

# Real Analysis HW #1

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### Question 1

Let  $n$  be a positive integer that is not a perfect square. Prove that  $\sqrt{n}$  is irrational.

**Solution:** Assume, for contradiction, that  $\sqrt{n}$  is a rational. Then  $\sqrt{n}$  can be written in the form  $\frac{a}{b}$  where  $a, b \in \mathbb{Z}$ . We have

$$\frac{a}{b} = \sqrt{n} \implies \left(\frac{a}{b}\right)^2 = n \implies a^2 = nb^2 \quad (1)$$

This means that  $n$  divides  $a^2$ . By the Fundamental Theorem of Arithmetic we can write  $a$ ,  $n$ , and  $b$  as unique product of primes.

Thus,

$$a^2 = nb^2 \implies \left(\prod_{i=1}^k P_i^{n_i}\right)^2 = \prod_{j=1}^l P_j^{m_j} \left(\prod_{k=1}^t P_k^{l_k}\right)^2 \quad (2)$$

After simplification we have

$$\prod_{i=1}^k P_i^{2n_i} = \prod_{j=1}^l P_j^{m_j} \prod_{k=1}^t P_k^{2l_k} \quad (3)$$

In both expressions of  $a^2$  and  $b^2$ , as a product of primes, we have an even number of each prime in the product. Because  $n$  is not a perfect square, there must be at least 1 prime that is expressed an odd number of times. We are then guaranteed that by expressing  $nb^2$  as a product of primes there must be at least 1 prime which appears an odd number of times. However, the left hand side of (3) clearly shows this is not the case  $\rightarrow \times$ .

With this contradiction we have no choice but to overturn our assumption and conclude that  $\sqrt{n} \in \mathbb{R} \setminus \mathbb{Q}$  ☺

### Question 2

Use the Principle of Mathematical Induction to prove:

$$1^2 + 3^2 + \dots + (2n-1)^2 = \frac{4n^3 - n}{3} \quad \forall n \in \mathbb{N}$$

**Solution:**

Let  $n = 1 \in \mathbb{N}$ . Then  $1^2 = \frac{4(1)^3 - 1}{3} = 1$ , showing that the equality holds for  $n = 1$ . We assume that

$$1^2 + 3^2 + \dots + (2n-1)^2 = \frac{4n^3 - n}{3},$$

is true and we proceed with induction on  $n$ . We want to show  $P(n+1) = \frac{4(n+1)^3 - (n+1)}{3}$ .

Consider

$$1^2 + 3^2 + \dots + (2n-1)^2 = \frac{4n^3 - n}{3}.$$

Then, adding  $(2(n+1)-1)^2$  to both sides of the equation,

$$\begin{aligned} 1^2 + 3^2 + \dots + (2n-1)^2 + (2(n+1)-1)^2 &= \frac{4n^3 - n}{3} + (2(n+1)-1)^2 \\ &= \frac{4n^3 - n}{3} + (4n^2 + 4n + 1) \\ &= \frac{4n^3 - n + 12n^2 + 12n + 3}{3} \\ &= \frac{4n^3 + 8n^2 + 4n + 4n^2 + 8n + 4 - n - 1}{3} \end{aligned}$$

$$\begin{aligned}
&= \frac{4[n^3 + 2n^2 + n + n^2 + 2n + 1] - (n + 1)}{3} \\
&= \frac{4[(n^2 + 2n + 1)(n + 1)] - (n + 1)}{3} \\
&= \frac{4(n + 1)^3 - (n + 1)}{3}.
\end{aligned}$$

Thus,  $P(n + 1) = \frac{4(n+1)^3 - (n+1)}{3}$ , proving  $1^2 + 3^2 + \dots + (2n - 1)^2 = \frac{4n^3 - n}{3} \quad \forall n \in \mathbb{N} \quad \ominus$

