

# The Real Numbers

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## 0.1 The Real Numbers

### 0.1.1 The irrationality of the square root of 2

#### 0.1.2 Preliminaries

##### Notation

- $\rightarrow$  For all/each/every
- $\exists \rightarrow$  There exists
- $\mathbb{R} \setminus \mathbb{Q} \rightarrow$  Irrationals
- $\mathbb{R} \rightarrow$  Real numbers
- $\mathbb{Z} \rightarrow$  Integers
- $\mathbb{Q} \rightarrow$  Rational numbers
- $\mathbb{N} \rightarrow$  Natural numbers
- $BWOC \rightarrow$  By way of contradiction
- $\rightarrow \times \rightarrow$  Contradiction
- $! \rightarrow$  Unique/factorial
- $\ominus \rightarrow$  End of proof (Quod Erat Demonstrandum)
- $\epsilon \rightarrow$  Epsilon, usually a small positive quantity
- $\ni \rightarrow$  Such that

##### Theorem 0.1.1

Two real numbers  $a$  and  $b$  are equal if and only if for every real number  $\epsilon > 0$  it follows that  $|a - b| < \epsilon$ .

**Proof:**  $\Rightarrow$  If  $a = b$  then we have  $|a - b| = 0$ . No matter which  $\epsilon > 0 \in \mathbb{R}$  is chosen we have that  $|a - b| = 0 < \epsilon$ . Thus,  $a = b$

$\Leftarrow$  Suppose, by way of contradiction, that  $a \neq b$ . We know that  $\forall \epsilon > 0$  and  $|a - b| < \epsilon$ . Let  $\epsilon_0 = \frac{|a-b|}{2}$ , then it is clear that  $|a - b| < \frac{|a-b|}{2} = \epsilon_0$  is false. Thus, with this contradiction we overturn our assumption and conclude  $a$  must equal  $b$ .  $\ominus$

### 0.1.3 The Axiom of Completeness

#### Definition 0.1.1: Bounded Above

A set  $A \subseteq \mathbb{R}$  is bounded above if there exists a number  $b \in \mathbb{R}$  such that  $a \leq b$  for all  $a \in A$ . The number  $b$  is called an upper bound for  $A$ . Similarly, the set  $A$  is bounded below if there exists a lower bound  $l \in \mathbb{R}$  satisfying  $l \leq a$  for every  $a \in A$ .

#### Definition 0.1.2: The Supremum of a Set

A real number  $s$  is the *Supremum* or the least upper bound for a set  $A \subseteq \mathbb{R}$  if:

- (i)  $s$  is an upper bound for  $A$ ;
- (ii) if  $b$  is any upper bound for  $A$ , then  $s \leq b$ .

##### Theorem 0.1.2 Lemma 1.3.8

Assume  $s \in \mathbb{R}$  is an upper bound for a set  $A \subseteq \mathbb{R}$ . Then,  $s = \sup A$  if and only if, for every choice of  $\epsilon > 0$ , there exists an element  $a \in A$  satisfying  $s - \epsilon < a$ .

**Proof:**  $\Rightarrow$  Let  $s = \sup A$  and arbitrarily choose any  $\epsilon > 0$ . Suppose, by way of contradiction, there does not exist  $a \in A$  with  $a > s - \epsilon$ . So for all  $a \in A$  we have that  $a \leq s - \epsilon < s$ . This means that  $s - \epsilon$  is an upper bound of  $A$ . However,  $s$  is the supremum or the least upper bound of  $A$ , so this is a contradiction. Thus, there must exist an element  $a \in A$  such that  $s - \epsilon < a$ .

$\Leftarrow$  Let  $\epsilon > 0$  and suppose there exists  $a \in A$  with  $s - \epsilon < a$  and we know that  $s$  is an upper bound of  $A$ . To show that  $s$  is the least upper bound, by way of contradiction, suppose that  $b < s$  and  $b$  is another upper bound of  $A$ .

Consider  $\epsilon_0 = s - b > 0$ . By hypothesis there exists  $a$  with  $s - \epsilon_0 < a \implies s - (s - b) < a \implies b < a$ . As we know that  $b$  is an upper bound of  $A$  this is impossible. Therefore it must be that  $b \geq s$ . Hence,  $s \leq b$  and  $s = \sup A$ .

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## 0.1.4 Consequences of Completeness

### Theorem 0.1.3 The Archimedes Property

- (a) For all  $x \in \mathbb{R}$  there exists  $n \in \mathbb{N}$  such that  $n > x$ .
- (b) For all  $y \in \mathbb{R}$  with  $y \neq 0$ , there exists  $n \in \mathbb{N}$  such that  $\frac{1}{n} < y$ .

