Real Analysis HW #4

Jack Krebsbach

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Question 1

Let A be a nonempty bounded set. The maximum value is a number $x \in A$ such that $a \leqslant x\ \forall a \in A$. Prove that a nonempty bounded set has a maximum value if and only if it contains its supremum.

Solution:

Proof: $\Rightarrow$ If a nonempty bounded set has a maximum value $x = Max(A)$ then for all $a \in A$ we have $a \leqslant x$. Then x is an upperbound of the set A. Let $b = \sup A$. Then $b \leqslant x$ and $b \geqslant a$ for all $a \in A$. Thus, $a \leqslant b \leqslant x$. Therefore the suprememum of A a is contained within the set, $b \in A$.

$\Leftarrow$ Let $b = \sup A$ with $b \in A$. Then for all $a \in A$ we have that $a \leqslant b$. Thus b is the maximum of A.

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Question 2

Let A be a non-empty set and let $\mathcal{P}(\mathcal{A})$ represent the collection of all subsets of A; this set is known as the power set of A.

(a) Suppose that A has n elements. Prove that $\mathcal{P}(\mathcal{A})$ has 2^n elements.

Solution:

Let $n = 1$. Take the set A with 1 element to be denoted set $A_1 = \{a_1\}$ and the power set $P_1 = \mathcal{P}(\mathcal{A}_1) = \{\{a_1\}, \emptyset\}$. Then $|P_1| = |\mathcal{P}(\mathcal{A}_1)| = 2^n = 2^1 = 2$. We have shown that this works for n. We would like to show, through proof by induction, that this works for $n + 1$.

Consider

$$\mathcal{P}(\mathcal{A}_{n+1}) = \mathcal{P}(\mathcal{A}_n) \cup \{\{p_{ni} \cup a_{n+1}\} : i \in \mathbb{N}_{2^n}\}.$$

Because the right side of the union is disjoint from the left side of the union we can add the cardinalities together.

Hence,

$$|\mathcal{P}(\mathcal{A}_{n+1})| = |\mathcal{P}(\mathcal{A}_n)| + |\{\{p_{ni} \cup a_{n+1}\} : i \in \mathbb{N}_{2^n}\}|.$$

Then,

$$|\mathcal{P}(\mathcal{A}_{n+1})| = 2^n + 2^n = 22^n = 2^{n+1}.$$

Thus, $|\mathcal{P}(\mathcal{A}_{n+1})| = 2^{n+1}$. We have shown that $\mathcal{P}(\mathcal{A})$ has 2^n elements.

(b) Suppose that A is countable. Prove that $\mathcal{P}(\mathcal{A})$ is uncountable.

Solution:

Assume, by way of contradiction, that $\mathcal{P}(\mathcal{A})$ is countable. Then there exists an onto function $f : \mathbb{N} \to \mathcal{P}(\mathcal{A})$. Construct $B = \{n : n \notin f(n)\}$. Since f is onto there must exist some $n_0 \in \mathbb{N}$ such that $f(n_0) = B$.

We consider two cases:

1. $n_0 \in B$. Then by construction of the set B, $n_0 \notin f(n_o)$. However, $f(n_0) = B$ so $n_0 \notin B$ $\rightarrow\!\leftarrow$.
2. $n_0 \notin B$. So $n_0 \in f(n_0)$. This implies $n_0 \in B$ $\rightarrow\!\leftarrow$

Thus, f can not be onto. If it is not onto then $|\mathbb{N}| \neq |\mathcal{P}(\mathcal{A})|$ and thus $\mathcal{P}(\mathcal{A})$ can not be countable.

(c) Suppose that A is uncountable. Prove that there is no bijection between A and $\mathcal{P}(\mathcal{A})$.

Proof: Assume, by way of contradiction, that there exists a bijection between A and $\mathcal{P}(\mathcal{A})$, hence $f : A \to \mathcal{P}(\mathcal{A})$. Construct $B = \{a : a \notin f(n)\}$. Since f is onto there must exist some $a_0 \in \mathbb{N}$ such that $f(a_0) = B$.

We consider two cases:

1. $a_0 \in B$. Then by construction of the set B, $n_0 \notin f(a_o)$. However, $f(a_0) = B$ so $a_0 \notin B$ $\rightarrow\!\leftarrow$.
2. $a_0 \notin B$. So $a_0 \in f(a_0)$. This implies $a_0 \in B$ $\rightarrow\!\leftarrow$

Thus, there can not be a bijection between A and $\mathcal{P}(\mathcal{A})$.

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Question 3

Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7 . Is E countable? Explain.

Solution:

Proof: E is countable. Let f be the function from the natural numbers to the set of all permutations of 4 and 7 in decimal expansion with n elements. That is $1 \longrightarrow \{0.4, 0.7\}$, $2 \longrightarrow \{0.47, 0.74\}$. Each of these sets is countable as $n \to \infty$. The union of of countable sets are countable (Theorem 1.5.8). Thus, E is countable.

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Question 4

Consider the function h defined by

$$h(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1/q & \text{if } x = p/q \end{cases}$$

Here it is assumed that the rational number p/q is in reduced form and that $q > 0$.

Solution: (a) Find $h(n)$ for each integer n

(b) Find three solutions to the equation $h(x) = 1/3$.

(c) Find all of the solutions to the equation $h(x) = 1/7$ that lie in the interval $(3, 4)$.

(d) Prove that the set of all solutions to the equation $h(x) = 1/5$ is countable infinite.

(e) Let (a, b) be any interval and let $\epsilon > 0$. Prove that $\{x \in (a, b) : h(x) \geqslant \epsilon\}$ is a finite set.

Question 5

ChatGPT Challenge: Find a sequence is bounded below, not bounded above, and contains an infinite number of negative terms.

Proof: Certainly, here's an example of a sequence that is bounded below, not bounded above, and contains an infinite number of negative terms: Consider the sequence defined as follows: $-1, -2, -3, -4, -5, \ldots$ This sequence consists of negative integers and is strictly decreasing. It is bounded below by negative infinity (there is no lower bound as it goes to negative infinity), not bounded above (it becomes more negative without bound), and it contains an infinite number of negative terms.