Real Analysis HW#1

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Question 1

Let n be a positive integer that is not a perfect square. Prove that $\sqrt{n}$ is irrational.

Solution: Assume, for contradiction, that $\sqrt{n}$ is a rational. Then $\sqrt{n}$ can be written in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$.
We have

$$\frac{a}{b} = \sqrt{n} \implies \left(\frac{a}{b}\right)^2 = n \implies a^2 = nb^2 \tag{1}$$

This means that n divides a^2. By the Fundamental Theorem of Arithmetic we can write a, n, and b as unique product of primes.
Thus,

$$a^2 = nb^2 \implies \left(\prod_{i=1}^{k} P_i^{n_i}\right)^2 = \prod_{j=1}^{l} P_j^{m_j} \left(\prod_{k=1}^{t} P_k^{l_k}\right)^2 \tag{2}$$

After simplification we have

$$\prod_{i=1}^{k} P_i^{2n_i} = \prod_{j=1}^{l} P_j^{m_j} \prod_{k=1}^{t} P_k^{2l_k} \tag{3}$$

In both expressions of a^2 and b^2, as a product of primes, we have an even number of each prime in the product. Because n is not a perfect square, there must be at least 1 prime that is expressed an odd number of times. We are then guaranteed that by expressing nb^2 as a product of primes there must be at least 1 prime which appears an odd number of times. However, the left hand side of (3) clearly shows this is not the case $\rightarrow\leftarrow$.
With this contradiction we have no choice but to overturn our assumption and conclude that $\sqrt{n} \in \mathbb{R}\setminus\mathbb{Q}$ ☺

Question 2

Use the Principle of Mathematical Induction to prove:

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3} \quad \forall n \in \mathbb{N}$$

Solution:
Let $n = 1 \in \mathbb{N}$. Then $1^2 = \frac{4(1)^3-1}{3} = 1$, showing that the equality holds for $n = 1$. We assume that

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3},$$

is true and we proceed with induction on n. We want to show $P(n+1) = \frac{4(n+1)^3-(n+1)}{3}$.
Consider

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}.$$

Then, adding $(2(n+1)-1)^2$ to both sides of the equation,

$$1^2 + 3^2 + \cdots + (2n-1)^2 + (2(n+1)-1)^2 = \frac{4n^3 - n}{3} + (2(n+1)-1)^2$$

$$= \frac{4n^3 - n}{3} + (4n^2 + 4n + 1)$$

$$= \frac{4n^3 - n + 12n^2 + 12n + 3}{3}$$

$$= \frac{4n^3 + 8n^2 + 4n + 4n^2 + 8n + 4 - n - 1}{3}$$

$$= \frac{4[n^3 + 2n^2 + n + n^2 + 2n + 1] - (n+1)}{3}$$

$$= \frac{4[(n^2 + 2n + 1)(n+1)] - (n+1)}{3}$$

$$= \frac{4(n+1)^3 - (n+1)}{3}.$$

Thus, $P(n+1) = \frac{4(n+1)^3-(n+1)}{3}$, proving $1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3-n}{3} \quad \forall n \in \mathbb{N}$ 😄

Question 3

Let $n > 1$ be a positive integer and let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that

$$\left|\sum_{k=1}^{n} a_k\right| \leqslant \sum_{k=1}^{n} |a_k|$$

Solution: Let $n = 2 \in \mathbb{Z}^+$. Then

$$\left|\sum_{k=1}^{2} a_k\right| \leqslant \sum_{k=1}^{2} |a_k| \implies |a_1 + a_2| \leqslant |a_1| + |a_2|,$$

which we know is true by the Triangle Inequality Theorem. We then want to show that $\left|\sum_{k=1}^{n+1} a_k\right| \leqslant \sum_{k=1}^{n+1} |a_k|$. We assume for proof by induction that

$$\left|\sum_{k=1}^{n} a_k\right| \leqslant \sum_{k=1}^{n} |a_k|$$

is true. Expanding yields

$$|a_1 + a_2 + \cdots + a_n| \leqslant |a_1| + |a_2| + \cdots + |a_n|.$$

Adding $|a_{n+1}|$ to both sides results in

$$|a_1 + a_2 + \cdots + a_n| + |a_{n+1}| \leqslant |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|,$$

and by the Triangle Inequality Theorem we have

$$|a_1 + a_2 + \cdots + a_n + a_{n+1}| \leqslant |a_1 + a_2 + \cdots + a_n| + |a_{n+1}| \leqslant |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|.$$

Therefore,

$$\left|\sum_{k=1}^{n+1} a_k\right| \leqslant \sum_{k=1}^{n+1} |a_k|$$

and we have indeed shown

$$\left|\sum_{k=1}^{n} a_k\right| \leqslant \sum_{k=1}^{n} |a_k| \quad 😄$$

Question 4

Exercise 1.2.8

(a) $f : \mathbb{N} \to \mathbb{N}$ where f is 1-1 but not onto. ***Solution:*** $f(x) = 2x$

(b) $f : \mathbb{N} \to \mathbb{N}$ where f is not 1-1 but onto. ***Solution:*** If f is onto then for every $y \in \mathbb{N}$ there exists $x \in \mathbb{N}$ where $f(x) = y$. If we assume f is *not* 1-1 then we know that there exists $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. As f is a function, we can not map $x \in \mathbb{N}$ to more than one $y \in \mathbb{N}$, then it must be that we are missing at least one $y \in \mathbb{N}$. Therefore, f can not be onto, which means this is impossible.

