

Real Analysis HW #3

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Question 1

1. Let S be a non-empty set of real numbers that is bounded above and let $\beta = \sup S$. Suppose that $\beta \notin S$. Prove that for each $\epsilon > 0$, the set $\{x \in S : x > \beta - \epsilon\}$ is infinite.

Solution:

Proof: Assume, by way of contradiction, that $(\beta - \epsilon, \beta)$ is finite. Then we have that $(\beta - \epsilon, \beta) \sim \mathbb{N}_N$ where $N \in \mathbb{N}$. Consider the value

$$\frac{\beta - \epsilon}{2N}.$$

This partitions the interval $(\beta - \epsilon, \beta)$ into $2N$ bins.

Next, we take the collection $\{\beta - \epsilon + \frac{\beta - \epsilon}{2N^*} : N^* \in \mathbb{N}_N\}$. We can start listing off elements of this set until we terminate at $N^* = N$, which is $\beta - \epsilon + \frac{\beta - \epsilon}{2N}$.

However,

$$N + 1 \notin \mathbb{N}_N$$

but

$$\beta - \epsilon + \frac{\beta - \epsilon}{2(N + 1)} \in (\beta - \epsilon, \beta).$$

By pigeon hole principle $|(\beta - \epsilon, \beta)| \neq |\mathbb{N}_N|$. Thus, the set $\{x \in S : x > \beta - \epsilon\}$ is infinite. ☺

Question 2

2. Prove that the union of a countable set and an uncountable set is uncountable.

Proof: Assume, by way of contradiction, that the union of a countable set A and an uncountable set B is countable. Then, $A \cup B$ is countable. By Theorem 1.5.7, then $B \subset A \cup B$ is countable or finite $\rightarrow \times$. Thus, $A \cup B$ must be uncountable. ☺

Question 3

3. Exercise 1.5.4:

(a) Show $|(a, b)| = |\mathbb{R}|$.

Solution:

$$f(x) = \tan\left(\frac{\pi}{b-a} \left[x - \frac{a+b}{2}\right]\right)$$

We have domain f is (a, b) , the range is $(-\infty, \infty) = \mathbb{R}$. The function f is both onto and 1-1. Thus, $(a, b) \sim \mathbb{R}$.

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality at \mathbb{R} as well.

Solution:

$$f(x) = \log(x - a)$$

(c) Show that $[0, 1)$ has the same cardinality as $(0, 1)$.

Solution:

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{x}{2} & x \in \{\frac{1}{2^n} : n \in \mathbb{N}\} \\ x & \text{else} \end{cases}$$

Question 4

4. Exercise 1.5.6:

Solution:

(a) Give an example of a countable collection of disjoint open intervals.

$$A = \{(a_n, a_n + 1) : n \in \mathbb{N}\} = \{(1, 2), (2, 3), (3, 4), \dots, (a_n, a_n + 1)\}$$

$$f: \mathbb{N} \rightarrow A$$

$$f(n) = (n, n + 1)$$

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

No such collection exists.

Proof: Let X be a collection of disjoint intervals. For each $x \in X$ denote $x = (a_i, b_i)$ where $i \in \mathbb{N}$. Because each interval is open they must contain at least two real numbers. By the density of \mathbb{Q} in \mathbb{R} we know that there exists $q_i \in \mathbb{Q}$ such that $a_i < q_i < b_i$. Thus we obtain a new set of rational numbers, Y . Note that while each a_i, b_i may not be unique between intervals, each r_i is. Combining Theorem 1.5.6 (\mathbb{Q} is countable) and Theorem 1.5.7 (A subset of a countable set is either countable or finite) Y must be countable. Hence, any collection of disjoint open intervals is countable. ☺

Question 5

5. ChatGPT Challenge: Find an example of a sequence of closed bounded intervals I_1, I_2, \dots with the property that $\bigcap_{k=1}^n I_k \neq \emptyset$ for all $n \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$

Proof: Let $I_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ for $n \in \mathbb{N}$. For each natural number n , the interval I_n is a closed bounded interval, and their intersection is non-empty:

$$\bigcap_{k=1}^n I_k = \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \neq \emptyset$$

Your proposition seems reasonable at first but be careful with your intervals. When $k = 1$, we have $I_k = [1, 0]$ which is the empty set. Thus, any intersection after that must be the empty set and your example does not hold.

You may consider $I_n = \left[\frac{1}{n+1}, 1 - \frac{1}{n+1}\right]$, but you still have problems. The next interval is $\left[\frac{1}{2}, \frac{1}{2}\right]$, and every interval after that is a superset of this - the infinite intersection can not be the empty set.

However, when we consider the infinite intersection:

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

The infinite intersection is empty because as n approaches infinity, the left endpoint $\frac{1}{n}$ approaches zero, and the right endpoint $1 - \frac{1}{n}$ approaches 1.

Even if we consider my suggested intervals, this is still false. We have that

$$\frac{1}{2} \in \bigcap_{n=1}^{\infty} I_n.$$

Make sure to do some sanity checks while you are going through examples. Make it concrete. Your logic does not hold.

So, the infinite intersection becomes the empty set.