The Real Numbers

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0.1 The Real Numbers

0.1.1 The irrationality of the square root of 2

0.1.2 Preliminaries

`Notation`

- $\forall \rightarrow$ For all/each/every
- $\exists \rightarrow$ There exists
- $\mathbb{R} \setminus \mathbb{Q} \rightarrow$ Irrationals
- $\mathbb{R} \rightarrow$ Real numbers
- $\mathbb{Z} \rightarrow$ Integers
- $\mathbb{Q} \rightarrow$ Rational numbers
- $\mathbb{N} \rightarrow$ Natural numbers
- $BWOC \rightarrow$ By way of contradiction
- $\rightarrow\!\leftarrow \rightarrow$ Contradiction
- $! \rightarrow$ Unique/factorial
- 😄 $\rightarrow$ End of proof (Quod Erat Demonstrandum)
- $\epsilon \rightarrow$ Epsilon, usually a small positive quantity
- $\ni \rightarrow$ Such that

Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.

Proof. $\rightarrow$ If $a = b$ then we have $|a - b| = 0$. No matter which $\epsilon > 0 \in \mathbb{R}$ is chosen we have that $|a - b| = 0 < \epsilon$. Thus, $a = b$

$\leftarrow$ Suppose, by way of contradiction, that $a \neq b$ and $\forall \epsilon > 0$, we have $|a - b| < \epsilon$. Let $\epsilon_0 = \frac{|a-b|}{2}$, then it is clear that $|a - b| < \frac{|a-b|}{2} = \epsilon_0$ is false. Thus, with this contradiction we overturn our assumption and conclude a must equal b. 😄

0.1.3 The Axiom of Completeness

Definition 0.1.1: Bounded Above

A set $A \subseteq \mathbf{R}$ is bounded above if there exists a number $b \in \mathbf{R}$ such that $a \leqslant b$ for all $a \in A$. The number b is called an upper bound for A. Similarly, the set A is bounded below if there exists a lower bound $l \in \mathbf{R}$ satisfying $l \leqslant a$ for every $a \in A$.

Definition 0.1.2: Supremum of a Set

A real number s is the *Supremum* or the least upper bound for a set $A \subseteq \mathbf{R}$ if:
(i) s is an upper bound for A;
(ii) if b is any upper bound for A, then $s \leqslant b$.

Lemma 1.3.8. Assume $s \in \mathbf{R}$ is an upper bound for a set $A \subseteq \mathbf{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

Proof. $\rightarrow$ Let $s = \sup A$ and arbitralily choose any $\epsilon > 0$. Suppose, by way of contradiction, there does not exist $a \in A$ with $a > s - \epsilon$. So for all $a \in A$ we have that $a \leqslant s - \epsilon < s$. This means that $s - \epsilon$ is an upperbound of A. Thus, there must exist an element $a \in A$ such that $s - \epsilon < a$.

$\leftarrow$ Let $\epsilon > 0$ and suppose there exists $a \in A$ with $s - \epsilon < a$ and we know that s is an upperbound of A. To show that s is the least upperbound, by way of contradiction, suppose that $b < s$ and b is another upperbound of A.

Consider $\epsilon_0 = s - b > 0$. By hypothesis there exists a with $s - \epsilon_0 < a \implies s - (s - b) < a \implies b < a$. As we know that b is an upperbound of A this is impossible. Therefore it must be that $b \geqslant s$. Hence, $s \leqslant b$ and $s = \sup A$. 😄