The Proof Is In The ~~Pudding~~ Probability

An introduction to the probabilistic method

Imagine you're waiting in line to grab an early morning coffee when you overhear a patron exclaim "The probability that aliens exist is greater than zero!". With your interest piqued, you continue listening in, only for them to conclude "…therefore they must exist!"

Confused? I would be! How could we possibly conclude something does exist, based only on the premise that there's a positive probability that it might? While you might not have any luck using this kind of reasoning to convince someone that aliens exist, a similar form of argument is often employed by mathematicians to rigorously prove the existence of mathematical objects!

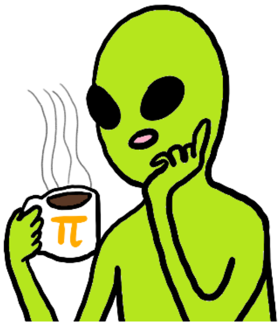

Figure 1. This alien is also confused by the patron's argument.

Perhaps one of the most interesting types of proof in all of mathematics, *the probabilistic method* (TPM) is based upon a simple, yet counterintuitive, idea:

If a mathematical object exists with positive probability, then it exists.

Pioneered by the prolific mathematician Paul Erdős[1], TPM involves using probability to prove all sorts of theorems that have nothing to do with probability!

But what exactly is TPM, and how can we use it? As with many abstract mathematical ideas, an example can go a long way to painting the bigger picture. We'll get to know the method by considering the following problem:

What is the smallest number of edges in a k-uniform hypergraph that is not 2-colourable?

Make sense? No? Don't worry if this sounds like gibberish now; over the next few pages we'll walk step by step through what this problem is asking, we'll see how TPM can help us in finding an answer, and we'll finally learn what TPM is all about.

But before then, let us first discuss why we need such a technique in the first place.

Proving the existence of mathematical objects

Many interesting mathematical problems involve proving an object with certain properties exists. Loosely speaking, mathematical objects are what we talk and write about when doing mathematics. The number 42, the set $\{1,2,3,\dots\}$ and the function $f:\mathbb{R}\to\mathbb{R}, f(x)=\sin(x)$ are all examples of mathematical objects. We've already come across the objects called graphs in MTH2132 in exploring Euler's formula and planarity. Recall a graph is an object that consists of vertices and edges, where each edge joins two vertices. We often represent graphs as diagrams like those in Figure 2.

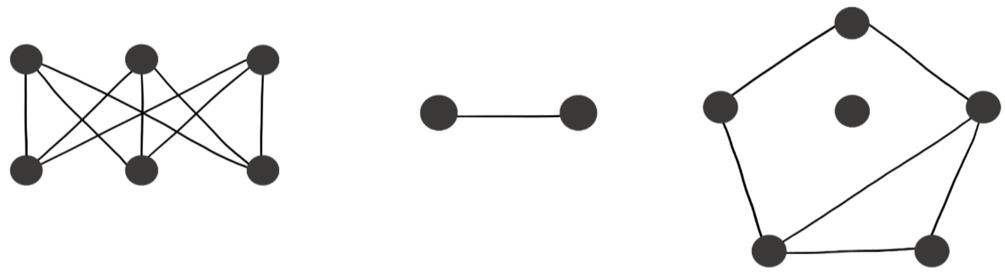

Figure 2. Three graph diagrams, the leftmost of which is the utility graph.

Say we wanted to prove that there exists a graph with 6 vertices that is non-planar. One straightforward way to do this would be to construct such a graph. In fact, we already know one like this, the utility graph! Such is an example of a *constructive proof*; one that demonstrates the existence of objects by creating them or providing a method to create them.

While constructive proofs are incredibly useful and intuitive, it is often difficult to construct more complicated objects, and sometimes, we don't even really care to know how such an object could be constructed, we just want to know that it exists. It is in such situations that *non-constructive proofs* shine. As the name suggests, non-constructive proofs don't involve constructing objects, instead, they prove an object must exist indirectly, through means which can vary. TPM is a non-constructive method, proving the existence of objects by taking an indirect route through probability theory.

