Ultimate Problem Set

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Question 1

Suppose that $x > -1$ and that $x \neq 0$. Prove that

$$(1+x)^n > 1+nx$$

for each integer $n > 1$. This result is know as Bernoulli's inequality.

Proof: We will show that this inequality holds for $x > -1$ and $x \neq 0$ by induction. First, we see when $n = 2$ that

$$(1+x)^2 = 1+2x+x^2 > 1+2x.$$

Thus, our base case holds. Now we assume that

$$(1+x)^n > 1+nx$$

is true. We want to show that $(1+x)^{n+1} > 1+(n+1)x$ is also true. We have that

$$(1+x)(1+x)^n > (1+nx)(1+x)$$

$$\implies (1+x)^{n+1} > 1+x+nx+nx^2 \geqslant 1+nx+x = 1+(n+1)x.$$

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Question 2

Show that e is irrational by supposing that $e = \frac{m}{n}$ and deriving a contradiction. Use the fact that $e = \sum_{j=0}^{\infty} \frac{1}{j!}$. Let $s_k = \sum_{j=0}^{k} \frac{1}{j!}$.

(a) Prove that

$$e - s_k < \frac{1}{(k+1)!}\left\{1 + \frac{1}{k+1} + \left(\frac{1}{k+1}\right)^2 + \cdots\right\}.$$

$$\begin{aligned} e - s_k &= \sum_{j=0}^{\infty} \frac{1}{j!} - \sum_{j=0}^{k} \frac{1}{j!} = \sum_{j=k+1}^{\infty} \frac{1}{j!} = \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \frac{1}{(k+3)!} + \cdots \\ &= \frac{1}{(k+1)!}\left[1 + \frac{1}{(k+2)} + \frac{1}{(k+2)(k+3)} + \cdots\right] \\ &< \frac{1}{(k+1)!}\left[1 + \frac{1}{(k+1)} + \frac{1}{(k+1)^2} + \cdots\right] \end{aligned}$$

(b) Prove that $e - s_k < \frac{1}{k(k!)}$ for all $k \in \mathbb{N}$.

Proof: Let

$$y_n = \sum_{n=0}^{m} \frac{1}{(k+1)^n}$$

. Then

$$\begin{aligned} y_n - \frac{1}{(k+1)^n} y_n &= \sum_{n=0}^{m} \frac{1}{(k+1)^n} - \sum_{n=1}^{m+1} \frac{1}{(k+1)^n} \\ \implies y_n\left(1 - \frac{1}{k+1}\right) &= \frac{1}{k+1} - \frac{1}{(k+1)^{m+1}} \\ \implies y_n\left(1 - \frac{1}{k+1}\right) &= \frac{1}{k+1} - \frac{1}{(k+1)^{m+1}} \\ \implies y_n &= \frac{\frac{1}{k+1} - \frac{1}{(k+1)^{m+1}}}{(1 - \frac{1}{k+1})}. \end{aligned}$$

Now let

$$\begin{aligned} \lim_{m\to\infty} y_n &= \lim_{m\to\infty} \frac{\frac{1}{k+1} - \frac{1}{(k+1)^{m+1}}}{(1 - \frac{1}{k+1})} \\ \implies \lim_{m\to\infty} y_n &= \frac{1}{k+1} \frac{1}{(1 - \frac{1}{k+1})} \\ &= \frac{1}{k+1} \frac{k+1}{k+1-1} \\ &= \frac{1}{k+1} \frac{k+1}{k} = 1. \end{aligned}$$

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(c) If $e = \frac{m}{n}$, prove that $n!e$ and $n!s_n$ are integers.

Proof: We have

$$n!e = n!\frac{m}{n} = (n-1)!m.$$

Since the integers are closed under multiplication then $n!e$ must be an integer.

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Proof:

$$n!s_n = n!s_k = \sum_{j=0}^{k} \frac{1}{j!}(n-1)! = n!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right).$$

$$= n! + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + 1.$$

This is just a sum of integers, so it must be that $n!s_n$ is also an integer.

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(d) If $e = \frac{m}{n}$, prove that $n!\,(e - s_n)$ is an integer between 0 and 1 , which is absurd.

Proof: Consider that

$$n!\,(e - s_n) < n!\frac{1}{n!n} = 1/n$$

Since $n \in \mathbb{N}$ and $e - s_n$ is an integer we have encountered a contradiction, $1/n < 1$. Thus, e can not be a rational number.

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Question 3

Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leqslant c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous.

(b) Pick some $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim_{n\to\infty} y_n$.

(c) Prove that y is a fixed point of f (i.e. $f(y) = y$) and that it is unique in this regard.

(d) Finally, prove that if x is any arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y (as defined in (b)).

Question 4

Let $\{r_n\}$ be a listing of all the rational numbers. Define a function f by $f(x) = 0$ if x is irrational and $f(r_n) = 1/n$ for all n. Show that f is continuous everywhere except for the set of rational numbers.

Question 5

Using the $\delta - \epsilon$ definition of a limit, show

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3.$$