Problem 79

Jack Krebsbach, Valen Feldmann , Makayla Henline, Grace Kirschbaum

Question 1: 79 (a)

An insurance policy pays 100 per day for up to 3 days of hospitalization and 50 per day for each day of hospitalization thereafter. The number of days of hospitalization, X, is a discrete random variable with probability function

$$p(k) = \begin{cases} \frac{6-k}{15} & k = 1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases}$$

Determine the expected payment for hospitalization under this policy.

Solution:

First we define the function to give us the payment from the hospitalization for x days under this policy.

$$m(x) = \begin{cases} 100 & x = 1 \\ 200 & x = 2 \\ 300 & x = 3 \\ 300 + 50(x-3) & x > 3 \end{cases}$$

Next, using the given probability function we create a table containing the probabilities and payout for 1 through 5 days. The probability of staying more than 5 days is 0. Let M denote a discrete random variable representing the payout.

x	1	2	3	4	5
m(x)	100	200	300	350	400
p(x)	5/15	4/15	3/15	2/15	1/15

Table 1: Probability p(x) and payout m(x) of staying x days.

Next, we calculate the expected value by finding the average payout expected weighted by the probability of staying x days in the hospital.

$$\begin{aligned} \mathrm{E}[M] &= 100(5/15) + 200(4/15) + 300(3/15) + 350(2/15) + 400(1/15) \\ &= 3300/15 \\ &= 220. \end{aligned}$$

Thus, the expected payout for hospitalization under this policy is \$220.

Question 2: 79 (b)

An insurance policy pays 100 per day for up to 3 days of hospitalization and $100(0.9)^i$ for the ith day of hospitalization thereafter. The number of days of hospitalization, X, is a discrete random variable with probability function

$$p(x) = \begin{cases} k\left(\frac{1}{200}\right)^x & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases},$$

where k is a positive constant.
Determine the expected payment for hospitalization under this policy.

Solution:

Let M denote the random variable representing the payout from the hospital. First, we find the normalizing constant k to make $p(x)$ a valid discrete probability distribution.

Let

$$\{p_n\} = \sum_{i=1}^{n} k\frac{1}{200^i}.$$

Next,

$$\{p_n\} - \frac{1}{200}\{p_n\} = \sum_{i=1}^{n} k\frac{1}{200^i} - \sum_{i=2}^{n+1} k\frac{1}{200^i}$$

$$\implies \{p_n\}(1 - \frac{1}{200}) = k\left[1/200 - 1/(200^{n+1})\right]$$

$$\implies \{p_n\} = k\left[\frac{1/200 - 1/(200^{n+1})}{199/200}\right]$$

Now we take the limit as $n \longrightarrow \infty$ and set our equation equal to 1. Hence,

$$\{p_n\} = \lim_{n\to\infty} k\left[\frac{1/200 - 1/(200^{n+1})}{199/200}\right] = \frac{k/200}{199/200} = k/199 = 1.$$

Thus, k must be equal to 199 for this to be a valid probability distribution.

Now, let us define $m(x)$ to represent the payout as a function of the number of days under hospitalization.

$$m(x) = \begin{cases} 100 & x = 1 \\ 200 & x = 2 \\ 300 & x = 3 \\ 300 + 100\sum_{i=1}^{x-3} 0.9^i & x > 3 \end{cases}.$$

We know that $\sum_{i=1}^{x-3} 0.9^i$ is a geometric series that has a closed form of

$$\sum_{i=1}^{x-3} 0.9^i = \frac{0.9 - 0.9^{x-2}}{1 - 0.9} = \frac{0.9(1 - 0.9^{x-3})}{1/10} = 9(1 - 0.9^{x-3}).$$

Thus, our payout function looks like:

$$m(x) = \begin{cases} 100 & x = 1 \\ 200 & x = 2 \\ 300 & x = 3 \\ 300 + 900(1 - 0.9^{x-3}) & x > 3 \end{cases}.$$

Next, we find the expected payout by taking an average of the payout over all days weighted by the probability of being hospitalized for x days.

Hence,

$$
\begin{aligned}
\mathrm{E}[M] &= p(1)m(1) + p(2)m(2) + p(3)m(3) + \sum_{x=4}^{\infty} p(x)m(x) \\
&= 100\left(\frac{k}{200}\right) + 200\left(\frac{k}{200^2}\right) + 300\left(\frac{k}{200^3}\right) + \sum_{x=4}^{\infty} \frac{k}{200^x}(300 + 900(1 - 0.9^{x-3})) \\
&= \frac{k}{2} + \frac{k}{200} + \frac{300k}{200^3} + \sum_{x=4}^{\infty}\left[\frac{300k}{200^x} + \frac{900k}{200^x}(1 - 0.9^{x-3})\right] \\
&= \frac{k}{2} + \frac{k}{200} + \frac{300k}{200^3} + \sum_{x=4}^{\infty}\frac{300k}{200^x} + \sum_{x=4}^{\infty}\frac{900k}{200^x} - \sum_{x=4}^{\infty}\frac{900k}{200^x}\frac{0.9^x}{0.9^3} \\
&= \frac{k}{2} + \frac{k}{200} + \frac{300k}{200^3} + 1200k\sum_{x=4}^{\infty}\frac{1}{200^x} - \frac{900k}{0.9^3}\sum_{x=4}^{\infty}\left(\frac{9}{2000}\right)^x \\
&= \frac{k}{2} + \frac{k}{200} + \frac{300k}{200^3} + 1200k\frac{(1/200)^4}{1 - 1/200} - \frac{900k}{0.9^3}\frac{(9/2000)^4}{1 - 9/2000} \\
&= \frac{k}{2} + \frac{k}{200} + \frac{300k}{200^3} + 1200k\frac{1/200^4}{199/200} - \frac{900k}{0.9^3}\frac{9^4/2000^4}{1991/2000} \\
&= \frac{k}{2} + \frac{k}{200} + \frac{300k}{200^3} + 1200k\frac{1/200^3}{199} - \frac{900k}{0.9^3}\frac{9^4/2000^3}{1991}.
\end{aligned}
$$

Now, let us make the substitution $k = 199$.

$$
\mathrm{E}[M] = \frac{199}{2} + \frac{199}{200} + \frac{300(199)}{200^3} + 1200(1/200^3) - \frac{900(199)}{0.9^3}\frac{9^4/2000^3}{1991}.
$$

$$
\begin{aligned}
\mathrm{E}[M] &\approx 99.5 + 0.995 + 0.0074625 + 0.00015 - 0.0001011991462 \\
&\approx 100.5025113009.
\end{aligned}
$$

Thus, the expected payout for hospitalization is approximately \$100.5025.