Expected Value of the Geometric Distribution

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Question 1

What is the expected value of the Geometric Distribution? Use a calculus based argument.

Proof. Let $X \sim$ Geometric(p). Then $Pr(X = x) = p(1-p)^{x-1}$ for $x = 1, 2, 3, \ldots$.

Now,

$$E[x] = \sum_{i=1}^{\infty} iPr(X = i) = \sum_{i=1}^{\infty} ip(1-p)^{i-1}.$$

Note that $d/dp(1-p)^i = -i(1-p)^{i-1}$.

We can now substitute and rearrange the terms so that we take the limit as the sum approaches infinity. We know the derivative of a finite sum is the sum of the derivatives, so this enables us to bring the derivative operator to the front of the sum.

$$E[x] = -p \sum_{i=1}^{\infty} d/dp(1-p)^i = -p \lim_{N\to\infty} d/dp \sum_{i=1}^{N} (1-p)^i.$$

Simplifying the finite sum in closed form yields:

$$\begin{aligned}
E[x] &= -p \lim_{N\to\infty} d/dp \left[(1-p)\frac{1-(1-p)^N}{1-(1-p)}\right] \\
&= -p \lim_{N\to\infty} d/dp \left[(1-p)\frac{1-(1-p)^N}{p}\right] \\
&= -p \lim_{N\to\infty} d/dp \left[\frac{(1-p)-(1-p)(1-p)^N}{p}\right] \\
&= -p \lim_{N\to\infty} d/dp \left[\frac{1-p-(1-p)^N+p(1-p)^N}{p}\right] \\
&= -p \lim_{N\to\infty} d/dp \left[\frac{1}{p} - 1 - \frac{(1-p)^N}{p} + (1-p)^N\right] \\
&= -p \lim_{N\to\infty} \left[-\frac{1}{p^2} - \left(-p^{-2}(1-p)^N - p^{-1}N(1-p)^{N-1}\right) - N(1-p)^{N-1}\right] \\
&= -p \lim_{N\to\infty} \left[-\frac{1}{p^2} + p^{-2}(1-p)^N + p^{-1}N(1-p)^{N-1} - N(1-p)^{N-1}\right] \\
&= -p \left(-\frac{1}{p^2} + \lim_{N\to\infty} \left[p^{-1}N(1-p)^{N-1} - N(1-p)^{N-1}\right]\right) \\
&= -p \left(-\frac{1}{p^2} + \lim_{N\to\infty} \left[p^{-1}N(1-p)^{N-1} - N(1-p)^{N-1}\right]\right) \\
&= -p \left(-\frac{1}{p^2} + \left(\frac{1}{p} - 1\right) \lim_{N\to\infty} N(1-p)^{N-1}\right) \\
&= -p \left(-\frac{1}{p^2} + \frac{\frac{1}{p}-1}{1-p} \lim_{N\to\infty} N(1-p)^{N}\right)
\end{aligned}$$

Note that $\lim_{N\to\infty} N(1-p)^N = \lim_{N\to\infty} \frac{N}{(1-p)^{-N}} = \lim_{N\to\infty} \frac{\frac{d}{dN}N}{\frac{d}{dN}(1-p)^{-N}} = \lim_{N\to\infty} \frac{1}{-\ln(1-p)(1-p)^{-N}} = \lim_{N\to\infty} \frac{-(1-p)^N}{\ln(1-p)} = 0$.

Hence,

$$E[x] = -p\left(-\frac{1}{p^2} + \frac{p-1}{1-p}0\right) = -p\left(\frac{-1}{p^2}\right) = \frac{1}{p}$$

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