The Real Numbers

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0.1 The Real Numbers

0.1.1 The irrationality of the square root of 2

0.1.2 Preliminaries

`Notation`

- $\longrightarrow$ For all/each/every
- $\exists \longrightarrow$ There exists
- $\mathbb{R} \setminus \mathbb{Q} \longrightarrow$ Irrationals
- $\mathbb{R} \longrightarrow$ Real numbers
- $\mathbb{Z} \longrightarrow$ Integers
- $\mathbb{Q} \longrightarrow$ Rational numbers
- $\mathbb{N} \longrightarrow$ Natural numbers
- $BWOC \longrightarrow$ By way of contradiction
- $\longrightarrow\!\!\times\!\!\longleftarrow$ Contradiction
- $! \longrightarrow$ Unique/factorial
- ☺ $\longrightarrow$ End of proof (Quod Erat Demonstrandum)
- $\epsilon \longrightarrow$ Epsilon, usually a small positive quantity
- $\ni\longrightarrow$ Such that

Theorem 0.1.1

Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.

Proof: $\Rightarrow$ If $a = b$ then we have $|a - b| = 0$. No matter which $\epsilon > 0 \in \mathbb{R}$ is chosen we have that $|a - b| = 0 < \epsilon$. Thus, $a = b$

$\Leftarrow$ Suppose, by way of contradiction, that $a \neq b$. We know that $\forall \epsilon > 0$ and $|a - b| < \epsilon$. Let $\epsilon_0 = \frac{|a-b|}{2}$, then it is clear that $|a - b| < \frac{|a-b|}{2} = \epsilon_0$ is false. Thus, with this contradiction we overturn our assumption and conclude a must equal b. ☺

0.1.3 The Axiom of Completeness

Definition 0.1.1: Bounded Above

A set $A \subseteq \mathbf{R}$ is bounded above if there exists a number $b \in \mathbf{R}$ such that $a \leqslant b$ for all $a \in A$. The number b is called an upper bound for A. Similarly, the set A is bounded below if there exists a lower bound $l \in \mathbf{R}$ satisfying $l \leqslant a$ for every $a \in A$.

Definition 0.1.2: The Supremum of a Set

A real number s is the *Supremum* or the least upper bound for a set $A \subseteq \mathbf{R}$ if:
(i) s is an upper bound for A;
(ii) if b is any upper bound for A, then $s \leqslant b$.

Theorem 0.1.2 Lemma 1.3.8

Assume $s \in \mathbf{R}$ is an upper bound for a set $A \subseteq \mathbf{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

Proof: $\Rightarrow$ Let $s = \sup A$ and arbitrarily choose any $\epsilon > 0$. Suppose, by way of contradiction, there does not exist $a \in A$ with $a > s - \epsilon$. So for all $a \in A$ we have that $a \leqslant s - \epsilon < s$. This means that $s - \epsilon$ is an upper bound of A. However, s is the supremum of or the least upper bound of A , so this is a contradiction. Thus, there must exist an element $a \in A$ such that $s - \epsilon < a$.

$\Leftarrow$ Let $\epsilon > 0$ and suppose there exists $a \in A$ with $s - \epsilon < a$ and we know that s is an upper bound of A. To show that s is the least upper bound, by way of contradiction, suppose that $b < s$ and b is another upper bound of A.

Consider $\epsilon_0 = s - b > 0$. By hypothesis there exists a with $s - \epsilon_0 < a \implies s - (s - b) < a \implies b < a$. As we know that b is an upper bound of A this is impossible. Therefore it must be that $b \geqslant s$. Hence, $s \leqslant b$ and $s = \sup A$.

😊

0.1.4 Consequences of Completeness

Theorem 0.1.3 The Archimedes Property

(a) For all $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n > x$.
(b) For all $y \in \mathbb{R}$ with $y \neq 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y$.

Proof: (a) Suppose, by way of contradiction, that the natural numbers are bounded. Let $\alpha \in \mathbb{R}$ be an upper bound, so $n \leqslant \alpha$ for all $n \in \mathbb{N}$. Let $\beta = \sup \mathbb{N}$ [exists by completeness]. Now $\beta - 1$ is not an upper bound. By our theorem, there exists $n_0 \in \mathbb{N}$ such that $\beta - 1 < n_0$.

So $\beta < n_0 + 1 \in \mathbb{N}$. This is a contradiction because we assumed that β was the supremum of the natural numbers. Thus, $\mathbb{N}$ is unbounded. For any $\alpha \in \mathbb{R}$, we can find a natural number that is larger than α.

