Ch 7. PARAMETER ESTIMATION

Time Series Analysis

For an observed series $\{Z_t\}$, assuming that we have specified the model ARIMA(p,d,q) by Chapter 6, and taken the d-th difference to get an stationary ARMA(p,q) series $\{Y_t = \nabla^d Z_t\}$, we will estimate the parameters of the ARMA(p,q) model $\{Y_t\}$ in this chapter.

7.1 The Method of Moments (MM)

The method of moments is one of the easiest ways to estimate the parameters. We equate sample moments to corresponding theoretical moments and solve the equations to obtain estimates of unknown parameters.

7.1.1. AR(p) Models

Examples.

1. **AR(1).** Theoretically $\rho_1 = \phi$. Then ρ_1 is estimated by r_1 in the method of moments. So ϕ can be estimated by:

$$\hat{\phi} = r_1.$$

2. **AR(2).** Replace ρ_k by r_k in Yule-Walker equations:

$$r_1 = \phi_1 + r_1\phi_2, \qquad r_2 = r_1\phi_1 + \phi_2.$$

Solve the system and we get the estimation

$$\hat{\phi}_1 = \frac{r_1(1 - r_2)}{1 - r_1^2}, \qquad \hat{\phi}_2 = \frac{r_2 - r_1^2}{1 - r_1^2}.$$

AR(p). For the general AR(p) case, we replace ρ_k by r_k in Yule-Walker equations to get

$$\begin{cases} \phi_1 + r_1\phi_2 + r_2\phi_3 + \cdots + r_{p-1}\phi_p & = r_1 \\ r_1\phi_1 + \phi_2 + r_1\phi_3 + \cdots + r_{p-2}\phi_p & = r_2 \\ r_2\phi_1 + r_1\phi_2 + \phi_3 + \cdots + r_{p-3}\phi_p & = r_3 \\ \qquad\qquad \vdots & \vdots \\ r_{p-1}\phi_1 + r_{p-2}\phi_2 + r_{p-3}\phi_3 + \cdots + \phi_p & = r_p \end{cases} \tag{1}$$

Then solve the linear system to get the **Yule-Walker estimates** $\hat{\phi}_1, \cdots, \hat{\phi}_p$.

7.1.2 MA(q) Models and ARMA(p,q) Models

Examples

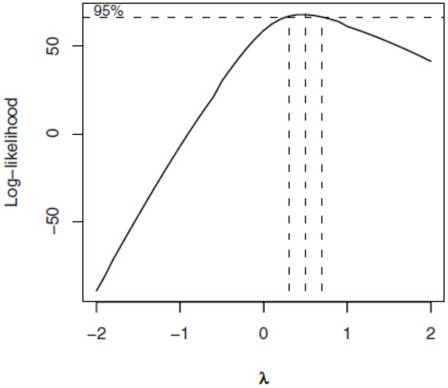
1. **MA(1).** Theoretically $\rho_1 = -\frac{\theta}{\theta^2+1}$. We replace ρ_1 by r_1 and solve the quadratic equation for θ. The only invertible solution is $\hat{\theta} = \dfrac{-1+\sqrt{1-4r_1^2}}{2r_1}$.

2. **ARMA(1,1).** Theoretically $\rho_k = \dfrac{(1-\theta\phi)(\phi-\theta)}{1-2\theta\phi+\theta^2}\phi^{k-1}$ for $k \geq 1$. So ϕ can be estimated by $\hat{\phi} = r_2/r_1$. Then we use $r_1 = \dfrac{(1-\theta\hat{\phi})(\hat{\phi}-\theta)}{1-2\theta\hat{\phi}+\theta^2}$ to find an invertible solution $\hat{\theta}$.

- For ARMA(p,q) models, the method of moments results in solving nonlinear numerical equations.
- In general, the estimators are very inefficient for models containing MA terms.

Now consider the **Canadian hare abundance series**.

