VI. Optional Technical Appendix

Theorem A-1 (Ville [24]): Let $\mathcal{Q}$ be a probabilistic transition system, let σ be an initialization of $\mathcal{Q}$, and let $\mu = \mu_{\mathcal{Q},\sigma}$. For every set $E \subseteq \mathbb{A}[\mathcal{Q}](\sigma)$, the following two conditions are equivalent.

(1) $\mu(E) = 0$.
(2) There is a $(\mathcal{Q},\sigma)$-martingale d such that $E \subseteq S^\infty[d]$.

Theorem A-2: If $\boldsymbol{\lambda}$ is a rate sequence, then, for each set $E \subseteq \boldsymbol{D}_{\boldsymbol{\lambda}}$, the following two conditions are equivalent.

1. $\mu_{\boldsymbol{\lambda}}(E) = 0$
2. There is a $\boldsymbol{\lambda}$-martingale d such that $E \subseteq S^\infty[d]$.

Proof: Suppose $\mu_{\boldsymbol{\lambda}}(E) = 0$. We wish to show that there exists a $\boldsymbol{\lambda}$-martingale, d, such that $E \subseteq S^\infty[d]$.

Assume the hypothesis. Then, for every $k \in \mathbb{N}$ there exists $C_k \subseteq (\{0,1\}^*)^{<|\boldsymbol{\lambda}|}$ such that

$$E \subseteq \bigcup_{w \in C_k} \Omega_w$$

and

$$\sum_{w \in C_k} \mu(\Omega_w) \leq 2^{-k}$$

Let, $g : \mathbb{N} \times \mathbb{N} \to (\{0,1\}^*)^{<|\boldsymbol{\lambda}|} \cup \{\emptyset\}$ be a function enumerating the elements of C_k, with the property that

- $E \subseteq \bigcup_{n=0}^{\infty} \Omega_{g(k,n)}$
- $\sum_{n=0}^{\infty} \mu(g(k,n)) \leq 2^{-k}$

We must define a $\boldsymbol{\lambda}$-martingale which succeeds on every $\boldsymbol{t} \in E \cap \Omega_{g(k,n)}$. Let $\boldsymbol{t} \in E \cap \Omega_{g(k,n)}$. Define the function $d_k : (\{0,1\}^*)^* \to [0,\infty)$ by

$$d_k(\lambda) = 2^{-k}$$

$$d_k(w) = \frac{\sum_{n=0}^{\infty} \mu(g(k,n) \wedge w)}{\mu(w)}$$

and where

$$\wedge : (\{0,1\}^*)^* \times (\{0,1\}^*)^* \to (\{0,1\}^*)^* \cup \emptyset$$

is defined by

$$x \wedge y = \begin{cases} x & \text{if } y \sqsubseteq x \\ y & \text{if } x \sqsubseteq y \\ \emptyset & \textit{otherwise} \end{cases} \tag{VI.1}$$

Let $k \in \mathbb{N}, w = (w_0, ..., w_{k-1}) \in (\{0,1\}^*)^*$. To see that the martingale condition is satisfied,

$$\begin{aligned}
&\sum_{b \in \{0,1\}} d_k((w_0, ..., w_{k-1}b))\mu((w_0, ..., w_{k-1}b)) \\
&= \sum_{b \in \{0,1\})} [\frac{\sum_{n=0}^{\infty} \mu(g(k,n) \wedge (w_0, ..., w_{k-1}b))}{\mu((w_0, ..., w_{k-1}b))}]\mu((w_0, ..., w_{k-1}b)) \\
&= \sum_{b \in \{0,1\}} \sum_{n=0}^{\infty} \mu(g(k,n) \wedge (w_0, ..., w_{k-1}b)) \\
&= \sum_{n=0}^{\infty} \sum_{b \in \{0,1\}} \mu(g(k,n) \wedge (w_0, ..., w_{k-1}b)) \\
&= \sum_{n=0}^{\infty} \mu(g(k,n) \wedge (w_0, ..., w_{k-1})) \\
&= d_k(w)\mu(w)
\end{aligned} \tag{VI.2}$$

Hence, $\forall k \in \mathbb{N}, d_k$ is a $\boldsymbol{\lambda}$-martingale.

Define the unitary success set of a $\boldsymbol{\lambda}$-martingale d to be

$$S^1[d] = \{\boldsymbol{t} \in [0,\infty)^\infty | (\exists w \sqsubseteq \boldsymbol{t}) d(w) \geq 1\}$$

Let $n \in \mathbb{N}, \boldsymbol{t} \in \Omega_{g(k,n)}$. Then, $g(k,n) \sqsubseteq \boldsymbol{t}$ and

$$d_k(g(k,n)) \geq \frac{\mu(g(k,n) \wedge g(k,n))}{\mu(g(k,n))} = 1$$

Thus, $\boldsymbol{t} \in S^1[d_k]$, and $\Omega_{g(k,n)} \subseteq S^1[d_k]$.

For each $k \in \mathbb{N}$, define $\hat{d}_k : (\{0,1\}^*)^* \to [0,\infty)$ by

$$\hat{d}_k(\lambda) = d_k(\lambda)$$

$$\hat{d}_k(wa) = \begin{cases} d_k(wa) & \text{if } \hat{d}_k(w) < 1 \\ \hat{d}_k(w) & \text{if } \hat{d}_k(w) \geq 1 \end{cases} \tag{VI.3}$$

$\hat{d}_k$ is a $\boldsymbol{\lambda}$-martingale. Define $\hat{d} : (\{0,1\}^*)^* \to [0,\infty)$ by

$$\hat{d}(w) = \sum_{k=0}^{\infty} \hat{d}_k(w)$$

$\hat{d}$ is a $\boldsymbol{\lambda}$-martingale with the property that $X \subseteq S^\infty[d]$. To see this, let $\boldsymbol{t} \in X, \alpha \in \mathbb{Z}^+$. It suffices to show that there exists $x \sqsubseteq \boldsymbol{t}, \hat{d}(x) \geq \alpha$.

