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NucE 155 Slaybaugh

Assignment 4

1. (10 points) Determine which of the following matrices are non-singular and compute the inverse of these matrices:

$$\text{a.} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \text{b.} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & -4 & -2 \\ 2 & 1 & 1 & 5 \\ -1 & 0 & -2 & -4 \end{pmatrix}$$

a) Determinant: 2(-3)(1/2) = -3: nonsingular, hence invertible.

Inverse: $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right]$

Hence the inverse matrix is: $\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

b) Determinant: $\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & -1 & 3 & 7 \\ 0 & 1 & -3 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$ → multiply by main diagonal:

1*1*0*(-4) = 0; the matrix is non-invertible, hence singular and cannot be inverted.

2. (10 points) Determine the eigenvalues and associated eigenvectors of the following matrices; also indicate what the spectral radius is:

$$\text{a.} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \qquad \text{b.} \begin{pmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{pmatrix}$$

a) $\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} \rightarrow (3-\lambda)(3-\lambda) - 1 = 0 \rightarrow 9 - 6\lambda + \lambda^2 - 1 = 0 \rightarrow$

$$\lambda^2 - 6\lambda + 8 = 0 \rightarrow (\lambda - 2)(\lambda - 4) = 0,$$
$$\lambda = 2, 4$$

for $\lambda = 2$: $\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $x_1 = x_2 \rightarrow x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda = 4$: $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 = x_2 \rightarrow x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Spectral Radius: $\rho(A) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$
Which, in this case, is 4, rho(A) = 4

b) $\begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 3-\lambda & 2 & -1 \\ 1 & -2-\lambda & 3 \\ 2 & 0 & 4-\lambda \end{bmatrix} \rightarrow$

$$(3-\lambda)(-2-\lambda)(4-\lambda) - 2[(4-\lambda) - 6)] - [0 - 2(-2-\lambda)] \rightarrow$$
$$(-6 - \lambda + \lambda^2)(4 - \lambda) \rightarrow$$

$$\lambda = -2, 3, 4$$

For $\lambda = 3$: $\begin{bmatrix} 0 & 2 & -1 \\ 1 & -5 & 3 \\ 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & -5 & 2.5 \\ 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{matrix} x_3 & = & x_3 \\ x_3 & = & -2x_1 \\ x_3 & = & 2x_2 \end{matrix} \rightarrow x_3 \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$

For $\lambda = -2$: $\begin{bmatrix} 5 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 0 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 2 & -1 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 = -3x_3 \\ 5(-3x_3) + 2x_2 - x_3 = 0 \\ 2x_2 - 16x_3 = 0 \rightarrow x_2 = 8x_3 \end{matrix} \rightarrow x_3 \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$

For $\lambda = 4$: $\begin{bmatrix} -1 & 2 & -1 \\ 1 & -6 & 3 \\ 2 & 0 & 0 \end{bmatrix} : x_1 = 0 \rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & -6 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow x_2 = \frac{1}{2}x_3 \rightarrow \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$

Eigenvalues: 3, -2, 4

Eigenvectors: $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}; \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$

The spectral radius is also 4.

3. (15 points) We have three systems of linear equations that are similar but different. Of them, one has an exact solution, one has infinitely many solutions, and one has no solution.

   (a) (3 points) Determine which system is which.

   (b) (4 points) Discuss the approach(es) you would use to solve these systems by hand.

   (c) (8 points) Find the solutions (as applicable). You may use a numerical program to solve these systems; submit the code and output that you use.

1.

$$\begin{aligned} 4x_1 + -1x_2 + 2x_3 + 3x_4 &= 10 \\ -2x_2 + 7x_3 + -4x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6 \end{aligned}$$

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2.

$$
\begin{aligned}
4x_1 + -1x_2 + 2x_3 + 3x_4 &= 10 \\
0x_2 + 7x_3 + -4x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6
\end{aligned}
$$

3.

$$
\begin{aligned}
4x_1 + -1x_2 + 2x_3 + 3x_4 &= 10 \\
0x_2 + 7x_3 + 0x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6
\end{aligned}
$$

a) (1) $\begin{bmatrix} 4 & -1 & 2 & 3 \\ 0 & -2 & 7 & -4 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \\ 4 \\ 6 \end{bmatrix}$ (2) $\begin{bmatrix} 4 & -1 & 2 & 3 \\ 0 & 0 & 7 & -4 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \\ 4 \\ 6 \end{bmatrix}$

(3) $\begin{bmatrix} 4 & -1 & 2 & 3 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \\ 4 \\ 6 \end{bmatrix}$

