Real Analysis HW #4

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Question 1

Let A be a nonempty bounded set. The maximum value is a number $x \in A$ such that $a \leqslant x \; \forall a \in A$. Prove that a nonempty bounded set has a maximum value if and only if it contains its supremum.

Solution:

Proof: $\Rightarrow$ If a nonempty bounded set has a maximum value $x = Max(A)$ then for all $a \in A$ we have $a \leqslant x$. Then x is an upperbound of the set A. Let $b = \sup A$. Then $b \leqslant x$ and $b \geqslant a$ for all $a \in A$. Thus, $a \leqslant b \leqslant x$. Therefore the suprememum of A (b) is contained within the set, that is $b \in A$.

$\Leftarrow$ Let $b = \sup A$ with $b \in A$. Then for all $a \in A$ we have that $a \leqslant b$. Thus, b is the maximum of A.

Question 2

Let A be a non-empty set and let $\mathcal{P}(\mathcal{A})$ represent the collection of all subsets of A; this set is known as the power set of A.

(a) Suppose that A has n elements. Prove that $\mathcal{P}(\mathcal{A})$ has 2^n elements.

Solution:

Let $n = 1$. Take the set A with 1 element to be denoted set $A_1 = \{a_1\}$ and the power set $P_1 = \mathcal{P}(\mathcal{A}_1) = \{\{a_1\}, \emptyset\}$. Then $|P_1| = |\mathcal{P}(\mathcal{A}_1)| = 2^n = 2^1 = 2$. We have shown that this works for n. We would like to show, through proof by induction, that this works for $n+1$.

Consider

$$\mathcal{P}(\mathcal{A}_{n+1}) = \mathcal{P}(\mathcal{A}_n) \cup \{\{p_{ni} \cup a_{n+1}\} : i \in \mathbb{N}_{2^n}\}.$$

Because the right side of the union is disjoint from the left side of the union we can add the cardinalities together.

Hence,

$$|\mathcal{P}(\mathcal{A}_{n+1})| = |\mathcal{P}(\mathcal{A}_n)| + |\{\{p_{ni} \cup a_{n+1}\} : i \in \mathbb{N}_{2^n}\}|.$$

Then,

$$|\mathcal{P}(\mathcal{A}_{n+1})| = 2^n + 2^n = 22^n = 2^{n+1}.$$

Thus, $|\mathcal{P}(\mathcal{A}_{n+1})| = 2^{n+1}$. We have shown that $\mathcal{P}(\mathcal{A})$ has 2^n elements.

(b) Suppose that A is countable. Prove that $\mathcal{P}(\mathcal{A})$ is uncountable.

Solution:

Assume, by way of contradiction, that $\mathcal{P}(\mathcal{A})$ is countable. Then there exists an onto function $f : \mathbb{N} \to \mathcal{P}(\mathcal{A})$. Construct $B = \{n : n \notin f(n)\}$. Since f is onto there must exist some $n_0 \in \mathbb{N}$ such that $f(n_0) = B$.

We consider two cases:

1. $n_0 \in B$. Then by construction of the set B, $n_0 \notin f(n_o)$. However, $f(n_0) = B$ so $n_0 \notin B$ $\rightarrow\!\leftarrow$.
2. $n_0 \notin B$. So $n_0 \in f(n_0)$. This implies $n_0 \in B$ $\rightarrow\!\leftarrow$

Thus, f can not be onto. If it is not onto then $|\mathbb{N}| \neq |\mathcal{P}(\mathcal{A})|$ and thus $\mathcal{P}(\mathcal{A})$ can not be countable.

(c) Suppose that A is uncountable. Prove that there is no bijection between A and $\mathcal{P}(\mathcal{A})$.

Proof: Assume, by way of contradiction, that there exists a bijection between A and $\mathcal{P}(\mathcal{A})$, hence $f : A \to \mathcal{P}(\mathcal{A})$. Construct $B = \{a : a \notin f(n)\}$. Since f is onto there must exist some $a_0 \in \mathbb{N}$ such that $f(a_0) = B$.

