PHYS 410: Homework 2

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Question 1

To find the derivative of a function $f(x)$ using the 7-points centered formula, we use the idea of differentiating the Lagrange interpolation of $f(x)$ at $x = x^*$. We start with the the Lagrange interpolation of $f(x)$ at $x = x^*$ using the seven points $x_j = x^* + jh$ for $j = -3, -2, -1, 0, 1, 2, 3$:

$$f(x^*) = L(x^*) = \sum_{j=-3}^{3} f_j l_j(x^*)$$

Where $f_j = f(x_j)$ and $l_j(x)$ are the Lagrange basis polynomials defined as:

$$l_j(x) = \prod_{m=-3,\ldots,3;\ m\neq j} \frac{x - x_m}{x_j - x_m}$$

The derivative, $f'(x)$, can be approximated by $L'(x)$, which can be calculated by taking the $log(L(x))$ and taking the derivative:

$$f'(x^*) \approx L'(x^*) = \sum_{j=-3}^{3} f_j l_j'(x^*)$$

where

$$l_j'(x^*) = l_j(x^*) \sum_{m=-3, m\neq j}^{3} \frac{1}{x^* - x_m}$$

Note: The "required weights" (from bullet 1) that we are looking for are the $l_j'(x^*)$ defined above.

The rest of question 1 will be hand written as I am running out of time...

Continued from LaTex Part:

$$f'(x) \approx L'(x) = \sum_{j=-3}^{3} f_j \, \ell_j'(x)$$

$$\ell_j'(x) = \ell_j(x) \cdot \sum_{m \neq 0,j} \frac{1}{x - x_m} = \prod_{m \neq 0,j} \frac{x - x_m}{x_j - x_m} \sum_{m \neq 0,j} \frac{1}{x - x_m}$$

$$\ell_j'(x^*) = \prod_{m \neq 0,j} \frac{x^* - x_m}{x_j - x_m} \sum_{m \neq 0 j} \frac{1}{x^* - x_m} = \frac{1}{jh} \prod_{m \neq j,0} \frac{-m}{m-j}, \text{ let } a_j = \ell_j'(x^*)$$

$$\boxed{a_j = \frac{1}{jh} \prod_{m \neq j,0} \frac{m}{m-j}}$$

$$a_{-3} = \frac{1}{-3h} \left(\frac{-2}{-2+3}\right)\left(\frac{-1}{-1+3}\right)\left(\frac{1}{1+3}\right)\left(\frac{2}{2+3}\right)\left(\frac{3}{3+3}\right)$$

$$= \frac{1}{-3h} \cancel{(-2)}\left(\cancel{-}\frac{1}{\cancel{2}}\right)\left(\frac{1}{4}\right)\left(\frac{2}{5}\right)\left(\frac{1}{2}\right)$$

$$= \frac{-1}{3h}\left(\frac{1}{20}\right) = \boxed{\frac{-1}{60h}}$$

$$a_{-2} = \frac{1}{-2h}\left(\frac{-3}{-3+2}\right)\left(\frac{-1}{-1+2}\right)\left(\frac{1}{1+2}\right)\left(\frac{2}{2+2}\right)\left(\frac{3}{3+2}\right)$$

$$= \frac{1}{+2h} \cancel{(3)}\cancel{(-1)}\left(\cancel{\frac{1}{3}}\right)\left(\frac{1}{2}\right)\left(\frac{3}{5}\right)$$

$$= \frac{1}{2h}\left(\frac{1}{2}\right)\left(\frac{3}{5}\right) = \boxed{\frac{3}{20h}}$$

$$a_j = \frac{1}{jh} \prod_{m \neq j,0} \frac{m}{m-j}$$

$$a_{-1} = \frac{-1}{h}\left(\frac{-3}{-3+1}\right)\left(\frac{-2}{-2+1}\right)\left(\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\right)\left(\frac{\cancel{3}}{4}\right)$$

$$= \frac{-1}{h}\left(\frac{3}{\cancel{2}}\right)\left(\cancel{2}\right)\left(\frac{1}{4}\right) = \boxed{\frac{-3}{4h}}$$

$$a_1 = \frac{1}{h}\left(\frac{-3}{-3-1}\right)\left(\frac{+\cancel{2}}{+\cancel{3}}\right)\left(\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{1}\right)\left(\frac{\cancel{3}}{\cancel{2}}\right)$$

$$= \boxed{\frac{3}{4h}}$$

$$a_2 = \frac{1}{\cancel{2}h}\left(\frac{+3}{+5}\right)\left(\frac{-\cancel{2}}{-4}\right)\left(\frac{+1}{+\cancel{3}}\right)\left(\frac{\cancel{1}}{\cancel{1}}\right)\left(\frac{\cancel{3}}{1}\right)$$

