1 Review of Set Theory

1.1 Definitions

- A set is a collection of objects. The objects in a set are frequently called **elements**.
- If x is an element of set A, we write $x \in A$. If x is not an element of set A, we write $x \notin A$.
- A set with no elements is called the empty set, symbolised by ϕ or $\{\}$.
- The elements of a set must be **unique**
  - Eg 1
- Two sets A and B are **equal** $\iff$ A and B contain the same elements. When two sets are equal, we write $A = B$.
  - Eg 2
- If A is a set with a finite number of elements (i.e. A is a finite set), we write $n(A) =$ number of elements in A
  - Eg 3
- We say a set A is a **subset** of set B $\iff$ every element of A is in B. We write $A \subseteq B$
  - Egs 4 - 8
  - In class Problem 1

1.2 Set Operations

- Let $U =$ universal set $=$ set whose subsets are of interest. Let $A \subseteq U$ and $B \subseteq U$.
- We define the **complement of** A, A^c or A' to be the set $A' = \{x \mid x \notin A\}$
- We define the **relative complement of** A with respect to B to be the set $B - A = \{x \mid x \in B \text{ and } x \notin A\}$
  - Eg 9
- The **union** of sets A and B, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
  - Eg 10
  - We can **extend** this definition to a countable sequence of sets $A_1, A_2, A_3, \ldots$ with the definition$\cup_{i=1}^{\infty} A_i = \{x \mid x \in A_i \text{ for some } i \in \mathbb{N}\}$
  - Eg 11
- The **intersection** of sets A and B, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
  - We can **extend** this definition to a countable sequence of sets $A_1, A_2, A_3, \ldots$ with the definition$\cap_{i=1}^{\infty} A_i = \{x \mid x \in A_i \text{ for all } i \in \mathbb{N}\}$
  - Eg 12 - 14
- If $A \cap B = \phi$, we say that A and B are **disjoint**.

1.3 Some properties about sets

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
3. $(A \cup B)' = A' \cap B'$.
   - In general, $(\cup_{i=1}^{\infty} A_i)' = \cap_{i=1}^{\infty} A_i'$
4. $(A \cap B)' = A' \cup B'$.
   - In general, $(\cap_{i=1}^{\infty} A_i)' = \cup_{i=1}^{\infty} A_i'$
5. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
6. If $A \subseteq B$, then $n(A) \leq n(B)$.

- Egs 15 - 17
- In class Problem 2
- Eg 18

2 Counting

- **Multiplication Principle**: If a compound action can be broken into a series of k component actions and each of these can be performed in $n_1, n_2, \ldots, n_k$ ways respectively, then the compound action can be performed in $n_1 n_2 \ldots n_k$ ways.
  - Egs 19 - 21
  - In class Problem 3
- **Permutation Principle**: Suppose I have n **distinct** objects and I wish to select r of them, and the **order** in which I select them is important. Then there are
  - Specifically, the number of ways to arrange n **distinct** objects where the **order** in which they are selected is important is
  - Eg 22 - 24
- **Combination Principle**: Suppose I have n **distinct** objects and I wish to select r of them, and the **order** in which I select them is **NOT** important. Then there are
  - Eg 25, 26
  - Eg 27, 28
- The **Binomial Theorem**: If x and y are variables, and n is a non-negative integer, then $(x+y)^n \equiv \Sigma_{k=0}^{n} C_k^n x^{n-k} y^k$.
  - Eg 29

3 Probability

1. A random experiment or simply an experiment is an experiment whose outcomes cannot be predicted with certainty.
2. The sample space S of an experiment is the set of all possible outcomes for the experiment.
3. An **event** is a **subset** of the sample space.
   - Eg 30
4. Probability is the measure of occurrence of an event.
5. We denote the probability of event A happening $\Pr(A)$
6. The function Pr satisfies the following axioms, known as Kolmogorov axioms:
   (a) Axiom 1: For any event E, $0 \leq \Pr(E) \leq 1$.
   (b) Axiom 2: $\Pr(S) = 1$.
   (c) Axiom 3: For any sequence of **mutually exclusive** events $\{E_n\}$ $n \geq 1$, that is $E_i \cap E_j = \phi$ for $i \neq j$, we have $\Pr\left(\cup_{i=1}^{\infty} E_i\right) = \Sigma_{i=1}^{\infty} \Pr(E_i)$. This axiom is known as **countable additivity**.
      i. Specifically, for **mutually exclusive** events A, B and C, we have
7. When each outcome of an experiment is just as likely as another, as in the example of tossing a fair coin, the outcomes are said to be **equally likely**. In which case, we use the formula $\Pr(E) =$ number of outcomes favourable to event E/total number of outcomes $= \frac{n(E)}{n(S)}$.
8. It is very important to keep in mind that this definition of probability applies only to a sample space that has **equally likely outcomes**. Applying the definition to a space with outcomes that are not equally likely leads to incorrect conclusions.
   (a) Eg 31 - 34