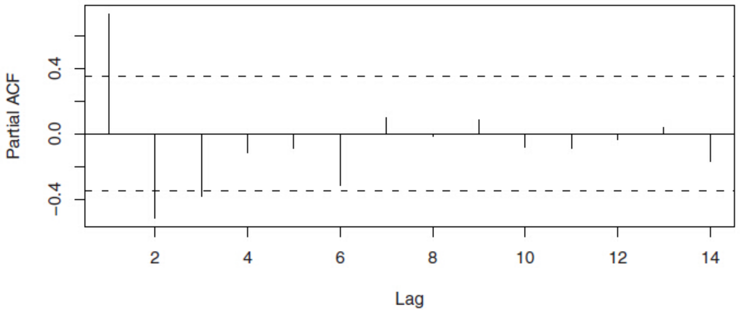
Exhibit 6.27 Box-Cox Power Transformation Results for Hare Abundance




```
> win.graph(width=3,height=3,pointsize=8)
> data(hare); BoxCox.ar(hare)
```

Exhibit 6.27 showed that `hare`$^{.5}$ is close to a stationary series.

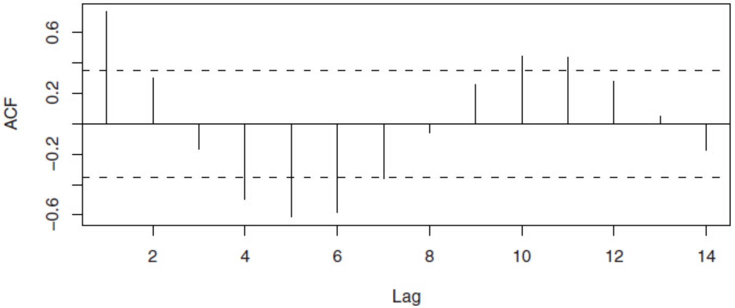
Exhibit 6.29 Sample Partial ACF for Square Root of Hare Abundance




```
> pacf(hare^.5)
```

Exhibit 6.29 showed that hare$^{.5}$ may be modeled by AR(2) or AR(3). Let us model it as AR(2) here.

Exhibit 6.28 Sample ACF for Square Root of Hare Abundance




```
> acf(hare^.5)
```

Exhibit 6.28 showed that the first two sample ACF of hare$^{.5}$ are $r_1 = 0.736$ and $r_2 = 0.304$.

Exhibit 6.28 showed that the first two sample ACF of `hare`$^{.5}$ are $r_1 = 0.736$ and $r_2 = 0.304$.

The method-of-moments estimates of ϕ_1 and ϕ_2 are

$$\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2} = 1.1178, \qquad \hat{\phi}_2 = \frac{r_2 - r_1^2}{1-r_1^2} = -0.519.$$

The sample mean is 5.82; the sample variance is $s^2 = 5.88$. So the noise variance is

$$\hat{\sigma}_e^2 = (1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2)s^2 = 1.97.$$

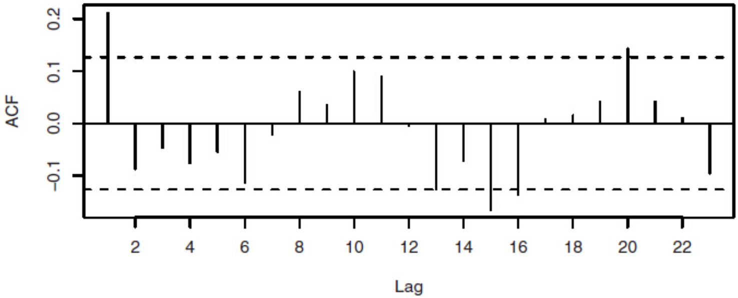
The estimated model of `hare` is then

$$\sqrt{Y_t} - 5.82 = 1.1178(\sqrt{Y_{t-1}} - 5.82) - 0.519(\sqrt{Y_{t-2}} - 5.82) + e_t$$

with $\sigma_e^2 = 1.97$.

