

# Real Analysis HW #7

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### Question 1

Let  $(x_n)$  be a sequence and suppose that the sequence  $(x_{n+1} - x_n)$  converges to 0. Give an example to show that the sequence  $(x_n)$  may not converge. (See CHATBOT Challenge)

**Solution:** Let

$$x_n = \sum_{k=1}^n 1/k.$$

This is the harmonic series, which converges to infinity. Let  $\epsilon > 0$ . By Archimedes Principle there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $1/n < \epsilon$ . Then

$$|(x_{n+1} - x_n) - 0| = \left| \frac{1}{n+1} \right| < \epsilon.$$

Thus,  $(x_{n+1} - x_n)$  converges to 0, but  $(x_n)$  converges to infinity.

### Question 2

Let  $(x_k)$  and  $(y_k)$  be two sequences and let  $(r_k)$  be a sequence of positive numbers that converges to 0. Suppose that  $0 < |y_k - x_k| < r_k \forall k \in \mathbb{N}$ .

(a) Give an example to show that the sequences  $(x_k)$  and  $(y_k)$  may not converge.

**Solution:** Let

$$y_k = k + \frac{1}{k}$$

and let

$$x_k = k + \frac{1}{k+1}.$$

(b) Suppose that  $(x_k)$  converges to  $L$ . Prove that the sequence  $(y_k)$  converges to  $L$ .

**Proof:** Let  $\epsilon > 0$ . Because  $(r_k)$  converges to 0 then there exists  $N_1 \in \mathbb{N}$  such that for all  $k > N_1$ ,  $|r_k - 0| < \frac{\epsilon}{2}$ .

Because  $(x_k)$  converges to  $L$  there exists  $N_2 \in \mathbb{N}$  such that for all  $k > N_2$ ,  $|x_k - L| < \frac{\epsilon}{2}$ . Let  $N = \max\{N_1, N_2\}$  and  $k > N$ .

Then,

$$0 < |y_k - x_k| < |r_k - 0| < \frac{\epsilon}{2}.$$

$$\implies |y_k - L + L - x_k| < \frac{\epsilon}{2}.$$

$$\implies |y_k - L| - |L - x_k| \leq |y_k - L + L - x_k| < \frac{\epsilon}{2}.$$

$$\implies |y_k - L| - \underbrace{|L - x_k|}_{< \frac{\epsilon}{2}} < \frac{\epsilon}{2}.$$

And we know that  $|x_k - L| < \frac{\epsilon}{2}$ ,

$$\implies |y_k - L| < \epsilon$$

Thus,  $y_k$  converges to  $L$ .



### Question 3

Assume that  $(x_n)$  is a bounded sequence with the property that every convergent subsequence of  $(x_n)$  converges to the same limit  $x \in \mathbb{R}$ . Show that  $(x_n)$  must converge to  $x$ .

**Proof:** To show this we will prove the contrapositive of the following: If every convergent subsequence of  $x_n$  converges to  $x$ , then  $x_n$  must converge to  $x$  as well.

We must show that if  $x_n \not\rightarrow x$ , then either (1)  $x_n$  is not bounded or (2) there exists a subsequence  $x_{n_k}$  such that  $x_{n_k} \rightarrow y$  where  $y \neq x$ .

1. We know that  $(x_n)$  is bounded so we must consider case (2).
2. Since  $x_n$  does not converge to  $x$  there exists  $\epsilon_0$  such that for all  $N \in \mathbb{N}$  there exists an  $n_N > N$  where  $|x_{n_N} - x| > \epsilon_0$ . Because this works for every  $N \in \mathbb{N}$  there exists a new sequence outside the radius of  $\epsilon_0$  around  $x$ . Consider the sequence consisting of all the terms  $x_n \notin B(x, \epsilon_0)$ , denoted  $x_{n^*}$ . By Bolzano Weierstrass, there exists a subsequence,  $x_{n_k}$ , where  $x_{n_k} \rightarrow y$ . We know  $y \neq x$  because  $x_{n^*} \notin B(x, \epsilon_0)$ .

Thus, if  $(x_n)$  is a bounded sequence such that every convergent subsequence converges to the same limit  $x$ , then  $(x_n)$  must also converge to  $x$ . ☺

### Question 4

Let  $(x_n)$  be a Cauchy sequence. Show directly that  $(x_n)$  is bounded.