Since $\boldsymbol{t} \in X, \forall k \in \mathbb{N}, \boldsymbol{t} \in S^1[d_k]$. Then $\forall w \sqsubseteq \boldsymbol{t}, 0 \leq k < \alpha, \hat{d}_k(w) \geq 1$. Then, $\forall w \sqsubseteq \boldsymbol{t}$,

$$\hat{d}(w) \geq \sum_{k=0}^{\alpha-1} d_k(w) \geq \alpha$$

so there must exist $x \sqsubseteq \boldsymbol{t}$ such that $\hat{d}(x) \geq \alpha$.

Now assume there exists $\boldsymbol{\lambda}$-martingale, d such that $E \subseteq S^\infty[d]$. Then, $\forall \boldsymbol{t} \in E, \alpha > 0, \exists w \in (\{0,1\}^*)^*$ such that $w \sqsubseteq \boldsymbol{t}$ and $d(w) > \alpha$. We wish to show that $\mu(E) = 0$.

We will show that there exists $g : \mathbb{N} \times \mathbb{N} \to (\{0,1\}^*)^* \cup \{\emptyset\}$, with the property that

(1) $E \subseteq \bigcup_{n=0}^{\infty} \Omega_{g(k,n)}$

(2) $\sum_{n=0}^{\infty} \mu(g(k,n)) \leq 2^{-k}$

For each $k \in \mathbb{N}$, define

$$A_k = \{w \in (\{0,1\}^*)^* | d(w) \geq 2^k d(\lambda)\}$$

and

$$B_k = \{w \in A_k | \forall v \sqsubseteq w, v \notin A_k\}$$

B_k is thus a set of all partial specifications "by which" d has accumulated 2^k value for the first time along the unique (with respect to the $\sqsubseteq$ relation) path that is each $w \in B_k$.

For all $k \in \mathbb{N}$, define $B_k(i)$ to be the ith element of B_k in standard enumeration of strings and define the function $g : \mathbb{N} \times \mathbb{N} \to (\{0,1\}^*)^* \cup \emptyset$ by

$$g(k,n) = \begin{cases} B_k(n) & \text{if } |B_k| \geq n \\ \emptyset & \text{otherwise} \end{cases} \tag{VI.4}$$

Let $k \in \mathbb{N}, \boldsymbol{t} \in E$, and let d_k be defined as in the previous section. Since $\boldsymbol{t} \in S^\infty[d_k]$, $\exists w \in B_k$ s.t. $w \sqsubseteq \boldsymbol{t}$. Then, $\exists n \in \mathbb{N}$ s.t. $g(k,n) \sqsubseteq \boldsymbol{t}$, whence

$$\boldsymbol{t} \in \bigcup_{n=0}^{\infty} \Omega_{g(k,n)}$$

and we have that

$$E \subseteq \bigcup_{n=0}^{\infty} \Omega_{g(k,n)}$$

By Lemma A-3,

$$\begin{aligned} d(\lambda) &\geq \sum_{w \in B_k} d(w)\mu(w) \\ &\geq 2^k d(\lambda) \sum_{w \in B_k} \mu(w) \\ &= 2^k d(\lambda) \sum_{n=0}^{\infty} \mu(g(k,n)) \end{aligned} \tag{VI.5}$$

and

$$\sum_{n=0}^{\infty} \mu(g(k,n)) \leq 2^{-k}$$

Thus, $\mu(E) = 0$. ■

Theorem 4: For every CTMC C and every set $X \subseteq \Omega[C]$, the following two conditions are equivalent.

(1) $\mu(X) = 0$

(2) There is a C-martingale d such that $X \subseteq S^\infty[d]$.

Proof: Suppose $\mu(X) = 0$. We wish to show that there exists a C-martingale, d, such that $X \subseteq S^\infty[d]$.

Assume the hypothesis. Then, $\forall k \in \mathbb{N} \exists C_k \subseteq (Q \times \{0,1\}^*)^*$ such that

$$X \subseteq \bigcup_{w \in C_k} \Omega_w$$

and

$$\sum_{w \in C_k} \mu(\Omega_w) \leq 2^{-k}$$

Let $k \in \mathbb{N}$. Suppose there exists $C_k \subseteq (Q \times \{0,1\}^*)^*$ satisfying the above conditions. Then, there exists $k \in \mathbb{N}$ and $g : \mathbb{N} \times \mathbb{N} \to (Q \times \{0,1\}^*)^* \cup \{\emptyset\}$, with the property that

- $X \subseteq \bigcup_{n=0}^{\infty} \Omega_{g(k,n)}$
- $\sum_{n=0}^{\infty} \mu(g(k,n)) \leq 2^{-k}$