The Oil Price series `oil.price` $\sim \{Y_t\}$

Exhibit 6.32 Sample ACF of Difference of Logged Oil Prices




```
> acf(as.vector(diff(log(oil.price))),xaxp=c(0,22,11))
```

Exhibit 6.32 showed that `diff(log(oil.price))` $\approx$ MA(1). The sample ACF value $r_1 = 0.212$. So the method-of-moments estimate of θ is

$$\hat{\theta} = \frac{-1 + \sqrt{1 - 4(0.212)^2}}{2(0.212)} = -0.222.$$

The sample mean and the sample variance of `diff(log(oil.price))` is 0.004 and 0.0072, respectively. Therefore, the estimated model is

$$\nabla \log(Y_t) = 0.004 + e_t + 0.222 e_{t-1},$$

with estimated noise variance

$$\hat{\sigma}_e^2 = \frac{s^2}{1 + \hat{\theta}^2} = \frac{0.0072}{1 + (-0.222)^2} = 0.00686.$$

The standard error of the sample mean of `diff(log(oil.price))` can be estimated by $\frac{\hat{\gamma}_0}{n}[1 + 2(1 - 1/n)r_1] \approx 0.0065$. So the observed sample mean 0.004 is not significantly different from 0. We remove the intercept term and get the final model:

$$\log(Y_t) = \log(Y_{t-1}) + e_t + 0.222 e_{t-1}.$$

7.2 Least Squares Estimation (CSS)

We introduce a possibly nonzero mean, μ, into our stationary models and treat it as another parameter to be estimated by least squares.

7.2.1 AR(p) Model

Ex. The AR(1) case becomes $Y_t - \mu = \phi(Y_{t-1} - \mu) + e_t$. View it as a regression model with predictor variable Y_{t-1} and response variable Y_t. Least squares estimation then proceeds by minimizing the sum of squares of the differences $Y_t - \mu - \phi(Y_{t-1} - \mu)$. We make the **conditional sum-of-squares (CSS) function**

$$S_c(\phi, \mu) = \sum_{t=2}^{n} [(Y_t - \mu) - \phi(Y_{t-1} - \mu)]^2$$

and estimate ϕ and μ to minimize $S_c(\phi, \mu)$. After computing $\partial S_c / \partial \phi$, $\partial S_c / \partial \mu$, and doing approximation, we get about the same estimation as in method of moments:

$$\hat{\mu} \approx \overline{Y}, \qquad \hat{\phi} \approx r_1.$$

Similar process works for stationary AR(p) models. The least squares estimates are about the same as those obtained by the method of moments — $\hat{\mu} \approx \overline{Y}$ and the conditional least squares estimates of the ϕ's are approximately obtained by solving the sample Yule-Walker equations.

7.2.2 MA(q) Models

For an invertible MA(q) model:

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q},$$

assuming

$$e_0 = e_{-1} = \cdots = e_{-q} = 0,$$

we compute $e_t = e_t(\theta_1, \cdots, \theta_q)$ recursively by

$$e_t = Y_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q}, \quad t = 1, 2, \cdots, n.$$

Then we use a multivariate numerical method (e.g. grid search for MA(1) models) to minimize

$$S_c(\theta_1, \cdots, \theta_q) = \sum_{t=1}^{n} (e_t)^2.$$

7.2.3 ARMA(p,q) Models

Similarly to the MA(q) case, for the ARMA(p,q) model:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q},$$

assuming $e_p = e_{p-1} = \cdots = e_{p+1-q} = 0$, we compute

$$e_t = Y_t - \phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p} + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q}, \quad t = p+1, \cdots, n,$$

then minimize

$$S_c(\phi_1, \cdots, \phi_p, \theta_1, \cdots, \theta_q) = \sum_{t=p+1}^{n} e_t^2$$

numerically to obtain the conditional least squares estimates of all the parameters.

For stationary invertible models, the start-up values e_p, e_{p-1}, $\cdots$, e_{p+1-q} will have very little influence on the final estimates of the parameters for large samples.

7.3 Maximum Likelihood (MLE) and Unconditional Least Squares (Unconditional SS)

For series of moderate length and for stochastic seasonal models, the start-up values $e_p = e_{p-1} = \cdots = e_{p+1-q} = 0$ will have great impact on the final estimates for the parameters. So we consider the maximum likelihood estimation. The advantage of the method of maximum likelihood is that all of the information in the data is used rather than just the first and second moments.

Definition 1

For any fixed observation $Y_1, \cdots, Y_n$, the **likelihood function** L is the joint probability density (pdf) of obtaining the observed data, considered as a function of the unknown parameters.

