Control variates for Monte Carlo valuation of American options

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This paper considers two applications of control variates to the Monte Carlo valuation of American options. The main contribution of the paper lies in the particular choice of a control variate for American or Bermudan options. It is shown that for any martingale process used as a control variate, it is optimal to sample no later than the time of exercise of the American option, as opposed to the time of expiry. The first application is to the valuation. Numerical examples show that standard errors can be dramatically reduced, allowing for faster valuation using fewer paths. Second, the control variate technique is used for improving the least-squares Monte Carlo (LSM) approach for determining exercise strategies. The suggestions made allow for more efficient estimation of the continuation value, used in determining the strategy. An additional suggestion is made in order to improve the stability of the LSM approach. It is suggested to generate paths for the LSM estimation from an initial distribution rather than the single initial point. Numerical examples show that the two LSM modifications improve the accuracy and stability of the exercise strategies, which may now be estimated using a lower number of paths.

1 Introduction

Recent research has contradicted the long-standing statement that Monte Carlo simulation was ill-suited for American options. Indeed, several successful methods have been proposed. The methods can be divided into two categories, of which the first tries to estimate the continuation value of the American option and determine the exercise strategy from it, whereas the second tries to globally maximize the value of the American option over a parametrized set of exercise strategies.

Most research has only paid little attention to the issue of variance reduction. Only a few articles go beyond applying antithetic variates. In this paper we consider the application of control variates to the valuation of American or Bermudan options. As a first choice for control variates for American options one would consider the control variates that work best for the corresponding

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European options. Naturally, the payoff of the European option itself is the optimal choice, provided the option value can be easily computed. In Broadie and Glasserman (1997, Tables 1–6) the European option payoff is used as a control variate in three examples of American options; a single asset call, a two-asset max-call, and a five-asset max-call, all valued in the Black–Scholes model. They find that the control variates work quite well for out-of-the-money options, and are good for at-the-money options. However, they are less effective for deep-in-the-money options. As noticed in Broadie and Glasserman (1997, footnote 7): "A control variate is most effective when it is highly correlated with the quantity being estimated. The European option and American option payoffs are less correlated when the options is deep-in-the-money". When the American option is initially out-of-the-money it is more likely to be exercised near expiry, or even expire unexercised, ie, out-of-the-money. In either case, a payoff close to the corresponding European option makes the European option payoff a good control variate. However, from an option holders point of view the accuracy of value estimates is of greatest importance for deep-in-the-money options, since this is where the critical exercise decisions have to be taken.

The first part of this paper shows that a simple but effective improvement can be made by sampling the control variates no later than the random time of exercise, as opposed to the fixed expiry of the option. This increases the correlation between the payoff of the American option and the control variate. In particular, for options with closed form expressions for the European version, we suggest sampling the discounted European option value at the stopping time corresponding to exercise of the American or Bermudan option. Numerical examples show that this use of control variates in some cases reduces up to 96% of the standard errors, corresponding to a 625-fold reduction in the required number of Monte Carlo paths.

In both Broadie and Glasserman (1997) and in the first part of this paper, the application of control variates is aimed at improving the valuation rather than the exercise strategy. However, it should be noted that exercise strategies, determined by optimizing a parametrized strategy as in, for instance, Andersen (2000), would benefit from any improvement in accuracy of the value estimate. Furthermore, as we will show, the increased accuracy of the value estimate helps us reduce some of the bias in upper-bound estimates as suggested by Andersen and Broadie (2004).

In the second part of this paper, we turn our attention to the least-squares Monte Carlo (LSM) approach for determining the exercise strategy as suggested by Carrie (1996) and, later, Longstaff and Schwartz (2001). Using a small and financially intuitive set of basis functions we investigate the LSM approach. We find that with the increased accuracy of valuation caused by the use of control variates, additional computational work in determining the LSM exercise strategy is required. This is both in order to balance the overall accuracy, as well as to obtain lower bound estimates that are not significantly different from the true value. The obvious choice, which we do not follow, would be to increase the number of paths and basis functions used in determining the exercise strategy.

The properties of the LSM approach have been studied carefully in several papers. The type and number of basis functions used are analyzed by Moreno and Navas (2002), who question the advice given in Longstaff and Schwartz (2001, p. 124) as a result of their Proposition 1 "... simply increase M until the value implied by the LSM algorithm no longer increases", where M is the number of basis functions used. They do, however, find that for the single-asset American option, the algorithm is stable with respect to the choice of basis functions, a conclusion that they unfortunately cannot extend to more complex settings. A thorough analysis of the convergence of the LSM algorithm is given by Clement *et al* (2002). They provide theoretical results on the convergence of the projection to the true conditional expectation as the number of orthogonal basis functions goes to infinity. Furthermore, for a given number of basis functions they provide convergence results showing that the estimated projection using regression converges to the true projection as the number of simulated paths goes to infinity. In total, they provide convergence results of the LSM algorithm as the number of basis functions and the number of simulated paths goes to infinity. Finally, they provide a central limit result on the estimate of the American option value.

Recent research by Haugh and Kogan (2004), Rogers (2002) and Andersen and Broadie (2004) has developed upper bounds on the American option value using a dual formulation of the optimal stopping problem. As shown by Andersen and Broadie (2004), the duality gap, the difference between the upper and lower bound, is a good measure of the optimality of the strategy. More importantly in the context of this paper, any improvements in the determination of the exercise strategy can be measured by the reduction in the duality gap.

The suggestions made in the second part of this paper focus on improving the accuracy and stability of the LSM approach by applying two modifications. The first modification is to replace the LSM projection of the sampled discounted payoffs from continuing the option with the projection of a random variable with the same conditional expectation, but with a smaller conditional variance. This produces a more efficient estimator of the unknown conditional expectation of the discounted payoff of the option. By applying control variates to the discounted payoff from continuing the option, a random variable with the two mentioned properties is obtained. However, unlike in the valuation of the option, the optimal coefficient of the control variate must be approximated across the different states for which the projection is needed. We suggest determining the first, second, and cross-moments involved in the computation of the control variate coefficients using projections onto the same basis functions as used for estimating the continuation value. Numerical examples show that the accuracy of the exercise boundary is highly improved, allowing fewer paths to be used in the regression while still obtaining satisfactory exercise strategies. In the paper we refer to this approach as the control variate improved LSM (CV-LSM) approach.

In both the original LSM and the CV-LSM approach, we may have difficulties in determining the exercise boundary for out-of-the-money options, in particular when time to expiry is long, ie, for the early part of the paths. This comes as no surprise, since this is where we find the smallest dispersion in the data generated by the Monte Carlo paths and, hence, the regression performs poorly in the neighborhood close to the exercise boundary. To accommodate this short-coming we suggest, secondly, to disperse the initial state variables, so that the Monte Carlo paths are generated from a distribution of initial points. We note that this does not change the validity of the exercise strategy which is independent of the distribution of the initial state. However, having determined the exercise strategy we might subsequently have to simulate several paths originating from the single initial state values for which the value estimate is sought in order to accurately estimate the lower bound value. Combining the two suggestions, we term the algorithm the dispersion improved CV-LSM (DCV-LSM).

Numerical examples are given for the single asset American put option investigated in Table 1 of Longstaff and Schwartz (2001). Using 50 000 pairs of antithetic paths as in Longstaff and Schwartz (2001), the DCV-LSM approach produces an exercise boundary that is very close to the benchmark computed using a finite-difference method. Furthermore, lower and upper bounds are very close to the benchmark estimate also using a finite-difference method. To obtain results that are comparable to Longstaff and Schwartz (2001), only 1000 pairs of antithetic paths are required for the DCV-LSM approach to determine the exercise strategy and to determining the lower bound estimate. Additional numerical results are given for two-, three-, and five-asset max-call options as analyzed in both Broadie and Glasserman (1997) and Andersen and Broadie (2004).

The paper is organized as follows. In Section 2 the American option valuation problem is formulated. The application and, in particular, the choice of control variates in Monte Carlo valuation are investigated in Section 3. The second part of the paper is introduced with Section 4, where the LSM approach is summarized. The control variate improved version of the LSM approach is introduced in Section 5. In Section 6 the additional suggestion of dispersion of initial state variables is made. Finally, conclusions are made in Section 7.

2 The American option valuation problem

The problem of valuing an American option consists of finding an optimal exercise strategy and valuing the expected discounted payoff from this strategy under the equivalent martingale measure.[1] We let V_t denote the time t solution to this problem. In mathematical terms it corresponds to the following optimal stopping

[1] In what follows we assume that the financial market considered is defined for the finite horizon $[0, T]$ on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$. Here the state space Ω is the set of all realizations of the financial market, $\mathcal{F}$ is the sigma algebra of events at time T, and $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$. The filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is assumed to be generated by the price processes of the financial market and augmented with the null sets of $\mathcal{F}$, and assuming $\mathcal{F}_T = \mathcal{F}$. We furthermore assume that using the numeraire process $\{\beta_t\}_{0\leq t\leq T}$ there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which all asset prices relative to the numeraire are martingales.

problem

$$\frac{V_t}{\beta_t} = \sup_{\tau \in \mathcal{T}(t,T)} E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right] \tag{1}$$

where $\{X_t\}_{0 \leq t \leq T}$ is the payoff process assumed adapted to the filtration, and $\mathcal{T}(t, T)$ denotes the set of stopping times τ satisfying

$$t \leq \tau \leq T$$

In general X_t may depend on the entire path of one or several underlying assets up to time t, for instance in the form of an average over time or across assets, or as the maximum or minimum value attained along the path. We can easily define a lower bound on the American option price at time t by L_t, since for any given exercise strategy or stopping time τ we have

$$\frac{L_t}{\beta_t} \equiv E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right] \leq \frac{V_t}{\beta_t} \tag{2}$$

To define an upper bound we refer the reader to Theorem 1 of Andersen and Broadie (2004), and leave out the details here.

2.1 Case study: numerical examples background

In this and the following case study sections we give numerical examples based on the single-asset put option, using the same combinations of underlying asset prices, time to expiry and volatilities as in Table 1 of Longstaff and Schwartz (2001). We furthermore give examples based on the n-asset max-call option as analyzed in Tables 3–6 of Broadie and Glasserman (1997) and in Table 2 of Andersen and Broadie (2004), for $n = 2, 3, 5$. Both sets of options are analyzed in the one- and n-dimensional Black–Scholes model, respectively.

2.2 Case study: payoff processes and benchmark exercise strategies

The payoff process $\{X_t\}_{0 \leq t \leq T}$ of the single-asset put option for strike K is given by

$$X_t = (K - S_t)^+$$

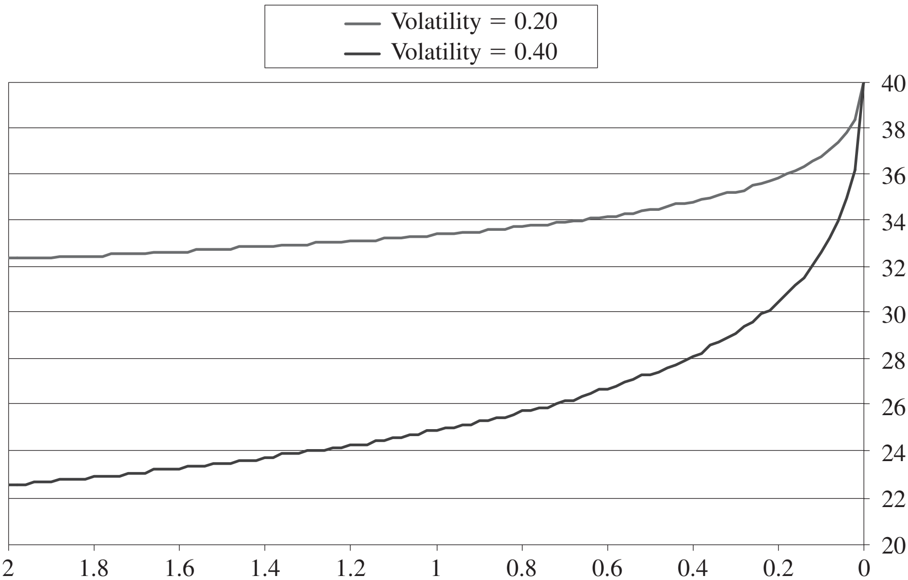
The payoff process $\{X_t^{\max}\}_{0 \leq t \leq T}$ of the n-asset max-call option is defined by

$$X_t^{\max} = \left(\max_{1 \leq j \leq n} S_t^j - K\right)^+$$

In both cases the holder can only exercise on the following set of equidistant points, $t_e = (e/d)T$, for $e = 0, 1, \dots, d$, and is hence a Bermudan rather than an American option.

