

2011 年全国硕士研究生入学统一考试数学二试题答案

一、选择题(1~8 小题, 每小题 4 分, 共 32 分. 下列每题给出的四个选项中, 只有一个选项符合题目要求的, 请将所选项前的字母填在答题纸指定位置上.)

(1) 【答案】(C).

$$\begin{aligned}
 \text{【解析】} & \text{因为 } \lim_{x \rightarrow 0} \frac{3 \sin x - \sin 3x}{cx^k} = \lim_{x \rightarrow 0} \frac{3 \sin x - \sin x \cos 2x - \cos x \sin 2x}{cx^k} \\
 &= \lim_{x \rightarrow 0} \frac{\sin x (3 - \cos 2x - 2 \cos^2 x)}{cx^k} = \lim_{x \rightarrow 0} \frac{3 - \cos 2x - 2 \cos^2 x}{cx^{k-1}} \\
 &= \lim_{x \rightarrow 0} \frac{3 - (2 \cos^2 x - 1) - 2 \cos^2 x}{cx^{k-1}} = \lim_{x \rightarrow 0} \frac{4 - 4 \cos^2 x}{cx^{k-1}} = \lim_{x \rightarrow 0} \frac{4 \sin^2 x}{cx^{k-1}} \\
 &= \lim_{x \rightarrow 0} \frac{4}{cx^{k-3}} = 1.
 \end{aligned}$$

所以 $c=4, k=3$, 故答案选(C).

(2) 【答案】(B).

$$\begin{aligned}
 \text{【解析】} & \lim_{x \rightarrow 0} \frac{x^2 f(x) - 2f(x^3)}{x^3} \\
 &= \lim_{x \rightarrow 0} \frac{x^2 f(x) - x^2 f(0) - 2f(x^3) + 2f(0)}{x^3} \\
 &= \lim_{x \rightarrow 0} \left[\frac{f(x) - f(0)}{x} - 2 \frac{f(x^3) - f(0)}{x^3} \right] \\
 &= f'(0) - 2f'(0) = -f'(0).
 \end{aligned}$$

故答案选(B).

(3) 【答案】(C).

$$\text{【解析】} f(x) = \ln|x-1| + \ln|x-2| + \ln|x-3|$$

$$\begin{aligned}
 f'(x) &= \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} \\
 &= \frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}
 \end{aligned}$$

令 $f'(x)=0$, 得 $x_{1,2} = \frac{6 \pm \sqrt{3}}{3}$, 故 $f(x)$ 有两个不同的驻点.

(4) 【答案】(C).

【解析】微分方程对应的齐次方程的特征方程为 $r^2 - \lambda^2 = 0$, 解得特征根 $r_1 = \lambda, r_2 = -\lambda$.

所以非齐次方程 $y'' - \lambda^2 y = e^{\lambda x}$ 有特解 $y_1 = x \cdot a \cdot e^{\lambda x}$,

非齐次方程 $y'' - \lambda^2 y = e^{-\lambda x}$ 有特解 $y_2 = x \cdot b \cdot e^{-\lambda x}$,

故由微分方程解的结构可知非齐次方程 $y'' - \lambda^2 y = e^{\lambda x} + e^{-\lambda x}$ 可设特解 $y = x(ae^{\lambda x} + be^{-\lambda x})$.

(5) 【答案】(A).

【解析】由题意有 $\frac{\partial z}{\partial x} = f'(x)g(y)$, $\frac{\partial z}{\partial y} = f(x)g'(y)$

所以, $\frac{\partial z}{\partial x}\bigg|_{(0,0)} = f'(0)g(0) = 0$, $\frac{\partial z}{\partial y}\bigg|_{(0,0)} = f(0)g'(0) = 0$, 即 $(0,0)$ 点是可能的极值点.

又因为 $\frac{\partial^2 z}{\partial x^2} = f''(x)g(y)$, $\frac{\partial^2 z}{\partial x \partial y} = f'(x)g'(y)$, $\frac{\partial^2 z}{\partial y^2} = g''(y)f(x)$,

所以, $A = \frac{\partial^2 z}{\partial x^2}\bigg|_{(0,0)} = f''(0) \cdot g(0)$, $B = \frac{\partial^2 z}{\partial x \partial y}\bigg|_{(0,0)} = f'(0) \cdot g'(0) = 0$,

$C = \frac{\partial^2 z}{\partial y^2}\bigg|_{(0,0)} = f(0) \cdot g''(0)$,

根据题意由 $(0,0)$ 为极小值点, 可得 $AC - B^2 = A \cdot C > 0$, 且 $A = f''(0) \cdot g(0) > 0$, 所以有

$C = f(0) \cdot g''(0) > 0$. 由题意 $f(0) > 0, g(0) < 0$, 所以 $f''(0) < 0, g''(0) > 0$, 故选 (A).

