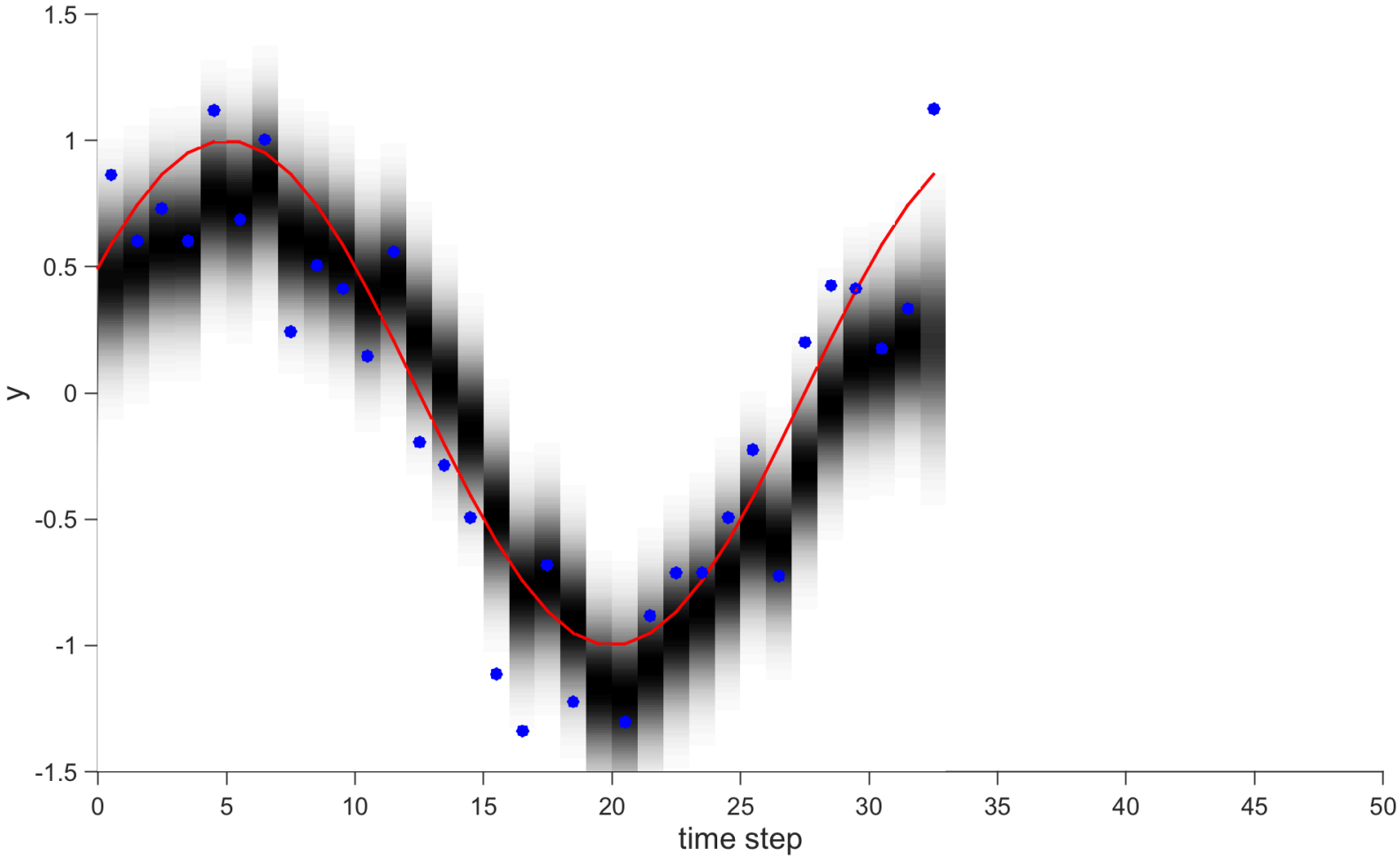

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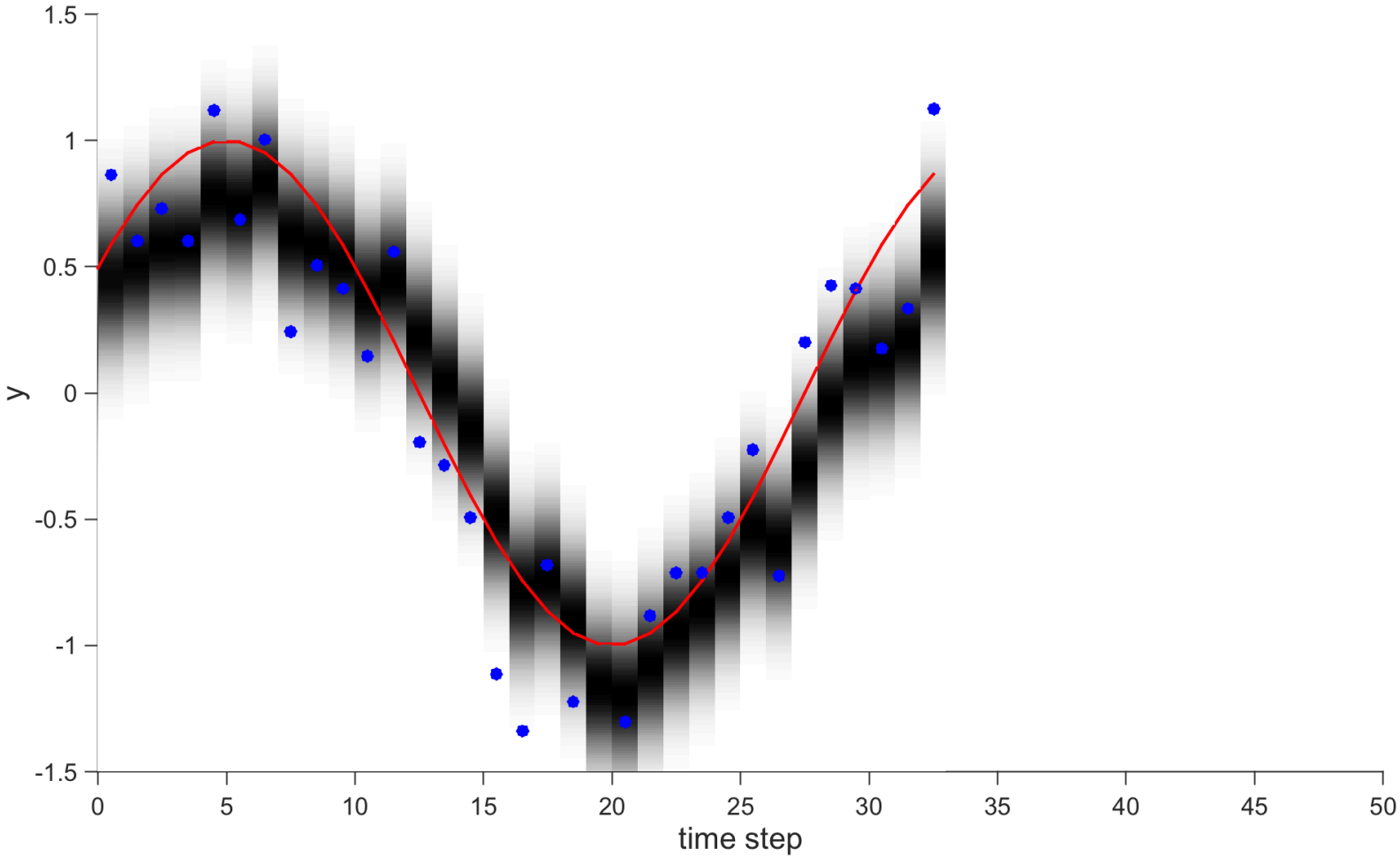

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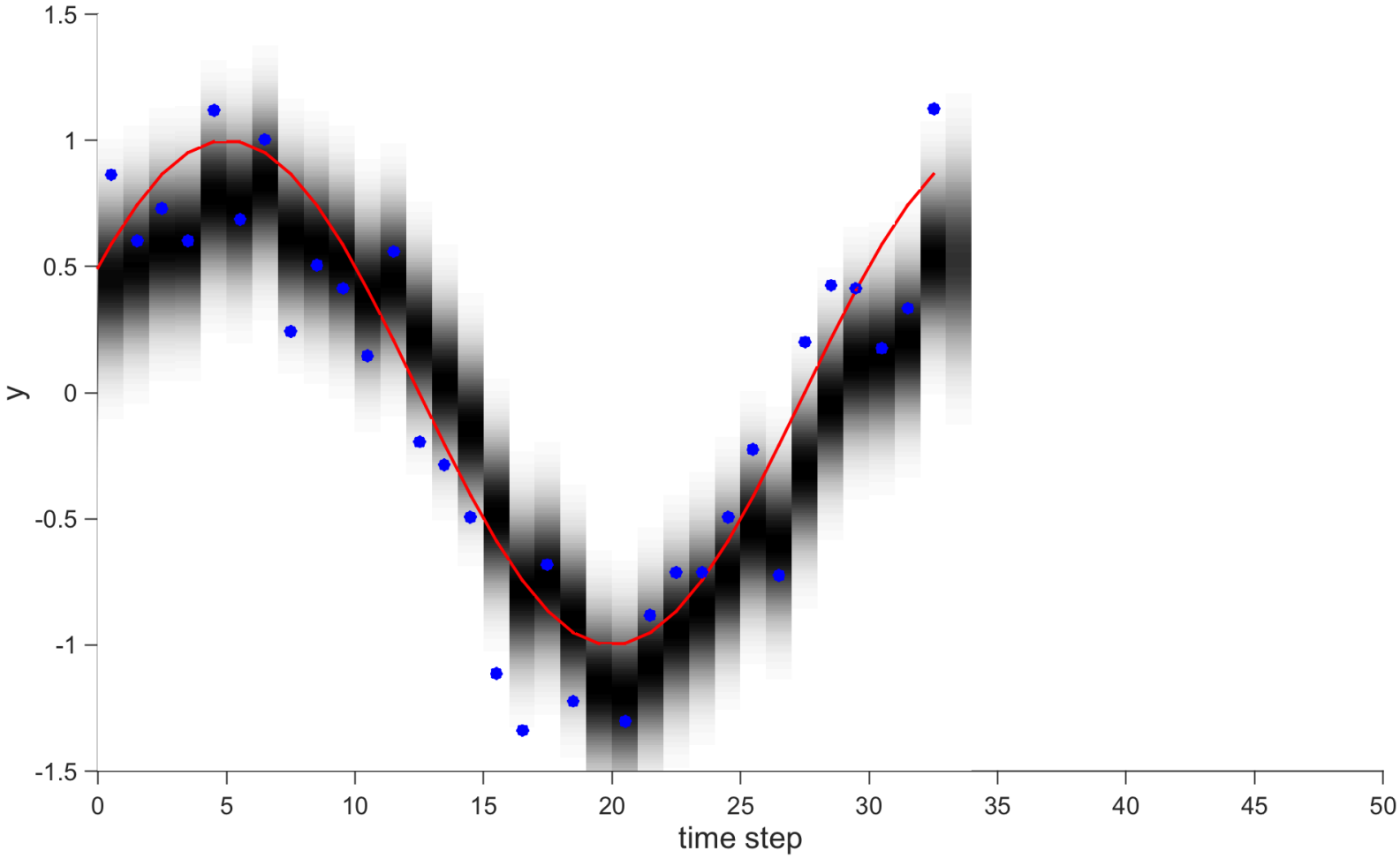

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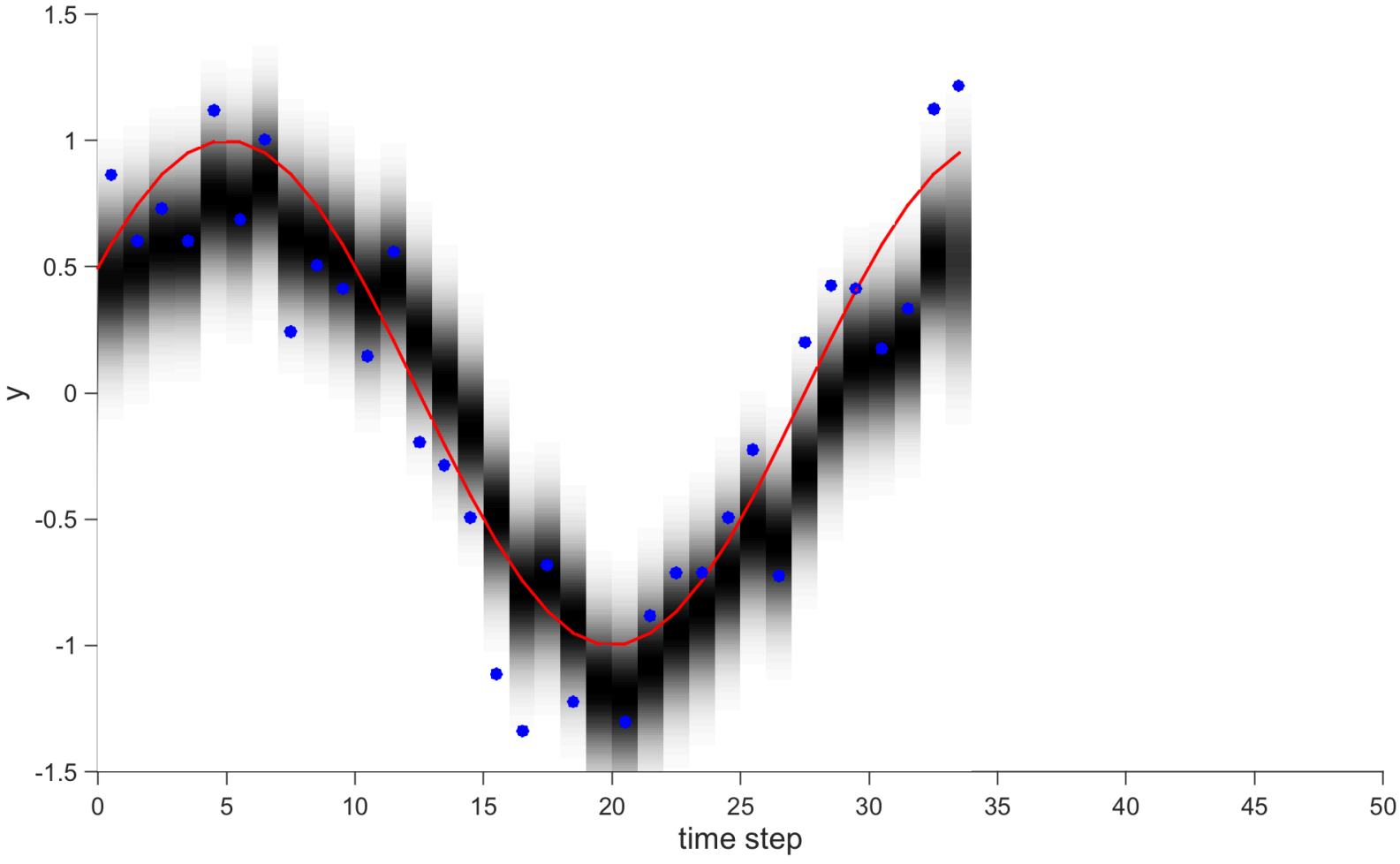

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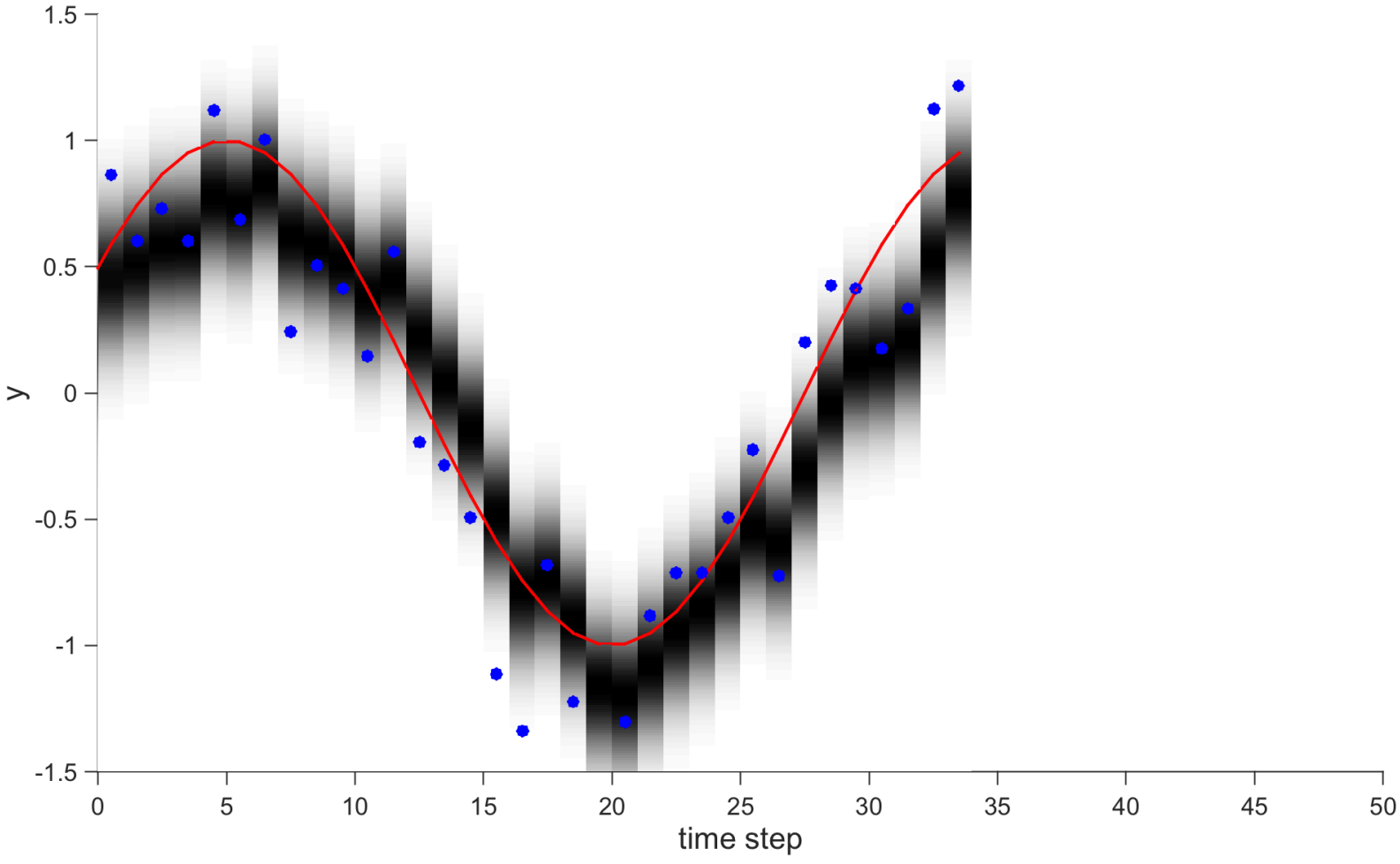

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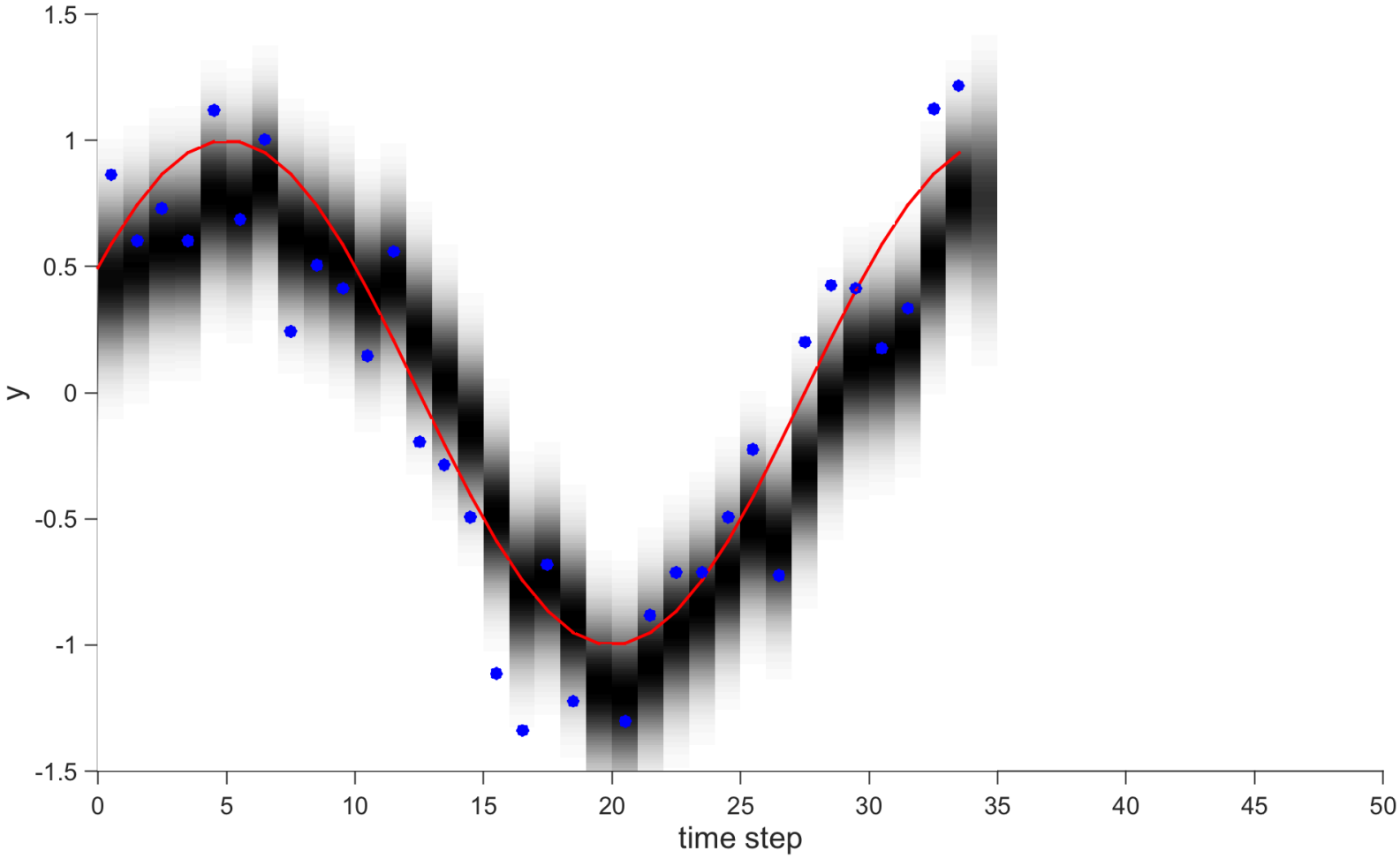

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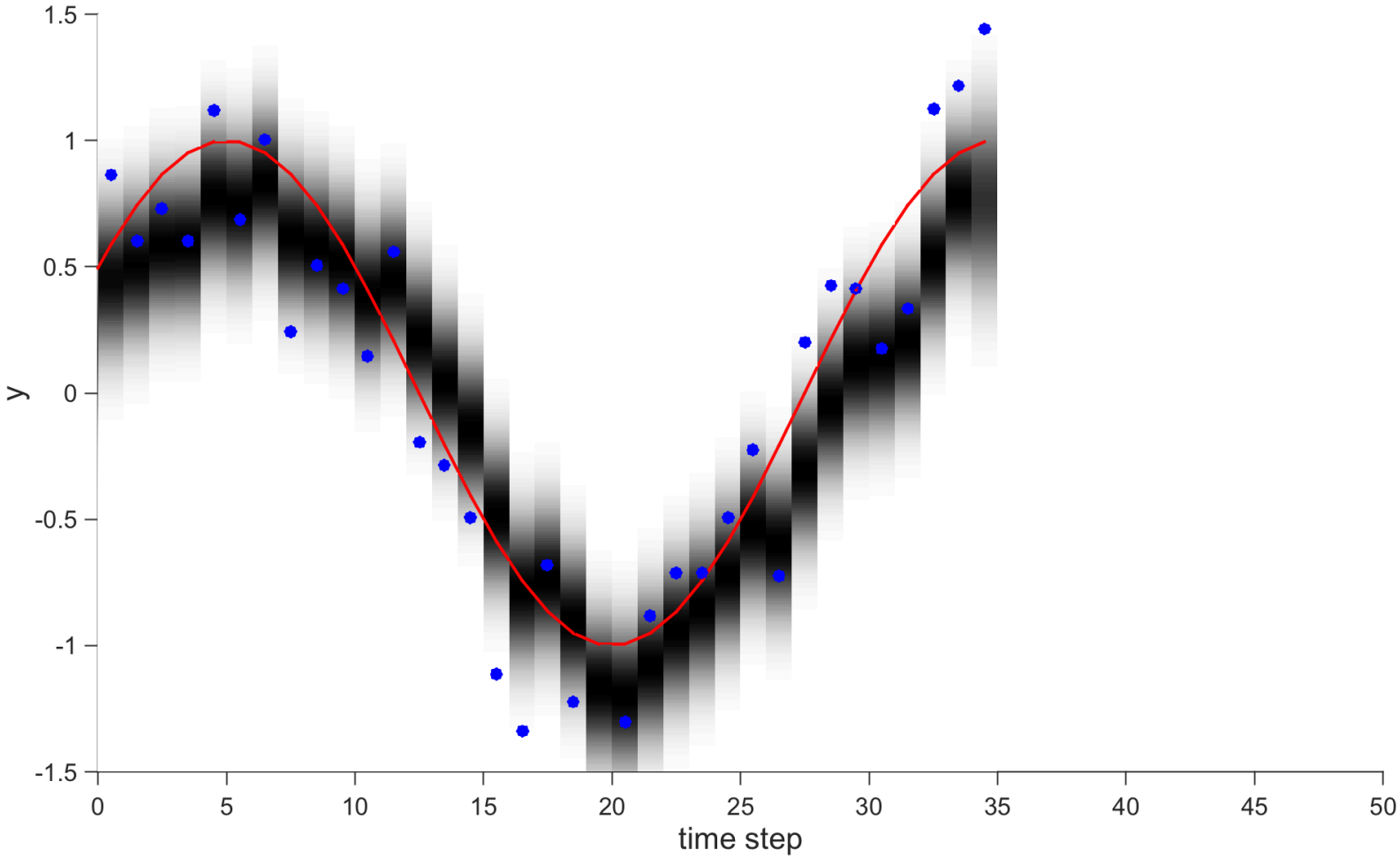
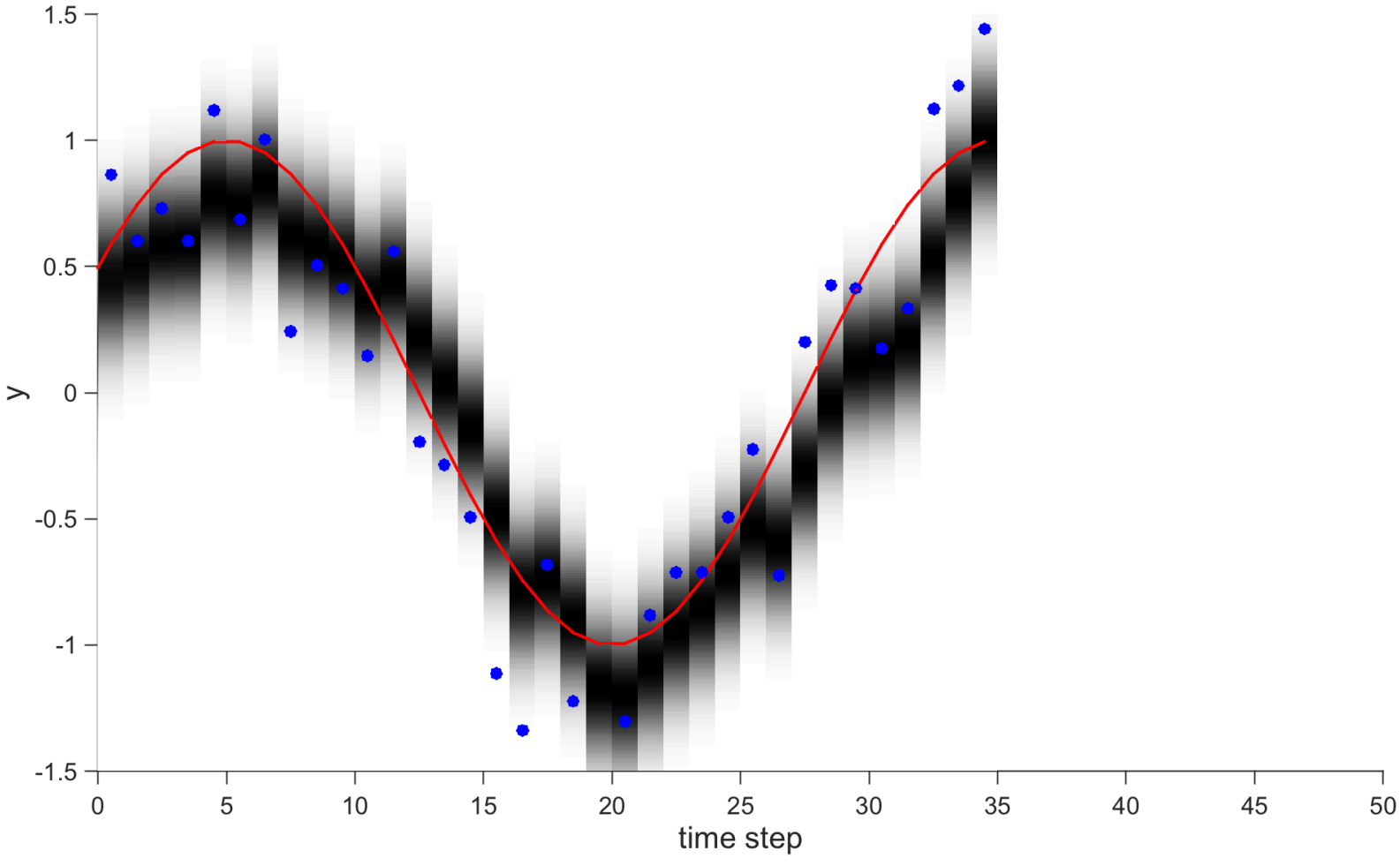


