

# Ultimate Problem Set

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### Question 1

Suppose that  $x > -1$  and that  $x \neq 0$ . Prove that

$$(1 + x)^n > 1 + nx$$

for each integer  $n > 1$ . This result is known as Bernoulli's inequality.

**Proof:** We will show that this inequality holds for  $x > -1$  and  $x \neq 0$  by induction. First, we see when  $n = 2$  that

$$(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x.$$

Thus, our base case holds. Now we assume that

$$(1 + x)^n > 1 + nx$$

is true. We want to show that  $(1 + x)^{n+1} > 1 + (n + 1)x$  is also true. First, we multiply both sides by  $(1 + x)$ , then

$$(1 + x)(1 + x)^n > (1 + x)(1 + nx)$$

$$\implies (1 + x)^{n+1} > 1 + x + nx + nx^2 \geq 1 + nx + x = 1 + (n + 1)x.$$

Hence,  $(1 + x)^n > 1 + nx$  for each integer  $n > 1$ .



## Question 2

Show that  $e$  is irrational by supposing that  $e = \frac{m}{n}$  and deriving a contradiction. Use the fact that  $e = \sum_{j=0}^{\infty} \frac{1}{j!}$ .  
Let  $s_k = \sum_{j=0}^k \frac{1}{j!}$ .

(a) Prove that

$$e - s_k < \frac{1}{(k+1)!} \left\{ 1 + \frac{1}{k+1} + \left( \frac{1}{k+1} \right)^2 + \cdots \right\}.$$

**Proof:** We have that

$$\begin{aligned} e - s_k &= \sum_{j=0}^{\infty} \frac{1}{j!} - \sum_{j=0}^k \frac{1}{j!} = \sum_{j=k+1}^{\infty} \frac{1}{j!} = \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \frac{1}{(k+3)!} + \cdots \\ &= \frac{1}{(k+1)!} \left[ 1 + \frac{1}{(k+2)} + \frac{1}{(k+2)(k+3)} + \cdots \right] \\ &< \frac{1}{(k+1)!} \left[ 1 + \frac{1}{(k+1)} + \frac{1}{(k+1)(k+1)} + \cdots \right] \\ &= \frac{1}{(k+1)!} \left[ 1 + \frac{1}{(k+1)} + \left( \frac{1}{k+1} \right)^2 + \cdots \right] \end{aligned}$$

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(b) Prove that  $e - s_k < \frac{1}{k(k!)}$  for all  $k \in \mathbb{N}$ .

**Proof:** Let

$$y_n = \sum_{n=0}^m \frac{1}{(k+1)^n}.$$

Then

$$\begin{aligned} y_n - \frac{1}{k+1} y_n &= \sum_{n=0}^m \frac{1}{(k+1)^n} - \sum_{n=1}^{m+1} \frac{1}{(k+1)^n} \\ \Rightarrow y_n \left( 1 - \frac{1}{k+1} \right) &= \frac{1}{k+1} - \frac{1}{(k+1)^{m+1}} \\ \Rightarrow y_n &= \frac{\frac{1}{k+1} - \frac{1}{(k+1)^{m+1}}}{\left( 1 - \frac{1}{k+1} \right)}. \end{aligned}$$

Now let

$$\begin{aligned} \lim_{m \rightarrow \infty} y_n &= \lim_{m \rightarrow \infty} \frac{\frac{1}{k+1} - \frac{1}{(k+1)^{m+1}}}{\left( 1 - \frac{1}{k+1} \right)} \\ \Rightarrow (y_n) &\rightarrow \frac{1}{k+1} \frac{1}{\left( 1 - \frac{1}{k+1} \right)} \\ &= \frac{1}{k+1} \frac{k+1}{k+1-1} \\ &= \frac{1}{k+1} \frac{k+1}{k} = \frac{1}{k}. \end{aligned}$$

Finally,

$$e - s_k < \frac{1}{(k+1)!} \left[ 1 + \frac{1}{(k+1)} + \frac{1}{(k+1)^2} + \cdots \right] = \frac{1}{(k+1)!} \{y_k\} \leq \frac{1}{(k+1)!} \frac{1}{k} < \frac{1}{k(k!)}$$

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(c) If  $e = \frac{m}{n}$ , prove that  $n!e$  and  $n!s_n$  are integers.

