HW #1

Jack Krebsbach

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Question 1

Let n be a positive integer that is not a perfect square. Prove that $\sqrt{n}$ is irrational.

Solution: Assume, for contradiction, that $\sqrt{n}$ is a rational. Then $\sqrt{n}$ can be written in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and a, b are cooprime, or have no common factors.

We have

$$\frac{a}{b} = \sqrt{n} \implies \left(\frac{a}{b}\right)^2 = n \implies a^2 = nb^2 \tag{1}$$

Clearly n divides a^2. By the Fundemental Theorem of arithmetic we can write a and n as a product of primes.

Thus,

$$\frac{a^2}{n} = \frac{\left(\prod_{i=1}^{k} P_i^{n_i}\right)^2}{\prod_{j=1}^{k} P_j^{m_j}} = \frac{\left(\prod_{i=1}^{k} P_i^{n_i}\right)\left(\prod_{i=1}^{k} P_i^{n_i}\right)}{\prod_{j=1}^{k} P_j^{m_j}} = b^2 \tag{2}$$

.

Because n divides a^2 we can re-write b^2 as the product

$$n\left(\prod_{l=1}^{t} P_l^{m_l}\right) = b^2. \tag{3}$$

This means $b^2 \geqslant n$. From the equation 2 we have $a(a) = nb^2$ and it follows that $a \geqslant n$. Therefore we can rearrange (2), which yields

$$a^2 = (n)(a)\left(\prod_{i=1}^{z} P_i^{m_i}\right) \implies a = n\left(\prod_{i=1}^{z} P_i^{m_i}\right).$$

Thus, n divides a in addition to a^2. Because of this we know that we can rewrite a in terms of n, or $a = t(n)$ where $t \in \mathbb{Z}$. Then

$$(tn)^2 = nb^2 \implies t^2n^2 = nb^2 \implies nt^2 = b^2 \tag{4}$$

which means n divides b^2, and by the preceding logic also divides b. We have the n divides a and b. Thus, a and b can not be be cooprime $\rightarrow\!\leftarrow$. With this contradiction we have no choice but to conclude that $\sqrt{n} \in \mathbb{R} \setminus \mathbb{Q}$ ☺

Question 2

Use the Principle of Mathematical Induction to prove:

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3} \quad \forall n \in \mathbb{N}$$

Solution:

Let $n = 1 \in \mathbb{N}$. Then $1^2 = \frac{4(1)^3-1}{3} = 1$, showing that the equality holds for $n = 1$. We asssume that the induction hypothesis,

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}$$

, is correct and we procceed with induction on n. We want to show $P(n+1) = \frac{4(n+1)^3-(n+1)}{3}$.

Consider

$$\begin{aligned}1^2 + 3^2 + \cdots + (2n-1)^2 + (2(n+1)-1)^2 &= \frac{4n^3 - n}{3} + (2(n+1)-1)^2 \\ &= \frac{4n^3 - n}{3} + (2n+1)^2\end{aligned}$$

$$= \frac{4n^3 - n}{3} + (4n^2 + 4n + 1)$$

$$= \frac{4n^3 - n + 12n^2 + 12n + 3}{3}$$

$$= \frac{4n^3 + 8n^2 + 4n + 4n^2 + 8n + 4 - n - 1}{3}$$

$$= \frac{4[n^3 + 2n^2 + n + n^2 + 2n + 1] - (n + 1)}{3}$$

$$= \frac{4[(n^2 + 2n + 1)(n + 1)] - (n + 1)}{3}$$

$$= \frac{4(n + 1)^3 - (n + 1)}{3}.$$

Thus, $P(n + 1) = \frac{4(n+1)^3-(n+1)}{3}$, proving $1^2 + 3^2 + \cdots + (2n - 1)^2 = \frac{4n^3-n}{3} \quad \forall n \in \mathbb{N}$ 😄

Question 3

Let $n > 1$ be a positive integer and let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that

$$\left|\sum_{k=1}^{n} a_k\right| \leqslant \sum_{k=1}^{n} |a_k|$$

Let $n = 1 \in \mathbb{R}$. Then

$$\left|\sum_{k=1}^{1} a_k\right| \leqslant \sum_{k=1}^{1} |a_k| \implies |a_1| \leqslant |a_1|.$$

Thus the equality holds in this base case where $n = 1$. We then want to show that $\left|\sum_{k=1}^{n+1} a_k\right| \leqslant \sum_{k=1}^{n+1} |a_k|$. We assume, in the proccess of proof by induction, that

$$\left|\sum_{k=1}^{n} a_k\right| \leqslant \sum_{k=1}^{n} |a_k|$$

is true. Expansion yields

$$|a_1 + a_2 + \cdots + a_n| \leqslant |a_1| + |a_2| + \cdots + |a_n|.$$

Adding $|a_{n+1}|$ to both sides gives us

$$|a_1 + a_2 + \cdots + a_n| + |a_{n+1}| \leqslant |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|,$$

and by the Triangle Inequality Theorem we have

$$|a_1 + a_2 + \cdots + a_n + a_{n+1}| \leqslant |a_1 + a_2 + \cdots + a_n| + |a_{n+1}| \leqslant |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|.$$

Therefore,

$$\left|\sum_{k=1}^{n+1} a_k\right| \leqslant \sum_{k=1}^{n+1} |a_k|$$

and thus

$$\left|\sum_{k=1}^{n} a_k\right| \leqslant \sum_{k=1}^{n} |a_k| \quad 😄$$

Question 4

Exercise 1.2.8

Question 5

Exercise 1.2.12

Question 6

Grade the following proof: Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

Proof:To prove that $\sqrt{3}$ is irrational, we can use a proof by contradiction. The idea is to assume that $\sqrt{3}$ is rational, and then derive a contradiction.

Assumption: $\sqrt{3}$ is rational.

If $\sqrt{3}$ is rational, it can be expressed as a fraction $\frac{a}{b}$ in its simplest form, where a and b are integers with no common factors other than 1 (i.e., they are coprime), and b is not equal to 0.

So, we have: $\sqrt{3} = \frac{a}{b}$

Squaring both sides of the equation: $3 = \frac{a^2}{b^2}$

Rearranging the equation: $a^2 = 3 \cdot b^2$

From this equation, we can see that a^2 is a multiple of 3 , which implies that a must also be a multiple of 3. Let's represent a as $3k$ where k is an integer:

$$(3k)^2 = 3 \cdot b^2$$
$$9k^2 = 3 \cdot b^2$$
$$3k^2 = b^2$$

Now, b^2 is also a multiple of 3 , which means b must be a multiple of 3 . However, this contradicts our initial assumption that a and b have no common factors other than 1 (i.e., they are coprime). This contradiction arises from assuming that $\sqrt{3}$ is rational.

Hence, our initial assumption that $\sqrt{3}$ is rational must be false, which means that $\sqrt{3}$ is irrational. Now, to address whether a similar argument works for $\sqrt{6}$:

No, a similar argument does not work for $\sqrt{6}$. The proof for the irrationality of $\sqrt{3}$ relies on the fact that it leads to a contradiction with respect to the common factors of a and b. However, for $\sqrt{6}$, this argument does not apply directly because the factors of 6 are not the same as the factors of 3. In fact, $\sqrt{6}$ is not irrational; it is a rational number. It can be expressed as the fraction $\frac{2\sqrt{3}}{3}$, which is a ratio of two integers.