(6) 【答案】(B).

【解析】因为 $0 < x < \frac{\pi}{4}$ 时, $0 < \sin x < \cos x < 1 < \cot x$,

又因 $\ln x$ 是单调递增的函数, 所以 $\ln \sin x < \ln \cos x < \ln \cot x$.

故正确答案为 (B).

(7) 【答案】(D).

【解析】由于将 A 的第 2 列加到第 1 列得矩阵 B , 故

$$A \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B,$$

即 $AP_1 = B$, $A = BP_1^{-1}$.

由于交换 B 的第 2 行和第 3 行得单位矩阵, 故

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} B = E,$$

即 $P_2 B = E$, 故 $B = P_2^{-1} = P_2$. 因此, $A = P_2 P_1^{-1}$, 故选 (D).

(8) 【答案】(D).

【解析】由于 $(1, 0, 1, 0)^T$ 是方程组 $Ax = 0$ 的一个基础解系, 所以 $A(1, 0, 1, 0)^T = 0$, 且 $r(A) = 4 - 1 = 3$, 即 $\alpha_1 + \alpha_3 = 0$, 且 $|A| = 0$. 由此可得 $A^* A = |A| E = O$, 即 $A^*(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T = O$, 这说明 $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 是 $A^* x = 0$ 的解.

由于 $r(A) = 3$, $\alpha_1 + \alpha_3 = 0$, 所以 $\alpha_2, \alpha_3, \alpha_4$ 线性无关. 又由于 $r(A) = 3$, 所以 $r(A^*) = 1$, 因此 $A^* x = 0$ 的基础解系中含有 $4 - 1 = 3$ 个线性无关的解向量. 而 $\alpha_2, \alpha_3, \alpha_4$ 线性无关, 且为 $A^* x = 0$ 的解, 所以 $\alpha_2, \alpha_3, \alpha_4$ 可作为 $A^* x = 0$ 的基础解系, 故选 (D).

二、填空题(9~14 小题, 每小题 4 分, 共 24 分. 请将答案写在答题纸指定位置上.)

(9) 【答案】 $\sqrt{2}$.

【解析】原式 $= e^{\lim_{x \rightarrow 0} (\frac{1+2^x}{2} - 1) \frac{1}{x}} = e^{\lim_{x \rightarrow 0} \frac{2^x - 1}{2x}} = e^{\lim_{x \rightarrow 0} \frac{2^x \cdot \ln 2}{2}} = e^{\frac{1}{2} \ln 2} = \sqrt{2}$.

(10) 【答案】 $y = e^{-x} \sin x$.

【解析】由通解公式得

$$\begin{aligned} y &= e^{-\int dx} \left(\int e^{-x} \cos x \cdot e^{\int dx} dx + C \right) \\ &= e^{-x} \left(\int \cos x dx + C \right) \\ &= e^{-x} (\sin x + C). \end{aligned}$$

由于 $y(0) = 0$, 故 $C = 0$. 所以 $y = e^{-x} \sin x$.

(11) 【解析】选取 x 为参数, 则弧微元 $ds = \sqrt{1 + (y')^2} dx = \sqrt{1 + \tan^2 x} dx = \sec x dx$

所以 $s = \int_0^{\frac{\pi}{4}} \sec x dx = \ln |\sec x + \tan x| \Big|_0^{\frac{\pi}{4}} = \ln(1 + \sqrt{2})$.

(12) 【答案】 $\frac{1}{\lambda}$.

