Valuing American Options by Simulation

- Least Square and Machine learning Approaches

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AMERICAN OPTION

OVERVIEW

- A general class of continuous-time American option pricing problems can be formulated by specifying a process $U(t)$, $0 < t < T$, representing the discounted payoff from exercise at time t, and a class of admissible stopping times τ_0 with values in $[0, T]$. The problem, then, is to find the optimal expected discounted payoff

$$sup_{\{\tau \in \tau_0\}} E[U(\tau)]$$

- We focus on American put options. In this case, consider a put with strike price K on a single underlying asset $S(t)$. The risk-neutral dynamics of S are modeled as geometric Brownian motion $GBM(r, \sigma^2)$, with r a constant risk-free interest rate. Suppose the option expires at T, its value at time 0 is then

$$sup_{\{\tau \in \tau_0\}} E[e^{-r\tau} \left(K - S(\tau)\right)^+]$$

OPTIMAL STOPPING PROBLEM

OVERVIEW

- Optimal Stopping problem for Stochastic Process X_t

$$V(x) = sup_{\{\tau \in \tau_0\}} E[U(X_\tau)|X_0 = x]$$

- American option pricing is an optimal stopping problem.
  - X_t is the stochastic process for underlying security's price
  - x is underlying security's current price
  - τ is set of exercise times corresponding to various stopping policies
  - $V(\cdot)$ is American option price as function of underlying current price
  - $U(\cdot)$ is the discounted option payoff.

PRICE AMERICAN OPTIONS

OVERVIEW

- Traditional ways
  1. Black-Scholes: Due to the optimal stopping problem, this is technically impossible.
  2. Lattice Tree(LT): Binomial or binary (recombining) tree pricing.
  3. Finite difference(FD): solving the PDE with free boundary conditions.
  4. Monte Carlo: Least Squares MC (**LSM**) model, etc.
- Machine Learning methods
  1. Gaussian Process Regression (**GPR**)
  2. Lease Square Policy Iteration (**LSPI**)
  3. Fitted Q-iteration (**FQI**)

LEAST SQUARE APPROACH

- Introduced and developed by Longstaff-Schwartz
- Approximate the continuation value using a least square approach
- the resulting price of the option is a lower bound on the option's true price

input : $SP[0:m, 0:n+1]$
output: option price at $t = 0$

CF$[0:m] \leftarrow [Payoff(s_{i,n})$ for i in range$(m)]$;
for $j \leftarrow n-1$ **to** 1 **do**
    CF$[0:m] \leftarrow CF[0:m] * e^{-r_{t_j}(t_{j+1}-t_j)}$;
    $X \leftarrow [\phi(s_{i,j})$ for i in range(m) if $Payoff(s_{i,j}) > 0]$;
    $Y \leftarrow [$CF$[i]$ for i in range(m) if $Payoff(s_{i,j}) > 0]$;
    $w \leftarrow (X^T \cdot X)^{-1} \cdot X^T \cdot Y$;
    for $i \leftarrow 0$ **to** $m-1$ **do**
        if $Payoff(s_{i,j}) > w \cdot \phi(s_{i,j})$ **then**
            CF$[i] \leftarrow Payoff(s_{i,j})$
        end
    end
    Return $e^{-r_{t_0}}(t_1 - t_0) \cdot \text{mean}(CF[0:M])$
end

Algorithm 1: Least Square Approach

GAUSSIAN PROCESS REGRESSION

APPROACHES

- Proposed by Gang Mu, Teodor Godina, Antonio Maffia, Yong Chao Sun.
- Approximate the continuation value using Gaussian process regression, instead of least square method.
- The authors claim a lower standard errors than the LSM.

input : $SP[0:m, 0:n+1]$

output: option price at $t = 0$

$\mathsf{CF}[0:m] \leftarrow [Payoff(s_{i,n})$ for i in range$(m)]$;

for $j \leftarrow n-1$ **to** 1 **do**

  $\mathsf{CF}[0:m] \leftarrow CF[0:m] * e^{-r_{t_j}(t_{j+1}-t_j)}$;

