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**Lifetime in a Flat Circle**

Recently, Ho Yin simulated the diffusion of particles in the absence of a well, and found that the log-binned dwell time distribution of the particles also exhibited the secondary peak. Here, we analytically solve the lifetime of the diffusing particles escaping from a flat circle. Consider the diffusion equation of the survival probability in polar coordinates

The boundary condition is and the initial condition is

Separation of variables: Let With and we obtain

Let Then the differential equation reduces to the Bessel equation

Hence, the solution can be expressed in terms of Bessel functions

and are Bessel functions of the first and second kinds, respectively. Since diverges at we have

**A graph of a function

AI-generated content may be incorrect.A graph of a function

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To satisfy the boundary conditions at we need to know the zeros of the Bessel function given in the following table.

A table of numbers and symbols

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Denoting as the th zero of i.e., the solution satisfying the boundary condition at is given by

The corresponding decay rate is

Combining all components, the survival probability is given by

To find the coefficients , we make use of the Sturm-Liouville theory to establish that the Bessel functions are orthogonal.

Using the completeness property for the initial condition,

Multiplying both sides by and integrating we get

Multiplying both sides by and integrating from to

The full solution becomes

The flux density at the boundary is given by

The lifetime distribution is equal to the total flux out of the circle at time Noting that

To compare with experimental results, we average over the initial positions

As derived at the end of this document,

Hence, we have

If the initial position is measured in units of , and is measured in units of then

To plot the lifetime distribution using log-binning, we note that

A graph with a line

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The log-binned lifetime distribution computed using the first 5 values of is measured in unit of

Remarks:

Bessel functions can be computed in Scipy.

The zeros of Bessel functions can be computed using jn\_zeros() in Scipy.

The profile of the survival distribution in space is similar to the vibration modes of a circular membrane, except that the oscillation dynamics is replaced by the relaxation dynamics. See, for example, <https://www.acs.psu.edu/drussell/demos/membranecircle/circle.html>.

A side product of this result is that since the total lifetime distribution should be normalized, we arrive at the identity

**Sturm-Liouville theory of the Bessel functions**

The orthogonal property of the Bessel functions can be proved by considering two solutions of the Bessel equations with

Multiplying the equations by and respectively and subtracting,

Integrating from to and integrating by parts, the first two terms vanish due to the boundary condition,

We are left with the last term,

Thus, we obtain the orthogonal property

The integral for is difficult to derive. After consulting a textbook (Mathematical Methods for Physics, H. W. Wyld), I found the following derivation. Consider the expression

The last term vanishes as is a solution of the Bessel equation. Integrating,

Using the fact that either or for we get

Furthermore, from the recursion relations of the Bessel functions,

The orthogonal property becomes

We also need to evaluate the integral For this purpose, we start with the series expansion of