Ex. Consider the AR(1) model. Assume that the i.i.d. r.v.s $e_t \sim N(0, \sigma_e^2)$. The pdf of each e_t is

$$(2\pi\sigma_e^2)^{-1/2} \exp\left(-\frac{e_t^2}{2\sigma_e^2}\right), \qquad -\infty < e_t < \infty.$$

By independence, the joint pdf of $e_2, \cdots, e_n$ is

$$(2\pi\sigma_e^2)^{-(n-1)/2} \exp\left(-\frac{1}{2\sigma_e^2}\sum_{t=2}^{n} e_t^2\right). \tag{2}$$

Given $Y_1 = y_1$, the linear transformation between $e_2, \cdots, e_n$ and $Y_2, \cdots, Y_n$ is

$$\begin{aligned} Y_2 - \mu &= \phi(Y_1 - \mu) + e_2 \\ Y_3 - \mu &= \phi(Y_2 - \mu) + e_2 \\ &\vdots \\ Y_n - \mu &= \phi(Y_{n-1} - \mu) + e_2 \end{aligned}$$

(Ex. cont.) The Jacobian of this transformation equals to 1. Thus the joint pdf of $Y_2, Y_3, \cdots, Y_n$ given $Y_1 = y_1$ can be obtained by substituting $e_t = (y_t - \mu) - \phi(y_{t-1} - \mu)$ in (2), namely

$$f(y_2, \cdots, y_n \mid y_1) =$$
$$(2\pi\sigma_e^2)^{-(n-1)/2} \exp\left\{-\frac{1}{2\sigma_e^2}\sum_{t=2}^{n}[(y_t - \mu) - \phi(y_{t-1} - \mu)]^2\right\}.$$

Now consider the (marginal) distribution of Y_1. By the linear process representation $Y_1 - \mu = \sum_{k=0}^{\infty} \phi^k e_{1-k}$, we have

$\mathrm{Var}(\mathrm{Y}_1) = \sum_{\mathrm{k}=0}^{\infty} \phi^{2\mathrm{k}} \sigma_{\mathrm{e}}^2 = \frac{\sigma_{\mathrm{e}}^2}{1 - \phi^2}$ and thus $Y_1 \sim N\left(\mu, \frac{\sigma_e^2}{1-\phi^2}\right)$.

The marginal pdf of Y_1 is

$$f(y_1) = \left(\frac{2\pi\sigma_e^2}{1-\phi^2}\right)^{-1/2} \exp\left(-\frac{(1-\phi^2)(y_1-\mu)^2}{2\sigma_e^2}\right).$$

(Ex. cont.) Multiplying the conditional pdf $f(y_2, \cdots, y_n \mid y_1)$ by the marginal pdf of Y_1 gives us the joint pdf of $Y_1, Y_2, \cdots, Y_n$, that is, the likelihood function for an AR(1) model (interpreted as a function of ϕ, μ and σ_e^2):

$$L(\phi, \mu, \sigma_e^2) = (2\pi\sigma_e^2)^{-n/2}(1-\phi^2)^{1/2} \exp\left[-\frac{1}{2\sigma_e^2} S(\phi, \mu)\right], \quad (3)$$

where $S(\phi, \mu)$ is called the **unconditional sum-of-squares function** and is given by

$$\begin{aligned} S(\phi, \mu) &= \sum_{t=2}^{n} [(Y_t - \mu) - \phi(Y_{t-1} - \mu)]^2 + (1-\phi^2)(Y_1 - \mu) \\ &= S_c(\phi, \mu) + (1-\phi^2)(Y_1 - \mu)^2. \end{aligned} \quad (4)$$

(Ex. cont.) The difference between $S(\phi, \mu)$ and $S_c(\phi, \mu)$ is only the rightmost term $(1-\phi^2)(Y_1-\mu)^2$. Since $S_c(\phi, \mu)$ involves a sum of $n-1$ components, we have $S(\phi, \mu) \approx S_c(\phi, \mu)$ for large sample size n. The effect of the rightmost term in estimating ϕ and μ will be more substantial when the minimum for ϕ occurs near the stationarity boundary of ± 1.