**Proof:** (a) Suppose, by way of contradiction, that the natural numbers are bounded. Let  $\alpha \in \mathbb{R}$  be an upper bound, so  $n \leq \alpha$  for all  $n \in \mathbb{N}$ . Let  $\beta = \sup \mathbb{N}$  [exists by completeness]. Now  $\beta - 1$  is not an upper bound. By our theorem, there exists  $n_0 \in \mathbb{N}$  such that  $\beta - 1 < n_0$ .

So  $\beta < n_0 + 1 \in \mathbb{N}$ . This is a contradiction because we assumed that  $\beta$  was the supremum of the natural numbers. Thus,  $\mathbb{N}$  is unbounded. For any  $\alpha \in \mathbb{R}$ , we can find a natural number that is larger than  $\alpha$ .

(b) To show why be is the case you can consider  $x = \frac{1}{y}$  where  $y \neq 0$ .

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## 0.1.5 Examples

1. Show that  $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$ .

**Proof:** Suppose otherwise, that the intersection is not the empty set, and let  $x \in \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$ . Then it follows  $x \in (0, \frac{1}{n})$  for all  $n \in \mathbb{N}$ . By the corollary of the Archimedes Property, there exists  $n_{\star} \in \mathbb{N}$  with  $\frac{1}{n_{\star}} < x$ . Then there does not exist  $x$  such that  $x \in \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$ . Therefore,  $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$ .

⊖

2. Show there does not exist a smallest positive number

**Proof:** To show that there does not exist a smallest positive number, suppose otherwise. Let  $x \in \mathbb{R}^+$ . By Archimedes property there exists  $n_0 \in \mathbb{N}$  such that  $\frac{1}{n_0} < x$ . So  $x$  cannot be the smallest.

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**Theorem 0.1.4 The Nested Cells Property**

For each  $n \in \mathbb{N}$ , assume we are given a closed interval  $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$ . Assume also that each  $I_n$  contains  $I_{n+1}$ . Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a non-empty intersection; that is,  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**Proof:** Let

$$A = \{a_1, a_2, a_3, \dots\}$$

and

$$B = \{b_1, b_2, b_3, \dots\}.$$

$A$  is bounded above by one element of  $B$  and  $B$  is bounded below by any  $a \in A$ . Let  $x = \sup A$  which implies that  $a_n \leq x$  for all  $n \in \mathbb{N}$ . Since  $b$  is an upper bound of  $A$  and  $x = \sup A$ , we have that  $x \leq b_n$  for all  $n$  and that  $a_n \leq x \leq b_n$  for all  $n$ . Thus,  $x \in [a_n, b_n]$  for all  $n$  and  $x \in \bigcap_{n=1}^{\infty} I_n$ .  $\odot$

**0.1.6 Intersection Examples**

- $\bigcap(0, \frac{1}{n}) = \emptyset$
- $\bigcap[0, \frac{1}{n}] = \{0\}$
- $\bigcap(0, \frac{1}{n}] = \emptyset$
- $\bigcap(-\frac{1}{n}, \frac{1}{n}) = \{0\}$

**Theorem 0.1.5 Density of  $\mathbb{Q}$  in  $\mathbb{R}$** 

Let  $a, b \in \mathbb{R}$ . Then there exist  $n \in \mathbb{Q}$  with  $a < r < b$ .

**Proof:** Since  $a, b \in \mathbb{R}$ , without loss of generality let  $a < b$ . Now, we have that  $b - a > 0$ . By the corollary to the Archimedes principle there exist  $n_{\star} \in \mathbb{N}$  with  $\frac{1}{n_{\star}} < b - a$ .

Consider  $n_{\star}a \in \mathbb{R}$ . Pick  $m \in \mathbb{N}$  so that

$$m - 1 \leq n_{\star}a < m.$$

In other words, we choose the smallest of natural numbers greater than  $n_{\star}a$ . By chance, it may be that one less than that number is  $n_{\star}a$ . so we end up with the equality  $m - 1 \leq n_{\star}a$ .

We have that

$$n_{\star}a < m \implies a < \frac{m}{n_{\star}}$$

and,

$$m \leq n_{\star}a + 1,$$

because we know that

$$m - 1 \leq n_{\star}a.$$

Next,

$$\frac{1}{n_{\star}} < (b - a) \implies 1 < n_{\star}(b - a) \implies 1 < n_{\star}b - n_{\star}a \implies n_{\star}a < n_{\star}b - 1 \implies a < \frac{n_{\star}b - 1}{n_{\star}} \implies a < b - \frac{1}{n_{\star}}.$$

We take

$$m \leq n_{\star}a + 1 < n_{\star}[b - \frac{1}{n_{\star}}] + 1 = n_{\star}b - 1 + 1 = n_{\star}b$$

So,  $m < n_{\star}b$  which means  $\frac{m}{n_{\star}} < b$ .

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## 0.1.7 Cardinality

### Definition 0.1.3: 1-1 and onto

- $f : a \rightarrow b$  is 1-1 if  $a_1 \neq a_2$  implies that  $f(a_1) \neq f(a_2)$ .
- $f : a \rightarrow b$  is onto if for every  $b \in B$  there exists  $a \in A$  such that  $f(a) = b$ .