【解析】原式 $= \int_0^{+\infty} x \lambda e^{-\lambda x} dx = - \int_0^{+\infty} x d e^{-\lambda x}$

$$\begin{aligned}
 &= -xe^{-\lambda x} \Big|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x} dx = -\lim_{x \rightarrow +\infty} \frac{x}{e^{\lambda x}} + 0 - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^{+\infty} \\
 &= -\lim_{x \rightarrow +\infty} \frac{1}{\lambda e^{\lambda x}} - \frac{1}{\lambda} \left(\lim_{x \rightarrow +\infty} \frac{1}{e^{\lambda x}} - e^0 \right) = \frac{1}{\lambda}.
 \end{aligned}$$

(13) 【答案】 $\frac{7}{12}$.

$$\begin{aligned}
 \text{【解析】原式} &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{2\sin\theta} r \cos\theta \cdot r \sin\theta dr = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r \cos\theta \cdot \sin\theta d\theta \int_0^{2\sin\theta} r^3 dr \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta \cdot \cos\theta \cdot \frac{1}{4} \cdot 16 \sin^4\theta d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4 \cos\theta \cdot \sin^5\theta d\theta = 4 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^5\theta d\sin\theta \\
 &= \frac{4}{6} \sin^6\theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{7}{12}.
 \end{aligned}$$

(14) 【答案】 2.

【解析】方法 1: f 的正惯性指数为所对应矩阵的特征值中正的个数.

$$\text{二次型 } f \text{ 对应矩阵为 } A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

$$\begin{aligned}
 |\lambda E - A| &= \begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-3 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} = \begin{vmatrix} \lambda & 0 & -\lambda \\ -1 & \lambda-3 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} = \begin{vmatrix} \lambda & 0 & 0 \\ -1 & \lambda-3 & -2 \\ -1 & -1 & \lambda-2 \end{vmatrix} \\
 &= \lambda \begin{vmatrix} \lambda-3 & -2 \\ -1 & \lambda-2 \end{vmatrix} = \lambda(\lambda-1)(\lambda-4),
 \end{aligned}$$

故 $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 4$. 因此 f 的正惯性指数为 2.

方法 2: f 的正惯性指数为标准形中正的平方项个数.

$$\begin{aligned}
 f(x_1, x_2, x_3) &= x_1^2 + 3x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \\
 &= (x_1 + x_2 + x_3)^2 - x_2^2 - 2x_2x_3 - x_3^2 + 3x_2^2 + x_3^2 + 2x_2x_3 \\
 &= (x_1 + x_2 + x_3)^2 + 2x_2^2,
 \end{aligned}$$

$$\text{令 } \begin{cases} y_1 = x_1 + x_2 + x_3, \\ y_2 = x_2, \\ y_3 = x_3, \end{cases} \quad \text{则 } f = y_1^2 + 2y_2^2, \text{ 故 } f \text{ 的正惯性指数为 2.}$$

三、解答题(15~23 小题, 共 94 分. 请将解答写在答题纸指定位置上. 解答应写出文字说明、证明过程或演算步骤.)

(15) (本题满分 10 分)

【解析】如果 $a \leq 0$ 时, $\lim_{x \rightarrow +\infty} \frac{\int_0^x \ln(1+t^2)dt}{x^a} = \lim_{x \rightarrow +\infty} x^{-a} \cdot \int_0^x \ln(1+t^2)dt = +\infty$,

显然与已知矛盾, 故 $a > 0$.

当 $a > 0$ 时, 又因为 $\lim_{x \rightarrow 0^+} \frac{\int_0^x \ln(1+t^2)dt}{x^a} = \lim_{x \rightarrow 0^+} \frac{\ln(1+x^2)}{ax^{a-1}} = \lim_{x \rightarrow 0^+} \frac{x^2}{ax^{a-1}} = \lim_{x \rightarrow 0^+} \frac{1}{a} \cdot x^{3-a} = 0$.

所以 $3-a > 0$ 即 $a < 3$.

又因为 $0 = \lim_{x \rightarrow +\infty} \frac{\int_0^x \ln(1+t^2)dt}{x^a} = \lim_{x \rightarrow +\infty} \frac{\ln(1+x^2)}{ax^{a-1}} = \lim_{x \rightarrow +\infty} \frac{\frac{2x}{1+x^2}}{a(a-1)x^{a-2}} = \frac{2}{a(a-1)} \lim_{x \rightarrow +\infty} \frac{x^{3-a}}{1+x^2}$

所以 $3-a < 2$, 即 $a > 1$, 综合得 $1 < a < 3$.

(16) (本题满分 11 分)

【解析】因为 $y'(x) = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t^2-1}{t^2+1}$,

$$y''(x) = \frac{d(\frac{t^2-1}{t^2+1})}{dt} \cdot \frac{1}{\frac{dx}{dt}} = \frac{2t(t^2+1) - (t^2-1) \cdot 2t}{(t^2+1)^2} \cdot \frac{1}{t^2+1} = \frac{4t}{(t^2+1)^3},$$

令 $y'(x) = 0$ 得 $t = \pm 1$,

当 $t = 1$ 时, $x = \frac{5}{3}$, $y = -\frac{1}{3}$, 此时 $y'' > 0$, 所以 $y = -\frac{1}{3}$ 为极小值.

