PHYS 410 - Tutorial 9: Differential Equations

The goal of this tutorial is to become familiar with Runge-Kutta methods for solving ordinary differential equations numerically.

Runge-Kutta methods

Suppose we want to solve a differential equation of the form

$$\frac{dy}{dt} = f(t, y(t)). \tag{1}$$

The most naive way to find the solution to (1) numerically is to divide the integration region into steps of size h and find the value of $y_{i+1} = y(t_{i+1})$ by using the information at step i:

$$y_{i+1} = y(t_i + h) \approx y(t_i) + hf(t_i, y_i). \tag{2}$$

This method is known as the *simple Euler method*. It works in practice, but not miraculously. Indeed, the simple Euler method assumes that the derivative at the beginning of the interval remains constant over the entire step, which results in poor accuracy. The insight of Runge and Kutta was to improve the approximation to the derivative by sampling the function $f(t, y)$ at many *intermediate* steps between (t_i, y_i) and (t_{i+1}, y_{i+1}). Using such intermediate quantities effectively corrects the overshoot we make when using other simpler methods. The *fourth-order* explicit Runge-Kutta method, often called RK4, uses 4 intermediate quantities to reach fourth-order accuracy[1]. Given information at some time (t_i, y_i), we find the solution at the next time step t_{i+1} to be

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \tag{3}$$

where

$$k_1 = f(t_i, y_i); \tag{4}$$

$$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right); \tag{5}$$

$$k_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right); \tag{6}$$

$$k_4 = f\left(t_i + h, y_i + hk_3\right). \tag{7}$$

The procedure is then repeated at every time step until we have solved the differential equation on the desired interval $[t_0, t_f]$.

[1] The local truncation error of RK4 goes like $\mathcal{O}(h^5)$, whereas the total error is of order $\mathcal{O}(h^4)$.

The van der Pol oscillator

The van der Pol oscillator is described by the following nonlinear differential equation:

$$x'' = -x - \alpha(x^2 - 1)x', \tag{8}$$

where $x = x(t)$.

1. Write (8) as a system of two first-order differential equations.
2. Use RK4 to numerically integrate the solution corresponding to the nonlinear case $\alpha = 1$ and initial conditions $x(0) = 0.5$ and $x'(0) = 0$. Use $tfinal = 30$.
3. Explore the phase space $\{(x(t), x'(t)), \forall t\}$ of the van der Pol oscillator for various initial conditions. What do you observe?

The N-particle linear chain model

Consider a chain of length L made of N particles with individual masses m_i located at positions x_i. Assume that all particles are fixed in the x-direction but move transversally according to Hooke's law. This could be a way to model waves propagating inside a crystal. For simplicity, take the chain's endpoints to remain fixed at all times: $y_1 = y_N = v_1 = v_N = 0 \;\; \forall t$. Our aim is to obtain the displacements $y_i = y(x_i)$ for all the masses. Doing so requires us to solve the coupled system of $2(N-2)$ first-order differential equations

$$\frac{dy_i}{dt} = v_i, \tag{9}$$

$$\frac{dv_i}{dt} = \frac{f_i}{m_i} = \frac{k}{m_i}(y_{i+1} - 2y_i + y_{i-1}) \tag{10}$$

for the remaining $N-2$ particles.

1. Write a function that describes the coupled system of differential equations above. Be careful with the fixed endpoints!
2. Use RK4 to find the numerical solution for a chain of length $L = 10$ m with $N = 50$ particles, spring constant $k = 10$ N/m and masses $m_i = 1$ kg corresponding to the initial conditions:
   - $y(t=0,x) = \sin(K_j x)$ and $v(t=0,x) = 0$, where $K_j = j\pi/L$ represents the j-th mode of vibration. Try $j = 2, 5, 9$.
   - $y(t=0,x) = \exp\left(-3\left(x - \frac{L}{a}\right)^2\right)$ and $v(t=0,x) = 0$ for $a = 2$ and 4.
   - Repeat the above for a chain of alternating masses: $m_{2i} = 1$ kg and $m_{2i+1} = M$ kg, with $M > 1$. What happens as you increase M?

   To visualize your solutions, create a *for* loop that plots $y(t,x)$ every 10 time steps. *Hint:* Use MATLAB's *drawnow* feature.