  $X \leftarrow [\phi(s_{i,j})$ for i in range(m) if $Payoff(s_{i,j}) > 0]$;

  $Y \leftarrow [\mathsf{CF}[i]$ for i in range(m) if $Payoff(s_{i,j}) > 0]$;

  $GPR \leftarrow \mathsf{fit}(X, Y)$;

  for $i \leftarrow 0$ **to** $m-1$ **do**

    if $Payoff(s_{i,j}) > GPR(s_{i,j})$ **then**

      $\mathsf{CF}[i] \leftarrow Payoff(s_{i,j})$

    end

  end

  Return $e^{-r_{t_0}}(t_1 - t_0) \cdot \text{mean}(CF[0:M])$

end

Algorithm 2: Kriging Approach

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REINFORCEMENT LEARNING APPROACHES

- American Option Pricing is an Optimal Stopping problem, and hence a Markov decision process, so can be tackled with reinforcement learning algorithms.
  - $State$ is a suitable function of the history of stock prices X_t
  - $Action$ is Boolean: Stop or Continue
  - $Reward$ always 0, except upon Stopping (when it is $= h(X_\tau)$, the payoff)
  - $State - transitions$ governed by Underlying Price's Stochastic Process
- We outline two RL Algorithms:
  - Least Squares Policy Iteration (LSPI)
  - Fitted Q-Iteration (FQI)

INGREDIENTS FOR APPROACHES

- m Monte-Carlo paths indexed $i = 0,1, \dots, m-1$
- $n+1$ time steps indexed $j = n, n-1, \dots, 1,0$
- Infinitesimal risk-free rate at time t_j denoted r_{t_j}
- Simulation paths of prices of underlying as input 2-dim array $SP[i, j]$
- At each time step, $CF[i]$ is PV of current+future cashflow for path i
- $s_{i,j}$ denotes state for $(i, j) \stackrel{\text{def}}{=}$ (time t_j, price history $SP[i, :(j+1)]$)
- $Payoff(s_{i,j})$ denotes Option payoff at (i, j)
- $\phi_0(s_{i,j}), \dots, \phi_{r-1}(s_{i,j})$ represents feature functions (of state $s_{i,j}$)
- $w_0, \dots, w_{r-1}$ are the regression weights
- Regression function $f(s_{i,j}) = w \cdot \phi(s_{i,j}) = \sum_{l=0}^{r-1} w_l \cdot \phi_l(s_{i,j})$
- $f(\cdot)$ is estimate of continuation value for in-the-money states

REINFORCEMENT LEARNING APPROACHES

APPROACHES

input : $SP[0:m, 0:n+1]$
output: option price at $t = 0$

`Comment:` $s_{i,j}$ is short-hand for state at (i, j);
`Comment:` A is an $r \times r$ matrix, b and w are r-length vectors;
`Comment:` $A \leftarrow \phi(s_{i,j}) \cdot (\phi(s_{i,j}) - \gamma \mathbb{1}_{w \cdot \phi(s_{i,j+1}) \geq Payoff(s_{i,j+1})} * \phi(s_{i,j+1}))^T$;
`Comment:` $b_{i,j} \leftarrow \gamma \mathbb{1}_{w \cdot \phi(s_{i,j+1}) < Payoff(s_{i,j+1})} * Payoff(s_{i,j+1}) * \phi(s_{i,j})$;

$A \leftarrow 0, B \leftarrow 0, w \leftarrow 0$;
for $i \leftarrow 0$ **to** $m-1$ **do**
  for $j \leftarrow 0$ **to** $n-1$ **do**
    $Q \leftarrow Payoff(s_{i,j+1})$;
    if $j < n-1$ & $Q \leq w \cdot \phi(s_{i,j+1})$ **then**
      $P \leftarrow \phi(s_{i,j+1})$;
    else
      $P \leftarrow 0$;
    end
    if $Q > w \cdot P$ **then**
      $R \leftarrow Q$;
    else
      $R \leftarrow 0$;
    end
    $A \leftarrow A + \phi(s_{i,j}) \cdot (\phi(s_{i,j}) - e^{-r_{t_j}(t_{j+1}-t_j)} * P)$;
    $B \leftarrow B + e^{-r_{t_j}(t_{j+1}-t_j)} * R * \phi(s_{i,j})$
  end
  if $(i+1)$ % $Batch\ Size == 0$ **then**
    $w \leftarrow A^{-1} \cdot b, A \leftarrow 0, b \leftarrow 0$
  end
end