当 $t = -1$ 时, $x = -1$, $y = 1$, 此时 $y'' < 0$, 所以 $y = 1$ 为极大值.

令 $y''(x) = 0$ 得 $t = 0$, $x = y = \frac{1}{3}$.

当 $t < 0$ 时, $x < \frac{1}{3}$, 此时 $y'' < 0$; 当 $t > 0$ 时, $x > \frac{1}{3}$, 此时 $y'' > 0$.

所以曲线的凸区间为 $(-\infty, \frac{1}{3})$, 凹区间为 $(\frac{1}{3}, +\infty)$, 拐点为 $(\frac{1}{3}, \frac{1}{3})$.

(17) (本题满分 9 分)

【解析】 $z = f[xy, yg(x)]$

$$\frac{\partial z}{\partial x} = f'_1[xy, yg(x)] \cdot y + f'_2[xy, yg(x)] \cdot yg'(x)$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= f_1'[xy, yg(x)] + y[f_{11}''(xy, yg(x))x + f_{12}''(xy, yg(x))g(x)] \\ &\quad + g'(x) \cdot f_2'[xy, yg(x)] + yg'(x)\{f_{12}''[xy, yg(x)] \cdot x + f_{22}''[xy, yg(x)]g(x)\}.\end{aligned}$$

因为 $g(x)$ 在 $x=1$ 可导, 且为极值, 所以 $g'(1)=0$, 则

$$\left. \frac{d^2 z}{dx dy} \right|_{\substack{x=1 \\ y=1}} = f_1'(1, 1) + f_{11}''(1, 1) + f_{12}''(1, 1).$$

(18) (本题满分 10 分)

【解析】由题意可知当 $x=0$ 时, $y=0$, $y'(0)=1$, 由导数的几何意义得 $y'=\tan \alpha$,

即 $\alpha = \arctan y'$, 由题意 $\frac{d}{dx}(\arctan y') = \frac{dy}{dx}$, 即 $\frac{y''}{1+y'^2} = y'$.

令 $y' = p$, $y'' = p'$, 则 $\frac{p'}{1+p^2} = p$, $\int \frac{dp}{p^3+p} = \int dx$, 即

$$\int \frac{dp}{p} - \int \frac{p}{p^2+1} dp = \int dx, \quad \ln |p| - \frac{1}{2} \ln(p^2+1) = x + c_1, \quad \text{即 } p^2 = \frac{1}{ce^{-2x}-1}.$$

当 $x=0$, $p=1$, 代入得 $c=2$, 所以 $y' = \frac{1}{\sqrt{2e^{-2x}-1}}$,

$$\begin{aligned}y(x) - y(0) &= \int_0^x \frac{dt}{\sqrt{2e^{-2t}-1}} = \int_0^x \frac{e^t dt}{\sqrt{2-e^{2t}}} \\ &= \int_0^x \frac{d\left(\frac{e^t}{\sqrt{2}}\right)}{\sqrt{1-\left(\frac{e^t}{\sqrt{2}}\right)^2}} = \arcsin \frac{e^t}{\sqrt{2}} \Big|_0^x = \arcsin \frac{e^x}{\sqrt{2}} - \frac{\pi}{4}.\end{aligned}$$

又因为 $y(0)=0$, 所以 $y(x) = \arcsin \frac{\sqrt{2}}{2} e^x - \frac{\pi}{4}$.

(19) (本题满分 10 分)

【解析】(I) 设 $f(x) = \ln(1+x)$, $x \in \left[0, \frac{1}{n}\right]$

显然 $f(x)$ 在 $\left[0, \frac{1}{n}\right]$ 上满足拉格朗日的条件,

$$f\left(\frac{1}{n}\right) - f(0) = \ln\left(1 + \frac{1}{n}\right) - \ln 1 = \ln\left(1 + \frac{1}{n}\right) = \frac{1}{1+\xi} \cdot \frac{1}{n}, \xi \in \left(0, \frac{1}{n}\right)$$

所以 $\xi \in \left(0, \frac{1}{n}\right)$ 时,

$$\frac{1}{1+\frac{1}{n}} \cdot \frac{1}{n} < \frac{1}{1+\xi} \cdot \frac{1}{n} < \frac{1}{1+0} \cdot \frac{1}{n}, \text{ 即: } \frac{1}{n+1} < \frac{1}{1+\xi} \cdot \frac{1}{n} < \frac{1}{n},$$

$$\text{亦即: } \frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$

结论得证.