(c) $f : \mathbb{N} \to \mathbb{Z}$ where f is 1-1 and onto.

Solution:

$$f(n) = \begin{cases} odd & \frac{n+1}{2}(-1)^{n+1} \\ even & \frac{n}{2}(-1)^{n+1} \end{cases}$$

Question 5

Exercise 1.2.12

(a) ***Solution:*** For proof by induction first consider $y_1 = 6 > -6$. Thus our base case holds. We assume the induction hypothesis

$$y_{n+1} = \frac{(2y_n - 6)}{3} \tag{4}$$

is true. We want to show that if $y_n > -6$ then it follows $y_{n+1} > -6$. First we subtract 3 from both sides.

$$y_n > -6 \implies y_n - 3 > -6 - 3 \implies 2(y_n - 3) > -18$$

and we have

$$\frac{2y_n - 6}{3} > -6.$$

This means that $y_{n+1} > -6$, and therefore $y_n > -6\ \forall n \in \mathbb{N}$ 😄

(b) ***Solution:*** We want to show that the sequence $(y_1, y_2, \dots)$ is decreasing, i.e $y_n > y_{n+1}\ \forall n \in \mathbb{N}$. We know $y_1 = 6$ and $y_2 = 2$. Clearly $y_1 > y_2$. Continuing on we assume that our induction hypothesis,

$$y_{n+1} = \frac{(2y_n - 6)}{3},$$

is true. We want to show if we have $y_n > y_{n+1}$ then we are guaranteed to have $y_{n+1} > y_{n+2}$. First we consider

$$y_n > y_{n+1} \implies y_n - 3 > y_{n+1} - 3 \implies 2y_n - 6 > 2y_{n+1} - 6 \implies \frac{2y_n - 6}{3} > \frac{2y_{n+1} - 6}{3}.$$

Then, we can substitute using induction hypothesis,

$$\frac{2y_n - 6}{3} > \frac{2y_{n+1} - 6}{3} \implies y_{n+1} > y_{n+1+1} \implies y_{n+1} > y_{n+2},$$

which is is exactly what we needed to show. Thus, $y_n > y_{n+1}\ \forall n \in \mathbb{N}$ 😄

Question 6

Grade the following proof: Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

Proof:To prove that $\sqrt{3}$ is irrational, we can use a proof by contradiction. The idea is to assume that $\sqrt{3}$ is rational, and then derive a contradiction.

Good job setting the stage for the rest of the proof.

Assumption: $\sqrt{3}$ is rational.

If $\sqrt{3}$ is rational, it can be expressed as a fraction $\frac{a}{b}$ in its simplest form, where a and b are integers with no common factors other than 1 (i.e., they are co prime), and b is not equal to 0.

So, we have: $\sqrt{3} = \frac{a}{b}$

Squaring both sides of the equation: $3 = \frac{a^2}{b^2}$

Rearranging the equation: $a^2 = 3 \cdot b^2$

From this equation, we can see that a^2 is a multiple of 3 , which implies that a must also be a multiple of 3. Let's represent a as $3k$ where k is an integer:

How do we know for certain that if a^2 is a multiple of 3 then a must be a multiple of 3? It is not immediately obvious why this is the case, so you should make some justification why this must be true.

$$(3k)^2 = 3 \cdot b^2$$
$$9k^2 = 3 \cdot b^2$$
$$3k^2 = b^2$$

Now, b^2 is also a multiple of 3 , which means b must be a multiple of 3 . However, this contradicts our initial assumption that a and b have no common factors other than 1 (i.e., they are co prime). This contradiction arises from assuming that $\sqrt{3}$ is rational.

I like the way you walk through exactly how the contradiction arises and then hint at what this leads us to conclude.

Hence, our initial assumption that $\sqrt{3}$ is rational must be false, which means that $\sqrt{3}$ is irrational. Now, to address whether a similar argument works for $\sqrt{6}$:

No, a similar argument does not work for $\sqrt{6}$. The proof for the irrationality of $\sqrt{3}$ relies on the fact that it leads to a contradiction with respect to the common factors of a and b. However, for $\sqrt{6}$, this argument does not apply directly because the factors of 6 are not the same as the factors of 3. In fact, $\sqrt{6}$ is not irrational; it is a rational number. It can be expressed as the fraction $\frac{2\sqrt{3}}{3}$, which is a ratio of two integers.

Your argument starts of alright, but you need to specify what about the difference in factors between 6 and 3 means that you preceding argument needs to be changed to prove that $\sqrt{6}$ is irrational. The second part of your statement, $\frac{2\sqrt{3}}{3}$, is false. This is not a ratio of two integers, in fact you just proved before that $\sqrt{3}$ itself is irrational!

Overall, you articulate most of your reasoning well, but make sure that you are thorough in the lines of your argument. Each step should clearly follow the preceding one - and make sure what you are saying is indeed true!

(2.5/5)