1a) $E\left(\frac{1}{1+x}\right) = \int_0^{\infty} \frac{f(x)}{1+x}\,dx = \frac{F(x)}{1+x}\Big|_0^{\infty} - \int_0^{\infty} \frac{-F(x)}{(1+x)^2}$

$u = \frac{1}{1+x}$ $\quad dv = f(x)$

$du = \frac{-1}{(1+x)^2}$ $\quad v = F(x)$

$= \frac{\lim_{x\to\infty} F(x)}{1+\infty} - \frac{0}{1} + \int_0^{\infty} \frac{F(x)}{(1+x)^2}$

$= \frac{1}{\infty} - 0 + \int_0^{\infty} \frac{F(x)}{(1+x)^2} = \boxed{\int_0^{\infty} \frac{F(x)}{(1+x)^2}}$

1b) $X_i \sim Exp(rate=1)$ $\quad$ Find $M_{X_{(n)}}(t)$

$Y_n = X_1 + \frac{1}{2}X_2 + \ldots + \frac{1}{n}X_n$ $\quad$ $M_{Y_n}(t)$

$P(X_{(n)} < x) = P(\text{all } X < x) = P(X_i < x)^n = F(x)^n$

$f_{X_{(n)}}(x) = nF(x)^{n-1}f(x) = n(1-e^{-x})^{n-1}e^{-x}$

$M_{X_{(n)}}(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}n(1-e^{-x})^{n-1}e^{-x}$

$= \int_0^{\infty} n(1-e^{-x})^{n-1}e^{-x}e^{tx}\,dx$

$u = 1-e^{-x}$

$du = e^{-x}dx$

$= \int_0^1 nu^{n-1}e^{tx}\,du$

$e^{-x} = 1-u$

$x = -\ln(1-u)$

$\int_0^1 nu^{n-1}e^{-t\ln(1-u)}\,du$

$e^{\ln(1-u)^{-t}}$

$$\int_0^1 n u^{n-1}(1-u)^{-t}\,du$$

$$n\int_0^1 u^{n-1}(1-u)^{1-t-1}\,du$$

$$= nB(n, 1-t) = \frac{n\,\Gamma(n)\,\Gamma(1-t)}{\Gamma(n+1-t)} = \frac{n!\,\Gamma(1-t)}{\Gamma(n+1-t)}$$

$$\frac{1}{2}X_2 \sim Exp\left(\frac{1}{1/2}\right) = Exp(2)$$

$$\frac{1}{3}X_3 \sim Exp(3)$$

$$\frac{1}{n}X_n \sim Exp(n)$$

If $Y = aX + b$,

$$M_Y(t) = E(e^{tY}) = E(e^{t(aX+b)}) = E(e^{atX}e^{tb})$$

$$= e^{tb}E(e^{atX}) = e^{tb}M_X(at)$$

$$M_{X_i}(t) = \frac{1}{1-t},\ t<1 \qquad M_{\frac{1}{K}X_K}(t) = M_{X_i}\left(\frac{t}{K}\right)\quad t<K$$

$$\frac{1}{K} = a \qquad = \frac{1}{1-\frac{t}{K}} = \frac{K}{K-t}$$

$X_1, \dots, X_n$ ind RVs w/ mgfs $M_{X_1}(t), \dots, M_{X_n}(t)$

then $Y = \sum_{i=1}^{n} X_i$ has mgf $\prod_{i=1}^{n} M_{X_i}(t)$

$Y_n = \sum_{i=1}^{n} \frac{1}{i} X_i$ has mgf $\prod_{i=1}^{n} M_{X_i}\left(\frac{t}{i}\right) = \prod_{i=1}^{n} \frac{i}{i-t}, \quad t<1$

	n		
MGF	1	2	3
$X_{(n)}: \frac{n!\,\Gamma(1-t)}{\Gamma(n+1-t)}$	$\frac{\Gamma(1-t)}{\Gamma(2-t)}$	$\frac{2\Gamma(1-t)}{\Gamma(3-t)}$	$\frac{6\Gamma(1-t)}{\Gamma(4-t)}$
	$\approx$	$\approx$	$\approx$
$Y_n: \prod_{i=1}^{n} \frac{i}{i-t}$	$\frac{1}{1-t}$	$\frac{2}{(1-t)(2-t)}$	$\frac{6}{(1-t)(2-t)(3-t)}$
	$\frac{\Gamma(13)}{\Gamma(14)} = \frac{1}{13}$		

$X_{(n)}$ & Y_n have the same MGFs

$\forall n$ and $\forall t$, so they have the same distributions ie $X_{(n)} \overset{d}{=} Y_n$

