Real Analysis HW #5

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Question 1

1. Let $x_n \geqslant 0$ for all $n \in \mathbb{N}$.

(a) If $(x_n) \to 0$, show that $\sqrt{x_n} \to 0$.

Proof: Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n > \mathbb{N}$, $|x_n - 0| = |x_n| < \epsilon^2$.
We have

$$|\sqrt{x_n} - 0| = |\sqrt{x_n}| = \sqrt{|x_n|} < \sqrt{\epsilon^2} = \epsilon.$$

Thus, $\sqrt{x_n} \to 0$. 😆

(b) If $(x_n) \to x$, show that $\sqrt{x_n} \to \sqrt{x}$.

Proof: We consider two cases:

1. If $x = 0$ then see proof of (a).
2. Let $\epsilon > 0$ and $x > 0$. There exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - x| < \epsilon\sqrt{x}$.
   We have

   $$\left|\sqrt{x_n} - \sqrt{x}\right| = \left|\sqrt{x_n} - \sqrt{x}\right| \left(\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}\right) = |x_n - x| \frac{1}{\sqrt{x_n} + \sqrt{x}} \leqslant |x_n - x| \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{x}} \epsilon \sqrt{x} = \epsilon.$$

   Hence, $\sqrt{x_n} \to \sqrt{x}$.

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Question 2

2. Let (a_n) and (b_n) be sequences of real numbers.

(a) Show that if $(a_n) \to 0$ and $(a_n - b_n) \to 0$, then $(b_n) \to 0$.

Proof: If $(a_n) \to 0$ then for all $\epsilon > 0$ there exists N_1 such that for all $n > N_1$, $|a_n - 0| < \frac{\epsilon}{2}$. Similarly, for all $\epsilon > 0$ there exists N_2 such that for all $n > N_2$, $|(a_n - b_n) - 0| < \frac{\epsilon}{2}$. Consider the sum of these two quantities with $n > N^\star = \max\{N_1, N_2\}$. Then

$$|a_n - 0| + |(a_n - b_n) - 0| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

and it follows

$$|0 - a_n| + |a_n - b_n| < \epsilon.$$

By the Triangle Inequality Theorem,

$$|0 - a_n + a_n - b_n| = |0 - b_n| \leqslant |0 - a_n| + |a_n - b_n| < \epsilon.$$

Thus, $|0 - b_n| = |b_n - 0| < \epsilon \implies (b_n) \to 0$.

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(b) Show that if $(a_n) \to 0$ and $|b_n - b| \leqslant a_n$, then $(b_n) \to b$

Proof: If $(a_n) \to 0$ then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - 0| < \frac{\epsilon}{2}$. We know that

$$|b_n - b| \leqslant |a_n| \implies |b_n - b| - |a_n| \leqslant 0 < \frac{\epsilon}{2}.$$

After the summing the two quantities,

$$\begin{aligned} |b_n - b| - |a_n| + |a_n - 0| &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ \implies |b_n - b| - |a_n| + |a_n| &< \epsilon \\ \implies |b_n - b| &< \epsilon \\ \implies (b_n) &\to b. \end{aligned}$$

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Question 3

3. Consider $y_1 = 1$, $y_{n+1} = (2y_n + 3)/4$ for all $n \in \mathbb{N}$. Show by direct calculations that $y_1 < y_2 < 2$. Then, show that if $y_{n-1} < y_n < 2$ that $y_n < y_{n+1} < 2$. Use this to show that $\{y_n\}$ converges and find its limit.

Proof: If $y_1 = 1$ then $y_{1+1} = [2(1) + 3]/4 = \frac{5}{4} < 2$. So $y_1 < y_2 < 2$.
If $y_{n-1} < y_n < 2$ then

$$\begin{aligned} y_{n-1} < y_n < 2 \implies 2y_{n-1} < 2y_n < 2(2) &\implies 2y_{n-1} + 3 < 2y_n + 3 < 2(2) + 3 \\ &\implies \frac{2y_{n-1} + 3}{4} < \frac{2y_n + 3}{4} < \frac{2(2) + 3}{4}. \\ &\implies y_n < y_{n+1} < \frac{7}{4} < 2 \end{aligned}$$

We see that $y_n < y_{n+1}$ so the sequence is monotone. We also see that it is bounded by 2. Therefore, by M.C.T, y_n is convergent.

