Real Analysis HW #8

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Definition 0.0.1: Fibonacci Sequence

The Fibonacci sequence, $1, 2, 3, 5, 8, 13, \ldots$ is given by the recursive formula

$$F_{n+1} = F_n + F_{n-1}$$

where $F_1 = 1$ and $F_2 = 2$. Let $a_n = \frac{F_n}{F_{n-1}}$.

Question 1

Suppose that $\{a_n\}$ converges to a limit. What must that limit be? Hint: Divide the above equation by F_n to find an equation relating a_{n+1} to a_n.

Solution: From the recursive formula dividing by F_n yields

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}.$$

Then,

$$a_{n+1} = 1 + \frac{F_{n-1}}{F_n}$$

$$\implies a_{n+1} = 1 + \frac{1}{a_n}.$$

Let $L = \lim_{n\to\infty} a_n$, then

$$L = 1 + \frac{1}{L}$$

$$\implies L^2 = L + 1$$

$$\implies L^2 - L - 1 = 0.$$

By the quadratic formula,

$$L = \frac{1 \pm \sqrt{5}}{2}.$$

Since $a_2 = 2$, a positive number, and the terms are defined recursively by $a_{n+1} = 1 + \frac{1}{a_n}$, every term is positive. Thus, we want the positive solution. Therefore

$$L = \frac{1 + \sqrt{5}}{2}.$$

Question 2

Show that $\frac{3}{2} \leqslant a_n \leqslant 2 \; \forall n \geqslant 2$.

Proof: Let $n \in \mathbb{N}$. We have that $a_2 = 2$ and $a_3 = 3/2$. Thus, our base case holds. We want to show that if this is true for a_n this is also true for a_{n+1}.

We assume that

$$\frac{3}{2} \leqslant a_n \leqslant 2$$

is true. Then

$$\begin{aligned} \frac{2}{3} \geqslant \frac{1}{a_n} \geqslant \frac{1}{2} \\ \implies 1 + \frac{2}{3} \geqslant 1 + \frac{1}{a_n} \geqslant 1 + \frac{1}{2} \\ \implies 1 + \frac{1}{2} \leqslant 1 + \frac{1}{a_n} \leqslant 1 + \frac{2}{3} \\ \implies \frac{3}{2} \leqslant a_{n+1} \leqslant \frac{5}{3} < 2. \end{aligned}$$

Thus, $\frac{3}{2} \leqslant a_n \leqslant 2$ for all $n \geqslant 2$. 😄

Question 3

For each $n > 2$, prove that $|a_{n+1} - a_n| \leqslant \left(\frac{2}{3}\right)^2 |a_n - a_{n-1}|$.

Proof: Let $n > 2$. Then

$$|a_{n+1} - a_n| = \left|1 + \frac{1}{a_n} - 1 - \frac{1}{a_{n-1}}\right| = \left|\frac{1}{a_n} - \frac{1}{a_{n-1}}\right| = \left|\frac{a_{n-1} - a_n}{a_{n-1}a_n}\right|.$$

Since for all $n \geqslant 2$ we have that $a_n \geqslant \frac{3}{2}$,

$$\begin{aligned} |a_{n+1} - a_n| = \left|\frac{a_{n-1} - a_n}{a_{n-1}a_n}\right| \leqslant \left|\frac{a_{n-1} - a_n}{\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)}\right| \\ \implies |a_{n+1} - a_n| \leqslant \left(\frac{2}{3}\right)^2 |a_{n-1} - a_n|. \end{aligned}$$

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Question 4

Prove that for each $m > 2$, $|a_{m+1} - a_m| \leqslant \left(\frac{2}{3}\right)^{2(m-2)} |a_3 - a_2|$.

Solution:

Proof: We see that $a_2 = 2$, $a_3 = 3/2$, and $a_4 = 5/3$. Thus, when $n = 3$

$$\left|\frac{5}{3} - \frac{3}{2}\right| \leqslant \left(\frac{2}{3}\right)^2 \left|\frac{3}{2} - \frac{4}{2}\right|$$

$$\implies \left|\frac{1}{6}\right| \leqslant \left(\frac{2}{3}\right)^2 \left|-\frac{1}{2}\right|$$

$$\implies \left|\frac{1}{6}\right| \leqslant \left|\frac{2}{9}\right|$$

$$\implies \left|\frac{9}{54}\right| \leqslant \left|\frac{12}{54}\right|$$

$$\implies |a_4 - a_3| \leqslant \left(\frac{2}{3}\right)^2 |a_3 - a_2|$$

Since $|a_{n+1} - a_n| \leqslant \left(\frac{2}{3}\right)^2 |a_n - a_{n-1}|$ for $n > 2$ it follows that

$$\implies |a_4 - a_3| \leqslant \left(\frac{2}{3}\right)^2 |a_3 - a_2|$$

$$\implies |a_5 - a_4| \leqslant \left(\frac{2}{3}\right)^4 |a_3 - a_2|$$

$$\implies |a_6 - a_5| \leqslant \left(\frac{2}{3}\right)^6 |a_3 - a_2|$$

and generally when $m > 2$,

$$|a_{m+1} - a_m| \leqslant \left(\frac{2}{3}\right)^{2(m-2)} |a_3 - a_2|.$$

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Question 5

Use the inequality in (4) to show that $\{a_n\}$ is a Cauchy sequence and therefore converges to a limit.