Now that we've covered a little bit about why we might need non-constructive proof methods like TPM, let's get back to deciphering all that jibber-jabber about hypergraphs and colouring.

Hypergraphs and colouring: A combinatoric detour

Here's the problem we stated earlier:

What is the smallest number of edges in a k-uniform hypergraph that is not 2-colourable?

To understand it, we need grasp three key concepts: hypergraphs, k-uniformity and (hyper)graph colouring.

Hypergraphs

Graphs are super interesting objects, but they aren't where the fun ends! *Hypergraphs* are generalisations of graphs in which edges can join *any* number of vertices. Edges in hypergraphs are often illustrated as areas enclosing all the vertices in the edge as shown in Figure 3 below.

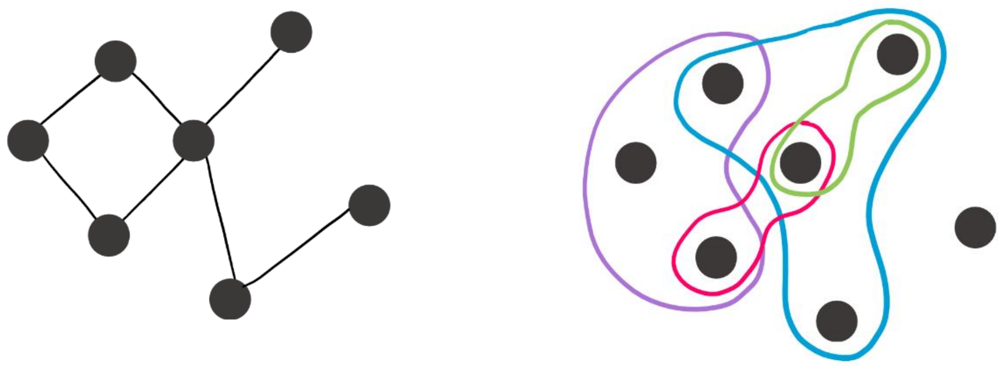

Figure 3. A graph (left) and a hypergraph on the same vertices (right).

k-uniform hypergraphs

If all edges in a hypergraph join exactly k vertices, we say that hypergraph is *k-uniform*. Figure 4 shows a 4-uniform hypergraph; one in which all edges join four vertices. Notice that a plain old ordinary graph is 2-uniform hypergraph in disguise!

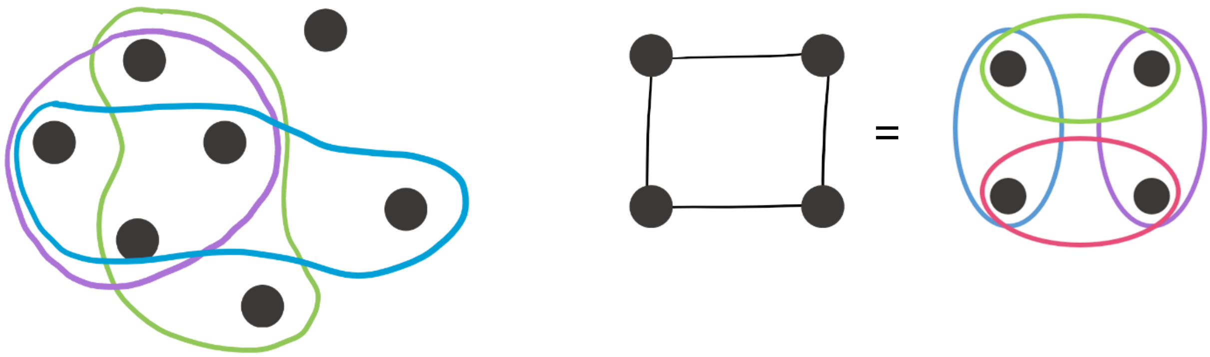

Figure 4. A 4-uniform hypergraph (left) and a 2-uniform hypergraph (right).

