Sequences and Series

Jack Krebsbach

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0.1 Sequences and Series

Definition 0.1.1: Sequence

A sequence is a function from $f : \mathbb{N} \to \mathbb{R}$.
Examples:

1. (a_n)
2. $(a_1, a_2, a_3, \ldots, a_n)$

Definition 0.1.2: Convergence

A sequence, (a_n), converges to a point, x, if for all $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - x| < \epsilon$.

Theorem 0.1.1 Uniqueness of Limits.
The limit of a sequence, when it exists, must me unique.

Proof: Let (x_n) be a convergent series that converges to x. By way of contradiction, suppose that $(x_n) \to y$ where $x \neq y$ and $x < y$. Let $\epsilon = \frac{1}{3}(y - x)$. Since (x_n) converges to x there exists $N_x \in \mathbb{N}$ such that for all $n > N_x$, $|x_n - x| < \epsilon$. Similarly, since (x_n) converges to y there exists $N_Y \in \mathbb{N}$ such that for all $n > N_y$, $|x_n - y| < \epsilon$.

Let $N = \max\{N_x, N_y\}$. Then $x_{N+2} \in \mathcal{B}(x, \epsilon) \cap \mathcal{B}(y, \epsilon)$. This is a contradiction, $x_{N+2} \notin \mathcal{B}(x, \epsilon) \cap \mathcal{B}(y, \epsilon)$. Thus, $x = y$ and limits are unique! 😄

Theorem 0.1.2 Convergent Sequences are Bounded

Proof: Let (x_n) be a convergent sequence converging to x. Let $\epsilon = 1$. Since (x_n) converges there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - x| < \epsilon$. By the triangle inequality theorem,$|x_n| - |x| \leqslant |x_n - x| < 1$. So $|x_n| < |x| + 1$ for all $n > N$.

Now consider the set $\{|x_1|, |x_2|, |x_3|, \ldots, |x_N|\}$. All, the elements outside the ball of convergence. Let $B = \{|x_1|, |x_2|, |x_3|, \ldots, |x_N|, |x| + 1\}$. Thus, $|x_n| \leqslant B$ for all $n \in \mathbb{N}$ and (x_n) is bounded.

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Theorem 0.1.3 Bounded and monotone sequences are convergent.

Proof: Let (a_n) be monotone and bounded. To prove (a_n) converges using the definition of convergence, we are going to need a candidate for the limit. Let's assume the sequence is increasing (the decreasing case is handled similarly), and consider the set of points $\{a_n : n \in \mathbf{N}\}$. By assumption, this set is bounded, so we can let

$$s = \sup\{a_n : n \in \mathbf{N}\}.$$

It seems reasonable to claim that $\lim a_n = s$. 2.4. The Monotone Convergence Theorem and Infinite Series 57 To prove this, let $\epsilon > 0$. Because s is the least upper bound for $\{a_n : n \in \mathbf{N}\}$, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \epsilon < a_N$. Now, the fact that (a_n) is increasing implies that if $n \geqslant N$, then $a_N \leqslant a_n$. Hence,

$$s - \epsilon < a_N \leqslant a_n \leqslant s < s + \epsilon,$$

which implies $|a_n - s| < \epsilon$, as desired. The Monotone Convergence Theorem is extremely useful for the study of infinite series, largely because it asserts the convergence of a sequence without explicit mention of the actual limit. This is a good moment to do some preliminary investigations, so it is time to formalize the relationship between sequences and series. 😄

Theorem 0.1.4 Algebraic Limit Theorem

Let $\lim a_n = a$, and $\lim b_n = b$. Then,

1. $\lim (ca_n) = ca$, for all $c \in \mathbb{R}$;
2. $\lim (a_n + b_n) = a + b$;
3. $\lim (a_n b_n) = ab$;
4. $\lim (a_n / b_n) = a/b$, provided $b \neq 0$.