Algorithm 3: Least Squares Policy Iteration Approach

input : $SP[0:m, 0:n+1]$
output: option price at $t = 0$

`Comment:` $s_{i,j}$ is short-hand for state at (i, j);
`Comment:` A is an $r \times r$ matrix, b and w are r-length vectors;
`Comment:` $A \leftarrow \phi(s_{i,j}) \cdot (\phi(s_{i,j}))^T$;
`Comment:` $b_{i,j} \leftarrow \gamma \max(Payoff(s_{i,j+1}), w \cdot \phi(s_{i,j+1})) * \phi(s_{i,j}))$;

$A \leftarrow 0, B \leftarrow 0, w \leftarrow 0$;
for $i \leftarrow 0$ **to** $m-1$ **do**
  for $j \leftarrow 0$ **to** $n-1$ **do**
    $Q \leftarrow Payoff(s_{i,j+1})$;
    if $j < n-1$ **then**
      $P \leftarrow \phi(s_{i,j+1})$;
    else
      $P \leftarrow 0$;
    end
    $A \leftarrow A + \phi(s_{i,j}) \cdot (\phi(s_{i,j}))^T$;
    $B \leftarrow B + \max(Payoff(s_{i,j+1}), w \cdot P) * \phi(s_{i,j})$
  end
  if $(i+1)$ % $Batch\ Size == 0$ **then**
    $w \leftarrow A^{-1} \cdot b, A \leftarrow 0, b \leftarrow 0$
  end
end

Algorithm 4: Fitted Q-Iteration Approach

NUMERICAL EXAMPLES

We validate and evaluate the above approaches under two cases:

1. Simple American Put
2. Heston Model with one underlier

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_{1t}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_{2t}$$

and dW_{1t}, dW_{2t} are Brownian motions with correlation ρ, i.e.,

$$dW_{1t} dW_{2t} = \rho dt$$

- The parameters in the above equations represent the following:
  - r is the domestic risk-free interest rate.
  - θ is the long variance, or long run average price variance; as t tends to infinity, the expected value of V_t tends to θ.
  - κ is the rate at which V_t reverts to θ.
  - σ is the volatility of the volatility
  - If the parameters obey the following Feller condition $2\kappa\theta > \sigma^2$ and the initial variance $V_0 > 0$, then V_t is strictly positive.

