

Real Analysis HW #8

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Definition 0.0.1: Fibonacci Sequence

The Fibonacci sequence, $1, 2, 3, 5, 8, 13, \dots$ is given by the recursive formula

$$F_{n+1} = F_n + F_{n-1}$$

where $F_1 = 1$ and $F_2 = 2$. Let $a_n = \frac{F_n}{F_{n-1}}$.

Question 1

Suppose that $\{a_n\}$ converges to a limit. What must that limit be? Hint: Divide the above equation by F_n to find an equation relating a_{n+1} to a_n .

Solution: From the recursive formula, dividing by F_n yields:

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$$

Then,

$$\begin{aligned} a_{n+1} &= 1 + \frac{F_{n-1}}{F_n} \\ \implies a_{n+1} &= 1 + \frac{1}{a_n} \end{aligned}$$

Let $L = \lim_{n \rightarrow \infty} a_n$, then

$$\begin{aligned} L &= 1 + \frac{1}{L} \\ \implies L^2 &= L + 1 \\ \implies L^2 - L - 1 &= 0. \end{aligned}$$

By the quadratic formula,

$$L = \frac{1 \pm \sqrt{5}}{2}.$$

Since this sequence is positive for all $n \in \mathbb{N}$ we want the positive solution. Thus,

$$L = \frac{1 + \sqrt{5}}{2}.$$

Question 2

Show that $\frac{3}{2} \leq a_n \leq 2 \forall n \geq 2$.

Proof: Let $n \in \mathbb{N}$. We have that $a_1 = 1, a_2 = 2, a_3 = 3/2$. Thus,

$$\frac{3}{2} \leq a_n \leq 2$$

for $1, 2, 4 \in \mathbb{N}$. We want to show that if this is true for a_n this is also true for a_{n+1} .

We assume that

$$\frac{3}{2} \leq a_n \leq 2$$

is true. Then,

$$\frac{2}{3} \geq \frac{1}{a_n} \geq \frac{1}{2}$$

$$\Rightarrow 1 + \frac{2}{3} \geq 1 + \frac{1}{a_n} \geq 1 + \frac{1}{2}$$

$$\Rightarrow 1 + \frac{1}{2} \leq 1 + \frac{1}{a_n} \leq 1 + \frac{2}{3}$$

$$\Rightarrow \frac{3}{2} \leq a_{n+1} \leq \frac{5}{3} < 2.$$

Thus, $\frac{3}{2} \leq a_n \leq 2$ for all $n \geq 2$.



Question 3

For each $n > 2$, prove that $|a_{n+1} - a_n| \leq \left(\frac{2}{3}\right)^2 |a_n - a_{n-1}|$.

Question 4

Prove that for each $m \geq 2$, $|a_{m+1} - a_m| \leq \left(\frac{2}{3}\right)^{2(m-2)} |a_3 - a_2|$.

Question 5

Use the inequality in (4) to show that $\{a_n\}$ is a Cauchy sequence and therefore converges to a limit.

Question 6

CHAT GPT CHALLENGE: To prove that a bounded sequence (x_n) with the property that every convergent subsequence of (x_n) converges to the same limit $x \in \mathbb{R}$ must converge to x , you can use the following proof:

Let $\epsilon > 0$ be given. Since (x_n) is bounded, it has a limit point, say y , which means that there exists a subsequence (x_{n_k}) that converges to y . Since every convergent subsequence of (x_n) converges to x , we have $y = x$.

Now, consider $\epsilon > 0$ and $y = x$. Since (x_n) is a bounded sequence, it has a limit point, and there exists a subsequence (x_{n_k}) that converges to x .

By the definition of the limit, there exists an N_1 such that for all $k \geq N_1$, $|x_{n_k} - x| < \epsilon/2$. Additionally, because every convergent subsequence of (x_n) converges to x , we can choose another N_2 such that for all $k \geq N_2$, $|x_{n_k} - x| < \epsilon/2$.

Now, let $N = \max(N_1, N_2)$. For all $n \geq N$, there exists $k_1 \geq N$ and $k_2 \geq N$ such that:

$$\begin{aligned} |x_{n_{k_1}} - x| &< \frac{\epsilon}{2} \\ |x_{n_{k_2}} - x| &< \frac{\epsilon}{2} \end{aligned}$$

By the triangle inequality:

$$\begin{aligned} |x_n - x| &\leq |x_n - x_{n_{k_1}}| + |x_{n_{k_1}} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So, for all $n \geq N$, $|x_n - x| < \epsilon$, which means that (x_n) converges to x . Therefore, we've shown that if (x_n) is a bounded sequence with the property that every convergent subsequence converges to the same limit $x \in \mathbb{R}$, then (x_n) itself must converge to x .