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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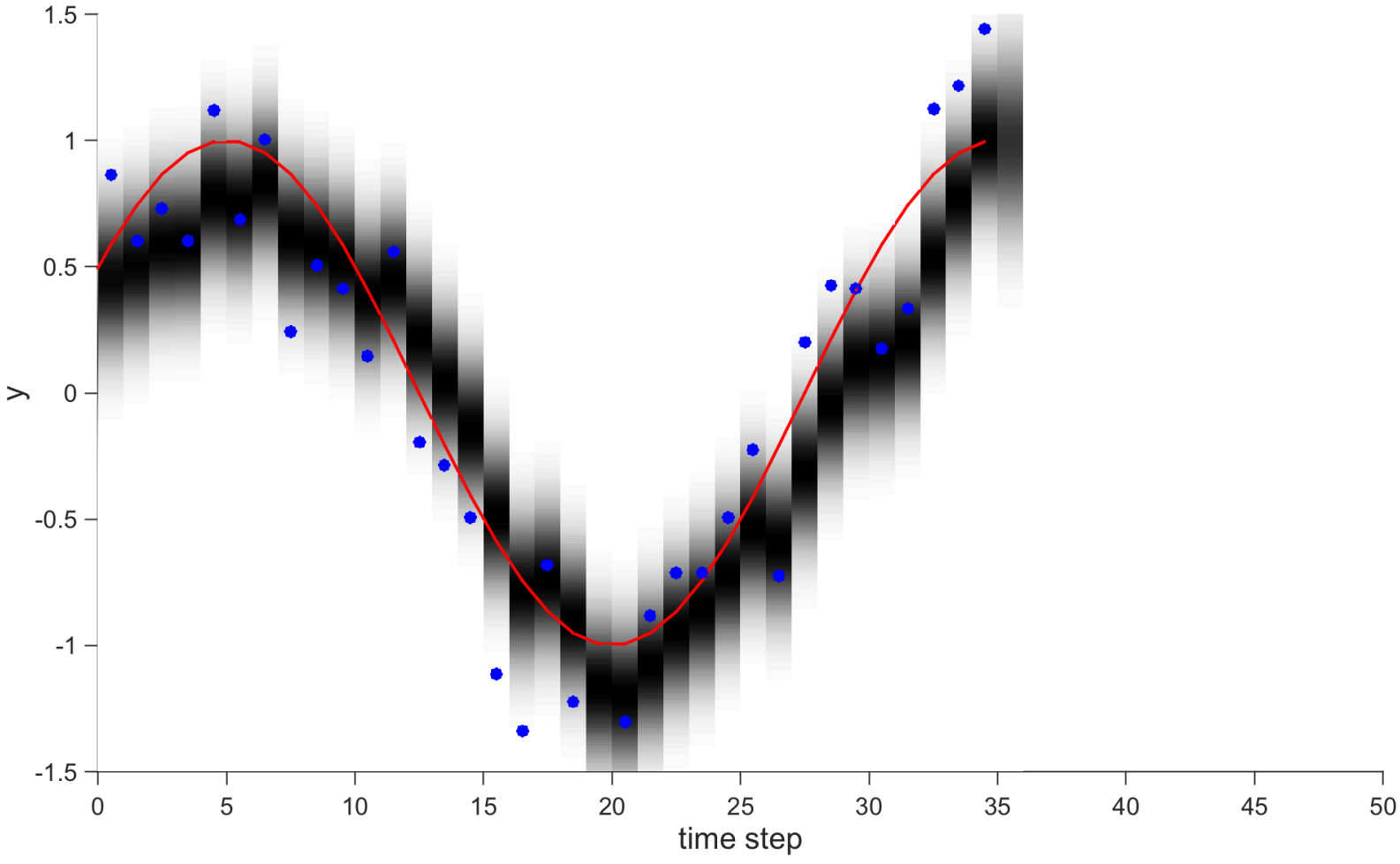

Kalman Filter Demo

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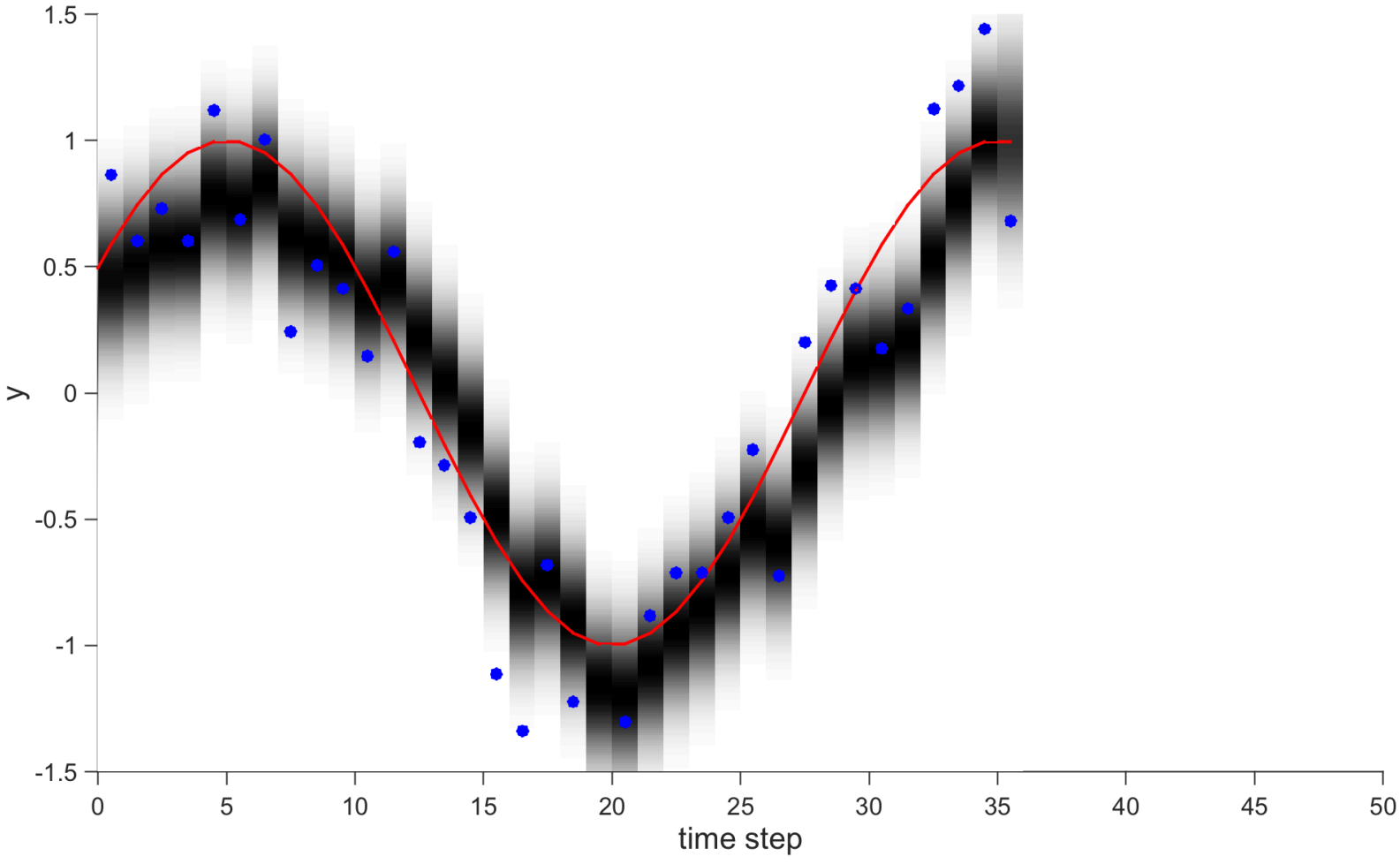

Kalman Filter Demo

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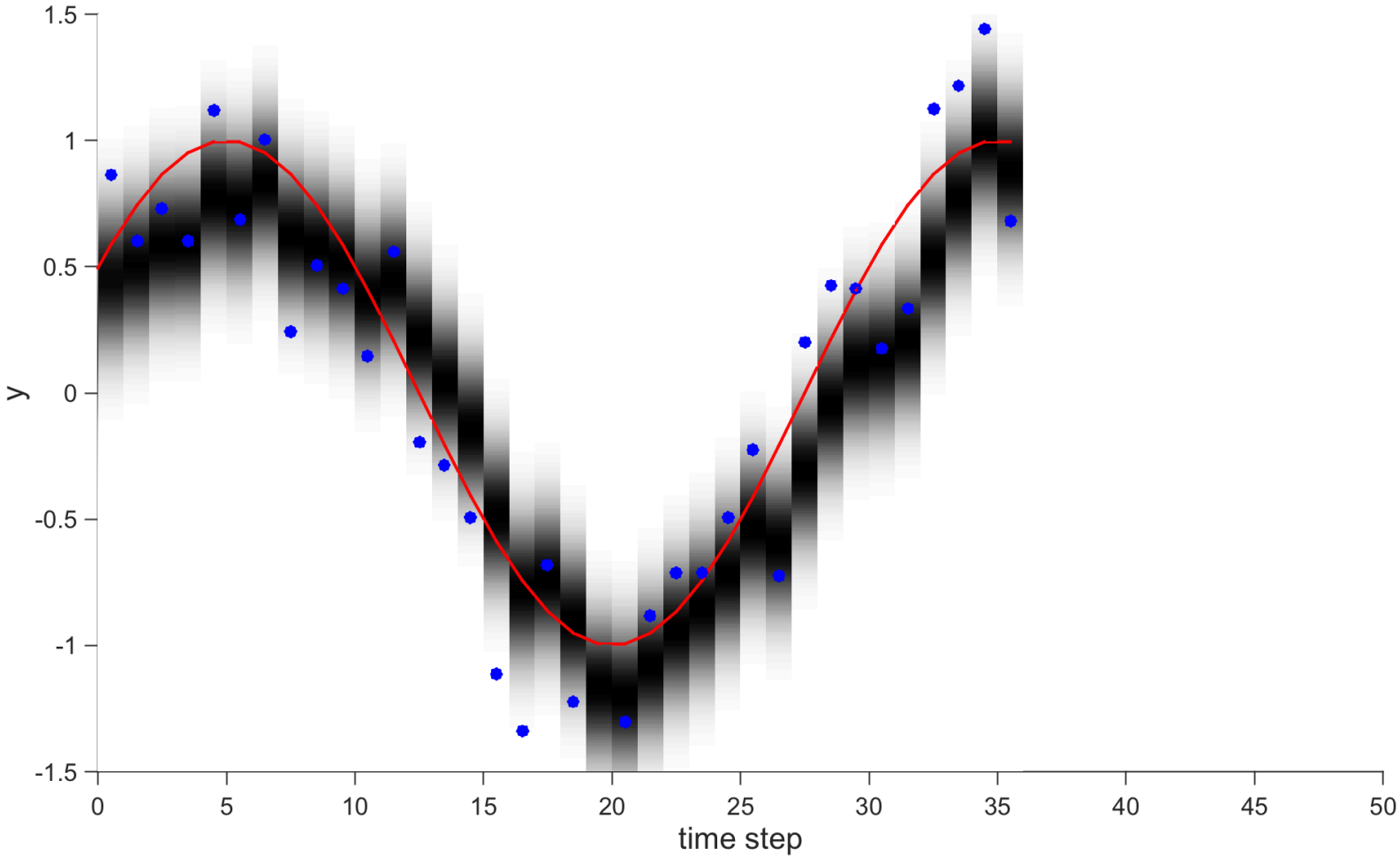

Kalman Filter Demo

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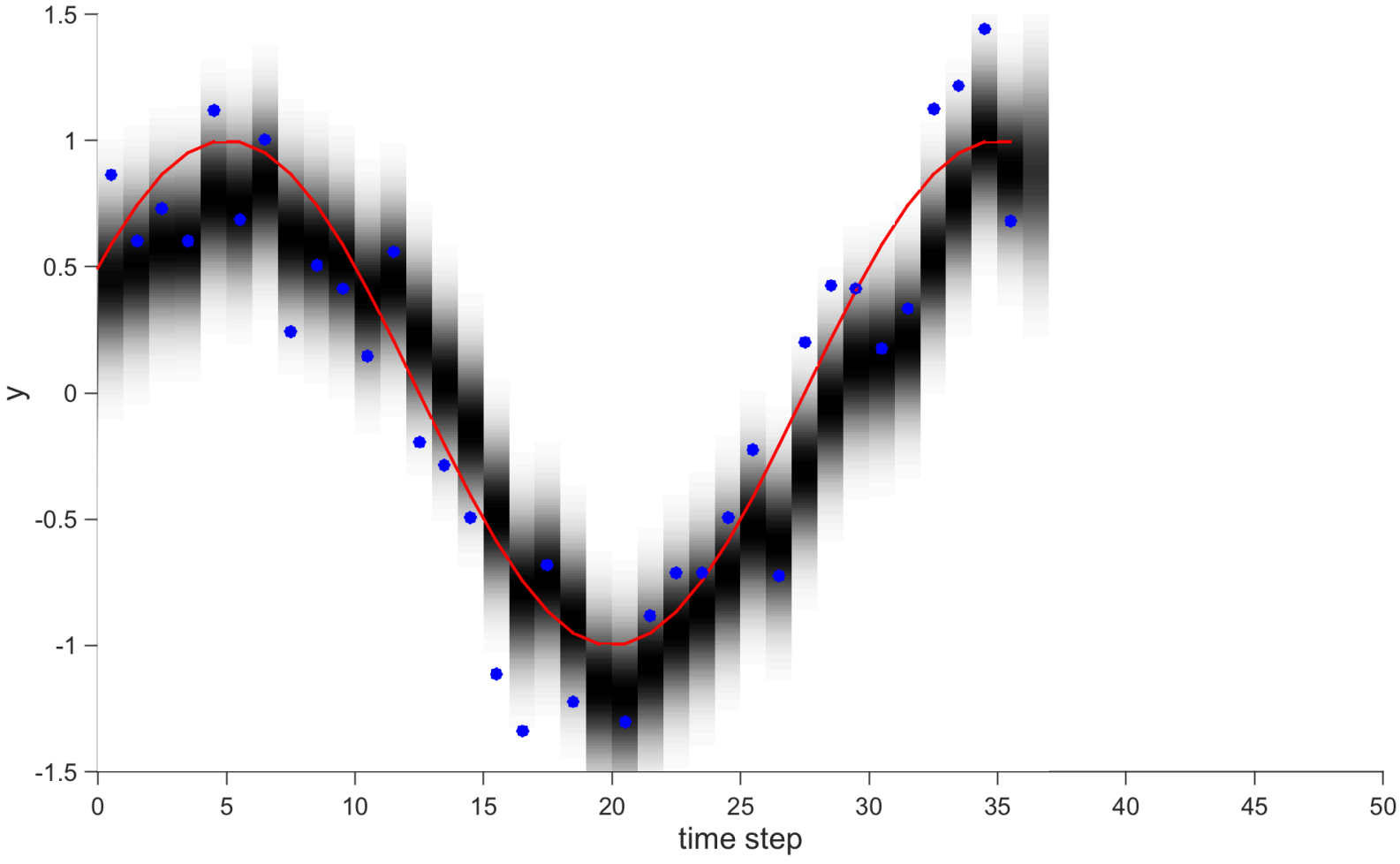

Kalman Filter Demo

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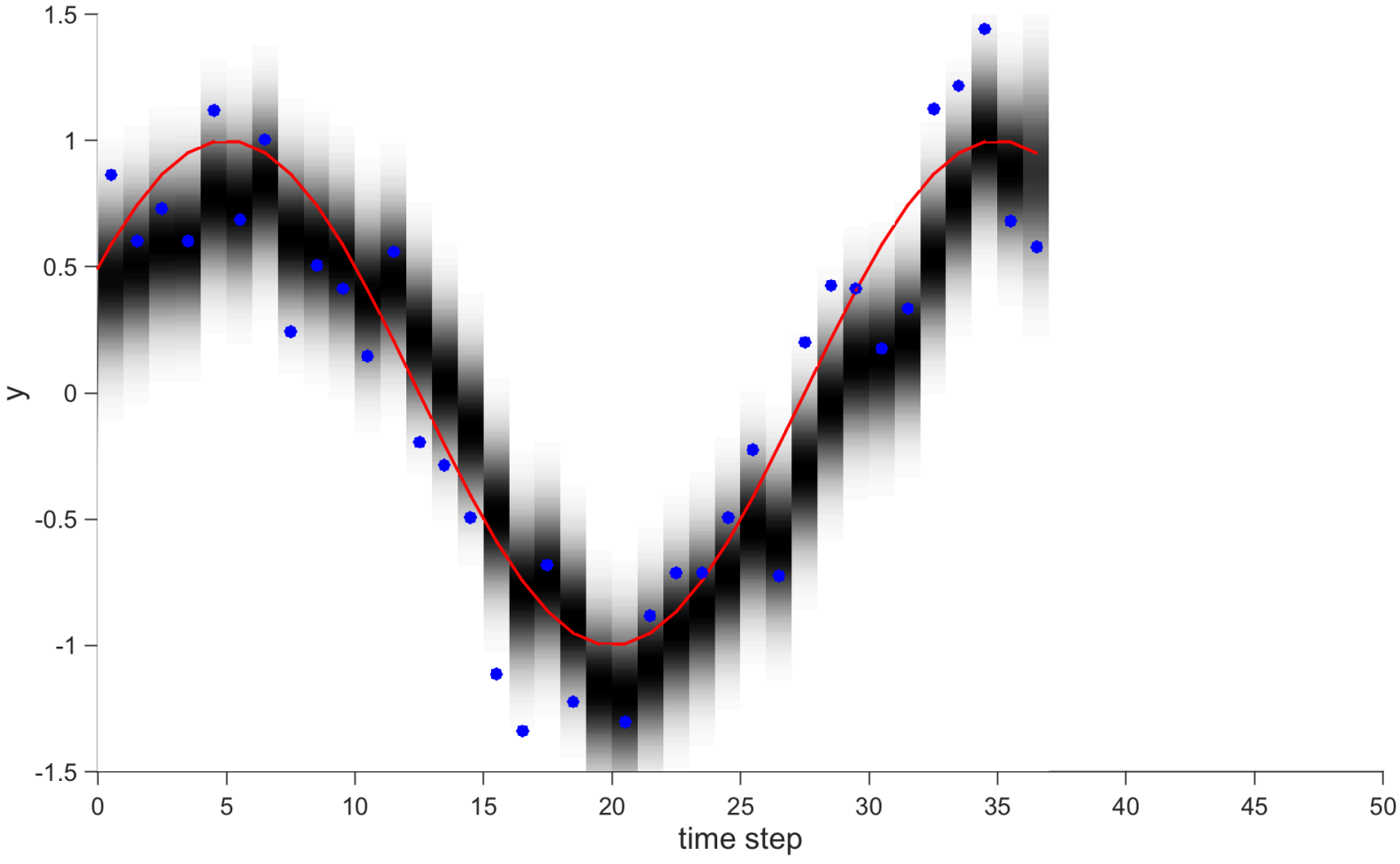
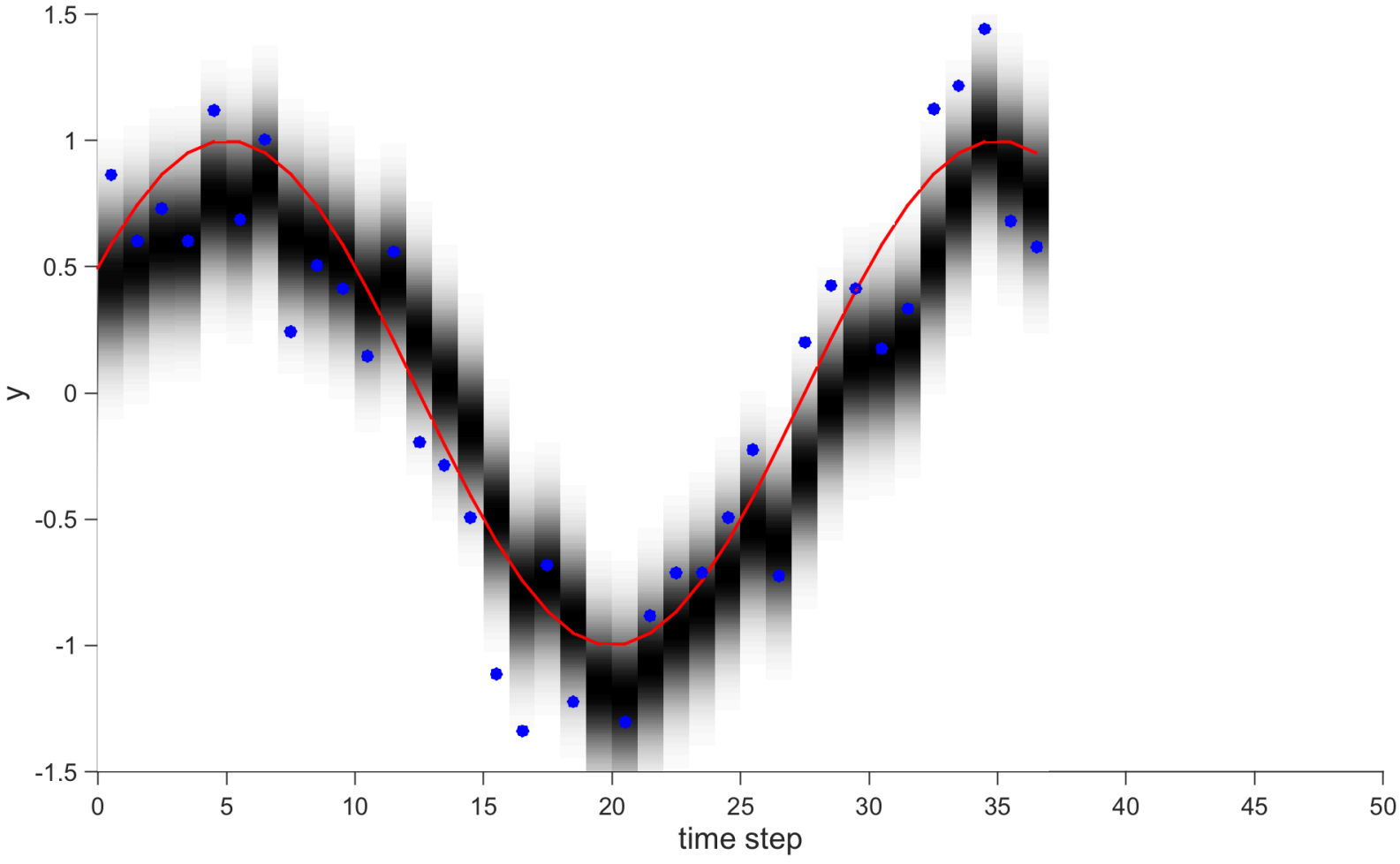


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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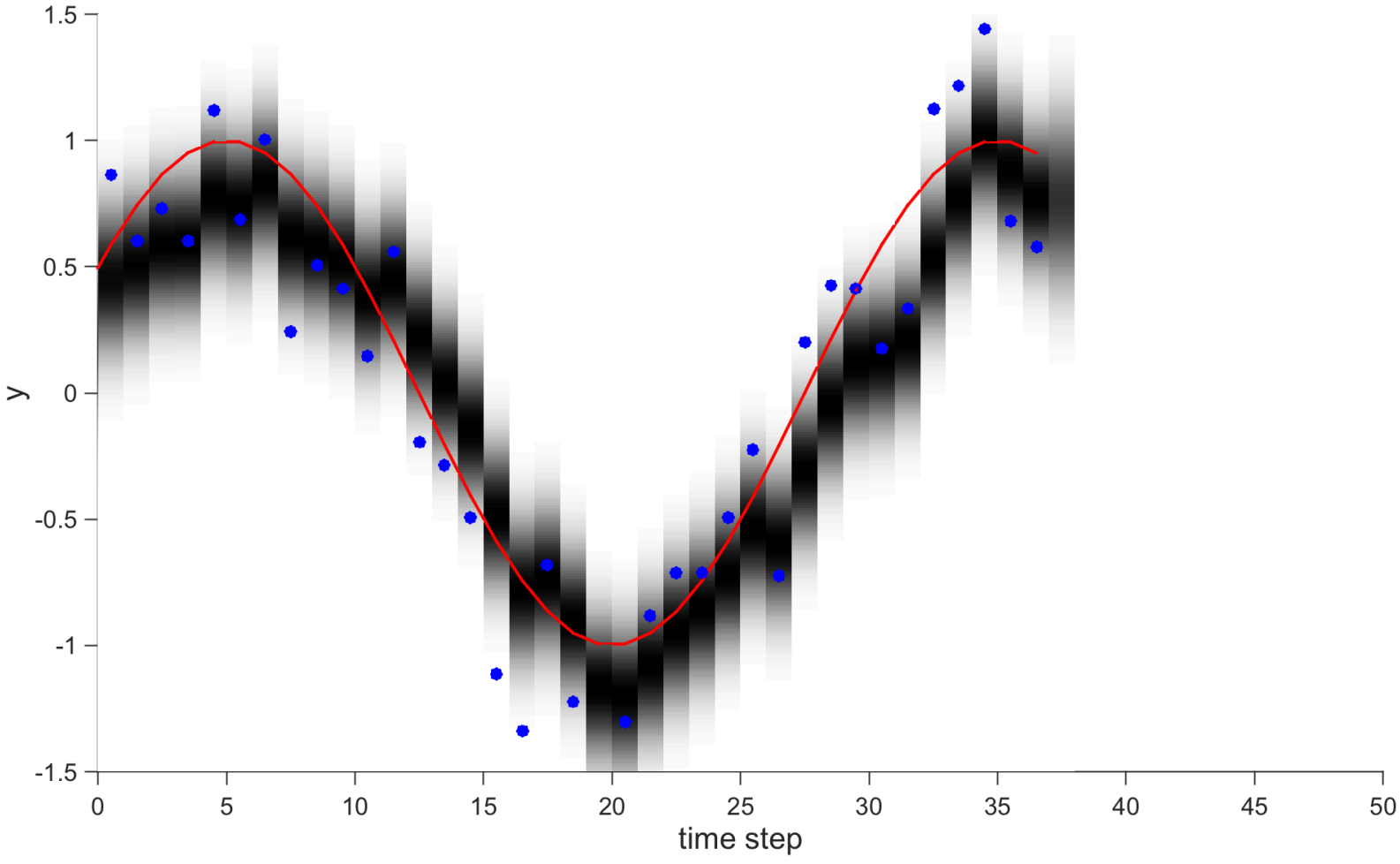

