Real Analysis HW #5

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Question 1

1. Let $x_n \geqslant 0$ for all $n \in \mathbb{N}$. (a) If $(x_n) \to 0$, show that $\sqrt{x_n} \to 0$. (b) If $(x_n) \to x$, show that $\sqrt{x_n} \to \sqrt{x}$.

Question 2

2. Let (a_n) and (b_n) be sequences of real numbers. (a) Show that if $(a_n) \to 0$ and $(a_n - b_n) \to 0$, then $(b_n) \to 0$. (b) Show that if $(a_n) \to 0$ and $|b_n - b| \leqslant a_n$, then $(b_n) \to b$

Question 3

3. Consider $y_1 = 1, y_{n+1} = (2y_n + 3)/4$ for all $n \in \mathbb{N}$. Show by direct calculations that $y_1 < y_2 < 2$. Then, show that if $y_{n-1} < y_n < 2$ that $y_n < y_{n+1} < 2$. Use this to show that $\{y_n\}$ converges and find its limit.

Question 4

4. (Cesaro Means). Show that if (x_n) is a convergent sequence, then the sequence given by the averages:

$$y_n = \frac{x_1 + x_2 + \ldots x_n}{n}$$

also converges to the same limit.

Question 5

5. ChatGPT Challenge: To prove that if $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$, you can use the definition of a limit and the properties of limits. Here's a proof:

Proof: Let $\epsilon > 0$ be arbitrary. Since $\lim(a_n - b_n) = 0$, there exists an N_1 such that for all $n \geqslant N_1$, we have $|a_n - b_n| < \epsilon/2$. Now, consider the inequality $|a_n - b_n| < \epsilon/2$. We can add $|b_n|$ to both sides of this inequality: $|a_n - b_n| + |b_n| < \epsilon/2 + |b_n|$

By the triangle inequality, we know that $|a_n - b_n + b_n| \leqslant |a_n - b_n| + |b_n|$, so: $|a_n| < \epsilon/2 + |b_n|$ Now, we have the following inequalities: $|a_n - b_n| < \epsilon/2$ (from our choice of N_1) $|a_n| < \epsilon/2 + |b_n|$ Now, choose N_2 such that for all $n \geqslant N_2$, we have $|b_n| < \epsilon/2$. Let $N = \max(N_1, N_2)$. For $n \geqslant N$, we have: $|a_n - b_n| < \epsilon/2$ (from the choice of N_1) $|b_n| < \epsilon/2$ (from the choice of N_2) Now, let's use these inequalities to bound $|a_n|$: $|a_n| = |a_n - b_n + b_n| \leqslant |a_n - b_n| + |b_n| < \epsilon/2 + \epsilon/2 = \epsilon$ This shows that for all $n \geqslant N$, we have $|a_n| < \epsilon$, which means that $\lim a_n = 0$. Since ϵ was arbitrary, we have shown that for any $\epsilon > 0$, there exists an N such that for all $n \geqslant N$, $|a_n| < \epsilon$, which is the definition of $\lim a_n = 0$. Therefore, we have proved that if $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.