(Hyper)graph colouring

We've come across a notion of colouring graphs in MTH2132 in the proof of The Art Gallery Theorem, where we coloured triangulations of n-gons so that each triangle had no two vertices of the same colour. Mathematicians usually define graph colouring a little differently, ensuring instead that no edge is *monochromatic*, that is, no edge joins two vertices of the same colour. A colouring of a graph that has no monochromatic edges is called a *valid colouring*. If there exists a valid colouring on a graph using k distinct colours, we say that graph is *k-colourable*.

Figure 5 a) shows a valid three colouring of a graph, so we say this graph is 3-colourable. However, the graph is not 2-colourable, as we cannot produce any valid 2-colouring. Figure 5 b) helps illustrate why; as we have an odd number of vertices arranged in a cycle, no matter what we try we'll always have at least one monochromatic edge.

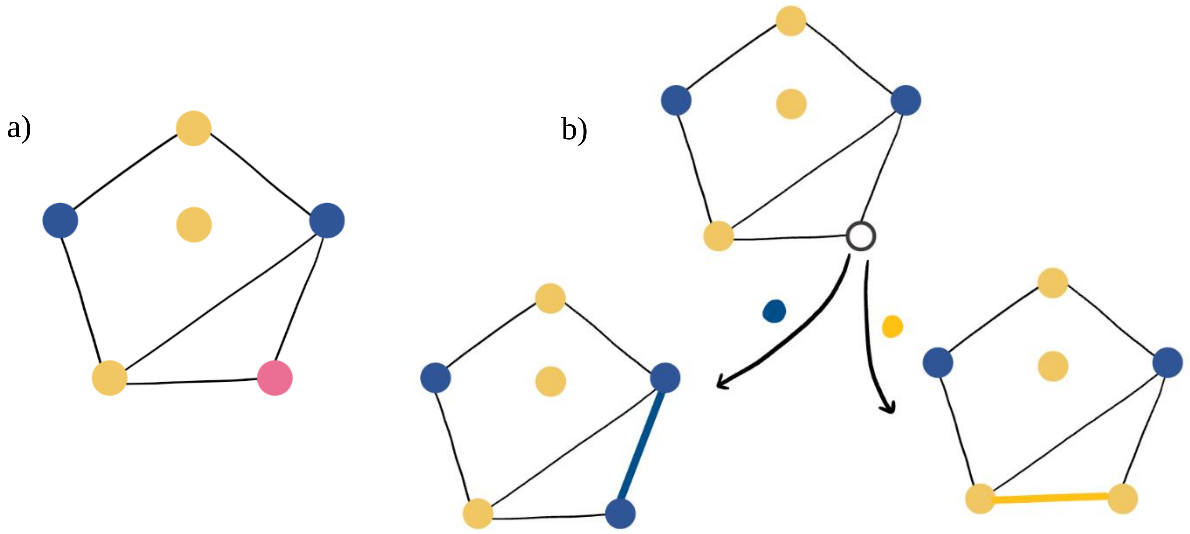

Figure 5. a) A valid 3-colouring of the graph, b) two invalid 2-colourings.

These notions of graph colouring naturally extend to colouring hypergraphs; a valid colouring of a hypergraph assigns colours to the vertices such that no edge joins vertices of only one colour, that is no edge is monochromatic, and a hypergraph is k-colourable if there exists a valid colouring for the graph using k colours.

Figure 6 shows that this hypergraph is 2-colourable; we have a valid 2-colouring of a hypergraph.