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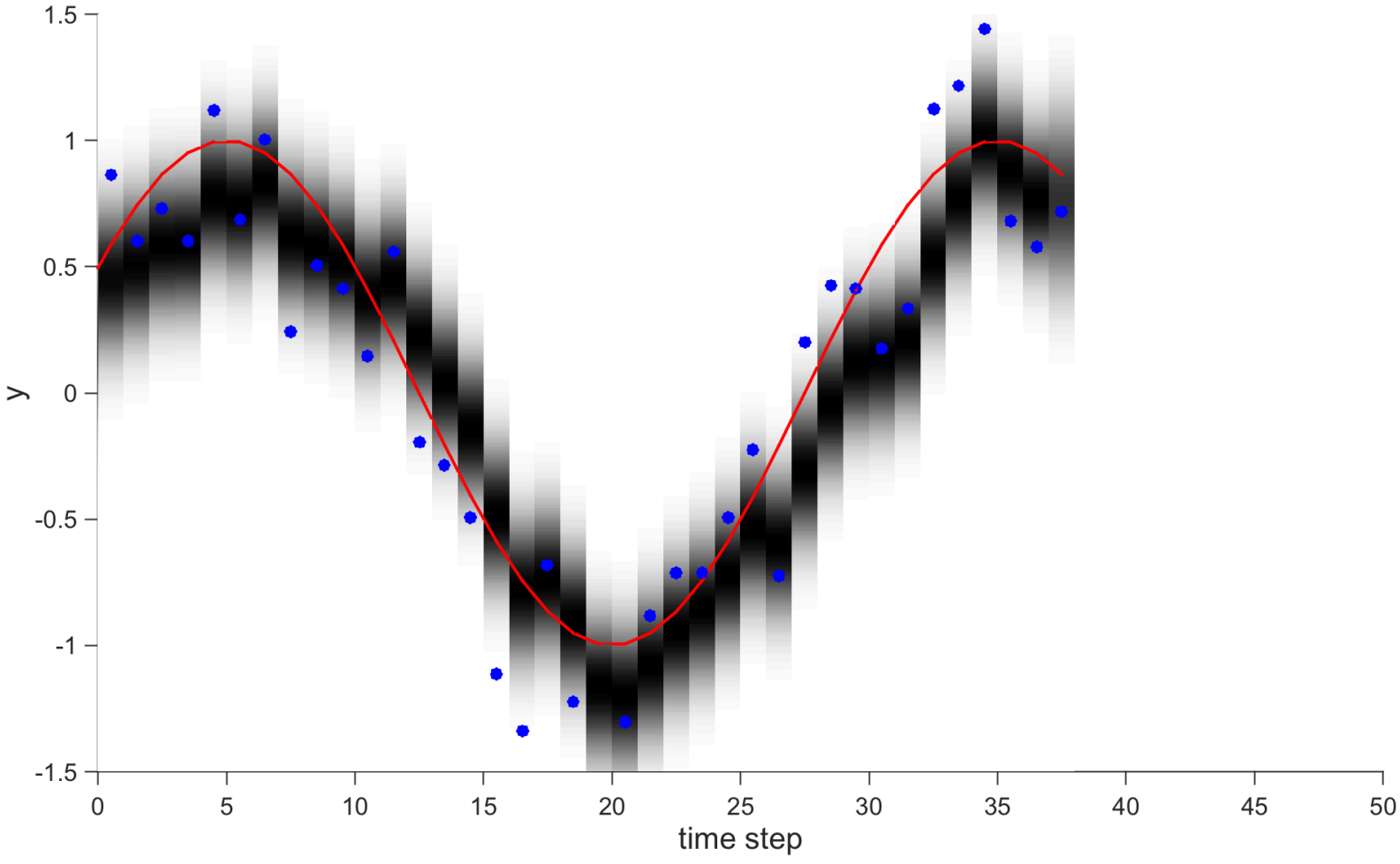
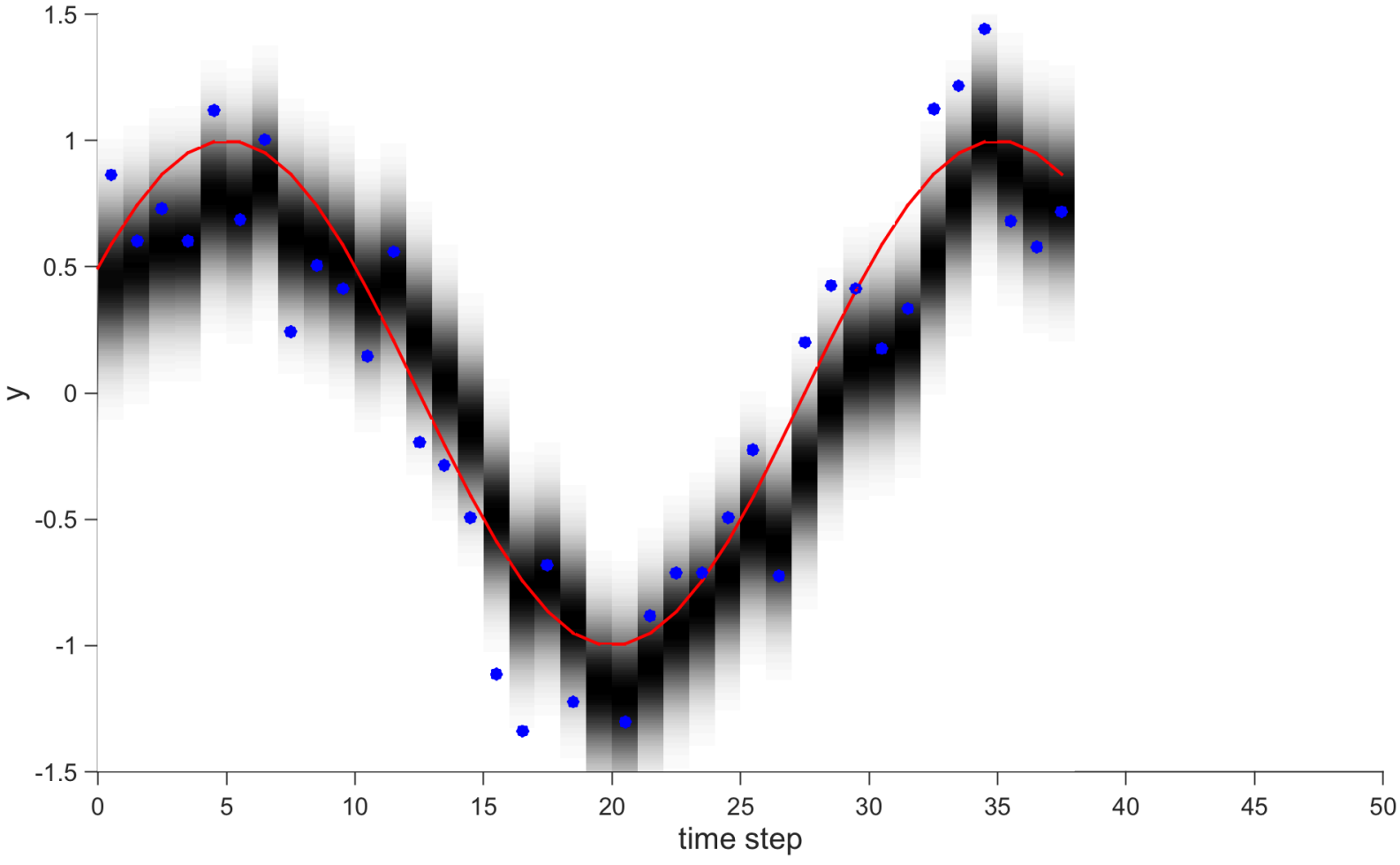


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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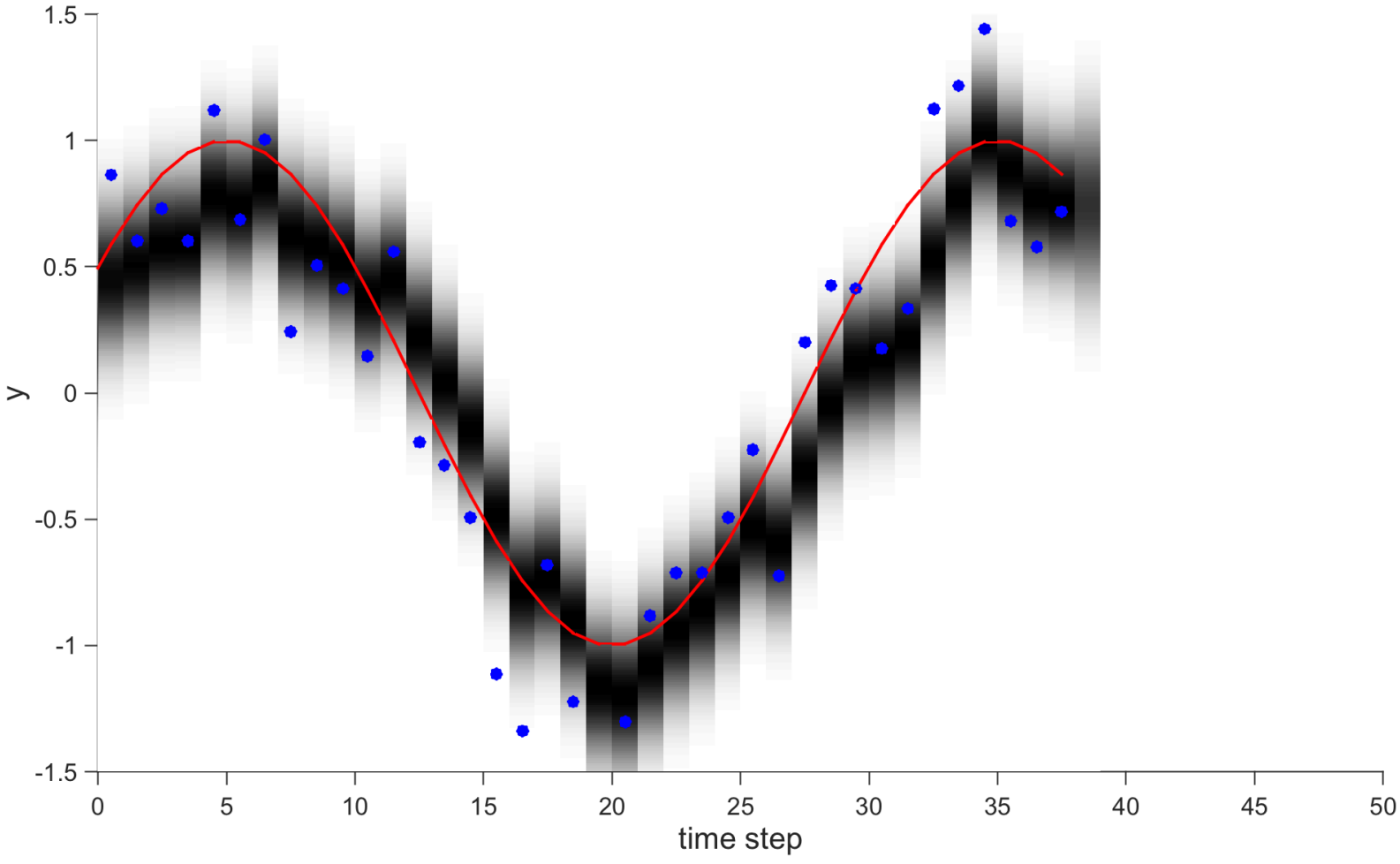

Kalman Filter Demo

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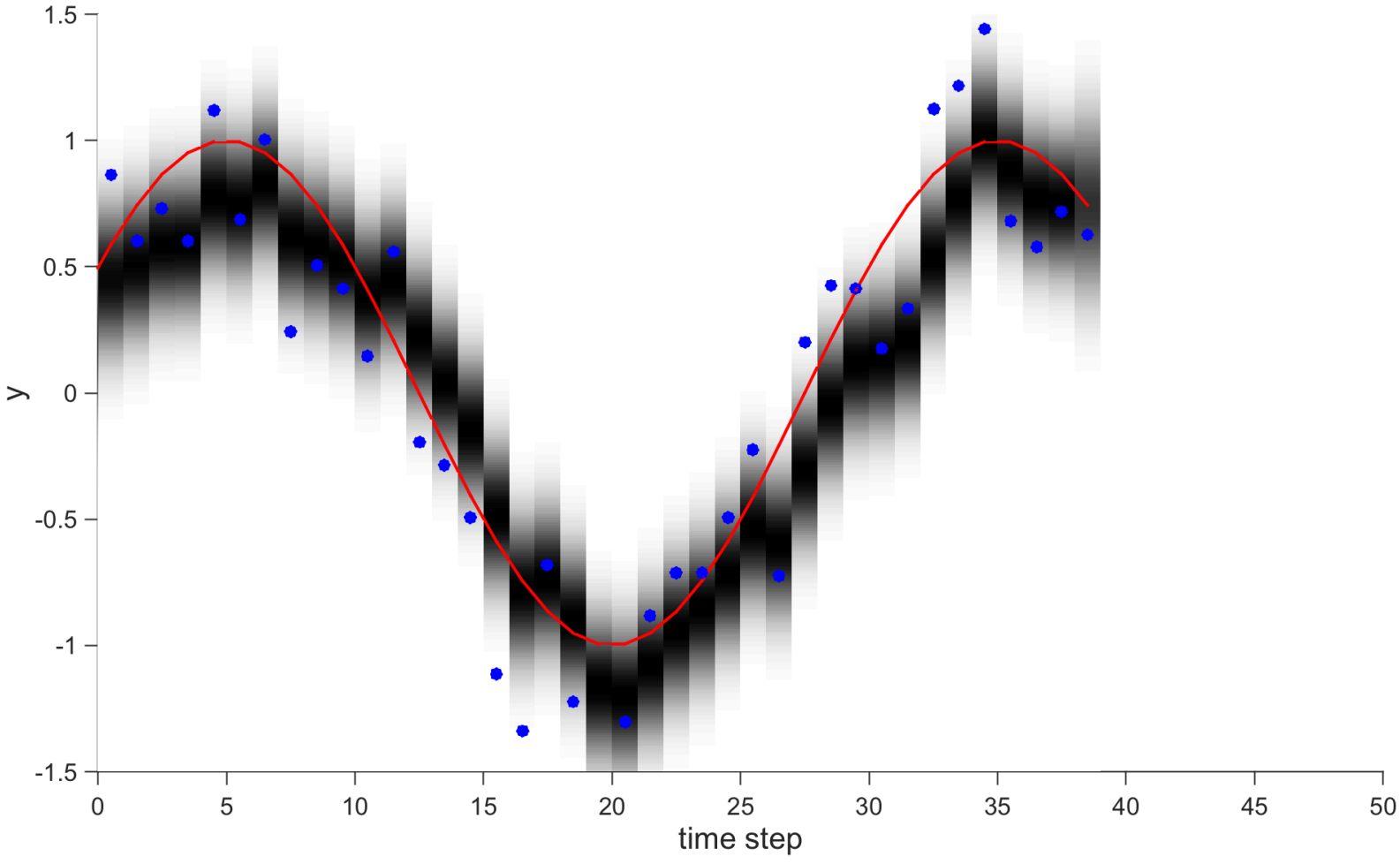

Kalman Filter Demo

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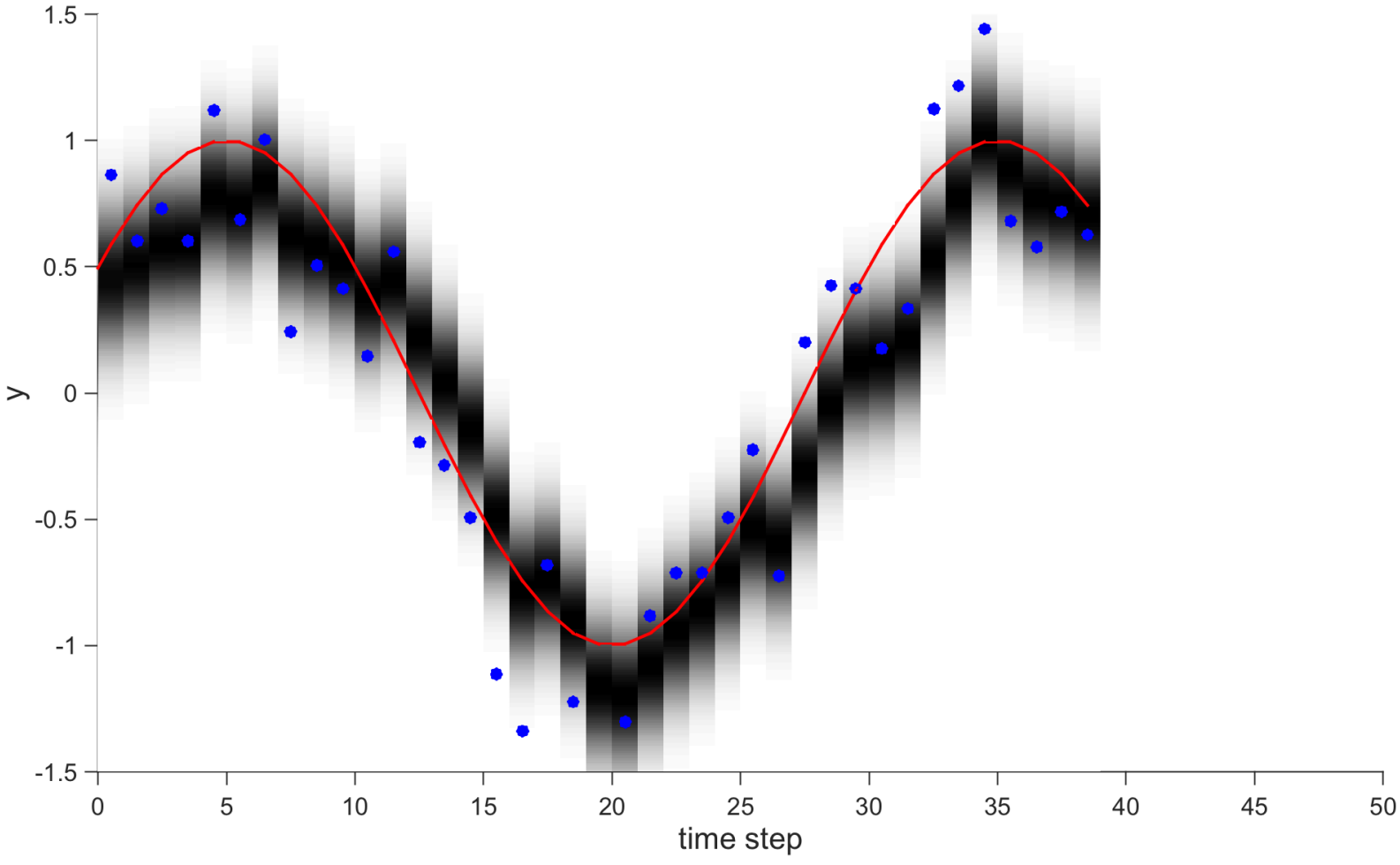

Kalman Filter Demo

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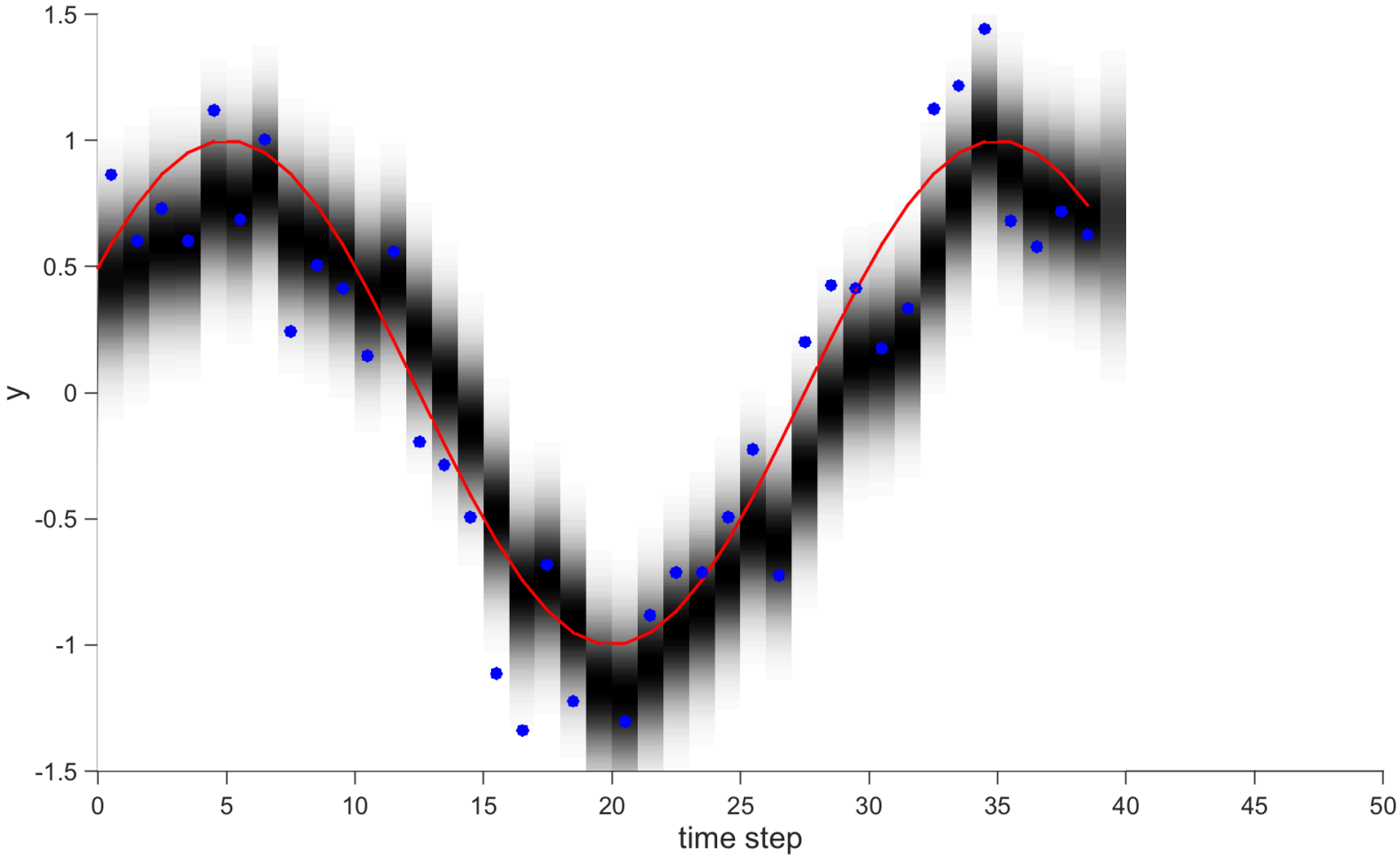

Kalman Filter Demo

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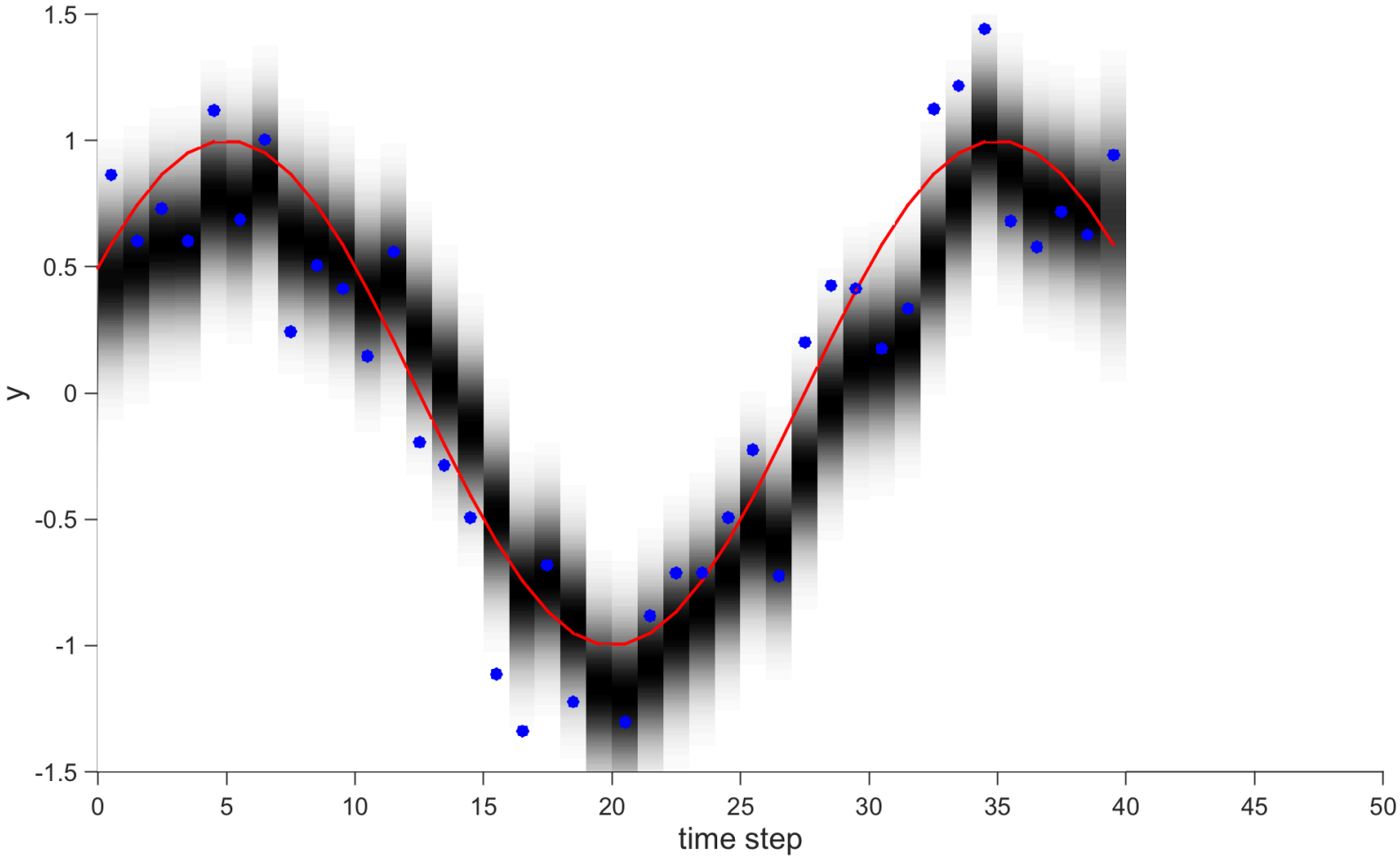

