Real Analysis HW #4

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Question 1

Let A be a nonempty bounded set. The maximum value is a number $x \in A$ such that $a \leqslant x\ \forall a \in A$. Prove that a nonempty bounded set has a maximum value if and only if it contains its supremum.

Solution:

Proof: $\Rightarrow$ Let $x \in A$ be the maximum value of A. For all $a \in A$ we have $a \leqslant x$. Then x is an upperbound of A. Let $b = \sup A$. Then $b \leqslant x$ and $a \leqslant b$ for all $a \in A$. Because $x \in A$ then $x \leqslant b$. Thus $x = b = \sup A$. As $x \in A$ the suprememum of A, is thus contained within the set.

$\Leftarrow$ Let $b = \sup A$ with $b \in A$. Then for all $a \in A$ we have that $a \leqslant b$ with $b \in A$. Thus, b is the maximum of A.

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Question 2

Let A be a non-empty set and let $\mathcal{P}(\mathcal{A})$ represent the collection of all subsets of A; this set is known as the power set of A.

(a) Suppose that A has n elements. Prove that $\mathcal{P}(\mathcal{A})$ has 2^n elements.

Solution:

Let $n = 1$. Take the set A with 1 element to be denoted set $A_1 = \{a_1\}$ and the power set $P_1 = \mathcal{P}(\mathcal{A}_1) = \{\{a_1\}, \emptyset\}$. Then $|P_1| = |\mathcal{P}(\mathcal{A}_1)| = 2^n = 2^1 = 2$. We have shown that this works for n. If $|A| = n$ then $|\mathcal{P}(\mathcal{A})| = 2^n$. We would like to show, through proof by induction, that this works for $n + 1$.

Consider

$$\mathcal{P}(\mathcal{A}_{n+1}) = \mathcal{P}(\mathcal{A}_n) \cup \{\{p_{ni} \cup a_{n+1}\} : i \in \mathbb{N}_{2^n}\}$$

where p_{ni} denotes the ith element of the power set of A_n. Because the right side of the union is disjoint from the left side of the union we can add the cardinalities together.

Hence,

$$|\mathcal{P}(\mathcal{A}_{n+1})| = |\mathcal{P}(\mathcal{A}_n)| + |\{\{p_{ni} \cup a_{n+1}\} : i \in \mathbb{N}_{2^n}\}|.$$

Then,

$$|\mathcal{P}(\mathcal{A}_{n+1})| = 2^n + 2^n = 2^n 2^1 = 2^{n+1}.$$

Thus, $|\mathcal{P}(\mathcal{A}_{n+1})| = 2^{n+1}$. We have shown that $\mathcal{P}(\mathcal{A})$ has 2^n elements.

(b) Suppose that A is countable. Prove that $\mathcal{P}(\mathcal{A})$ is uncountable.

Solution:

Assume, by way of contradiction, that $\mathcal{P}(\mathcal{A})$ is countable. Then there exists an onto function $f : \mathbb{N} \to \mathcal{P}(\mathcal{A})$. Construct $B = \{n : n \notin f(n)\}$. Since f is onto there must exist some $n_0 \in \mathbb{N}$ such that $f(n_0) = B$.

We consider two cases:

1. $n_0 \in B$. Then by construction of the set B, $n_0 \notin f(n_o)$. However, $f(n_0) = B$ so $n_0 \notin B$ $\rightarrow\!\leftarrow$.
2. $n_0 \notin B$. So $n_0 \in f(n_0)$. This implies $n_0 \in B$ $\rightarrow\!\leftarrow$

Thus, f can not be onto. If it is not onto then $|\mathbb{N}| \neq |\mathcal{P}(\mathcal{A})|$ and thus $\mathcal{P}(\mathcal{A})$ can not be countable.

(c) Suppose that A is uncountable. Prove that there is no bijection between A and $\mathcal{P}(\mathcal{A})$.

Proof: Assume, by way of contradiction, that there exists a bijection between A and $\mathcal{P}(\mathcal{A})$, hence $f : A \to \mathcal{P}(\mathcal{A})$. Construct $B = \{a : a \notin f(n)\}$. Since f is onto there must exist some $a_0 \in \mathbb{N}$ such that $f(a_0) = B$.

We consider two cases:

1. $a_0 \in B$. Then by construction of the set B, $a_0 \notin f(a_o)$. However, $f(a_0) = B$ so $a_0 \notin B$ $\rightarrow\!\leftarrow$.
2. $a_0 \notin B$. So $a_0 \in f(a_0)$. This implies $a_0 \in B$ $\rightarrow\!\leftarrow$

Thus, there can not be a bijection between A and $\mathcal{P}(\mathcal{A})$.

