

Real Analysis HW #6

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Question 1

Ex 2.4 .2

(a) Consider the recursively defined sequence $y_1 = 1$, $y_{n+1} = 3 - y_n$ and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y , we conclude $\lim y_n = 3/2$. What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence? Why or why not. Justify your answer.

Question 2

For each natural number n , let

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Prove that the sequence (x_n) converges.

Question 3

Consider the sequence

$$x_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}.$$

Show that (x_n) converges.

Question 4

Prove that (x_n) converges where $x_n = \sum_{k=1}^n k^2 2^{-k}$.

Question 5

Consider the sequence $x_n = \sum_{k=1}^n \ln\left(\frac{n+1}{n}\right)$. Does the sequence converge? Explain.

Question 6

ChatGPT Challenge: Consider the sequence

$$x_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}.$$

Show that (x_n) converges.

To show that the sequence $\{x_n\}$ converges, we can use the limit properties of sequences. We will prove that the limit exists by finding the limit of x_n as n approaches infinity.

The sequence x_n is defined as:

$$x_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}$$

We want to find $\lim_{n \rightarrow \infty} x_n$. First, let's break it down into two parts and consider their limits separately:
1. The sum $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$. 2. The term $-2\sqrt{n}$.

For the first part, we can use the fact that the limit of a sum is the sum of the limits (if they exist):

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \right) = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2}} + \lim_{n \rightarrow \infty} \frac{1}{\sqrt{3}} + \cdots + \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}}$$

Each term in the sum approaches zero as n goes to infinity because $\frac{1}{\sqrt{n}}$ converges to zero:

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

So, the limit of the sum of these terms is:

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \right) = 0$$

For the second part, the limit of $-2\sqrt{n}$ as n approaches infinity is also straightforward:

$$\lim_{n \rightarrow \infty} (-2\sqrt{n}) = -\infty$$

Now, we can find the limit of the entire sequence x_n by combining the limits of its two parts:

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} - 2\sqrt{n} \right) = 0 - \infty = -\infty$$

Since $\lim_{n \rightarrow \infty} x_n$ is a finite value ($-\infty$), we can conclude that the sequence $\{x_n\}$ converges to $-\infty$.