$$= \boxed{\frac{-3}{20h}}$$

$$a_3 = \frac{1}{3h}\left(\frac{1}{\cancel{2}}\right)\left(\frac{+\cancel{2}}{+5}\right)\left(\frac{+1}{4}\right)\left(\frac{+1}{\cancel{2}}\right)\left(\frac{+\cancel{2}}{1}\right)$$

$$= \frac{1}{60h}$$

$$f'(x) \approx L'(x) = \frac{-1}{60h} f(x^*-3h) + \frac{3}{20h} f(x^*-2h) - \frac{3}{4h} f(x^*-h)$$

$$+ \frac{3}{4h} f(x^*+h) - \frac{3}{20h} f(x^*+2h) + \frac{1}{60h} f(x^*+3h)$$

$x^* = x_0$

$$L'(x) = \frac{3}{4}\,\frac{f(x_0+h)-f(x_0-h)}{h} - \frac{3}{20}\,\frac{f(x_0+2h)-f(x_0-2h)}{h} + \frac{1}{60}\,\frac{f(x+3h)-f(x-3h)}{h}$$

Truncation error → 7^{th} order taylor expansion term:

$$\frac{(mh)^7}{7!} f^7(x)$$

$$\frac{h^7 f^7(x)}{7!}\left[(-3)^7\left(\frac{-1}{60h}\right) + (-2)^7\left(\frac{3}{20h}\right) + (-1)^7\left(\frac{-3}{4h}\right) + \frac{3}{4h} + 2^7\left(\frac{-3}{20h}\right) + 3^7\left(\frac{1}{60h}\right)\right]$$

$$= \frac{h^7 f^7(x)}{h\, 7!}\left[\frac{3^7}{60} - 2^7\left(\frac{3}{20}\right) + \frac{3}{4} + \frac{3}{4} - 2^7\left(\frac{3}{20}\right) + \frac{3^7}{60}\right]$$

$$= \frac{h^6 f^7(x)}{7!}\left[\frac{6}{4} - 2^8\left(\frac{3}{2^2 \cdot 5}\right) + \frac{2 \cdot 3^7}{60}\right]$$

$$= \frac{h^6 f^7(x)}{7!}\left[\frac{6}{4} - \frac{(64)(3)}{5} + \frac{3^6}{10}\right]$$

$$= \frac{h^6 f^7(x)}{7!}\left[36\right]$$

$$= h^6 f^7(x)\left(\frac{36}{7!}\right)$$

$$= \boxed{h^6 f^7(x)\left(\frac{1}{140}\right)} \longrightarrow f^7(x) = \frac{d^7 \sin(x)}{dx^7}$$

$$\max_{0 \le x \le 1} |f^7(x)| = 2181.29$$

Wolfram Alpha

$$\text{MAX Truncation Error} = \frac{2181.29}{140} h^6$$

Rounding error:

From lecture: round off error ~ eps/h

Every function evaluation introduces rounding error

$$RErr = \frac{3}{4}\left(\frac{2eps}{h}\right) + \frac{3}{20}\left(\frac{2eps}{h}\right) + \frac{1}{60}\left(\frac{2eps}{h}\right)$$