For the single-asset Bermudan put option in Figure 1, we have shown the benchmark exercise boundary determined using the finite-difference method, based on the numerical example given in Table 1 of Longstaff and

FIGURE 1 Exercise boundary for the Bermudan put option with 50 exercise points per year, determined using the finite-difference method. The strike of the put option equals $K = 40$. The short-term rate is 0.06, the upper curve is for $\sigma = 0.20$, and the lower curve is $\sigma = 0.40$. The horizontal axis indicates time to expiry.



Schwartz (2001). For each possible exercise time it is determined by finding the lowest value of the stock in the grid for which the continuation value is greater than the intrinsic value

$$h^{\mathrm{FD}}(t_e) = \min\{S_m : C^{\mathrm{FD}}(S_m, t_e) > X_{t_e}\}$$

where C^{FD} denotes the finite-difference solution of the Black–Scholes partial differential equation (PDE) for the Bermudan put on a grid with asset values S_m, $m = 1, \dots, M$. For numerical solutions of the Black–Scholes PDE consult Wilmott *et al* (1993). This gives us the following exercise strategy for the Bermudan put option

$$\begin{aligned}\tau^{\mathrm{FD}} &= \min\{e : C^{\mathrm{FD}}_{t_e} \leq X_{t_e}\}\\ &= \min\{e : S_{t_e} \leq h^{\mathrm{FD}}(t_e)\}\end{aligned}$$

3 Monte Carlo valuation with control variates

Given a stopping time $\tau \in \mathcal{T}(t, T)$ we want to determine the following conditional expectation given the information at time t

$$\frac{L_t}{\beta_t} = E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right]$$

Using the underlying model to generate N independent paths of the variables determining the payoff process $\{X_t\}_{0\le t\le T}$ and the numeraire process $\{\beta_t\}_{0\le t\le T}$, a crude Monte Carlo estimate is given by

$$\frac{L_t^{(N)}}{\beta_t} \equiv \frac{1}{N}\sum_{i=1}^{N}\frac{X_\tau^i}{\beta_\tau^i}$$

where X_τ^i/β_τ^i is the discounted payoff from the ith path[2] using the exercise strategy given by the stopping time τ.

To reduce the variance of the Monte Carlo estimate of the American option for a given strategy $\tau \in \mathcal{T}(t, T)$ we can replace the path estimate X_τ^i/β_τ^i with the following path estimate

$$\frac{Z_\tau^i}{\beta_\tau^i} \equiv \frac{X_\tau^i}{\beta_\tau^i} + \theta_t(Y^i - E_t^{\mathbb{Q}}[Y^i]) \tag{3}$$

for some appropriately chosen $\mathcal{F}_t$-measurable random variable θ_t, where Y^i is the ith observation of a random variable[3] for which we can easily compute the time t conditional expectation. The observations of Y are generated using the same paths used for generating the observations of X_τ/β_τ, hence they are mutually independent but not independent of the discounted payoffs. The Monte Carlo estimate using control variates is then given by

$$\begin{aligned}\frac{L_t^{(N)\text{CV}}}{\beta_t} &\equiv \frac{1}{N}\sum_{i=1}^{N}\frac{Z_\tau^i}{\beta_\tau^i}\\ &= \frac{L_t^{(N)}}{\beta_t} + \theta_t(Y_t^{(N)} - E_t^{\mathbb{Q}}[Y])\end{aligned} \tag{4}$$

where $L_t^{(N)}/\beta_t$ and $Y_t^{(N)}$ are given by

$$\frac{L_t^{(N)}}{\beta_t} \equiv \frac{1}{N}\sum_{i=1}^{N}\frac{X_\tau^i}{\beta_\tau^i}, \qquad Y_t^{(N)} \equiv \frac{1}{N}\sum_{i=1}^{N} Y^i$$

The unbiasedness of (4) is easily shown since

$$E_t^{\mathbb{Q}}[\theta_t(Y^i - E_t^{\mathbb{Q}}[Y^i])] = \theta_t E_t^{\mathbb{Q}}[Y^i - E_t^{\mathbb{Q}}[Y^i]] = 0 \tag{5}$$

by the $\mathcal{F}_t$-measurability of θ_t.

[2]Note that without loss of generality we could assume that X_τ^i/β_τ^i is generated using antithetic variates.

[3]Note that the derivations in this section easily carry over to the multi-dimensional setting of several control variates. However, for the sake of simplicity we state the basic setup in the one-dimensional case of a single control variate.

By standard ordinary least-squares theory, the optimal choice of θ_t is given by

$$\theta_t^* = -\frac{\mathrm{Cov}_t^{\mathbb{Q}}[X_\tau/\beta_\tau, Y]}{\mathrm{Var}_t^{\mathbb{Q}}[Y]} \tag{6}$$

which results in an optimal variance given by

$$\mathrm{Var}_t^{\mathbb{Q}}\left[\frac{L_t^{(N)\mathrm{CV}}}{\beta_t}\right] = \frac{1}{N}\,\mathrm{Var}_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right]\left(1 - \rho_t^2\left(\frac{X_\tau}{\beta_\tau}, Y\right)\right)$$

where

$$\rho_t\left(\frac{X_\tau}{\beta_\tau}, Y\right) = \frac{\mathrm{Cov}_t^{\mathbb{Q}}[X_\tau/\beta_\tau, Y]}{\sqrt{\mathrm{Var}_t^{\mathbb{Q}}[X_\tau/\beta_\tau]\,\mathrm{Var}_t^{\mathbb{Q}}[Y]}}$$

The most effective control variate Y is therefore obtained by having the largest possible correlation (either positive or negative) with the discounted payoff from the Bermudan option.

In practice, we can only seldom determine θ_t^* analytically, and hence we have to estimate it as part of the Monte Carlo valuation. To ensure that (5) is still satisfied we can estimate θ_t^* using an independent set of paths. However, for the sake of numerical efficiency we might choose to ignore the possible bias and estimate it using the same set as used in the valuation. To summarize, we estimate θ_t^* using N independent paths by

$$\theta_t^{(N)} \equiv -\frac{(LY)_t^{(N)}/\beta_t - (L_t^{(N)}/\beta)Y_t^{(N)}}{(Y_t^2)^{(N)} - (Y_t^{(N)})^2} \tag{7}$$

where $(Y_t^2)^{(N)}$ and $(LY)_t^{(N)}$ are defined by

$$(Y_t^2)^{(N)} \equiv \frac{1}{N}\sum_{i=1}^{N}(Y^i)^2, \qquad \frac{(LY)_t^{(N)}}{\beta_t} \equiv \frac{1}{N}\sum_{i=1}^{N}\frac{X_\tau^i}{\beta_\tau^i}Y^i$$

3.1 Choosing control variates for American options

A good control variate should have the following two properties: it should be highly correlated with the payoff of the option in question and its conditional expectation should be easy to compute. From a financial viewpoint, a good control variate should be a good hedge of the option. In models of a complete market there exists by definition a self-financing portfolio that perfectly replicates the payoff of the option. Such a portfolio would be the optimal control variate in a Monte Carlo valuation, providing a zero standard error estimate. However, knowing the initial portfolio weights and the subsequent dynamic strategy amounts to knowing the value of the option, which is only the case for the few combinations of models and options with closed form expressions. In general, for complex options we should look for a self-financing portfolio that matches the payoff as closely possible, for instance by using static portfolios of simpler options.

When looking for a control variate for the Bermudan option, the best possible control variate for Monte Carlo valuation of the corresponding European option would be our first choice. Let W_T be the value of a self-financing portfolio at expiry of the Bermudan option. Using the discounted value of this portfolio, the European option control variate Y_T is then defined by

$$Y_T = \frac{W_T}{\beta_T} \tag{8}$$

By construction of the equivalent martingale measure $\mathbb{Q}$, the process $\{Y_t\}_{0\leq t\leq T}$ defined by

$$Y_t = E_t^{\mathbb{Q}}[Y_T] \tag{9}$$

is a martingale.

The optimal[4] European option control variate is given by

$$Y_T = \frac{X_T}{\beta_T}$$

for which we have

$$Y_t = E_t^{\mathbb{Q}}\left[\frac{X_T}{\beta_T}\right] \equiv \frac{C_t}{\beta_t}$$

the discounted European option price. Such a control variate clearly satisfies the first of the two properties mentioned, however it is only a valid control variate if the model and the payoff process allow the second property to be satisfied, ie, that C_t is easily computed. Nevertheless, in the following it serves as a useful benchmark.

To gain some intuition we now use the control variate (8), with a θ fixed at the suboptimal level of -1, to get a path estimate allowing the following decomposition

$$\begin{aligned}\frac{X_\tau}{\beta_\tau} - (Y_T + Y_t) &= \frac{X_\tau}{\beta_\tau} - \frac{W_T}{\beta_T} + E_t^{\mathbb{Q}}\left[\frac{W_T}{\beta_T}\right] \\ &= \left(\frac{X_\tau}{\beta_\tau} - \frac{X_T}{\beta_T}\right) + \left(\frac{X_T}{\beta_T} - \frac{W_T}{\beta_T}\right) + E_t^{\mathbb{Q}}\left[\frac{W_T}{\beta_T}\right]\end{aligned}$$

We now measure the time t conditional variance of the decomposition. The variance of the first term on the right-hand side is increasing in the distance between exercise of the Bermudan option and expiry of the option $(T - \tau)$. This implies that for options that are far from the exercise region and hence exercised late or never, this term contributes very little to the variance, whereas the converse is true for options close to the exercise region. It furthermore depends on the magnitude of the early exercise value corresponding to the exercise strategy given by τ. The variance of the second term depends on how close the

[4] In the sense of perfectly replicating the European option payoff.

self-financing portfolio W_T is to replicating the European option payoff, in the optimal case it vanishes. In general it depends on the distance between the time of valuation and expiry of the option $(T - t)$. Finally, the last term contributes nothing to the time t conditional variance.

The above analysis leads to the main idea of this paper which is to replace the control variate (8) with the control variate

$$Y_\tau = \frac{W_\tau}{\beta_\tau} = E_\tau^{\mathbb{Q}}\left[\frac{W_T}{\beta_T}\right] \tag{10}$$

Thus, rather than sampling the discounted payoff process at expiry of the option we suggest sampling the discounted value process at the time of exercise of the Bermudan option. By the optional sampling theorem we get that the time t conditional expectation of the control variate (10)

$$E_t^{\mathbb{Q}}[Y_\tau] = Y_t$$

is identical to the expectation of the control variate (8).

In the few cases where we have the control variate that is optimal to the European option, the corresponding Bermudan option control variate is given by

$$Y_\tau = E_\tau^{\mathbb{Q}}\left[\frac{X_T}{\beta_T}\right] \equiv \frac{C_\tau}{\beta_\tau}$$

which is the discounted European option price measured at the exercise time of the Bermudan option.

Using the control variate (10), once again with θ fixed at -1, we now get the following decomposition of the path estimate

$$\begin{aligned}\frac{X_\tau}{\beta_\tau} - (Y_\tau + Y_t) &= \frac{X_\tau}{\beta_\tau} - \frac{W_\tau}{\beta_\tau} + E_t^{\mathbb{Q}}\left[\frac{W_\tau}{\beta_\tau}\right] \\ &= \left(\frac{X_\tau}{\beta_\tau} - \frac{C_\tau}{\beta_\tau}\right) + \left(\frac{C_\tau}{\beta_\tau} - \frac{W_\tau}{\beta_\tau}\right) + E_t^{\mathbb{Q}}\left[\frac{W_\tau}{\beta_\tau}\right]\end{aligned}$$

We now see that the time t conditional variance of the first term on the right-hand side no longer depends on the distance between exercise and expiry of the option, it depends on the distance between valuation and exercise of the option $(\tau - t)$. Thus, for options close to the exercise region it contributes less than for options far from the exercise region. As for the control variate (8) the variance of the first term also depends on the magnitude of the early exercise value. The variance of the second term still depends on the replicating properties of the self-financing portfolio now at the time of exercise W_τ, again vanishing in the optimal case. However, in general it does now only depend on the distance between the time of exercise and the time of valuation $(\tau - t)$.

In the following Theorem we give a more formal analysis of the problem showing that the control variate (10) has a higher absolute correlation with the discounted Bermudan payoff than any control variate sampled later than the time of exercise, including the time of expiry as exemplified by the control variate (8).

