

Real Analysis HW #9

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Question 1

Let $g : A \rightarrow \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$. Suppose that $\lim_{x \rightarrow a} f(x) = 0$.


- (a) Show that $\lim_{x \rightarrow a} f(x)g(x) = 0$ for any function g as above IS NOT TRUE.
- (b) Assume that g is a bounded function on A . Show that $\lim_{x \rightarrow c} g(x)f(x) = 0$.

Question 2

Let a and b be real numbers with $a \neq 0$. Use the definition of continuity to prove that the function f defined by $f(x) = ax + b$ is continuous at every real number.

Proof: Let $\epsilon > 0$ and $c \in \mathbb{R}$. Choose $\delta = \epsilon/a$. If we have $|x - c| < \delta$ it follows

$$|f(x) - f(c)| = |ax + b - (ac + b)| = |ax + b - ac - b| = |ax - ac| = |a(x - c)| < |a| \frac{\epsilon}{a} < \epsilon.$$


Thus, $f(x) = ax + b$ is continuous at every real number. 

Question 3

Use the definition of limit to prove that $\lim_{x \rightarrow c} x^2 = c^2$ for every real number c .

Proof: Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/(2c + 1)\}$. If we have $0 < |x - c| < \delta$ it follows that,

$$|f(x) - L| = |x^2 - c^2| = |x + c||x - c| < (2c + 1) \frac{\epsilon}{2c + 1} = \epsilon$$

Thus, $\lim_{x \rightarrow c} x^2 = c^2$ for every real number c . 

Question 4

Find constants a and b so that the function f defined by

$$f(x) = \begin{cases} 3ax^2 + 1 & 0 \leq x \leq 1 \\ ax + b & 1 < x < 2 \\ 2bx + a & 2 \leq x \leq 4 \end{cases}$$

has a limit at each point of $[0, 4]$. Be sure to show the limit exists.

Solution:

First we find constants a and b so that $f(x)$ has a limit defined at each point $[0, 4]$. Plugging in 1 and 2 in each of the equations defined in the piecewise function $f(x)$ yields a system of equations:

$$2a + 1 = a + b \implies 2a + 1 = b$$

and

$$2a + b = 4b + a \implies a = 3b.$$

Substituting $a = 3b$ into the first equation yields

$$2(3b) = b \implies 6b + 1 = 6 \implies 5b = -1 \implies b = \frac{-1}{5}.$$

Finally, solving for $a = 3b = 3(-1/5) = -3/5$. Thus,

$$b = \frac{-1}{5} \text{ and } a = \frac{-3}{5}$$

and f becomes

$$f(x) = \begin{cases} -9/5x^2 + 1 & 0 \leq x \leq 1 \\ -3/5x - 1/5 & 1 < x < 2 \\ -2/5x - 3/5 & 2 \leq x \leq 4 \end{cases}$$

Question 5

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(x) = 0$ for each irrational number x in $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Question 6

CHAT GPT CHALLENGE: Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function on $[a, b]$, and suppose that $f(x) = 0$ for each rational number x in $[a, b]$. We aim to show that $f(x) = 0$ for all $x \in [a, b]$.

Assume, for the sake of contradiction, that there exists $c \in [a, b]$ such that $f(c) \neq 0$. Since f is continuous, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in [a, b]$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Consider the case when $\epsilon = |f(c)|$. By continuity, there exists $\delta > 0$ such that for all $x \in [a, b]$, if $|x - c| < \delta$, then $|f(x) - f(c)| < |f(c)|$. This implies

$$-f(c) < f(x) - f(c) < f(c).$$

Adding $f(c)$ to all parts of the inequality yields $0 < f(x)$ for x in the interval $(c - \delta, c) \cap [a, b]$. Similarly, subtracting $f(c)$ from all parts of the inequality yields $f(x) < 0$ for x in the interval $(c, c + \delta) \cap [a, b]$. Now, consider the set of rational numbers in $[a, b]$. Since \mathbb{Q} is dense in \mathbb{R} , there exist rational numbers in both $(c - \delta, c) \cap [a, b]$ and $(c, c + \delta) \cap [a, b]$. However, we have shown that for any rational x in these intervals, $f(x)$ has opposite signs, which is a contradiction.

Therefore, our assumption that there exists c such that $f(c) \neq 0$ is false, and we conclude that $f(x) = 0$ for all $x \in [a, b]$.