We must define a martingale which succeeds on every $\boldsymbol{\tau} \in X \cap \Omega_{g(k,n)}$. Let $\boldsymbol{\tau} \in X \cap \Omega_{g(k,n)}$. Define the function $d_k : (Q \times \{0,1\}^*)^* \to [0, \infty)$ by

$$d_k(\lambda) = 2^{-k}$$

$$d_k(w) = \frac{\sum_{n=0}^{\infty} \mu(g(k,n) \wedge w)}{\mu(w)}$$

and where

$$\wedge : (Q \times \{0,1\}^*)^* \times (Q \times \{0,1\}^*)^* \to (Q \times \{0,1\}^*)^* \cup \emptyset$$

is defined by

$$x \wedge y = \begin{cases} x & \text{if } y \sqsubseteq x \\ y & \text{if } x \sqsubseteq y \\ \emptyset & \textit{otherwise} \end{cases} \tag{VI.6}$$

d_k is a C-martingale if it satisfies the conditions:

$$1. \forall w \in (Q \times \{0,1\}^*)^*,$$

$$d(w)\mu(w) = \sum_{q \in Q} d(w(q,\lambda))\mu(w(q,\lambda))$$

$$2. \forall w \in (Q \times \{0,1\}^*)^*, q \in Q, u \in \{0,1\}^*,$$

$$d(w(q,u))\mu(w(q,u)) = \sum_{b \in \{0,1\}} d(w(q,ub))\mu(w(q,ub))$$

Let $k \in \mathbb{N}, q \in Q, u \in \{0,1\}, w \in (Q \times \{0,1\}^*)^*$. To see that (1) is satisfied,

$$\begin{aligned} &\sum_{q \in Q} d_k(w(q,\lambda))\mu(w(q,\lambda)) \\ &= \sum_{q \in Q} [\frac{\sum_{n=0}^{\infty} \mu(g(k,n) \wedge (w(q,\lambda)))}{\mu(w(q,\lambda))}]\mu(w(q,\lambda)) \\ &= \sum_{q \in Q} \sum_{n=0}^{\infty} \mu(g(k,n) \wedge w(q,\lambda)) \\ &= \sum_{n=0}^{\infty} \sum_{q \in Q} \mu(g(k,n) \wedge w(q,\lambda)) \\ &= \sum_{n=0}^{\infty} \mu(g(k,n) \wedge w) \\ &= d_k(w)\mu(w) \end{aligned} \tag{VI.7}$$

To see that (2) is satisfied,

$$\sum_{b\in\{0,1\}} d_k(w(q,ub))\mu(w(q,ub))$$
$$= \sum_{b\in\{0,1\}} [\frac{\sum_{n=0}^{\infty}\mu(g(k,n)\wedge(w(q,ub)))}{\mu(w(q,ub))}]\mu(w(q,ub))$$
$$= \sum_{b\in\{0,1\}}\sum_{n=0}^{\infty}\mu(g(k,n)\wedge w(q,ub))$$
$$= \sum_{n=0}^{\infty}\sum_{b\in\{0,1\}}\mu(g(k,n)\wedge w(q,ub))$$
$$= \sum_{n=0}^{\infty}\mu(g(k,n)\wedge w(q,u))$$
$$= d_k(w(q,u))\mu(w(q,u)) \tag{VI.8}$$

Hence, $\forall k\in\mathbb{N}, d_k$ is a C-martingale.

Define the unitary success set of a martingale d to be

$$S^1[d] = \{\boldsymbol{\tau}\in(Q\times(0,\infty))^{\infty}|(\exists w\sqsubseteq\boldsymbol{\tau})d(w)\geq 1\}$$

Let $n\in\mathbb{N},\boldsymbol{\tau}\in\Omega_{g(k,n)}$. Then, $g(k,n)\sqsubseteq\boldsymbol{\tau}$ and

$$d_k(g(k,n))\geq\frac{\mu(g(k,n)\wedge g(k,n))}{\mu(g(k,n))}=1$$

Thus, $\boldsymbol{\tau}\in S^1[d_k]$, and $\Omega_{g(k,n)}\subseteq S^1[d_k]$.

For each $k\in\mathbb{N}$, define $\hat{d}_k:(Q\times\{0,1\}^*)^*\to[0,\infty)$ by

$$\hat{d}_k(\lambda)=d_k(\lambda)$$
$$\hat{d}_k(wa)=\begin{cases} d_k(wa) & \text{if } \hat{d}_k(w)<1\\ \hat{d}_k(w) & \text{if } \hat{d}_k(w)\geq 1\end{cases} \tag{VI.9}$$

$\hat{d}_k$ is a C-martingale. Define $\hat{d}:(Q\times\{0,1\}^*)^*\to[0,\infty)$ by

$$\hat{d}(w)=\sum_{k=0}^{\infty}\hat{d}_k(w)$$

$\hat{d}$ is a C-martingale with the property that $X\subseteq S^{\infty}[d]$. To see this, let $\boldsymbol{\tau}\in X,\alpha\in\mathbb{Z}^+$. It suffices to show that there exists $x\sqsubseteq\boldsymbol{\tau},\hat{d}(x)\geq\alpha$.

Since $\boldsymbol{\tau}\in X,\forall k\in\mathbb{N},\boldsymbol{\tau}\in S^1[d_k]$. Then $\forall w\sqsubseteq\boldsymbol{\tau}, 0\leq k<\alpha,\hat{d}_k(w)\geq 1$. Then, $\forall w\sqsubseteq\boldsymbol{\tau}$,

$$\hat{d}(w)\geq\sum_{k=0}^{\alpha-1}d_k(w)\geq\alpha$$

so there must exist $x\sqsubseteq\boldsymbol{\tau}$ such that $\hat{d}(x)\geq\alpha$. Thus, one direction is proven.

