Real Analysis HW #8

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Definition 0.0.1: Fibonacci Sequence

The Fibonacci sequence, $1, 2, 3, 5, 8, 13, \ldots$ is given by the recursive formula

$$F_{n+1} = F_n + F_{n-1}$$

where $F_1 = 1$ and $F_2 = 2$. Let $a_n = \frac{F_n}{F_{n-1}}$.

Question 1

Suppose that $\{a_n\}$ converges to a limit. What must that limit be? Hint: Divide the above equation by F_n to find an equation relating a_{n+1} to a_n.

Solution: From the recursive formula, dividing by F_n yields:

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$$

Then,

$$a_{n+1} = 1 + \frac{F_{n-1}}{F_n}$$

$$\implies a_{n+1} = 1 + \frac{1}{a_n}$$

Let $L = \lim_{n\to\infty} a_n$, then

$$L = 1 + \frac{1}{L}$$

$$\implies L^2 = L + 1$$

$$\implies L^2 - L - 1 = 0.$$

By the quadratic formula,

$$L = \frac{1 \pm \sqrt{5}}{2}.$$

Since this sequence is positive for all $n \in \mathbb{N}$ we want the positive solution. Thus,

$$L = \frac{1 + \sqrt{5}}{2}.$$

Question 2

Show that $\frac{3}{2} \leqslant a_n \leqslant 2 \; \forall n \geqslant 2$.

Proof: Let $n \in \mathbb{N}$. We have that $a_1 = 1$, $a_2 = 2$, $a_3 = 3/2$. Thus,

$$\frac{3}{2} \leqslant a_n \leqslant 2$$

for $1, 2, 4 \in \mathbb{N}$. We want to show that if this is true for a_n this is also true for a_{n+1}.

We assume that

$$\frac{3}{2} \leqslant a_n \leqslant 2$$

is true. Then,

$$\frac{2}{3} \geqslant \frac{1}{a_n} \geqslant \frac{1}{2}$$

$$\implies 1 + \frac{2}{3} \geqslant 1 + \frac{1}{a_n} \geqslant 1 + \frac{1}{2}$$

$$\implies 1 + \frac{1}{2} \leqslant 1 + \frac{1}{a_n} \leqslant 1 + \frac{2}{3}$$

$$\implies \frac{3}{2} \leqslant a_{n+1} \leqslant \frac{5}{3} < 2.$$

Thus, $\frac{3}{2} \leqslant a_n \leqslant 2$ for all $n \geqslant 2$.

Question 3

For each $n > 2$, prove that $|a_{n+1} - a_n| \leqslant \left(\frac{2}{3}\right)^2 |a_n - a_{n-1}|$.

Question 4

Prove that for each $m \geqslant 2$, $|a_{m+1} - a_m| \leqslant \left(\frac{2}{3}\right)^{2(m-2)} |a_3 - a_2|$.

Question 5

Use the inequality in (4) to show that $\{a_n\}$ is a Cauchy sequence and therefore converges to a limit.

Question 6

CHAT GPT CHALLENGE: To prove that a bounded sequence (x_n) with the property that every convergent subsequence of (x_n) converges to the same limit $x \in \mathbb{R}$ must converge to x, you can use the following proof:

Let $\epsilon > 0$ be given. Since (x_n) is bounded, it has a limit point, say y, which means that there exists a subsequence (x_{n_k}) that converges to y.

How do you know that (x_n) converges at all? The Monotone Convergence Theorem only works if you know that the sequence is bounded and monotone. You can not assume that (x_n) converges. Since every convergent subsequence of (x_n) converges to x, we have $y = x$.

Now, consider $\epsilon > 0$ and $y = x$. Since (x_n) is a bounded sequence, it has a limit point, and there exists a subsequence (x_{n_k}) that converges to x.

By the definition of the limit, there exists an N_1 such that for all $k \geqslant N_1$, $\left|x_{n_k} - x\right| < \epsilon/2$. Additionally, because every convergent subsequence of (x_n) converges to x, we can choose another N_2 such that for all $k \geqslant N_2$, $\left|x_{n_k} - x\right| < \epsilon/2$.

Now, let $N = \max(N_1, N_2)$. For all $n \geqslant N$, there exists $k_1 \geqslant N$ and $k_2 \geqslant N$ such that:

$$\left|x_{n_{k_1}} - x\right| < \frac{\epsilon}{2}$$
$$\left|x_{n_{k_2}} - x\right| < \frac{\epsilon}{2}$$

By the triangle inequality:

$$\begin{aligned} |x_n - x| &\leqslant \left|x_n - x_{n_{k_1}}\right| + \left|x_{n_{k_1}} - x\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So, for all $n \geqslant N$, $|x_n - x| < \epsilon$, which means that (x_n) converges to x. Therefore, we've shown that if (x_n) is a bounded sequence with the property that every convergent subsequence converges to the same limit $x \in \mathbb{R}$, then (x_n) itself must converge to x.