$$= \frac{eps}{h}\left(\frac{3}{2} + \frac{3}{10} + \frac{1}{30}\right)$$

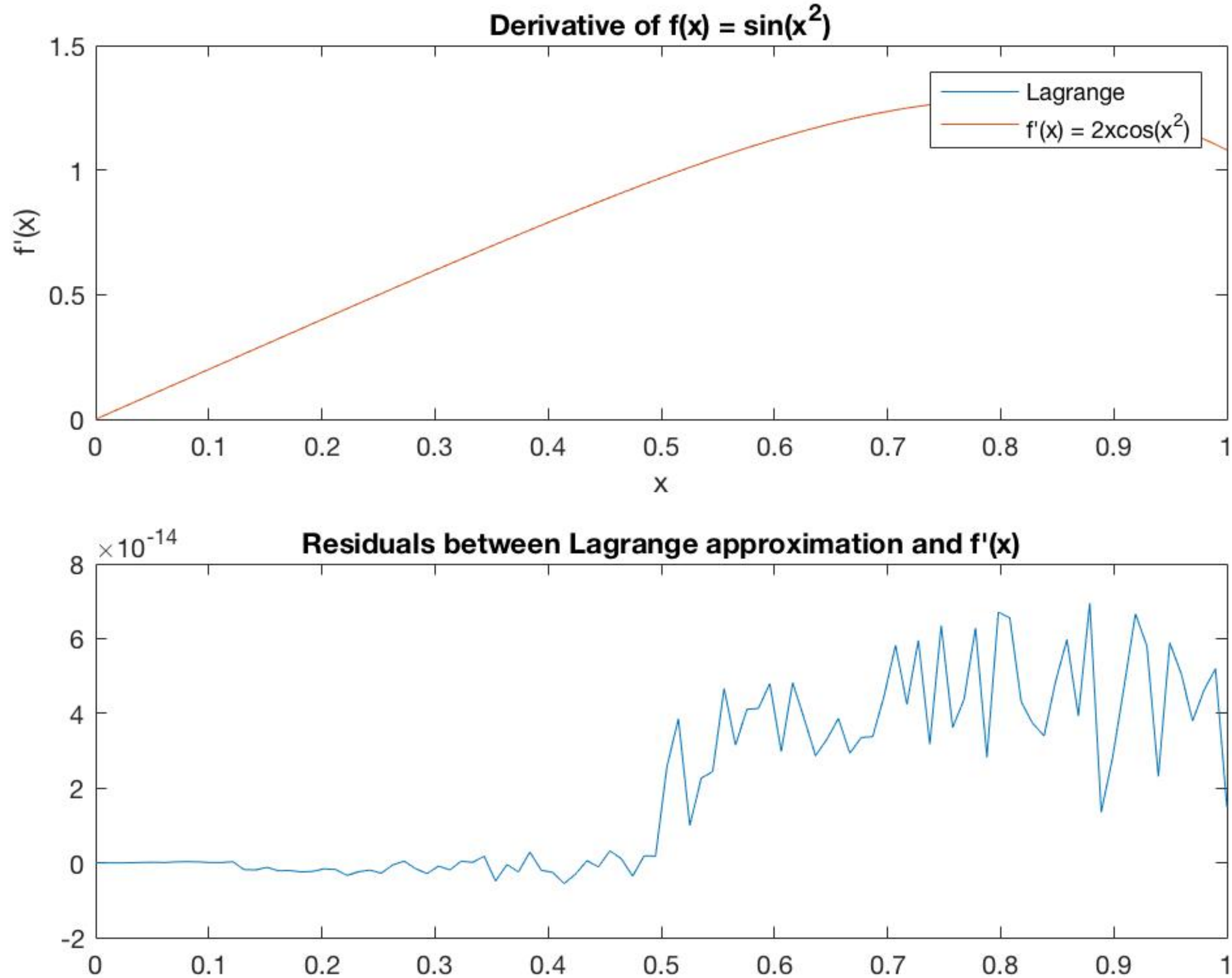
$$= \frac{11}{6}\frac{eps}{h}$$

$$\boxed{\underset{\text{(upper bound)}}{\text{Total Error}} = \frac{2181.29}{140} h^6 + \frac{11}{6}\frac{eps}{h}}$$

