# Real Analysis HW #7

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## Question 1

Let $(x_n)$ be a sequence and suppose that the sequence $(x_{n+1} - x_n)$ converges to 0 . Give an example to show that the sequence $(x_n)$ may not converge. (See CHATBOT Challenge)

***Solution:*** Let

$$x_n = \sum_{k=1}^{n} 1/k.$$

This is the harmonic series, which converges to infinity. Let $\epsilon > 0$. By Archimedes Principle there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Let $n > N$. Then

$$|(x_{n+1} - x_n) - 0| = \left|\frac{1}{n+1} - \frac{1}{n}\right| < \left|\frac{1}{N}\right| < \epsilon.$$

Thus, $(x_{n+1} - x_n)$ converges to 0, but $(x_n)$ converges to infinity.

## Question 2

Let $(x_k)$ and $(y_k)$ be two sequences and let $(r_k)$ be a sequence of positive numbers that converges to 0 . Suppose that $0 < |y_k - x_k| < r_k \forall k \in \mathbb{N}$.

(a) Give an example to show that the sequences $(x_k)$ and $(y_k)$ may not converge.

***Solution:*** Let

$$y_k = k + \frac{1}{k}$$

and let

$$x_k = k + \frac{1}{(k+1)}.$$

(b) Suppose that $(x_k)$ converges to $L$. Prove that the sequence $(y_k)$ converges to $L$.

***Proof:*** Let $\epsilon > 0$. Because $(r_k)$ converges to 0 then there exists $N_1 \in \mathbb{N}$ such that for all $k > N_1$, $|r_k - 0| < \frac{\epsilon}{2}$. Because $(x_k)$ converges to $L$ there exists $N_2 \in \mathbb{N}$ such that for all $k > N_2$, $|x_k - L| < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$ and $k > N$.

Then,

$$0 < |y_k - x_k| < |r_k - 0| < \frac{\epsilon}{2}.$$

$$\implies |y_k - L + L - x_k| < \frac{\epsilon}{2}.$$

$$\implies |y_k - L| - |L - x_k| \leqslant |y_k - L + L - x_k| < \frac{\epsilon}{2}.$$

$$\implies |y_k - L| - \underbrace{|L - x_k|}_{<\frac{\epsilon}{2}} < \frac{\epsilon}{2}.$$

And we know that $|x_k - L| < \frac{\epsilon}{2}$,

$$\implies |y_k - L| < \epsilon$$

Thus, $y_k$ converges to $L$. ☺

## Question 3

Assume that $(x_n)$ is a bounded sequence with the property that every convergent subsequence of $(x_n)$ converges to the same limit $x \in \mathbb{R}$. Show that $(x_n)$ must converge to $x$.

***Proof:*** To show this we will prove the contrapositive of the following: If every convergent subsequence of $x_n$ converges to $x$, then $x_n$ must converge to $x$ as well.

We must show that if $x_n \nrightarrow x$, then either (1) $x_n$ is not bounded or (2) there exists a subsequence $x_{n_k}$ such that $x_{n_k} \longrightarrow y$ where $y \neq x$.

1. We know that $(x_n)$ is bounded so we must consider case (2).

2. Since $x_n$ does not converge to $x$ there exists $\epsilon_0$ such that for all $N \in \mathbb{N}$ there exists an $n_N > N$ where $|x_{n_N} - x| > \epsilon_0$. Now consider the set of all the terms $x_n \notin B(x, \epsilon_0)$, denoted $x_{n^\star}$. By Bolzano Weierstrass, there exists a subsequence of $x_{n^\star}$, where $x_{n_k} \longrightarrow y$. We know $y \neq x$ because $x_{n^\star} \notin B(x, \epsilon_0)$.

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## Question 4

Let $(x_n)$ be a Cauchy sequence. Show directly that $(x_n)$ is bounded.

Let $\epsilon > 0$ and $(x_n)$ be a Cauchy sequence. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geqslant N$, $|x_n - x_m| < 1$. Then,

$$|x_m - x_N| < 1.$$

$$\implies |x_m| - |x_N| < |x_m - x_N| < 1.$$

$$\implies |x_m| - |x_N| < 1.$$

$$\implies |x_m| < 1 + |x_N|.$$

Thus, we have a bound for all the terms $N$ and beyond. Let

$$B = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_N|, |x_N| + 1\}.$$

Then $|x_n| < B$ for all $n \in \mathbb{N}$ and therefore $(x_n)$ must be bounded.

