PHYS 410: Project 3
Driven Damped Pendulum Motion

Jack Hong
30935134

November 23, 2016

All the plots included in this report can be generated by running the included *pendulum.m* file.

NOTE:

- All phase portraits are v vs θ, and not θ vs v as stated in the assignment because v vs θ is the convention and the patterns are clearer that way.
- The bottom (resting position) of the pendulum is $\theta = \pi$, and θ is constrained to the range $[0, 2\pi]$.

We can rearrange equation (1) of the assignment to solve for $\frac{d^2\theta}{dt^2}$:

$$m\frac{d^2\theta}{dt^2} + \nu\frac{d\theta}{dt} + mgsin(\theta) = Asin(\omega t) \tag{1}$$

$$\frac{d^2\theta}{dt^2} = \frac{A}{m}sin(\omega t) - \frac{\nu}{m}\frac{d\theta}{dt} - gsin(\theta) \tag{2}$$

Now we can make substitute $y_1 = \theta, y_2 = \theta'$ and write equation (2) as a pair of first order equations:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \theta' \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} \theta' \\ \theta'' \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} y_2 \\ \frac{A}{m}sin(\omega t) - \frac{\nu}{m}y_2 - gsin(y_1) \end{bmatrix} \tag{5}$$

And now, we can use this to solve equation (2) with RK4.m.

1 Unforced Damping

Referring to Figure 1, we see that:

- $\nu = 1$ corresponds to under-damped motion since the pendulum oscillates around $\theta = 0$ a couple of times before reaching a steady-state.
- $\nu = 5$ corresponds to close to critically-damped motion. (It is evidently still over-damped because $\nu = 3$ reaches steady-state faster.)
- $\nu = 10$ corresponds to over-damped motion and takes longer to reach steady-state.

2 Unstable manifold of chaotic motion

- Figure 2 is A=0.5. It is in the stable manifold, and we see that the motion is a simple sine wave (after it settled from the initial position).