Let  $\epsilon > 0$  and  $(x_n)$  be a Cauchy sequence. Then there exists  $N \in \mathbb{N}$  such that for all  $n, m \geq N$ ,  $|x_n - x_m| < 1$ . Then,

$$|x_m - x_N| < 1.$$

$$\implies |x_m| - |x_N| < |x_m - x_N| < 1.$$

$$\implies |x_m| - |x_N| < 1.$$

$$\implies |x_m| < 1 + |x_N|.$$

Thus, we have a bound for all the terms  $N$  and beyond. Let

$$B = \max\{|x_1|, |x_2|, |x_3|, \dots, |x_N|, |x_N| + 1\}.$$

Then  $|x_n| < B$  for all  $n \in \mathbb{N}$  and therefore  $(x_n)$  must be bounded.

### Question 5

Let  $(x_n)$  be a sequence and let  $r$  be a real number satisfying  $0 < r < 1$ . Suppose that  $|x_{n+1} - x_n| < r |x_n - x_{n-1}|$  for all  $n \geq 2$ . Prove that  $(x_n)$  is a Cauchy sequence and hence convergent.

**Proof:** We know that  $|x_{n+1} - x_n| < r |x_n - x_{n-1}|$ . Thus,

$$\begin{aligned} |x_3 - x_2| &< r |x_2 - x_1| \\ \implies |x_4 - x_3| &< r^2 |x_2 - x_1| \\ \implies |x_5 - x_4| &< r^3 |x_2 - x_1| \\ \implies |x_{n+1} - x_n| &< r^{n-1} |x_2 - x_1|. \end{aligned}$$

Now let

$$B = |x_2 - x_1|.$$

Thus, for  $n \geq 2$ ,  $|x_{n+1} - x_n| < Br^{n-1}$ . Let  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that for all  $n > N$ ,

$$B \frac{r^n}{1-r} < \epsilon.$$

Let  $m > n > N$ . We have,

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - x_{m-3} + \cdots + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + |x_{m-2} - x_{m-3}| + \cdots + |x_{n+1} - x_n| \\ &\leq \sum_{k=n+1}^m |x_k - x_{k-1}| \\ &\leq \sum_{k=n+1}^m Br^{k-1}. \end{aligned}$$

Now, if  $(x_n) = \sum_{k=n+1}^m Br^{k-1}$ , then

$$\begin{aligned} x_n - rx_n &= B \left[ \sum_{k=n+1}^m r^{k-1} - \sum_{k=n+2}^{m+1} r^{k-1} \right] \\ \implies x_n(1-r) &= B[r^n - r^m] \\ \implies x_n &= B \left[ \frac{r^n - r^m}{1-r} \right] \\ &\leq B \frac{r^n}{1-r} < \epsilon. \end{aligned}$$

Thus,  $(x_n)$  is Cauchy and hence convergent.



### Question 6

CHAT GPT CHALLENGE: Let  $(x_n)$  be a sequence such that  $(x_{n+1} - x_n)$  converges to 0. We want to show that  $(x_n)$  also converges.

By the definition of a convergent sequence, we need to show that for any  $\epsilon > 0$ , there exists a positive integer  $N$  such that for all  $n \geq N$ ,  $|x_n - L| < \epsilon$  for some real number  $L$ .

**Good job setting the stage for your proof and clearly stating what you need to show to substantiate your claim. This is correct.**

Since  $(x_{n+1} - x_n)$  converges to 0, we can choose  $\epsilon > 0$  and find a positive integer  $M$  such that for all  $m \geq M$ ,  $|x_{m+1} - x_m| < \epsilon$ . This implies that for all  $m \geq M$ ,  $|x_{m+1} - x_m| < \epsilon$ . Now, let  $N = M$ . For any  $n \geq N$ , we have:

$$\begin{aligned} |x_n - x_N| &= |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \dots + (x_{N+1} - x_N)| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \dots + |x_{N+1} - x_N| \\ &< \epsilon + \epsilon + \dots + \epsilon \quad (\text{Since } |x_{m+1} - x_m| < \epsilon \text{ for all } m \geq M) \\ &= (n - N)\epsilon. \end{aligned}$$

Dividing both sides by  $(n - N)$ , we get:

$$|x_n - x_N| < \epsilon$$

**This is incorrect. When you divide both sides by  $(n - N)$  you get**

$$\frac{1}{(n - N)} |x_n - x_N| < \epsilon$$

**which does not show that  $|x_n - x_N| < \epsilon$ .**

This shows that for any  $\epsilon > 0$  and  $n \geq N$ , we have  $|x_n - x_N| < \epsilon$ , where  $N = M$ . Therefore,  $(x_n)$  converges to  $x_N$ .

**This does not prove what you wanted to show. You are actually trying to prove a false statement. What happens when  $(x_n)$  is the harmonic series?**

Hence, we have shown that if  $(x_{n+1} - x_n)$  converges to 0, then  $(x_n)$  also converges.