SIMPLE AMERICAN PUT

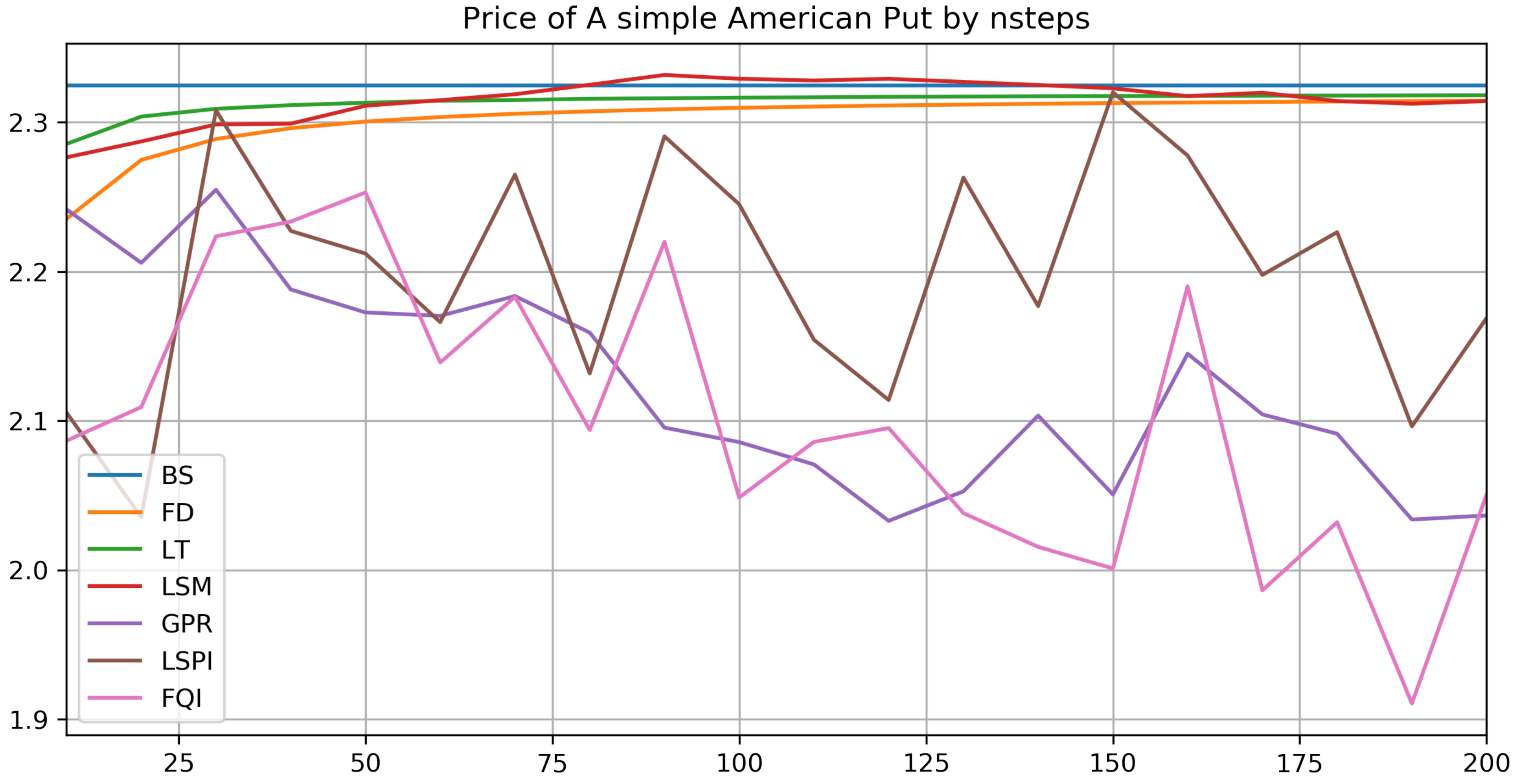
EXPERIMENTS

S	sigma	T	LT-CV	LT	FD	LSM	GPR	FQI	LSPI
36	0.2	1	4.4840	4.4845	4.4644	4.48 (0.02%)	4.52 (0.87%)	4.38 (-2.37%)	4.37 (-2.53%)
		2	4.8501	4.8462	4.8288	4.83 (-0.31%)	4.75 (-2.12%)	4.85 (0.01%)	4.79 (-1.15%)
	0.4	1	7.1098	7.0997	7.0765	7.09 (-0.21%)	6.86 (-3.53%)	6.87 (-3.39%)	6.78 (-4.64%)
		2	8.5195	8.5079	8.4854	8.51 (-0.13%)	7.93 (-6.95%)	7.94 (-6.77%)	8.22 (-3.56%)
38	0.2	1	3.2568	3.2529	3.2359	3.25 (-0.24%)	3.19 (-2.08%)	3.17 (-2.78%)	3.01 (-7.46%)
		2	3.7540	3.7482	3.7336	3.74 (-0.26%)	3.57 (-4.99%)	3.57 (-4.90%)	3.67 (-2.17%)
	0.4	1	6.1533	6.1805	6.1221	6.13 (-0.30%)	6.16 (0.08%)	5.89 (-4.28%)	5.86 (-4.73%)
		2	7.6713	7.6891	7.6456	7.69 (0.20%)	7.22 (-5.85%)	6.89 (-10.13%)	7.41 (-3.35%)
40	0.2	1	2.3245	2.3130	2.3004	2.31 (-0.58%)	2.23 (-4.02%)	2.25 (-3.08%)	2.21 (-4.84%)
		2	2.8965	2.8851	2.8741	2.89 (-0.23%)	2.69 (-7.25%)	2.55 (-11.94%)	2.77 (-4.21%)
	0.4	1	5.3260	5.3028	5.2865	5.32 (-0.21%)	5.09 (-4.52%)	5.08 (-4.68%)	5.1 (-4.30%)
		2	6.9351	6.9137	6.8937	6.94 (0.04%)	6.3 (-9.09%)	6.86 (-1.06%)	7.32 (5.51%)
42	0.2	1	1.6204	1.6239	1.6051	1.62 (-0.10%)	1.54 (-4.97%)	1.59 (-1.94%)	1.6 (-1.13%)
		2	2.2187	2.2172	2.2029	2.23 (0.31%)	2.08 (-6.27%)	1.85 (-16.70%)	2.06 (-7.24%)
	0.4	1	4.5884	4.6137	4.5578	4.59 (0.10%)	4.22 (-8.00%)	4.41 (-3.91%)	4.43 (-3.36%)
		2	6.2464	6.2643	6.2200	6.27 (0.38%)	5.85 (-6.31%)	5.3 (-15.17%)	5.96 (-4.63%)
44	0.2	1	1.1118	1.1212	1.1005	1.12 (0.30%)	1.03 (-7.56%)	1.13 (1.30%)	1.09 (-1.82%)
		2	1.6921	1.6973	1.6816	1.69 (0.17%)	1.46 (-13.49%)	1.33 (-21.44%)	1.47 (-13.36%)
