Ultimate Problem Set

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Question 1

Suppose that $x > -1$ and that $x \neq 0$. Prove that

$$(1+x)^n > 1 + nx$$

for each integer $n > 1$. This result is know as Bernoulli's inequality.

Proof: We will show that this inequality holds for $x > -1$ and $x \neq 0$ by induction. First, we see when $n = 2$ that

$$(1+x)^2 = 1 + 2x + x^2 > 1 + 2x.$$

Thus, our base case holds. Now we assume that

$$(1+x)^n > 1 + nx$$

is true. We want to show that $(1+x)^{n+1} > 1 + (n+1)x$ is also true. First, we multiply both sides by $(1+x)$, then

$$(1+x)(1+x)^n > (1+x)(1+nx)$$

$$\implies (1+x)^{n+1} > 1 + x + nx + nx^2 \geqslant 1 + nx + x = 1 + (n+1)x.$$

Hence, $(1+x)^n > 1 + nx$ for each integer $n > 1$.

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Question 2

Show that e is irrational by supposing that $e = \frac{m}{n}$ and deriving a contradiction. Use the fact that $e = \sum_{j=0}^{\infty} \frac{1}{j!}$. Let $s_k = \sum_{j=0}^{k} \frac{1}{j!}$.

(a) Prove that

$$e - s_k < \frac{1}{(k+1)!}\left\{1 + \frac{1}{k+1} + \left(\frac{1}{k+1}\right)^2 + \cdots\right\}.$$

Proof: We have that

$$\begin{aligned} e - s_k = \sum_{j=0}^{\infty} \frac{1}{j!} - \sum_{j=0}^{k} \frac{1}{j!} = \sum_{j=k+1}^{\infty} \frac{1}{j!} &= \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \frac{1}{(k+3)!} + \cdots \\ &= \frac{1}{(k+1)!}\left[1 + \frac{1}{(k+2)} + \frac{1}{(k+2)(k+3)} + \cdots\right] \\ &< \frac{1}{(k+1)!}\left[1 + \frac{1}{(k+1)} + \frac{1}{(k+1)(k+1)} + \cdots\right] \\ &= \frac{1}{(k+1)!}\left[1 + \frac{1}{(k+1)} + \left(\frac{1}{k+1}\right)^2 + \cdots\right] \end{aligned}$$

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(b) Prove that $e - s_k < \frac{1}{k(k!)}$ for all $k \in \mathbb{N}$.

Proof: Let

$$y_n = \sum_{n=0}^{m} \frac{1}{(k+1)^n}.$$

Then

$$\begin{aligned} y_n - \frac{1}{k+1} y_n &= \sum_{n=0}^{m} \frac{1}{(k+1)^n} - \sum_{n=1}^{m+1} \frac{1}{(k+1)^n} \\ \implies y_n\left(1 - \frac{1}{k+1}\right) &= 1 - \frac{1}{(k+1)^{m+1}} \\ \implies y_n &= \frac{1 - \frac{1}{(k+1)^{m+1}}}{(1 - \frac{1}{k+1})}. \end{aligned}$$

Now let

$$\begin{aligned} \lim_{m\to\infty} y_n &= \lim_{m\to\infty} \frac{1 - \frac{1}{(k+1)^{m+1}}}{(1 - \frac{1}{k+1})} \\ \implies (y_n) &\to \frac{1}{(1 - \frac{1}{k+1})} \\ &= \frac{k+1}{k} \end{aligned}$$

Finally,

$$e - s_k < \frac{1}{(k+1)!}\left[1 + \frac{1}{(k+1)} + \frac{1}{(k+1)^2} + \cdots\right] = \frac{1}{(k+1)!}\{y_k\} \leqslant \frac{1}{(k+1)!}\frac{k+1}{k!} = \frac{1}{k(k!)}$$

(c) If $e = \frac{m}{n}$, prove that $n!e$ and $n!s_n$ are integers.

Proof: We have

$$n!e = n!\frac{m}{n} = (n-1)!m.$$

Since the integers are closed then $n!e$ must be an integer.

Proof:

$$n!s_n = n!s_n = n!\sum_{j=0}^{n}\frac{1}{j!} = n!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right).$$

$$= n! + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + 1$$

$$= n! + n(n-1)(n-2)\cdots(4)(3) + n(n-1)(n-2)\cdots(5)(4) + \cdots + 1.$$

Again, because the integers are closed it must be that $n!s_n$ is also an integer.

(d) If $e = \frac{m}{n}$, prove that $n!(e - s_n)$ is an integer between 0 and 1 , which is absurd.

Proof: Consider that

$$n!(e - s_n) < n!\frac{1}{n!n} = 1/n,$$

which means that $0 < n!(e - s_n) < 1$. Because $n!(e - s_n)$ must be an integer we have encountered a contradiction, this is impossible. Thus, e can not be a rational number.

Question 3

Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leqslant c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous.

Proof: Let $\epsilon > 0$ and $x, y \in \mathbb{R}$. Choose $\delta = \epsilon/c$. Now if we have

$$|x - y| < \delta$$

then

$$|f(x) - f(y)| \leqslant c|x - y| < c\frac{\epsilon}{c} = \epsilon.$$

Hence, f must be continuous. ☺

(b) Pick some $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim_{n\to\infty} y_n$.

