HW #1

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Question 1

Let n be a positive integer that is not a perfect square. Prove that $\sqrt{n}$ is irrational.

Solution: Assume, for contradiction, that $\sqrt{n}$ is a rational. Then $\sqrt{n}$ can be written in the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$.

We have

$$\frac{a}{b} = \sqrt{n} \implies \left(\frac{a}{b}\right)^2 = n \implies a^2 = nb^2 \tag{1}$$

This means that n divides a^2. By the Fundemental Theorem of Arithmetic we can write a, n, and b as unique product of primes.

Thus,

$$a^2 = nb^2 \implies \left(\prod_{i=1}^{k} P_i^{n_i}\right)^2 = \prod_{j=1}^{l} P_j^{m_j} \left(\prod_{k=1}^{t} P_k^{l_k}\right)^2 \tag{2}$$

After simplification of (2) we have

$$\prod_{i=1}^{k} P_i^{2n_i} = \prod_{j=1}^{l} P_j^{m_j} \prod_{k=1}^{t} P_k^{2l_k} \tag{3}$$

In both expressions of a^2 and b^2, as a product of primes, we have an even number of each prime in the product. Because n is not a perfect square, there must be at least 1 prime that is expressed an odd number of times. We are then guaranteed that by expressing nb^2 as a product of primes there must be at least 1 prime which appears an odd number of times. However, the left hand side of (3) clearly shows this is not the case $\rightarrow\!\leftarrow$.

With this contradiction we have no choice but to overturn our assumption and conclude that $\sqrt{n} \in \mathbb{R} \setminus \mathbb{Q}$ 😊

Question 2

Use the Principle of Mathematical Induction to prove:

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3} \quad \forall n \in \mathbb{N}$$

Solution:

Let $n = 1 \in \mathbb{N}$. Then $1^2 = \frac{4(1)^3 - 1}{3} = 1$, showing that the equality holds for $n = 1$. We asssume that

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3},$$

is true and we procceed with induction on n. We want to show $P(n+1) = \frac{4(n+1)^3 - (n+1)}{3}$.

Consider

$$\begin{aligned}
1^2 + 3^2 + \cdots + (2n-1)^2 + (2(n+1)-1)^2 &= \frac{4n^3 - n}{3} + (2(n+1)-1)^2 \\
&= \frac{4n^3 - n}{3} + (4n^2 + 4n + 1) \\
&= \frac{4n^3 - n + 12n^2 + 12n + 3}{3} \\
&= \frac{4n^3 + 8n^2 + 4n + 4n^2 + 8n + 4 - n - 1}{3} \\
&= \frac{4[n^3 + 2n^2 + n + n^2 + 2n + 1] - (n+1)}{3} \\
&= \frac{4[(n^2 + 2n + 1)(n+1)] - (n+1)}{3}
\end{aligned}$$

$$= \frac{4(n+1)^3 - (n+1)}{3}.$$

Thus, $P(n+1) = \frac{4(n+1)^3-(n+1)}{3}$, proving $1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3-n}{3} \quad \forall n \in \mathbb{N}$ 😊

Question 3

Let $n > 1$ be a positive integer and let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that

$$\left|\sum_{k=1}^{n} a_k\right| \leqslant \sum_{k=1}^{n} |a_k|$$

Solution: Let $n = 2 \in \mathbb{Z}^+$. Then

$$\left|\sum_{k=1}^{2} a_k\right| \leqslant \sum_{k=1}^{2} |a_k| \implies |a_1 + a_2| \leqslant |a_1| + |a_2|,$$

which we know is true by the Triangle Inequality Theorem. We then want to show that $\left|\sum_{k=1}^{n+1} a_k\right| \leqslant \sum_{k=1}^{n+1} |a_k|$. We assume for proof by induction that

$$\left|\sum_{k=1}^{n} a_k\right| \leqslant \sum_{k=1}^{n} |a_k|$$

is true. Expanding yields

$$|a_1 + a_2 + \cdots + a_n| \leqslant |a_1| + |a_2| + \cdots + |a_n|.$$

Adding $|a_{n+1}|$ to both sides results in

$$|a_1 + a_2 + \cdots + a_n| + |a_{n+1}| \leqslant |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|,$$

and by the Triangle Inequality Theorem we have

$$|a_1 + a_2 + \cdots + a_n + a_{n+1}| \leqslant |a_1 + a_2 + \cdots + a_n| + |a_{n+1}| \leqslant |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|.$$