Now let $C=(Q,\sigma,\boldsymbol{\lambda})$ be a CRN ($|Q|<\infty$). Let $X\subseteq\Omega[C]$. Suppose there exists a C-martingale, d such that $X\subseteq S^{\infty}[d]$. Then, $\forall\boldsymbol{\tau}\in X,\alpha>0,\exists w\in(Q\times\{0,1\}^*)^*$ such that $w\sqsubseteq\boldsymbol{\tau}$ and $d(w)>\alpha$. We wish to show that $\mu(X)=0$.

We will show that there exists $g:\mathbb{N}\times\mathbb{N}\to(Q\times\{0,1\}^*)^*\cup\{\emptyset\}$, with the property that

(1) $X\subseteq\bigcup_{n=0}^{\infty}\Omega_{g(k,n)}$

(2) $\sum_{n=0}^{\infty}\mu(\Omega_{g(k,n)})\leq 2^{-k}$

For each $k\in\mathbb{N}$, define

$$A_k=\{w\in(Q\times\{0,1\}^*)^*|d(w)\geq 2^k d(\lambda)\}$$

and

$$B_k=\{w\in A_k|\forall v\sqsubseteq w, v\notin A_k\}$$

B_k is thus the set of all partial specifications "by which" d has accumulated 2^k value for the first time along the unique path that is each $w\in B_k$.

For all $k\in\mathbb{N}$, define $B_k(i)$ to be the i-th element of B_k in standard enumeration of strings and define the function $g:\mathbb{N}\times\mathbb{N}\to(Q\times\{0,1\}^*)^*\cup\emptyset$ by

$$g(k,n)=\begin{cases} B_k(n) & \text{if } |B_k|\geq n\\ \emptyset & \text{otherwise}\end{cases} \tag{VI.10}$$

To see that (1) is satisfied, let $k\in\mathbb{N},\boldsymbol{\tau}\in X$, and let d_k be defined as in the previous section. Since $\boldsymbol{\tau}\in S^{\infty}[d_k]$, $\exists w\in B_k$ s.t. $w\sqsubseteq\boldsymbol{\tau}$. Then, $\exists n\in\mathbb{N}$ s.t. $g(k,n)\sqsubseteq\boldsymbol{\tau}$, whence

$$\boldsymbol{\tau}\in\bigcup_{n=0}^{\infty}\Omega_{g(k,n)}$$

and we have that

$$X\subseteq\bigcup_{n=0}^{\infty}\Omega_{g(k,n)}$$

To see that (2) is satisfied, by Lemma A-3,

$$\begin{aligned} d(\lambda) &\geq\sum_{w\in B_k}d(w)\mu(w)\\ &\geq 2^k d(\lambda)\sum_{w\in B_k}\mu(w)\\ &=2^k d(\lambda)\sum_{n=0}^{\infty}\mu(g(k,n))\end{aligned} \tag{VI.11}$$

and

$$\sum_{n=0}^{\infty}\mu(g(k,n))\leq 2^{-k}$$

Thus, $\mu(X)=0$. ■

Theorem 5: For every CTMC C and every set $X\subseteq\Omega[C]$, the following two conditions are equivalent.

(1) $\mu_{\text{constr}}(X)=0$.

(2) There is a lower semi-computable C-martingale d such that $X\subseteq S^{\infty}[d]$.

Proof: This proof follows the structure of the above proof with some adjustments:

Assume $\mu_{\text{constr}}(X)=0$. Then, $\exists g:\mathbb{N}\times\mathbb{N}\to(Q\times\{0,1\}^*)^*$ such that g is computable and

$$X\subseteq\bigcup_{n=0}^{\infty}g(k,n) \text{ and } \sum_{n=0}^{\infty}\mu(g(k,n))\leq 2^{-k}$$

Consider the same construction as before, and fix some $k \in \mathbb{N}$. Let M_k be the machine enumerating $g(k,0), g(k,1), \ldots$. To show that $d = \sum d_k$ is lower semicomputable, define

$$\hat{d}(w,t) = \sum_{k=0}^{\infty} \frac{\mu((C_k)_t \cap C_w)}{\mu(C_w)}$$

where

$$(C_k)_t = \bigcup_{n=0}^{t} g(k,n) \text{ and } \lim_{t\to\infty} (C_k)_t = C_k$$

for each $k \in \mathbb{N}$.

Clearly, $\hat{d}(w,t) \leq \hat{d}(w,t+1) < d(w)$ and $\lim_{t\to\infty} \hat{d}(w,t) = d(w)$ for all w, t.

Assume instead that there exists a constructive martingale d, with $X \subseteq S^{\infty}[d]$ and a function $\hat{d}$ testifying to the lower semi-computability of d. We wish to show that for each $k \in \mathbb{N}$, the set A_k is computably enumerable.

Define an enumerator M_k: For each $(w,t) \in (Q \times \{0,1\}^*)^* \times \mathbb{N}$, dovetailing, compute $\hat{d}(w,t)$. If $\hat{d}(w,t) \geq 2^k d(\lambda)$, output w.