Proof: Let $\epsilon > 0$. First we will show that

$$|y_{n+1} - y_n| \leqslant c^{n-1}|y_2 - y_1| = c^n\frac{|y_2 - y_1|}{c}$$

is true by induction:

1. Base case: From the given, when $n = 2$,

$$|y_3 - y_2| = |f(y_2) - f(y_1)| \leqslant c^1|y_2 - y_1|.$$

2. Inductive step: Now we want to show that if this holds true for n, this also holds true for $n + 1$. We assume

$$|y_{n+1} - y_n| \leqslant c^{n-1}|y_2 - y_1|$$

is true. We have

$$c|y_{n+1} - y_n| \leqslant c^n|y_2 - y_1|.$$

It follows

$$|y_{n+2} - y_{n+1}| = |f(y_{n+1}) - f(y_n)| \leqslant c|y_{n+1} - y_n| \leqslant c^n|y_2 - y_1|.$$

Finally, we conclude that

$$|y_{n+1} - y_n| \leqslant c^{n-1}|y_2 - y_1| = c^n\frac{|y_2 - y_1|}{c}$$

for all $n > 1$.

Now we may continue in the proof showing that (y_n) is Cauchy. Let

$$B = \frac{|y_2 - y_1|}{c}.$$

Choose $N \in \mathbb{N}$ such that for all $n > N$,

$$B\frac{c^n}{1-c} < \epsilon.$$

Let $m > n > N$. We have that

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} + \cdots + x_{n+1} - x_n|$$

$$\leqslant |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| = \sum_{n}^{m-1} |x_{n+1} - x_n| \leqslant \sum_{n}^{m-1} Bc^n.$$

If

$$z_n = \sum_{n}^{m-1} Bc^n,$$

then

$$z_n - cz_n = \sum_{n}^{m-1} Bc^n - \sum_{n+1}^{m} Bc^n$$

$$\implies z_n(1-c) = Bc^n - Bc^m$$

$$\implies z_n = \frac{B(c^n - c^m)}{1-c} = B\frac{c^n - c^m}{1-c}$$

$$< B\frac{c^n}{1-c} < \epsilon.$$

Hence, $|x_m - x_n| < \epsilon$ and (y_n) is thus Cauchy. We may let $y = \lim_{n\to\infty} y_n$.

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(c) Prove that y is a fixed point of f (i.e. $f(y) = y$) and that it is unique in this regard.

Proof: Consider that $y_{n+1} = f(y_n)$. Then,

$$\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} f(y_n)$$

$$\implies y = f\left(\lim_{n\to\infty} y_n\right)$$

$$\implies y = f(y).$$

Thus, y is a fixed point. Consider, by way of contradiction, that y is not the only fixed point and there exists another fixed point x where $y \neq x$.

Then,

$$|f(y) - f(x)| = |x - y| \leqslant c|x - y|$$

which is a contradiction since $0 < c < 1$. Thus, it must be that y is a unique fixed point.

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(d) Finally, prove that if x is any arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y (as defined in (b)).

Proof: Because we proved previously that for any arbitrary $x \in \mathbb{R}$ the sequence $(y_n) \to y$ and y is a unique fixed point it must be the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y (as defined in (b)). 😊

Question 4

Let $\{r_n\}$ be a listing of all the rational numbers. Define a function f by $f(x) = 0$ if x is irrational and $f(r_n) = 1/n$ for all n. Show that f is continuous everywhere except for the set of rational numbers.

Proof: Let $\epsilon > 0$. We consider (1) if f is continuous at c, an irrational number and (2) if is continuous at c, a rational number (2). Note that $f(x) \geqslant 0$ for all $x \in \mathbb{R}$.

1. By the Archimedes Principle there exists $N \in \mathbb{N}$ such that for all $n > N$, $1/n < \epsilon$. Consider all the rational numbers where $n \leqslant N$, and choose $\delta = \min\{|r_n - c|\}/2$. Note that we have chosen $N \in \mathbb{N}$ such that for all rational numbers where $n > N$, $f(r_n) < \epsilon$ and also for all $x \in \mathbb{R} \setminus \mathbb{Q}$, $f(x) = 0 < \epsilon$. Thus, when $x \in \mathbb{R}$ we have chosen δ such that when $|x - c| < \delta$, we automatically have that $|f(x) - f(c)| = |f(x) - 0| = |f(x)| = f(x) < \epsilon$. Hence, f is continuous on the irrationals.

2. Consider, by way of contradiction, that f is continuous on the rational numbers. Then there exists δ such that when $x \in \mathbb{R}$ and $|x - y| < \delta$ we automatically have $|f(x) - f(c)| < f(c)/2$. By the density of the irrational numbers in $\mathbb{R}$ there exists an irrational number , x_I, such that $|x_I - y| < \delta$. It follows $|f(x_I) - f(c)| = |0 - f(c)| = f(c) < f(c)/2 \implies 1 < 1/2$, a contradiction. Thus, f can not be continuous on the rational.

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Question 5

Using the $\delta - \epsilon$ definition of a limit, show

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3.$$

Proof: Consider,

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \left[(x - 1) \frac{x^2 + x + 1}{x - 1} \right] = \lim_{x \to 1} x^2 + x + 1 = 3.$$

Let $\epsilon > 0$ and choose $\delta = \min\{1, \epsilon/4\}$. Note, if restrict δ to be a maximum of 1 then $|x + 2| \leqslant |x| + 2| \leqslant |2| + 2 = 4$. If we have $0 < |x - 1| < \delta$ then

$$|f(x) - L| = |x^2 + x + 1 - 3| = |x^2 + x - 2| = |(x + 2)(x - 1)| = |x + 2||x - 1|.$$

Altogether,

$$|f(x) - L| = |x + 2||x - 1| < 4\frac{\epsilon}{4} = \epsilon.$$

Hence,

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3.$$

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