	0.4	1	3.9529	3.9616	3.9243	3.96 (0.17%)	3.39 (-14.31%)	3.86 (-2.29%)	3.87 (-1.98%)
		2	5.6511	5.6516	5.6158	5.67 (0.26%)	4.57 (-19.05%)	4.64 (-17.83%)	5.29 (-6.41%)

Estimated American put option price computed by multiple numerical methods. Lattice method with control variate is used as the benchmark to compared with. The option is exercisable 50 times per year. In this experiment, the strike price of the put is 40, the short-term interest rate is 0.06. and the underlying stock price S, the volatility of returns σ, and the number of years until the final expiration of the option T are as indicated. 100,000 paths are simulated for LT-CV, LT, FD and LSM. The three machine learning methods, each simulated 1,000 paths because of limit of time.

SIMPLE AMERICAN PUT

EXPERIMENTS



$$S_0 = 40, \sigma = 0.2, T = 1$$

SIMPLE AMERICAN PUT

EXPERIMENTS

S	sigma	T	GPR	FQI	LSPI	LSM
36	0.2	1	0.0966	0.0455	0.1527	0.7777
		2	0.1195	0.0261	0.0416	0.6849
	0.4	1	0.1926	0.3042	0.4242	0.5161
		2	0.2980	0.6438	1.1146	0.6494
38	0.2	1	0.0852	0.2275	0.2124	0.1770
		2	0.1450	0.1178	0.2959	0.1332
	0.4	1	0.2167	0.9136	1.0117	0.1291
		2	0.2233	0.8447	0.9468	0.3068
40	0.2	1	0.0766	1.5490	1.2054	0.2250
		2	0.1114	0.9847	0.9639	0.2112
	0.4	1	0.1771	0.4988	0.7561	0.2251
		2	0.2447	1.0665	1.2114	0.0274
42	0.2	1	0.0666	0.0905	0.3514	0.4079
		2	0.0959	0.5022	0.5380	0.4005
	0.4	1	0.1805	0.9311	0.8266	0.4738
		2	0.2010	0.5504	1.0005	0.2225
44	0.2	1	0.0522	0.4689	0.4798	0.4732
		2	0.0966	0.3125	0.4816	0.5238
	0.4	1	0.1943	0.6032	0.7278	0.6419
		2	0.2926	0.0931	0.5602	0.4323

Standard deviation by 50 independent runs each.