THEOREM 1 *Let $\{Y_t\}_{0\le t\le T}$ be a martingale process, and let the stopping times τ, $\sigma \in \mathcal{T}(t, T)$ be given such that $\tau \le \sigma$. Then*

$$\rho_t^2\left(\frac{X_\tau}{\beta_\tau}, Y_\tau\right) \ge \rho_t^2\left(\frac{X_\tau}{\beta_\tau}, Y_\sigma\right)$$

PROOF We first consider the conditional covariances. By using iterated expectations, the martingale property of Y, and the optional sampling theorem we get

$$\begin{aligned}
\mathrm{Cov}_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}, Y_\sigma\right] &= E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}Y_\sigma\right] - E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right]E_t^{\mathbb{Q}}[Y_\sigma] \\
&= E_t^{\mathbb{Q}}\left[E_\tau^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}Y_\sigma\right]\right] - E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right]E_t^{\mathbb{Q}}[E_\tau^{\mathbb{Q}}[Y_\sigma]] \\
&= E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}Y_\tau\right] - E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right]E_t^{\mathbb{Q}}[Y_\tau] \\
&= \mathrm{Cov}_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}, Y_\tau\right]
\end{aligned}$$

We next consider the conditional variances. By using the same arguments as for the conditional covariance we get

$$\begin{aligned}
\mathrm{Var}_t^{\mathbb{Q}}[Y_\sigma] &= E_t^{\mathbb{Q}}[Y_\sigma^2] - E_t^{\mathbb{Q}}[Y_\sigma]^2 \\
&= E_t^{\mathbb{Q}}[E_\tau^{\mathbb{Q}}[Y_\sigma^2 - Y_\tau^2 + Y_\tau^2]] - E_t^{\mathbb{Q}}[E_\tau^{\mathbb{Q}}[Y_\sigma]]^2 \\
&= E_t^{\mathbb{Q}}[E_\tau^{\mathbb{Q}}[Y_\sigma^2] - E_\tau^{\mathbb{Q}}[Y_\sigma]^2] + E_t^{\mathbb{Q}}[Y_\tau^2] - E_t^{\mathbb{Q}}[Y_\tau]^2 \\
&= E_t^{\mathbb{Q}}[\mathrm{Var}_\tau^{\mathbb{Q}}[Y_\sigma]] + \mathrm{Var}_t^{\mathbb{Q}}[Y_\tau] \\
&\ge \mathrm{Var}_t^{\mathbb{Q}}[Y_\tau]
\end{aligned}$$

from which we conclude that

$$\frac{\mathrm{Cov}_t^{\mathbb{Q}}[X_\tau/\beta_\tau, Y_\tau]^2}{\mathrm{Var}_t^{\mathbb{Q}}[X_\tau/\beta_\tau]\,\mathrm{Var}_t^{\mathbb{Q}}[Y_\tau]} \ge \frac{\mathrm{Cov}_t^{\mathbb{Q}}[X_\tau/\beta_\tau, Y_\sigma]^2}{\mathrm{Var}_t^{\mathbb{Q}}[X_\tau/\beta_\tau]\,\mathrm{Var}_t^{\mathbb{Q}}[Y_\sigma]}$$

as claimed. □

3.2 Case study: sampling control variates at expiry versus at exercise

Using the finite-difference-based exercise boundary as a benchmark, thereby reducing the uncertainty of the strategy to the numerical error of the scheme, we can compare different applications of variance reduction to the valuation of the Bermudan single-asset put option in Table 1. All columns are based on 50 000 pairs of antithetic paths. The early exercise value is determined by subtracting the closed form valuation of the European put option from the lower bound

TABLE 1 Monte Carlo valuation of the Bermudan option using the exercise strategy obtained by the finite-difference computation. The simulations are based on 50 000 pairs of antithetic paths. For all of the options the strike $K = 40$ and the short-term rate $r = 0.06$, and the current stock price S, the volatility σ, and the time to expiry T are given. The early exercise value is computed by subtracting the closed form European option value from the lower bound Monte Carlo estimate. The early exercise difference is computed by subtracting the Monte Carlo estimate from the finite-difference estimate of the Bermudan option.

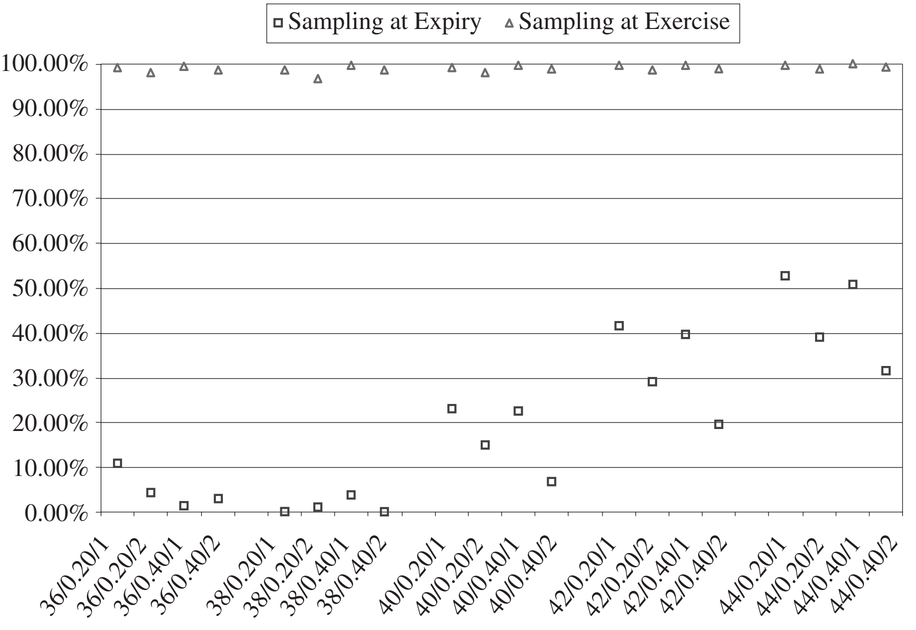
						Control variates at expiry			Control variates at exercise		
S	σ	T	Early exercise value	S.E.	Early exercise diff.	Early exercise value	S.E.	Early exercise diff.	Early exercise value	S.E.	Early exercise diff.
36	0.20	1	0.6282	0.0059	0.0053	0.6321	0.0056	0.0014	0.6332	0.0005	0.0003
36	0.20	2	1.0785	0.0067	−0.0013	1.0696	0.0065	0.0076	1.0767	0.0010	0.0005
36	0.40	1	0.3846	0.0083	0.0052	0.3966	0.0082	−0.0068	0.3893	0.0006	0.0005
36	0.40	2	0.7963	0.0102	0.0104	0.8215	0.0101	−0.0148	0.8043	0.0013	0.0023
38	0.20	1	0.3927	0.0046	0.0055	0.3985	0.0046	−0.0004	0.3974	0.0005	0.0008
38	0.20	2	0.7520	0.0059	0.0022	0.7551	0.0058	−0.0009	0.7536	0.0010	0.0006
38	0.40	1	0.3056	0.0079	0.0076	0.3049	0.0078	0.0083	0.3122	0.0005	0.0010
38	0.40	2	0.6867	0.0097	0.0024	0.6995	0.0097	−0.0104	0.6900	0.0011	−0.0009
40	0.20	1	0.2392	0.0052	0.0084	0.2482	0.0046	−0.0006	0.2481	0.0004	−0.0005
40	0.20	2	0.5273	0.0063	0.0013	0.5264	0.0058	0.0022	0.5297	0.0009	−0.0011
40	0.40	1	0.2533	0.0088	−0.0011	0.2434	0.0078	0.0088	0.2523	0.0004	−0.0001
40	0.40	2	0.5894	0.0098	0.0016	0.6095	0.0095	−0.0185	0.5908	0.0010	0.0002
42	0.20	1	0.1541	0.0057	−0.0017	0.1609	0.0043	−0.0085	0.1524	0.0004	0.0001
42	0.20	2	0.3637	0.0068	0.0073	0.3740	0.0057	−0.0030	0.3702	0.0008	0.0008
42	0.40	1	0.2124	0.0100	−0.0087	0.2059	0.0078	−0.0022	0.2044	0.0004	−0.0007
42	0.40	2	0.4971	0.0107	0.0115	0.5091	0.0095	−0.0004	0.5094	0.0009	−0.0008
44	0.20	1	0.0922	0.0055	0.0007	0.0963	0.0038	−0.0034	0.0927	0.0003	0.0002
44	0.20	2	0.2591	0.0068	0.0015	0.2592	0.0053	0.0014	0.2613	0.0007	−0.0007
44	0.40	1	0.1494	0.0109	0.0155	0.1590	0.0076	0.0059	0.1652	0.0004	−0.0003
44	0.40	2	0.4499	0.0116	−0.0106	0.4453	0.0096	−0.0061	0.4394	0.0008	−0.0002

Monte Carlo estimate. The early exercise difference is measured relative to the finite difference estimate of the lower bound of the Bermudan put option.

In the first three columns of results the only variance reduction technique applied is the method of antithetic variates. In the following three columns the discounted European put option payoff is used as a control variate, corresponding to sampling at expiry of the Bermudan put option. Finally, in the last three columns the discounted European option value is used as a control variate, sampled at the time of exercise of the Bermudan option, as suggested in Section 3.1.

It is observed from comparing the standard errors that the reduction obtained from using the European option payoff, compared with using only antithetic variates, is almost unobservable for in-the-money and even at-the-money levels of the underlying asset ($S = 36$, 38, 40). The reduction for the in-the-money levels is also very moderate. However, the reduction obtained from using the discounted European option value is quite significant, both compared with using

FIGURE 2 Comparison of ρ^2 $(X_\tau/\beta_\tau, Y_T)$ and ρ^2 $(X_\tau/\beta_\tau, Y_\tau)$. Squared correlations of control variates sampled at expiry versus at exercise of the Bermudan put option. The different scenarios are indicated along the horizontal axis with spot/volatility/expiry.



only antithetic variates as well as compared with using the European option payoff.

The difference between the two control variates can be analyzed by observing the squared correlations from Figure 2. It is noticed that, when sampled at expiry, the squared correlation increases as the option goes out-of-money, whereas when sampled at exercise the squared correlations are all in the high nineties. The first phenomenon is explained by the fact that the Bermudan option is closer to being exercised late, near expiry, making the European option payoff bear closer resemblance to the Bermudan payoff.

3.3 Case study: implications for upper bound estimates

The reductions of standard errors allow us to decrease the number of antithetic pairs of paths, while still obtaining a satisfactory level of accuracy. In Table 2 the number of antithetic pairs of paths have been reduced by a factor 50 down to 1000 pairs of paths. As observed from the standard errors in the first block of results, all but one are less than or equal to those obtained using the European option

TABLE 2 As Table 1, now using 1000 pairs of antithetic paths.

			1000 pairs of antithetic paths, control variates			150 pairs of antithetic paths, 1000 subpaths		50 pairs of antithetic paths, 1000 subpaths with control variates	
S	σ	T	Early exercise value	S.E.	Early exercise diff.	Duality gap	S.E.	Duality gap	S.E.
36	0.20	1	0.6338	0.0036	−0.0004	0.0088	0.0011	0.0000	0.0000
36	0.20	2	1.0820	0.0065	−0.0048	0.0116	0.0012	0.0002	0.0001
36	0.40	1	0.3944	0.0044	−0.0046	0.0118	0.0013	0.0001	0.0001
36	0.40	2	0.7963	0.0091	0.0104	0.0191	0.0016	0.0002	0.0001
38	0.20	1	0.3942	0.0037	0.0040	0.0043	0.0006	0.0000	0.0000
38	0.20	2	0.7418	0.0070	0.0123	0.0093	0.0008	0.0001	0.0001
38	0.40	1	0.3151	0.0035	−0.0020	0.0134	0.0016	0.0002	0.0001
38	0.40	2	0.6964	0.0082	−0.0073	0.0181	0.0015	0.0001	0.0000
40	0.20	1	0.2503	0.0030	−0.0027	0.0044	0.0007	0.0001	0.0000
40	0.20	2	0.5357	0.0063	−0.0070	0.0051	0.0008	0.0001	0.0001
40	0.40	1	0.2496	0.0028	0.0026	0.0108	0.0016	0.0002	0.0001
40	0.40	2	0.5856	0.0068	0.0054	0.0155	0.0015	0.0001	0.0001
42	0.20	1	0.1524	0.0026	0.0001	0.0022	0.0005	0.0000	0.0000
42	0.20	2	0.3725	0.0056	−0.0015	0.0043	0.0007	0.0001	0.0001
42	0.40	1	0.2007	0.0028	0.0030	0.0073	0.0010	0.0001	0.0001
42	0.40	2	0.5069	0.0062	0.0018	0.0130	0.0015	0.0001	0.0001
44	0.20	1	0.0962	0.0021	−0.0033	0.0021	0.0005	0.0000	0.0000
44	0.20	2	0.2659	0.0047	−0.0053	0.0033	0.0006	0.0000	0.0000
44	0.40	1	0.1698	0.0027	−0.0050	0.0066	0.0011	0.0000	0.0000
44	0.40	2	0.4394	0.0060	−0.0002	0.0128	0.0014	0.0001	0.0000

payoff as a control variate. We have hence effectively reduced the work involved in estimating the lower bound of the Bermudan put option given an exercise strategy.

