Real Analysis HW #2

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Question 1

Exercise 1.3.7. Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Proof: Suppose that $b = \sup A$. Let a be an upper bound for A and $a \in A$. We know b is an upper bound so for every $a \in A$ we have $a \leqslant b$. Since $b = \sup A$ and a is an upper bound of A we also know $b \leqslant a$. Thus, $a = b = \sup A$. 😊

Question 2

Exercise 1.4.1. Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well. ***Solution:*** If $a, b \in \mathbb{Q}$ then we can write a and b as a ratio of integers, $a = \frac{z}{k}$ and $b = \frac{l}{t}$ with $t, l, z, k \in \mathbb{Z}$. Consider

$$a + b = \frac{z}{k} + \frac{l}{t} = \frac{zt + lk}{kt}.$$

Thus, we can write $a + b$ as a ratio of two integers $(zt + lk, kt \in \mathbb{Z}) : a + b \in \mathbb{Q}$.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.
Solution: If $a \in \mathbb{Q}$ then we can define a as a ratio of two integers $m, n \in \mathbb{Z}$ such that $a = \frac{m}{n}$ where $a \neq 0$. If $t \in \mathbb{R} \setminus \mathbb{N}$. Then we can not an expression of t as a ratio of two integers. Then it is impossible to have $a + t$ and at as a ratio of two integers.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is I closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st ?
Solution: For any two irrational numbers $s, t \in \mathbb{R} \setminus \mathbb{Q}$ which respectively can not be expressed as the ratio of two integers, it is impossible to find $l, m \in \mathbb{Z}$ where $s + t = \frac{l}{m}$. So the irrationals are closed under multiplication.
However, we have that for for some $i \in \mathbb{R} \setminus \mathbb{Q}$ we can construct a rational number by simply squaring i. For example, $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ and $\sqrt{2}^2 = 2 \in \mathbb{Q}$. So the irrationals are not closed under multiplication.

Question 3

Exercise 1.4.3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Assume that there exists some $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$. Then for every $n \in \mathbb{N}$ we have that $x \in (0, \frac{1}{n})$. However,by Archimedes Property for any real number $y, y \neq 0$ there exists $n \in \mathbb{N}$ such that $\frac{q}{n} < y$. So there must exist $n \in \mathbb{N}$ such that $\frac{1}{n} < x$, implying that $x \notin \bigcap_{n=1}^{\infty}(0, 1/n)$ and $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

Question 4

Exercise 1.4.5. Using Exercise 1.4.1, supply a proof for Corollary 1.4 .4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution: Consider two numbers $a < b$, it follows that $a - \sqrt{2} < b - \sqrt{2}$.By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists some $x \in \mathbb{Q}$, which can be expressed $x = \frac{l}{k}$, $a, b \in \mathbb{Z}$, such that $a - \sqrt{2} < \frac{l}{k} < b - \sqrt{2}$.
We can add $\sqrt{2}$ to the equality

$$a - \sqrt{2} < \frac{l}{k} < b - \sqrt{2},$$

yielding,

$$a < \frac{l}{k} + \sqrt{2} < b.$$

Thus, we have found a real number, x, such that $a < x < b$. For all real numbers $a, b \in \mathbb{R}$ where $a < b$ there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < t < b$.

Question 5

Exercise 1.4.8. Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.

$A = x : x = \sqrt{2} - 1/n, n \in N.$
$B = (\sqrt{2} - 1, \sqrt{2}) \setminus A$
We have $\sup A = \sup B = \sqrt{2}$ and $A \cap B = \emptyset$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

$J_n = (-\frac{1}{n}, \frac{1}{n})$.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in R : x \geqslant a\}$.)
$J_n = [n, \infty)$. Assume that there exists some $x \in \mathbb{R}$ that exists in every J_n. By Archimedes Property we know for any $x \in \mathbb{R}$ there exists $n^\star \in \mathbb{R}$ where $n < n^\star$, so we have $\bigcap_{n=1}^{\infty} J_n = \emptyset$.

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Question 6

6. ChatGPT Challenge: Is the following correct? Make comments as if you are grading it:

Proof: $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$
($\Rightarrow$) Direction Assume that $a < b$. We want to show that $a < b + \epsilon$ for every $\epsilon > 0$. Let $\epsilon > 0$ be arbitrary. Adding ϵ to both sides of the inequality $a < b$ gives: $a + \epsilon < b + \epsilon$ Since $\epsilon > 0$, we have $a < a + \epsilon$. Combining these inequalities, we get: $a < a + \epsilon < b + \epsilon$ Thus, $a < b + \epsilon$ for every $\epsilon > 0$, as desired.

($\Leftarrow$) Direction Assume that $a < b + \epsilon$ for every $\epsilon > 0$. We want to show that $a < b$. Suppose, for the sake of contradiction, that $a \geqslant b$. Then, the difference $a - b$ is non-negative or zero ($a - b \geqslant 0$). Choose $\epsilon = \frac{a-b}{2}$. Since $a \geqslant b$, we have $\epsilon > 0$. By assumption, $a < b + \epsilon$, which gives:

$$a < b + \frac{a-b}{2}$$

Simplifying the right side, we get:

$$a < \frac{a+b}{2}$$

Now, adding b to both sides:

$$a + b < \frac{a+b}{2} + b$$

Simplifying further:

$$a + b < \frac{a + b + 2b}{2}$$
$$a + b < \frac{a + 3b}{2}$$

Since $a + b$ is less than $\frac{a+3b}{2}$, this contradicts our assumption that $a < b + \epsilon$ for every $\epsilon > 0$. Therefore, our assumption that $a \geqslant b$ must be false, which means $a < b$.