In general, it is more convenient to work with the **log-likelihood function**

$$\begin{aligned}\ell(\phi, \mu, \sigma_e^2) &= \log L(\phi, \mu, \sigma_e^2) \qquad (5)\\ &= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma_e^2) + \frac{1}{2}\log(1-\phi^2) - \frac{1}{2\sigma_e^2}S(\phi, \mu)\end{aligned}$$

To maximize ℓ, we take partial derivative of $\ell(\phi, \mu, \sigma_e^2)$ w.r.t. σ_e^2 and get $\sigma_e^2 = \dfrac{S(\phi, \mu)}{n}$. Since we are estimating two parameters ϕ and μ, we obtain a less biased estimator

$$\hat{\sigma}_e^2 = \frac{S(\hat{\phi}, \hat{\mu})}{n-2}.$$

The derivation of the likelihood function for more general ARMA models is considerably more involved. The estimations are often done by numerical methods.

A compromise between conditional least squares estimates and full maximum likelihood estimates is the **unconditional least squares estimates**, that is, estimates minimizing $S(\phi, \mu)$ (instead of $L(\phi, \mu, \sigma_e^2)$).

7.4. Properties of the Estimates

The large-sample properties of the maximum likelihood and least squares (conditional or unconditional) estimators are identical and can be obtained by modifying standard maximum likelihood theory.

For large n, the estimators are approximately unbiased and normally distributed. The variances and correlations are as follows:

$$\text{AR(1)}: \; Var(\hat{\phi}) \approx \frac{1-\phi^2}{n} \tag{7.4.9}$$

$$\text{AR(2)}: \begin{cases} Var(\hat{\phi}_1) \approx Var(\hat{\phi}_2) \approx \dfrac{1-\phi_2^2}{n} \\ Corr(\hat{\phi}_1, \hat{\phi}_2) \approx -\dfrac{\phi_1}{1-\phi_2} = -\rho_1 \end{cases} \tag{7.4.10}$$

$$\text{MA(1)}: \; Var(\hat{\theta}) \approx \frac{1-\theta^2}{n} \tag{7.4.11}$$

$$\text{MA(2)}: \begin{cases} Var(\hat{\theta}_1) \approx Var(\hat{\theta}_2) \approx \dfrac{1-\theta_2^2}{n} \\ Corr(\hat{\theta}_1, \hat{\theta}_2) \approx -\dfrac{\theta_1}{1-\theta_2} \end{cases} \tag{7.4.12}$$

$$\text{ARMA(1,1)}: \begin{cases} Var(\hat{\phi}) \approx \left[\dfrac{1-\phi^2}{n}\right]\left[\dfrac{1-\phi\theta}{\phi-\theta}\right]^2 \\ Var(\hat{\theta}) \approx \left[\dfrac{1-\theta^2}{n}\right]\left[\dfrac{1-\phi\theta}{\phi-\theta}\right]^2 \\ Corr(\hat{\phi}, \hat{\theta}) \approx \dfrac{\sqrt{(1-\phi^2)(1-\theta^2)}}{1-\phi\theta} \end{cases} \tag{7.4.13}$$

Notice that, in the AR(1) case, the variance of the estimator of ϕ decreases as ϕ approaches ± 1. Also notice that even though an AR(1) model is a special case of an AR(2) model, the variance of $\hat{\phi}_1$ shown in Equations (7.4.10) shows that our estimation of ϕ_1 will generally suffer if we erroneously fit an AR(2) model when, in fact, $\phi_2 = 0$. Similar comments could be made about fitting an MA(2) model when an MA(1) would suffice or fitting an ARMA(1,1) when an AR(1) or an MA(1) is adequate.

For the ARMA(1,1) case, note the denominator of $\phi - \theta$ in the variances in Equations (7.4.13). If ϕ and θ are nearly equal, the variability in the estimators of ϕ and θ can be extremely large.

Note that in all of the two-parameter models, the estimates can be highly correlated, even for very large sample sizes.

Ex. Exhibit 7.2 gives numerical values for the large-sample approximate standard deviations of the estimates of ϕ in an AR(1) model for some ϕ and n.

Thus, in estimating an AR(1) model with, for example, $n = 100$ and $\phi = 0.7$, we can be about 95% confident that our estimate of ϕ is in error by no more than $\pm 2(0.07) = \pm 0.14$.