(b) To show why be is the case you can consider $x = \frac{1}{y}$ where $y \neq= 0$. 😊

0.1.5 Examples

1. Show that $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$.

   Proof: Suppose otherwise, that the intersection is no the empty set, and let $x \in \bigcap_{n=1}^{\infty}$. Then it follows $x \in (0, \frac{1}{n})$ for all $n \in \mathbb{N}$. By the corollary of the Archimedes Property, there exists $n_{\star} \in \mathbb{N}$ with $\frac{1}{n_{\star}} < x$. Then there does not exist x such that $x \in \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$. Therefore, $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$. 😊

2. Show there does not exist a smallest positive number

   Proof: To show that there does not exist a smallest positive number, suppose otherwise. Let $x \in \mathbb{R}^{+}$. By Archimedes property there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$. So x cannot be the smallest. 😊

Theorem 0.1.4 The Nested Cells Property

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leqslant x \leqslant b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a non-empty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof: Let

$$A = \{a_1, a_2, a_3, \dots\}$$

and

$$B = \{b_1, b_2, b_3, \dots\}.$$

A is bounded above by one element of B and B is bounded below by any $a \in A$. Let $x = \sup A$ which implies that $a_n \leqslant x$ for all $n \in \mathbb{N}$. Since b is an upper bound of A and $x = \sup A$, we have that $x \leqslant b_n$ for all n and that $a_n \leqslant x \leqslant b_n$ for all n.Thus, $x \in [a_n, b_n]$ for all n and $x \in \bigcap_{n=1}^{\infty} I_n$. 😆

0.1.6 Intersection Examples

- $\bigcap(0, \frac{1}{n}) = \emptyset$
- $\bigcap[0, \frac{1}{n}] = \{0\}$
- $\bigcap(0, \frac{1}{n}] = \emptyset$
- $\bigcap(-\frac{1}{n}, \frac{1}{n}) = \{0\}$

Theorem 0.1.5 Density of $\mathbb{Q}$ in $\mathbb{R}$

Let $a, b \in \mathbb{R}$. Then there exist $n \in \mathbb{Q}$ with $a < r < b$.

Proof: Since $a, b \in \mathbb{R}$, without loss of generality let $a < b$. Now, we have that $b - a > 0$. By the corollary to the Archimedes principle there exist $n_\star \in \mathbb{N}$ with $\frac{1}{n_\star} < b - a$.

Consider $n_\star a \in \mathbb{R}$. Pick $m \in \mathbb{N}$ so that

$$m - 1 \leqslant n_\star a < m.$$

In other words, we choose the smallest of natural numbers greater than $n_\star a$. By chance, it may be that one less than that number is $n_\star a$. so we end up with the equality $m - 1 \leqslant n_\star a$.

We have that

$$n_\star a < m \implies a < \frac{m}{n_\star}$$

and,

$$m \leqslant n_\star a + 1,$$

because we know that

$$m - 1 \leqslant n_\star a.$$

Next,

$$\frac{1}{n_\star} < (b - a) \implies 1 < n_\star(b - a) \implies 1 < n_\star b - n_\star a \implies n_\star a < n_\star b - 1 \implies a < \frac{n_\star b - 1}{n_\star} \implies a < b - \frac{1}{n_\star}.$$

We take

$$m \leqslant n_\star a + 1 < n_\star[b - \frac{1}{n_\star}] + 1 = n_\star b - 1 + 1 = n_\star b$$

So, $m < n_\star b$ which means $\frac{m}{n_\star} < b$.

😆

0.1.7 Cardinality

Definition 0.1.3: 1-1 and onto

- $f : a \longrightarrow b$ is 1-1 if $a_1 \neq a_2$ implies that $f(a_1) \neq f(a_2)$.
- $f : a \longrightarrow b$ is onto if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.

Definition 0.1.4: Cardinality

A set A has the same cardinality as B if there exists a 1-1 and onto function, $f : A \longrightarrow B$. We have $|A| = |B|$ or A B.

Examples

1. Show that $\mathbb{N} \sim \mathbb{Z}$.

   Proof:

   $$f(n) = \begin{cases} odd & \frac{n-1}{2} \\ even & -\frac{n}{2} \end{cases}$$

2. Show that $[0, 1] \sim [\pi, 5]$.

   Proof:

   $$y = (5 - \pi)x + \pi$$

   Is $(0, 1] \sim (\pi, 5)$? Yes but we need to be careful how we show.

Definition 0.1.5: Infinite

- A set is *finite* if $|A| = |\mathbb{N}_n| n \in \mathbb{N}$
- A set is *infinite* if it is not *finite*.

Definition 0.1.6: Countable

- An infinite set is *countable* if $|A| = |\mathbb{N}|$.
- An infinite set is *uncountable* if it is not *countable*.

Examples

1. $\mathbb{Q}$?
2. $\mathbb{R} \setminus \mathbb{Q}$?
3. $\mathbb{R} \setminus \mathbb{Q}$?
4. Is the union of countable sets countable?

Theorem 0.1.6
Let $|A| = n$ and $|B| = m$. Then $A \cup B$ is finite.