In the same table we have analyzed the effect of using the discounted European option value as a control variate in the subpaths used in estimating the duality gap. The possible upward bias coming from the uncertainty of the lower-bound estimates was briefly analyzed in Andersen and Broadie (2004). In Table 2 we observe that even with the benchmark finite-difference exercise boundary significant duality gaps are estimated, when no control variates are used in the inner paths. In contrast, the duality gaps are estimated at quite insignificant levels when the control variate is applied in the inner valuations. Notice that the only variance reduction applied to the duality-gap estimates, in the outer loops, is the method of antithetic sampling.

4 The LSM approach

The main assumption of the LSM approach is that the conditional expectation in (2), the time t value of following the strategy given by τ until expiry, can be expressed as a linear combination of a countable set of $\mathcal{F}_t$-measurable basis functions, ie,[5]

$$\frac{L_t}{\beta_t} = \sum_{j=0}^{\infty} a_t^j F_t^j \tag{11}$$

To implement this assumption in the LSM approach, (11) is approximated with a finite sum at a given level. We let L_t^M/β_t denote this approximation when $M+1$ terms are used. We then have

$$\frac{L_t^M}{\beta_t} \equiv \sum_{j=0}^{M} a_t^j F_t^j \tag{12}$$

Using cross-sectional observations of the Monte Carlo generated state variables, the coefficients a_t^j, $j = 0, 1, \ldots, M$, are determined by least-squares regression, where the current basis functions F_t^j, $j = 0, 1, \ldots, M$, are the independent variables and the discounted payoff X_τ/β_τ is the dependent variable. Since we only need the approximation where the option is in-the-money at time t, ie, when $X_t/\beta_t > 0$, we only include these observations in the regression. As argued by Longstaff and Schwartz (2001) this increases the numerical efficiency as less terms are needed in (12) to fit the observations. With a total of N Monte Carlo generated paths, we let $a_t^{j(N)}$, $j = 0, 1, \ldots, M$, denote the time t coefficients determined from the regression using the in-the-money paths. Hence, the approximation used in the implementation denoted by $L_t^{M(N)}$ is given by

$$\frac{L_t^{M(N)}}{\beta_t} \equiv \sum_{j=0}^{M} a_t^{j(N)} F_t^j \tag{13}$$

Given the time t values of the state variables summarized by the basis functions F_t^j, $j = 0, 1, \ldots, M$, we thus exercise whenever the following inequality is satisfied

$$\frac{L_t^{M(N)}}{\beta_t} \le \frac{X_t}{\beta_t}$$

4.1 Case study: LSM-based exercise strategies

We now apply the LSM approach to estimating the exercise strategy of the single-asset Bermudan put option analyzed in the previous examples. For the sake of

[5] As argued by Longstaff and Schwartz (2001, p. 122), this assumption can be justified by assuming that L_t/β_t is an element of the L^2 space of square-integrable functions relative to some measures. Since L^2 is a Hilbert space, it has a countable orthonormal basis and the conditional expectations can be represented as a linear function of the elements of the basis.

simplicity and to maintain some financial intuition we use the following basis functions corresponding to $M = 3$. All functions depend on the current time t and the current stock price S_t, ie, $F_t^j = f_j(S_t, t)$, where

$$\begin{aligned} f_0(s, t) &= K \\ f_1(s, t) &= s \\ f_2(s, t) &= C(s, t) \\ f_3(s, t) &= sC(s, t) \end{aligned}$$

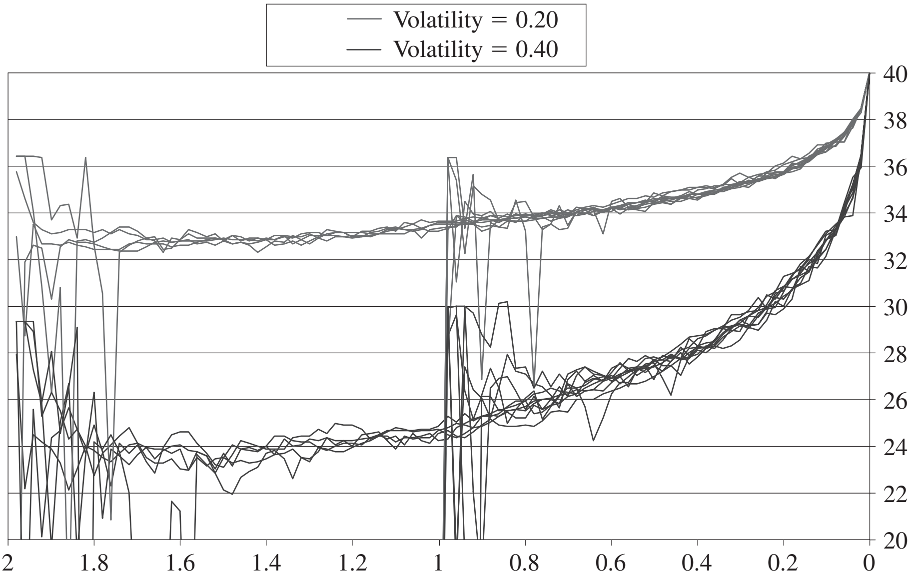
and K is the strike of the Bermudan put option, s is the current asset price, and $C(s, t)$ is the European option price for the put option expiring at time T, with the same parameters as the Bermudan option. The financial intuition lies in the interpretation of the continuation value estimate as a portfolio in the cash instrument, the underlying stock and the corresponding European option, and finally a less intuitive mixed term as the product of the stock and the European option.

Using 50 000 pairs of antithetic paths we have estimated an exercise strategy for each of the 20 combinations of initial stock price, time to expiry, and volatility. The resulting exercise boundaries are determined by finding the minimal value of s, for which $\sum_{j=0}^{M} f_j(s, t)$ is greater than or equal to the intrinsic value X_t. As a sub-optimality control, we have used the European option value and selected no critical level for the stock s for which the European option value $C(s, t)$ exceeds the intrinsic value X_t. This control can naturally be used in all situations where the corresponding European option value can be evaluated or approximated. Since the intrinsic value is linear in s in the in-the-money region and the derivatives of the chosen basis functions are easily derived we can use the Newton–Raphson procedure to determine the exercise boundary by equating the two expressions. The exercise boundaries found are graphically demonstrated in Figure 3.

It is observed that the exercise boundaries fluctuate wildly at the longest time to expiry, corresponding to the initial time. Notice furthermore, that since all scenarios are represented in the figure, a new set of fluctuations start at time to expiry equal to 1. It should also be observed that for both the one- and two-year time-to-expiry cases the exercise boundaries tend to behave nicely when less than two thirds of the time to expiry remains.

Early exercise values are shown in the first three columns of results in Table 3 using an independent set of 50 000 pairs of antithetic paths. The important observation is that using the discounted European option value as a control variate we have to reject the hypothesis that the early exercise difference is zero for all but four of the early exercise values. However, had we only used 1000 pairs of antithetic paths with control variates, corresponding to the case of using only antithetic variates, this would not be the case. This can be observed from the next three columns. Finally, the suboptimality of the LSM exercise boundary is verified

FIGURE 3 Exercise boundary for the Bermudan put option with 50 exercise points per year, determined using the LSM approach with 50 000 pairs of antithetic paths. The short-term rate is 0.06, the upper curve is for $\sigma = 0.20$, and the lower for curve is for $\sigma = 0.40$. The horizontal axis indicates time to expiry.



by inspecting the duality gaps in the last two columns. Accounting for standard errors of both lower bound estimates as well as duality gaps (see Andersen and Broadie (2004) for the definition of a conservative confidence interval), we can contain the finite-difference benchmark in a 95% confidence interval. These observations clearly call for an improvement of the exercise strategy, as dealt with in the following examples.

4.1.1 The Bermudan max-call option

To investigate the application of control variates to a more complex example than the single-asset put option investigated in the previous numerical examples, we now investigate the n-asset Bermudan max-call option for the case of $n = 2$, 3, and 5 assets. The same example has been studied in Andersen and Broadie (2004) without any use of variance reduction.

Andersen and Broadie (2004) use a set of 13 basis functions involving the highest and second highest asset prices, as well as polynomials of these, together with the value of the European max-call option on the two largest assets and polynomials of this. Inspired by their choice we use a smaller set of basis functions.

TABLE 3 As Table 2, now using the LSM approach for determining the exercise strategy based on 50 000 pairs of antithetic paths, and independent paths for the valuation. Early exercise differences marked with * indicates a statistically significant value at a 5% confidence level.

			50 000 pairs of antithetic paths, control variates			1000 pairs of antithetic paths, control variates			50 pairs of antithetic paths, 1000 subpaths with control variates	
S	σ	T	Early exercise value	S.E.	Early exercise diff.	Early exercise value	S.E.	Early exercise diff.	Duality gap	S.E.
36	0.20	1	0.6323	0.0005	0.0012*	0.6358	0.0036	−0.0024	0.0010	0.0003
36	0.20	2	1.0725	0.0009	0.0047*	1.0675	0.0067	0.0097	0.0043	0.0017
36	0.40	1	0.3850	0.0006	0.0048*	0.3907	0.0043	−0.0009	0.0059	0.0051
36	0.40	2	0.8036	0.0012	0.0031*	0.8031	0.0087	0.0035	0.0034	0.0011
38	0.20	1	0.3974	0.0005	0.0008	0.3952	0.0038	0.0029	0.0004	0.0002
38	0.20	2	0.7543	0.0010	−0.0002	0.7616	0.0070	−0.0074	0.0008	0.0003
38	0.40	1	0.3120	0.0005	0.0011*	0.3138	0.0035	−0.0006	0.0027	0.0009
38	0.40	2	0.6873	0.0011	0.0018	0.6852	0.0082	0.0039	0.0022	0.0008
40	0.20	1	0.2455	0.0004	0.0021*	0.2482	0.0030	−0.0006	0.0006	0.0002
40	0.20	2	0.5266	0.0009	0.0020*	0.5294	0.0062	−0.0008	0.0027	0.0010
40	0.40	1	0.2500	0.0004	0.0021*	0.2515	0.0028	0.0007	0.0006	0.0003
40	0.40	2	0.5886	0.0009	0.0024*	0.5898	0.0064	0.0012	0.0029	0.0016
42	0.20	1	0.1516	0.0004	0.0009*	0.1534	0.0025	−0.0010	0.0012	0.0007
42	0.20	2	0.3688	0.0008	0.0022*	0.3754	0.0055	−0.0044	0.0003	0.0002
42	0.40	1	0.2027	0.0004	0.0010*	0.2028	0.0027	0.0010	0.0005	0.0003
42	0.40	2	0.5056	0.0009	0.0031*	0.5012	0.0061	0.0075	0.0031	0.0014
44	0.20	1	0.0926	0.0003	0.0003	0.0928	0.0021	0.0001	0.0005	0.0003
44	0.20	2	0.2539	0.0007	0.0067*	0.2547	0.0048	0.0059	0.0052	0.0047
44	0.40	1	0.1588	0.0004	0.0060*	0.1617	0.0027	0.0032	0.0019	0.0007
44	0.40	2	0.4316	0.0008	0.0076*	0.4428	0.0062	−0.0036	0.0023	0.0010

For the case of two assets, the European max-call option price can easily be computed (for implementational details see Haug (1997)). However, for the case of three or five assets this computation becomes more involving, requiring numerical integration. To circumvent this shortcoming we therefore choose to use single-asset call options on the individual assets, as well as combinations of two-asset max-call options in both our regression and as control variates. For the regression part we choose the following nine basis functions

$$
\begin{aligned}
b_0^{\max}(s^*, s^{**}, t) &= K \\
b_1^{\max}(s^*, s^{**}, t) &= s^* \\
b_2^{\max}(s^*, s^{**}, t) &= (s^*)^2 \\
b_3^{\max}(s^*, s^{**}, t) &= C(s^*, t) \\
b_4^{\max}(s^*, s^{**}, t) &= (C(s^*, t))^2 \\
b_5^{\max}(s^*, s^{**}, t) &= C^{\max}(s^*, s^{**}, t) \\
b_6^{\max}(s^*, s^{**}, t) &= (C^{\max}(s^*, s^{**}, t))^2
\end{aligned}
$$

$$b_7^{\max}(s^*, s^{**}, t) = s^{**}$$
$$b_8^{\max}(s^*, s^{**}, t) = s^* s^{**}$$

where s^* and s^{**} are the highest and second highest asset prices respectively, and $C(s^*, t)$ is the European call option price on the asset with the highest price, and finally $C^{\max}(s^*, s^{**}, t)$ is the two-asset max-call option on assets with the highest and second highest prices. As control variates we use the discounted values of all of the single-asset European call options as well as the discounted values of all the two-asset max-call options, both sampled at the time of exercise of the max-call options. This results in a total of $n + n * (n - 1)/2$ controls, where n is the number of assets.