**Proof:** We have

$$n!e = n! \frac{m}{n} = (n-1)!m.$$

Since the integers are closed then  $n!e$  must be an integer.

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**Proof:**

$$n!s_n = n! \sum_{j=0}^n \frac{1}{j!} = n! \left( 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \right).$$

$$= n! + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + 1$$

$$= n! + n(n-1)(n-2) \cdots (4)(3) + n(n-1)(n-2) \cdots (5)(4) + \cdots + 1.$$

Again, because the integers are closed it must be that  $n!s_n$  is also an integer.

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(d) If  $e = \frac{m}{n}$ , prove that  $n!(e - s_n)$  is an integer between 0 and 1, which is absurd.

**Proof:** Consider that

$$n!(e - s_n) < n! \frac{1}{n!n} = 1/n,$$

which means that  $0 < n!(e - s_n) < 1$ . Because  $n!(e - s_n)$  must be an integer we have encountered a contradiction, this is impossible. Thus,  $e$  can not be a rational number.

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### Question 3

Let  $f$  be a function defined on all of  $\mathbb{R}$ , and assume there is a constant  $c$  such that  $0 < c < 1$  and

$$|f(x) - f(y)| \leq c|x - y|$$

for all  $x, y \in \mathbb{R}$ .

(a) Show that  $f$  is continuous.

**Proof:** Let  $\epsilon > 0$  and  $x, y \in \mathbb{R}$ . Choose  $\delta = \epsilon/c$ . Now if we have

$$|x - y| < \delta$$

then

$$|f(x) - f(y)| \leq c|x - y| < c \frac{\epsilon}{c} = \epsilon.$$

Hence,  $f$  must be continuous. ☺

(b) Pick some  $y_1 \in \mathbb{R}$  and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \dots).$$

In general, if  $y_{n+1} = f(y_n)$ , show that the resulting sequence  $(y_n)$  is a Cauchy sequence. Hence we may let  $y = \lim_{n \rightarrow \infty} y_n$ .

**Proof:** Let  $\epsilon > 0$ . First we will show that

$$|y_{n+1} - y_n| \leq c^{n-1}|y_2 - y_1| = c^n \frac{|y_2 - y_1|}{c}$$

is true by induction:

1. Base case: From the given, when  $n = 2$ ,

$$|y_3 - y_2| = |f(y_2) - f(y_1)| \leq c^1|y_2 - y_1|.$$

2. Inductive step: Now we want to show that if this holds true for  $n$ , this also holds true for  $n + 1$ . We assume

$$|y_{n+1} - y_n| \leq c^{n-1}|y_2 - y_1|$$

is true. We have

$$c|y_{n+1} - y_n| \leq c^n|y_2 - y_1|.$$

It follows

$$|y_{n+2} - y_{n+1}| = |f(y_{n+1}) - f(y_n)| \leq c|y_{n+1} - y_n| \leq c^n|y_2 - y_1|.$$

Finally, we conclude that

$$|y_{n+1} - y_n| \leq c^{n-1}|y_2 - y_1| = c^n \frac{|y_2 - y_1|}{c}$$

for all  $n > 1$ .

Now we may continue in the proof showing that  $(y_n)$  is Cauchy. Let

$$B = \frac{|y_2 - y_1|}{c}.$$

Choose  $N \in \mathbb{N}$  such that for all  $n > N$ ,

$$B \frac{c^n}{1 - c} < \epsilon.$$

Let  $m > n > N$ . We have that

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} + \cdots + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| = \sum_n^{m-1} |x_{n+1} - x_n| \leq \sum_n^{m-1} Bc^n. \end{aligned}$$

If

$$y_n = \sum_n^{m-1} Bc^n,$$

then

$$\begin{aligned} y_n - cy_n &= \sum_n^{m-1} Bc^n - \sum_{n+1}^m Bc^n \\ \implies y_n(1 - c) &= Bc^n - Bc^m \\ \implies y_n &= \frac{B(c^n - c^m)}{1 - c} = B \frac{c^n - c^m}{1 - c} \\ &< B \frac{c^n}{1 - c} < \epsilon. \end{aligned}$$

Hence,  $|x_m - x_n| < \epsilon$  and  $(y_n)$  is thus Cauchy. We may let  $y = \lim_{n \rightarrow \infty} y_n$ .

