Real Analysis CW #7

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Question 1

Let (x_n) be a sequence and suppose that the sequence $(x_{n+1} - x_n)$ converges to 0 . Give an example to show that the sequence (x_n) may not converge. (See CHATBOT Challenge)

Solution: Let

$$x_n = \sum_{k=1}^{n} 1/k.$$

This is the harmonic series, which converges to infinity. Let $\epsilon > 0$. By Archimedes Principle there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Let $n > N$. Then

$$|(x_{n+1} - x_n) - 0| = \left|\frac{1}{n+1} - \frac{1}{n}\right| < \left|\frac{1}{N}\right| < \epsilon.$$

Thus, $(x_{n+1} - x_n)$ converges to 0, but (x_n) converges to infinity.

Question 2

Let (x_k) and (y_k) be two sequences and let (r_k) be a sequence of positive numbers that converges to 0 . Suppose that $0 < |y_k - x_k| < r_k \forall k \in \mathbb{N}$.

(a) Give an example to show that the sequences (x_k) and (y_k) may not converge.

Solution: Let

$$y_k = k + \frac{1}{k}$$

and let

$$x_k = k + \frac{1}{(k+1)}.$$

(b) Suppose that (x_k) converges to L. Prove that the sequence (y_k) converges to L.

Proof: Let $\epsilon > 0$. Because (r_k) converges to 0 then there exists $N_1 \in \mathbb{N}$ such that for all $k > N_1$, $|r_k - 0| < \frac{\epsilon}{2}$. Because (x_k) converges to L there exists $N_2 \in \mathbb{N}$ such that for all $k > N_2$, $|x_k - L| < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$ and $k > N$.

Then,

$$0 < |y_k - x_k| < |r_k - 0| < \frac{\epsilon}{2}.$$

$$\implies |y_k - L + L - x_k| < \frac{\epsilon}{2}.$$

$$\implies |y_k - L| - |L - x_k| \leqslant |y_k - L + L - x_k| < \frac{\epsilon}{2}.$$

$$\implies |y_k - L| - \underbrace{|L - x_k|}_{<\frac{\epsilon}{2}} < \frac{\epsilon}{2}.$$

And we know that $|x_k - L| < \frac{\epsilon}{2}$,

$$\implies |y_k - L| < \epsilon$$

Thus, y_k converges to L. 😄

Question 3

Assume that (x_n) is a bounded sequence with the property that every convergent subsequence of (x_n) converges to the same limit $x \in \mathbb{R}$. Show that (x_n) must converge to x.

Question 4

Let (x_n) be a Cauchy sequence. Show directly that (x_n) is bounded.

Let $\epsilon > 0$ and (x_n) be a Cauchy sequence. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geqslant N$, $|x_n - x_m| < 1$. Then,

$$|x_m - x_N| < 1.$$

$$\implies |x_m| - |x_N| < |x_m - x_N| < 1.$$

$$\implies |x_m| - |x_N| < 1.$$

$$\implies |x_m| < 1 + |x_N|.$$

Then, we have a bound for all the terms N and beyond. Let

$$B = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_N| + 1\}.$$

Then $|x_n| < B$ for all $n \in \mathbb{N}$ and therefore (x_n) must be bounded.

Question 5

Let (x_n) be a sequence and let r be a real number satisfying $0 < r < 1$. Suppose that $|x_{n+1} - x_n| < r\,|x_n - x_{n-1}|$ for all $n > 2$. Prove that (x_n) is a Cauchy sequence and hence convergent.

Let $\epsilon > 0$. Since $|x_{n+1} - x_n| < r\,|x_n - x_{n-1}|$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $r|x_n - x_n - 1| < \frac{1}{r}\epsilon$.

Then,

$$|x_{n+1} - x_n| < r\,|x_n - x_{n-1}| < \frac{1}{r}\epsilon$$

$$\implies |x_{n+1} - x_n| < \frac{1}{r}\epsilon$$

for all $n > N$. Let $n, m > N$, and without loss of generality let $m > n$.
Then,

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - x_{m-3} + \cdots + x_{n+1} - x_n|$$

$$\leqslant |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + |x_{m-2} - x_{m-3}| + \cdots + |x_{n+1} - x_n|$$

$$< \frac{\epsilon}{r} + \frac{\epsilon}{r} + \frac{\epsilon}{r} + \cdots + \frac{\epsilon}{r} < \epsilon.$$

Hence, (x_n) is Cauchy, and thus convergent.

Question 6

CHAT GPT CHALLENGE: Let (x_n) be a sequence such that $(x_{n+1} - x_n)$ converges to 0 . We want to show that (x_n) also converges.

By the definition of a convergent sequence, we need to show that for any $\epsilon > 0$, there exists a positive integer N such that for all $n \geqslant N$, $|x_n - L| < \epsilon$ for some real number L.

Good job setting the stage for your proof and clearly stating what you need to show to substantiate your claim. This is correct.

Since $(x_{n+1} - x_n)$ converges to 0 , we can choose $\epsilon > 0$ and find a positive integer M such that for all $m \geqslant M$, $|(x_{m+1} - x_m) - 0| < \epsilon$. This implies that for all $m \geqslant M$, $|x_{m+1} - x_m| < \epsilon$. Now, let $N = M$. For any $n \geqslant N$, we have:

$$\begin{aligned} |x_n - x_N| &= |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \ldots + (x_{N+1} - x_N)| \\ &\leqslant |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \ldots + |x_{N+1} - x_N| \\ &< \epsilon + \epsilon + \ldots + \epsilon \quad (\text{Since } |x_{m+1} - x_m| < \epsilon \text{ for all } m \geqslant M) \\ &= (n - N)\epsilon. \end{aligned}$$

Dividing both sides by $(n - N)$, we get:

$$|x_n - x_N| < \epsilon$$

This is incorrect. When you divide both sides by $(n - N)$ you get

$$\frac{1}{(n - N)} |x_n - x_N| < \epsilon$$

which does not show that $|x_n - x_N| < \epsilon$

This shows that for any $\epsilon > 0$ and $n \geqslant N$, we have $|x_n - x_N| < \epsilon$, where $N = M$. Therefore, (x_n) converges to x_N.

This does not prove what you wanted to show.

Hence, we have shown that if $(x_{n+1} - x_n)$ converges to 0 , then (x_n) also converges.