In a similar manner to Andersen and Broadie (2004), we use 200 000 independent paths for the regression involved in determining the exercise strategy, and we use 2 000 000 independent paths for the valuation of the Bermudan max-call option, combined with the use of control variates as outlined above. From the resulting estimates of the Bermudan max-call options in Table 4 we see that the use of control variates reduces between 72 and 83% of the standard errors reported by Andersen and Broadie (2004). The effect gradually reduces as the number of assets increases, and as the options come more in-the-money.

The early exercise difference compares the Monte Carlo estimates of the Bermudan option with the corresponding binomial implementation reported by Andersen and Broadie (2004). However, in the five-asset case the early exercise value has been estimated using a Monte Carlo estimate of the European option, as no efficient method exists for evaluating the five-asset max-call option. We see that using the exercise strategy found, we have significant deviations between the two. We should, however, keep in mind that the binomial implementation has an approximate error of 0.003 for $n = 2$ assets and an approximate error of 0.015 for $n = 3$ assets. Comparing with the lower bound (Monte Carlo) estimates in Table 2 of Andersen and Broadie (2004), our Monte Carlo estimates are within their 95% confidence interval.

Reducing the number of paths used for valuing the Bermudan max-call options to 100 000 independent paths, while keeping the number of paths used for determining the exercise strategy fixed, we obtain standard errors that are more comparable to those reported by Andersen and Broadie (2004). From Table 4 we see that for $n = 2$ assets none of the deviations from the binomial model benchmark are significant, and for $n = 3$ assets only one is significantly different. All of the differences are below 1 cent for the $n = 2$ assets case and for $n = 3$ assets the scenario $S = 110$ has the highest deviation of 0.0338.

By inspecting the duality-gap estimates of Table 4 we see for the case of $n = 2$ assets, none would fall outside the 95% confidence interval of the early exercise value with 100 000 independent paths. However, for the case of $n = 3$ or 5 assets the duality gaps are significant with respect to the standard errors of the early exercise values. For $n = 3$ they are of magnitudes between 0.0167 and 0.0309, and for $n = 5$ they range between 0.0415 and 0.0676. Thus, in the

TABLE 4 Monte Carlo valuation of two-, three- and five-asset Bermudan max-call options using the LSM approach for determining the exercise strategy. The regressions are based on 200 000 independent paths. The valuations are based on 2 000 000 independent paths. A combination of discounted European options all sampled at the time of exercise are used as control variates. For all the options the strike $K = 100$. The short-term rate $r = 0.05$, the dividend rate $\delta = 0.10$, the volatility of each asset $\sigma = 0.20$, and the correlation $\rho = 0$. For all of the max-call options time to expiry $T = 3$ years with exercise points at $t_e = (e/d)T$, for $e = 0, 1, \ldots, d$ with $d = 9$. The binomial model estimates are taken from Andersen and Broadie (2004). The early exercise values are computed by subtracting the European option estimate from the binomial model and lower bound Monte Carlo estimates, respectively. The early exercise differences (for $n = 2, 3$) are computed by subtracting the Monte Carlo estimates from the binomial model estimates of the early exercise value.

		2 000 000 independent paths, control variates			**100 000 independent paths, control variates**			**1500 independent paths, 10 000 subpaths with control variates**	
n	S	**Early exercise value**	**S.E.**	**Early exercise diff.**	**Early exercise value**	**S.E.**	**Early exercise diff.**	**Duality gap**	**S.E.**
2	90	1.4130	0.0010	0.0069	1.4147	0.0047	0.0052	0.0208	0.0079
	100	2.7019	0.0014	0.0044	2.7058	0.0062	0.0004	0.0062	0.0024
	110	4.4119	0.0016	0.0046	4.4154	0.0074	0.0010	0.0027	0.0007
3	90	1.7384	0.0012	0.0130	1.7304	0.0055	0.0226	0.0220	0.0055
	100	2.9958	0.0017	0.0138	2.9811	0.0075	0.0227	0.0167	0.0027
	110	4.4108	0.0022	0.0397	4.4152	0.0096	0.0234	0.0309	0.0041
5	90	2.0380	0.0017	–	2.0349	0.0074	–	0.0415	0.0084
	100	3.0669	0.0024	–	3.0522	0.0106	–	0.0659	0.0179
	110	4.0361	0.0031	–	4.0257	0.0140	–	0.0676	0.0059

two latter cases, the LSM approach does not lead to sufficiently accurate exercise strategies using the chosen set of basis functions. However, it should be noted that none of the early exercise values are outside the 95% confidence intervals of the Bermudan max-call values in Table 2 of Andersen and Broadie (2004).

4.2 Case study: correlations and speed-up factors of control variates

To sum up the effects of control variates for American options, we show in Table 5 the squared correlation coefficients of the control variates. Two additional test cases have been investigated, the Bermudan geometric and arithmetic

TABLE 5 Squared correlations and speed-up factors of the control variates used for the Bermudan options. Exercise strategy is τ^{LSM}; $\rho^2(X_\tau/\beta_\tau, Y)$–$(1/(1-\rho^2(X_\tau/\beta_\tau, Y)))$.

S	σ	T	Bermudan put option		Bermudan geometric Asian option		European arithmetic Asian option		Bermudan arithmetic Asian option	
36	0.20	1	0.9925	133	0.8938	9	0.9946	185	0.8968	10
36	0.20	2	0.9807	52	0.8871	9	0.9956	227	0.8745	8
36	0.40	1	0.9948	192	0.9158	12	0.9851	67	0.9348	15
36	0.40	2	0.9860	71	0.9245	13	0.9757	41	0.9326	15
38	0.20	1	0.9867	75	0.9306	14	0.9983	588	0.9323	15
38	0.20	2	0.9664	30	0.9189	12	0.9981	526	0.9170	12
38	0.40	1	0.9960	250	0.9507	20	0.9915	118	0.9514	21
38	0.40	2	0.9865	74	0.9436	18	0.9852	68	0.9457	18
40	0.20	1	0.9936	156	0.9501	20	0.9991	1111	0.9424	17
40	0.20	2	0.9801	50	0.9325	15	0.9983	588	0.9288	14
40	0.40	1	0.9975	400	0.9619	26	0.9951	204	0.9521	21
40	0.40	2	0.9902	102	0.9578	24	0.9895	95	0.9389	16
42	0.20	1	0.9961	256	0.9725	36	0.9987	769	0.9658	29
42	0.20	2	0.9862	72	0.9569	23	0.9979	476	0.9508	20
42	0.40	1	0.9984	625	0.9738	38	0.9949	196	0.9651	29
42	0.40	2	0.9932	147	0.9656	29	0.9904	104	0.9458	18
44	0.20	1	0.9972	357	0.9867	75	0.9982	556	0.9803	51
44	0.20	2	0.9900	100	0.9763	42	0.9973	370	0.9649	28
44	0.40	1	0.9988	833	0.9807	52	0.9942	172	0.9651	29
44	0.40	2	0.9949	196	0.9746	39	0.9894	94	0.9493	20

	Bermudan max-call option					
S	$n = 2$ assets		$n = 3$ assets		$n = 5$ assets	
90	0.9847	65	0.9845	65	0.9787	47
100	0.9833	60	0.9807	52	0.9703	34
110	0.9821	56	0.9763	42	0.9593	25

Asian options.[6] In both cases the discounted value of the European geometric Asian option has been used as a control variate.[7] As for the LSM exercise strategy the choice of basis functions is equivalent to the case of the single-asset put option, except that the European geometric Asian option has been used where the single-asset European option was used.

In parentheses we show the corresponding speed-up factors. The speed-up factor is computed as the reciprocal of $(1 - \rho^2(X_\tau/\beta_\tau, Y))$ and measures the

[6]Results on early exercise values, standard errors, and duality gaps have been omitted to save space, however, numerical results can be obtained from the author upon request.

[7]Note that the European geometric Asian option value should be based on the average already sampled along the path.

equivalent multiplication in number of paths required in a brute-force Monte Carlo valuation to obtain the same accuracy. From Table 5 we see that the use of control variates in the Monte Carlo valuation of Bermudan options corresponds to an increase in the number of paths in the range from a factor of 8 to a factor of 833.

4.3 Accuracy, stability and convergence of the LSM approach

The accuracy of the LSM exercise strategy is solely determined by the accuracy of the following two approximations

$$\frac{L_t}{\beta_t} \approx \frac{L_t^M}{\beta_t} \tag{14}$$

and

$$\frac{L_t^M}{\beta_t} \approx \frac{L_t^{M(N)}}{\beta_t} \tag{15}$$

For an infinite computational budget, we should require that $L_t^{M(N)} \to L_t$ as $N \to \infty$ and $M \to \infty$. As discussed in the introduction, the convergence results of Longstaff and Schwartz (2001) are based on a very restrictive set of assumptions. However, in Clement *et al* (2002), the convergence of $L_t^M \to L_t$ as $M \to \infty$ is established when an orthogonal set of basis functions is used. They furthermore considered the convergence of $L_t^{M(N)} \to L_t^M$ as $N \to \infty$. Finally, they established a central limit result for the rate of convergence of the LSM algorithm. These results are very important, in particular from a theoretical point of view. However, from a practical point of view we are more concerned with the performance for a finite sample of N paths, and a finite number of basis functions M.

With respect to the approximation (14), it is argued in Longstaff and Schwartz (2001) that M should be increased until the lower bound estimate no longer increases. In a similar manner to Moreno and Navas (2002) we warn against this argument for a finite sample of N paths, as it reduces the accuracy of the approximation (15). Numerical examples show that the inclusion of too many terms in the approximation (12) can cause instability in the regression and produce oscillations in the estimated continuation value, which in turn deteriorates the exercise strategy in contrast to the intentions. Rather, we focus in the following section on improving the approximation (15) for a given number of paths and basis functions. The objective is to replace (13) with a more accurate approximation.

5 The control variate improved LSM approach

In this section we reuse and generalize the concept of control variates now in the context of the regressions involved in the LSM approach. For the sake of simplicity, in the following we denote this approach as the CV-LSM approach.

We suggest replacing the discounted payoff X_τ/β_τ from following the strategy τ for the remaining life of the option, with a random variable Z_τ/β_τ with the same time t conditional expectation, and with a smaller time t conditional variance. As the following proposition shows, this improves the efficiency of the least-squares projection without changing the bias.