M_k enumerates

$$A_k = \{w \in (Q \times \{0,1\}^*)^* | d(w) \geq 2^k d(\lambda)\}$$

A prefix set $B_k \subseteq A_k$ can be enumerated by running the enumerator for A_k and not enumerating any element for which a prefix has been printed or which would prefix and already printed element. The resulting function

$$g(k,n) = \begin{cases} B_k(n) & \text{if } |B_k| \geq n \\ \emptyset & \text{otherwise} \end{cases} \tag{VI.12}$$

produces a constructive null cover of X. ■

*Lemma A-3 (*Generalized Kraft Inequality): Let $C = (Q, \boldsymbol{\lambda}, \sigma)$ be a CRN, d a C-martingale (resp. $\boldsymbol{\lambda}$-martingale or $\mathscr{Q}$-martingale), and $B \subseteq (Q \times \{0,1\}^*)^*$ (resp. $(\{0,1\}^*)^*$ or Q^*) a prefix set. Then,

$$d(\lambda)\mu(\lambda) = d(\lambda) \geq \sum_{w\in B} d(w)\mu(w)$$

Proof: * If $d(\lambda) = 0$, this is immediate. Assume $d(\lambda) > 0$. Note that μ is a probability measure on $(Q \times \{0,1\}^*)^{\infty}$ because it satisfies the following conditions:

1. $\mu : (Q \times \{0,1\}^*)^* \to [0,1]$
2. $\mu(\lambda) = 1$.
3. If $|w| = n$ and $w = (q_0,u_0)\ldots(q_{n-1},u_{n-1})$ then,

$$\begin{aligned} &\sum_{q\in Q} \mu(w(q,\lambda)) \\ &= \sum_{q\in Q} \sigma(q_0)\Pi_{i=0}^{n-2}(q_i,q_{i+1})\Pi_{i=0}^{n-1} 2^{-|u_i|} p(q_{n-1},q) \\ &= \mu(w) \end{aligned} \tag{VI.13}$$

4. If $|w| = n, u \in \{0,1\}^*$ and $w = (q_0,u_0)\ldots(q_{n-2},u_{n-2})(q_{n-1},u)$ then,

$$\begin{aligned} \sum_{b\in\{0,1\}^*} \mu(wb) &= \sum_{b\in\{0,1\}^*} \sigma(q_0)(\Pi_{i=0}^{n-2}(q_i,q_{i+1}))(\Pi_{i=0}^{n-1} 2^{-|u_i|})2^{-|ub|} \\ &= \mu(w) \end{aligned} \tag{VI.14}$$

where wb is shorthand for $(q_0,u_0)\ldots(q_{n-2},u_{n-2})(q_{n-1},ub)$.

Define $\pi : (Q \times \{0,1\}^*)^* \to [0,1]$ by

$$\sigma(w) = \frac{d(w)\mu(w)}{d(\lambda)}$$

It is straightforward to show that this is a probability measure on $(Q \times \{0,1\}^*)^{\infty}$. Write

$$d(w) = d(\lambda)\frac{\pi(w)}{\mu(w)}$$

where π is a "strategy" and μ is the "environment".

Then, choose $\omega \in (Q \times \{0,1\}^*)^{\infty}$ according to π and let E be the event that $\exists w \in (Q \times \{0,1\}^*)^*)$ such that $w \sqsubseteq \omega$ for some $w \in B$ in this experiment. Then,

$$\begin{aligned} 1 &\geq Pr(E) \\ &= \sum_{w\in B} \pi(w) \\ &= \frac{1}{d(\lambda)} \sum_{w\in B} d(w)\mu(w) \end{aligned} \tag{VI.15}$$

So,

$$d(\lambda) \geq \sum_{w\in B} d(w)\mu(w)$$

■

Lemma A-4: Let $d_0, d_1, d_2, d_3, \ldots$ be a sequence of C-martingales (resp. $\boldsymbol{\lambda}$-martingales or $\mathscr{Q}$-martingales) such that

$$\sum_{n=0}^{\infty} d_n(\lambda) < \infty$$

Then, the function $d : (Q \times \{0,1\}^*)^* \to [0,\infty)$ (resp. $(\{0,1\}^*)^*$ or Q^*) defined by: $\forall w$,

$$d(w) = \sum_{n=0}^{\infty} d_n(w)$$

is a C-martingale (resp. $\boldsymbol{\lambda}$-martingale or $\mathscr{Q}$-martingale).

Proof: Let $d_0, d_1, \ldots$ and d be as given.

$\forall w \in (Q \times \{0,1\}^*)^*, q \in Q, u \in \{0,1\}^*$

*Kraft inequalities corresponding to $\boldsymbol{\lambda}$-martingales and $\mathscr{Q}$-martingales have nearly identical proofs and we omit these.

$$
\begin{aligned}
&\sum_{b\in\{0,1\}} d(w(q,ub))\mu(w(q,ub)) \\
&= \mu(w(q,u)) \sum_{b\in\{0,1\}} (\sum_{n=0}^{\infty} d_n(w(q,ub))) \\
&= \mu(w(q,u)) \sum_{n=0}^{\infty} \sum_{b\in\{0,1\}} d_n(w(q,ub)) \\
&= \mu(w(q,u)) \sum_{n=0}^{\infty} d_n(w(q,u)) \\
&= \mu(w(q,u)) d(w(q,u))
\end{aligned} \tag{VI.16}
$$

and

$$
\begin{aligned}
\sum_{q\in Q} d(w(q,\lambda))\mu(w(q,\lambda)) &= \mu(w) \sum_{q\in Q} (\sum_{n=0}^{\infty} d_n(w(q,\lambda))) \\
&= \mu(w) \sum_{n=0}^{\infty} \sum_{q\in Q} d_n(w(q,\lambda)) \\
&= \mu(w) \sum_{n=0}^{\infty} d_n(w) \\
&= \mu(w) d(w)
\end{aligned} \tag{VI.17}
$$

Since $d(\lambda)$ is finite and the martingale conditions hold, it follows by simple induction that $\forall w, d(w)$ is also finite. Thus, d is a C-martingale. ∎

Lemma 6: Let C be a CTMC and $\boldsymbol{\tau} = (q_0,t_0)(q_1,t_1).... \in \Omega[C]$ be random. Then, the subsequence consisting of all states in $\boldsymbol{\tau}$, $\boldsymbol{q} = q_0, q_1, q_2, ... \in Q^\infty$ is random with respect to $(\mathscr{Q}, \sigma)$.