Exhibit 7.2 AR(1) Model Large-Sample Standard Deviations of $\hat{\phi}$

	n		
ϕ	**50**	**100**	**200**
0.4	0.13	0.09	0.06
0.7	0.10	0.07	0.05
0.9	0.06	0.04	0.03

Comparision of Parameter Estimation Methods:

1. For stationary AR(p) models with large samples, the method of moments yields estimators equivalent to least squares and maximum likelihood.
2. For ARMA(p,q) models, the variance for the method-of-moments estimator is always larger than the variance of the maximum likelihood estimator.

Ex. Exhibit 7.3 displays the ratio of the large-sample standard deviations for Method of Moments (MM) vs. Maximum Likelihood (MLE) for some θ. These ratios indicate that the method-of-moments estimator should not be used for the MA(1) model. The same advice applies to all models that contain moving average terms.

Exhibit 7.3 Method of Moments (MM) vs. Maximum Likelihood (MLE) in MA(1) Models

θ	SD_{MM}/SD_{MLE}
0.25	1.07
0.50	1.42
0.75	2.66
0.90	5.33

7.5. Illustrations of Parameter Estimation

Ex. *(TS-ch7.R) Consider the simulated MA(1) series* `ma1.2.s` *with* $\theta = -0.9$*:*

- *MM:* $\hat{\theta} = -0.554$ *(poor, see Exhibit 7.1);*
- *conditional SS:* -0.879*;*
- *unconditional SS:* -0.923*;*
- *MLE:* -0.915 *(closest).*

By Equation (7.4.11), the estimated standard error of $\hat{\theta}$ *is*

$$\hat{\mathrm{Var}}\left(\hat{\theta}\right) = \sqrt{\frac{1-\hat{\theta}^2}{\mathrm{n}}} = \sqrt{\frac{1-(0.915)^2}{120}} \approx 0.04.$$

Both the maximum likelihood and conditional/unconditional sum-of-squares estimates are not significantly far from the true value of -0.9*.*

The `arima` function estimates an ARIMA(p,d,q) model for the time series passed to it as the first argument.

1. `order=c(p,d,q)`: The ARIMA order is specified by the order argument.
2. `method= 'CSS'`: Estimate by conditional sum-of-squares method.
3. `method='ML'`: Estimate by maximum likelihood method. The default estimation method is maximum likelihood, with initial values determined by the CSS method.
4. The intercept term reported in the output of the `arima` function is in fact the mean μ (instead of θ_0).
5. We may fix the values of some elements by `fixed` argument. For example, for an ARMA(1,2) model, `fixed=c(NA,0.2,NA,0)` sets $ma1 = 0.2$ (or $\theta_1 = -0.2$) and intercept $\mu = 0$.

The output of the `arima` function is a `list` structure. Try the following commands:

```
arima(ma1.2.s, order=c(0,0,1),
method='ML',fixed=c(NA,0))
ma1.2.f=arima(ma1.2.s, order=c(0,0,1),
method='ML')
str(ma1.2.f) # show the content of ma1.2.f
fitted(ma1.2.f) # values of the fitted model
residuals(ma1.2.f) # residuals for the fitted
model
```

Ex. *(TS-ch7.R) For the MA(1) simulation* `ma1.1.s` *with* $\theta = 0.9$*:*

- *MM:* $\hat{\theta} = 0.719$ *(Exhibit 7.1);*
- *conditional SS:* 0.958*;*
- *unconditional SS:* 0.983*;*
- *MLE is* 1*.*

The estimated standard error is about 0.04*. The MLE* $\hat{\theta} = 1$ *corresponds to a noninvertible model. We should perform further investigation.*

Ex. *We simulate an MA(2) series* `ma2.my.s` *with* $\theta_1 = 0.4, \theta_2 = 0.21$, *and* $n = 200$. *Then we may work to specify the model, find the best subsets ARMA models, and estimate the parameters of this series. (See TS-ch7.R)*

By equation (7.4.12), the standard error of the parameter estimations are

$$\sqrt{\mathrm{Var}\left(\hat{\theta}_1\right)} \approx \sqrt{\mathrm{Var}\left(\hat{\theta}_2\right)} \approx \sqrt{\frac{1-\theta_2^2}{n}} \approx 0.069.$$

Both CSS and ML estimates of the parameters are within the confidence intervals.