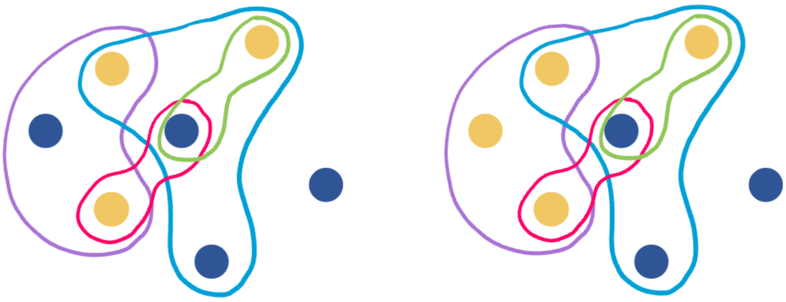

Figure 6. A valid (left) 2-colouring of a hypergraph, and an invalid 2-colouring (right).

Take a minute to let all these ideas sink in, and when you're ready, revisit our problem statement.

What is the smallest number of edges in a k-uniform hypergraph that is not 2-colourable?

All that jargon should make a little more sense!

The smallest number of edges is…

Before jumping all the way in, let's start with $k = 2$. Figure 7 makes it clear that any graph with only one or two edges is 2-colourable. But we can't 2-colour a graph with an odd cycle, so three is the smallest number of edges such that a 2-uniform hypergraph is not 2-colourable!

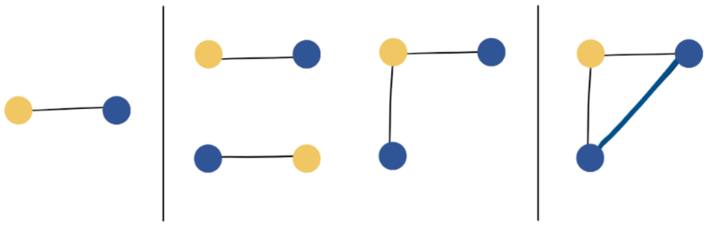

Figure 7. All graphs with 1 or 2 edges are 2-colourable, but the 3-cycle is not!

If we set $k = 3$ it gets trickier, but it can be shown the minimum number of edges is seven. The 3-uniform hypergraph with seven edges that can't be 2-coloured is the Fano plane[2], shown in Figure 8 below. Cool, isn't it?!

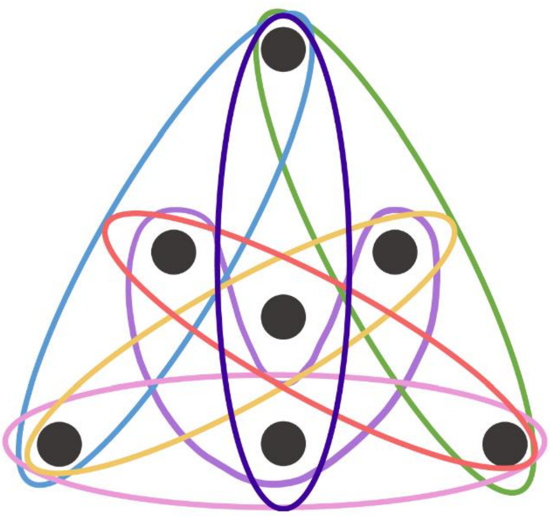

Figure 8. Fano plane hypergraph.

Okay, enough of the small stuff, what about for arbitrary k? Well, as it turns out, for $k \geq 5$ onwards no-one knows the answer exactly![1] What we can do though is prove a lower bound on the number of edges:

For any $k \geq 2$, the smallest number of edges in a k-uniform hypergraph that is not 2-colourable is at least 2^{k-1}.

How? Its time put TPM to work!

Proof [1]:

- Let H be a k-uniform hypergraph with $|E|$ edges.
- Pick two colours. Randomly assign each vertex one of these colours independently with equal probability.
- Consider an arbitrary edge in H. Since H is k-uniform, the edge contains k vertices. Further, since the vertices are coloured independently:

$$\begin{aligned}\mathbb{P}(\text{this edge is monochromatic}) &= \mathbb{P}(\text{all vertices in edge are red}) + \mathbb{P}(\text{all vertices in edge are blue}) \\ &= \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k \\ &= \frac{1}{2^{k-1}}\end{aligned}$$

  Recalling that from high-school probability, the probability of independent events occurring together is the product of their individual probabilities.

