Real Analysis HW #7

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Question 1

Let (x_n) be a sequence and suppose that the sequence $(x_{n+1} - x_n)$ converges to 0 . Give an example to show that the sequence (x_n) may not converge. (See ChatGPT Challenge)

Question 2

Let (x_k) and (y_k) be two sequences and let (r_k) be a sequence of positive numbers that converges to 0 . Suppose that $0 < |y_k - x_k| < r_k \forall k \in \mathbb{N}$.

(a) Give an example to show that the sequences (x_k) and (y_k) may not converge.

(b) Suppose that (x_k) converges to L. Prove that the sequence (y_k) converges to L.

Question 3

Assume that (x_n) is a bounded sequence with the property that every convergent subsequence of (x_n) converges to the same limit $x \in \mathbb{R}$. Show that (x_n) must converge to x.

Question 4

Let (x_n) be a Cauchy sequence. Show directly that (x_n) is bounded.

Question 5

Let (x_n) be a sequence and let r be a real number satisfying $0 < r < 1$. Suppose that $|x_{n+1} - x_n| < r\,|x_n - x_{n-1}|$ for all $n > 2$. Prove that (x_n) is a Cauchy sequence and hence convergent.

Question 6

CHAT GPT CHALLENGE: Let (x_n) be a sequence such that $(x_{n+1} - x_n)$ converges to 0 . We want to show that (x_n) also converges.

By the definition of a convergent sequence, we need to show that for any $\epsilon > 0$, there exists a positive integer N such that for all $n \geqslant N, |x_n - L| < \epsilon$ for some real number L. Since $(x_{n+1} - x_n)$ converges to 0 , we can choose $\epsilon > 0$ and find a positive integer M such that for all $m \geqslant M, |(x_{m+1} - x_m) - 0| < \epsilon$. This implies that for all $m \geqslant M, |x_{m+1} - x_m| < \epsilon$. Now, let $N = M$. For any $n \geqslant N$, we have:

$$\begin{aligned}
|x_n - x_N| &= |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \ldots + (x_{N+1} - x_N)| \\
&\leqslant |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \ldots + |x_{N+1} - x_N| \\
&< \epsilon + \epsilon + \ldots + \epsilon \quad (\text{ Since } |x_{m+1} - x_m| < \epsilon \text{ for all } m \geqslant M) \\
&= (n - N)\epsilon.
\end{aligned}$$

Dividing both sides by $(n - N)$, we get:

$$|x_n - x_N| < \epsilon$$

This shows that for any $\epsilon > 0$ and $n \geqslant N$, we have $|x_n - x_N| < \epsilon$, where $N = M$. Therefore, (x_n) converges to x_N.

Hence, we have shown that if $(x_{n+1} - x_n)$ converges to 0 , then (x_n) also converges.