Real Analysis HW #3

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Question 1

1. Let S be a nonempty set of real numbers that is bounded above and let $\beta = \sup S$. Suppose that $\beta \notin S$. Prove that for each $\epsilon > 0$, the set $\{x \in S : x > \beta - \epsilon\}$ is infinite.

Question 2

2. Prove that the union of a countable set and an uncountable set is uncountable.

Question 3

3. Exercise 1.5.4:

(a) Show $|(a, b)| = |\mathbb{R}|$.

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality at $\mathbb{R}$ as well.

(c) Show that $[0, 1)$ has the same cardinality as $(0, 1)$.

Question 4

4. Exercise 1.5.6:

Solution:

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue thatno such collection exists.

Question 5

5. ChatGPT Challenge: Find an example of a sequence of closed bounded intervals $I_1, I_2, \ldots$ with the property that $\bigcap_{k=1}^{n} I_k \neq \emptyset$ for all $n \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$

Proof: Let $I_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ for $n \in \mathbb{N}$. For each natural number n, the interval I_n is a closed bounded interval, and their intersection is non-empty:

$$\bigcap_{k=1}^{n} I_k = \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \neq \emptyset$$

However, when we consider the infinite intersection:

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

The infinite intersection is empty because as n approaches infinity, the left endpoint $\frac{1}{n}$ approaches zero, and the right endpoint $1 - \frac{1}{n}$ approaches 1 . So, the infinite intersection becomes the empty set.