Ultimate Problem Set

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Question 1

Suppose that $x > -1$ and that $x \neq 0$. Prove that

$$(1+x)^n > 1 + nx$$

for each integer $n > 1$. This result is know as Bernoulli's inequality.

Question 2

Show that e is irrational by supposing that $e = \frac{m}{n}$ and deriving a contradiction. Use the fact that $e = \sum_{j=0}^{\infty} \frac{1}{j!}$. Let $s_k = \sum_{j=0}^{k} \frac{1}{j!}$.

(a) Prove that

$$e - s_k < \frac{1}{(k+1)!}\left\{1 + \frac{1}{k+1} + \left(\frac{1}{k+1}\right)^2 + \cdots\right\}.$$

(b) Prove that $e - s_k < \frac{1}{k(k+)}$ for all $k \in \mathbb{N}$. (c) If $e = \frac{m}{n}$, prove that $n!e$ and $n!s_n$ are integers. (d) If $e = \frac{m}{n}$, prove that $n!\,(e - s_n)$ is an integer between 0 and 1 , which is absurd.

Question 3

Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leqslant c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous. (b) Pick some $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim_{n\to\infty} y_n$. (c) Prove that y is a fixed point of f (i.e. $f(y) = y$) and that it is unique in this regard. (d) Finally, prove that if x is any arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y (as defined in (b)).

Question 4

Let $\{r_n\}$ be a listing of all the rational numbers. Define a function f by $f(x) = 0$ if x is irrational and $f(r_n) = 1/n$ for all n. Show that f is continuous everywhere except for the set of rational numbers.

Question 5

Using the $\delta - \epsilon$ definition of a limit, show

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = 3.$$