Kalman Filter Demo

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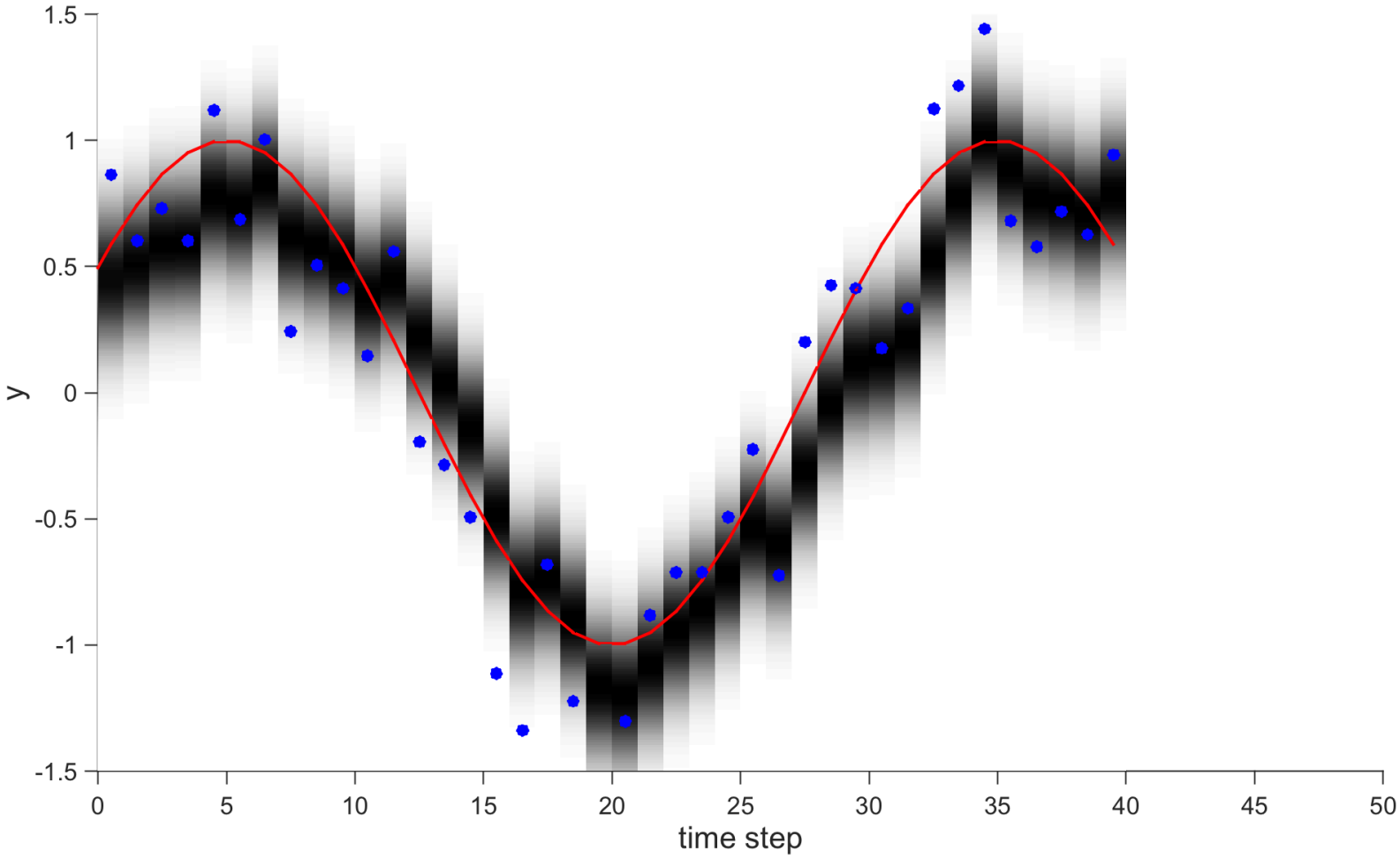

Kalman Filter Demo

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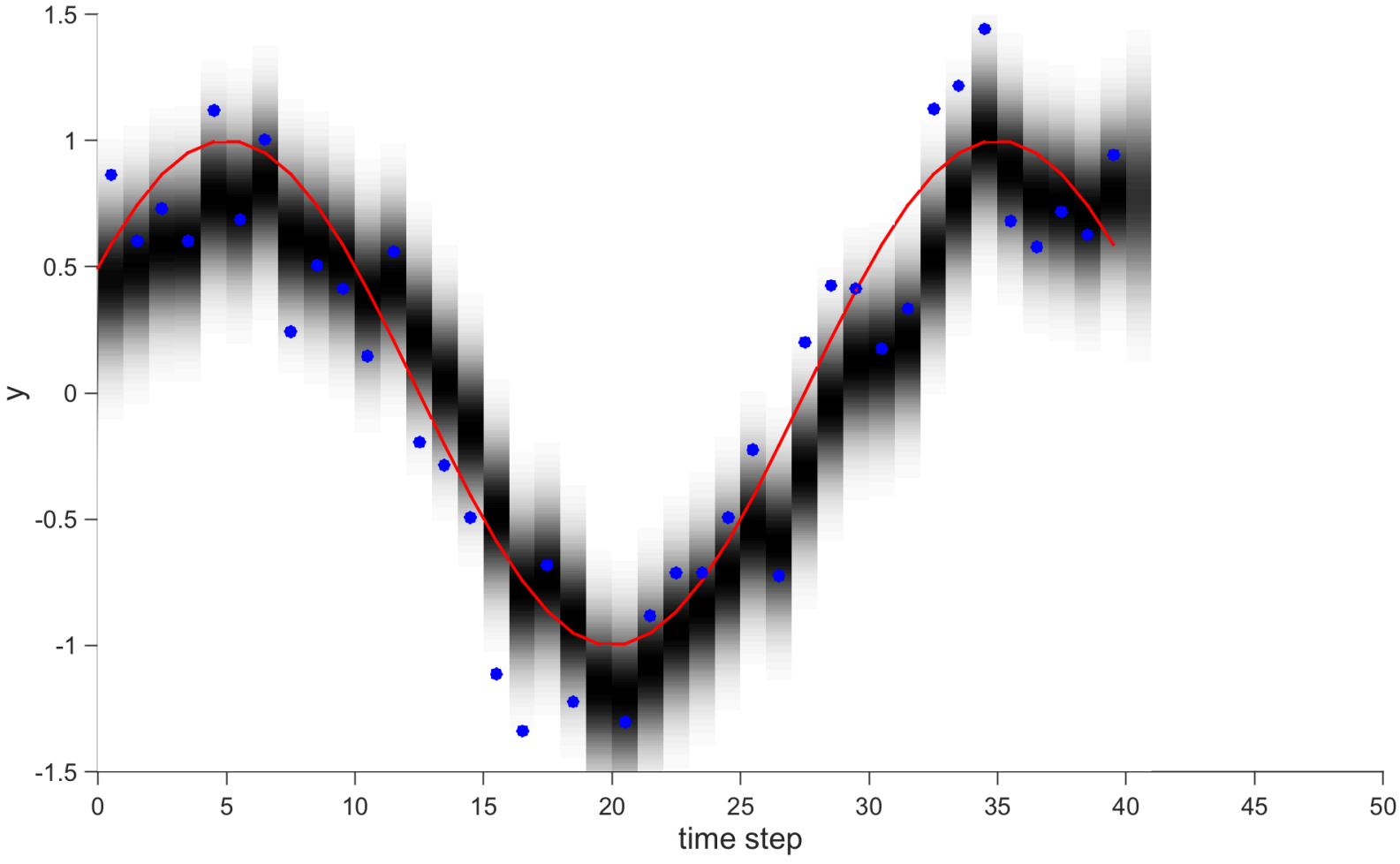

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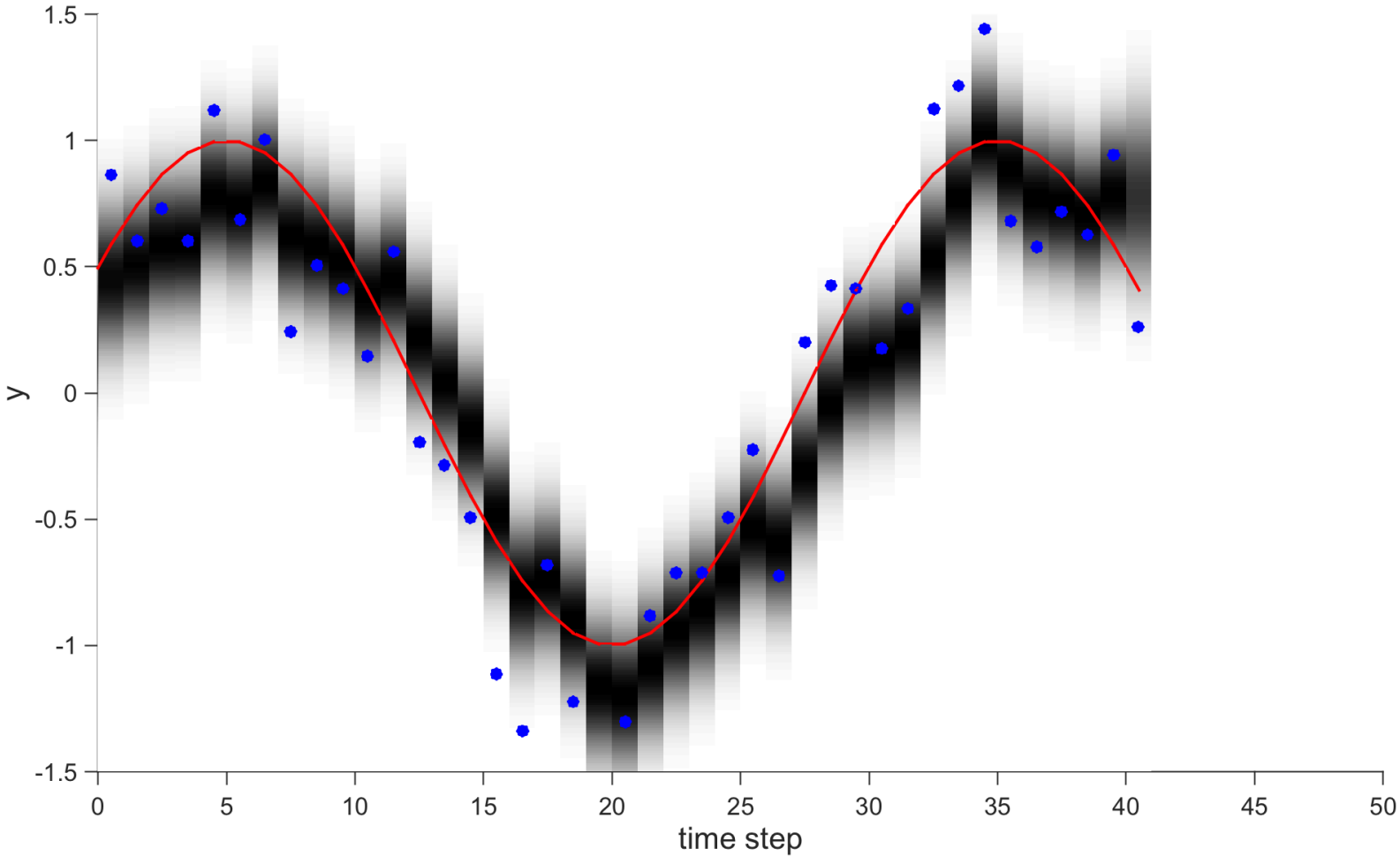

Kalman Filter Demo

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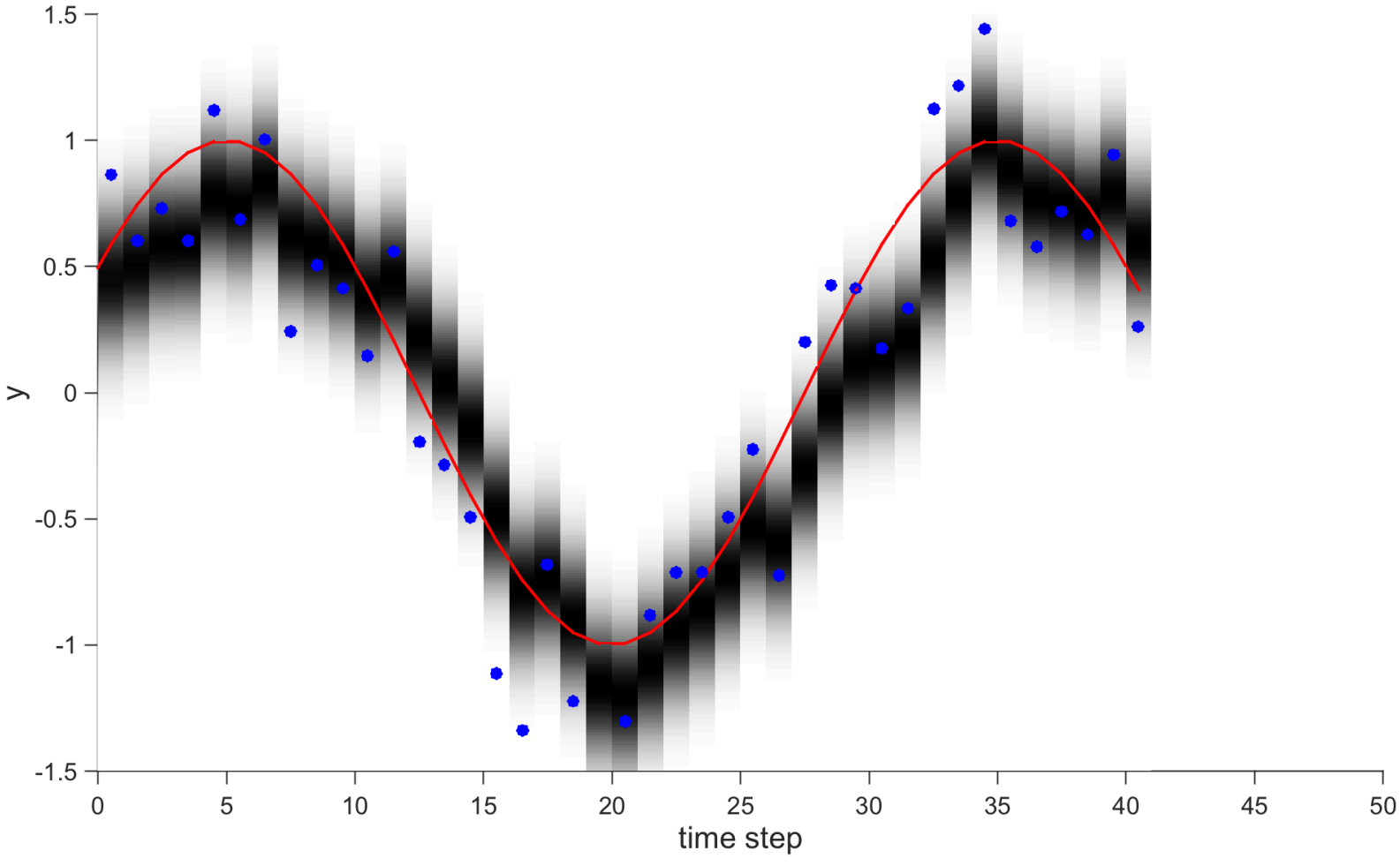

Kalman Filter Demo

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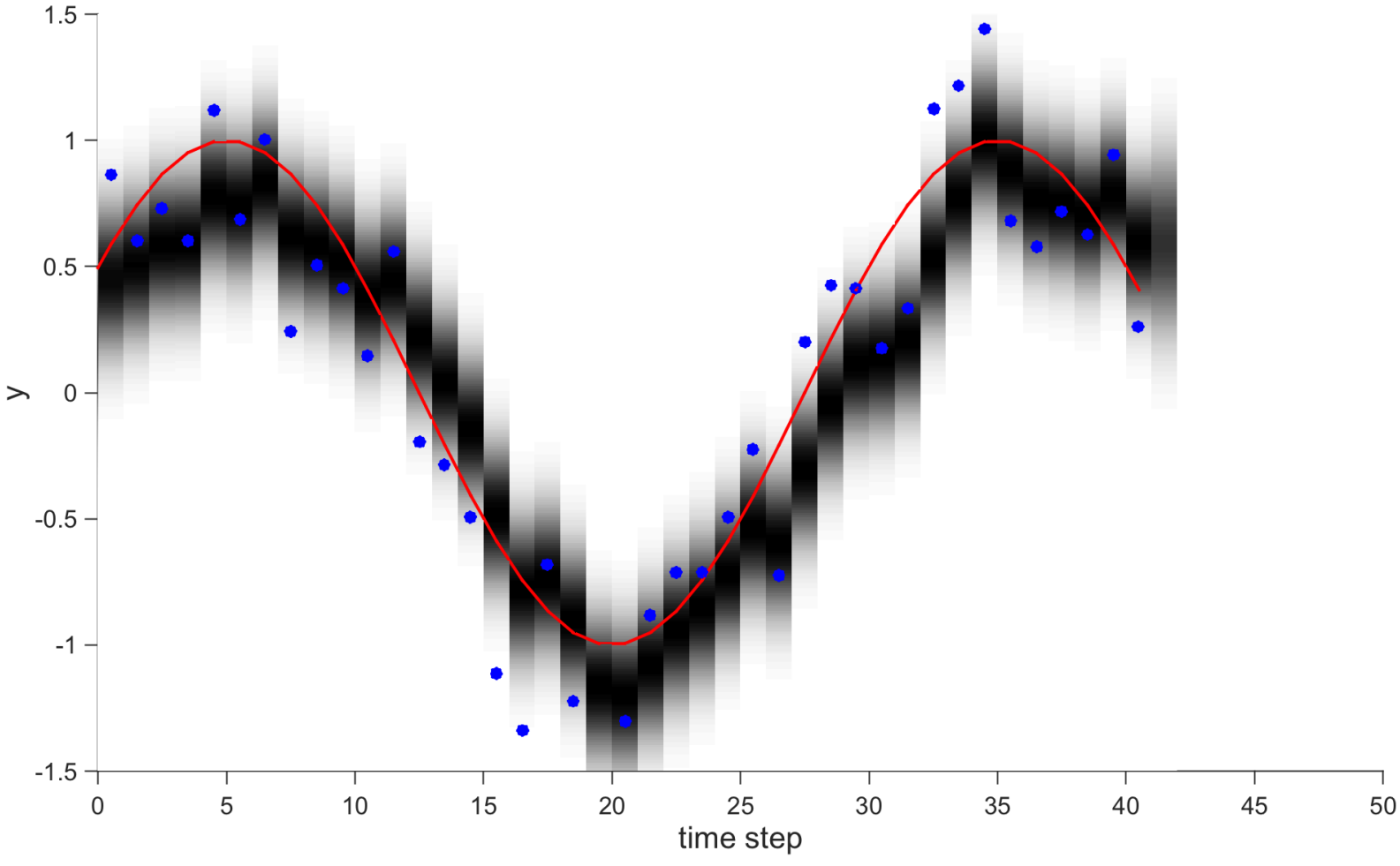

Kalman Filter Demo

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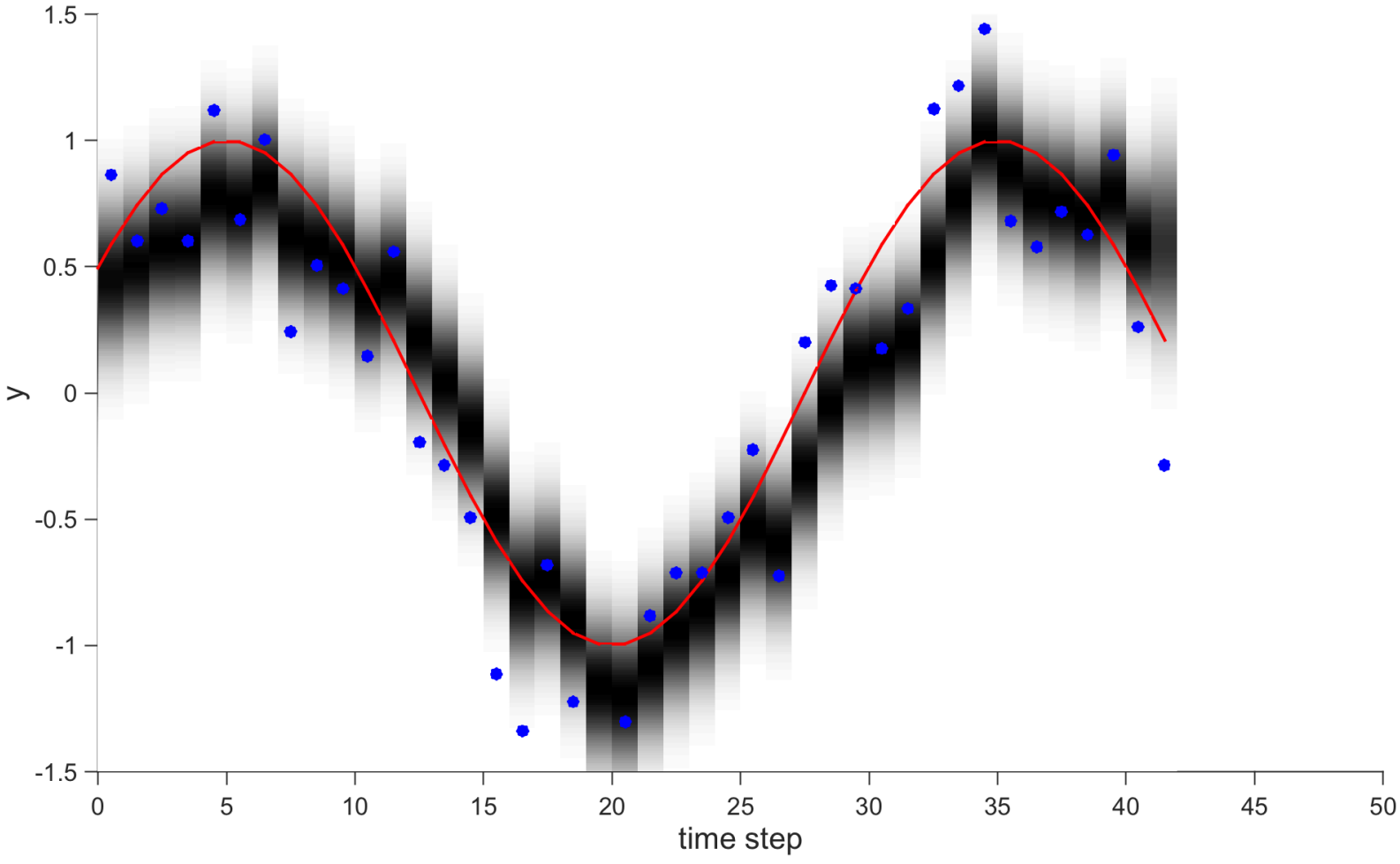

Kalman Filter Demo

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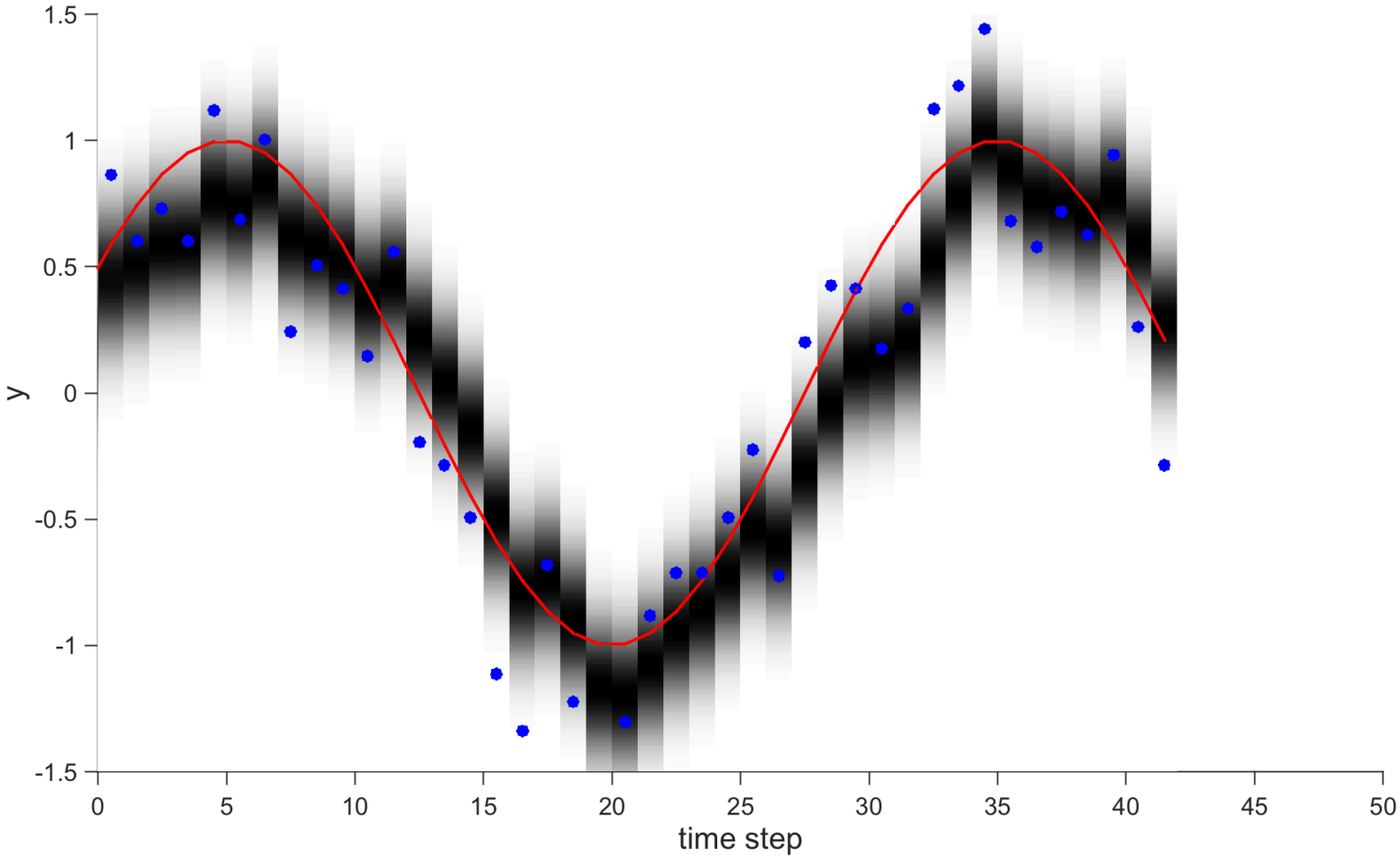

Kalman Filter Demo

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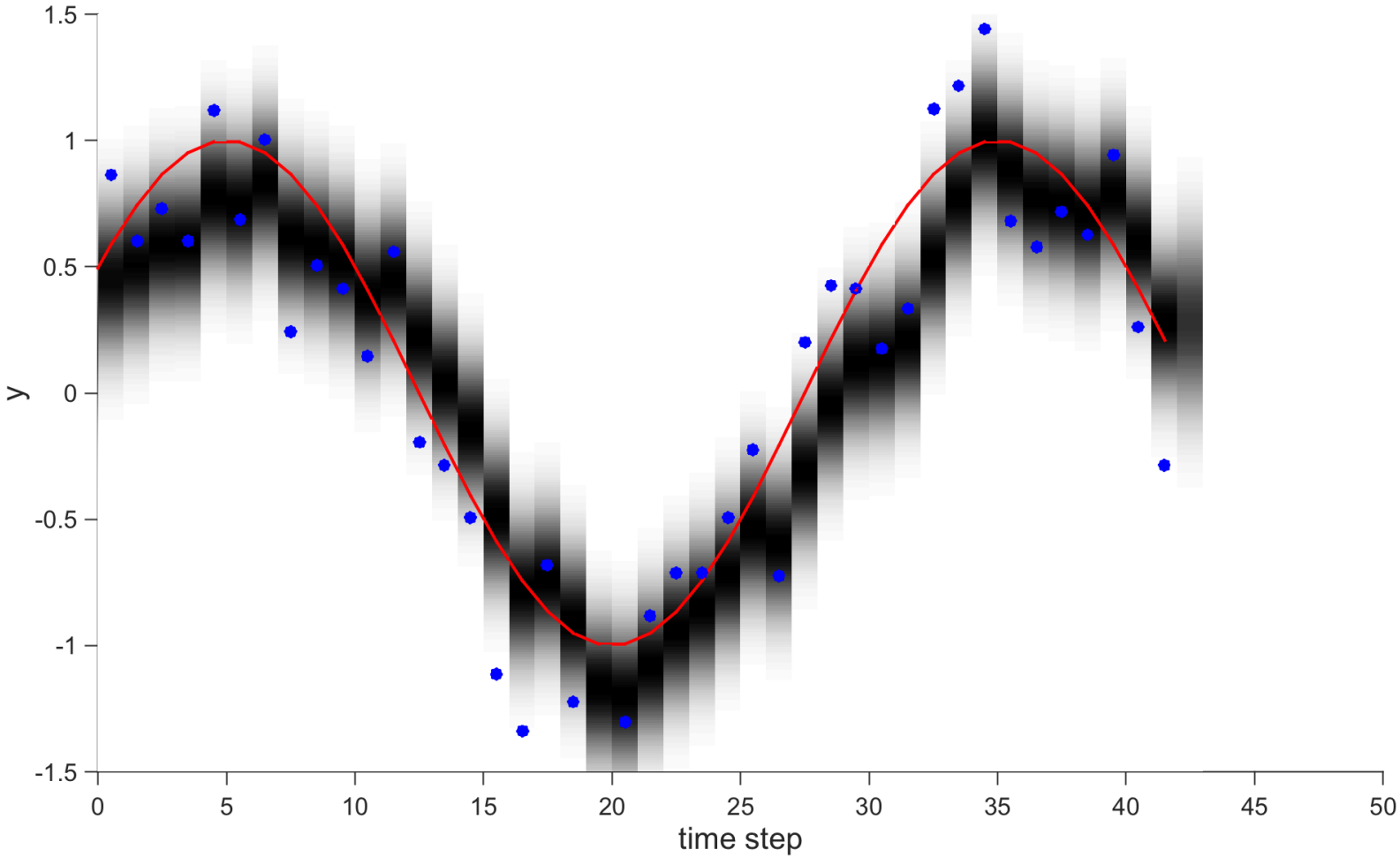