- Also recall that for two events A, B that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, so if $\mathbb{P}(A \cap B) > 0$, then $\mathbb{P}(A \cup B) < \mathbb{P}(A) + \mathbb{P}(B)$.
- Using a similar argument, applied to $|E|$ events instead of just two (known as Boole's inequality[3]) we get:

$$\mathbb{P}(\textit{any} \text{ edge is monochromatic}) < |E| \cdot \mathbb{P}(\textit{one} \text{ edge is monochromatic}) = |E| \cdot \frac{1}{2^{k-1}}$$

- But since

$$\mathbb{P}(\textit{no} \text{ edge is monochromatic}) = 1 - \mathbb{P}(\textit{any} \text{ edge is monochromatic})$$

  and

$$\mathbb{P}(\textit{any} \text{ edge is monochromatic}) < |E| \cdot \frac{1}{2^{k-1}}$$

  we have:

$$\mathbb{P}(\textit{no} \text{ edge is monochromatic}) > 1 - |E| \cdot \frac{1}{2^{k-1}}$$

- Here's the magic moment, if $|E| < 2^{k-1}$, then $0 < |E| \cdot \frac{1}{2^{k-1}} < 1$ and so $\mathbb{P}(\textit{no}$ edge is monochromatic$) > 0$.
- Thus, we have shown that if we randomly 2-colour k-uniform hypergraph with less than 2^{k-1} edges, then the probability the random colouring has no monochromatic edges is greater than 0. So we must be able to create *at least one valid 2-colouring* in this way, otherwise we would have got $\mathbb{P}(\textit{no}$ edge is monochromatic$) = 0$.
- So every k-uniform hypergraph with less than 2^{k-1} edges must be 2-colourable! Thus, the smallest number of edges in a k-uniform hypergraph that is not 2-colourable must be at least 2^{k-1}. ■

The Probabilistic Method

The proof above is a classical application of TPM. Most often applied to problems involving combinatoric objects, the method can be broken down into three steps:

1. Consider all possible objects of interest.
2. Introduce randomness by placing probabilities on each of these objects.
3. Given these probabilities, show that the probability of the object(s) that meet have the properties you are interested in is greater than 0.

Notice how we applied these steps in our problem:

1. We considered all possible k-uniform hypergraphs by focusing on an arbitrary k-uniform hypergraph.
2. We placed an even probability on generating each possible 2-colouring.
3. Given these probabilities, we showed that the probability of a 2-colourable k-uniform hypergraph was greater than zero if the number of edges was less than 2^{k-1}.

Here, we've only explored a basic application of the TPM. In some more advanced proofs, other tools from probability theory, such as expected value and variance, are used in place of direct probabilities. And there you have it! While it might not be able to prove the existence of aliens, I hope you've enjoyed learning about the unique and surprisingly powerful probabilistic method.

References

Inline citations

[1] Alon, Noga; Spencer, Joel H. (2008). *The probabilistic method* (3ed). New York: Wiley-Interscience, Pages xiii, 7

[2] Jithin Mathews, Manas Kumar Panda, Saswata Shannigrahi (2015*), On the construction of non-2-colorable uniform hypergraphs*, Discrete Applied Mathematics, Volume 180, Pages 181-187.

[3] *Boole's Inequality*, Wikipedia page, https://en.wikipedia.org/wiki/Boole%27s_inequality

Other reference materials

- J. Matoušek, J. Vondrak (2008). *The Probabilistic Method Lecture notes, https://www.cs.cmu.edu/~15850/handouts/matousek-vondrak-prob-ln.pdf*
- Robert Kübler (2021), *The Probabilistic Method, https://www.cantorsparadise.com/the-probabilistic-method-6e949f62ae09*

All diagrams and pictures drawn by me.