Real Analysis HW #9

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Question 1

Let $g : A \longrightarrow \mathbb{R}$ and $f : A \longrightarrow \mathbb{R}$. Suppose that $\lim_{x\to a} f(x) = 0$.

(a) Show that $\lim_{x\to a} f(x)g(x) = 0$ for any function g as above IS NOT TRUE.

Proof: Assume for the sake of contradiction that $\lim_{x\to a} f(x)g(x) = 0$ for any $g(x)$. Consider the case when $g(x) = 1/x^2$. Let $\epsilon > 0$. Then there exists δ such that whenever $0 < |x - 0| < \delta$ we have $|f(x)g(x) - 0| < \epsilon$.

By the Archimedes principle there exists $N \in \mathbb{N}$ such that for all $n > N$, $0 < |1/n| < \delta$. Thus,

$$|f(1/n)g(1/n) - 0| = |f(1/n)n^2 - 0| < \epsilon.$$

We consider three cases:

1. If f approaches zero at a faster rate than n^2 increases then $\lim_{x\to a} f(x)g(x) = 0$ is true.
2. If f approaches zero at the same rate that n^2 increases then $\lim_{x\to a} f(x)g(x) = c$ for $c \in \mathbb{R} \setminus \{0\}$ and the assumption is false.
3. If f approaches zero slower then n^2 increases then we choose $n_* > N$ such that $|f(1/n_*)n_*^2| > \epsilon$, yielding a contradiction and thus the assumption is false.

Hence, $\lim_{x\to a} f(x)g(x) = 0$ for any function g is not true. 😄

(b) Assume that g is a bounded function on A. Show that $\lim_{x\to a} g(x)f(x) = 0$.

Proof: Let $\epsilon > 0$ and g be bounded by $B \in \mathbb{R}^+$. So $|g(x)| < B$ for all $x \in \mathbb{R}$. Because $\lim_{x\to a} f(x) = 0$ then there exists δ such that if $c \in \mathbb{R}$ and $0 < |x - c| < \delta$ we automatically have $|f(x) - 0| < \epsilon/B$. Now,

$$|g(x)f(x) - 0| < |g(x)|\left|\frac{\epsilon}{B}\right| \leqslant |B|\left|\frac{\epsilon}{B}\right| = \epsilon.$$

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Question 2

Let a and b be real numbers with $a \neq 0$. Use the definition of continuity to prove that the function f defined by $f(x) = ax + b$ is continuous at every real number.

Proof: Let $\epsilon > 0$ and $c \in \mathbb{R}$. Choose $\delta = \epsilon/a$. If we have $|x - c| < \delta$ it follows

$$|f(x) - f(c)| = |ax + b - (ac + b)| = |ax + b - ac - b| = |ax - ac| = |a(x - c)| < |a\frac{\epsilon}{a}| < \epsilon.$$

Thus, $f(x) = ax + b$ is continuous at every real number. 😁

Question 3

Use the definition of limit to prove that $\lim_{x \to c} x^2 = c^2$ for every real number c.

Proof: Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/(2c + 1)\}$. If we have $0 < |x - c| < \delta$ it follows that,

$$|f(x) - L| = |x^2 - c^2| = |x + c||x - c| < (2c + 1)\frac{\epsilon}{2c + 1} = \epsilon$$

.

Thus, $\lim_{x \to c} x^2 = c^2$ for every real number c. 😁

Question 4

Find constants a and b so that the function f defined by

$$f(x) = \begin{cases} 3ax^2+1 & 0 \leqslant x \leqslant 1 \\ ax+b & 1 < x < 2 \\ 2bx+a & 2 \leqslant x \leqslant 4 \end{cases}$$

has a limit at each point of $[0,4]$. Be sure to show the limit exists.

Solution:

First we find constants a and b so that $f(x)$ has a limit defined at each point $[0,4]$. Plugging in 1 and 2 in each of the equations defined in the piecewise function $f(x)$ yields a system of equations:

$$2a+1 = a+b \implies 2a+1 = b$$

and

$$2a+b = 4b+a \implies a = 3b.$$

Substituting $a = 3b$ into the first equation yields

$$2(3b) = b \implies 6b+1 = 6 \implies 5b = -1 \implies b = \frac{-1}{5}.$$

Finally, solving for $a = 3b = 3(-1/5) = -3/5$. Thus,

$$b = \frac{-1}{5} \text{ and } a = \frac{-3}{5}$$

and f becomes

$$f(x) = \begin{cases} -9/5x^2+1 & 0 \leqslant x \leqslant 1 \\ -3/5x-1/5 & 1 < x < 2 \\ -2/5x-3/5 & 2 \leqslant x \leqslant 4 \end{cases}$$

Now, we show that the limit exists at 2 and 1 from the left and the right.