Kalman Filter Demo

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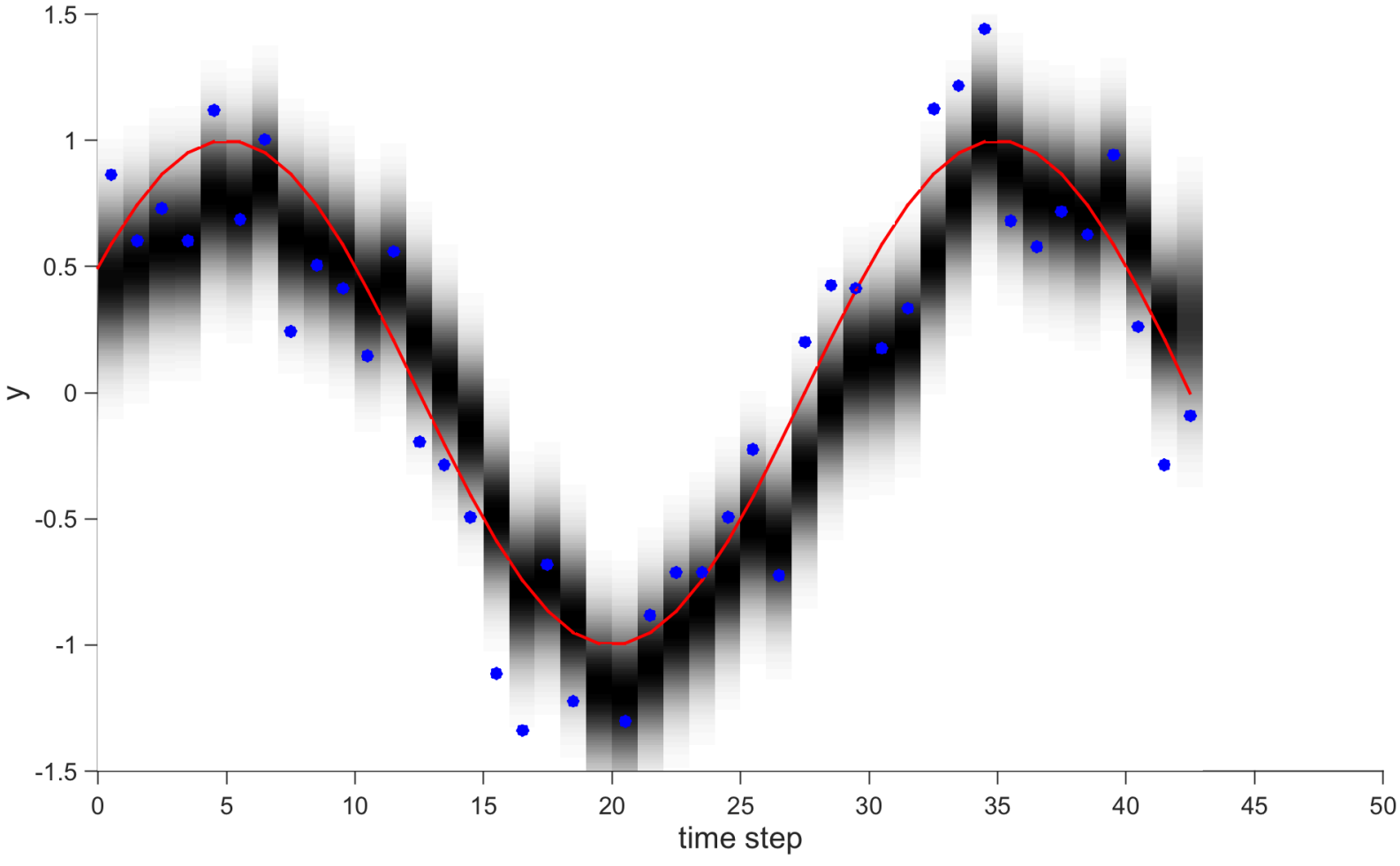

Kalman Filter Demo

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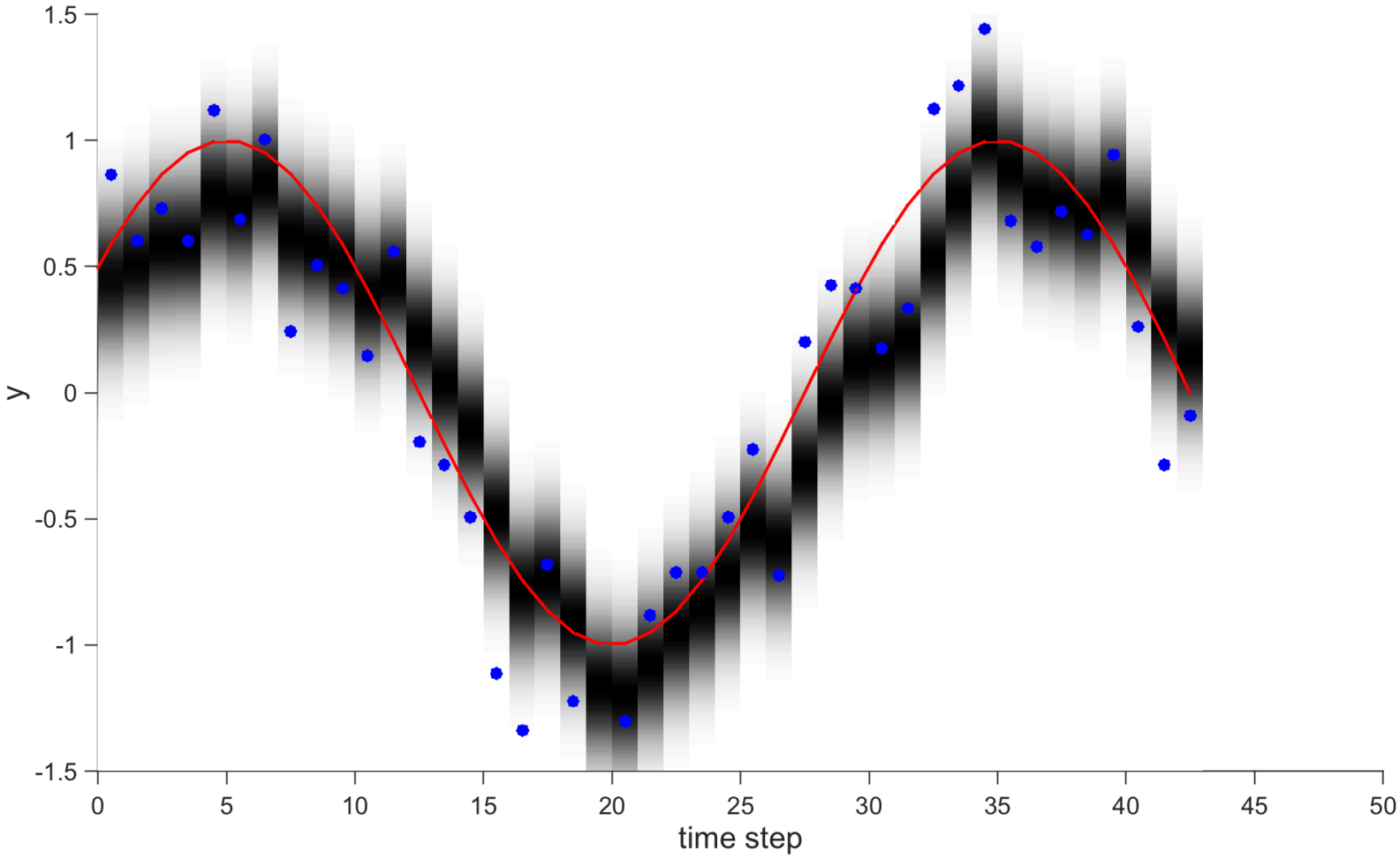

Kalman Filter Demo

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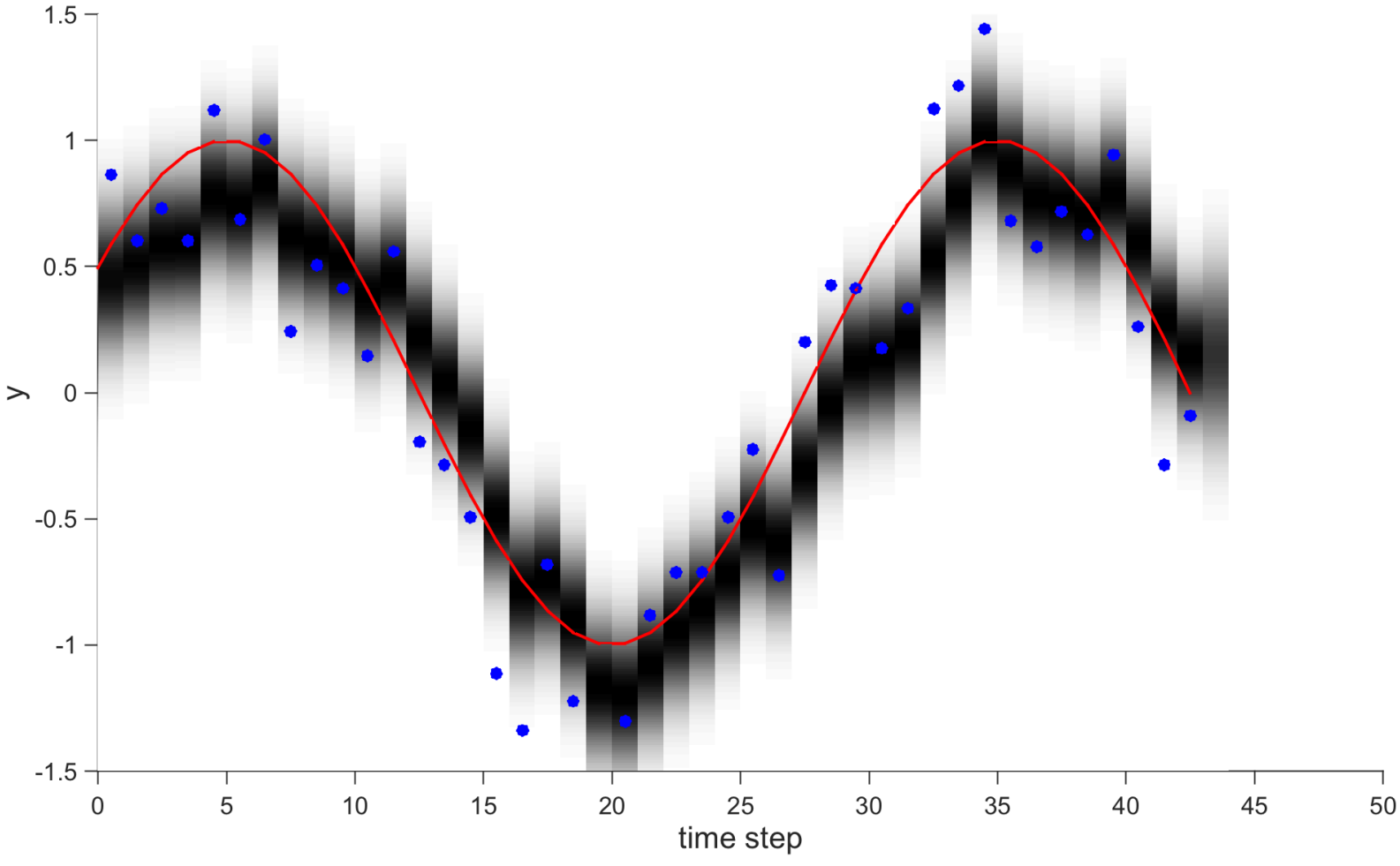
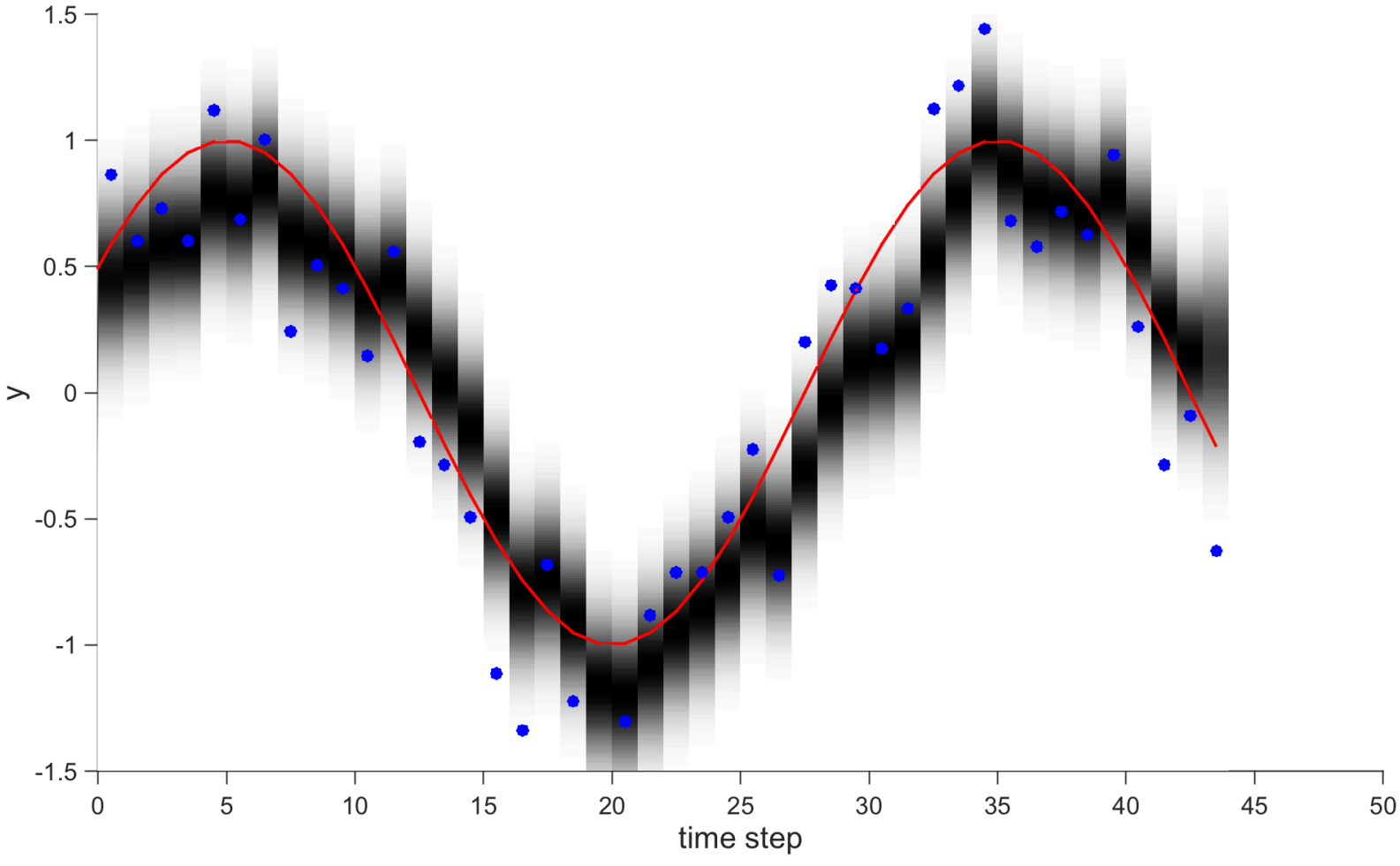


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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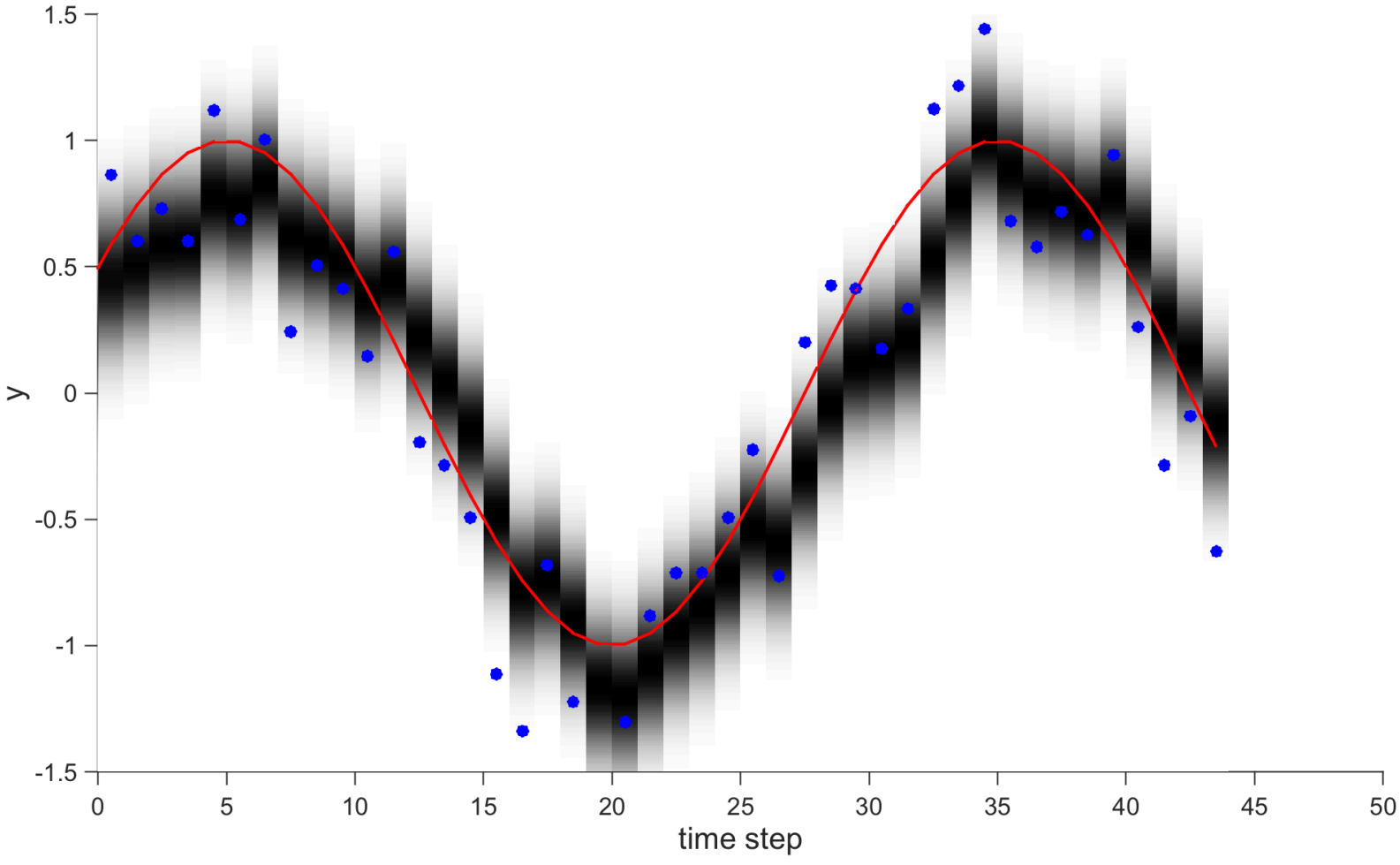

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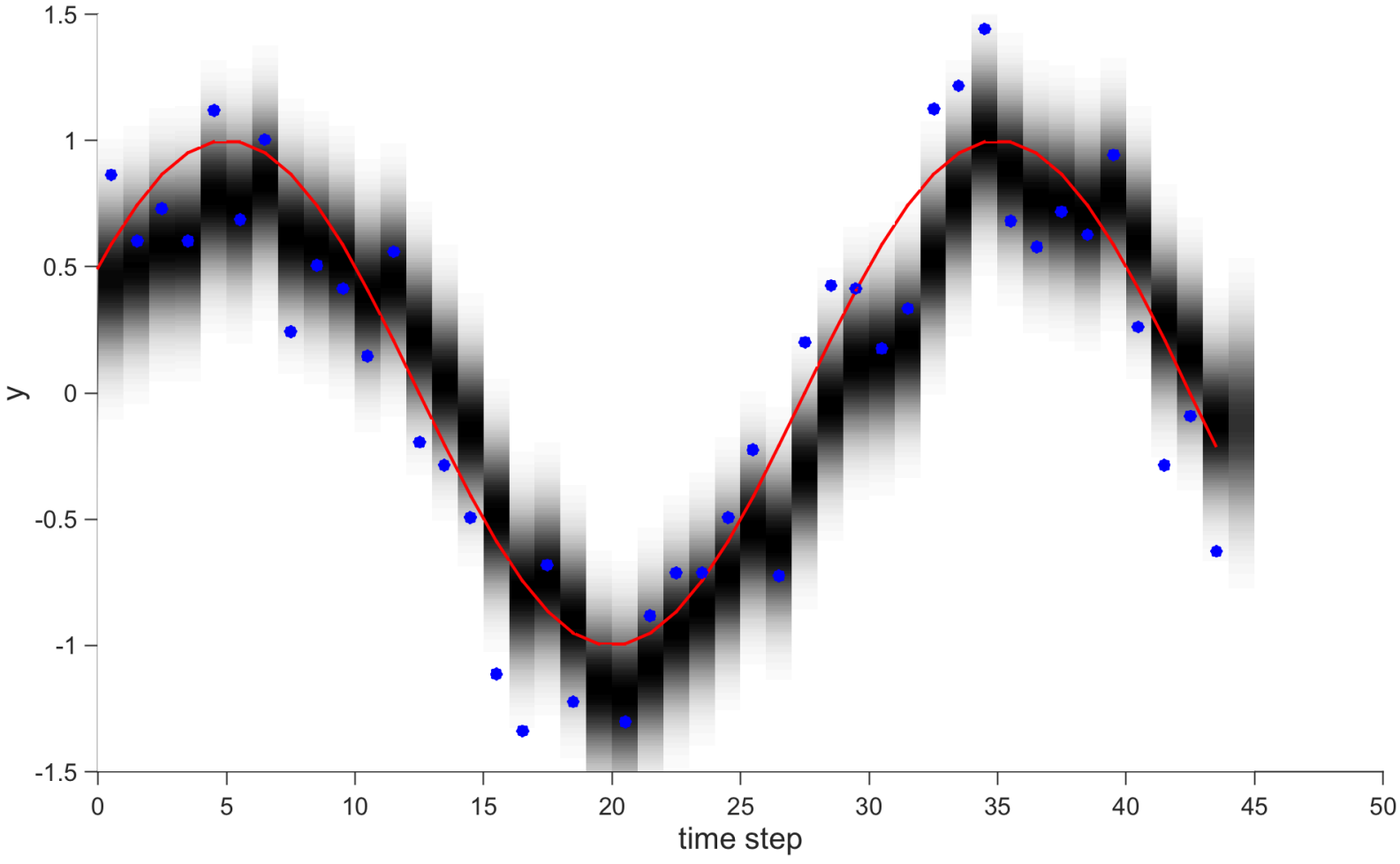

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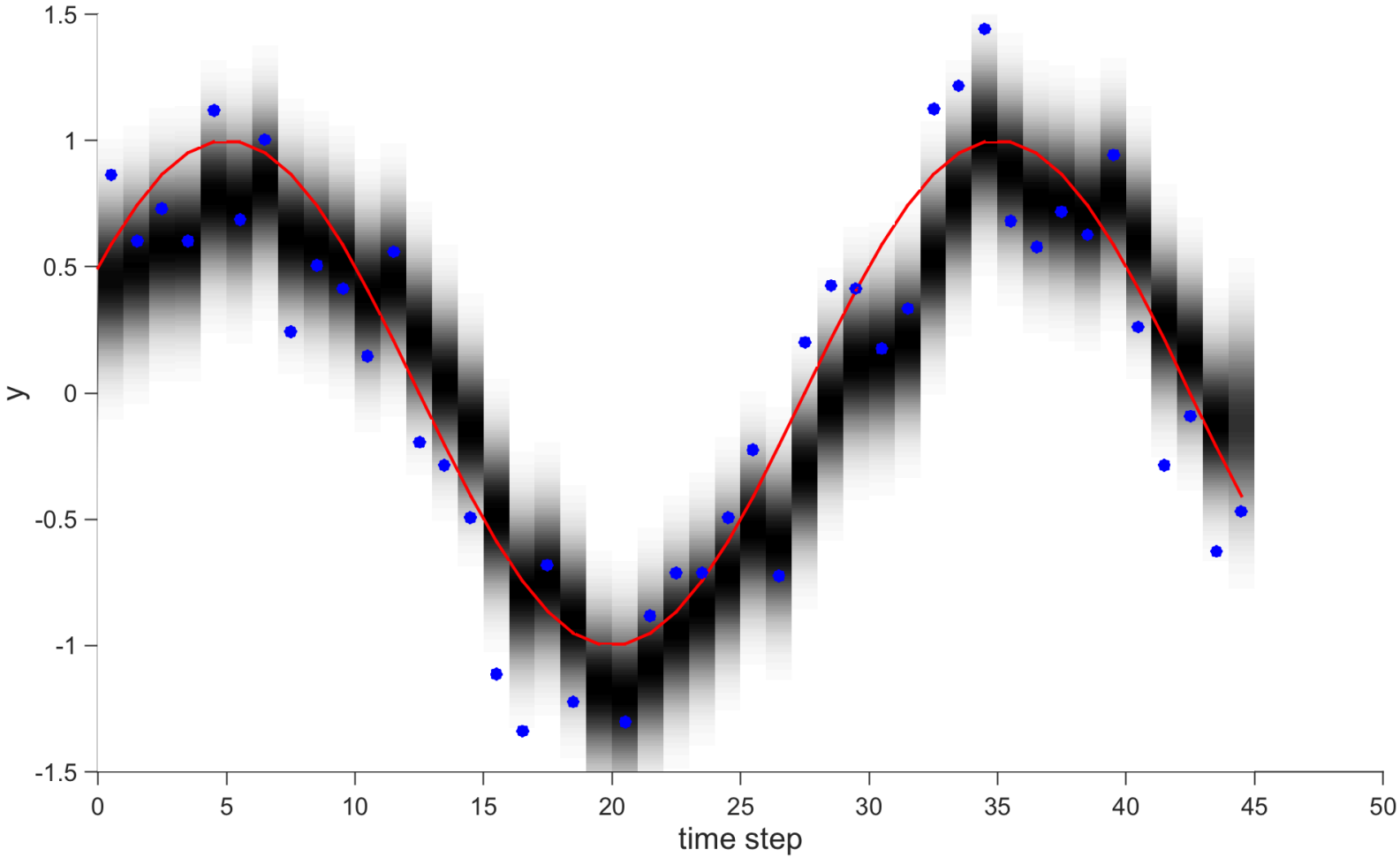