2a) $X_i \sim Pois(\lambda)$ $\quad f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$Y_i = X_i X_{2i} \quad S_n = \sum_{i=1}^n Y_i = \sum_{i=1}^n X_i X_{2i}$

$$E(S_n) = E\left(\sum_{i=1}^n X_i X_{2i}\right) = \sum E(X_i X_{2i})$$

$$= \sum_{i=1}^n E(X_i) E(X_{2i})$$

$$= n(\lambda \cdot \lambda) = \boxed{n\lambda^2}$$

2b) $Var(S_n) = \sum_{i=1}^n Var(Y_i) = n\, Var(Y_i)$

$\leq Cn$

$\forall n \geq 1$

$$= n\left[E(Y_i^2) - E(Y_i)^2\right]$$

$$n\left[E(X_i^2 X_{2i}^2) - (\lambda^2)^2\right]$$

$$n\left[E(X_i^2)E(X_{2i}^2) - \lambda^4\right]$$

$$n\left[(\lambda+\lambda^2)^2 - \lambda^4\right]$$

use

$E(S_n - E(S_n))^2$

$E(S_n - n\lambda^2)^2$

$E(S_n^2 - 2n\lambda^2 S_n + n^2\lambda^4)$

$E\left[\left(\sum X_i X_{2i}\right)^2\right] - 2n\lambda^2(n\lambda^2) + n^3\lambda^4$

$E\left(\sum_{i=1}^n \sum_{j=1}^n X_i X_{2i} X_j X_{2i}\right)$

use $E(X^2) = Var(X) + E(X)^2 = \lambda + \lambda^2$

$$n(\lambda^2 + 2\lambda^3 + \lambda^4 - \lambda^4) = n(\lambda^2 + 2\lambda^3) = Var(S_n)$$

use $c = \lambda^2 + 2\lambda^3 + 1$ $\therefore$ $Var(S_n) \le n(\lambda^2 + 2\lambda^3 + 1)$

$a_n = n\lambda^2$ $\quad E\left(\frac{S_n}{n\lambda^2}\right) = \frac{E(S_n)}{n\lambda^2} = \frac{n\lambda^2}{n\lambda^2} = 1$

asym unbiased

$$Var\left(\frac{S_n}{n\lambda^2}\right) = \frac{1}{n^2\lambda^4} Var(S_n) = \frac{n(\lambda^2 + 2\lambda^3)}{n^2\lambda^4}$$

$$= \frac{\lambda^2 + 2\lambda^3}{n\lambda^4} \to 0 \text{ as } n \to \infty$$

$\therefore$ by thm w/ asy unbiased it is consistent

$$\therefore \frac{S_n}{n\lambda^2} \xrightarrow{P} 1$$

$$MSE\left(\frac{S_n}{n\lambda^2}\right) = Var\left(\frac{S_n}{n\lambda^2}\right) + \underbrace{0}_{\text{unbiased}} = \frac{\lambda^2 + 2\lambda^3}{n\lambda^4} \to 0$$

$= E\left(\left(\frac{S_n}{n\lambda^2} - 1\right)^2\right) \to 0$ $\therefore$ conv in rth mean

$X_i \sim Exp\left(rate = \frac{\theta_i}{\lambda}\right)$ $\qquad$ $Y_i \sim Exp\left(rate = \frac{1}{\theta_i \lambda}\right)$

$f_x = \frac{\theta_i}{\lambda} e^{-\frac{\theta_i}{\lambda} x}$ $\qquad$ $f_y = \frac{1}{\theta_i \lambda} e^{-\frac{y}{\theta_i \lambda}}$

$$f_{x,y} = \prod_{i=1}^{n} \frac{\theta_i}{\lambda} e^{-\frac{\theta_i}{\lambda} x_i} \mathbb{1}_{(0,\infty)}(x_i) \prod_{i=1}^{n} \frac{1}{\theta_i \lambda} e^{-\frac{y_i}{\theta_i \lambda}} \mathbb{1}_{(0,\infty)}(y_i)$$

$$= \frac{\prod \theta_i}{\prod \theta_i} \frac{1}{\lambda^{2n}} e^{-\sum \frac{\theta_i}{\lambda} x_i} e^{-\sum \frac{y_i}{\theta_i \lambda}} \prod \mathbb{1}(x_i) \mathbb{1}(y_i)$$

d(x) exp family class => comp & suff for λ

$$= \lambda^{-2n} \underbrace{e^{-\frac{1}{\lambda} \sum \left(\theta_i x_i + \frac{y_i}{\theta_i}\right)}} \prod \mathbb{1}(x_i) \mathbb{1}(y_i)$$

$$\ell = -2n \log \lambda - \frac{1}{\lambda} \sum_{i=1}^{n} \left(\theta_i x_i + \frac{y_i}{\theta_i}\right) + \log \sum \mathbb{1}(x_i) + \log \sum (y_i)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{-2n}{\lambda} + \frac{1}{\lambda^2} \sum \left(\theta_i x_i + \frac{y_i}{\theta_i}\right) \overset{set}{=} 0$$

$$\frac{\sum \left(\theta_i x_i + \frac{y_i}{\theta_i}\right)}{\lambda^2} = \frac{2n}{\lambda}$$

