

# Real Analysis HW #9

Jack Krebsbach

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### Question 1

Let  $g : A \rightarrow \mathbb{R}$  and  $f : A \rightarrow \mathbb{R}$ . Suppose that  $\lim_{x \rightarrow a} f(x) = 0$ .

(a) Show that  $\lim_{x \rightarrow a} f(x)g(x) = 0$  for any function  $g$  as above IS NOT TRUE.

(b) Assume that  $g$  is a bounded function on  $A$ . Show that  $\lim_{x \rightarrow c} g(x)f(x) = 0$ .

### Question 2

Let  $a$  and  $b$  be real numbers with  $a \neq 0$ . Use the definition of continuity to prove that the function  $f$  defined by  $f(x) = ax + b$  is continuous at every real number.

### Question 3

Use the definition of limit to prove that  $\lim_{x \rightarrow c} x^2 = c^2$  for every real number  $c$ .

### Question 4

Find constants  $a$  and  $b$  so that the function  $f$  defined by

$$f(x) = \begin{cases} 3ax^2 + 1 & 0 \leq x \leq 1 \\ ax + b & 1 < x < 2 \\ 2bx + a & 2 \leq x \leq 4 \end{cases}$$

has a limit at each point of  $[0, 4]$ . Be sure to show the limit exists.

### Question 5

Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and suppose that  $f(x) = 0$  for each irrational number  $x$  in  $[a, b]$ . Prove that  $f(x) = 0$  for all  $x \in [a, b]$ .

### Question 6

CHAT GPT CHALLENGE: Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function on  $[a, b]$ , and suppose that  $f(x) = 0$  for each rational number  $x$  in  $[a, b]$ . We aim to show that  $f(x) = 0$  for all  $x \in [a, b]$ .

Assume, for the sake of contradiction, that there exists  $c \in [a, b]$  such that  $f(c) \neq 0$ . Since  $f$  is continuous, for any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x \in [a, b]$ , if  $|x - c| < \delta$ , then  $|f(x) - f(c)| < \epsilon$ .

Consider the case when  $\epsilon = |f(c)|$ . By continuity, there exists  $\delta > 0$  such that for all  $x \in [a, b]$ , if  $|x - c| < \delta$ , then  $|f(x) - f(c)| < |f(c)|$ . This implies

$$-f(c) < f(x) - f(c) < f(c).$$

Adding  $f(c)$  to all parts of the inequality yields  $0 < f(x)$  for  $x$  in the interval  $(c - \delta, c) \cap [a, b]$ . Similarly, subtracting  $f(c)$  from all parts of the inequality yields  $f(x) < 0$  for  $x$  in the interval  $(c, c + \delta) \cap [a, b]$ . Now, consider the set of rational numbers in  $[a, b]$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there exist rational numbers in both  $(c - \delta, c) \cap [a, b]$  and  $(c, c + \delta) \cap [a, b]$ . However, we have shown that for any rational  $x$  in these intervals,  $f(x)$  has opposite signs, which is a contradiction.

Therefore, our assumption that there exists  $c$  such that  $f(c) \neq 0$  is false, and we conclude that  $f(x) = 0$  for all  $x \in [a, b]$ .