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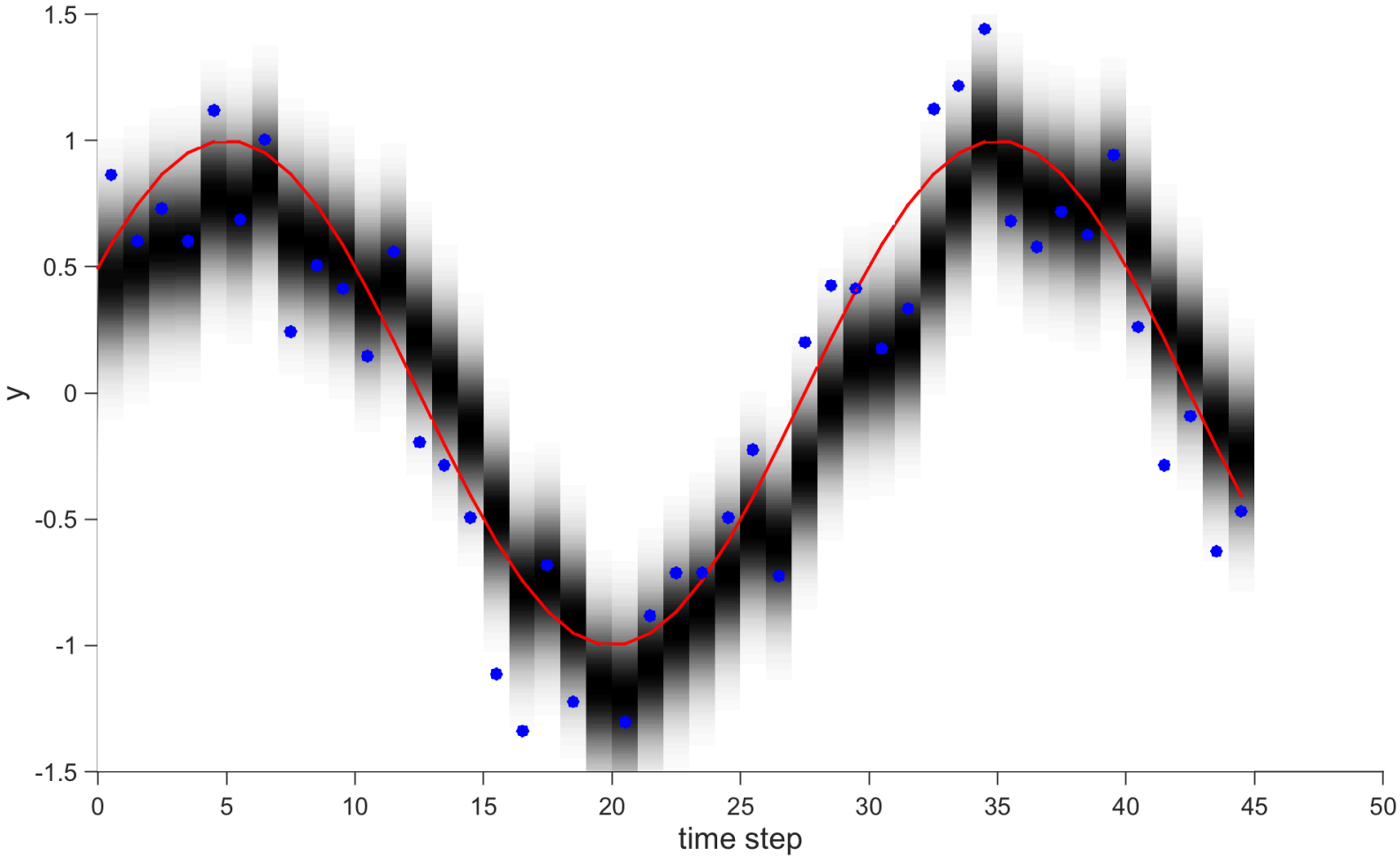

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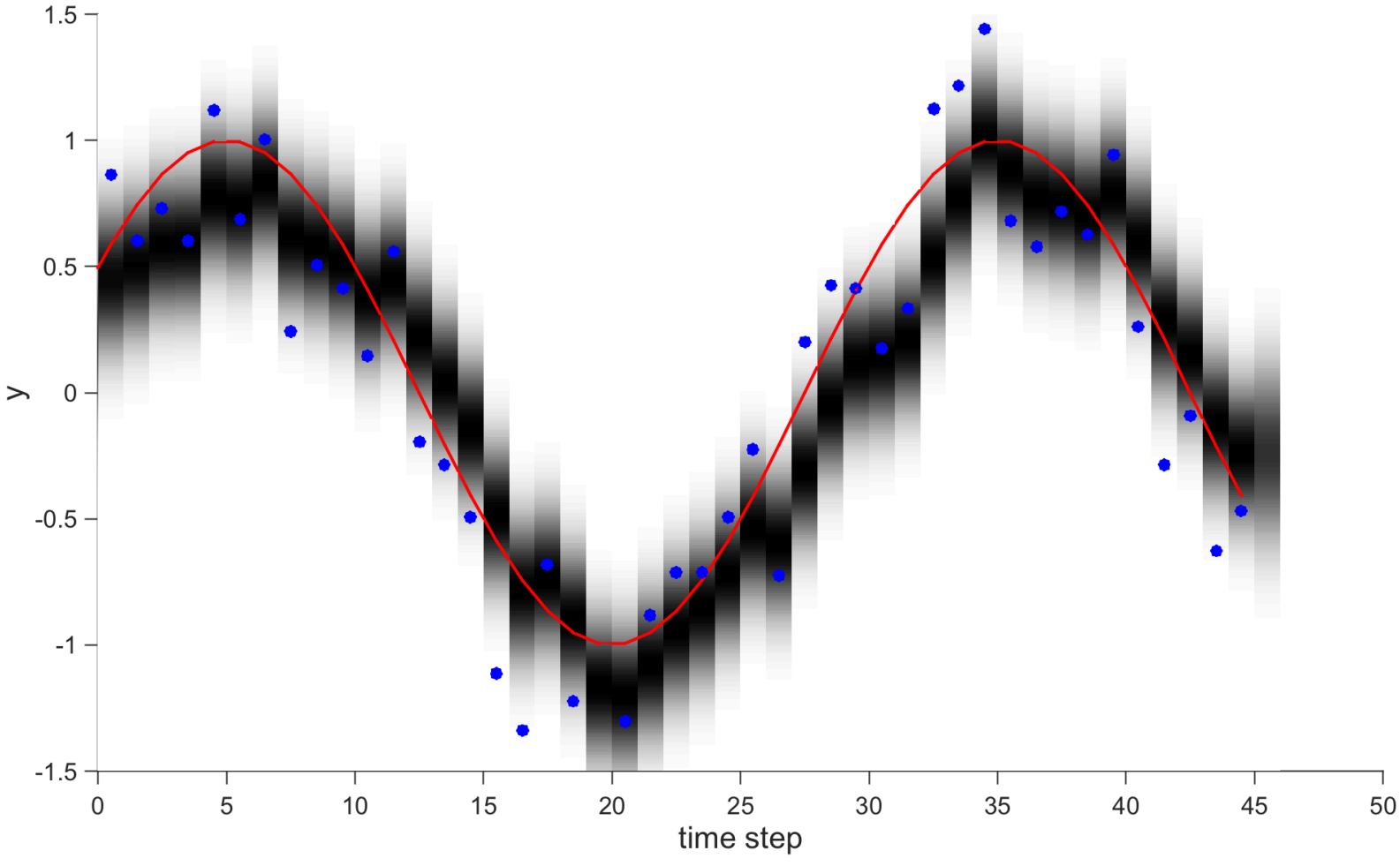

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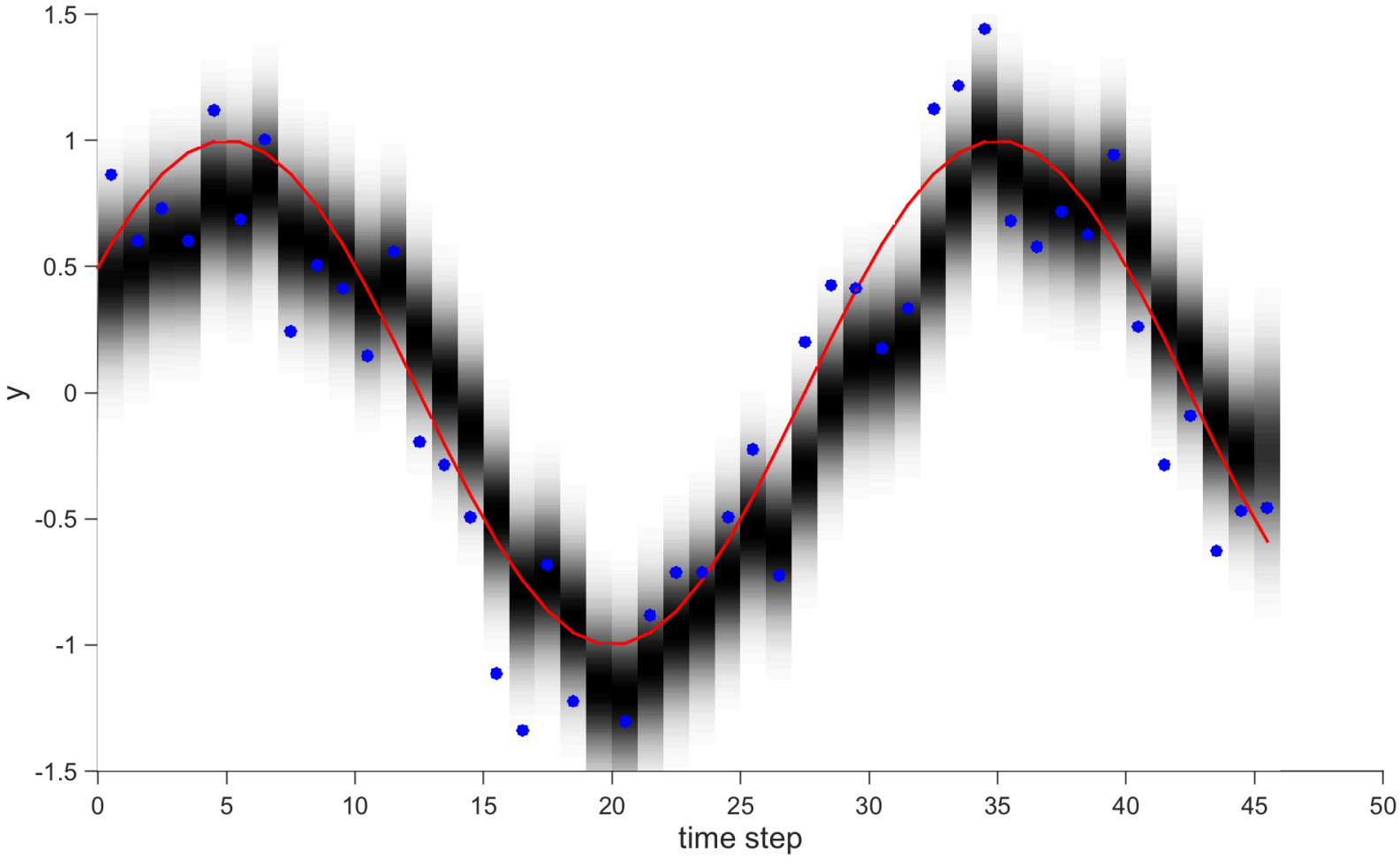

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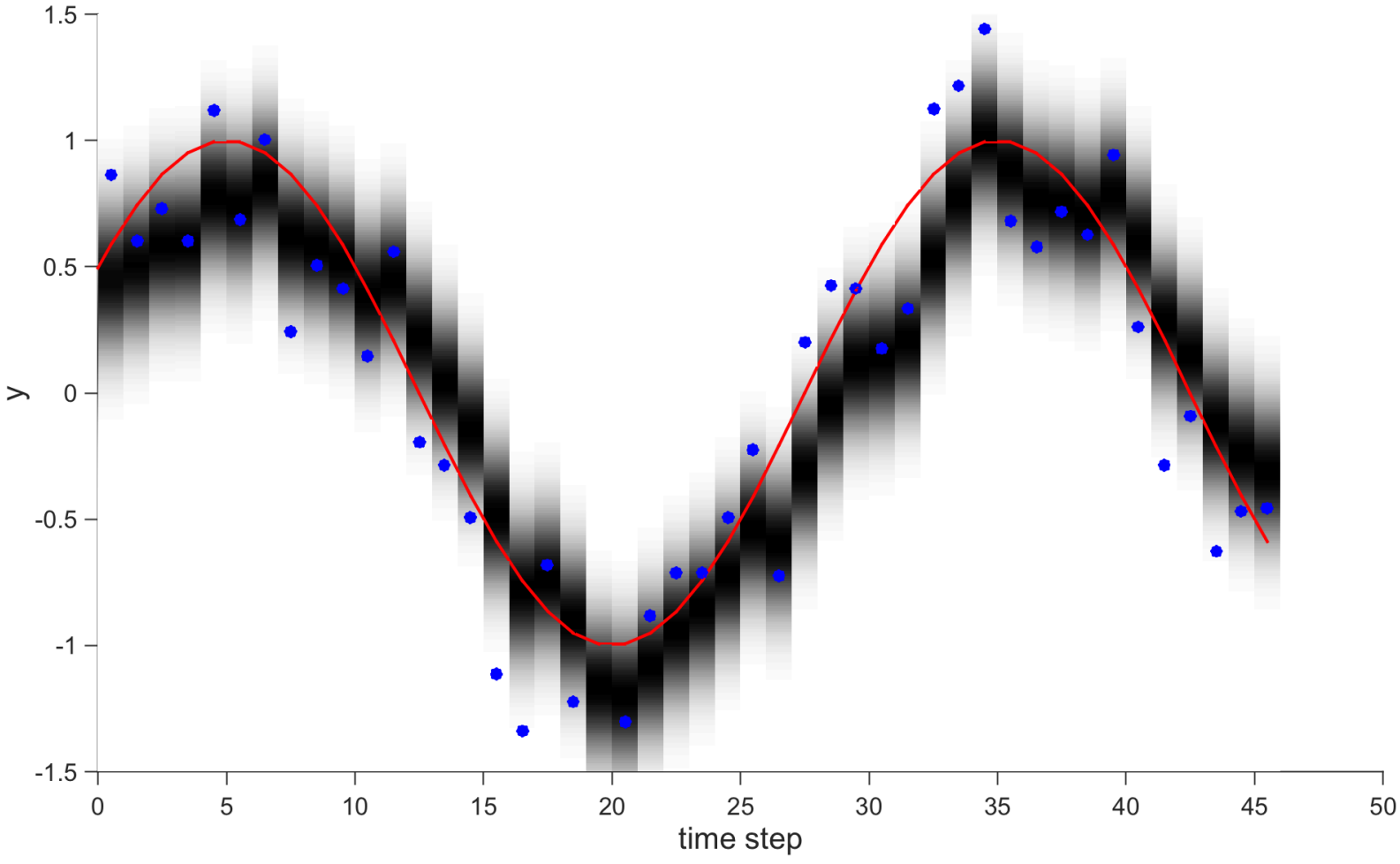

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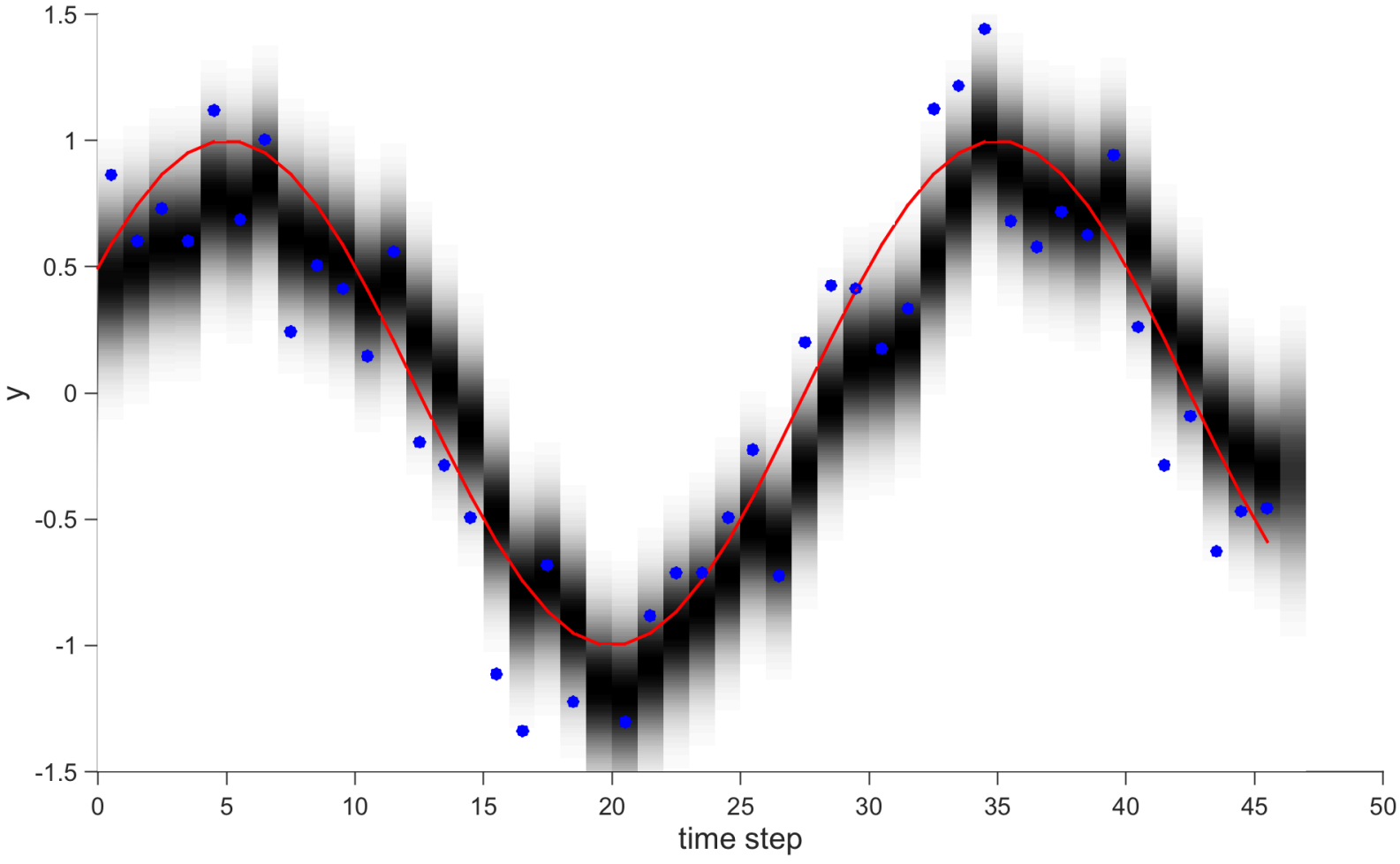

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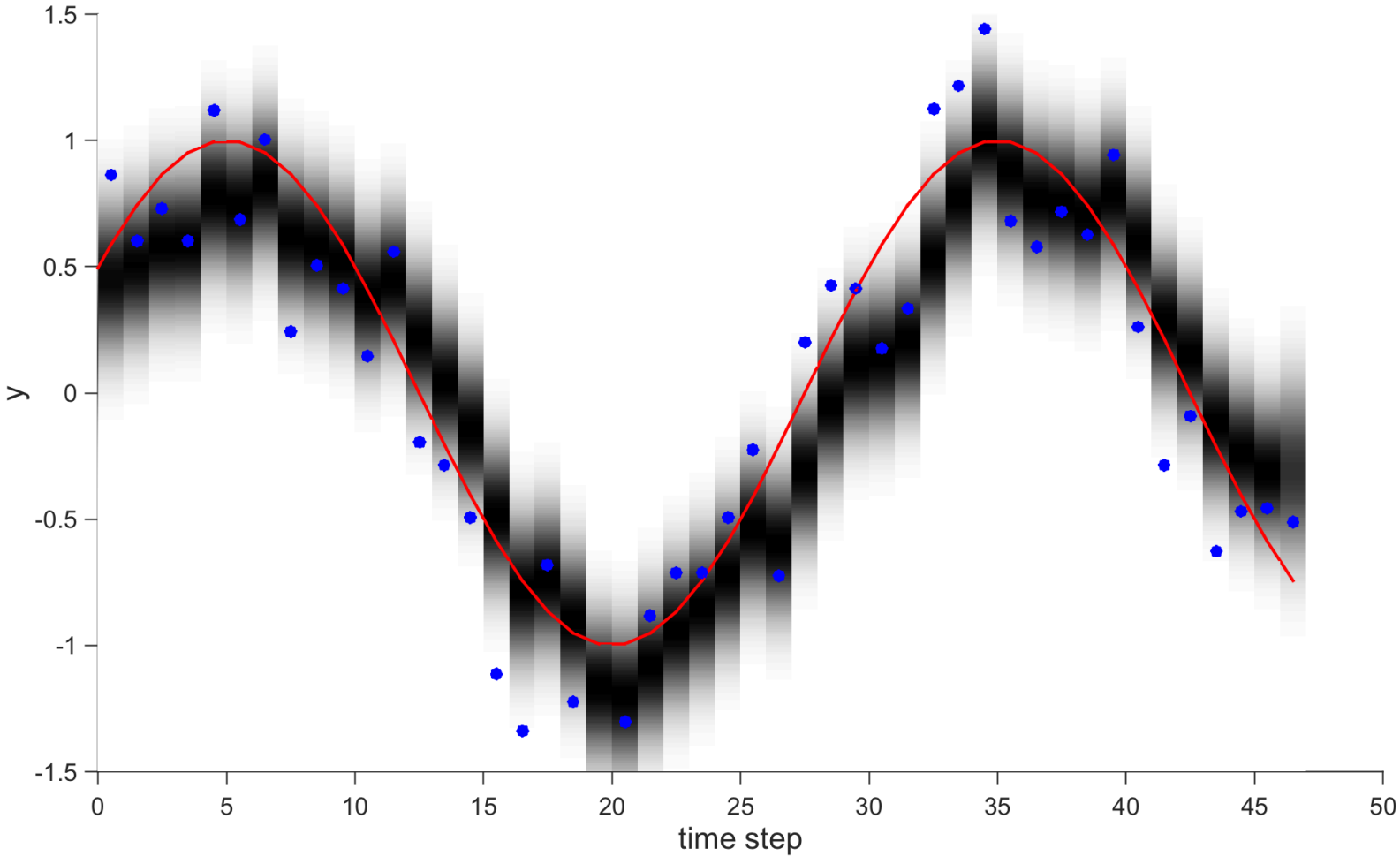

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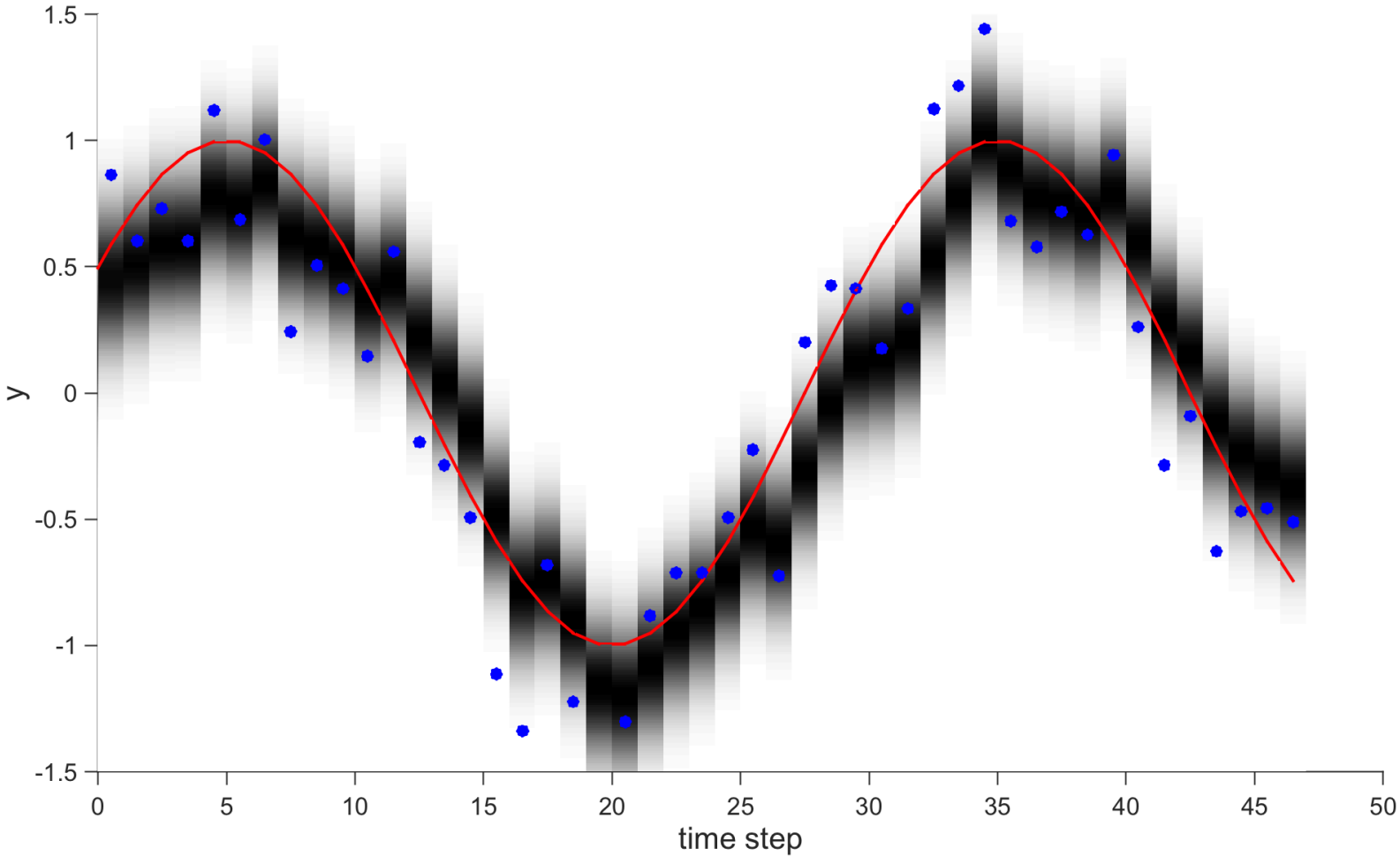

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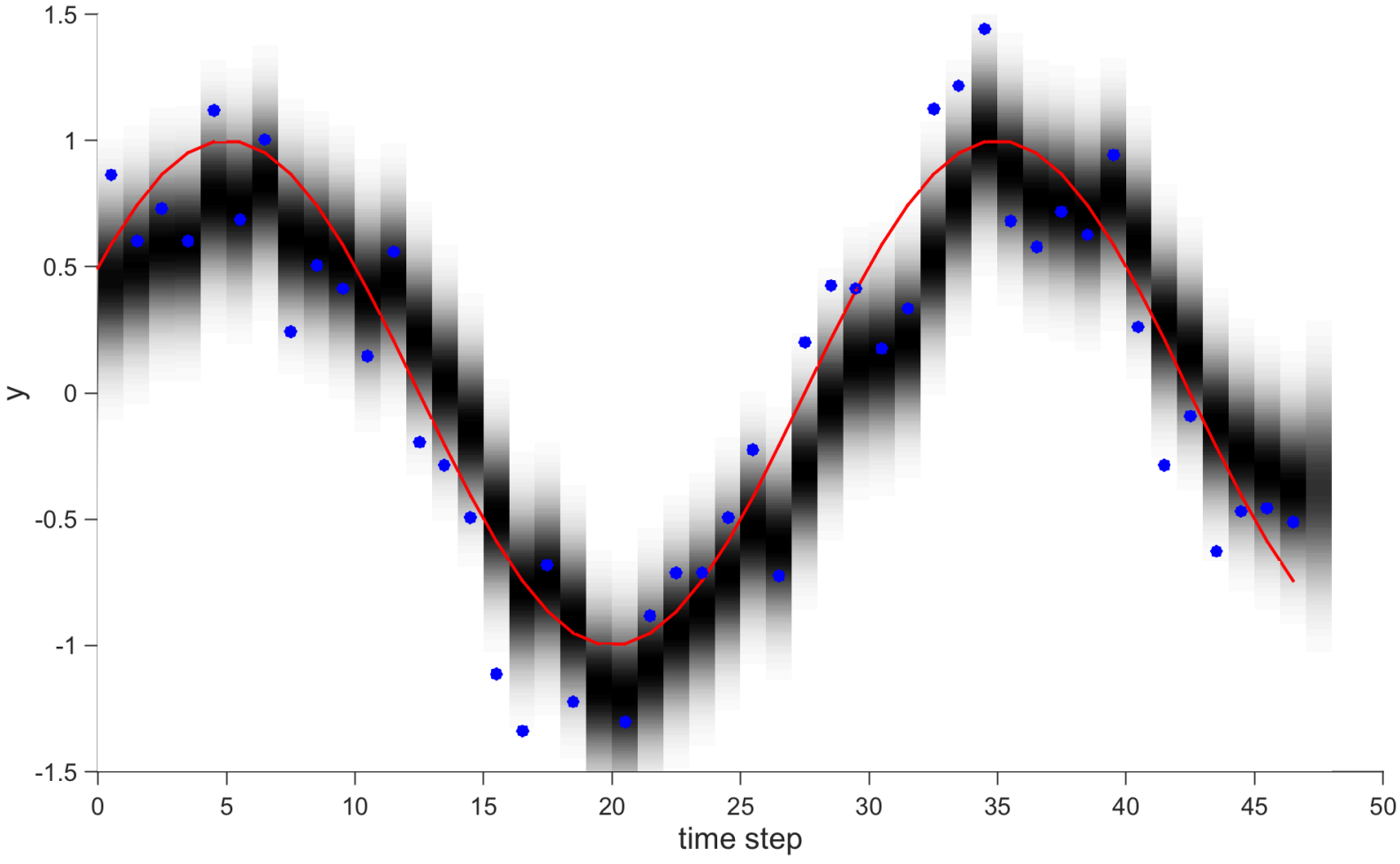