### Definition 0.1.4: Cardinality

A set  $A$  has the same cardinality as  $B$  if there exists a 1-1 and onto function,  $f : A \rightarrow B$ . We have  $|A| = |B|$  or  $A \sim B$ .

### Examples

1. Show that  $\mathbb{N} \sim \mathbb{Z}$ .

*Proof:*

$$f(n) = \begin{cases} \text{odd} & \frac{n-1}{2} \\ \text{even} & -\frac{n}{2} \end{cases}$$

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2. Show that  $[0, 1] \sim [\pi, 5]$ .

*Proof:*

$$y = (5 - \pi)x + \pi$$

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Is  $(0, 1] \sim (\pi, 5)$ ? Yes but we need to be careful how we show.

### Definition 0.1.5: Infinite

- A set is *finite* if  $|A| = |\mathbb{N}_n|, n \in \mathbb{N}$
- A set is *infinite* if it is not *finite*.

### Definition 0.1.6: Countable

- An infinite set is *countable* if  $|A| = |\mathbb{N}|$ .
- An infinite set is *uncountable* if it is not *countable*.

### Examples

1.  $\mathbb{Q}$ ?
2.  $\mathbb{R} \setminus \mathbb{Q}$ ?
3.  $\mathbb{R} \setminus \mathbb{Q}$ ?
4. Is the union of countable sets countable?

**Theorem 0.1.6**

Let  $|A| = n$  and  $|B| = m$ . Then  $A \cup B$  is finite.

*Proof:* Let

$$A = \{a_1, a_2, \dots, a_n\}$$

and

$$B = \{b_1, b_2, \dots, b_m\}.$$

Define  $n_1 = \min\{k : k \in \mathbb{N}, b_k \notin A\}$  and  $n_2 = \min\{k : b_k \notin A, k > n_1\}$ . Generally,  $n_j = \min\{k : b_k \in A, k > n_{j-1}\}$ . Then  $|A| = n$  and  $|B \setminus A| = j$ .

$$f(l) = \begin{cases} a_l & l \leq n \\ b_l & n+1 \leq l \leq n+j \end{cases}$$

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**Theorem 0.1.7**

The subset,  $A$ , of a finite set  $B$  is finite. That is  $A \subset B$  is finite if  $B$  is finite.

*Proof:* Let  $|B| = n$ ,  $B = \{b_1, b_2, \dots, b_n\}$ .  
Then let

$$n_1 = \min\{k : b_k \in A\},$$

$$n_2 = \min\{k : b_k \in A, k > n_1\},$$

and

$$n_j = \min\{k : b_k \in A, k > n_{j-1}\}.$$

Define the bijections  $f : \mathbb{N}_{n_j} \rightarrow A$ . Which is a finite bijection.

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**Theorem 0.1.8**

Let  $A$  and  $B$  be sets with  $A \subset B$ .

1. If  $B$  is countable, then  $A$  is countable or finite.
2. If  $A$  is uncountable, then  $B$  is uncountable.

*Proof:* 1. Since  $B$  is countable let  $B = \{b_j : j \in \mathbb{N}\}$ . To "count"  $A$  let  $n_1 = \min\{k : b_k \in A\}$  and  $n_2 = \min\{k : b_k \in A, k > n_1\}$ . Generally,

$$n_j = \min\{k : b_k \in A, k > n_{j-1}\}.$$

If there does not exist  $k \in b_k \in A$  then  $|A| = m$ , which is finite. We have a function  $f(m) = b_{n_m}$  which is onto  $A$  in a 1-1 manner.

2. Contrapositive of 1!

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**Theorem 0.1.9**

The countable union of countable sets is countable.

*Proof:* Let  $A_n$  where  $n \in \mathbb{N}$  be a collection of countable sets. So that  $A_n = \{A_1, A_2, \dots\}$ . We can list off elements in each set within the collection  $A_n = \{a_{nm} : m \in \mathbb{N}\}$ . We want to show that  $\bigcup_{n=1}^{\infty} \{a_{nm} : m \in \mathbb{N}\}$  is countable. Consider  $f(a_{nm}) = 2^n 3^m$ . This is 1-1 by prime factorization, known as the Fundamental Theorem of Arithmetic. The set  $\{2^n 3^m : n, m \in \mathbb{N}\} \subset \mathbb{N}$  and is countable.

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**Theorem 0.1.10**

$\mathbb{Q}$  is countable.

**Proof:** Consider  $A_n = \{\frac{m}{n} : m \in \mathbb{N}\}$ . Then  $\bigcup_{n=1}^{\infty} A_n$  is countable. The set  $B_n = \{-\frac{m}{n} : m \in \mathbb{N}\}$  is also countable. The set  $\{0\}$  is finite and thus countable. Altogether we have

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} B_n \cup \{0\}$$

is countable by the previous theorem.

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