Proof: Let $\boldsymbol{\tau}, \boldsymbol{q}$ be as described. To prove by contrapositive, suppose there exists a lower semicomputable $(\mathscr{Q}, \sigma)$-martingale $d : Q^* \to [0,\infty)$ which succeeds on $\boldsymbol{q}$ (that is, $\boldsymbol{q}$ is not random).

We use the shorthand wb, where $w = (q_0,u_0),...,(q_k,u_k)$, and $b \in \{0,1\}$, to denote $(q_0,u_0)...(q_k,u_k b)$ and wq, where $q \in Q$, to denote $(q_0,u_0)...(q_k,u_k),(q,\lambda)$.

Define the C-martingale $\hat{d} : (Q\times\{0,1\}^*)^* \to [0,\infty)$ as follows:

If $q \in Q$

$$\hat{d}(wq) = d(q_0,...,q_{n-1},q)$$

If $b \in \{0,1\}$

$$\hat{d}(wb) = \hat{d}(w)$$

That is, $\hat{d}$ only bets on states (and bets on them according to d's strategy), while hedging its bets on times. To see that $\hat{d}$ is in fact a C-martingale:

$\forall w \in (Q\times\{0,1\}^*)^*, q \in Q, u \in \{0,1\}^*$

$$
\begin{aligned}
\sum_{b\in\{0,1\}} \hat{d}(w(q,ub))\mu(w(q,ub)) &= \hat{d}(w(q,u)) \sum_{b\in\{0,1\}} \mu(w(q,ub)) \\
&= \hat{d}(w(q,u))\mu(w(q,u))
\end{aligned} \tag{VI.18}
$$

and $\forall w \in (Q\times\{0,1\}^*)^*, |w| = n$

$$
\begin{aligned}
\sum_{q\in Q} \hat{d}(w(q,\lambda))\mu(w(q,\lambda)) &= \sum_{q\in Q} d(q_0,...,q_{n-1},q)\mu(w(q,\lambda)) \\
&= d(q_0,...,q_{n-1}) \sum_{q\in Q} \mu(w(q,\lambda)) \\
&= \hat{d}(w) \sum_{q\in Q} \mu(w(q,\lambda)) \\
&= \hat{d}(w)\mu(w)
\end{aligned} \tag{VI.19}
$$

To see that $\hat{d}$ succeeds on $\boldsymbol{\tau}$ let $\alpha > 0$. Since d succeeds on $\boldsymbol{q}$, $\exists n \in \mathbb{N}$ and $w_n \sqsubseteq \boldsymbol{q}$ such that $d(w_n) > \alpha$. Then, since $\hat{d}$ does not bet on sojourn times and bets on states according to d,

$$\hat{d}((q_0,u_0)(q_1,u_1)...(q_{n-1},u_{n-1})) > \alpha$$

To see that $\hat{d}$ is lower semicomputable, let $d' : Q^* \times \mathbb{N} \to \mathbb{Q}$ be a function testifying to the fact that d is lower semicomputable. Define $\hat{d}'$ as $\hat{d}$ is defined above, replacing instances of d with instances of d'. Its limit behavior is as desired. ∎

Lemma 7: Let $\boldsymbol{\tau} = (q_0,t_0)(q_1,t_1)..... \in \Omega[C]$ where C is some CTMC. Suppose $\exists n \in \mathbb{N}$ such that t_n is not random. Then, $\boldsymbol{\tau}$ is not random.

Proof: Assume the hypothesis. For every $n \in \mathbb{N}$ define a C-martingale $\hat{d}_n$ which

1.Doesn't bet on states

2.Bets according to d on only the nth sojourn time $t_n (n = 0...\infty)$.

$$\hat{d}_n(\lambda) = 2^{-n},$$

$$\hat{d}_n(w(q,\lambda)) = \hat{d}_n(w)$$

If $|w| = n, w = (q_0,u_0)(q_1,u_1)...(q_{n-1},u), u \in \{0,1\}^*, b \in \{0,1\}$

$$\hat{d}(w[0...n-2](q_{n-1},ub)) = d(ub)$$

If $|w| = k \neq n, w = (q_0,u_0)(q_1,u_1)...(q_{k-1},u), u \in \{0,1\}^*, b \in \{0,1\}$

$$\hat{d}(w[0...k-2](q_{k-1},ub)) = \hat{d}(w[0...k-2](q_{k-1},u))$$

Let $n \in \mathbb{N}$. We must prove $\hat{d}_n$ is indeed a martingale. If $q \in Q$,

$$
\begin{aligned}
\sum_{q\in Q} \hat{d}_n(w(q,\lambda))\mu(w(q,\lambda)) &= \sum_{q\in Q} \hat{d}_n(w)\mu(w(q,\lambda)) \\
&= \hat{d}_n(w) \sum_{q\in Q} \mu(w(q,\lambda)) \\
&= \hat{d}_n(w)\mu(w)
\end{aligned} \tag{VI.20}
$$

If $|w| = k \neq n$,

$$\begin{aligned}
&\sum_{b\in\{0,1\}} \hat{d}_n(w[0...n-2](q,ub))\mu(w[0...n-2](q,ub)) \\
&= \sum_{b\in\{0,1\}} d(ub)\mu(w[0...n-2](q,ub)) \\
&= d(u)\mu(w[0...n-2](q,u)) \\
&= \hat{d}(w[0...n-2](q,u))\mu(w[0...n-2](q,u))
\end{aligned} \tag{VI.21}$$

