Real Analysis HW #2

Jack Krebsbach

Sep 13th

Question 1

Exercise 1.3.7. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Proof: Suppose that $b = \sup A$. Let a be an upper bound for A and $a \in A$. We know b is an upper bound so for every $a \in A$ we have $a \leqslant b$. Since $b = \sup A$ and a is an upper bound of A we also know $b \leqslant a$. Thus, $a = b = \sup A$. 😄

Question 2

Exercise 1.4.1. Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well. ***Solution:*** If $a, b \in \mathbb{Q}$ then we can write a and b as a ratio of integers, $a = \frac{z}{k}$ and $b = \frac{l}{t}$ with $t, l, z, k \in \mathbb{Z}$. Consider the sum

$$a + b = \frac{z}{k} + \frac{l}{t} = \frac{zt + lk}{kt}.$$

Thus, we can write $a + b$ as a ratio of two integers $(zt + lk, kt \in \mathbb{Z}) : a + b \in \mathbb{Q}$.

Consider the product

$$ab = \frac{z}{k}\frac{l}{t} = \frac{lt}{zk} \in \mathbb{Q}.$$

Both the denominator and the numerator are in the integers. So we can express ab as a ratio of two integers and $ab \in \mathbb{Q}$.

$\mathbb{Q}$ is thus closed under addition and multiplication.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.
Solution: If $a \in \mathbb{Q}$ then we can define a as a ratio of two integers $m, n \in \mathbb{Z}$ such that $a = \frac{m}{n}$ where $a \neq 0$. If $t \in \mathbb{R} \setminus \mathbb{N}$. Then we can not an expression of t as a ratio of two integers. Then it is impossible to have $a + t$ and at as a ratio of two integers.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is I closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st ?
Solution: Consider the sum of two irrational numbers $a = 1 + \pi$ and $b = 1 - \pi$. We have that $a + b = 2 \in \mathbb{N}$, showing that the irrationals are not closed under addition.

We have that for for some $i \in \mathbb{R} \setminus \mathbb{Q}$ we can construct a rational number by simply squaring i. For example, $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ and $\sqrt{2}^2 = 2 \in \mathbb{N}$. So the irrationals are not closed under multiplication.

Question 3

Exercise 1.4.3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Assume that there exists some $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$. Then for every $n \in \mathbb{N}$ we have that $x \in (0, \frac{1}{n})$. However,by Archimedes Property for any real number $y, y \neq 0$ there exists $n \in \mathbb{N}$ such that $\frac{q}{n} < y$. So there must exist $n \in \mathbb{N}$ such that $\frac{1}{n} < x$, implying that $x \notin \bigcap_{n=1}^{\infty}(0, 1/n)$ and $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

Question 4

Exercise 1.4.5. Using Exercise 1.4.1, supply a proof for Corollary 1.4 .4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution: Consider two numbers $a < b$, it follows that $a - \sqrt{2} < b - \sqrt{2}$.By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists some $x \in \mathbb{Q}$, which can be expressed $x = \frac{l}{k}$, $a, b \in \mathbb{Z}$, such that $a - \sqrt{2} < \frac{l}{k} < b - \sqrt{2}$.

We can add $\sqrt{2}$ to the equality

$$a - \sqrt{2} < \frac{l}{k} < b - \sqrt{2},$$

yielding,

$$a < \frac{l}{k} + \sqrt{2} < b.$$

Thus, we have found a real number, $x + \sqrt{2}$, such that $a < x + \sqrt{2} < b$. For all real numbers $a, b \in \mathbb{R}$ where $a < b$ there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < t < b$.

Question 5

Exercise 1.4.8. Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.

$A = x : x = \sqrt{2} - 1/n, n \in N.$
$B = (\sqrt{2} - 1, \sqrt{2}) \setminus A$
We have found sets satisfying the conditions A and B with $A \cap B = \emptyset$, $\sup A = \sup B = \sqrt{2}$, $\sqrt{2} \notin A$ and $\sqrt{2} \notin B$

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

$J_n = (-\frac{1}{n}, \frac{1}{n})$.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in R : x \geqslant a\}$.)

$J_n = [n, \infty)$. Assume that there exists some $x \in \mathbb{R}$ that exists in every J_n. By Archimedes Property we know for any $x \in \mathbb{R}$ there exists $n^\star \in \mathbb{N}$ where $n < n^\star$, so we know $x \notin \bigcap_{n=1}^{n^\star} J_n$. Thus we have $\bigcap_{n=1}^{\infty} J_n = \emptyset$.

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Question 6

6. ChatGPT Challenge: Is the following correct? Make comments as if you are grading it:

Proof: $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$
($\Rightarrow$) Direction Assume that $a < b$. We want to show that $a < b + \epsilon$ for every $\epsilon > 0$. Let $\epsilon > 0$ be arbitrary. Adding ϵ to both sides of the inequality $a < b$ gives: $a + \epsilon < b + \epsilon$ Since $\epsilon > 0$, we have $a < a + \epsilon$. Combining these inequalities, we get: $a < a + \epsilon < b + \epsilon$ Thus, $a < b + \epsilon$ for every $\epsilon > 0$, as desired.
Good job explaining what you are going to show and proving it with clear logical steps.

($\Leftarrow$) Direction Assume that $a < b + \epsilon$ for every $\epsilon > 0$. We want to show that $a < b$.
Good job showing the end goal of this direction.
Suppose, for the sake of contradiction, that $a \geqslant b$. Then, the difference $a - b$ is non-negative or zero ($a - b \geqslant 0$). Choose $\epsilon = \frac{a-b}{2}$. Since $a \geqslant b$, we have $\epsilon > 0$. By assumption, $a < b + \epsilon$, which gives:

$$a < b + \frac{a - b}{2}$$

Simplifying the right side, we get:

$$a < \frac{a+b}{2}$$

Now, adding b to both sides:

$$a + b < \frac{a+b}{2} + b$$

Simplifying further:

$$a + b < \frac{a+b+2b}{2}$$
$$a + b < \frac{a+3b}{2}$$

Everything up to here tracks algebraically, but it appears that your logic breaks down. You want to start with your assumption you made for the sake of contradiction and use the information to get to a logical fallacy (hint: if $a < b + \epsilon$, what happens when $\epsilon = a - b$?.

Since $a + b$ is less than $\frac{a+3b}{2}$, this contradicts our assumption that $a < b + \epsilon$ for every $\epsilon > 0$. Therefore, our assumption that $a \geqslant b$ must be false, which means $a < b$.

Your proof seems to have some gaps: why is it the case that since $a + b$ is less than $\frac{a+3b}{2}$, this contradicts our assumption that $a < b + \epsilon$ for every $\epsilon > 0$? It is also confusing that you refer to the assumption you made for contradiction ($a \geqslant b$) and a given information we know ($a < b + \epsilon$, $\epsilon > 0$) interchangeably.

Overall you made some nice arguments and the first part of your proof holds. If you are using proof by contradiction make sure you clearly show the logical fallacy that overturns the contradiction. The backwards direction of the if and only if proof needs work. (2.5/5)