

# Real Analysis HW #9

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### Question 1

Let  $g : A \rightarrow \mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$ . Suppose that  $\lim_{x \rightarrow a} f(x) = 0$ .

(a) Show that  $\lim_{x \rightarrow a} f(x)g(x) = 0$  for any function  $g$  as above IS NOT TRUE.

**Proof:** Assume for the sake of contradiction that  $\lim_{x \rightarrow a} f(x)g(x) = 0$  for any  $g(x)$ . Consider the case when  $g(x) = 1/x$ . Let  $\epsilon > 0$ . Then there exists  $\delta$  such that whenever  $0 < |x - 0| < \delta$  we have  $|f(x)g(x) - 0| < \epsilon$ . ☹

(b) Assume that  $g$  is a bounded function on  $A$ . Show that  $\lim_{x \rightarrow a} g(x)f(x) = 0$ .

**Proof:** Let  $\epsilon > 0$  and  $g$  be bounded by  $B \in \mathbb{R}^+$ . So  $|g(x)| < B$  for all  $x \in \mathbb{R}$ . Because  $\lim_{x \rightarrow a} f(x) = 0$  then there exists  $\delta$  such that if  $c \in \mathbb{R}$  and  $0 < |x - c| < \delta$  we automatically have  $|f(x) - 0| < \epsilon/B$ . Now,

$$|g(x)f(x) - 0| < |g(x)| \left| \frac{\epsilon}{B} \right| \leq |B| \left| \frac{\epsilon}{B} \right| = \epsilon.$$

☺

### Question 2

Let  $a$  and  $b$  be real numbers with  $a \neq 0$ . Use the definition of continuity to prove that the function  $f$  defined by  $f(x) = ax + b$  is continuous at every real number.

**Proof:** Let  $\epsilon > 0$  and  $c \in \mathbb{R}$ . Choose  $\delta = \epsilon/a$ . If we have  $|x - c| < \delta$  it follows

$$|f(x) - f(c)| = |ax + b - (ac + b)| = |ax + b - ac - b| = |ax - ac| = |a(x - c)| < |a| \frac{\epsilon}{a} < \epsilon.$$

Thus,  $f(x) = ax + b$  is continuous at every real number. ☺

### Question 3

Use the definition of limit to prove that  $\lim_{x \rightarrow c} x^2 = c^2$  for every real number  $c$ .

**Proof:** Let  $\epsilon > 0$ . Choose  $\delta = \min\{1, \epsilon/(2c + 1)\}$ . If we have  $0 < |x - c| < \delta$  it follows that,

$$|f(x) - L| = |x^2 - c^2| = |x + c||x - c| < (2c + 1) \frac{\epsilon}{2c + 1} = \epsilon$$

Thus,  $\lim_{x \rightarrow c} x^2 = c^2$  for every real number  $c$ . ☺

### Question 4

Find constants  $a$  and  $b$  so that the function  $f$  defined by

$$f(x) = \begin{cases} 3ax^2 + 1 & 0 \leq x \leq 1 \\ ax + b & 1 < x < 2 \\ 2bx + a & 2 \leq x \leq 4 \end{cases}$$

has a limit at each point of  $[0, 4]$ . Be sure to show the limit exists.

**Solution:**

First we find constants  $a$  and  $b$  so that  $f(x)$  has a limit defined at each point  $[0, 4]$ . Plugging in 1 and 2 in each of the equations defined in the piecewise function  $f(x)$  yields a system of equations:

$$2a + 1 = a + b \implies 2a + 1 = b$$

and

$$2a + b = 4b + a \implies a = 3b.$$

Substituting  $a = 3b$  into the first equation yields

$$2(3b) = b \implies 6b + 1 = 6 \implies 5b = -1 \implies b = \frac{-1}{5}.$$

Finally, solving for  $a = 3b = 3(-1/5) = -3/5$ . Thus,

$$b = \frac{-1}{5} \text{ and } a = \frac{-3}{5}$$

and  $f$  becomes

$$f(x) = \begin{cases} -9/5x^2 + 1 & 0 \leq x \leq 1 \\ -3/5x - 1/5 & 1 < x < 2 \\ -2/5x - 3/5 & 2 \leq x \leq 4 \end{cases}$$

### Question 5

Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and suppose that  $f(x) = 0$  for each irrational number  $x$  in  $[a, b]$ . Prove that  $f(x) = 0$  for all  $x \in [a, b]$ .

**Proof:** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  then it must be continuous on every point  $c \in [a, b]$ . Assume, for the sake of contradiction, that there exists  $x_0 \in [a, b]$  with  $x_0 \in \mathbb{R} \setminus \mathbb{Q}$  and  $f(x_0) \neq 0$ .

Then  $f$  must be continuous at  $x_0$ , and there must exist  $\delta$  such that  $|x - x_0| < \delta$  implies that  $|f(x) - f(x_0)| < |f(x_0)|/2$ .

By the density of rationals in the real numbers there must exist an irrational number,  $x_I$ , such that  $|x_I - x_0| < \delta$ .

Since  $x_I$  is irrational then  $f(x_I) = 0$  and we have  $|f(x_I) - f(x_0)| = |f(x_0)| < |f(x_0)|/2 \implies 1 < 1/2$ . Thus, we have arrived at a contradiction which means that  $f(x) = 0$  for all  $x \in [a, b]$ . ☺

### Question 6

CHAT GPT CHALLENGE: Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function on  $[a, b]$ , and suppose that  $f(x) = 0$  for each rational number  $x$  in  $[a, b]$ . We aim to show that  $f(x) = 0$  for all  $x \in [a, b]$ .

Assume, for the sake of contradiction, that there exists  $c \in [a, b]$  such that  $f(c) \neq 0$ . Since  $f$  is continuous, for any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x \in [a, b]$ , if  $|x - c| < \delta$ , then  $|f(x) - f(c)| < \epsilon$ .

Consider the case when  $\epsilon = |f(c)|$ . By continuity, there exists  $\delta > 0$  such that for all  $x \in [a, b]$ , if  $|x - c| < \delta$ , then  $|f(x) - f(c)| < |f(c)|$ . This implies

$$-f(c) < f(x) - f(c) < f(c).$$

Adding  $f(c)$  to all parts of the inequality yields  $0 < f(x)$  for  $x$  in the interval  $(c - \delta, c) \cap [a, b]$ . Similarly, subtracting  $f(c)$  from all parts of the inequality yields  $f(x) < 0$  for  $x$  in the interval  $(c, c + \delta) \cap [a, b]$ . Now, consider the set of rational numbers in  $[a, b]$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there exist rational numbers in both  $(c - \delta, c) \cap [a, b]$  and  $(c, c + \delta) \cap [a, b]$ . However, we have shown that for any rational  $x$  in these intervals,  $f(x)$  has opposite signs, which is a contradiction.

**You are almost there. Instead of saying there exists a rational number in the intervals you should say there exists an irrational number, say  $x_I$ , which we know  $f(x_I) = 0$ . From then you can continue with your contradiction argument. You also need to note that for any number in those intervals  $f(x)$  has opposite signs.**

Therefore, our assumption that there exists  $c$  such that  $f(c) \neq 0$  is false, and we conclude that  $f(x) = 0$  for all  $x \in [a, b]$ .