Therefore,

$$\left|\sum_{k=1}^{n+1} a_k\right| \leqslant \sum_{k=1}^{n+1} |a_k|$$

and we have indeed shown

$$\left|\sum_{k=1}^{n} a_k\right| \leqslant \sum_{k=1}^{n} |a_k| \quad 😊$$

Question 4

Exercise 1.2.8

(a) $f : \mathbb{N} \to \mathbb{N}$ where f is 1-1 but not onto. ***Solution:*** $f(x) = x^2$

(b) $f : \mathbb{N} \to \mathbb{N}$ where f is not 1-1 but onto. ***Solution:*** If f is onto then for every $y \in \mathbb{N}$ there exists $x \in \mathbb{N}$ where $f(x) = y$. If we assume f is *not* 1-1 then we know that there exists $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. As f is a function, we can not map $x \in \mathbb{N}$ to more than one $y \in \mathbb{N}$, then by pigenhole principle we must be missing at least one $y \in \mathbb{N}$. Therefore, f can not be onto, which means this is impossible.

(c) $f : \mathbb{N} \to \mathbb{N}$ where f is 1-1 and onto.

Solution:

$$f(n) = \begin{cases} odd & \frac{n+1}{2}(-1)^{n+1} \\ even & \frac{n}{2}(-1)^{n+1} \end{cases}$$

Question 5

Exercise 1.2.12

(a) ***Solution:*** For proof by induction first consider $y_1 = 6 > -6$. Thus our base case holds. We assume the induction hypothesis

$$y_{n+1} = \frac{(2y_n - 6)}{3} \tag{4}$$

is true. We want to show that if $y_n > -6$ then it follows $y_{n+1} > -6$. First we subtract 3 from both sides of (4).

$$y_n \leq -6 \implies y_n - 3 > -6 - 3 \implies 2(y_n - 3) > -18$$

Which means that

$$y_{n+1} > -6.$$

Thus $y_n > -6 \ \forall n \in \mathbb{N}$ 😊

(b) ***Solution:*** We want to show that the sequence $(y_1, y_2, \ldots)$ is decreasing, i.e $y_n > y_{n+1} \ \forall n \in \mathbb{N}$.
We know $y_1 = 6$ and $y_2 = 2$. Clearly $y_1 > y_2$. Continuing on we assume then that our induction hypothesis,

$$y_n > y_{n+1},$$

is correct. We want to show $y_{n+1} > y_{n+2}$. We have

$$y_n > y_{n+1} \implies y_n - 3 > y_{n+1} - 3 \implies 2y_n - 6 > 2y_{n+1} - 6.$$

Next,

$$2y_n - 6 > 2y_{n+1} - 6 \implies \frac{2y_n - 6}{3} > \frac{2y_{n+1} - 6}{3},$$

which means that

$$\frac{2y_n - 6}{3} > \frac{2y_{n+1} - 6}{3}.$$

Finally, we can substitue

$$y_{n+1} > y_{n+1+1} \implies y_{n+1} > y_{n+2}$$

which is is exactly what we needed to show. 😊

Question 6

Grade the following proof: Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

Proof:To prove that $\sqrt{3}$ is irrational, we can use a proof by contradiction. The idea is to assume that $\sqrt{3}$ is rational, and then derive a contradiction.

Assumption: $\sqrt{3}$ is rational.

If $\sqrt{3}$ is rational, it can be expressed as a fraction $\frac{a}{b}$ in its simplest form, where a and b are integers with no common factors other than 1 (i.e., they are coprime), and b is not equal to 0.

So, we have: $\sqrt{3} = \frac{a}{b}$

Squaring both sides of the equation: $3 = \frac{a^2}{b^2}$

Rearranging the equation: $a^2 = 3 \cdot b^2$

From this equation, we can see that a^2 is a multiple of 3 , which implies that a must also be a multiple of 3. Let's represent a as $3k$ where k is an integer:

$$(3k)^2 = 3 \cdot b^2$$
$$9k^2 = 3 \cdot b^2$$
$$3k^2 = b^2$$

Now, b^2 is also a multiple of 3 , which means b must be a multiple of 3 . However, this contradicts our initial assumption that a and b have no common factors other than 1 (i.e., they are coprime). This contradiction arises from assuming that $\sqrt{3}$ is rational.

Hence, our initial assumption that $\sqrt{3}$ is rational must be false, which means that $\sqrt{3}$ is irrational. Now, to address whether a similar argument works for $\sqrt{6}$:

No, a similar argument does not work for $\sqrt{6}$. The proof for the irrationality of $\sqrt{3}$ relies on the fact that it leads to a contradiction with respect to the common factors of a and b. However, for $\sqrt{6}$, this argument does not apply directly because the factors of 6 are not the same as the factors of 3. In fact, $\sqrt{6}$ is not irrational; it is a rational number. It can be expressed as the fraction $\frac{2\sqrt{3}}{3}$, which is a ratio of two integers.