Now we look at some examples of AR models.

Ex. (chap7.R) The dataset `ar1.s` is an AR(1) series with $\phi = 0.9$, and `ar1.2.s` is an AR(1) series with $\phi = 0.4$.

Exhibit 7.4 Parameter Estimation for Simulated AR(1) Models

Parameter ϕ	Method-of-Moments Estimate	Conditional SS Estimate	Unconditional SS Estimate	Maximum Likelihood Estimate	n
0.9	0.831	0.857	0.911	0.892	60
0.4	0.470	0.473	0.473	0.465	60

```
data(ar1.s); data(ar1.2.s)
ar(ar1.s,order.max=1,AIC=F,method='yw')
ar(ar1.s,order.max=1,AIC=F,method='ols')
ar(ar1.s,order.max=1,AIC=F,method='mle')
ar(ar1.2.s,order.max=1,AIC=F,method='yw')
ar(ar1.2.s,order.max=1,AIC=F,method='ols')
ar(ar1.2.s,order.max=1,AIC=F,method='mle')
```

Ex. (cont) The `ar` function estimates the AR model for the centered data (that is, mean-corrected data), so the intercept must be zero. Some arguments:

1. `order.max`: the maximum AR order, must be specified;
2. `AIC=T` (default): the AR order may be estimated by choosing the order, between 0 and the maximum order, whose model has the smallest AIC;
3. `AIC=F`: the AR order is set to the maximum AR order;
4. `method='yw'`: the parameters are estimated by the Yule-Walker equations;
5. `method='ols'`: the parameters are estimated by ordinary least squares;
6. `method='mle'`: the parameters are estimated by maximum likelihood estimation (assuming normally distributed white noise error terms).

Ex. (cont) By Equation (7.4.9), the standard errors for the estimated parameter of `ar1.s` is

$$\sqrt{\widehat{\mathrm{Var}}\left(\hat{\phi}\right)} \approx \sqrt{\frac{1-\hat{\phi}^2}{n}} = \sqrt{\frac{1-0.831^2}{60}} \approx 0.07.$$

Similarly, the standard errors for the estimated parameter of `ar1.2.s` is

$$\sqrt{\widehat{\mathrm{Var}}\left(\hat{\phi}\right)} \approx 0.11.$$

All four methods estimate reasonably well for AR(1) models.

Ex. *(chap7.R)*

Exhibit 7.5 Parameter Estimation for a Simulated AR(2) Model

Parameters	Method-of-Moments Estimates	Conditional SS Estimates	Unconditional SS Estimates	Maximum Likelihood Estimate	*n*
$\phi_1 = 1.5$	1.472	1.5137	1.5183	1.5061	120
$\phi_2 = -0.75$	–0.767	–0.8050	–0.8093	–0.7965	120

```
> data(ar2.s)
> ar(ar2.s,order.max=2,AIC=F,method='yw')
> ar(ar2.s,order.max=2,AIC=F,method='ols')
> ar(ar2.s,order.max=2,AIC=F,method='mle')
```

By Equation (7.4.10), the standard errors for the estimates are

$$\sqrt{\hat{\mathrm{Var}}\left(\hat{\phi}_1\right)} \approx \sqrt{\hat{\mathrm{Var}}\left(\hat{\phi}_2\right)} \approx \sqrt{\frac{1-\hat{\phi}_2^2}{n}} = \sqrt{\frac{1-0.75^2}{120}} \approx 0.06.$$

All four methods estimate reasonably well for AR(2) models.