PROPOSITION 2 *Let P_t^M denote the projection onto the $\mathcal{F}_t$-measurable $M+1$ basis functions. Let X_τ/β_τ denote the discounted payoff of the American or Bermudan option using the strategy $\tau \in \mathcal{T}(t, T)$. Assume we have a $\mathcal{F}_\tau$-measurable random variable Z_τ/β_τ with the following two properties*

$$E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right] = E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right] \tag{16}$$

$$\mathrm{Var}_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right] \leq \mathrm{Var}_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right] \tag{17}$$

Then

$$E_t^{\mathbb{Q}}\left[P_t^M \frac{Z_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right]\right] = E_t^{\mathbb{Q}}\left[P_t^M \frac{X_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right]\right] \tag{18}$$

and

$$E_t^{\mathbb{Q}}\left[\left(P_t^M \frac{Z_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right]\right)^2\right] \leq E_t^{\mathbb{Q}}\left[\left(P_t^M \frac{X_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right]\right)^2\right] \tag{19}$$

PROOF To show (18) we use the $\mathcal{F}_t$-measurability of P_t^M and property (16)

$$\begin{aligned}
E_t^{\mathbb{Q}}\left[P_t^M \frac{Z_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right]\right] &= P_t^M E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right] - E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right] \\
&= P_t^M E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right] - E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right] \\
&= E_t^{\mathbb{Q}}\left[P_t^M \frac{X_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right]\right]
\end{aligned}$$

To show (19) we also use the $\mathcal{F}_t$-measurability of P_t^M together with (18) and property (17):

$$\begin{aligned}
&E_t^{\mathbb{Q}}\left[\left(P_t^M \frac{Z_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right]\right)^2\right] \\
&\quad = E_t^{\mathbb{Q}}\left[\left(P_t^M \frac{Z_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[P_t^M \frac{Z_\tau}{\beta_\tau}\right]\right)^2\right] \\
&\qquad + 2E_t^{\mathbb{Q}}\left[\left(P_t^M \frac{Z_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[P_t^M \frac{Z_\tau}{\beta_\tau}\right]\right)\left(E_t^{\mathbb{Q}}\left[P_t^M \frac{Z_\tau}{\beta_\tau}\right] - E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right]\right)\right] \\
&\qquad + E_t^{\mathbb{Q}}\left[\left(E_t^{\mathbb{Q}}\left[P_t^M \frac{Z_\tau}{\beta_\tau}\right] - E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right]\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= P_t^M E_t^{\mathbb{Q}}\left[\left(\frac{Z_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right]\right)^2\right] \\
&\quad + \left(E_t^{\mathbb{Q}}\left[P_t^M \frac{Z_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right]\right]\right)^2 \\
&= P_t^M \operatorname{Var}_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right] + \left(E_t^{\mathbb{Q}}\left[P_t^M \frac{Z_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{Z_\tau}{\beta_\tau}\right]\right]\right)^2 \\
&\leq P_t^M \operatorname{Var}_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right] + \left(E_t^{\mathbb{Q}}\left[P_t^M \frac{X_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right]\right]\right)^2 \\
&= E_t^{\mathbb{Q}}\left[\left(P_t^M \frac{X_\tau}{\beta_\tau} - E_t^{\mathbb{Q}}\left[\frac{X_\tau}{\beta_\tau}\right]\right)^2\right] \qquad \square
\end{aligned}
$$

An obvious candidate for Z_τ/β_τ is given by the control variate adjusted discounted payoff. In the following let Z_τ/β_τ be given by

$$
\frac{Z_\tau}{\beta_\tau} = \frac{X_\tau}{\beta_\tau} + \theta_t(Y_\tau - E_t^{\mathbb{Q}}[Y_\tau]) \tag{20}
$$

where Y_τ is the control variate suggested in Section 3.1, and θ_t is an appropriately chosen $\mathcal{F}_t$-measurable random variable.

If we for a moment take θ_t as given, and let P_t^M be as defined in Proposition 2, we get by the linearity of projections that

$$
\begin{aligned}
P_t^M \frac{Z_\tau}{\beta_\tau} &= P_t^M\left(\frac{X_\tau}{\beta_\tau} + \theta_t(Y_\tau - E_t^{\mathbb{Q}}[Y_\tau])\right) \\
&= P_t^M \frac{X_\tau}{\beta_\tau} + \theta_t(P_t^M Y_\tau - P_t^M E_t^{\mathbb{Q}}[Y_\tau])
\end{aligned} \tag{21}
$$

Hence, we see that to find the projection of Z_τ/β_τ we find the projection of X_τ/β_τ, Y_τ, and $E_t^{\mathbb{Q}}[Y_\tau]$ and take the linear combination of the projections using θ_t.

If we make the simplifying assumption that the time t conditional expectation of the control variate $Y_t = E_t^{\mathbb{Q}}[Y_\tau]$ is in the span of the $M+1$ basis functions, which is equivalent to

$$
Y_t = P_t^M E_t^{\mathbb{Q}}[Y_\tau] \tag{22}
$$

we can modify (21) to

$$
P_t^M \frac{Z_\tau}{\beta_\tau} = P_t^M \frac{X_\tau}{\beta_\tau} + \theta_t(P_t^M Y_\tau - Y_t) \tag{23}
$$

To see that (22) is a reasonable condition, note that if Y_τ has a high covariation with X_τ/β_τ in the directions not spanned by the basis functions, to improve the projection of X_τ/β_τ its conditional expectation Y_t could itself be included in the basis functions, which would satisfy (22).

We now turn to estimating θ_t^*. Rather than a point estimate as used in the valuation we now need a functional estimate of θ_t. We therefore suggest an approach which is close in spirit to the original LSM approach. First, remember by the definitions of $\mathrm{Cov}_t^{\mathbb{Q}}[(X_\tau/\beta_\tau), Y_\tau]$ and $\mathrm{Var}_t^{\mathbb{Q}}[Y_\tau]$ we get

$$\begin{aligned}\theta_t^* &= -\frac{E_t^{\mathbb{Q}}[(X_\tau/\beta_\tau)Y_\tau] - E_t^{\mathbb{Q}}[X_\tau/\beta_\tau]E_t^{\mathbb{Q}}[Y_\tau]}{E_t^{\mathbb{Q}}[Y_\tau^2] - E_t^{\mathbb{Q}}[Y_\tau]^2}\\ &= -\frac{(LY)_t/\beta_t - (L_t/\beta_t)Y_t}{(Y^2)_t - (Y_t)^2}\end{aligned}$$

where L_t/β_t is defined in (2) and Y_t is defined in (9), and we let $(Y^2)_t$ and $(LY)_t/\beta_t$ be defined by

$$(Y^2)_t \equiv E_t^{\mathbb{Q}}[Y_\tau^2], \qquad \frac{(LY)_t}{\beta_t} \equiv E_t^{\mathbb{Q}}[(X_\tau/\beta_\tau)Y_\tau]$$

As in (11) of Section 4, we assume that the time t conditional expectations Y_t, $(Y^2)_t$, and $(LY)_t/\beta_t$ can also be expressed as a countable sum of the same set of $\mathcal{F}_t$-measurable basis functions, ie,

$$Y_t = \sum_{j=0}^{\infty} b_t^j F_t^j, \quad (Y^2)_t = \sum_{j=0}^{\infty} c_t^j F_t^j, \quad \frac{(LY)_t}{\beta_t} = \sum_{j=0}^{\infty} d_t^j F_t^j$$

Again as in (12) we approximate the conditional expectations by truncating the sums at the same level M. Letting Y_t^M, $(Y^2)_t^M$, and $(LY)_t^M/\beta_t$ denote the approximations, we have

$$Y_t^M \equiv \sum_{j=0}^{M} b_t^j F_t^j, \quad (Y^2)_t^M \equiv \sum_{j=0}^{M} c_t^j F_t^j, \quad \frac{(LY)_t^M}{\beta_t} \equiv \sum_{j=0}^{M} d_t^j F_t^j$$

Using L_t^M/β_t, Y_t^M, $(Y^2)_t^M$, and $(LY)_t^M/\beta_t$, we can approximate θ_t^* with θ_t^M given by

$$\theta_t^M \equiv -\frac{(LY)_t^M/\beta_t - (L_t^M/\beta_t)Y_t^M}{(Y^2)_t^M - (Y_t^M)^2}$$

and we can approximate the time t conditional expectation of the discounted payoff from following the strategy τ by $L_t^{M,\mathrm{CV}}$ given by the following expression

$$\frac{L_t^{M,\mathrm{CV}}}{\beta_t} \equiv \frac{L_t^M}{\beta_t} + \theta_t^M(Y_t^M - Y_t)$$

where we note that the first term of the right-hand side is simply (12), the original LSM approximation.

To implement the CV-LSM approach we use the same set of cross-sectional observations of the Monte Carlo generated state-variables that were used to determine (13). We let $b_t^{j(N)}$, $c_t^{j(N)}$, and $d_t^{j(N)}$, $j = 0, 1, \ldots, M$, denote the time t coefficients determined from the least-squares regression using the in-the-money paths. Thus, together with (13) we get the approximations $Y_t^{M(N)}$,

$(Y^2)_t^{M(N)}$, and $(LY)_t^{M(N)}/\beta_t$ given by

$$Y_t^{M(N)} \equiv \sum_{j=0}^{M} b_t^{j(N)} F_t^j, \quad (Y^2)_t^{M(N)} \equiv \sum_{j=0}^{M} c_t^{j(N)} F_t^j, \quad \frac{(LY)_t^{M(N)}}{\beta_t} \equiv \sum_{j=0}^{M} d_t^{j(N)} F_t^j$$

Using the estimated approximations $L_t^{M(N)}/\beta_t$, $Y_t^{M(N)}$, $(Y^2)_t^{M(N)}$, and $(LY)_t^{M(N)}/\beta_t$, we can approximate θ_t^* with $\theta_t^{M(N)}$ which is given by

$$\theta_t^{M(N)} \equiv -\frac{(LY)_t^{M(N)}/\beta_t - (L_t^{M(N)}/\beta_t)Y_t^{M(N)}}{(Y^2)_t^{M(N)} - (Y_t^{M(N)})^2}$$

In total the CV-LSM approximation of the continuation value $L_t^{M,\mathrm{CV}(N)}$ using $M+1$ basis functions and N paths is given by

$$\frac{L_t^{M,\mathrm{CV}(N)}}{\beta_t} \equiv \frac{L_t^{M(N)}}{\beta_t} + \theta_t^{M(N)}(Y_t^{M(N)} - Y_t) \tag{24}$$

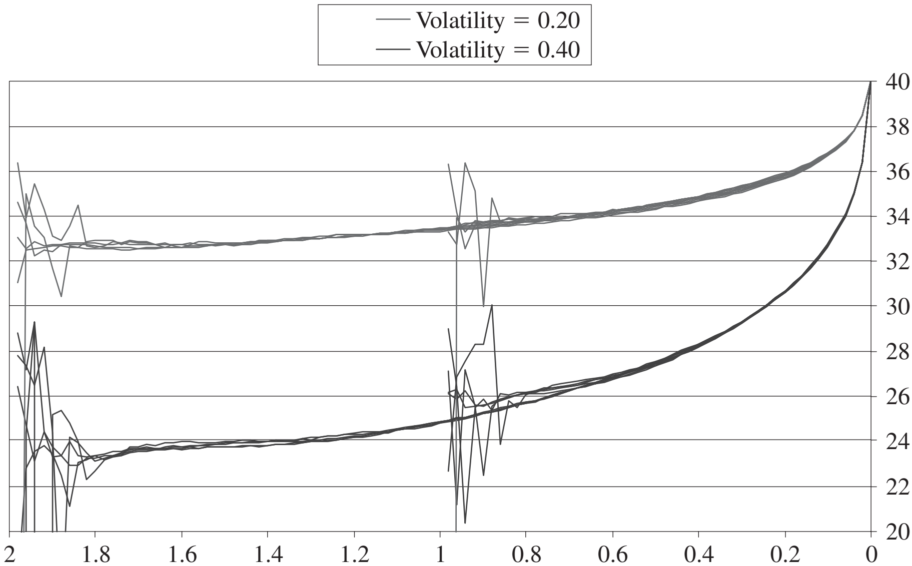
From a computational point of view, the extension of the LSM approach to the CV-LSM approach involves a few additional steps. The first step is to sample the control variate Y_τ whenever the American or Bermudan option is exercised. This adds no extra cost if the control variate is already used to improve the valuation. The second step is to evaluate the known time t conditional expectation of the control variate Y_t for each exercise time where the option is in-the-money. However, if Y_t is already in the set of basis functions, as argued below (23), this also involves no additional cost. The third step is to include four rather than one dependent variable in the least-squares regression. Since we have already computed the variance–covariance matrix of the independent variables, it only involves computing the additional covariances between the added dependent variables and the independent variables. The computational cost of solving the least-squares equations is considered fixed.

5.1 Case study: CV-LSM-based exercise strategies

For the single-asset Bermudan put option investigated in the previous examples, we now apply the CV-LSM approach to estimating the exercise strategy. As before we are now using 50 000 pairs of antithetic paths, performing regressions on both the discounted Bermudan option payoffs as well as the discounted European option value sampled at the time of exercise. Furthermore, the squared moments and the cross moment between the two are estimated using the same set of basis functions.

The resulting exercise boundary is displayed in Figure 4. We note that the boundary is much more smooth, however with some kinks and spikes for the longest times to expiry, ie, in the left-hand side of the figure. As for the LSM approach this is explained by the lack of in-the-money paths, which are the only ones used in the four regressions.

FIGURE 4 Exercise boundary for the Bermudan put option with 50 exercise points per year, determined using the CV-LSM approach with 50 000 pairs of antithetic paths. The short-term rate is 0.06, the upper curve is for $\sigma = 0.20$, and the lower curve is for $\sigma = 0.40$. The horizontal axis indicates time to expiry.



To assess the quality of the exercise boundary we use an independent set of 50 000 pairs of antithetic paths together with the control variate to estimate the early exercise value in Table 6. We note that the standard errors are close in line with those estimated using the LSM approach. Also, now the early exercise differences are much smaller, however with two outside a 95% confidence interval around the finite-difference benchmark. Using only 1000 pairs of antithetic paths only one of these would have been outside the confidence interval. Finally, the improved quality of the exercise strategy is confirmed by inspecting the duality gaps of Table 6, which have all been decreased by an amount almost equivalent to the decrease in early exercise differences.