### Question 3

Let  $n > 1$  be a positive integer and let  $a_1, a_2, \dots, a_n$  be real numbers. Prove that

$$\left| \sum_{k=1}^n a_k \right| \leq \sum_{k=1}^n |a_k|$$

**Solution:** Let  $n = 2 \in \mathbb{Z}^+$ . Then

$$\left| \sum_{k=1}^2 a_k \right| \leq \sum_{k=1}^2 |a_k| \implies |a_1 + a_2| \leq |a_1| + |a_2|,$$

which we know is true by the Triangle Inequality Theorem. We then want to show that  $\left| \sum_{k=1}^{n+1} a_k \right| \leq \sum_{k=1}^{n+1} |a_k|$ . We assume for proof by induction that

$$\left| \sum_{k=1}^n a_k \right| \leq \sum_{k=1}^n |a_k|$$

is true. Expanding yields

$$|a_1 + a_2 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|.$$

Adding  $|a_{n+1}|$  to both sides results in

$$|a_1 + a_2 + \dots + a_n| + |a_{n+1}| \leq |a_1| + |a_2| + \dots + |a_n| + |a_{n+1}|,$$

and by the Triangle Inequality Theorem we have

$$|a_1 + a_2 + \dots + a_n + a_{n+1}| \leq |a_1 + a_2 + \dots + a_n| + |a_{n+1}| \leq |a_1| + |a_2| + \dots + |a_n| + |a_{n+1}|.$$

Therefore,

$$\left| \sum_{k=1}^{n+1} a_k \right| \leq \sum_{k=1}^{n+1} |a_k|$$

and we have indeed shown

$$\left| \sum_{k=1}^n a_k \right| \leq \sum_{k=1}^n |a_k| \quad \ominus$$

### Question 4

Exercise 1.2.8

(a)  $f : \mathbb{N} \rightarrow \mathbb{N}$  where  $f$  is 1-1 but not onto. **Solution:**  $f(x) = 2x$

(b)  $f : \mathbb{N} \rightarrow \mathbb{N}$  where  $f$  is not 1-1 but onto. **Solution:** If  $f$  is onto then for every  $y \in \mathbb{N}$  there exists  $x \in \mathbb{N}$  where  $f(x) = y$ . If we assume  $f$  is *not* 1-1 then we know that there exists  $x_1 \neq x_2$  such that  $f(x_1) = f(x_2)$ . As  $f$

is a function, we can not map  $x \in \mathbb{N}$  to more than one  $y \in \mathbb{N}$ , then it must be that we are missing at least one  $y \in \mathbb{N}$ . Therefore,  $f$  can not be onto, which means this is impossible.

(c)  $f : \mathbb{N} \rightarrow \mathbb{Z}$  where  $f$  is 1-1 and onto.

**Solution:**

$$f(n) = \begin{cases} \text{odd} & \frac{n+1}{2}(-1)^{n+1} \\ \text{even} & \frac{n}{2}(-1)^{n+1} \end{cases}$$

### Question 5

#### Exercise 1.2.12

(a) **Solution:** For proof by induction first consider  $y_1 = 6 > -6$ . Thus our base case holds. We assume the induction hypothesis

$$y_{n+1} = \frac{(2y_n - 6)}{3} \quad (4)$$

is true. We want to show that if  $y_n > -6$  then it follows  $y_{n+1} > -6$ . First we subtract 3 from both sides.

$$y_n > -6 \implies y_n - 3 > -6 - 3 \implies 2(y_n - 3) > -18$$

and we have

$$\frac{2y_n - 6}{3} > -6.$$

This means that  $y_{n+1} > -6$ , and therefore  $y_n > -6 \forall n \in \mathbb{N} \quad \ominus$

(b) **Solution:** We want to show that the sequence  $(y_1, y_2, \dots)$  is decreasing, i.e.  $y_n > y_{n+1} \forall n \in \mathbb{N}$ .