Ex. *(TS-ch7.R) We simulate an AR(2) series with* $\phi_1 = -0.3, \phi_2 = 0.4$, *and* $n = 1000$. *Then we specify the model, find the best subset ARMA models, and estimate the parameters.*

Eq (7.4.10) shows that the standard errors for the estimates are

$$\sqrt{\hat{\mathrm{Var}}\left(\hat{\phi}_1\right)} \approx \sqrt{\hat{\mathrm{Var}}\left(\hat{\phi}_2\right)} \approx \sqrt{\frac{1-\hat{\phi}_2^2}{n}} = \sqrt{\frac{1-0.4^2}{1000}} \approx 0.029.$$

All three estimates (MM, CSS, ML) of the parameters are within the confidence intervals

Ex. *(chap7.R) Exhibit 7.6 estimates the parameters for the simulated ARMA(1,1) Model* `arma11.s` *with* $\phi = 0.6$, $\theta = -0.3$, $n = 100$.

Exhibit 7.6 Parameter Estimation for a Simulated ARMA(1,1) Model

Parameters	Method-of-Moments Estimates	Conditional SS Estimates	Unconditional SS Estimates	Maximum Likelihood Estimate	n
$\phi = 0.6$	0.637	0.5586	0.5691	0.5647	100
$\theta = -0.3$	−0.2066	−0.3669	−0.3618	−0.3557	100

```
> data(arma11.s)
> arima(arma11.s, order=c(1,0,1),method='CSS')
> arima(arma11.s, order=c(1,0,1),method='ML')
```

Let discuss some real time series.

Ex. *Consider the industrial chemical property time series* `color`*. The sample PACF strongly suggested an AR(1) model for this series. Here we show the various estimates of the parameter ϕ using four different methods of estimation.*

Exhibit 7.7 Parameter Estimation for the Color Property Series

Parameter	Method-of-Moments Estimate	Conditional SS Estimate	Unconditional SS Estimate	Maximum Likelihood Estimate	*n*
ϕ	0.5282	0.5549	0.5890	0.5703	35

The standard error of the estimates is about

$$\sqrt{\widehat{\mathrm{Var}}\left(\hat{\phi}\right)} \approx \sqrt{\frac{1-(0.57)^2}{35}} \approx 0.14,$$

so all of the estimates are comparable.

Ex. *Consider the Canadian hare abundance series* `hare`. *The sample PACF also suggested an AR(3) model. We use MLE for the parameters:*

Exhibit 7.8 Maximum Likelihood Estimates from R Software: Hare Series

Coefficients:	ar1	ar2	ar3	Intercept†
	1.0519	–0.2292	–0.3931	5.6923
s.e.	0.1877	0.2942	0.1915	0.3371

```
sigma^2 estimated as 1.066: log-likelihood = -46.54, AIC = 101.08
```

† The intercept here is the estimate of the process mean μ—not of θ_0.

```
> data(hare)
> arima(sqrt(hare),order=c(3,0,0))
```

Ex. *(cont.) The estimated AR(3) model for* `hare`$= \{Y_t\}$ *is*

$$\begin{aligned}\sqrt{Y_t} - 5.6923 &= 1.0519(\sqrt{Y_{t-1}} - 5.6923) \\ &\quad -0.2292(\sqrt{Y_{t-2}} - 5.6923) \\ &\quad -0.3930(\sqrt{Y_{t-3}} - 5.6923) + e_t\end{aligned}$$

or

$$\sqrt{Y_t} = 3.25 + 1.0519\sqrt{Y_{t-1}} - 0.2292\sqrt{Y_{t-2}} - 0.3930\sqrt{Y_{t-3}} + e_t.$$

Since the lag 2 term $\hat{\phi}_2$ *is insignificant from 0, we may drop the term and obtain new estimates of* ϕ_1 *and* ϕ_3 *with this subset model.*

Ex. *Consider the oil price series* `oil.price`. *The sample ACF suggested an MA(1) model on* `diff(log(oil.price))`$= \{Y_t\}$. *Here we estimate* θ *by the various methods.*

Exhibit 7.9 Estimation for the Difference of Logs of the Oil Price Series

Parameter	Method-of-Moments Estimate	Conditional SS Estimate	Unconditional SS Estimate	Maximum Likelihood Estimate	*n*
θ	–0.2225	–0.2731	–0.2954	–0.2956	241

```
> data(oil.price)
> arima(log(oil.price),order=c(0,1,1),method='CSS')
> arima(log(oil.price),order=c(0,1,1),method='ML')
```

The method-of-moments estimate differs quite a bit from the others. The others are nearly equal given their standard errors of about 0.07.