Proof: Let

$$m > 2 \text{ and } B = \left(\frac{3}{2}\right)^4 |a_3 - a_2|.$$

We know that

$$|a_{m+1} - a_m| \leqslant \left(\frac{2}{3}\right)^{2(m-2)} |a_3 - a_2|$$

$$\implies |a_{m+1} - a_m| \leqslant \left(\frac{2}{3}\right)^{2m} B.$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $n > N$,

$$B\frac{4}{5}\left(\frac{2}{3}\right)^{2n} < \epsilon.$$

Let $m > n > N$. Then

$$|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - a_{m-2} + a_{m-2} - a_{m-3} + \cdots + a_{n+1} - a_n|$$

$$\leqslant |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + |a_{m-2} - a_{m-3}| + \cdots + |a_{n+1} - a_n|$$

$$\leqslant \sum_{k=n+1}^{m} |a_k - a_{k-1}|$$

$$\leqslant \sum_{k=n+1}^{m} \left(\frac{2}{3}\right)^{2k} B.$$

Now, if $(x_n) = \sum_{k=n+1}^{m} \left(\frac{2}{3}\right)^{2k} B$,

$$x_n - \left(\frac{2}{3}\right)^2 x_n = B\left[\sum_{k=n+1}^{m} \left(\frac{2}{3}\right)^{2k} - \sum_{k=n+2}^{m+1} \left(\frac{2}{3}\right)^{2k}\right]$$

$$\implies x_n\left(1 - \frac{4}{9}\right) = B\left[\left(\frac{2}{3}\right)^{2(n+1)} - \left(\frac{2}{3}\right)^{2(m+1)}\right]$$

$$\implies x_n = B\left[\frac{\left(\frac{2}{3}\right)^{2(n+1)} - \left(\frac{2}{3}\right)^{2(m+1)}}{5/9}\right]$$

$$\leqslant B\frac{9}{5}\left(\frac{2}{3}\right)^{2(n+1)}$$

$$\leqslant B\frac{9}{5}\left(\frac{2}{3}\right)^{2}\left(\frac{2}{3}\right)^{2n}$$

$$\leqslant B\frac{4}{5}\left(\frac{2}{3}\right)^{2n} < \epsilon.$$

Thus, (x_n) is Cauchy and hence convergent.

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Question 6

CHAT GPT CHALLENGE: To prove that a bounded sequence (x_n) with the property that every convergent subsequence of (x_n) converges to the same limit $x \in \mathbb{R}$ must converge to x, you can use the following proof:

Let $\epsilon > 0$ be given. Since (x_n) is bounded, it has a limit point, say y, which means that there exists a subsequence (x_{n_k}) that converges to y.

How do you know that (x_n) converges at all? The Monotone Convergence Theorem only works if you know that the sequence is bounded and monotone. You can not assume that (x_n) converges. You are correct in saying that there exists a subsequence that converges to a number however this is by the Bolzano Weierstrass. What about the sequence $x_n = (-1)^n$? What can you say about its convergence and the convergence of its subsequences. Do they all converge to the same limit?

Since every convergent subsequence of (x_n) converges to x, we have $y = x$.

Now, consider $\epsilon > 0$ and $y = x$. Since (x_n) is a bounded sequence, it has a limit point, and there exists a subsequence (x_{n_k}) that converges to x. **Again, this only works if we know that the sequence is monotone and bounded. We just know that the sequence is bounded.**

By the definition of the limit, there exists an N_1 such that for all $k \geqslant N_1, \left|x_{n_k} - x\right| < \epsilon/2$. Additionally, because every convergent subsequence of (x_n) converges to x, we can choose another N_2 such that for all $k \geqslant N_2, \left|x_{n_k} - x\right| < \epsilon/2$.

Now, let $N = \max(N_1, N_2)$. For all $n \geqslant N$, there exists $k_1 \geqslant N$ and $k_2 \geqslant N$ such that:

$$\left|x_{n_{k_1}} - x\right| < \frac{\epsilon}{2}$$
$$\left|x_{n_{k_2}} - x\right| < \frac{\epsilon}{2}$$

By the triangle inequality:

$$\begin{aligned} |x_n - x| &\leqslant \left|x_n - x_{n_{k_1}}\right| + \left|x_{n_{k_1}} - x\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

The expansion of the Triangle Inequality is correct but your interpretation is not. Check your subscripts. You have not shown that $\left|x_n - x_{n_{k_1}}\right| < \frac{\epsilon}{2}$.

So, for all $n \geqslant N$, $|x_n - x| < \epsilon$, which means that (x_n) converges to x. Therefore, we've shown that if (x_n) is a bounded sequence with the property that every convergent subsequence converges to the same limit $x \in \mathbb{R}$, then (x_n) itself must converge to x.