To gain from the improvements in estimation of the exercise strategy we now repeat the exercise using only 1000 pairs of antithetic paths in the CV-LSM approach. The resulting exercise boundary can be observed from Figure 5. As the figure shows this is really a test of the stability of the CV-LSM approach. Qualitatively, the figure looks very similar to Figure 3 where the LSM approach was used with 50 000 pairs of antithetic paths.

To quantitatively assess the exercise boundary we show the early exercise values in Table 7. The values have been determined using an independent set of 1000 pairs of antithetic paths. The standard errors are comparable to the

TABLE 6 As Table 3, now using the CV-LSM approach for determining the exercise strategy based on 50 000 pairs of antithetic paths.

			50 000 pairs of antithetic paths, control variates			1000 pairs of antithetic paths, control variates			50 pairs of antithetic paths, 1000 subpaths with control variates	
S	σ	T	Early exercise value	S.E.	Early exercise diff.	Early exercise value	S.E.	Early exercise diff.	Duality gap	S.E.
36	0.20	1	0.6332	0.0005	0.0003	0.6361	0.0035	−0.0026	0.0001	0.0001
36	0.20	2	1.0766	0.0010	0.0006	1.0785	0.0067	−0.0013	0.0007	0.0002
36	0.40	1	0.3891	0.0006	0.0007	0.3910	0.0045	−0.0012	0.0003	0.0002
36	0.40	2	0.8061	0.0012	0.0006	0.7944	0.0089	0.0123	0.0011	0.0004
38	0.20	1	0.3981	0.0005	0.0000	0.3978	0.0038	0.0003	0.0002	0.0001
38	0.20	2	0.7522	0.0010	0.0020*	0.7595	0.0069	−0.0053	0.0004	0.0002
38	0.40	1	0.3136	0.0005	−0.0004	0.3111	0.0035	0.0021	0.0004	0.0002
38	0.40	2	0.6881	0.0011	0.0010	0.6846	0.0080	0.0045	0.0002	0.0001
40	0.20	1	0.2473	0.0004	0.0003	0.2482	0.0030	−0.0006	0.0001	0.0001
40	0.20	2	0.5270	0.0009	0.0016	0.5200	0.0061	0.0086	0.0003	0.0002
40	0.40	1	0.2525	0.0004	−0.0003	0.2495	0.0029	0.0027	0.0001	0.0001
40	0.40	2	0.5900	0.0010	0.0010	0.5909	0.0070	0.0001	0.0005	0.0003
42	0.20	1	0.1528	0.0004	−0.0004	0.1545	0.0025	−0.0021	0.0000	0.0000
42	0.20	2	0.3704	0.0008	0.0006	0.3710	0.0054	0.0000	0.0001	0.0000
42	0.40	1	0.2043	0.0004	−0.0006	0.2072	0.0027	−0.0035	0.0000	0.0000
42	0.40	2	0.5090	0.0009	−0.0003	0.5013	0.0062	0.0074	0.0003	0.0002
44	0.20	1	0.0932	0.0003	−0.0003	0.0940	0.0021	−0.0011	0.0000	0.0000
44	0.20	2	0.2605	0.0007	0.0001	0.2603	0.0046	0.0003	0.0001	0.0000
44	0.40	1	0.1633	0.0004	0.0016*	0.1571	0.0026	0.0078*	0.0000	0.0000
44	0.40	2	0.4397	0.0008	−0.0005	0.4445	0.0059	−0.0053	0.0001	0.0001

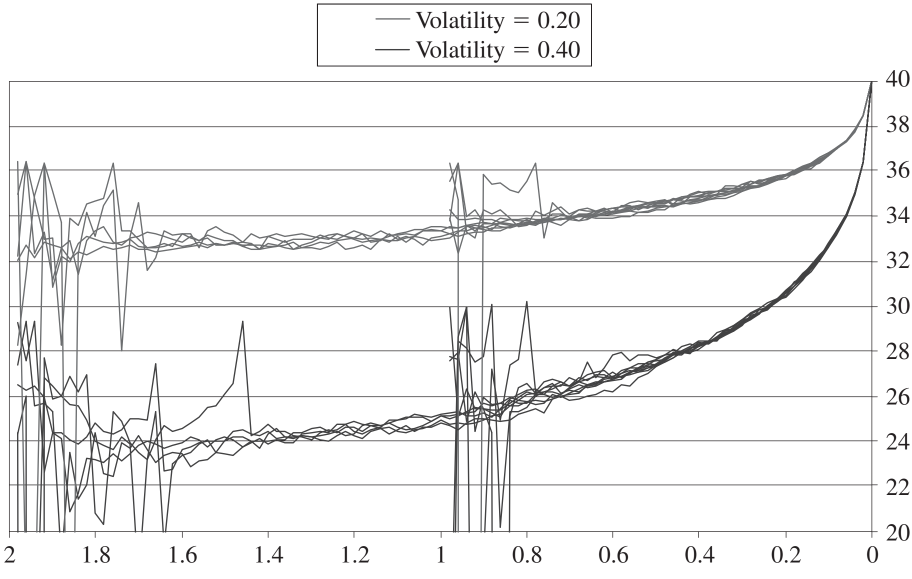
corresponding errors in Table 6, however among the early exercise differences five are now highly significantly outside the 95% confidence interval of the benchmark finite-difference estimate, which is not acceptable. Fortunately, all of the large deviations are captured by the duality gaps, which is important in cases where the Bermudan option value can be determined in no other way. Using independent re-runs is both an easier and more pragmatic way of determining the stability of the CV-LSM algorithm utilizing the fact that the regression pre-run as well as the valuation post-run can now be performed quite fast using only 1000 pairs of antithetic paths in each case.

6 The DCV-LSM approach

In this section we analyze the second of our suggested improvements to the LSM approach. Here we consider improving the LSM approach by applying a dispersion of the initial state variables from which the Monte Carlo paths used in the least-squares regression are generated. Combined with the first suggestion, we denote this suggestion as the DCV-LSM approach.

Given the variance of X_τ/β_τ the accuracy of (13) can only be improved by increasing the number of observations used in the least-squares regression. Remembering, that for a sample of N paths only the time t in-the-money states

FIGURE 5 Exercise boundary for the Bermudan put option with 50 exercise points per year, determined using the CV-LSM approach with 1000 pairs of antithetic paths. The short-term rate is 0.06, the upper curve is for $\sigma = 0.20$, and the lower curve is for $\sigma = 0.40$. The horizontal axis indicates time to expiry.



are used. Hence, we expect to use only the fraction $E_0^{\mathbb{Q}}[\mathbf{1}_{\{X_t/\beta_t>0\}}]$ of the paths at time t. The same applies to the accuracy of the other terms involved in estimating (24) in the CV-LSM approach outlined in the previous section.

Rather than increasing the number of paths, we suggest modifying the generation of paths, such that the fraction of time t in-the-money paths is increased. To this end, at least two suggestions often used in the context of Monte Carlo simulation spring to mind.

First, the method of importance sampling may be used to shift the distribution of the state variables generated at time t. Intuitively, the probability of the states in the neighborhood of the optimal exercise boundary should be increased in order to increase the accuracy of the LSM estimate where it is most critical. However, the boundary is inherently unknown, making this suggestion difficult to operationalize. Once the exercise boundary is determined, the importance sampling method may likely be a useful tool in the valuation step. This subject, which is closely related to the application of importance sampling for barrier options, is not considered here, however, several studies can be found.

Secondly, the method of stratification may be used to sample few paths in the time t out-of-the-money strata and many paths in the time t in-the-money strata.

TABLE 7 As Table 3, now using the CV-LSM approach for determining the exercise strategy based on 1000 pairs of antithetic paths.

			1000 pairs of antithetic paths, control variates			50 pairs of antithetic paths, 1000 subpaths with control variates	
S	σ	T	Early exercise value	S.E.	Early exercise diff.	Duality gap	S.E.
36	0.20	1	0.6365	0.0035	−0.0030	0.0012	0.0005
36	0.20	2	1.0730	0.0068	0.0042	0.0015	0.0005
36	0.40	1	0.3823	0.0045	0.0075	0.0110	0.0075
36	0.40	2	0.7883	0.0086	0.0184*	0.0102	0.0091
38	0.20	1	0.3991	0.0037	−0.0010	0.0002	0.0001
38	0.20	2	0.6399	0.0075	0.1143*	0.1172	0.0223
38	0.40	1	0.3077	0.0034	0.0055	0.0000	0.0000
38	0.40	2	0.7027	0.0077	−0.0136	0.0030	0.0012
40	0.20	1	0.2449	0.0031	0.0027	0.0017	0.0013
40	0.20	2	0.5194	0.0064	0.0092	0.0032	0.0014
40	0.40	1	0.2420	0.0028	0.0102*	0.0053	0.0039
40	0.40	2	0.5853	0.0064	0.0057	0.0050	0.0032
42	0.20	1	0.1519	0.0026	0.0005	0.0003	0.0002
42	0.20	2	0.3462	0.0055	0.0248*	0.0498	0.0160
42	0.40	1	0.2023	0.0025	0.0014	0.0009	0.0005
42	0.40	2	0.5031	0.0062	0.0056	0.0033	0.0019
44	0.20	1	0.0901	0.0019	0.0028	0.0034	0.0033
44	0.20	2	0.2544	0.0046	0.0062	0.0013	0.0008
44	0.40	1	0.1600	0.0025	0.0049	0.0131	0.0077
44	0.40	2	0.4115	0.0057	0.0277*	0.0171	0.0098

This method may also become difficult to generalize since stratification in high dimensions easily becomes inefficient.

Both of the above methods suggest sampling more in-the-money paths by altering the distribution of the paths generated. The suggestion made below does also involve changing the distribution, but using a simpler approach, which is easy to generalize. The idea applied is based on the fact that the optimal exercise strategy of an American or Bermudan option depends only on the information generated by the path leading to the current state, as summarized by $\mathcal{F}_t$, and not on the distribution of the paths. Hence, we suggest generating paths from an initial distribution of paths rather than the single point for which the option value is sought. The distribution should be chosen such that enough in-the-money paths are generated for all possible exercise time points. In particular for the longest time to expiry, ie, in the early parts of the paths, the number of observations should be increased. In contrast to the method of variance reduction, this approach actually implies an increase in the variance, however, with the goal of obtaining a globally better estimate of the continuation value and, in particular, the exercise strategy itself.

We suggest the following approach for satisfying the requirements above. By using a fictitious initial time point $-T_D < 0$ and the original initial state, we suggest simulating the state variables from this time point and forward. Since the underlying discounted asset prices of the model are martingales under $\mathbb{Q}$ by construction of the risk-neutral measure, the distribution at time 0 will reflect the volatility of the underlying assets while being centered around the values of the state variables for which the option value is sought.

6.1 Case study: DCV-LSM-based exercise strategies

Here we apply the DCV-LSM approach with an initial dispersion of the asset price given by

$$S_{t_0}^D = S_{t_0} \exp(-\tfrac{1}{2}\sigma^2 T_D + \sigma\sqrt{T_D}\varepsilon), \quad \varepsilon \sim N(0, 1) \tag{25}$$

We have determined the value of T_D by inspecting the LSM and CV-LSM exercise boundaries of Figures 3, 4 and 5. In all cases the irregularities of the exercise boundaries are concentrated in the region $t < T/2$. The dispersion of the state variables from $t = T/2$ and until expiry seems to provide sufficient in-the-money observations to estimate the exercise boundary. We therefore choose to set

$$T_D = \frac{T}{2}$$

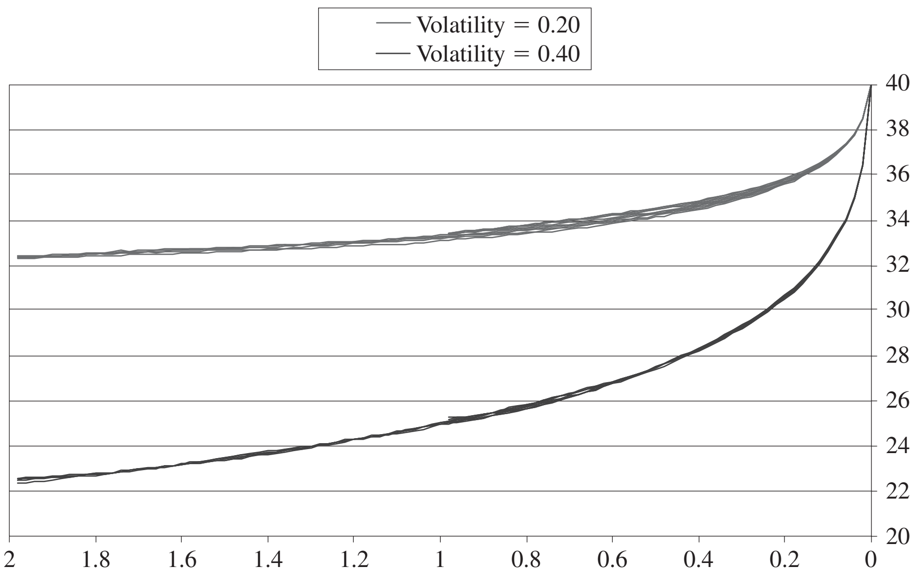
in the following examples.