## Question 5

Let $(x_n)$ be a sequence and let $r$ be a real number satisfying $0 < r < 1$. Suppose that $|x_{n+1} - x_n| < r\,|x_n - x_{n-1}|$ for all $n \geqslant 2$. Prove that $(x_n)$ is a Cauchy sequence and hence convergent.

***Proof:*** We know that $|x_{n+1} - x_n| < r\,|x_n - x_{n-1}|$. Thus,

$$
\begin{aligned}
&|x_3 - x_2| < r\,|x_2 - x_1| \\
\implies &|x_4 - x_3| < r^2\,|x_2 - x_1| \\
\implies &|x_5 - x_4| < r^3\,|x_2 - x_1| \\
\implies &|x_{n+1} - x_n| < r^{n-1}\,|x_2 - x_1|.
\end{aligned}
$$

Now let

$$B = |x_2 - x_1|.$$

Thus, for $n \geqslant 2$, $|x_{n+1} - x_n| < Br^{n-1}$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n > N$,

$$B\frac{r^n}{1-r} < \epsilon.$$

Let $m > n > N$. We have,

$$
\begin{aligned}
|x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - x_{m-3} + \cdots + x_{n+1} - x_n| \\
&\leqslant |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + |x_{m-2} - x_{m-3}| + \cdots + |x_{n+1} - x_n| \\
&\leqslant \sum_{k=n+1}^{m} |x_k - x_{k-1}| \\
&\leqslant \sum_{k=n+1}^{m} Br^{k-1}.
\end{aligned}
$$

Now, if $(x_n) = \sum_{k=n+1}^{m} Br^{n-1}$, then

$$
\begin{aligned}
x_n - rx_n &= B\left[\sum_{k=n+1}^{m} r^{k-1} - \sum_{k=n+2}^{m+1} r^{k-1}\right] \\
\implies x_n(1-r) &= B\left[r^n - r^m\right] \\
\implies x_n &= B\left[\frac{r^n - r^m}{1-r}\right] \\
&\leqslant B\frac{r^n}{1-r} < \epsilon.
\end{aligned}
$$

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**Question 6**

CHAT GPT CHALLENGE: Let $(x_n)$ be a sequence such that $(x_{n+1} - x_n)$ converges to 0 . We want to show that $(x_n)$ also converges.

By the definition of a convergent sequence, we need to show that for any $\epsilon > 0$, there exists a positive integer $N$ such that for all $n \geqslant N$, $|x_n - L| < \epsilon$ for some real number $L$.

**Good job setting the stage for your proof and clearly stating what you need to show to substantiate your claim. This is correct.**

Since $(x_{n+1} - x_n)$ converges to 0 , we can choose $\epsilon > 0$ and find a positive integer $M$ such that for all $m \geqslant M$, $|(x_{m+1} - x_m) - 0| < \epsilon$. This implies that for all $m \geqslant M$, $|x_{m+1} - x_m| < \epsilon$. Now, let $N = M$. For any $n \geqslant N$, we have:

$$\begin{aligned} |x_n - x_N| &= |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \ldots + (x_{N+1} - x_N)| \\ &\leqslant |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \ldots + |x_{N+1} - x_N| \\ &< \epsilon + \epsilon + \ldots + \epsilon \quad (\text{ Since } |x_{m+1} - x_m| < \epsilon \text{ for all } m \geqslant M) \\ &= (n - N)\epsilon. \end{aligned}$$

Dividing both sides by $(n - N)$, we get:

$$|x_n - x_N| < \epsilon$$

**This is incorrect. When you divide both sides by $(n - N)$ you get**

$$\frac{1}{(n-N)} |x_n - x_N| < \epsilon$$

**which does not show that $|x_n - x_N| < \epsilon$.**

This shows that for any $\epsilon > 0$ and $n \geqslant N$, we have $|x_n - x_N| < \epsilon$, where $N = M$. Therefore, $(x_n)$ converges to $x_N$.

**This does not prove what you wanted to show.**

Hence, we have shown that if $(x_{n+1} - x_n)$ converges to 0 , then $(x_n)$ also converges.