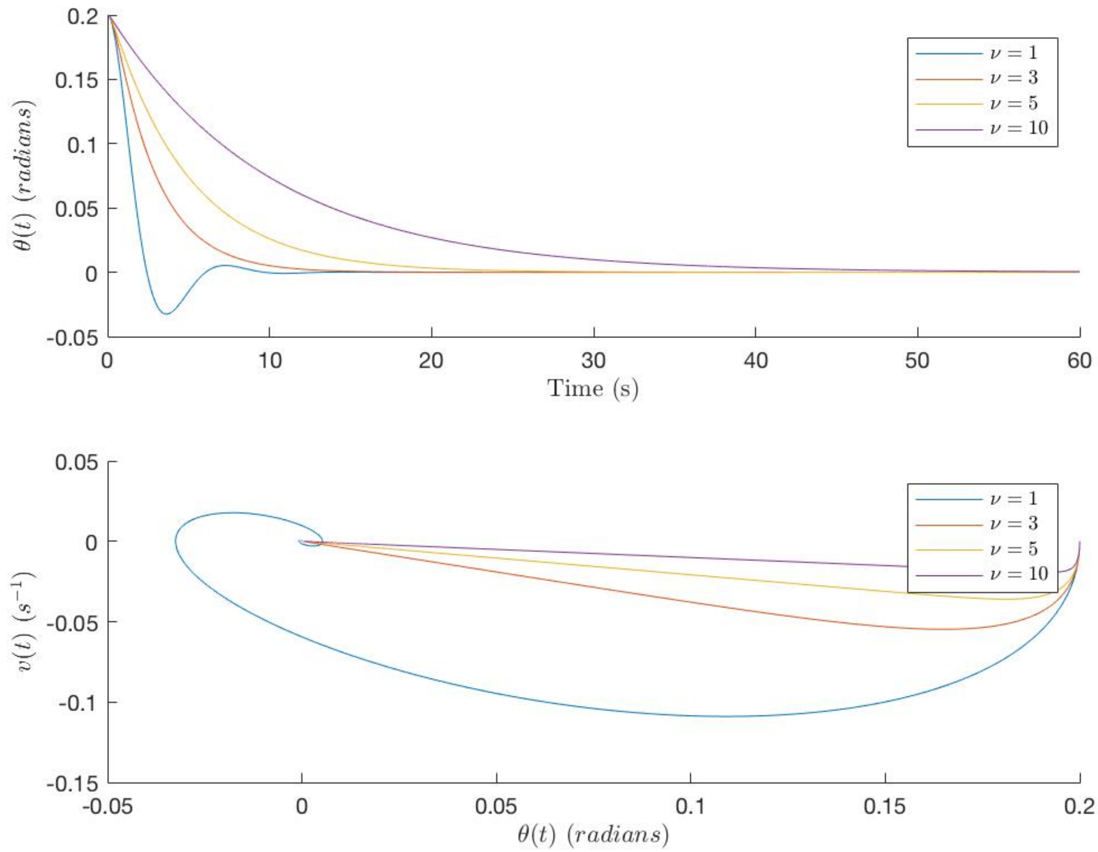

Figure 1: Unforced, damped pendulum motion. $A = 0$; Initial condition: $\theta_i = 0.2, v_i = 0$

- Figure 3 is A=1.2. It is in the unstable manifold and its motion is chaotic. Since it is in the unstable manifold, the error is time-step sensitive and grows exponentially over time. As such, I decided to use RK45.m instead of RK4.m because RK45.m uses variable step sizes and automatically assures that the error stays bounded between 1e-2 and 1e-6 at each step.

3 Poincare Section

For plots of $\theta(t)$:

- Figure 4 – A=1.35: Is periodic with $T = 3\pi$ (same period as driving force).
- Figure 5 – A=1.44: Is periodic with $T = 6\pi$ (twice the period of driving force).
- Figure 6 – A=1.465: Is periodic with $T = 12\pi$ (quadruple the period of the driving force).

For the plot of the Poincare section (Figure 7):

- A=1.35: Has only one point since the period of its motion is the same as the period of the forcing function. This is not chaotic behaviour.