$$(II) \text{ 设 } a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n = \sum_{k=1}^n \frac{1}{k} - \ln n.$$

先证数列 $\{a_n\}$ 单调递减.

$$a_{n+1} - a_n = \left[\sum_{k=1}^{n+1} \frac{1}{k} - \ln(n+1) \right] - \left[\sum_{k=1}^n \frac{1}{k} - \ln n \right] = \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right) = \frac{1}{n+1} - \ln\left(1 + \frac{1}{n}\right),$$

利用 (I) 的结论可以得到 $\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right)$, 所以 $\frac{1}{n+1} - \ln\left(1 + \frac{1}{n}\right) < 0$ 得到 $a_{n+1} < a_n$, 即

数列 $\{a_n\}$ 单调递减.

再证数列 $\{a_n\}$ 有下界.

$$a_n = \sum_{k=1}^n \frac{1}{k} - \ln n > \sum_{k=1}^n \ln\left(1 + \frac{1}{k}\right) - \ln n,$$

$$\sum_{k=1}^n \ln\left(1 + \frac{1}{k}\right) = \ln \prod_{k=1}^n \left(\frac{k+1}{k}\right) = \ln\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n}\right) = \ln(n+1),$$

$$a_n = \sum_{k=1}^n \frac{1}{k} - \ln n > \sum_{k=1}^n \ln\left(1 + \frac{1}{k}\right) - \ln n > \ln(n+1) - \ln n > 0.$$

得到数列 $\{a_n\}$ 有下界. 利用单调递减数列且有下界得到 $\{a_n\}$ 收敛.

(20) (本题满分 11 分)

【解析】(I) 容器的容积即旋转体体积分为两部分

$$V = V_1 + V_2 = \pi \int_{\frac{1}{2}}^2 (2y - y^2) dy + \pi \int_{-1}^{\frac{1}{2}} (1 - y^2) dy$$

$$= \pi \left(y^2 - \frac{y^3}{3} \right) \Big|_{\frac{1}{2}}^2 + \pi \left(y - \frac{y^3}{3} \right) \Big|_{-1}^{\frac{1}{2}} = \pi \left(5 + \frac{1}{4} - 3 \right) = \frac{9}{4} \pi .$$

(II) 所做的功为

$$dw = \pi \rho g (2-y)(1-y^2)dy + \pi \rho g (2-y)(2y-y^2)dy$$

$$w = \pi \rho g \int_{-1}^{\frac{1}{2}} (2-y)(1-y^2)dy + \pi \rho g \int_{\frac{1}{2}}^2 (2-y)(2y-y^2)dy$$

$$= \pi \rho g \left(\int_{-1}^{\frac{1}{2}} (y^3 - 2y^2 - y + 2)dy + \int_{\frac{1}{2}}^2 (-y^2 + y^3 - 4y^2 + 4y)dy \right)$$

$$= \pi \rho g \left(\frac{y^4}{4} \Big|_{-1}^{\frac{1}{2}} - \frac{2y^3}{3} \Big|_{-1}^{\frac{1}{2}} - \frac{y^2}{2} \Big|_{-1}^{\frac{1}{2}} + 2y \Big|_{-1}^{\frac{1}{2}} + \frac{y^2}{4} \Big|_{\frac{1}{2}}^2 - \frac{4y^3}{3} \Big|_{\frac{1}{2}}^2 + 2y^2 \Big|_{\frac{1}{2}}^2 \right)$$

$$= \frac{27 \times 10^3}{8} \pi g = 3375 g \pi .$$

(21) (本题满分 11 分)

【解析】因为 $f(x,1)=0$, $f(1,y)=0$, 所以 $f'_x(x,1)=0$.

$$\begin{aligned} I &= \int_0^1 x dx \int_0^1 y f''_{xy}(x,y) dy = \int_0^1 x dx \int_0^1 y df'_x(x,y) \\ &= \int_0^1 x dx \left[y f'_x(x,y) \Big|_0^1 - \int_0^1 f'_x(x,y) dy \right] = \int_0^1 x dx \left(f'_x(x,1) - \int_0^1 f'_x(x,y) dy \right) \\ &= - \int_0^1 x dx \int_0^1 f'_x(x,y) dy = - \int_0^1 dy \int_0^1 x f'_x(x,y) dx = - \int_0^1 dy \left[x f(x,y) \Big|_0^1 - \int_0^1 f(x,y) dx \right] \\ &= - \int_0^1 dy \left[f(1,y) - \int_0^1 f(x,y) dx \right] = \iint_D f(x,y) dx dy = a . \end{aligned}$$