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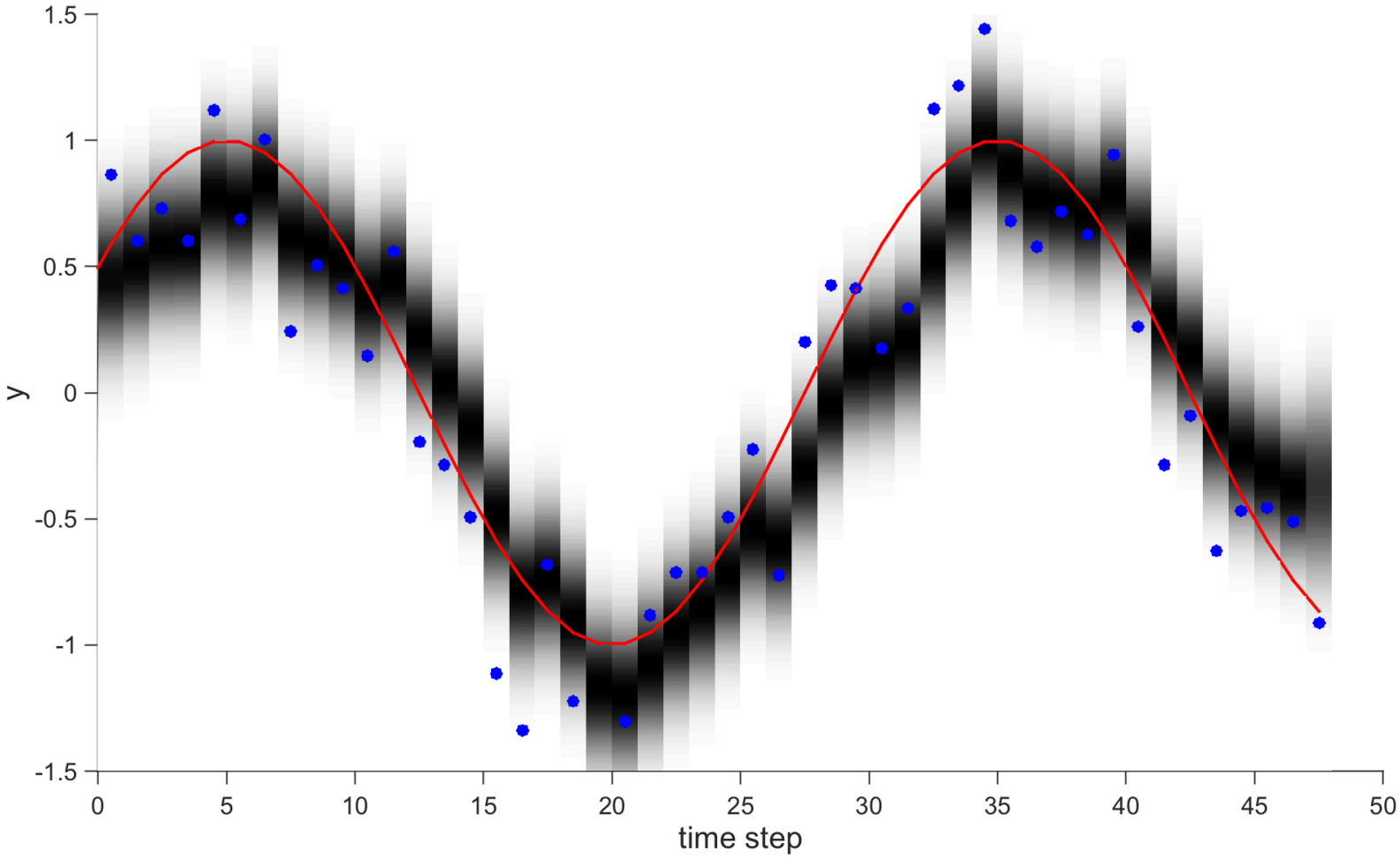

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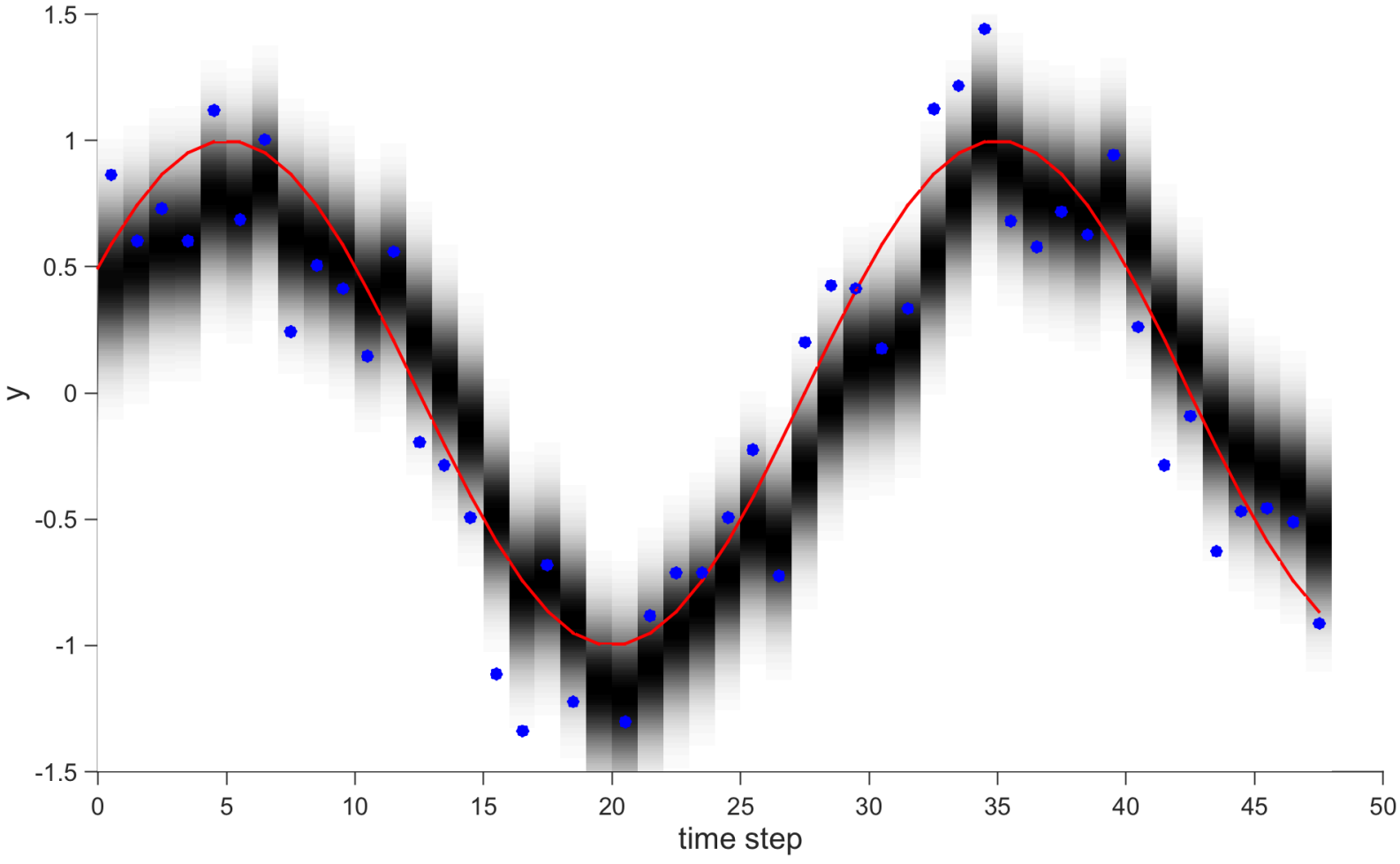

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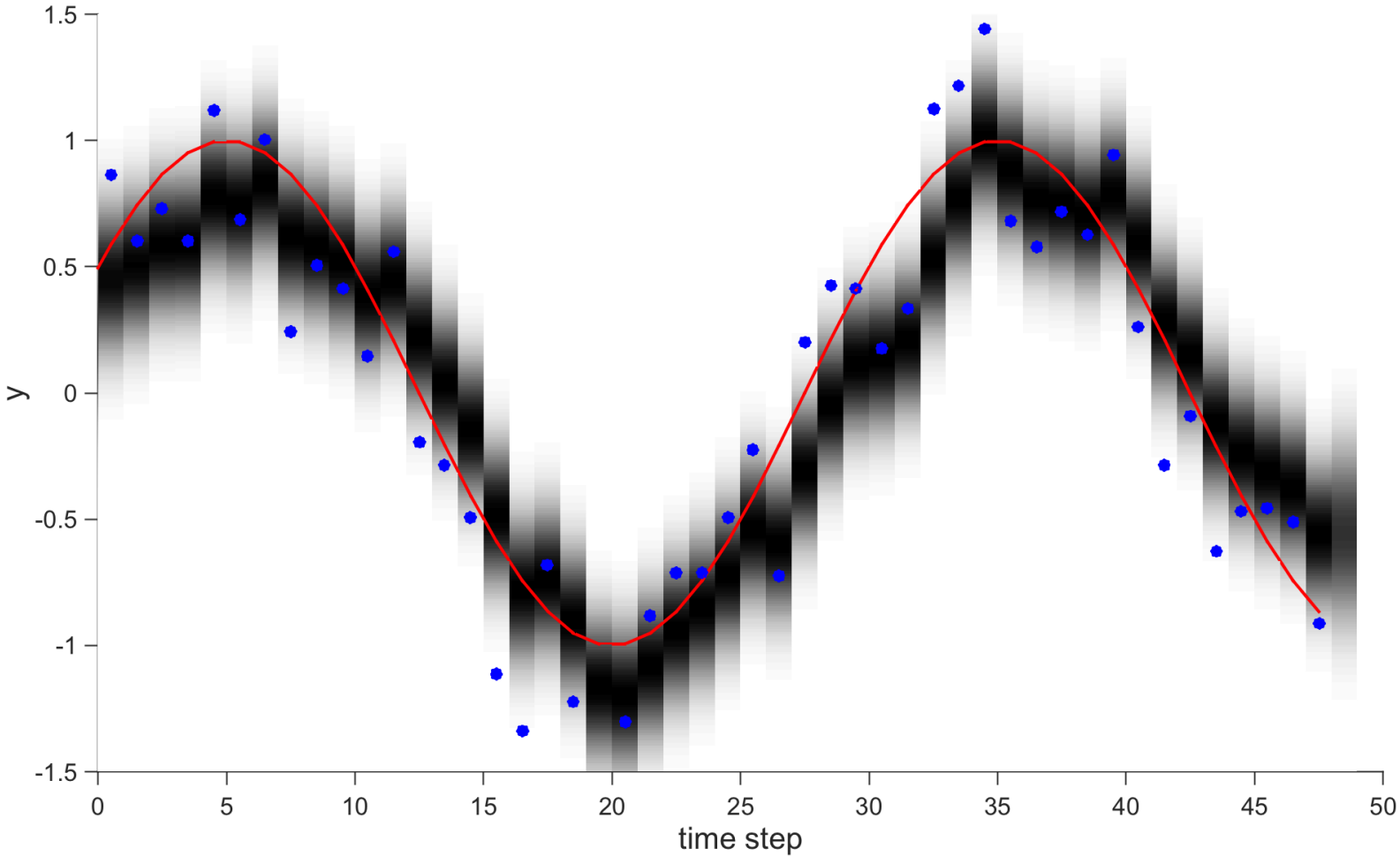

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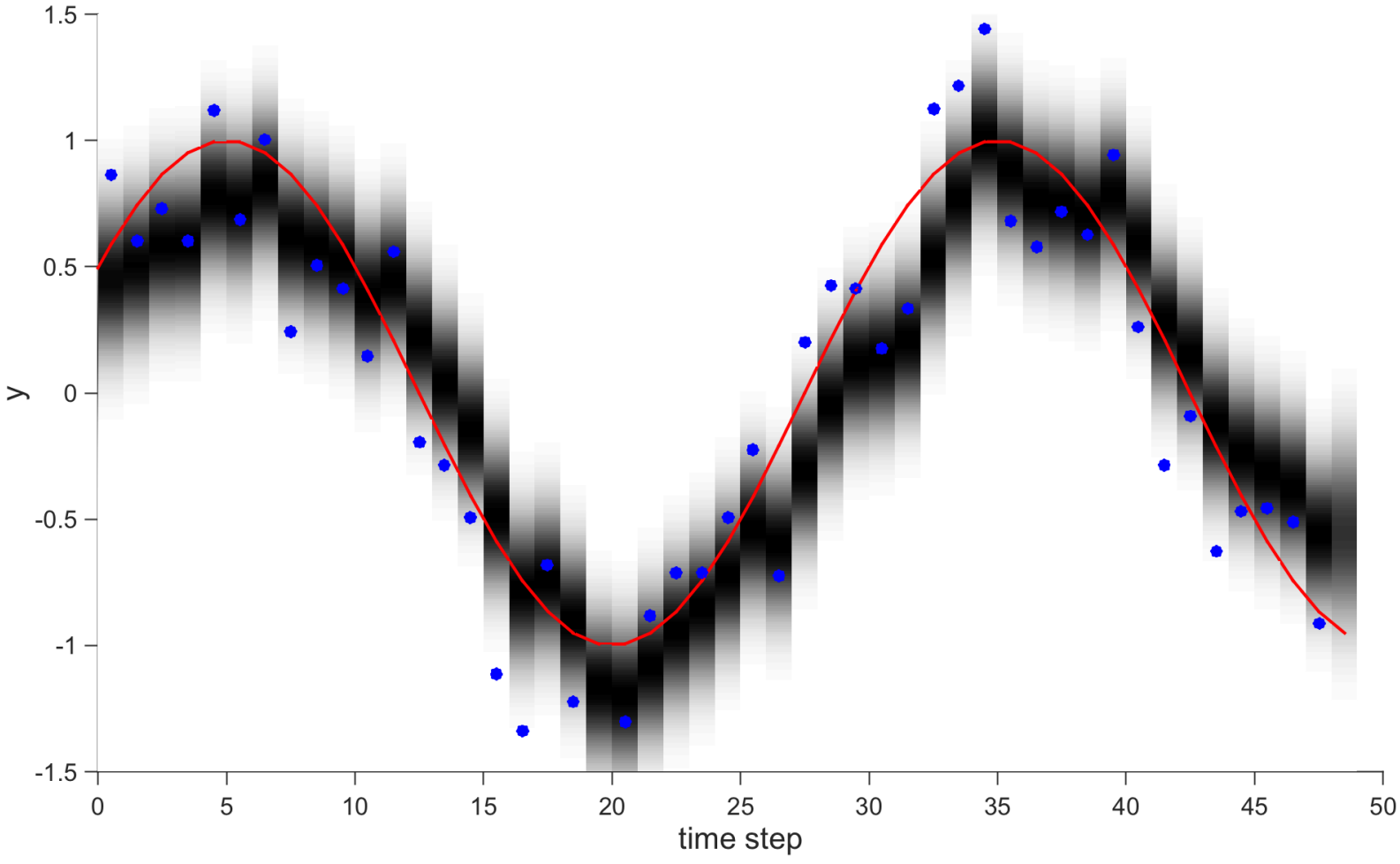

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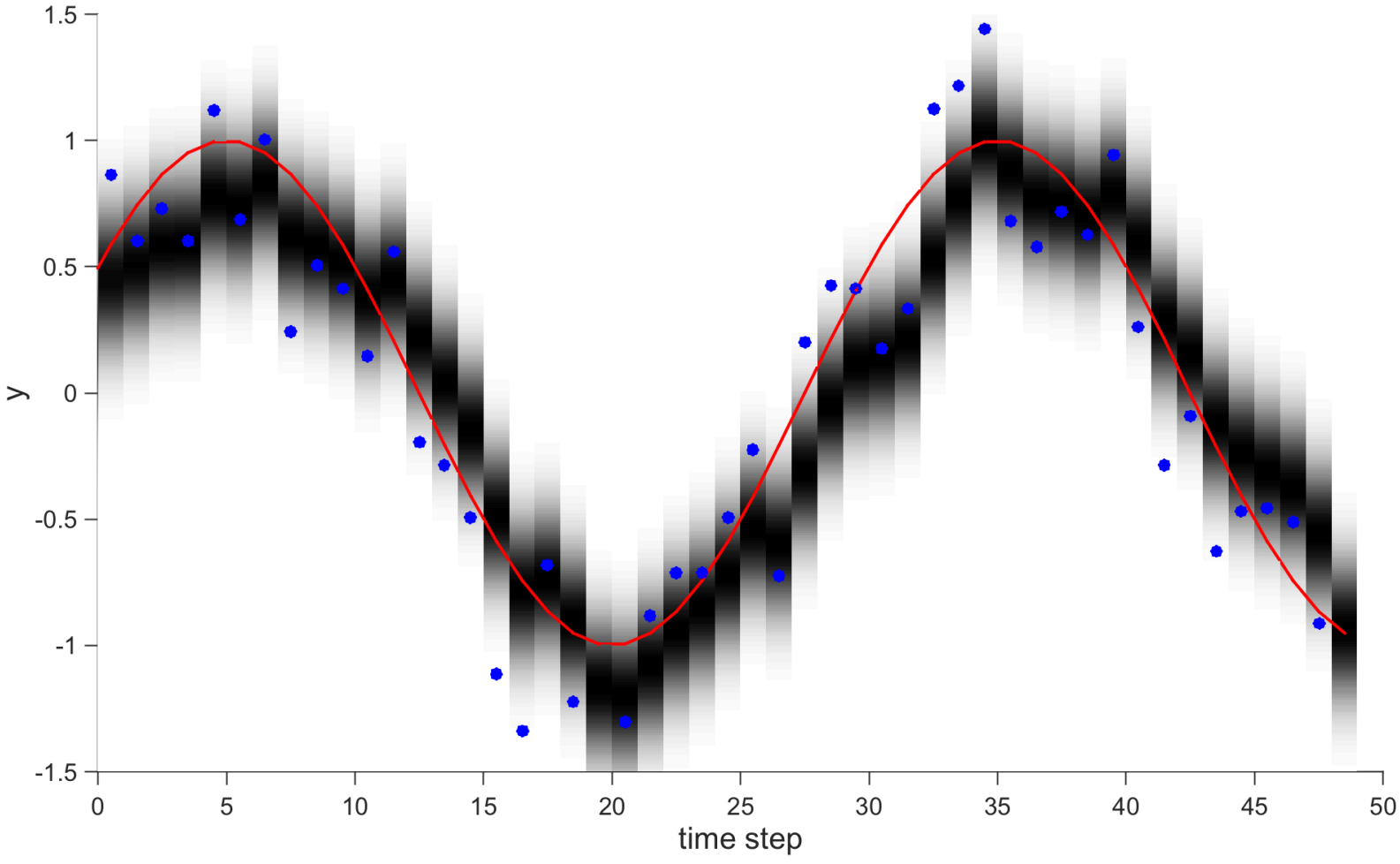

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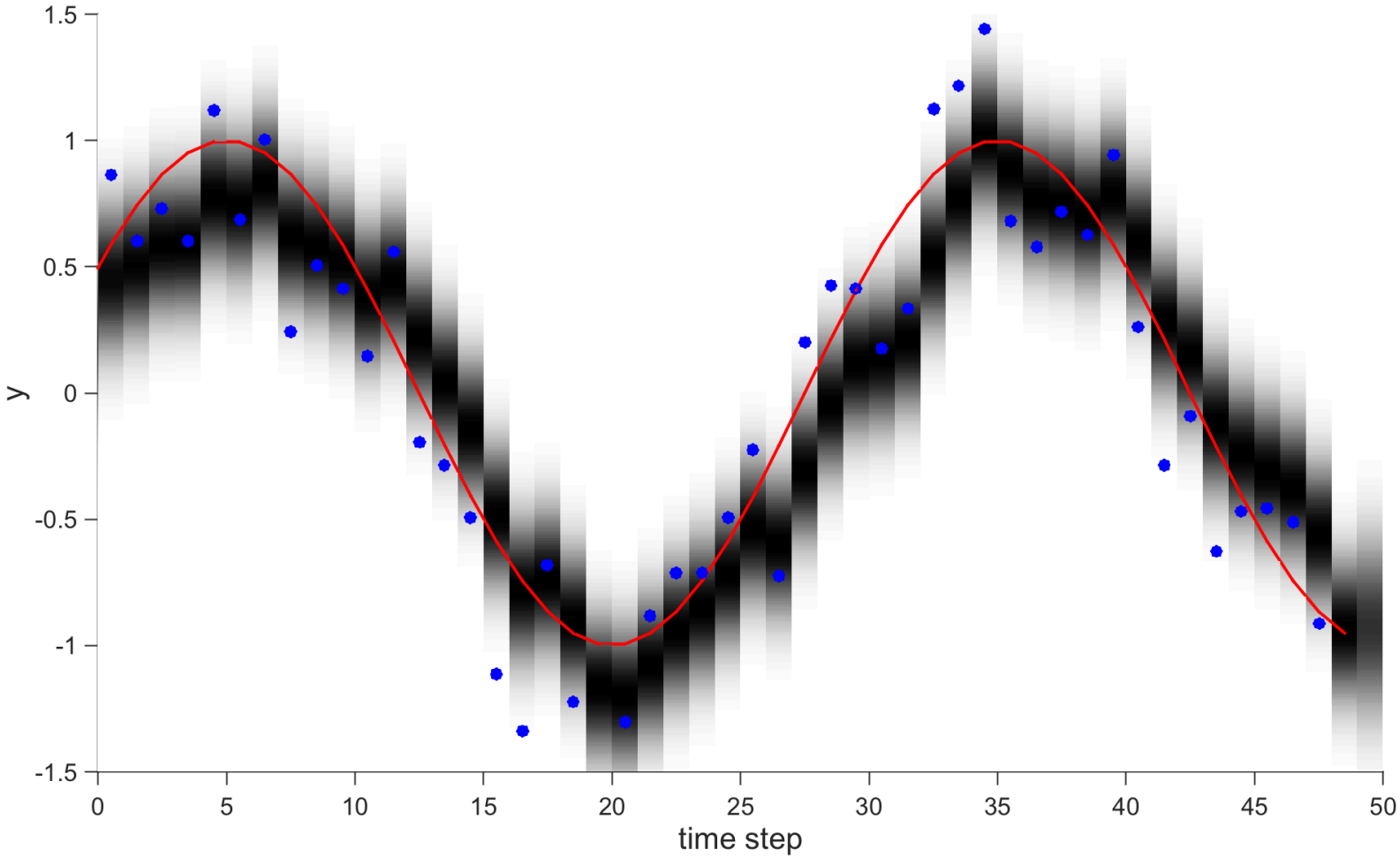

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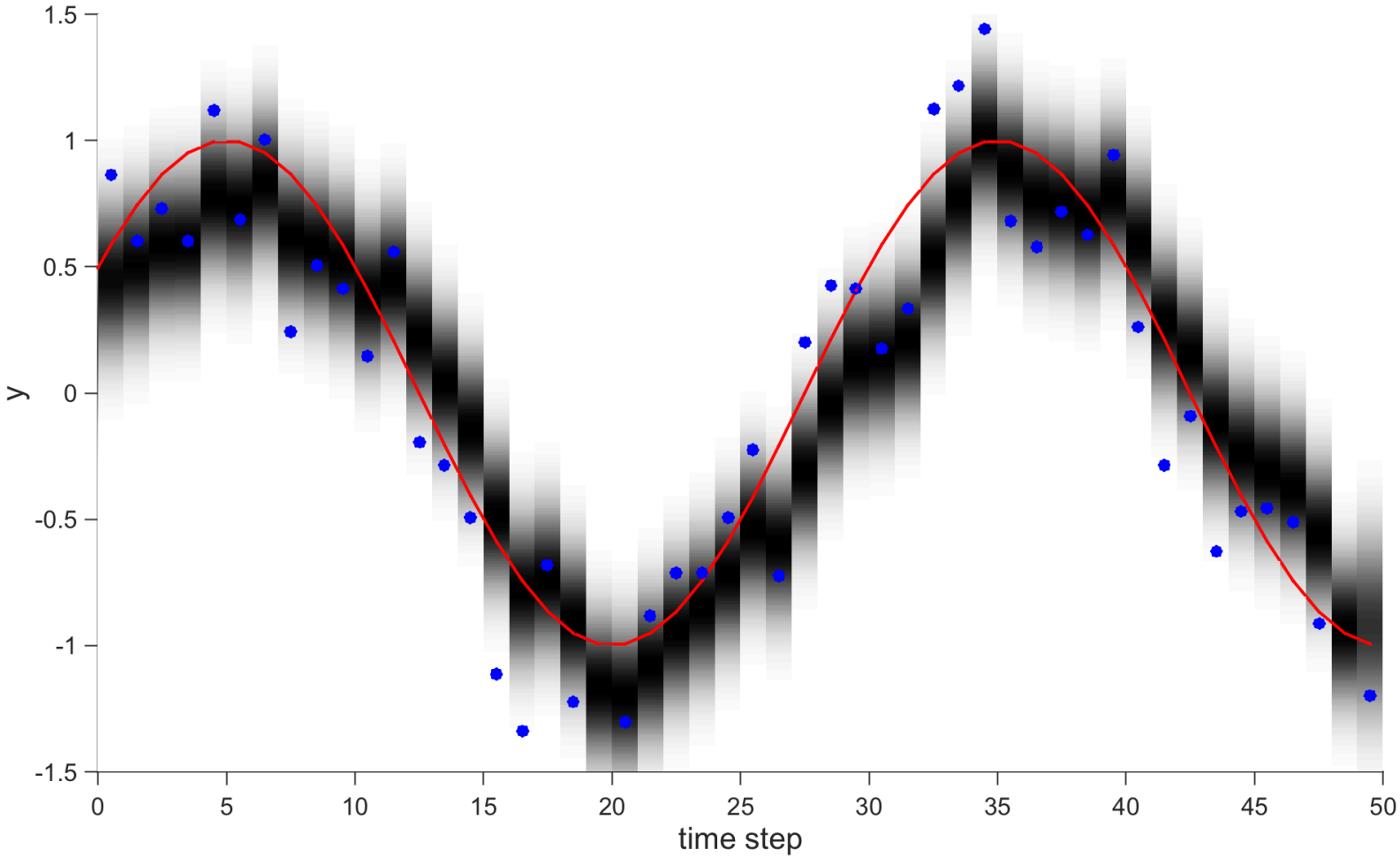

