

# Real Analysis HW #5

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### Question 1

1. Let  $x_n \geq 0$  for all  $n \in \mathbb{N}$ .

(a) If  $(x_n) \rightarrow 0$ , show that  $\sqrt{x_n} \rightarrow 0$ .

**Proof:** For all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $|x_n - 0| < \epsilon$ . This implies that  $|x_n| < \epsilon$ . Let  $\epsilon_0 = \epsilon^2$ . Then

$$|x_n| < \epsilon_0 = \epsilon^2 \implies |\sqrt{x_n}| < \epsilon \implies |\sqrt{x_n} - 0| < \epsilon.$$

Thus, we have shown  $\sqrt{x_n} \rightarrow 0$ . ☺

(b) If  $(x_n) \rightarrow x$ , show that  $\sqrt{x_n} \rightarrow \sqrt{x}$ .

**Proof:** We consider two cases:

1. If  $x = 0$  then see proof of (a).

2. If  $x > 0$ . Since  $(x_n) \rightarrow x$  we know that for every  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $|x_n - x| < \epsilon$ . Let  $\epsilon_0 = \epsilon\sqrt{x}$ .

Then

$$|x_n - x| < \epsilon\sqrt{x} \implies \left| \sqrt{x_n}^2 - \sqrt{x}^2 \right| < \epsilon\sqrt{x}.$$

This is a difference of squares so,

$$\begin{aligned} \left| \sqrt{x_n}^2 - \sqrt{x}^2 \right| &= |(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})| < \epsilon\sqrt{x} \\ \implies \left| (\sqrt{x_n} - \sqrt{x}) \left( \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x}} \right) \right| &< \epsilon \\ \implies \left| (\sqrt{x_n} - \sqrt{x}) \left( \frac{\sqrt{x_n}}{\sqrt{x}} + 1 \right) \right| &< \epsilon \end{aligned}$$

Since  $\left( \frac{\sqrt{x_n}}{\sqrt{x}} + 1 \right) > 1$  it must be that  $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ . Thus,  $\sqrt{x_n} \rightarrow \sqrt{x}$ . ☺

### Question 2

2. Let  $(a_n)$  and  $(b_n)$  be sequences of real numbers.

(a) Show that if  $(a_n) \rightarrow 0$  and  $(a_n - b_n) \rightarrow 0$ , then  $(b_n) \rightarrow 0$ .

**Proof:** If  $(a_n) \rightarrow 0$  then for all  $\epsilon > 0$  there exists  $N_1$  such that for all  $n > N_1$ ,  $|a_n - 0| < \frac{\epsilon}{2}$ . Similarly, for all  $\epsilon > 0$  there exists  $N_2$  such that for all  $n > N_2$ ,  $|(a_n - b_n) - 0| < \frac{\epsilon}{2}$ . Consider the sum of these two quantities with  $n > \max\{N_1, N_2\}$ .

$$|a_n - 0| + |(a_n - b_n) - 0| < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

By the Triangle Inequality Theorem,

$$|a_n - 0 + a_n - b_n - 0| \leq |(a_n - b_n) - 0| + |a_n - 0| < \epsilon.$$

Thus  $|0 - b_n| = |b_n - 0| < \epsilon$ . Hence  $(b_n) \rightarrow 0$ . ☺

(b) Show that if  $(a_n) \rightarrow 0$  and  $|b_n - b| \leq a_n$ , then  $(b_n) \rightarrow b$

**Proof:** If  $(a_n) \rightarrow 0$  then for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $|a_n - 0| < \frac{\epsilon}{2}$ . We know that

$$|b_n - b| \leq |a_n| \implies |b_n - b| - |a_n| \leq 0 < \frac{\epsilon}{2}.$$

After the summing the two quantities,

$$|b_n - b| - |a_n| + |a_n - 0| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

and finally,

$$|b_n - b| < \epsilon.$$

Thus,  $(b_n) \rightarrow b$ .

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### Question 3

3. Consider  $y_1 = 1$ ,  $y_{n+1} = (2y_n + 3)/4$  for all  $n \in \mathbb{N}$ . Show by direct calculations that  $y_1 < y_2 < 2$ . Then, show that if  $y_{n-1} < y_n < 2$  that  $y_n < y_{n+1} < 2$ . Use this to show that  $\{y_n\}$  converges and find its limit.