Proof: Let
$$A = \{a_1, a_2, \ldots, a_n\}$$
and
$$B = \{b_1, b_2, \ldots, b_m\}.$$

Define $n_1 = \min\{k : k \in \mathbb{N}, b_k \notin A\}$ and $n_2 = \min\{k : b_k \notin A, k > n_1\}$. Generally, $n_j = \min\{k : b_k \in A, k > n_{j-1}\}$. Then $|A| = n$ and $|B \setminus A| = j$.

$$f(l) = \begin{cases} a_l & l \leqslant n \\ b_l & n+1 \leqslant l \leqslant n+j \end{cases}$$

😄

Theorem 0.1.7
The subset, A, of a finite set B is finite. That is $A \subset B$ is finite if B is finite.l

Proof: Let $|B| = n$, $B = \{b_1, b_2, \ldots, b_n\}$.
Then let
$$n_1 = \min\{k : b_k \in A\},$$
$$n_2 = \min\{k : b_k \in A, k > n_1\},$$
and
$$n_j = \min\{k : b_k \in A, k > n_{j-1}\}.$$
Define the bijections $f : \mathbb{N}_{n_j} \to A$ where $f(j) = b_{n_j}$ is a finite bijection. 😄

Theorem 0.1.8
Let A and B be sets with $A \subset B$.

1. If B is countable, then A is countable or finite.
2. If A is uncountable, then B is uncountable.

Proof:

1. Since B is countable let $B = \{b_j : j \in \mathbb{N}$. To "count" A let $n_1 = \min\{k : b_k \in A\}$ and $n_2 = \min\{k : b_k \in A, k > n_1\}$. Generally,
$$n_j = \min\{k : b_k \in A, k > n_{j-1}\}.$$
If there does not exist $k \in b_k \in A$ then $|A| = m$, which is finite. We have a function $f(m) = b_{n_m}$ which is onto A in a 1-1 manner.
2. Contrapositive of 1!

😄

Theorem 0.1.9
The countable union of countable sets is countable.

Proof: Let A_n where $n \in \mathbb{N}$ be a collection of countable sets. So that $A_n = \{A_1, A_2, \ldots\}$. We can list off elements in each set within the collection $A_n = \{a_{nm} : m \in \mathbb{N}\}$. We want to show that $\bigcup_{n=1}^{\infty}\{a_{nm} : m \in \mathbb{N}\}$ is countable. Consider $f(a_{nm}) = 2^n 3^m$. This is 1-1 by prime factorization, known as the Fundamental Theorem of Arithmetic. The set $\{2^n 3^m : n, m \in \mathbb{N}\} \subset \mathbb{N}$ and is countable. 😄

Theorem 0.1.10

$\mathbb{Q}$ is countable.

Proof: Consider $A_n = \{\frac{m}{n} : m \in \mathbb{N}\}$. Then $\bigcup_{n=1}^{\infty} A_n$ is countable. The set $B_n = \{-\frac{m}{n} : m \in \mathbb{N}\}$ is also countable. The set $\{0\}$ is finite and thus countable. Altogether we have

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n \cup \{0\}$$

is countable by the previous theorem. ☺

0.1.8 Cantors Diagonilization

Theorem 0.1.11

$\mathbb{R}$ is uncountable. The open interval $(0,1) = \{x \in \mathbf{R} : 0 < x < 1\}$ is uncountable.

Proof: Suppose that the interval $(0,1)$ is countable. Then there exists a bijection $f : \mathbb{N} \to (0,1)$. We can express this like so.

N		$(0,1)$								
1	$\longleftrightarrow$	$f(1)$	=	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\cdots$
2	$\longleftrightarrow$	$f(2)$	=	$.a_{21}$	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	..
3	$\longleftrightarrow$	$f(3)$	=	$.a_{31}$	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	..
4	$\longleftrightarrow$	$f(4)$	=	$.a_{41}$	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	$\ldots$
5	$\longleftrightarrow$	$f(5)$	=	$.a_{51}$	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	.
6	$\longleftrightarrow$	$f(6)$	=	a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	$\cdots$
$\vdots$				$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Where $a_{nm} \in \{0, \ldots, 9\}$. We can define $y = y_1 y_2 \ldots$ where

$$y_{mm} = \begin{cases} 2 & a_{mm} \geqslant 5 \\ 7 & a_{mm} \leqslant 4 \end{cases}$$

Thus, we have constructed y that differs from every element in our list. This says that $(0,1) \subset \mathbb{R}$ is uncountable. Thus, $\mathbb{R}$ is uncountable.

☺

Second proof

Proof: To show that $\mathbb{R}$ is uncountable, suppose otherwise. Then $\mathbb{R}$ can be written $\mathbb{R} = \{x_1, x_2, x_3, \ldots\}$. We create a $1-1$ correspondance with the natural numnbers. Pick x_1 and choose an unbounded interval I_1 so that $x_1 \notin I_1$. Choose I_2 and x_2 so that $I_2 \subset I_1$ and $x_2 \notin I_2$. In similar fashion create $I_3 \subset I_2$ and choose $x_3 \notin I_3$. We are creating a sequence of nested, close bounded intervals with $x_n \notin I_n$.

By the nested inerval property $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. So there exists $\alpha \in \bigcap_{n=1}^{\infty} I_n$. Since α is a real number there exists $n_0 \in \mathbb{N}$ where $x_{n_0} = \alpha$. But we know $x_{n_0} \notin I_n$ by construction, and have found a contradiction. So $\mathbb{R}$ is uncountable. ☺