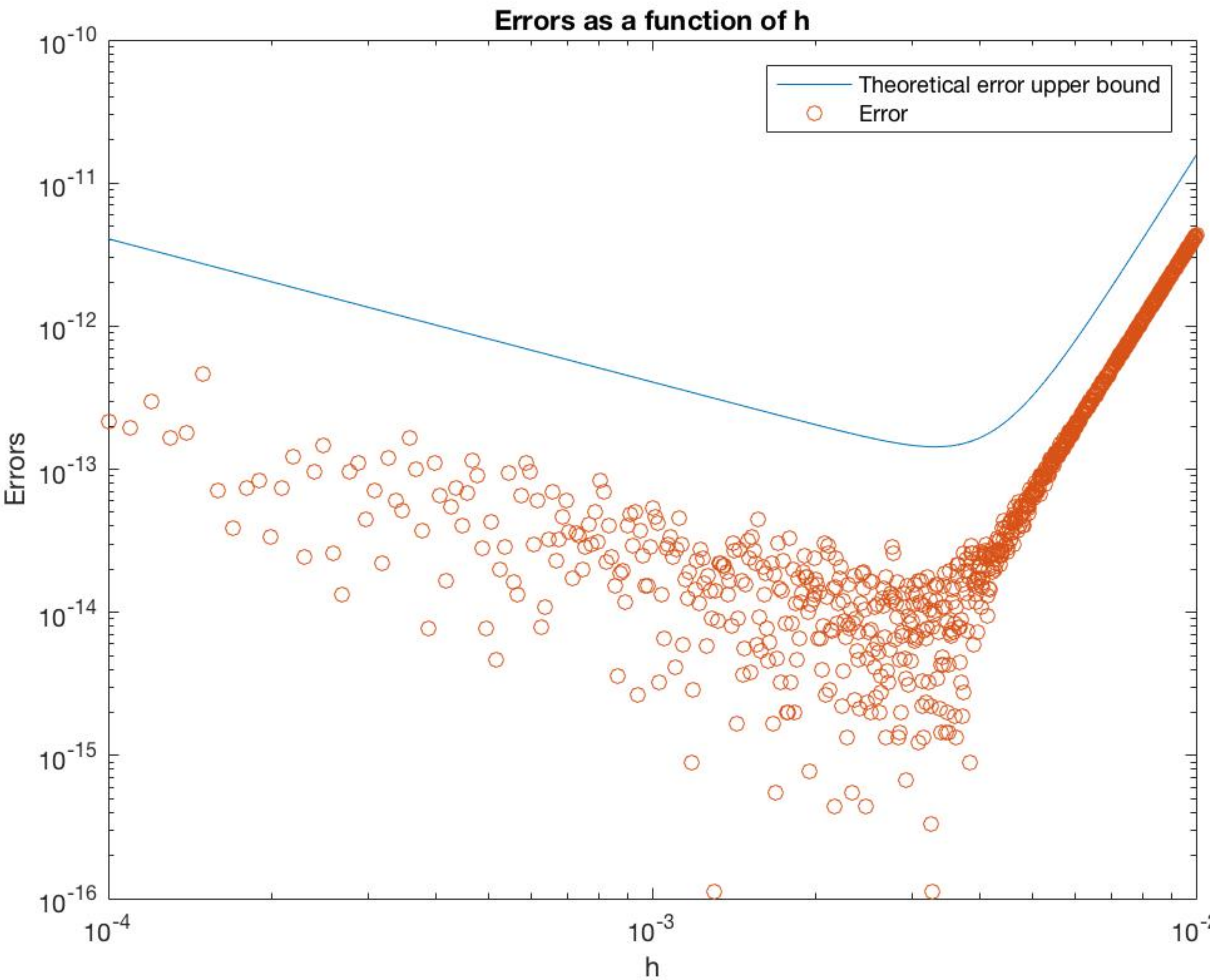
$$\frac{d(TE)}{dh} = \frac{2181.29}{140}(6h^5) - \frac{11eps}{6h^2} = 0$$

USING MATLAB fzero function to find the root closest to 0:

$$\boxed{h_{optimal} = 0.00331016232169}$$



Derivative of f(x) = sin(x^2)
Lagrange
f'(x) = 2xcos(x^2)
f'(x)
x
1.5
1
0.5
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Residuals between Lagrange approximation and f'(x)
×10^-14
8
6
4
2
0
-2
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1



Errors as a function of h
Theoretical error upper bound
Error
Errors
h
10^-10
10^-11
10^-12
10^-13
10^-14
10^-15
10^-16
10^-4
10^-3
10^-2

Question 2

The MATLAB code for this section is entirely contained in the file *romberg_integration.m*. To run the code for function $f(x) = sin(x)$ and interval $[a,b] = [0,1]$, call

$$romberg\_integration(@(x)\ sin(x), 0, 1)$$

in the MATLAB command window. This will return the result of the numerical integration of

$$\int_0^1 sin(x)dx$$

using the Romberg method with the composite trapezoid formula. Following the instructions in the question, the function starts $h = b - a$ with error of order h^2 and then use sub intervals of length $h2^{-m}$ etc. until all errors up to order h^6 are eliminated. This means that m goes from 0 to 2.

- $m = 0$ is the first estimate, which only uses 1 interval with $h = b - a = 1$. This has error on the order of h^2.
- $m = 1$ is the second estimate, which uses 2 intervals with $h = 1/2$. This now has error on the order of h^4 after using the linear combination $(4T_0(m+1) - T_0(m))/3$.
- $m = 2$ is the third estimate, which uses 4 intervals with $h = 1/4$. This now has error on the order of h^6 after using the linear combination $(16T_1(m+2) - T_1(m+1))/15$.

We can also construct a table similar to the table in section 3.9 by printing out the intermediate steps of the code. However, note that unlike the iterative code shown in section 3.9 of the textbook my code doesn't explicitly go through the iterations and just uses function calls to make the code a little simpler.

m	T_0	T_1	T_2
0	0.420735492403948		
1	0.450080515504076	0.459862189870785	
2	0.457300937571502	0.459707744927311	0.459697690389871

Table 1: Romberg integrations of $f(x) = sin(x)$

My code returns 0.459697690389871 which, considering that the function was only evaluated at five points, is really close to the correct answer of:

$$\int_0^1 sin(x)dx = [-cos(x)]_0^1 = 0.459697694131860$$

The absolute error is:

$$|0.459697694131860 - 0.459697690389871| = 3.741988829908394 \times 10^{-9}$$

The relative error is:

$$\frac{|0.459697694131860 - 0.459697690389871|}{0.459697694131860} \times 100\% = 8.140107896288549 \times 10^{-7}\%$$