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(c) Prove that  $y$  is a fixed point of  $f$  (i.e.  $f(y) = y$ ) and that it is unique in this regard.

**Proof:** Consider that  $y_{n+1} = f(y_n)$ . Then,

$$\begin{aligned} \lim_{n \rightarrow \infty} y_{n+1} &= \lim_{n \rightarrow \infty} f(y_n) \\ \implies y &= f\left(\lim_{n \rightarrow \infty} y_n\right) \\ \implies y &= f(y). \end{aligned}$$

Thus,  $y$  is a fixed point. Consider, by way of contradiction, that  $y$  is not the only fixed point and there exists another fixed point  $x$  where  $y \neq x$ .

Then,

$$|f(y) - f(x)| = |x - y| < c|x - y|$$

which is a contradiction since  $0 < c < 1$ . Thus, it must be that  $y$  is a unique fixed point.

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(d) Finally, prove that if  $x$  is any arbitrary point in  $\mathbb{R}$ , then the sequence  $(x, f(x), f(f(x)), \dots)$  converges to  $y$  (as defined in (b)).

**Proof:** Because we proved previously that for any arbitrary  $x \in \mathbb{R}$  the sequence  $(y_n) \rightarrow y$  and  $y$  is a unique fixed point it must be the sequence  $(x, f(x), f(f(x)), \dots)$  converges to  $y$  (as defined in (b)).

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#### Question 4

Let  $\{r_n\}$  be a listing of all the rational numbers. Define a function  $f$  by  $f(x) = 0$  if  $x$  is irrational and  $f(r_n) = 1/n$  for all  $n$ . Show that  $f$  is continuous everywhere except for the set of rational numbers.

**Proof:** Let  $\epsilon > 0$ . We consider (1) if  $f$  is continuous at  $c$ , an irrational number and (2) if  $f$  is continuous at  $c$ , a rational number (2). Note that  $f(x) \geq 0$  for all  $x \in \mathbb{R}$ .

1. By the Archimedes Principle there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $1/n < \epsilon$ . Consider all the rational numbers where  $n \leq N$ , and choose  $\delta = \min\{|r_n - c|\}/2$ . Note that we have chosen  $N \in \mathbb{N}$  such that for all rational numbers where  $n > N$ ,  $f(r_n) < \epsilon$  and also for all  $x \in \mathbb{R} \setminus \mathbb{Q}$ ,  $f(x) = 0 < \epsilon$ . Thus, when  $x \in \mathbb{R}$  we have chosen  $\delta$  such that when  $|x - c| < \delta$ , we automatically have that  $|f(x) - f(c)| = |f(x) - 0| = |f(x)| = f(x) < \epsilon$ . Hence,  $f$  is continuous on the irrationals.
2. Consider, by way of contradiction, that  $f$  is continuous on the rational numbers. Then there exists  $\delta$  such that when  $x \in \mathbb{R}$  and  $|x - y| < \delta$  we automatically have  $|f(x) - f(c)| < f(c)/2$ . By the density of the irrational numbers in  $\mathbb{R}$  there exists an irrational number,  $x_I$ , such that  $|x_I - y| < \delta$ . It follows  $|f(x_I) - f(c)| = |0 - f(c)| = f(c) < f(c)/2 \implies 1 < 1/2$ , a contradiction. Thus,  $f$  can not be continuous on the rational.



#### Question 5

Using the  $\delta - \epsilon$  definition of a limit, show

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3.$$

**Proof:** Consider,

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1} \left[ (x - 1) \frac{x^2 + x + 1}{x - 1} \right] = \lim_{x \rightarrow 1} x^2 + x + 1 = 3.$$

Let  $\epsilon > 0$  and choose  $\delta = \min\{1, \epsilon/4\}$ . Note, if restrict  $\delta$  to be a maximum of 1 then  $|x + 2| \leq |x| + 2 \leq |2| + 2 = 4$ . If we have  $0 < |x - 1| < \delta$  then

$$|f(x) - L| = |x^2 + x + 1 - 3| = |x^2 + x - 2| = |(x + 2)(x - 1)| = |x + 2||x - 1|.$$

Altogether,

$$|f(x) - L| = |x + 2||x - 1| < 4 \frac{\epsilon}{4} = \epsilon.$$

Hence,

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3.$$