HESTON MODEL – ONE UNDERLIER

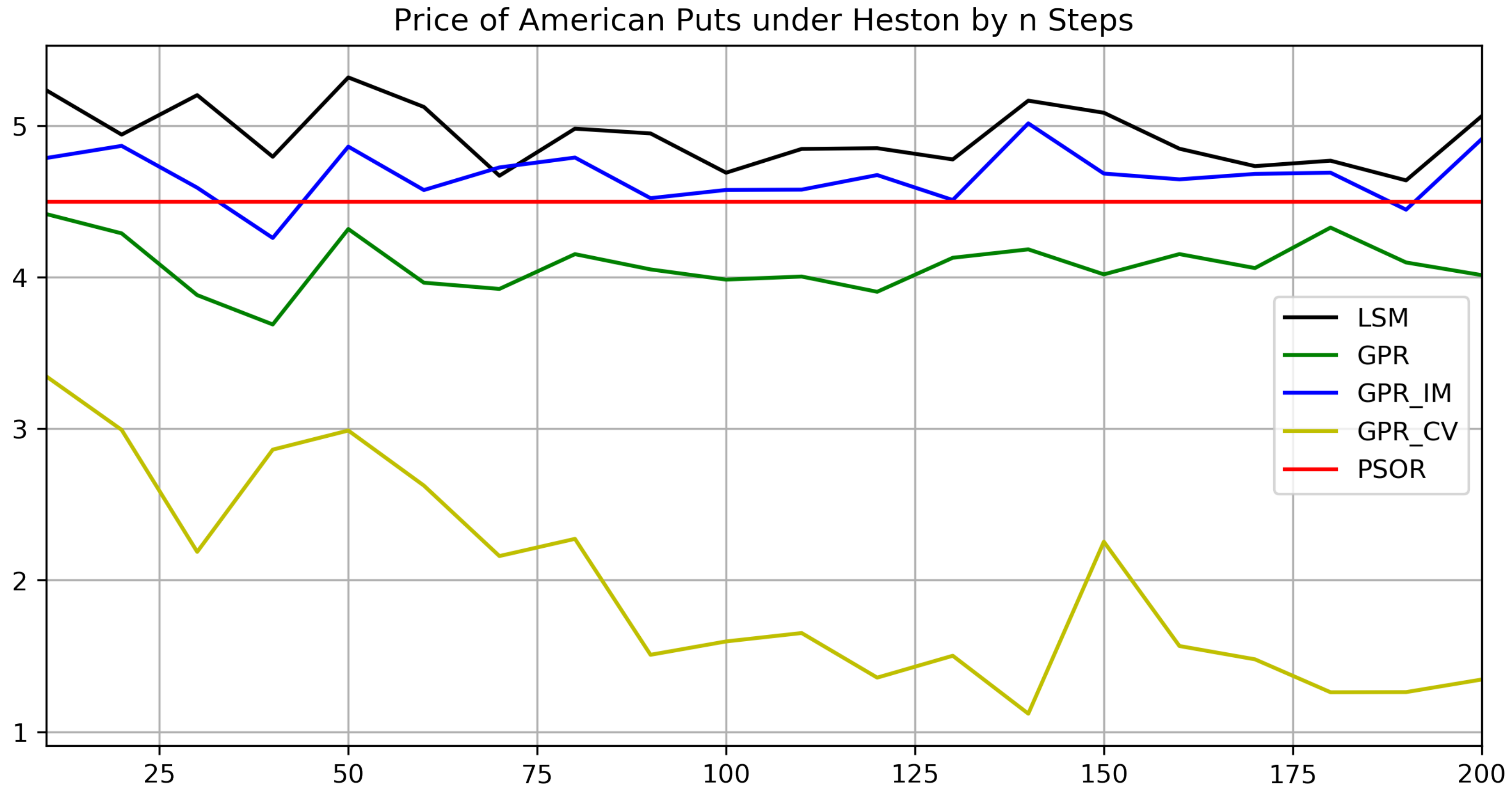
EXPERIMENTS

S0	sigma	T	PSOR	LSM	GPR	GPR_IM	GPR_CV
90	0.04	0.25	10.1229	10.76 (6.27%)	10.62 (4.88%)	10.25 (1.30%)	10.03 (-0.89%)
		0.5	10.5667	10.9 (3.19%)	10.38 (-1.79%)	10.41 (-1.47%)	9.86 (-6.67%)
	0.09	0.25	10.9573	11.5 (4.98%)	10.88 (-0.73%)	10.81 (-1.39%)	10.01 (-8.67%)
		0.5	11.7658	11.88 (0.99%)	11.58 (-1.61%)	11.51 (-2.14%)	9.92 (-15.66%)
	0.16	0.25	12.1200	12.22 (0.79%)	11.53 (-4.90%)	11.11 (-8.31%)	9.92 (-18.16%)
		0.5	13.2329	14.2 (7.34%)	12.07 (-8.77%)	12.02 (-9.15%)	10.07 (-23.93%)
100	0.04	0.25	3.4813	3.61 (3.60%)	3.4 (-2.37%)	3.72 (6.72%)	0.96 (-72.44%)
		0.5	4.6645	5.13 (9.99%)	3.8 (-18.54%)	5.05 (8.29%)	0.86 (-81.56%)
	0.09	0.25	4.4961	5.23 (16.22%)	3.93 (-12.68%)	4.78 (6.27%)	1.68 (-62.68%)
		0.5	6.2573	6.56 (4.80%)	5.3 (-15.34%)	5.99 (-4.23%)	2.73 (-56.41%)
	0.16	0.25	6.4933	6.69 (3.06%)	5.25 (-19.21%)	5.98 (-7.84%)	2.39 (-63.19%)
		0.5	8.0073	8.11 (1.25%)	6.39 (-20.25%)	7.21 (-9.92%)	3.7 (-53.82%)
110	0.04	0.25	0.8417	0.88 (5.04%)	0.76 (-9.14%)		0 (-99.54%)
		0.5	1.7875	1.41 (-20.91%)	1.14 (-36.11%)		0.13 (-92.73%)
	0.09	0.25	1.8641	1.55 (-17.08%)	1.27 (-31.81%)		0.06 (-97.00%)
		0.5	3.0673	2.66 (-13.20%)	2.07 (-32.54%)		0.04 (-98.76%)
	0.16	0.25	3.1470	3.47 (10.33%)	2.32 (-26.34%)		0.05 (-98.40%)
		0.5	4.6232	4.9 (5.90%)	3.48 (-24.64%)		0.72 (-84.43%)

- Convergence of American put option prices computed by Least Square Method, Gaussian process regression, Gaussian process regression only in the money path and Gaussian process regression with control variate.