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Question 3

Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7 . Is E countable? Explain.

Solution:

Proof: E is not countable. Assume, by way of contradiction, that E is countable. Then there exists a bijection from the natural numbers to E. Then we can list the correspondence where each $a_{nm} \in \{0,4,7\}$.

N		E									
1	$\longleftrightarrow$	$f(1)$	$=$	$.a_{11}$	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\cdots$
2	$\longleftrightarrow$	$f(2)$	$=$	$.a_{21}$	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	..
3	$\longleftrightarrow$	$f(3)$	$=$	$.a_{31}$	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	..
4	$\longleftrightarrow$	$f(4)$	$=$	$.a_{41}$	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	$\ldots$
5	$\longleftrightarrow$	$f(5)$	$=$	$.a_{51}$	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	$\ldots$
6	$\longleftrightarrow$	$f(6)$	$=$	$.a_{61}$	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	$\cdots$
$\vdots$				$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	:	

Construct $E_0 = .e_1e_2e_3...$ where

$$e_n = \begin{cases} 4 & a_{nn} = 7, 0 \\ 7 & a_{nn} = 4 \end{cases}$$

Thus, E_0 can not in the range of the surjection. Therefore E is uncountable.

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Question 4

Consider the function h defined by

$$h(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1/q & \text{if } x = p/q \end{cases}$$

Here it is assumed that the rational number p/q is in reduced form and that $q > 0$.

(a) Find $h(n)$ for each integer n

Solution: For each integer n, $f(n) = 1$

(b) Find three solutions to the equation $h(x) = 1/3$.

Solution: $x = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$

(c) Find all of the solutions to the equation $h(x) = 1/7$ that lie in the interval $(3,4)$.

Solution: $x = \frac{22}{7}, \frac{23}{7}, \frac{24}{7}, \frac{25}{7}, \frac{26}{7}, \frac{27}{7},$

(d) Prove that the set of all solutions to the equation $h(x) = 1/5$ is countable infinite.

Solution:

Proof: Let S be the set of solutions to the equation $h(x) = \frac{1}{5}$. Then for every $s \in S$ we can write as the ratio of two integers - a rational number. Then $S \subset \mathbb{Q}$. Since S is a subset of a countable set (the rational numbers) then it must be countable (Theorem 1.5.7.) 😄

(e) Let (a,b) be any interval and let $\epsilon > 0$. Prove that $\{x \in (a,b) : h(x) \geqslant \epsilon\}$ is a finite set.

Solution: If $f(x) \geqslant \epsilon > 0$ then x must be a rational number by construction of $h(x)$. Consider the solution set for every $N \in \mathbb{N}$ where $x = \frac{p}{q}$ and $\frac{1}{N} = \frac{1}{q}$ under the constraint that $x \in (a,b)$. Each of these sets is finite. For each N, $a < \frac{p}{N} < b \implies aN < p < bN$. So the set of numerators of x at each N is the integers between aN and bN. This is finite if we only consider reduced forms of x.

By Archy :) we can choose the minimum N such that $\frac{1}{N+1} < \epsilon$ and $\frac{1}{N} \geqslant \epsilon$. To find the number of elements in A count the number of *reduced form solutions* satisfying $h(x) = \frac{1}{n}$ for $n = 1, 2, \ldots, N$. Thus, A is finite.

Question 5

ChatGPT Challenge: Find a sequence is bounded below, not bounded above, and contains an infinite number of negative terms.

Proof: Certainly, here's an example of a sequence that is bounded below, not bounded above, and contains an infinite number of negative terms: Consider the sequence defined as follows: $-1, -2, -3, -4, -5, \ldots$ This sequence consists of negative integers and is strictly decreasing.

This is correct.

It is bounded below by negative infinity (there is no lower bound as it goes to negative infinity),

This doesn't mean anything. Infinity is not a number so this does not make sense. In addition, the problem asks for as sequence that *is* bounded from below and you have just stated that there is no lower bound. Consider the function $f(n) = e^{n(-1)^n} - 1$ where $n \in N$. The co domain of this function is bounded below by -1 but is not bounded from above. Does this satisfy the other conditions of the sequence?

not bounded above (it becomes more negative without bound), and it contains an infinite number of negative terms.

Make sure that you are precise in your definitions. Remember a set $A \subseteq \mathbf{R}$ is bounded above if there exists a number $b \in \mathbf{R}$ such that $a \leqslant b$ for all $a \in A$. Similarly, the set A is bounded below if there exists a lower bound $l \in \mathbf{R}$ satisfying $l \leqslant a$ for every $a \in A$.