We consider two cases:

1. $a_0 \in B$. Then by construction of the set B, $n_0 \notin f(a_o)$. However, $f(a_0) = B$ so $a_0 \notin B$ $\rightarrow\!\leftarrow$.
2. $a_0 \notin B$. So $a_0 \in f(a_0)$. This implies $a_0 \in B$ $\rightarrow\!\leftarrow$

Thus, there can not be a bijection between A and $\mathcal{P}(\mathcal{A})$.

Question 3

Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7 . Is E countable? Explain.

Solution:

Proof: E is countable. Let f be the function from the natural numbers to the set of all permutations of 4 and 7 in decimal expansion with n elements. That is $1 \longrightarrow \{0.4, 0.7\}$, $2 \longrightarrow \{0.47, 0.74, 0.44, 0.77\}$ and so forth. Each of these sets is countable for every $n \in \mathbb{N}$. Thus the union of of countable sets as $n \longrightarrow \infty$ is countable (Theorem 1.5.8). Therefore, E is countable.

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Question 4

Consider the function h defined by

$$h(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1/q & \text{if } x = p/q \end{cases}$$

Here it is assumed that the rational number p/q is in reduced form and that $q > 0$.

(a) Find $h(n)$ for each integer n

Solution: For each integer n, $f(n) = 1$

(b) Find three solutions to the equation $h(x) = 1/3$.

Solution: $x = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$

(c) Find all of the solutions to the equation $h(x) = 1/7$ that lie in the interval $(3, 4)$.

Solution: $x = \frac{22}{7}, \frac{23}{7}, \frac{24}{7}, \frac{25}{7}, \frac{26}{7}, \frac{27}{7},$

(d) Prove that the set of all solutions to the equation $h(x) = 1/5$ is countable infinite.

Solution:

Proof: Let S be the set of solutions to the equation $h(x) = \frac{1}{5}$. Then for every $s \in S$ we can write as the ratio of two integers - a rational number. Then $S \subset \mathbb{R} \setminus \mathbb{N}$. Since S is a subset of a countable set (the rational) then it must be countable (Theorem 1.5.7.) 😃

(e) Let (a, b) be any interval and let $\epsilon > 0$. Prove that $\{x \in (a, b) : h(x) \geqslant \epsilon\}$ is a finite set.

Solution: If $f(x) \geqslant \epsilon > 0$ then x must be a rational number by construction of $h(x)$. We want to show that the set of the reduced forms of the rational numbers within an interval (a, b) is finite. Let $A = \{x \in (a, b) : h(x) \geqslant \epsilon\}$.

Consider the solution set for every $N \in \mathbb{N}$ where $x = \frac{p}{q}$ and $\frac{1}{N} = \frac{1}{q}$ under the constraint that $x \in (a, b)$. Each of these sets is finite. By Archy :) we can choose the maximum N such that $\frac{1}{N+1} < \epsilon$ and $\frac{1}{N} \geqslant \epsilon$. Thus all we have to do to find the number of elements in A is to count the number of solutions satisfying $h(x) = \frac{1}{n}$ for $n = 1, 2, \ldots, N$. Therefore A is finite.

Question 5

ChatGPT Challenge: Find a sequence is bounded below, not bounded above, and contains an infinite number of negative terms.

Proof: Certainly, here's an example of a sequence that is bounded below, not bounded above, and contains an infinite number of negative terms: Consider the sequence defined as follows: $-1, -2, -3, -4, -5, \ldots$ This sequence consists of negative integers and is strictly decreasing. It is bounded below by negative infinity (there is no lower bound as it goes to negative infinity), **This doesn't mean anything? Infinity is not a number so this does not make sense.**

not bounded above (it becomes more negative without bound), and it contains an infinite number of negative terms.