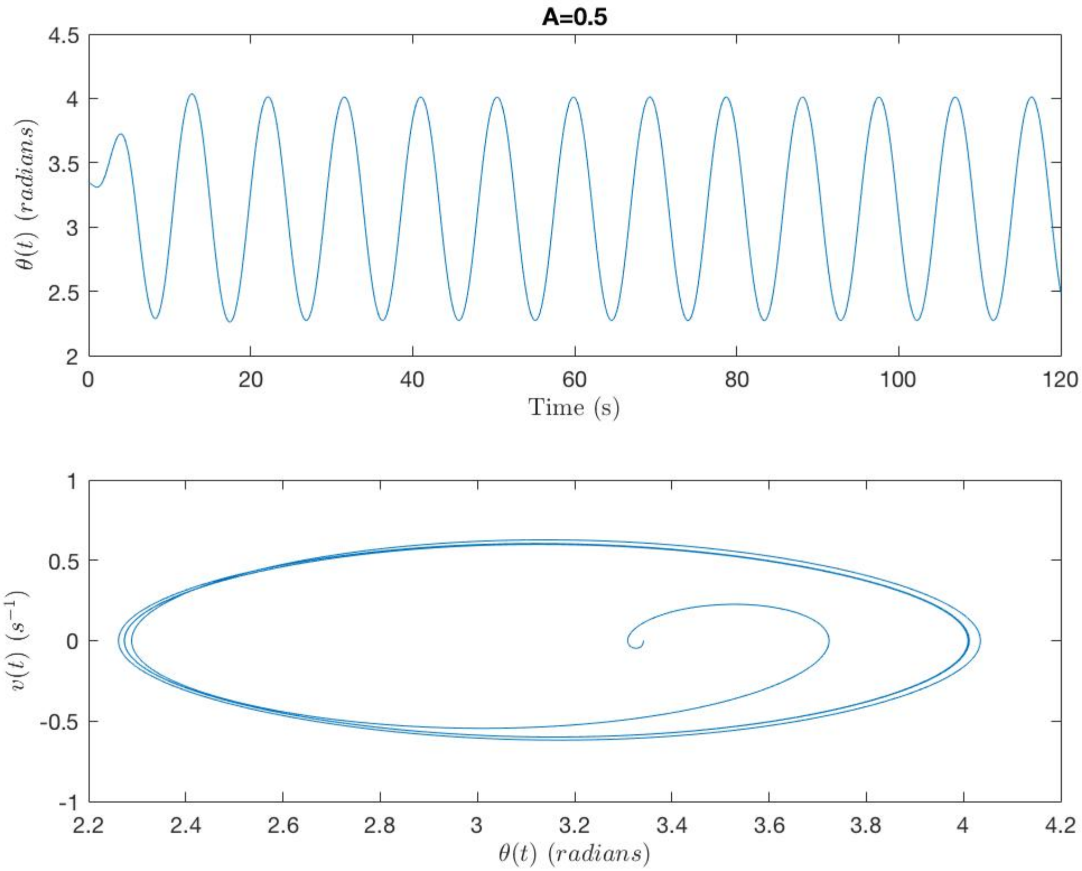

Figure 2: Forced, damped pendulum motion. Stable. $\nu = 1/2$, $\omega = 2/3$, $A = 0.5$. Initial conditions: $\theta_i = 0.2, v_i = 0$

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- A=1.44: Has two points because the period of its motion is twice the period of the forcing function. This is the start of the transition to chaotic behaviour (but it not yet chaotic).
- A=1.465: Has around 4 points (or at least 4 areas where the points lie). This is a further step into the transition to chaotic behaviour.

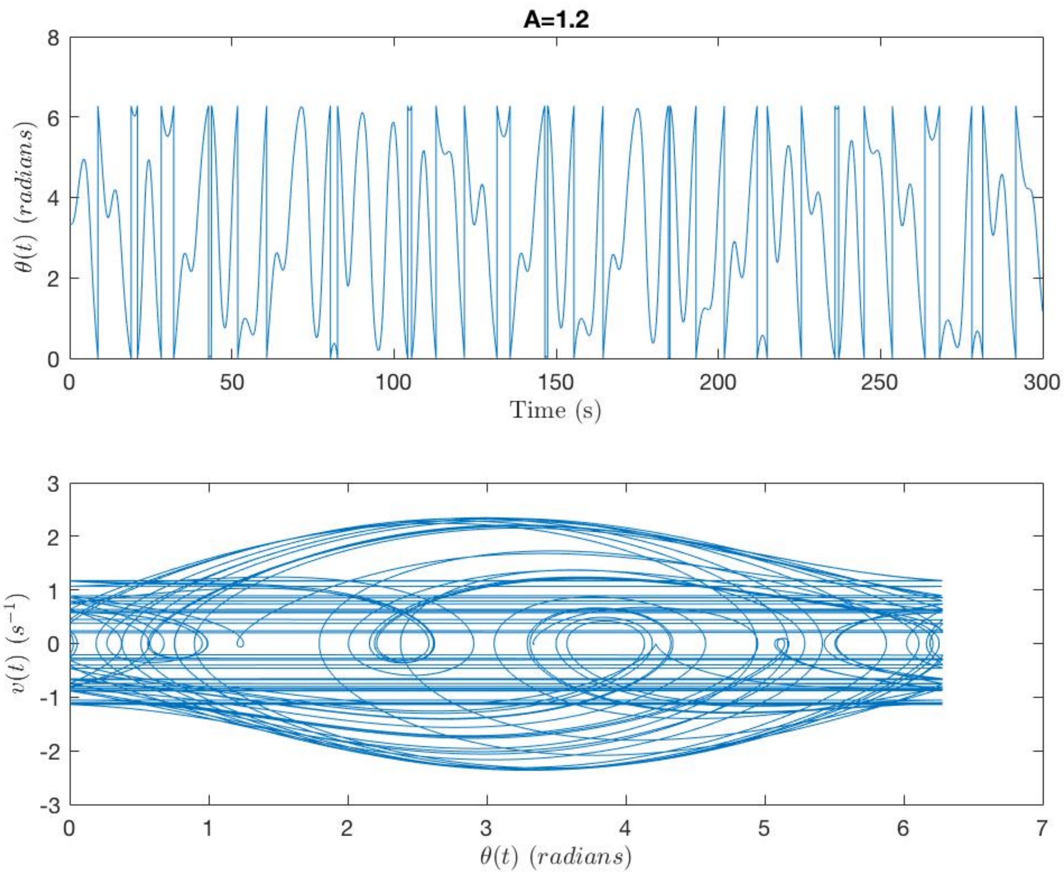

Figure 3: Forced, damped pendulum motion. Unstable. $\nu = 1/2$, $\omega = 2/3$, $A = 1.2$. Initial conditions: $\theta_i = 0.2, v_i = 0$. Generated with RK45.m (variable step size).