Proof: $\lim_{x\to 1^-} = -4/5$:

Let $\epsilon > 0$. Now, we restrict our δ to be a maximum of 1. Choose $\delta = \min\{1, \epsilon\sqrt{5}/6\}$. Then when $1-\delta < x < 1$ we have

$$|f(x)-L| = |-9/5x^2+1--4/5| = |-9/5x^2+9/5| = |9/5x^2-9/5| = |(3/\sqrt{5}x-3/\sqrt{5})(3/\sqrt{5}x+3/\sqrt{5})|$$

$$\leqslant 3/\sqrt{5}|x-1||x+1| < \frac{3}{\sqrt{5}}\epsilon\frac{\sqrt{5}}{6}2 = \epsilon$$

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Proof: $\lim_{x\to 1^+} = -4/5$

Let $\epsilon > 0$ and choose $\delta = \epsilon\frac{5}{3}$. Then when $1 < x < 1+\delta$ we have

$$|f(x)-L| = |-3/5x-1/5--4/5| = |-3/5x+3/5| = \frac{3}{5}|-x+1| = \frac{3}{5}|x-1| < \frac{3}{5}\epsilon\frac{5}{3} = \epsilon.$$

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Proof: $\lim_{x\to 2^-} = -7/5$

Let $\epsilon > 0$ and choose $\delta = \epsilon\frac{5}{3}$. Then when $2-\delta < x < 2$ we have

$$|f(x) - L| = |-3/5x - 1/5 - -7/5| = |-3/5x + 6/5| = \frac{3}{5}|-x+2| = \frac{3}{5}|x-2| < \frac{3}{5}\epsilon\frac{5}{3} = \epsilon.$$

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Proof: $\lim_{x\to 2^+} = -7/5$

Let $\epsilon > 0$ and choose $\delta = \epsilon\frac{5}{2}$. Then when $2 < x < 2+\delta$ we have

$$|f(x) - L| = |-2/5x - 3/5 - -7/5| = |-2/5x + 4/5| = \frac{2}{5}|-x+2| = \frac{2}{5}|x-2| < \frac{2}{5}\epsilon\frac{5}{2} = \epsilon.$$

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Question 5

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and suppose that $f(x)=0$ for each irrational number x in $[a,b]$. Prove that $f(x)=0$ for all $x\in[a,b]$.

Proof: If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ then it must be continuous on every point $c\in[a,b]$. Assume, for the sake of contradiction, that there exists $x_0\in[a,b]$ with $x_0\in\mathbb{R}\setminus\mathbb{Q}$ and $f(x_0)\neq 0$.

Then f must be continuous at x_0, and there must exist δ such that $|x-x_0|<\delta$ implies that $|f(x)-f(x_0)|<|f(x_0)|/2$.

By the density of rationals in the real numbers there must exist an irrational number, x_I, such that $|x_I-x_0|<\delta$.

Since x_I is irrational then $f(x_I)=0$ and we have $|f(x_I)-f(x_0)|=|f(x_0)|<|f(x_0)|/2 \implies 1<1/2 \rightarrow\leftarrow$. Thus, we have arrived at a contradiction which means that $f(x)=0$ for all $x\in[a,b]$.

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Question 6

CHAT GPT CHALLENGE: Let $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$, and suppose that $f(x)=0$ for each rational number x in $[a,b]$. We aim to show that $f(x)=0$ for all $x\in[a,b]$.

Assume, for the sake of contradiction, that there exists $c\in[a,b]$ such that $f(c)\neq 0$. Since f is continuous, for any $\epsilon>0$, there exists a $\delta>0$ such that for all $x\in[a,b]$, if $|x-c|<\delta$, then $|f(x)-f(c)|<\epsilon$.

Consider the case when $\epsilon=|f(c)|$. By continuity, there exists $\delta>0$ such that for all $x\in[a,b]$, if $|x-c|<\delta$, then $|f(x)-f(c)|<|f(c)|$. This implies

$$-f(c)<f(x)-f(c)<f(c).$$

Adding $f(c)$ to all parts of the inequality yields $0<f(x)$ for x in the interval $(c-\delta,c)\cap[a,b]$. Similarly, subtracting $f(c)$ from all parts of the inequality yields $f(x)<0$ for x in the interval $(c,c+\delta)\cap[a,b]$. Now, consider the set of rational numbers in $[a,b]$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist rational numbers in both $(c-\delta,c)\cap[a,b]$ and $(c,c+\delta)\cap[a,b]$. However, we have shown that for any rational x in these intervals, $f(x)$ has opposite signs, which is a contradiction.

You are almost there. Instead of saying there exists a rational number in the intervals you should say there exists an irrational number, say x_I, which we know $f(x_I)=0$. From then you can continue with your contradiction argument. You also need to note that for any number in those intervals $f(x)$ has opposite signs.

Therefore, our assumption that there exists c such that $f(c)\neq 0$ is false, and we conclude that $f(x)=0$ for all $x\in[a,b]$.