Using 50 000 pairs of antithetic paths, each pair generated from an initial point dispersed according to (25), we show the resulting exercise boundary in Figure 6.

The exercise boundary is now very close to the benchmark finite difference exercise boundary. Furthermore, the exercise strategies of all scenarios with the same volatility are very close to each other, independent of the initial state from which the dispersed states have been generated. Finally, the exercise strategies estimated using only one year to expiry are now hard to distinguish from those estimated using two years to expiry. Theoretically, they should coincide; however, as the previous numerical examples have shown, this is not always the case in practice.

We now expect to produce early exercise values very close to the benchmark finite-difference values. This is confirmed by observing Table 8. Although two of the early exercise differences are outside a 95% confidence interval of the finite-difference estimate, they are all very small, nothing compared to the corresponding differences in Table 6. Again, using only 1000 pairs of antithetic paths, we cannot distinguish early exercise differences from the noise as measured by the standard errors. As expected the duality gaps of Table 8 are very small, confirming the accuracy of the exercise strategy shown in Figure 6.

FIGURE 6 Exercise boundary for the Bermudan put option with 50 exercise points per year, determined using the DCV-LSM approach with 50 000 pairs of antithetic paths. The short-term rate is 0.06, the upper curve is for $\sigma = 0.20$, and the lower curve is for $\sigma = 0.40$. The horizontal axis indicates time to expiry.



We are now close to the goal of obtaining a stable method for determining sufficiently accurate Bermudan option values using only 1000 pairs of antithetic paths in the estimation of the exercise strategy as well as in the valuation. In Figure 7 we have used only 1000 pairs of antithetic paths dispersed according to (25), to obtain the exercise boundary.

As in Figure 6, in Figure 7 we see no sharp fluctuations and no jagged curves. The two figures display the same stability; however, the latter with the random noise coming from using only 1000 pairs of antithetic paths.

The accuracy of the exercise strategy estimated is investigated in Table 9, where all early exercise values are within a 95% confidence interval of the benchmark finite-difference estimates. Furthermore, the duality gaps are accordingly small. All in all, it is a robust and efficient estimation of the Bermudan put option value.

7 Conclusion

We have shown that the application of control variates to the valuation of American or rather Bermudan options can be very effective if sampled at the exercise time rather than at expiry. Furthermore, unlike in Broadie and Glasserman (1997), the control variates are effective for out-of-the-money as well as for in-the-money options. They are also efficient in the case where they lead to a reduction

TABLE 8 As Table 3, now using the DCV-LSM approach for determining the exercise strategy based on 50 000 pairs of antithetic paths.

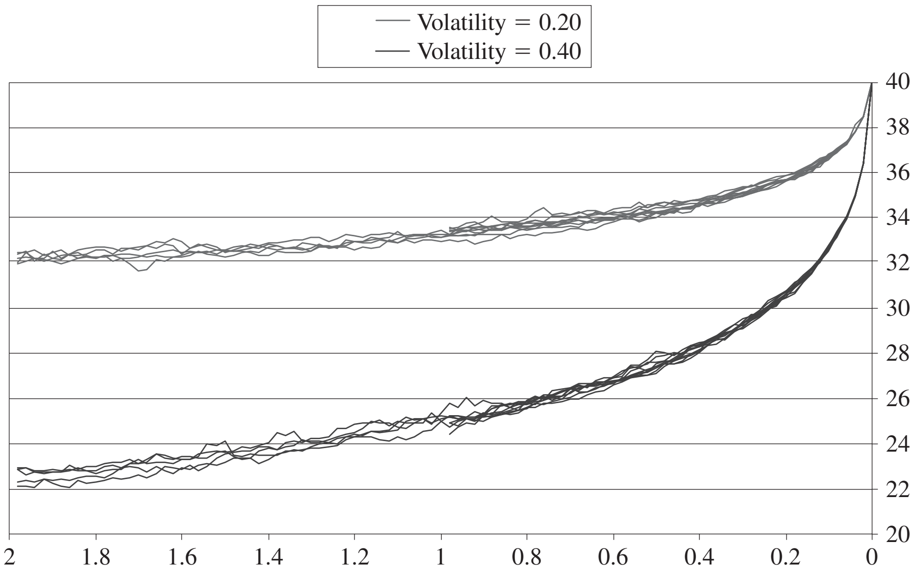
			50 000 pairs of antithetic paths, control variates			1000 pairs of antithetic paths, control variates			50 pairs of antithetic paths, 1000 subpaths with control variates	
S	σ	T	Early exercise value	S.E.	Early exercise diff.	Early exercise value	S.E.	Early exercise diff.	Duality gap	S.E.
36	0.20	1	0.6329	0.0005	0.0006	0.6319	0.0036	0.0016	0.0005	0.0002
36	0.20	2	1.0739	0.0010	0.0033*	1.0887	0.0067	−0.0115	0.0031	0.0006
36	0.40	1	0.3898	0.0006	0.0000	0.3925	0.0045	−0.0027	0.0001	0.0000
36	0.40	2	0.8074	0.0013	−0.0007	0.7933	0.0089	0.0134	0.0001	0.0001
38	0.20	1	0.3977	0.0005	0.0004	0.3998	0.0037	−0.0017	0.0003	0.0001
38	0.20	2	0.7528	0.0010	0.0014	0.7574	0.0070	−0.0032	0.0016	0.0004
38	0.40	1	0.3135	0.0005	−0.0003	0.3158	0.0037	−0.0026	0.0001	0.0000
38	0.40	2	0.6887	0.0011	0.0004	0.6952	0.0081	−0.0061	0.0001	0.0001
40	0.20	1	0.2475	0.0004	0.0001	0.2486	0.0030	−0.0010	0.0002	0.0001
40	0.20	2	0.5271	0.0009	0.0015	0.5386	0.0063	−0.0100	0.0004	0.0002
40	0.40	1	0.2527	0.0004	−0.0005	0.2568	0.0028	−0.0046	0.0000	0.0000
40	0.40	2	0.5929	0.0010	−0.0019	0.5919	0.0069	−0.0009	0.0001	0.0000
42	0.20	1	0.1521	0.0004	0.0003	0.1533	0.0025	−0.0009	0.0000	0.0000
42	0.20	2	0.3702	0.0008	0.0008	0.3695	0.0056	0.0015	0.0006	0.0003
42	0.40	1	0.2038	0.0004	−0.0001	0.1987	0.0026	0.0050	0.0001	0.0000
42	0.40	2	0.5070	0.0009	0.0017	0.5011	0.0065	0.0076	0.0002	0.0001
44	0.20	1	0.0928	0.0003	0.0001	0.0912	0.0020	0.0017	0.0000	0.0000
44	0.20	2	0.2590	0.0007	0.0016*	0.2576	0.0047	0.0030	0.0003	0.0001
44	0.40	1	0.1649	0.0004	0.0000	0.1678	0.0025	−0.0029	0.0000	0.0000
44	0.40	2	0.4399	0.0008	−0.0007	0.4390	0.0059	0.0002	0.0000	0.0000

in the number of paths that reduces the computational task by more than what is required to compute the control variates.

For the single asset Bermudan put option we have to compute the Black–Scholes values of the corresponding European options, for each path that is exercised early. This computational effort should be weighed against the reduction of the number of Monte Carlo paths needed to reach a given accuracy. This not only involves the generation of random variables and the construction of asset price paths and payoffs, but also the computation of continuation values to determine when to exercise or not. When using the LSM method, the latter part typically involves valuing certain polynomials, or as in our case valuing the European options. This feature enables us to reuse the value of the European option in the case that Bermudan option is exercised early. The computational cost of determining the coefficients is, in this context, considered fixed, as the use of control variates in our current setting does not improve the regression part of the Monte Carlo valuation, which is the subject of the second part of this paper.

For the Bermudan max-call option in the multi-asset Black–Scholes model, we have to compute several single asset European call options as well as several two-asset European max-call options in the case of early exercise. Note, that if we could have easily computed the multi-asset European max-call option for more

FIGURE 7 Exercise boundary for the Bermudan put option with 50 exercise points per year, determined using the DCV-LSM approach with 1000 pairs of antithetic paths. The short-term rate is 0.06, the upper curve is for $\sigma = 0.20$, and the lower curve is for $\sigma = 0.40$. The horizontal axis indicates time to expiry.



than two assets we would have used it rather than this combination of European options. For the Bermudan max-call option, this computation effort should be compared to generating multi-asset price paths, and computing the early exercise values, which now typically involves an even higher number of basis functions.

In the second part of this paper we have developed two improvements to the LSM approach of Longstaff and Schwartz (2001). The first is the CV-LSM approach, where we suggest applying control variates to the least-squares projections to obtain a more efficient estimate. This improvement is highly inspired by the application of control variates to the Monte Carlo valuation, given a predetermined exercise strategy, as investigated in the first part of the paper. The second is the DCV-LSM approach, where we suggest dispersing the initial state variables and generate paths from an initial distribution rather than a single point.

The convergence of the LSM approach as both the number of basis functions and the number of paths go to infinity is well established in Clement *et al* (2002). However, the improvements suggested here focus on the exercise strategy for a finite set of basis functions and a finite sample of paths.

We show that the improved accuracy obtained by the CV-LSM approach results in better exercise strategies, which in turn imply lower and upper bound

TABLE 9 As Table 3, now using the DCV-LSM approach for determining the exercise strategy based on 1000 pairs of antithetic paths.

			1000 pairs of antithetic paths, control variates			50 pairs of antithetic paths, 1000 subpaths with control variates	
S	σ	T	Early exercise value	S.E.	Early exercise diff.	Duality gap	S.E.
36	0.20	1	0.6295	0.0037	0.0040	0.0002	0.0001
36	0.20	2	1.0654	0.0073	0.0118	0.0062	0.0008
36	0.40	1	0.3820	0.0045	0.0078	0.0002	0.0001
36	0.40	2	0.7965	0.0086	0.0102	0.0009	0.0004
38	0.20	1	0.3980	0.0039	0.0001	0.0011	0.0004
38	0.20	2	0.7436	0.0074	0.0106	0.0044	0.0007
38	0.40	1	0.3137	0.0035	−0.0005	0.0004	0.0002
38	0.40	2	0.6991	0.0079	−0.0100	0.0003	0.0002
40	0.20	1	0.2538	0.0031	−0.0061	0.0004	0.0003
40	0.20	2	0.5274	0.0065	0.0012	0.0011	0.0004
40	0.40	1	0.2517	0.0028	0.0005	0.0001	0.0001
40	0.40	2	0.5856	0.0068	0.0054	0.0003	0.0001
42	0.20	1	0.1535	0.0025	−0.0011	0.0001	0.0001
42	0.20	2	0.3751	0.0054	−0.0041	0.0007	0.0003
42	0.40	1	0.2091	0.0027	−0.0053	0.0002	0.0001
42	0.40	2	0.5077	0.0059	0.0010	0.0014	0.0004
44	0.20	1	0.0900	0.0021	0.0029	0.0002	0.0002
44	0.20	2	0.2591	0.0050	0.0015	0.0008	0.0004
44	0.40	1	0.1618	0.0025	0.0031	0.0001	0.0001
44	0.40	2	0.4430	0.0061	−0.0038	0.0006	0.0002

Monte Carlo estimates closer to the true value. We furthermore show that the dispersion of paths implied by the DCV-LSM approach improves the stability of the projections and the overall accuracy of the exercise strategy.

In total, the combined improvements encapsulated in the DCV-LSM approach, provide a more accurate and stable method for determining the exercise strategies of American or Bermudan options in a Monte Carlo setup. The obtained accuracy and stability can be traded-off for increased speed in determining the exercise strategy. Combined with the application of control variates to the Monte Carlo valuation of the American or Bermudan option, we have produced a very fast method for valuing the early exercise feature of such options.

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