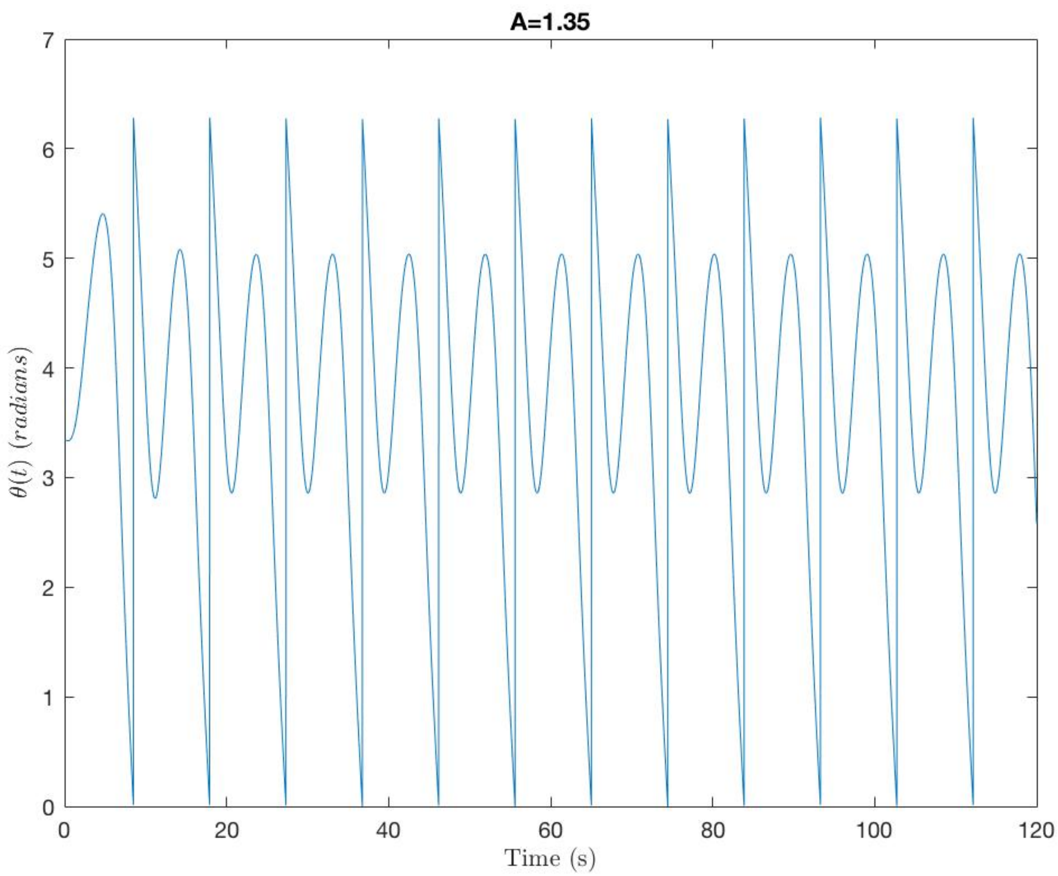

Figure 4: $\nu = 0.5$, $A = 1.35$. Initial conditions: $\theta_i = 0.2, v_i = 0$. Periodic with $T = 3\pi$.