Let $\lim_{n\to\infty} y_n = L$. Then

$$\begin{aligned} \lim_{n\to\infty} y_{n+1} &= \lim_{n\to\infty} (2y_n + 3)/4 \\ \implies L &= (2L + 3)/4 \\ \implies 4L &= 2L + 3 \\ \implies 2L &= 3 \\ \implies L &= 3/2. \end{aligned}$$

Thus, $y_n \to 3/2$.

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Question 4

4. (Cesaro Means). Show that if (x_n) is a convergent sequence, then the sequence given by the averages:

$$y_n = \frac{x_1 + x_2 + \dots x_n}{n}$$

also converges to the same limit.

Proof: Let $\epsilon > 0$ and $(x_n) \longrightarrow x$. There exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $|x_n - x| < \epsilon/2$. Then

$$|y_n - x| = \left|\frac{x_1 + x_1 + \cdots + x_n}{n}\right| =$$
$$\left|\frac{x_1 + x_1 + \cdots + x_{N_1} + \cdots + x_n}{n} - \frac{nx}{n}\right| =$$
$$\left|\frac{x_1 - x + x_1 - x + \cdots + x_{N_1} - x}{n} + \frac{x_{N_1+1} - x + \cdots + x_n - x}{n}\right|$$
$$\leqslant \underbrace{\left|\frac{x_1 - x + x_1 - x + \cdots + x_{N_1} - x}{n}\right|}_{K} + \underbrace{\left|\frac{x_{N_1+1} - x + \cdots + x_n - x}{n}\right|}_{Z}$$

Since for all $n > N_1$, $|x_n - x| < \epsilon/2$, it follows

$$Z < \frac{(n - N_1)}{n}\frac{\epsilon}{2} \leqslant \frac{\epsilon}{2}.$$

Because the numerator of K is a fixed number, by the Archimedes Principle there exists N_2 such that for all $n > N_2$, $K < \epsilon/2$. Let $N = \max\{N_1, N_2\}$. Then for all $n > N$, we have

$$|y_n - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, we conclude $(y_n) \longrightarrow x$ and also $(x_n) \longrightarrow x$. ☺

Question 5

5. ChatGPT Challenge: To prove that if $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$, you can use the definition of a limit and the properties of limits. Here's a proof:

Proof: Let $\epsilon > 0$ be arbitrary. Since $\lim(a_n - b_n) = 0$, there exists an N_1 such that for all $n \geqslant N_1$, we have $|a_n - b_n| < \epsilon/2$. Now, consider the inequality $|a_n - b_n| < \epsilon/2$. We can add $|b_n|$ to both sides of this inequality: $|a_n - b_n| + |b_n| < \epsilon/2 + |b_n|$

By the triangle inequality, we know that $|a_n - b_n + b_n| \leqslant |a_n - b_n| + |b_n|$, so: $|a_n| < \epsilon/2 + |b_n|$ Now, we have the following inequalities: $|a_n - b_n| < \epsilon/2$ (from our choice of N_1) $|a_n| < \epsilon/2 + |b_n|$

Everything up until this point tracks. You have clearly shown how we got to this point.

Now, choose N_2 such that for all $n \geqslant N_2$, we have $|b_n| < \epsilon/2$.

Where does this come from? This only works if we know (b_n) converges to 0. You are trying to prove that (b_n) and (a_n) converge to the same value. We do not know if one or both converge to zero.

Let $N = \max(N_1, N_2)$. For $n \geqslant N$, we have: $|a_n - b_n| < \epsilon/2$ (from the choice of N_1) $|b_n| < \epsilon/2$ (from the choice of N_2) Now, let's use these inequalities to bound $|a_n|$: $|a_n| = |a_n - b_n + b_n| \leqslant |a_n - b_n| + |b_n| < \epsilon/2 + \epsilon/2 = \epsilon$ This shows that for all $n \geqslant N$, we have $|a_n| < \epsilon$, which means that $\lim a_n = 0$. Since ϵ was arbitrary, we have shown that for any $\epsilon > 0$, there exists an N such that for all $n \geqslant N$, $|a_n| < \epsilon$, which is the definition of $\lim a_n = 0$. Therefore, we have proved that if $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

Again we are trying to show that (a_n) and (b_n) converge to the same number. You have not even proved that $\lim b_n = 0$, so this does not show that $\lim(a_n) = \lim(b_n)$. Make sure you have a clear goal of what you are trying to prove and lay clear logical steps to prove that assertion.