

# The Real Numbers

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## 0.1 The Real Numbers

### 0.1.1 The irrationality of the square root of 2

#### 0.1.2 Preliminaries

##### Notation

- $\rightarrow$  For all/each/every
- $\exists \rightarrow$  There exists
- $\mathbb{R} \setminus \mathbb{Q} \rightarrow$  Irrationals
- $\mathbb{R} \rightarrow$  Real numbers
- $\mathbb{Z} \rightarrow$  Integers
- $\mathbb{Q} \rightarrow$  Rational numbers
- $\mathbb{N} \rightarrow$  Natural numbers
- $BWOC \rightarrow$  By way of contradiction
- $\rightarrow \times \rightarrow$  Contradiction
- $! \rightarrow$  Unique/factorial
- $\odot \rightarrow$  End of proof (Quod Erat Demonstrandum)
- $\epsilon \rightarrow$  Epsilon, usually a small positive quantity
- $\ni \rightarrow$  Such that

##### Theorem 0.1.1

Two real numbers  $a$  and  $b$  are equal if and only if for every real number  $\epsilon > 0$  it follows that  $|a - b| < \epsilon$ .

**Proof:**  $\Rightarrow$  If  $a = b$  then we have  $|a - b| = 0$ . No matter which  $\epsilon > 0 \in \mathbb{R}$  is chosen we have that  $|a - b| = 0 < \epsilon$ . Thus,  $a = b$

$\Leftarrow$  Suppose, by way of contradiction, that  $a \neq b$ . We know that  $\forall \epsilon > 0$  and  $|a - b| < \epsilon$ . Let  $\epsilon_0 = \frac{|a-b|}{2}$ , then it is clear that  $|a - b| < \frac{|a-b|}{2} = \epsilon_0$  is false. Thus, with this contradiction we overturn our assumption and conclude  $a$  must equal  $b$ .  $\odot$

### 0.1.3 The Axiom of Completeness

#### Definition 0.1.1: Bounded Above

A set  $A \subseteq \mathbb{R}$  is bounded above if there exists a number  $b \in \mathbb{R}$  such that  $a \leq b$  for all  $a \in A$ . The number  $b$  is called an upper bound for  $A$ . Similarly, the set  $A$  is bounded below if there exists a lower bound  $l \in \mathbb{R}$  satisfying  $l \leq a$  for every  $a \in A$ .

#### Definition 0.1.2: The Supremum of a Set

A real number  $s$  is the *Supremum* or the least upper bound for a set  $A \subseteq \mathbb{R}$  if:

- (i)  $s$  is an upper bound for  $A$ ;
- (ii) if  $b$  is any upper bound for  $A$ , then  $s \leq b$ .

##### Theorem 0.1.2 Lemma 1.3.8

Assume  $s \in \mathbb{R}$  is an upper bound for a set  $A \subseteq \mathbb{R}$ . Then,  $s = \sup A$  if and only if, for every choice of  $\epsilon > 0$ , there exists an element  $a \in A$  satisfying  $s - \epsilon < a$ .

**Proof:**  $\Rightarrow$  Let  $s = \sup A$  and arbitrarily choose any  $\epsilon > 0$ . Suppose, by way of contradiction, there does not exist  $a \in A$  with  $a > s - \epsilon$ . So for all  $a \in A$  we have that  $a \leq s - \epsilon < s$ . This means that  $s - \epsilon$  is an upper bound of  $A$ . However,  $s$  is the supremum or the least upper bound of  $A$ , so this is a contradiction. Thus, there must exist an element  $a \in A$  such that  $s - \epsilon < a$ .

$\Leftarrow$  Let  $\epsilon > 0$  and suppose there exists  $a \in A$  with  $s - \epsilon < a$  and we know that  $s$  is an upper bound of  $A$ . To show that  $s$  is the least upper bound, by way of contradiction, suppose that  $b < s$  and  $b$  is another upper bound of  $A$ .

Consider  $\epsilon_0 = s - b > 0$ . By hypothesis there exists  $a$  with  $s - \epsilon_0 < a \implies s - (s - b) < a \implies b < a$ . As we know that  $b$  is an upper bound of  $A$  this is impossible. Therefore it must be that  $b \geq s$ . Hence,  $s \leq b$  and  $s = \sup A$ .

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