- Projected Successive Over Relaxation(PROR) is used as benchmark.
- $K = 100, r = 0.05, \kappa = 3, \theta = 0.04, \sigma = 0.1, \rho = -0.7, N = 100, M = 1000$

HESTON MODEL – ONE UNDERLIER

EXPERIMENTS



$$S_0 = 100, V_0 = 0.09, T = 0.25$$

Summary

- In this project, several numerical methods for American option pricing have been studied, namely the least square method, the Gaussian process regression approach, Least square policy iteration and Fitted Q-iteration method.
  - The LSM is the best among all by accuracy and computational time.
  - GPR performs different from what the original paper states, especially when it is with control variate.
  - LSPI and FPI performs fair enough, which is better than GPR but worse than LSM in both accuracy and running time.
- The machine learning techniques has a fluctuated estimation as time step increases, while the LSM converge quick to a fixed value.
- In terms of simulation error, LSM is the best. LSPI the second and then FQI. GPR is the worst.

FUTURE WORK

- Use low-discrepancy points to reduce variance.
- Try other regression models rather than Gaussian process regression, for example ANN. Discussed with Danny and Victor, they have some inspiring results.
- The LSM is more suitable for multi-dimension problems according to Professor Adam. May test the performance of different approaches under multi-dimensional cases, for example, American max call on multiple assets.

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