

HW #1

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### Question 1

Let  $n$  be a positive integer that is not a perfect square. Prove that  $\sqrt{n}$  is irrational.

**Solution:** Assume, for contradiction, that  $\sqrt{n}$  is a rational. Then  $\sqrt{n}$  can be written in the form  $\frac{a}{b}$  where  $a, b \in \mathbb{Z}$  and  $a, b$  are coprime, or have no common factors.

We have

$$\frac{a}{b} = \sqrt{n} \implies \left(\frac{a}{b}\right)^2 = n \implies a^2 = nb^2 \quad (1)$$

Clearly  $n$  divides  $a^2$ . By the Fundamental Theorem of arithmetic we can write  $a$  and  $n$  as a product of primes.

Thus,

$$\frac{a^2}{n} = \frac{\left(\prod_{i=1}^k p_i^{n_i}\right)^2}{\prod_{j=1}^k p_j^{m_j}} = \frac{\left(\prod_{i=1}^k p_i^{n_i}\right)\left(\prod_{i=1}^k p_i^{n_i}\right)}{\prod_{j=1}^k p_j^{m_j}} = b^2 \quad (2)$$

Because  $n$  divides  $a^2$  we can re-write  $b^2$  as the product

$$n \left( \prod_{l=1}^t p_l^{m_l} \right) = b^2 \quad (3)$$

Clearly,  $b^2 \geq n$ , and it follows that  $a \geq n$ . Therefore we can rearrange (2) yielding

$$a^2 = (n)(a) \left( \prod_{i=1}^z p_i^{m_i} \right) \implies a = n \left( \prod_{i=1}^z p_i^{m_i} \right) \quad (4)$$

Thus,  $n$  divides  $a$  in addition to  $a^2$ . Because of this we know that we can rewrite  $a$  in terms of  $n$ , or  $a = t(n)$  where  $t \in \mathbb{Z}$ . Then

$$(tn)^2 = nb^2 \implies t^2 n^2 = nb^2 \implies nt^2 = b^2 \quad (5)$$

which means  $n$  is a common factor of  $b^2$ , and by the preceding logic  $b$  in addition  $a$ . Thus  $a$  and  $b$  can not be coprime  $\rightarrow \times$ . This contradicts our initial assumption and we have no choice but to conclude that  $\sqrt{n} \in \mathbb{R} \setminus \mathbb{Q}$  ☺

### Question 2

Use the Principle of Mathematical Induction to prove:

$$1^2 + 3^2 + \dots + (2n-1)^2 = \frac{4n^3 - n}{3} \quad \forall n \in \mathbb{N}$$

### Question 3

Let  $n > 1$  be a positive integer and let  $a_1, a_2, \dots, a_n$  be real numbers. Prove that

$$\left| \sum_{k=1}^n a_k \right| \leq \sum_{k=1}^n |a_k|$$

$$\left| \sum_{k=1}^n a_k \right| \leq \sum_{k=1}^n |a_k|$$

#### Question 4

Exercise 1.2.8

#### Question 5

Exercise 1.2.12

#### Question 6

Grade the following proof: Prove that  $\sqrt{3}$  is irrational. Does a similar argument work to show  $\sqrt{6}$  is irrational?

Proof: To prove that  $\sqrt{3}$  is irrational, we can use a proof by contradiction. The idea is to assume that  $\sqrt{3}$  is rational, and then derive a contradiction.

Assumption:  $\sqrt{3}$  is rational.

If  $\sqrt{3}$  is rational, it can be expressed as a fraction  $\frac{a}{b}$  in its simplest form, where  $a$  and  $b$  are integers with no common factors other than 1 (i.e., they are coprime), and  $b$  is not equal to 0.

So, we have:  $\sqrt{3} = \frac{a}{b}$

Squaring both sides of the equation:  $3 = \frac{a^2}{b^2}$

Rearranging the equation:  $a^2 = 3 \cdot b^2$

From this equation, we can see that  $a^2$  is a multiple of 3, which implies that  $a$  must also be a multiple of 3. Let's represent  $a$  as  $3k$  where  $k$  is an integer:

$$(3k)^2 = 3 \cdot b^2$$

$$9k^2 = 3 \cdot b^2$$

$$3k^2 = b^2$$

Now,  $b^2$  is also a multiple of 3, which means  $b$  must be a multiple of 3. However, this contradicts our initial assumption that  $a$  and  $b$  have no common factors other than 1 (i.e., they are coprime). This contradiction arises from assuming that  $\sqrt{3}$  is rational.

Hence, our initial assumption that  $\sqrt{3}$  is rational must be false, which means that  $\sqrt{3}$  is irrational. Now, to address whether a similar argument works for  $\sqrt{6}$ :

No, a similar argument does not work for  $\sqrt{6}$ . The proof for the irrationality of  $\sqrt{3}$  relies on the fact that it leads to a contradiction with respect to the common factors of  $a$  and  $b$ . However, for  $\sqrt{6}$ , this argument does not apply directly because the factors of 6 are not the same as the factors of 3. In fact,  $\sqrt{6}$  is not irrational; it is a rational number. It can be expressed as the fraction  $\frac{2\sqrt{3}}{3}$ , which is a ratio of two integers.