APPM Statistics Preliminary Exams

December 1, 2024

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1. Let $\lambda > 0$ and $0 < p < 1$ be real constants. Suppose that $N \sim \text{Poisson}(\lambda)$ and that, conditioned on $N = n, B \sim \text{Binomial}(n, p)$. In particular, $B = 0$ when $N = 0$.

   (a) Show that $B \sim \text{Poisson}(\lambda p)$.

   (b) Without further calculations, what should the distribution of $(N - B)$ be?

   (c) Finally, show that B and $(N - B)$ are independent.

2. In what follows, $\xrightarrow{p}$ and $\xrightarrow{d}$ denote convergence in probability and distribution of random variables, respectively. Additionally, "a.s." stands for almost surely.

   (a) Is it TRUE that if $X_n \xrightarrow{d} 0$ then $X_n \xrightarrow{p} 0$? If so, show this using the definitions of convergence in probability and distribution, otherwise provide a counter-example.

   (b) Let $(X_i)_{i \geq 0}$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean 2 and variance 1 . Invoking well-known a.s. convergence results, which you must name explicitly as part of your solution, justify the existence of the following limit in the a.s. sense, and determine it explicitly.

$$\lim_{n \to \infty} \frac{\sum_{i=1}^n X_i^2}{\sqrt{n \cdot \sum_{i=1}^n \left(X_i - \bar{X}\right)^2}}, \text{ where } \bar{X} := \frac{1}{n} \sum_{i=1}^n X_i.$$

3. Suppose $X_1, \ldots, X_n$ with $n > 3$ are i.i.d. exponential with rate parameter $\lambda > 0$ (that is, with mean $1/\lambda$). We will consider estimation of λ.

   (a) Find the expectation of $1/\bar{X}$.

   (b) Based on (a), find an unbiased estimator for λ.

   (c) Find the mean squared error (MSE) of $1/\bar{X}$, and the MSE of your estimator from part (b). Which one is smaller?

4. Suppose $X_1, \ldots, X_n$ are i.i.d. with p.d.f.

$$f(x; \lambda) = \lambda^2 x e^{-\lambda x}, \quad x > 0,$$

   where $\lambda > 0$ is unknown.

   (a) Find a maximum likelihood estimator (MLE) for λ.

   (b) Find the asymptotic distribution of your MLE from (a).

   (c) Find an MLE for $e^{-\lambda}$.

   (d) Find the asymptotic distribution of your MLE from (c).

5. Let $Y_1, \ldots, Y_n$ be independent, with $Y_i \sim \text{Poisson}\left(a_i \cdot \nu\right)$ for some known constants a_i and unknown $\nu > 0$.

   (a) Find the joint p.m.f. of $Y_1, \ldots, Y_n$

   (b) Find a sufficient and complete statistic for ν.

   (c) Determine the unique UMVUE of ν.

   (d) Determine the unique UMVUE of ν^2.

6. Let $Y_1, \ldots, Y_n$ be independent, with $Y_i \sim \text{Poisson}\left(a_i \cdot \nu\right)$ for some known constants a_i and unknown $\nu > 0$.

   (a) Find the joint p.m.f. of $Y_1, \ldots, Y_n$

   (b) Find a sufficient and complete statistic for ν.

   (c) Determine the unique UMVUE of ν.

   (d) Determine the unique UMVUE of ν^2.