If $|w| = n$,

$$\begin{aligned}
&\sum_{b\in\{0,1\}} \hat{d}_n(w[0...n-2](q,ub))\mu(w[0...n-2](q,ub)) \\
&= \sum_{b\in\{0,1\}} d(ub)\mu(w[0...n-2](q,ub)) \\
&= d(u)\mu(w[0...n-2](q,u)) \\
&= \hat{d}(w[0...n-2](q,u))\mu(w[0...n-2](q,u))
\end{aligned} \tag{VI.22}$$

Define $\hat{d}$ to be a C-martingale obtained by applying Lemma A-4:

$$\hat{d} = \sum_{n=0}^{\infty} \hat{d}_n$$

$\hat{d}$ succeeds on $\boldsymbol{\tau}$.

Since d is lower semicomputable, let d' testify to this. Substituting d' in the above construction shows that $\hat{d_n}$ is lower semicomputable for all n, and thus that $\hat{d}$ is also lower semicomputable. Thus, $\boldsymbol{\tau}$ is not random. ∎

Lemma 8: Let $\boldsymbol{\tau} \in \Omega[C]$ be a trajectory in a CTMC, C. If $\boldsymbol{\tau}$ is random, then all sojourn times $t_0, t_1, t_2, ...$ in $\boldsymbol{\tau}$ are independently random.

Proof: We prove by contrapositive. Let $\boldsymbol{\tau} \in \Omega[C]$ and suppose there exists n such that $t_1, ..., t_n$ are not independently random. Then, there exists $d : \{0,1\}^*_n \to [0,\infty)$ (where $\{0,1\}^*_n$ denotes the set of all n-tuples of strings *of the same length*) such that $\forall w \in \{0,1\}^*_n$

$$d(w)\mu(w) = \sum_{a\in 0,1^1_n} d(wa)\mu(wa)$$

and

$$\limsup_{k\to\infty} d((t_1, ..., t_n) \restriction k) = \infty,$$

where μ refers to the probability measure on $\{0,1\}^*_n$ defined by

$$\mu((w_1, ..., w_n)) = \Pi_{i=1}^n \mu_i(w_i)$$

and d is lower semicomputable.

Define the martingale $d : \{0,1\}^* \to [0,\infty)$ by

$$d(w) = d(w, t_2, ..., t_3).$$

It's clear that is a martingale which succeeds on t_0, from which it follows that $\boldsymbol{t}$ and thus also $\boldsymbol{\tau}$ cannot be random. ∎

Lemma 9: There exists a rate sequence $\boldsymbol{\lambda}$ and a sequence $\boldsymbol{R} = (\boldsymbol{t_0}, \boldsymbol{t_1}, ...)$ of $\boldsymbol{\lambda}$-durations such that $\boldsymbol{t_0}, \boldsymbol{t_1}, ...$ are independently random but $\boldsymbol{R}$ is not random with respect to $\mu_{\boldsymbol{\lambda}}$.

Proof: Let $\boldsymbol{\lambda}$ be a rate sequence and let $S_0, S_1, ...$ be a sequence of elements of $\{0,1\}^\infty$ representing times $t_0, t_1, ...$ each of which are random with respect to the rates $\lambda_0, \lambda_1, ...$ Then, the times (qua binary sequences) in the $\boldsymbol{\lambda}$-duration sequence $(0S_0, 0S_1, 0S_2, ..)$ are not independently random since a lower-semicomputable $\boldsymbol{\lambda}$-martingale exists which can bet only on the first bit of each sequence and hedge on all other bits. ∎

*Theorem 10 (*Non-Zeno property): Let C be a CRN. Then, if $\boldsymbol{\tau} = (q_0, t_0), (q_1, t_1), ... \in \Omega[C]$ is random and has bounded molecular counts, then $\boldsymbol{\tau}$ satisfies the *non-Zeno property* that

$$\sum_{i=0}^{\infty} t_i = \infty.$$

Proof: By contrapositive. Let C be a CRN and $\boldsymbol{\tau} \in \Omega[C]$ a trajectory with bounded molecular counts. Since $\boldsymbol{\tau}$ has bounded molecular counts, there exists a constant $M \in \mathbb{R}$ which is the maximum reaction rate along $\boldsymbol{\tau}$. Since $\boldsymbol{\tau}$ has the Zeno property, there must exist $i \in \mathbb{N}$ such that $\forall k \geq i, t_k \in J(\lambda_{q_i}, 0)$. Define a C-martingale which only bets on the first bit of each sojourn time $t_i, t_{i+1}, ...$ as follows:

$$d_i(\lambda) = 1 \tag{VI.23}$$

$$d_i(w(q,\lambda)) = d_i(w) \tag{VI.24}$$

$$d_i(w(q,ub)) = \begin{cases} (2d_i(w(q,u)) & \text{if } |w| = i, u = \lambda, \text{ and } b = 0 \\ d_i(w(q,u)) & \text{if } |w| < i \\ 0 & \text{if } |w| = i, \text{ and } b \neq 0 \end{cases} \tag{VI.25}$$

Since i is a definite value, d_i does not begin to bet until it reaches the i-th sojourn time, and d_i bets only on the first bit of each sojourn time after the ith, d_i succeeds on $\boldsymbol{\tau}$. d_i is clearly lower semicomputable. Thus, $\boldsymbol{\tau}$ cannot be random. ∎

Lemma 12: For every cylinder, Ω_w of a CTMC C,

$$K(w) \leq l(w) + K(\text{prof}(w)) + O(1),$$

where $l(w) = \log \frac{1}{\mu_C(w)}$ is the "self-information" of w.