(22) (本题满分 11 分)

【解析】(I) 由于 $\alpha_1, \alpha_2, \alpha_3$ 不能由 $\beta_1, \beta_2, \beta_3$ 线性表示, 对 $(\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3)$ 进行初等行变换:

$$\begin{aligned} (\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3) &= \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 1 \\ 1 & 2 & 4 & 0 & 1 & 3 \\ 1 & 3 & a & 1 & 1 & 5 \end{array} \right) \\ &\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 & 2 \\ 0 & 2 & a-3 & 0 & 1 & 4 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 & 2 \\ 0 & 0 & a-5 & 2 & -1 & 0 \end{array} \right) . \end{aligned}$$

当 $a=5$ 时, $r(\beta_1, \beta_2, \beta_3) = 2 \neq r(\beta_1, \beta_2, \beta_3, \alpha_1) = 3$, 此时, α_1 不能由 $\beta_1, \beta_2, \beta_3$ 线性表示,

故 $\alpha_1, \alpha_2, \alpha_3$ 不能由 $\beta_1, \beta_2, \beta_3$ 线性表示.

(II) 对 $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ 进行初等行变换:

$$\begin{aligned}
 (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) &= \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 1 & 2 & 4 \\ 1 & 1 & 5 & 1 & 3 & 5 \end{array} \right) \\
 &\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 1 & 2 & 4 \\ 0 & 1 & 4 & 0 & 2 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 1 & 2 & 4 \\ 0 & 0 & 1 & -1 & 0 & -2 \end{array} \right) \\
 &\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 & 2 & 10 \\ 0 & 0 & 1 & -1 & 0 & -2 \end{array} \right),
 \end{aligned}$$

故 $\beta_1 = 2\alpha_1 + 4\alpha_2 - \alpha_3$, $\beta_2 = \alpha_1 + 2\alpha_2$, $\beta_3 = 5\alpha_1 + 10\alpha_2 - 2\alpha_3$.

(23) (本题满分 11 分)

【解析】(I) 由于 $A \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$, 设 $\alpha_1 = (1, 0, -1)^T$, $\alpha_2 = (1, 0, 1)^T$, 则

$A(\alpha_1, \alpha_2) = (-\alpha_1, \alpha_2)$, 即 $A\alpha_1 = -\alpha_1$, $A\alpha_2 = \alpha_2$, 而 $\alpha_1 \neq 0, \alpha_2 \neq 0$, 知 A 的特征值为 $\lambda_1 = -1, \lambda_2 = 1$, 对应的特征向量分别为 $k_1\alpha_1 (k_1 \neq 0)$, $k_2\alpha_2 (k_2 \neq 0)$.

由于 $r(A) = 2$, 故 $|A| = 0$, 所以 $\lambda_3 = 0$.

由于 A 是三阶实对称矩阵, 故不同特征值对应的特征向量相互正交, 设 $\lambda_3 = 0$ 对应的特征向量为 $\alpha_3 = (x_1, x_2, x_3)^T$, 则

$$\begin{cases} \alpha_1^T \alpha_3 = 0, \\ \alpha_2^T \alpha_3 = 0, \end{cases} \text{ 即 } \begin{cases} x_1 - x_3 = 0, \\ x_1 + x_3 = 0. \end{cases}$$

解此方程组, 得 $\alpha_3 = (0, 1, 0)^T$, 故 $\lambda_3 = 0$ 对应的特征向量为 $k_3\alpha_3 (k_3 \neq 0)$.

(II) 由于不同特征值对应的特征向量已经正交, 只需单位化:

$$\beta_1 = \frac{\alpha_1}{\|\alpha_1\|} = \frac{1}{\sqrt{2}}(1, 0, -1)^T, \beta_2 = \frac{\alpha_2}{\|\alpha_2\|} = \frac{1}{\sqrt{2}}(1, 0, 1)^T, \beta_3 = \frac{\alpha_3}{\|\alpha_3\|} = (0, 1, 0)^T.$$

$$\text{令 } Q = (\beta_1, \beta_2, \beta_3), \text{ 则 } Q^T A Q = \Lambda = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix},$$

$$A = Q \Lambda Q^T$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

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