C Loop Invariant for Horner's Rule

The following code fragment implements Horner's Rule for numerical solution of a polynomial of degree *n*:

```
1    y = 0;
2    i = n;
3    while (i >= 0) {
4         y = a[i] + x * y;
5         i = i - 1;
6    }
```

The problem statement proposes the following summation as a Loop Invariant of the statements in lines 3 – 5 of the code fragment:

$$y = \sum_{k=0}^{n-(i+1)} a_{k+i+1} x^k$$

According to pages 17-18 of the textbook, a valid loop invariant must meet three conditions:

A. It is true prior to the first iteration of the loop
B. If it is true before an iteration of the loop, it remains true before the next iteration.
C. When the loop terminates, the invariant gives a useful property that helps to show that the algorithm is correct.

<u>Condition A</u>

Before the first iteration of the loop, $y = 0$ and $i = n$. Therefore, the summation formula evaluates as follows:

$$y = \sum_{k=0}^{n-(i+1)} a_{k+i+1} x^k$$

$$0 = \sum_{k=0}^{n-(n+1)} a_{k+n+1} x^k$$

$$0 = \sum_{k=0}^{-1} a_{k+n+1} x^k$$

When the starting index of a summation is greater than the ending index, the summation contains no terms. So, the summation evaluates to

$$0 = 0$$

Since this is true, Condition A is satisfied.

Condition B

Condition A has already shown that the Loop Invariant is true before the first iteration. Condition B requires that it remain true before the next iteration.

Let: y_1 = Value of y before the current loop iteration
y_2 = Value of y before the next loop iteration

We have confirmed that the value of y before the loop is the loop invariant:

$$y_1 = \sum_{k=0}^{n-(i+1)} a_{k+i+1} x^k$$

Two operations are performed during the loop: Horner's Rule is applied, and i is decremented. Therefore:

$$y_2 = \sum_{k=0}^{n-((i-1)+1)} a_{k+(i-1)+1} x^k = a_i + x y_1$$

If we simplify the decrementing of i in the second term and substitute the value of y_1 into the third term of the previous equation, we get:

$$\sum_{k=0}^{n-i} a_{k+i} x^k = a_i + x \sum_{k=0}^{n-(i+1)} a_{k+i+1} x^k$$

Expand both sides of the equation into their corresponding series:

$$a_i + a_{i+1}x + a_{i+2}x^2 + \cdots + a_n x^{n-i} = a_i + a_{i+1}x + a_{i+2}x^2 + \cdots + a_n x^{n-i}$$

Since this equation is true, Condition B is satisfied

Condition C

Condition C requires that the loop invariant tell us something about the correctness of the solution.

We know that before the final loop iteration, $i = 0$. Condition B confirms that the loop invariant stays true through the loop. Let y_F be the final value of y after applying Horner's Rule:

$$y_F = \sum_{k=0}^{n} a_k x^k$$

So, after the final loop, the value of y is the same as the definition of the polynomial that we are trying to evaluate. This shows that Condition C is satisfied.