We know  $y_1 = 6$  and  $y_2 = 2$ . Clearly  $y_1 > y_2$ . Continuing on we assume that our induction hypothesis,

$$y_{n+1} = \frac{(2y_n - 6)}{3},$$

is true. We want to show if we have  $y_n > y_{n+1}$  then we are guaranteed to have  $y_{n+1} > y_{n+2}$ . First we consider

$$y_n > y_{n+1} \implies y_n - 3 > y_{n+1} - 3 \implies 2y_n - 6 > 2y_{n+1} - 6 \implies \frac{2y_n - 6}{3} > \frac{2y_{n+1} - 6}{3}.$$

Then, we can substitute using induction hypothesis,

$$\frac{2y_n - 6}{3} > \frac{2y_{n+1} - 6}{3} \implies y_{n+1} > y_{n+2} \implies y_{n+1} > y_{n+2},$$

which is exactly what we needed to show. Thus,  $y_n > y_{n+1} \forall n \in \mathbb{N} \quad \ominus$

### Question 6

Grade the following proof: Prove that  $\sqrt{3}$  is irrational. Does a similar argument work to show  $\sqrt{6}$  is irrational?

**Proof:** To prove that  $\sqrt{3}$  is irrational, we can use a proof by contradiction. The idea is to assume that  $\sqrt{3}$  is rational, and then derive a contradiction.

**Good job setting the stage for the rest of the proof.**

Assumption:  $\sqrt{3}$  is rational.

If  $\sqrt{3}$  is rational, it can be expressed as a fraction  $\frac{a}{b}$  in its simplest form, where  $a$  and  $b$  are integers with no common factors other than 1 (i.e., they are coprime), and  $b$  is not equal to 0.

So, we have:  $\sqrt{3} = \frac{a}{b}$

Squaring both sides of the equation:  $3 = \frac{a^2}{b^2}$

Rearranging the equation:  $a^2 = 3 \cdot b^2$

From this equation, we can see that  $a^2$  is a multiple of 3, which implies that  $a$  must also be a multiple of 3. Let's represent  $a$  as  $3k$  where  $k$  is an integer:

**How do we know for certain that if  $a^2$  is a multiple of 3 then  $a$  must be a multiple of 3? It is not immediately obvious why this is the case, so you should make some justification why this must be true.**

$$(3k)^2 = 3 \cdot b^2$$

$$9k^2 = 3 \cdot b^2$$

$$3k^2 = b^2$$

Now,  $b^2$  is also a multiple of 3, which means  $b$  must be a multiple of 3. However, this contradicts our initial assumption that  $a$  and  $b$  have no common factors other than 1 (i.e., they are coprime). This contradiction arises from assuming that  $\sqrt{3}$  is rational.

**I like the way you walk through exactly how the contradiction arises and then hint at what this leads us to conclude.**

Hence, our initial assumption that  $\sqrt{3}$  is rational must be false, which means that  $\sqrt{3}$  is irrational. Now, to address whether a similar argument works for  $\sqrt{6}$ :

No, a similar argument does not work for  $\sqrt{6}$ . The proof for the irrationality of  $\sqrt{3}$  relies on the fact that it leads to a contradiction with respect to the common factors of  $a$  and  $b$ . However, for  $\sqrt{6}$ , this argument does not apply directly because the factors of 6 are not the same as the factors of 3. In fact,  $\sqrt{6}$  is not irrational; it is a rational number. It can be expressed as the fraction  $\frac{2\sqrt{3}}{3}$ , which is a ratio of two integers.

**Your argument starts off alright, but you need to specify what about the difference in factors between 6 and 3 means that your preceding argument needs to be changed to prove that  $\sqrt{6}$  is irrational. The second part of your statement,  $\frac{2\sqrt{3}}{3}$ , is false. This is not a ratio of two integers, in fact you just proved before that  $\sqrt{3}$  itself is irrational!**

Overall, you articulate most of your reasoning well, but make sure that you are thorough in the lines of your argument. Each step should clearly follow the preceding one - and make sure what you are saying is indeed true!

(2.5/5)