Real Analysis HW #3

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Question 1

1. Let S be a non-empty set of real numbers that is bounded above and let $\beta = \sup S$. Suppose that $\beta \notin S$. Prove that for each $\epsilon > 0$, the set $\{x \in S : x > \beta - \epsilon\}$ is infinite.

Solution:

Proof: Assume, by way of contradiction, that $(\beta - \epsilon, \beta)$ is finite. Then we have that $(\beta - \epsilon, \beta) \sim \mathbb{N}_N$ where $N \in \mathbb{N}$. Consider in preparation for a contradiction that the value

$$\frac{\beta - \epsilon}{2N}$$

partitions our set into $2N$ bins.

We can start listing the elements in the collection $\{\beta - \epsilon + \frac{(\beta-\epsilon)N^\star}{2N} : N^\star \in \mathbb{N}_N\}$ where $\beta - \epsilon + \frac{(\beta-\epsilon)N^\star}{2N} \in (\beta - \epsilon, \beta)$ until we terminate at $N^\star = N$, which is $\beta - \epsilon + \frac{\beta - \epsilon}{2}$.

However,

$$N + 1 \notin \mathbb{N}_N$$

but

$$\beta - \epsilon + \frac{(N+1)(\beta - \epsilon)}{2N} \in (\beta - \epsilon, \epsilon).$$

By pigeon hole principle $|(\beta - \epsilon, \beta)| \neq |\mathbb{N}_N|$. We have counted more than N elements in the set $\{x \in S : x > \beta - \epsilon\}$, so it can not be finite. Thus, the set $\{x \in S : x > \beta - \epsilon\}$ is infinite.

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Question 2

2. Prove that the union of a countable set and an uncountable set is uncountable.

Proof: Assume, by way of contradiction, that the union of a countable set A and a uncountable set B is countable. Then, $A \cup B$ is countable. By Theorem 1.5.7, then $B \subset A \cup B$ is countable or finite $\rightarrow\!\leftarrow$. This is a problem because we know B is uncountable. Thus, $A \cup B$ must be uncountable.

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Question 3

3. Exercise 1.5.4:

(a) Show $|(a, b)| = |\mathbb{R}|$.

Solution:

$$f(x) = \tan\left(\frac{\pi}{b-a}\left[x - \frac{a+b}{2}\right]\right)$$

We have domain f is (a, b), the range is $(-\infty, \infty) = \mathbb{R}$. The function f is both onto and 1-1. Thus, $(a, b) \sim \mathbb{R}$.

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality at $\mathbb{R}$ as well.

Solution:

$$f(x) = log(x - a)$$

This is a 1-1 onto function of the real numbers. The domain of f is (a, ∞) and the co-domain is $\mathbb{R}$.

(c) Show that $[0, 1)$ has the same cardinality as $(0, 1)$.

Solution: Let the domain of f be $[0, 1)$.

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{x}{2} & x \in \{\frac{1}{2^n} : n \in \mathbb{N}\} \\ x & x \neq 0, x \notin \{\frac{1}{2^n} : n \in \mathbb{N}\} \end{cases}$$

Question 4

4. Exercise 1.5.6:

Solution:

(a) Give an example of a countable collection of disjoint open intervals.

$$A = \{(n, n+1) : n \in \mathbb{N}\} = \{(1,2), (2,3), (3,4), \dots\}$$
$$f : \mathbb{N} \to A$$
$$f(n) = (n, n+1)$$

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

No such collection exists.

Proof: Let X be a collection of disjoint intervals. Each interval $x = (a, b)$ where $a, b \in \mathbb{R}$ and $x \in X$ is open, and thus must contain at least two real numbers. By the density of $\mathbb{Q}$ in $\mathbb{R}$ we know that there exists $r \in \mathbb{Q}$ such that $a < r < b$. Thus, we obtain a new set of rational numbers, $Y \subset \mathbb{Q}$.

Note that while each a, b may not be unique between intervals, each r is, as it lies between the bounds of the *disjoint* open intervals. Combining Theorem 1.5.6 ($\mathbb{Q}$ is countable) and Theorem 1.5.7 (A subset of a countable set is either countable or finite) Y must be countable. Hence, any collection of disjoint open intervals is countable.

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Question 5

5. ChatGPT Challenge: Find an example of a sequence of closed bounded intervals $I_1, I_2, \dots$ with the property that $\bigcap_{k=1}^{n} I_k \neq \emptyset$ for all $n \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$

Proof: Let $I_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ for $n \in \mathbb{N}$. For each natural number n, the interval I_n is a closed bounded interval, and their intersection is non-empty:

$$\bigcap_{k=1}^{n} I_k = \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \neq \emptyset$$

Your proposition seems reasonable at first but be careful with your intervals. When $n = 1$, we have $I_n = [1, 0]$ which is the empty set. Thus, any intersection after that must be the empty set and your example does not hold.

You may consider $\mathbf{I_n} = \left[\frac{1}{n+1}, 1 - \frac{1}{n+1}\right]$, but you still have problems. The next interval is $\left[\frac{1}{2}, \frac{1}{2}\right]$, and every interval after that is a superset of this - the infinite intersection can not be the empty set.

However, when we consider the infinite intersection:

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

The infinite intersection is empty because as n approaches infinity, the left endpoint $\frac{1}{n}$ approaches zero, and the right endpoint $1 - \frac{1}{n}$ approaches 1 .

Even if we consider my suggested intervals, this is still false. We have that

$$\mathbf{\frac{1}{2} \in \bigcap_{n=1}^{\infty} I_n}.$$

Make sure to do some sanity checks while you are going through examples. Make it concrete. Your logic does not hold.

So, the infinite intersection becomes the empty set.