### Question 4

4. (Cesari Means). Show that if  $(x_n)$  is a convergent sequence, then the sequence given by the averages:

$$y_n = \frac{x_1 + x_2 + \dots + x_n}{n}$$

also converges to the same limit.

**Proof:** Let  $(x_n)$  converge to  $x$ . If  $(x_n)$  is convergent then for every  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $|x_n - x| < \epsilon$ .

Then

$$\begin{aligned} & \left| \frac{x_1 + x_1 + \dots + x_n}{n} - x \right| = \\ & \left| \frac{x_1 + x_1 + \dots + x_N + \dots + x_n}{n} - x \right| = \\ & \left| \frac{x_1 + x_1 + \dots + x_N}{n} + \frac{x_{N+1} + \dots + x_n}{n} - x \right| \leq \underbrace{\left| \frac{x_1 + x_1 + \dots + x_N}{n} \right|}_K + \underbrace{\left| \frac{x_{N+1} + \dots + x_n}{n} - x \right|}_Z \end{aligned}$$

The last step is a result of the Triangle Inequality Theorem. Since for ever  $n > N$ ,  $|x_n - x| < \epsilon$ , the average must be less than  $\epsilon$  as well. So  $Z < \epsilon$ . As  $n$  tends to infinity  $K \rightarrow 0$ . Thus,  $K + Z < \epsilon$  and

$$\left| \frac{x_1 + x_1 + \dots + x_n}{n} - x \right| < \epsilon.$$

This shows that  $(y_n) \rightarrow x$  and  $(x_n) \rightarrow x$ .

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### Question 5

5. ChatGPT Challenge: To prove that if  $\lim (a_n - b_n) = 0$ , then  $\lim a_n = \lim b_n$ , you can use the definition of a limit and the properties of limits. Here's a proof:

Proof: Let  $\epsilon > 0$  be arbitrary. Since  $\lim (a_n - b_n) = 0$ , there exists an  $N_1$  such that for all  $n \geq N_1$ , we have  $|a_n - b_n| < \epsilon/2$ . Now, consider the inequality  $|a_n - b_n| < \epsilon/2$ . We can add  $|b_n|$  to both sides of this inequality:  $|a_n - b_n| + |b_n| < \epsilon/2 + |b_n|$

By the triangle inequality, we know that  $|a_n - b_n + b_n| \leq |a_n - b_n| + |b_n|$ , so:  $|a_n| < \epsilon/2 + |b_n|$  Now, we have the following inequalities:  $|a_n - b_n| < \epsilon/2$  (from our choice of  $N_1$ )  $|a_n| < \epsilon/2 + |b_n|$

**Everything up until this point tracks. You have clearly shown how we got to this point.**

Now, choose  $N_2$  such that for all  $n \geq N_2$ , we have  $|b_n| < \epsilon/2$ .

**Where does this come from? This only works if we know  $(b_n)$  converges to 0. You are trying to prove that  $(b_n)$  and  $(a_n)$  converge to the same value. We do not know if one or both converge to zero.**

Let  $N = \max(N_1, N_2)$ . For  $n \geq N$ , we have:  $|a_n - b_n| < \epsilon/2$  (from the choice of  $N_1$ )  $|b_n| < \epsilon/2$  (from the choice of  $N_2$ ) Now, let's use these inequalities to bound  $|a_n|$ :  $|a_n| = |a_n - b_n + b_n| \leq |a_n - b_n| + |b_n| < \epsilon/2 + \epsilon/2 = \epsilon$  This shows that for all  $n \geq N$ , we have  $|a_n| < \epsilon$ , which means that  $\lim a_n = 0$ . Since  $\epsilon$  was arbitrary, we have shown that for any  $\epsilon > 0$ , there exists an  $N$  such that for all  $n \geq N$ ,  $|a_n| < \epsilon$ , which is the definition of  $\lim a_n = 0$ . Therefore, we have proved that if  $\lim (a_n - b_n) = 0$ , then  $\lim a_n = \lim b_n$ .

**Again we are trying to show that  $(a_n)$  and  $(b_n)$  converge to the same number. You have not even proved that  $\lim b_n = 0$ , so this does not show that  $\lim(a_n) = \lim(b_n)$ . Make sure you have a clear goal of what you are trying to prove and lay clear logical steps to prove that assertion.**