Some rules and theorems for Probability

1. Complementary Events
   For **any** event E, $\Pr(E') = 1 - \Pr(E)$
2. For **any two** events A and B, $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$. Similarly, $\Pr(B) = \Pr(B \cap A) + \Pr(B \cap A')$.
3. Union of Events
   For **any two** events A and B, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.
   - Note that if A and B are **mutually exclusive**, i.e. $\Pr(A \cap B) = 0$, then we have a specific version of Axiom 3, which is $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.
   - Note, also, that we can obtain an expression for $\Pr(A \cup B \cup C)$ using $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.

   Egs 35 - 41
   (a) In class Problem 4, In class Problem 5

Incorporating Counting Techniques into Probability
Egs 42 - 45

4 Conditional Probability and Independence

1. For **any two** events A and B with $\Pr(B) > 0$, we define the **conditional** probability of A happening, given that B **has occurred** by $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$.
2. Egs 46 - 50
3. In general, for n events $A_1, A_2, A_3 \ldots, A_n$, $\Pr(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = \Pr(A_1)\Pr(A_2|A_1)\Pr(A_3|A_1 \cap A_2) \ldots \Pr(A_n|A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{n-1})$
4. Egs 51, 52

Note:

1. Realize that $\Pr(A \cap B) = \Pr(A|B)\Pr(B)$ is also true.

2. All the results we have for **unconditional probability** are true for **conditional probability** as well, for example:

   (a) the Kolmogorov axioms

   (b) $\Pr(E'|C) = 1 - \Pr(E|C)$. [Please note, though, that it IS NOT NECESSARILY TRUE that $\Pr(E|C') = 1 - \Pr(E|C)$]

   (c) $\Pr((A \cup B)|C) = \Pr(A|C) + \Pr(B|C) - \Pr((A \cap B)|C)$

   (d) $\Pr(A|C) = \Pr((A \cap B)|C) + \Pr((A \cap B')|C)$

Bayes Formula

Essentially helps us to reverse the conditional probability order: $\Pr(B|A) = \frac{\Pr(A|B)\Pr(B)}{\Pr(A)}$. Bayes Formula in general states the following:

Suppose that the sample space S is the union of mutually exclusive events $H_1, H_2, \ldots, H_n$ with $\Pr(H_i) > 0$ for each i. Then for **any** event A and **any** H_i, we have $\Pr(H_i|A) = \frac{\Pr(A|H_i)\Pr(H_i)}{\Pr(A)}$

where $\Pr(A) = \Pr(A|H_1)\Pr(H_1) + \Pr(A|H_2)\Pr(H_2) + \cdots + \Pr(A|H_n)\Pr(H_n)$.

In-class Problem 6

Eg 53, 54

In-class Problem 7

Independence

1. In terms of conditional probability, two events A and B are said to be statistically **independent** if and only if $\Pr(A|B) = \Pr(A)$. [We also have $\Pr(B|A) = \Pr(B)$]

2. If A and B are independent, we then have $\Pr(A \cap B) = \Pr(A)\Pr(B)$. Why? We know that $\Pr(A \cap B) = \Pr(A|B)\Pr(B)$ is ALWAYS true. Therefore, with the independence of A and B, we have $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

   (a) More generally, if we have events $A, B, C, ..., N$ which are all mutually independent of each other, then $\Pr(A \cap B \cap C \cap \cdots \cap N) = \Pr(A)\Pr(B)\Pr(C)\ldots\Pr(N)$

3. If A and B are independent then so are A and B'.

4. If A and B are independent then so are A' and B'.

Eg 55 - 58

In-class Problem 8

Odds

The odds **in favour** of an event E are defined as $\frac{\Pr(E)}{1-\Pr(E)}$. Thus, if someone says "the odds in favour of E are $a : b$", then $\Pr(E) = \frac{a}{a+b}$.

Eg 59 - 61