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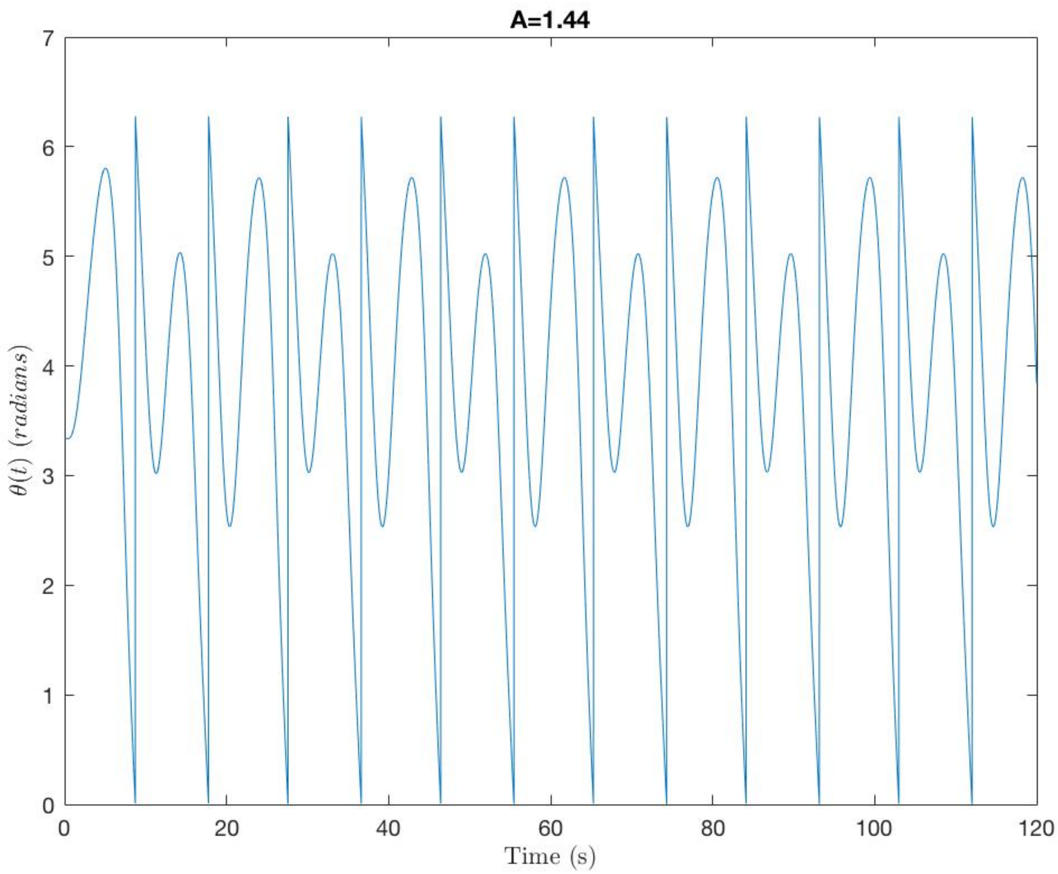

Figure 5: $\nu = 0.5$, $A = 1.44$. Initial conditions: $\theta_i = 0.2, v_i = 0$. Periodic with $T = 6\pi$.

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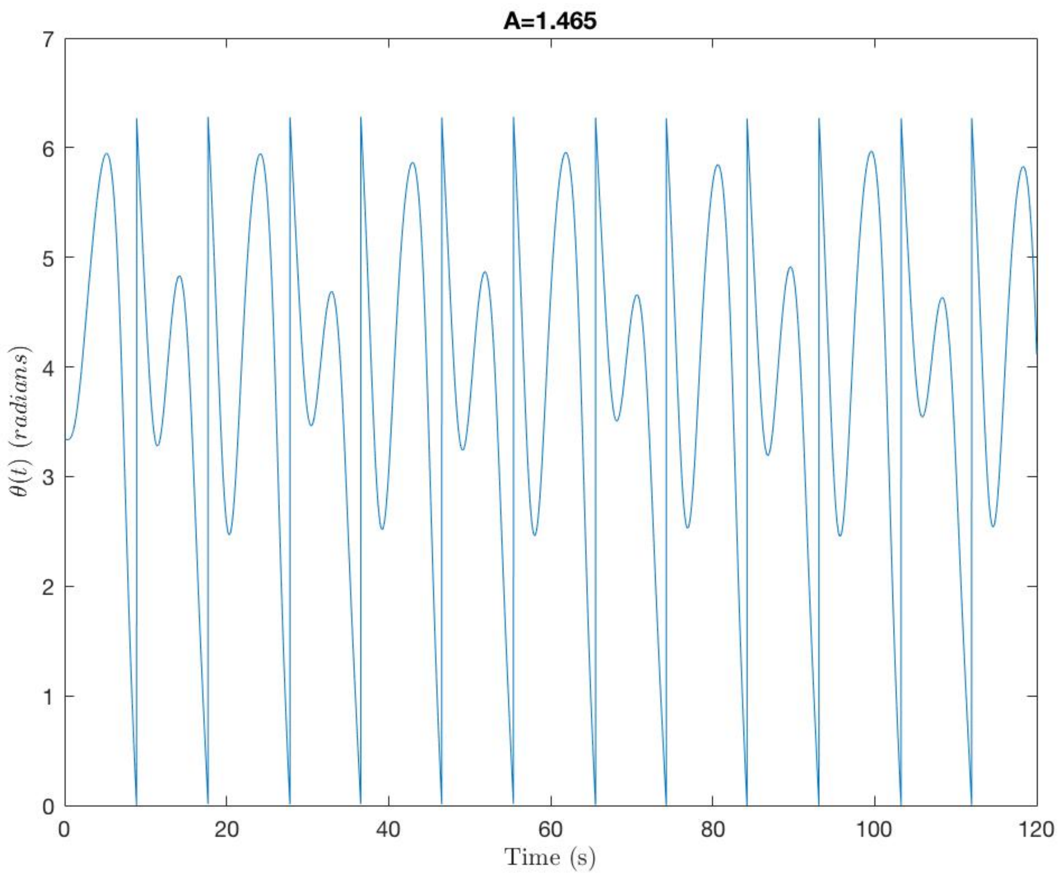