Number (1) has 1 solution, (2) has no solution, (3) has infinitely many solutions

b) To solve these by hand, I would use backward substitution. X_4 is pretty obvious. From there, I would plug in the result for x_4 into the third equation to solve for x_3. From there, the values for x_3 and x_4 would be plugged into equation 2 to solve for x_2. The same process applies to solve for x_1.

c) (1) $x_4 = 2, 6x_3 + 5(2) = 4 \to 6x_3 + 10 = 4 : 6x_3 = -6 : x_3 = -1$

$-2x_2 + 7(-1) - 4(2) = -7 \to -2x_2 - 7 - 8 = -7 \to -2x_2 - 15 = -7 \to -2x_2 = 8 \to x_2 = -4$

$4x_1 - (-4) + 2(-1) + 3(2) = 10 : 4x_1 + 4 - 2 + 6 = 10 \to 4x_1 = 2 \to x_1 = 0.5$

solution:

$$
\begin{bmatrix} \frac{1}{2} \\ -4 \\ -1 \\ 2 \end{bmatrix}
$$

(2) $x_4 = 2, x_3 = -1 : 7(-1) - 4(2) = -7 : -7 - 8 = 7 \to 15 \neq -7 \therefore \textit{no solution}$

(3) $x_4 = 2; x_3 = -1 : 4x_1 - x_2 + 2(-1) + 3(2) = 10 : 4x_1 - x_2 - 2 + 6 = 10$

solutions: $x_1 = x_1, x_4 = 2, x_3 = -1$
$x_2 = 4x_1 - 6$

4. (10 points) Find the parabola

$$y = a + bx + cx^2$$

that passes through the points $(1,1)$, $(2,-4)$, and $(3,1)$.

Use Gaussian elimination and backward substitution as your solution technique.

$$y = a + b + cx^2 \quad for\ points: (1,1); (2,-4); (3,1)$$

eq. 1: $1 = a + b + c$
eq. 2: $-4 = a + 2b + 4c$
eq. 3: $1 = a + 3x + 9c$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -5 \\ 0 & 2 & 8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 2 & 10 \end{array}\right] \rightarrow 2c = 10 \rightarrow c = 5$$

$$b + 3c = -5 \rightarrow b + 15 = -5 \therefore b = -20 \ \& \ 1 = a - 20 + 5 \ \therefore a = 16$$

a = 16, b= -20, c = 5

5. (15 points) Find the LU Decomposition of $\mathbf{A}$ using Gaussian elimination and use it to solve $\mathbf{A}\vec{x} = \vec{b}$.

$$\mathbf{A} = \begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 1 & -1 & 5 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

U = $\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 1 & -1 & 5 \end{bmatrix}\begin{bmatrix} R_2 - m_{21}R_1, m_{21} = -\frac{3}{10}\left(\frac{a_{21}}{a_{11}}\right) \\ R_3 - m_{31}R_2, m_{31} = \frac{1}{10}\left(\frac{a_{31}}{a_{11}}\right) \end{bmatrix} = \begin{bmatrix} 10 & -7 & 0 \\ 0 & -1/10 & 6 \\ 0 & -3/10 & 5 \end{bmatrix}\begin{bmatrix} R_3 - m_{32}R_2, m_{32} = -3 \end{bmatrix}$

U = $\begin{bmatrix} 10 & -7 & 0 \\ 0 & -\frac{1}{10} & 6 \\ 0 & 0 & -13 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ -3/10 & 1 & 0 \\ 1/10 & 3 & 1 \end{bmatrix} \rightarrow L\vec{y} = b$: $L = \begin{bmatrix} 1 & 0 & 0 \\ -3/10 & 1 & 0 \\ 1/10 & 3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$

$y_1 = 7 \rightarrow -\frac{3}{10}(7) + y_2 = 4 \rightarrow \ -\frac{21}{10} + y_2 = 4 \rightarrow \ y_2 = 6.1$
$\frac{1}{10}(7) + 3(6.1) + y_3 = 6$: $\frac{7}{10} + 18.3 + y_3 = 6 \rightarrow \ 19 + y_3 = 6 \rightarrow y_3 = -13$

U = $\begin{bmatrix} 10 & -7 & 0 \\ 0 & -\frac{1}{10} & 6 \\ 0 & 0 & -13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.1 \\ -13 \end{bmatrix} \rightarrow x_3 = 1$

$-\frac{1}{10}x_2 + 6(1) = 6.1, \ x_2 = -1$

$$10x_1 - 7(-1) = 7, \; x_1 = 0$$