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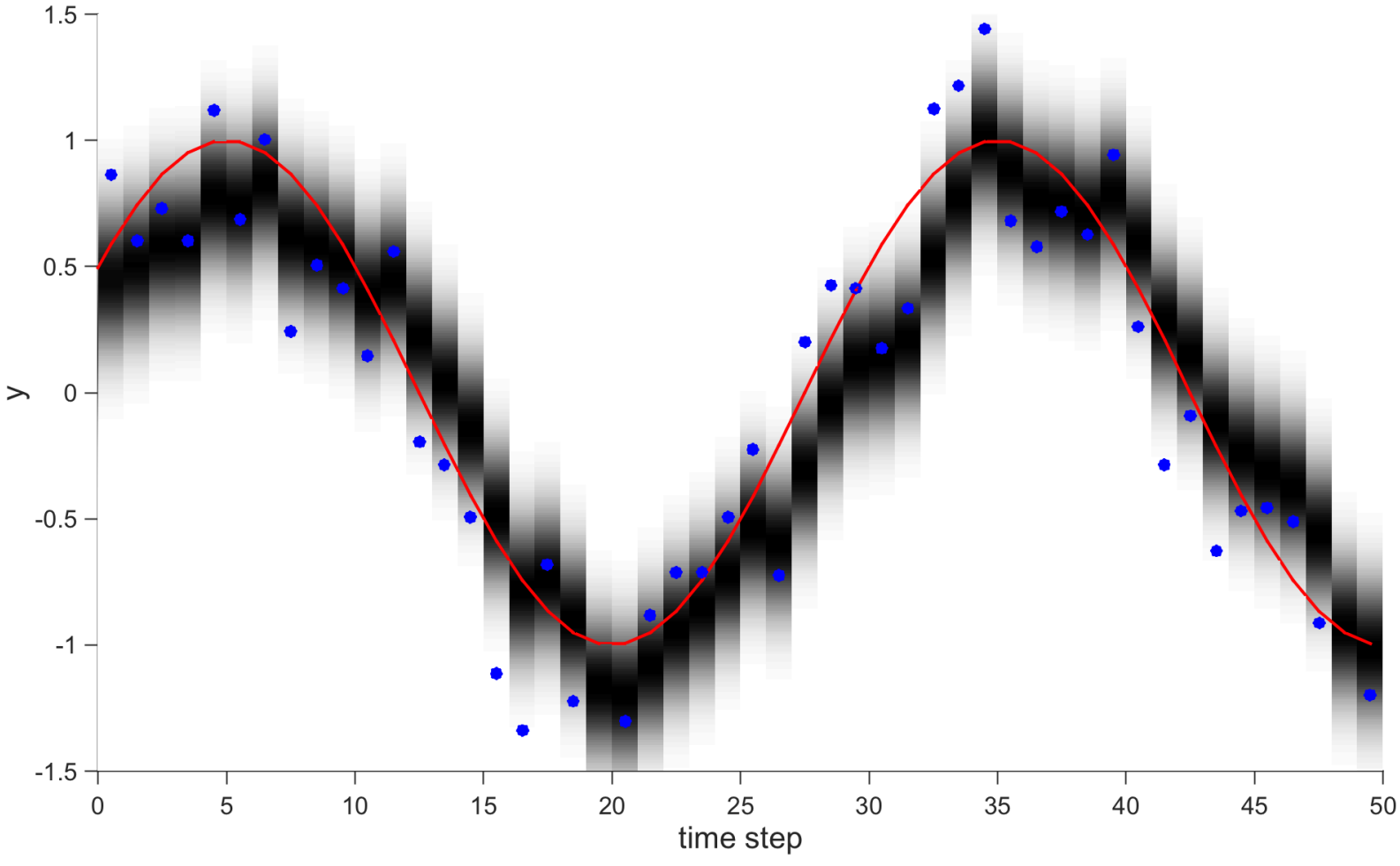

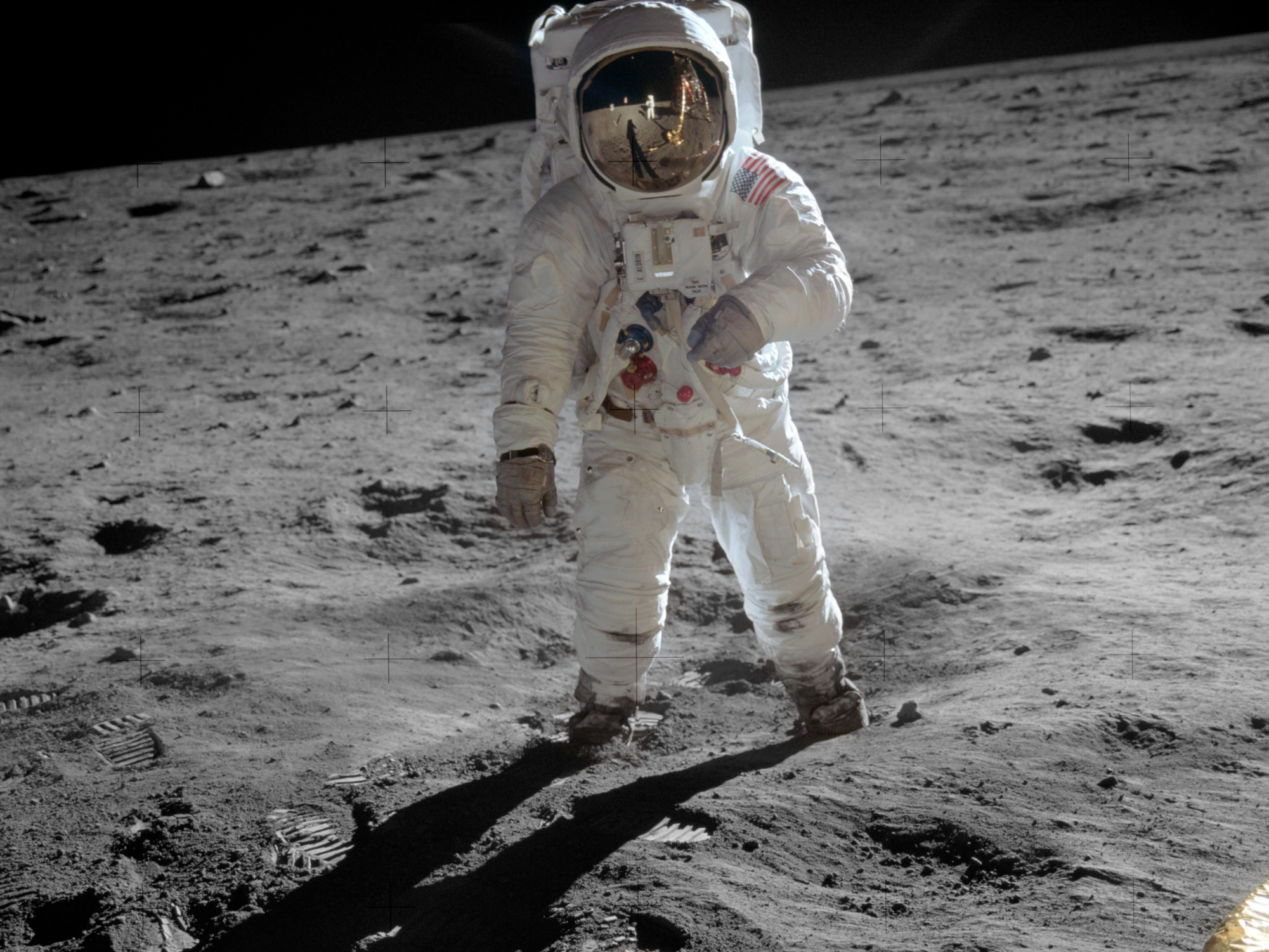

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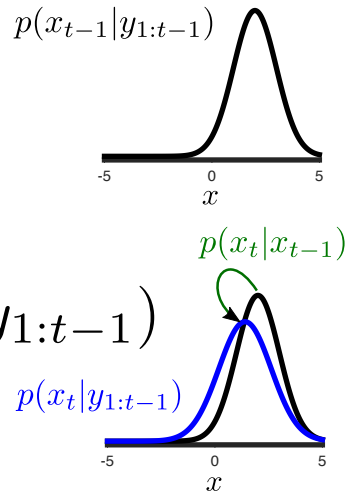
Inference: Forward Algorithm

$$p(x_{t-1} = k | y_{1:t-1})$$

diffuse via dynamics

$$p(x_t = k | y_{1:t-1}) = \sum_{l=1}^{K} p(x_t = k | x_{t-1} = l) p(x_{t-1} = l | y_{1:t-1})$$



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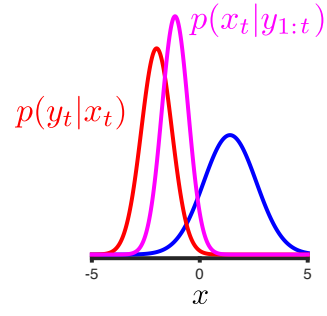

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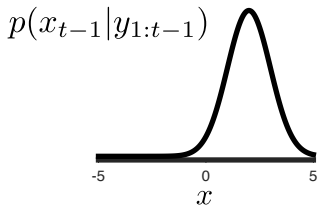

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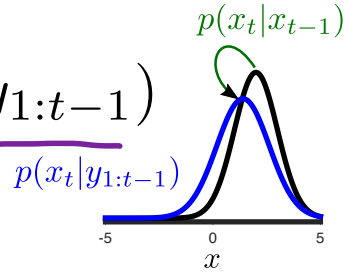

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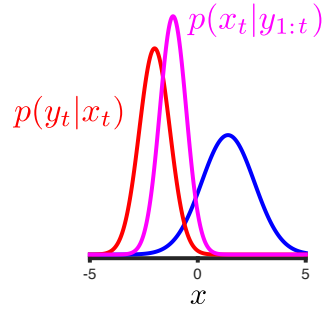

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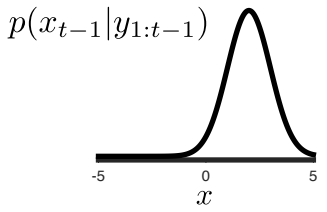

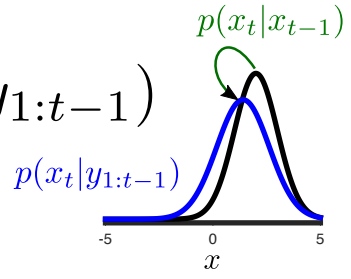

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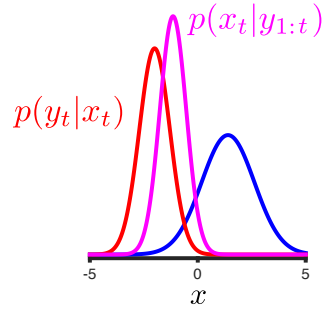

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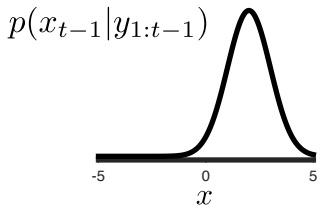

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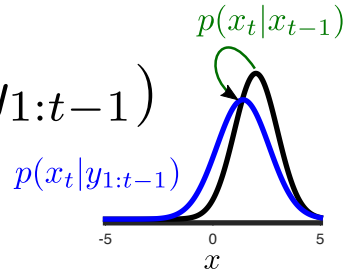

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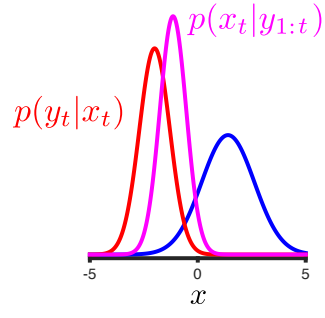

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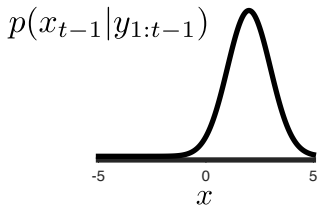

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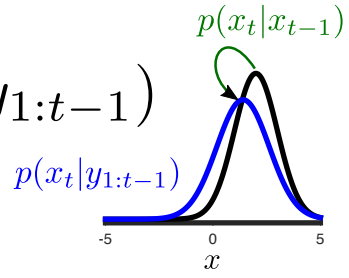

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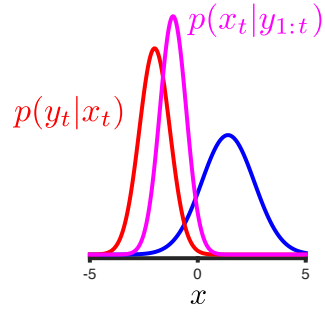

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When implementing, take care with numerical underflow/overflow.

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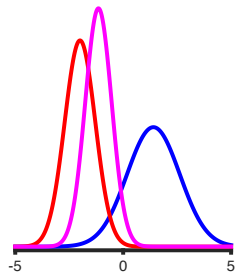

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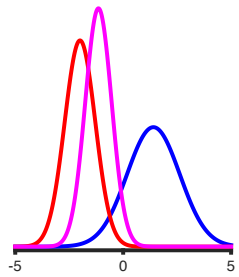

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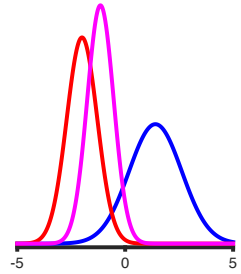

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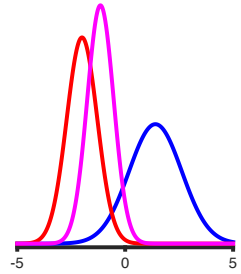

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The magic of the Forward Algorithm: Dynamic Programming

What's going on here?

In discrete case, likelihood involves sum over all sequences: $x_{1:T}^{(k)}$

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Trellis diagram represents possible sequences:

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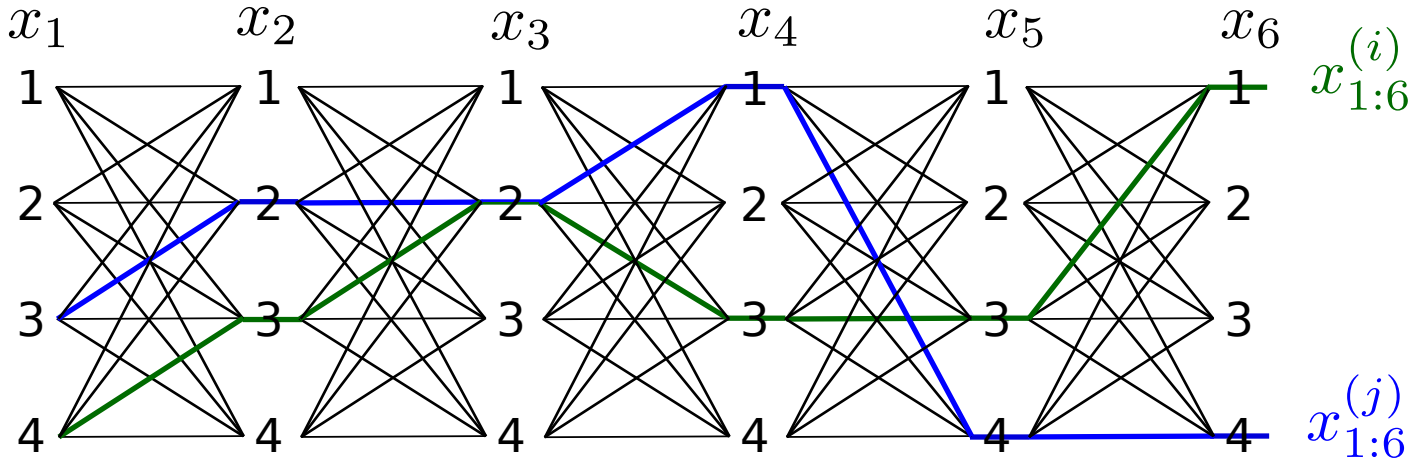

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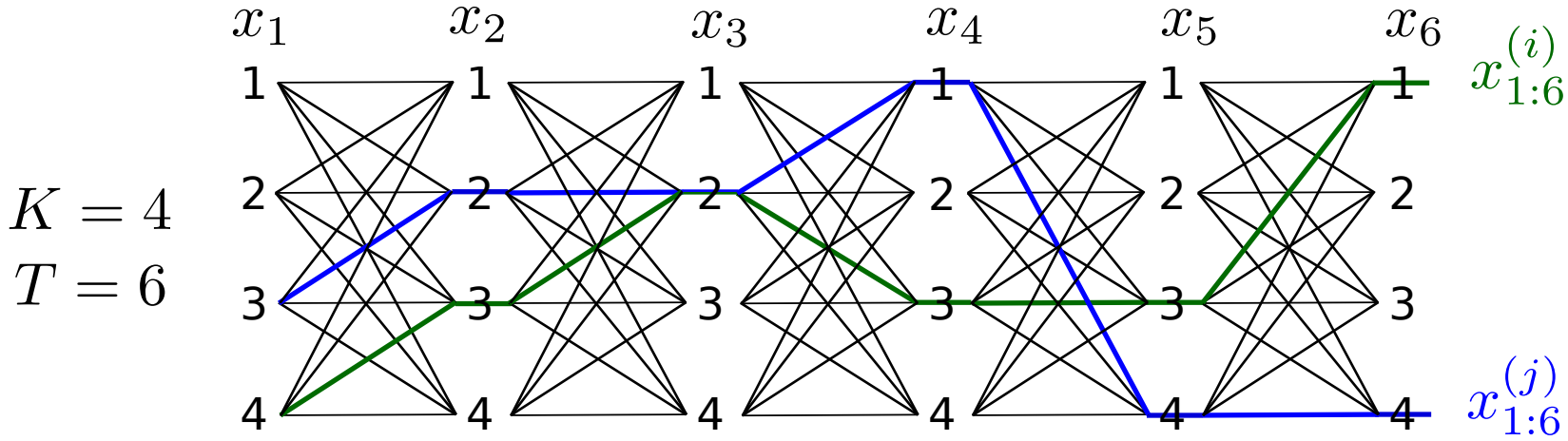

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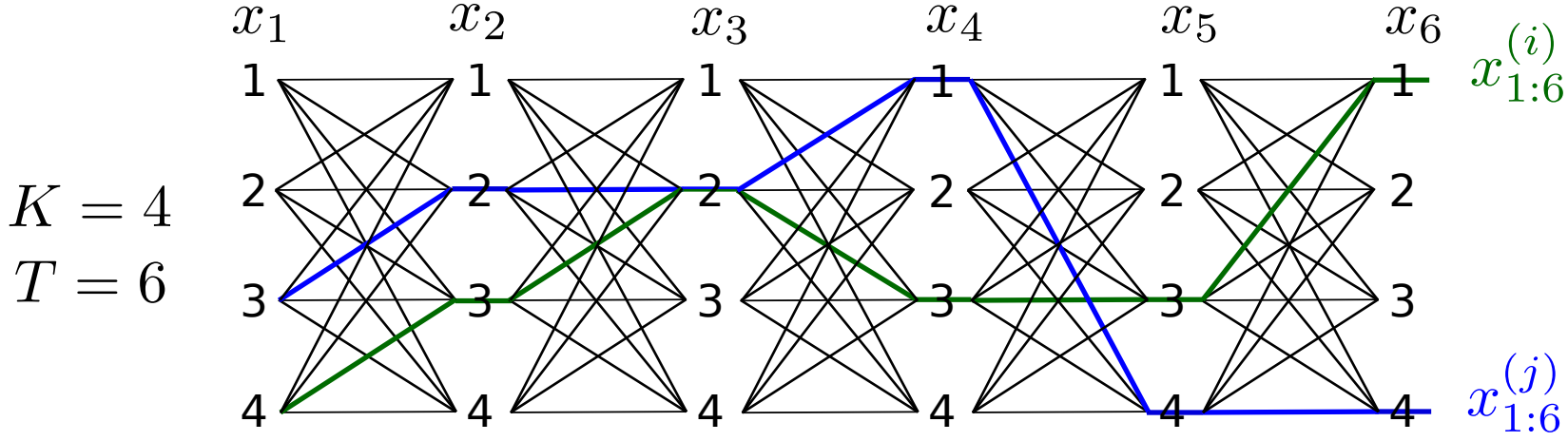

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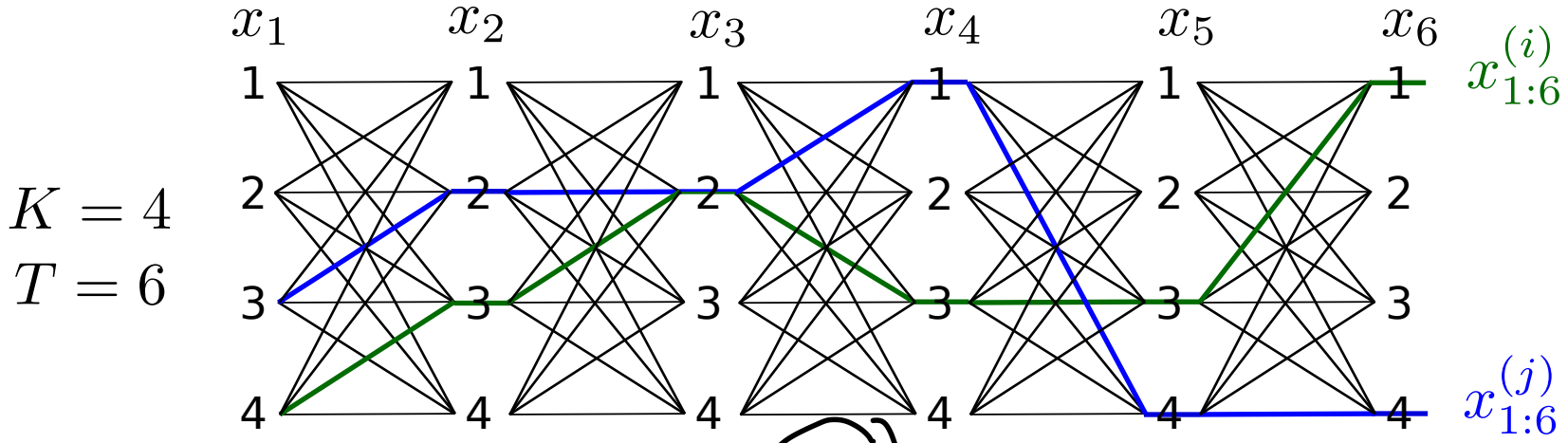

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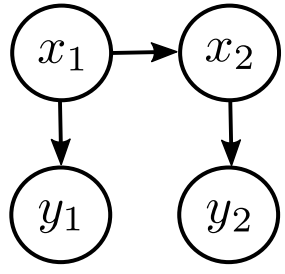

Markov property means we can forget history of previous states:
just remember last one (dynamic programming/belief propagation)

Maximum Likelihood Learning of HMMs: simple once inference is solved

log-likelihood: $$\log p(y_{1:T}|\theta) = \log \int p(y_{1:T}, x_{1:T}|\theta) dx_{1:T}$$

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$$E(\theta; x_{1:T}, y_{1:T}) = \sum_t \left[\log p(y_t|x_t, \theta) + \log p(x_t|x_{t-1}, \theta)\right]$$

show gradient depends on simple moments of posterior:

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \frac{1}{p(y_{1:T}|\theta)} \int \frac{\mathrm{d}}{\mathrm{d}\theta} \exp(\overbrace{\log p(y_{1:T}, x_{1:T}|\theta)}^{E(\theta; x_{1:T}, y_{1:T})}) dx_{1:T}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \frac{1}{p(y_{1:T}|\theta)} \int p(y_{1:T}, x_{1:T}|\theta) \frac{\mathrm{d}}{\mathrm{d}\theta} E(\theta; x_{1:T}, y_{1:T}) dx_{1:T}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \int p(x_{1:T}|y_{1:T}, \theta) \frac{\mathrm{d}}{\mathrm{d}\theta} E(\theta; x_{1:T}, y_{1:T}) dx_{1:T}$$

Maximum Likelihood Learning of HMMs: simple once inference is solved

log-likelihood:
$$\log p(y_{1:T}|\theta) = \log \int p(y_{1:T}, x_{1:T}|\theta) dx_{1:T}$$

gradient of log-likelihood:
$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \frac{1}{p(y_{1:T}|\theta)} \int \frac{\mathrm{d}}{\mathrm{d}\theta} p(y_{1:T}, x_{1:T}|\theta) dx_{1:T}$$

simple form: e.g. quadratic in x for LGSSMs

$$E(\theta; x_{1:T}, y_{1:T}) = \sum_t \left[\log p(y_t|x_t, \theta) + \log p(x_t|x_{t-1}, \theta)\right]$$

show gradient depends on simple moments of posterior:

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$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \frac{1}{p(y_{1:T}|\theta)} \int p(y_{1:T}, x_{1:T}|\theta) \frac{\mathrm{d}}{\mathrm{d}\theta} E(\theta; x_{1:T}, y_{1:T}) dx_{1:T}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \int p(x_{1:T}|y_{1:T}, \theta) \frac{\mathrm{d}}{\mathrm{d}\theta} E(\theta; x_{1:T}, y_{1:T}) dx_{1:T}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log p(y_{1:T}|\theta) = \left\langle \frac{\mathrm{d}}{\mathrm{d}\theta} E(\theta; x_{1:T}, y_{1:T}) \right\rangle_{p(x_{1:T}|y_{1:T}, \theta)}$$

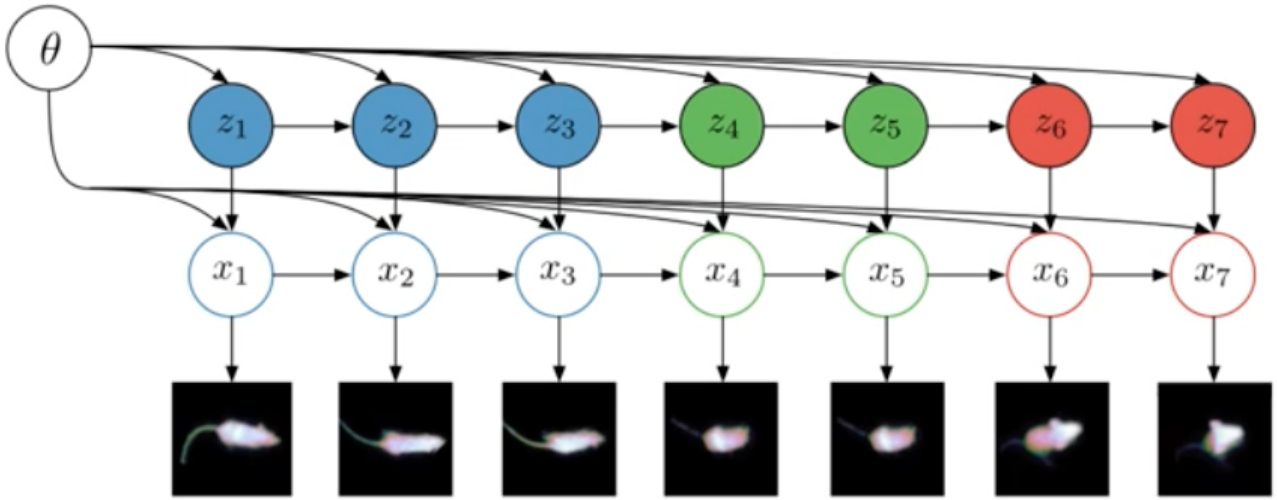
requires posterior moments: marginals and pairwise marginals

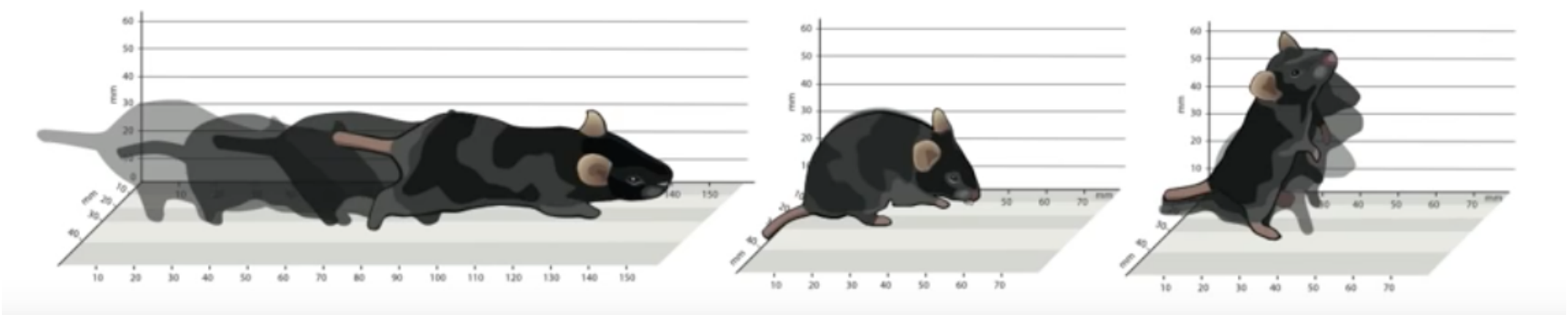
Summary of Sequence Modelling

- **Markov models**: class of probabilistic models for sequence data
  - **N-Gram models** (discrete data) and **Gaussian auto-regressive models** (continuous data)
  - simple to perform maximum likelihood fitting
  - unnatural for many tasks e.g. removing additive noise, separating signals, data containing latent variables
- **Hidden Markov Models**: more flexible class of probabilistic model
  - generalise Markov models and naturally support a wider range of tasks (removal of noise, source separation, representation learning)
  - different varieties: discrete vs. continuous latent variables (**discrete HMMs** and **linear Gaussian state space models**)
  - inference in these models requires **dynamic programming** / message passing e.g. smoothing via the **forwards-backwards** recursions or **Kalman filtering-smoothing** recursions
  - maximum-likelihood fitting requires smoothing as a subroutine

Hidden Markov Models for unsupervised high dimensional video understanding

Hidden Markov Models for unsupervised high dimensional video understanding





https://www.youtube.com/watch?v=btr1poCYIzw