Proof: In the following proof, we let p range over all profiles, and assume there is some natural encoding (enumerating process) between natural numbers and profiles, and also between natural numbers and cylinders.

$$\begin{aligned}
\Omega &= \sum_p 2^{-K(p)}, \quad \text{for some constant } c \\
&= \sum_p \left(2^{-K(p)} \sum_{\text{prof}(w)=p} 2^{-l(w)} \right) \\
&= \sum_w 2^{-(K(\text{prof}(w))+l(w))} < \infty.
\end{aligned}$$

Then, by the minimality of K and the coding relation between cylinders and natural numbers, we have

$$K(w) \le l(w) + K(\mathrm{prof}(w)) + O(1).$$

■

Lemma 13: There is a constant $c \in \mathbb{N}$ such that, for every profile p of a CTMC C and every $k \in \mathbb{N}$,

$$\mu_C\left(\bigcup_{\substack{w \\ prof(w)=p \\ K(w)<l(w)+K(p)-k}} \Omega_w \right) < 2^{c-k}.$$

Substituting $k + K(\mathrm{prof}(w))$ for k here gives

$$\mu_C\left(\bigcup_{\substack{w \\ prof(w)=p \\ K(w)<l(w)-k}} \Omega_w \right) < 2^{c-k-K(p)}.$$

Proof: We only need to note that

$$\sum_p \sum_{\mathrm{prof}(w)=p} 2^{-K(w)} = \sum_w 2^{-K(w)} < \infty.$$

Then by the minimality [2] of K, we have,

$$\begin{aligned} 2^{-K(p)+c} &\ge \sum_{\mathrm{prof}(w)=p} 2^{-K(w)} \\ &= \sum_{\mathrm{prof}(w)=p} \mu(w)\frac{1}{\mu(w)}2^{-K(w)} \\ &= \sum_{\mathrm{prof}(w)=p} \mu(w)2^{\log\frac{1}{\mu(w)}}2^{-K(w)} \\ &= \sum_{\mathrm{prof}(w)=p} \mu(w)2^{l(w)-K(w)} \\ &= \mathbb{E}_\mu[2^{l(w)-K(w)}] \end{aligned}$$

Therefore,

$$\begin{aligned} &\mu\left\{w \mid K(w) < l(w) + K(\mathrm{prof}(w)) - k\right\} \\ &= \mu\left\{w \mid l(w) - K(w) > k - K(\mathrm{prof}(w)))\right\} \\ &= \mu\left\{w \mid 2^{l(w)-K(w)} > 2^{k-K(\mathrm{prof}(w)))}\right\} \\ &< \frac{\mathbb{E}_\mu[2^{l(w)-K(w)}]}{2^{k-K(\mathrm{prof}(w)))}} \le \frac{2^{-K(p)+c}}{2^{k-K(\mathrm{prof}(w)))}} = 2^{c-k} \end{aligned}$$

The first inequality in the last row follows by the Markov inequality. ■

Theorem 14: A trajectory $\boldsymbol{\tau}$ is Martin Löf random if and only if there exists $k \in \mathbb{N}$, such that for every $w \sqsubseteq \boldsymbol{\tau}$, $K(w) \ge l(w) - k$.

Proof: "Only if": We prove the contrapositive. Suppose that for every k, there is at least one $w \sqsubseteq \boldsymbol{\tau}$, such that $K(w) < l(w) - k$. We let

$$U_k = \{w \mid K(w) < l(w) - k\}.$$

Note that w ranges over all cylinders in the above definition. Therefore, it it clear that $\boldsymbol{\tau}$ is covered by the U_k.

Next, we are going to estimate the measure of U_k. First we consider the t-slice of U_k, U_k^t, defined as:

$$U_k^p = \{w \mid \mathrm{prof}(w) = p \text{ and } w \in U_k\}.$$

Note that by Lemma 13, we have $\mu[U_k^p] < 2^{c-k-K(p)}$, therefore

$$\mu[U_k] = \sum_p \mu[U_k^p] \le \sum_t 2^{c-k-K(p)} \le 2^{c-k}\Omega \le 2^{c-k}.$$

Also note that each U_k is recursively enumerable, and $V_k = U_{c+k}$ is a Martin Löf test.

"If": Again by contrapositive: Assume $\boldsymbol{\tau}$ is not Martin Löf random, and let $\{U_k\}$ be a Martin Löf test. We construct the following (output, size-of-program) requirement pairs as follows:

$$\{(w, l(w) - k) \mid w \in U_{k^2}, k \ge 2\}$$

It can be checked this requirement satisfies Kraft's inequality, since the measure of the size-of-program is bounded from above by

$$\sum_{k\ge 2} 2^{-(k^2-k)} = 1/2^2 + 1/2^6 + 1/2^{12} \cdots < 1$$

Then by Levin's coding lemma [11], [12], this requirement can be fulfilled.

Note that $\boldsymbol{\tau}$ can be covered by U_{k^2}, and therefore for each $k \ge 2$ there are prefixes w of $\boldsymbol{\tau}$ for which $K(w) \le l(w) - k < l(w) - (k-1)$.

That is, for every $k' = k - 1 > 0$, there is some $w \sqsubseteq \boldsymbol{\tau}$, such that $K(w) < l(w) - k'$, Hence $\boldsymbol{\tau}$ is not random in the Kolmogorov sense.

■

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