Figure 6: $\nu = 0.5$, $A = 1.465$. Initial conditions: $\theta_i = 0.2, v_i = 0$. Periodic with $T = 12\pi$.

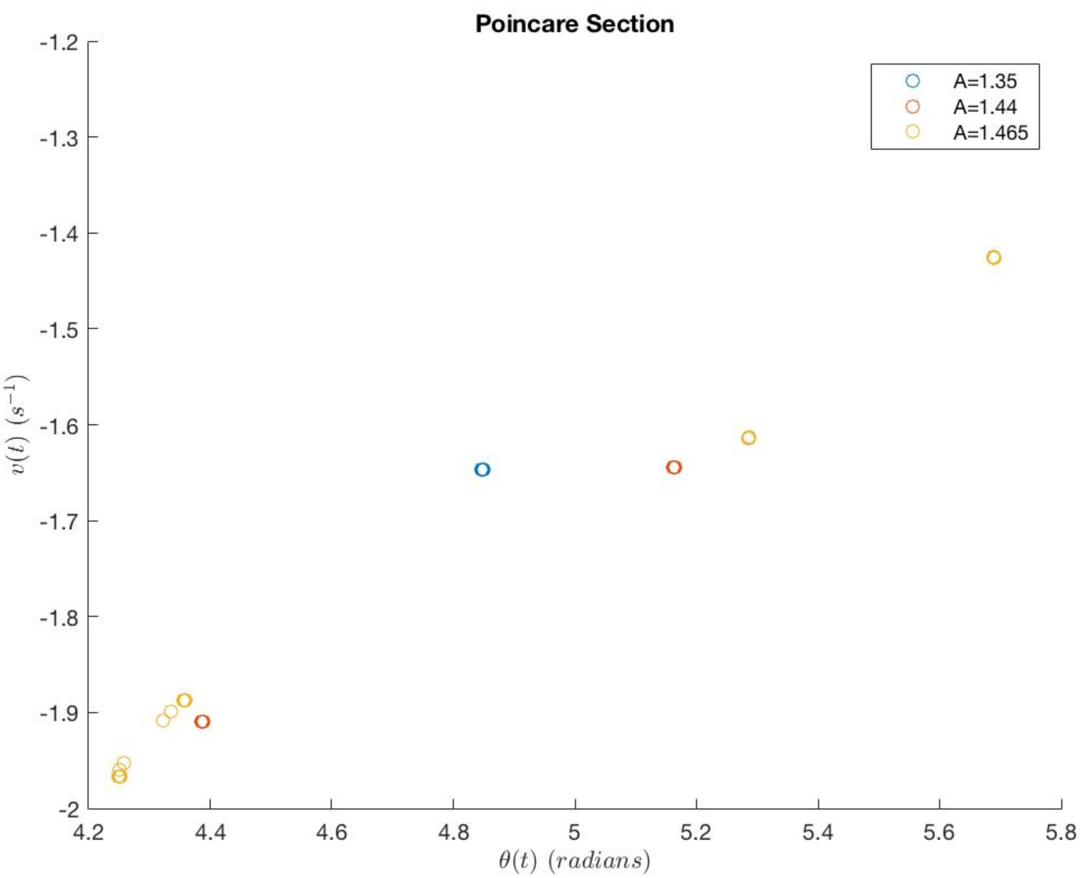

Figure 7: Poincare section. A=1.35 has a single point. A=1.44 has two points, and A=1.465 has around 4 "points".