b) when θ_i are known

$$\frac{\sum \left(\theta_i x_i + \frac{y_i}{\theta_i}\right)}{2n} = \hat{\lambda}_n$$

$$\frac{\partial}{\partial \theta_i} \left(\theta_i x_i + \frac{y_i}{\theta_i}\right) = x_i - \frac{y_i}{\theta_i^2} = 0$$

when not known, we minimize $\theta_i x_i + \frac{y_i}{\theta_i}$

$$\theta_i^2 = \frac{y_i}{x_i} \qquad \theta_i = \sqrt{\frac{y_i}{x_i}}$$

$$\sqrt{\frac{y_i}{x_i}}\, x_i + \frac{y_i}{\sqrt{y_i/x_i}} = 2\sqrt{y_i x_i} \quad \text{so} \quad MLE = \frac{2\sum\sqrt{x_i y_i}}{2n}$$

$$= \frac{\sum\sqrt{x_i y_i}}{n} = \overline{\sqrt{x_i y_i}}$$

b)
Find UMVUE

$S(\underline{x}) = \sum\left(\theta_i x_i + \frac{y_i}{\theta_i}\right)$ comp & suff for λ by exp family class

since $\hat{\lambda}_n$ is a function of $S(\underline{x})$, I just need it to be unbiased for it to be the UMVUE

$$E\left[\frac{\sum\left(\theta_i x_i + \frac{y_i}{\theta_i}\right)}{2n}\right] = \frac{1}{2n}\left(\sum E(\theta_i x_i) + \sum E\left(\frac{y_i}{\theta_i}\right)\right)$$

$$= \frac{1}{2n}\left(\sum_{i=1}^{n}\lambda + \sum\lambda\right) = \frac{1}{2n}(2n\lambda)$$

$$= \lambda$$

unbiased ✓

$X \sim Exp(\lambda) \quad \mu = \frac{1}{\lambda}$

$cX \sim Exp\left(\frac{\lambda}{c}\right) \quad \mu = \frac{c}{\lambda}$

$\theta_i x_i \sim Exp\left(\frac{\theta_i}{\lambda}\frac{1}{\theta_i}\right)$

$= Exp\left(\frac{1}{\lambda}\right) \quad \mu = \lambda$

or use $S(\underline{X}) \sim Gamma\left(2n, \frac{1}{\lambda}\right)$

sum of 2n ind x_i / y_i

$\frac{y_i}{\theta_i} \sim Exp\left(\frac{1}{\lambda\theta_i}\theta_i\right)$

$= Exp\left(\frac{1}{\lambda}\right) \quad \mu = \lambda$

$\therefore E(S) = \frac{2n}{1/\frac{1}{\lambda}} = 2n\lambda$

and $E\left(\frac{S}{2n}\right) = \lambda$

c) $\text{efficiency} = \dfrac{CRLB}{Var}$

$$\frac{\partial \ell}{\partial \lambda} = \frac{-2n}{\lambda} + \frac{1}{\lambda^2}\sum\left(\theta_i x_i + \frac{y_i}{\theta_i}\right)$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = \frac{2n}{\lambda^2} - \frac{2}{\lambda^3}\sum\left(\theta_i x_i + \frac{y_i}{\theta_i}\right)$$

$$I_n(\lambda) = -E\left[\frac{2n}{\lambda^2} - \frac{2}{\lambda^3}\sum\left(\theta_i x_i + \frac{y_i}{\theta_i}\right)\right]$$

$$= \frac{-2n}{\lambda^2} + \frac{2}{\lambda^3}\left(2n\lambda\right) = \frac{2n}{\lambda^2}$$

$$CRLB = \frac{1}{I_n} = \frac{\lambda^2}{2n}$$

$$Var\left(\frac{S}{2n}\right) = \frac{1}{4n^2}Var(S) = \frac{1}{4n^2}\,2n\lambda^2 = \frac{\lambda^2}{2n}$$

$\uparrow$

$\text{Gamma}(2n, \frac{1}{\lambda})$

$Var = \dfrac{\alpha}{\beta^2}$

ratio of MSE

relative efficiency ... MLE is the UMVUE

so $\nearrow$ = 1 $\forall n$ and their variances decrease

and both achieve the CRLB

5a) $X, Y \sim N(0,1)$

$W = \min(X, Y)$

$P(W < w) = 1 - P(W > w)$

$= 1 - P(X \text{ and } Y > w)$

$= 1 - P(X > w) P(Y > w)$

iid $= 1 - P(X > w)^2$

$= 1 - [1 - P(X < w)]^2$

$F_W(w) = 1 - [1 - F_X(w)]^2$

$f_W = 2(1 - F_X(w))(-f_X(w))$

$= 2(F_X(w) - 1) f_X(w)$

$\chi^2(1) = \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

$\frac{1}{\Gamma(\frac{1}{2})} \frac{1}{2}^{1/2} x^{\frac{1}{2}-1} e^{-\frac{1}{2}x}$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{x}} e^{-x/2}$

$P(W^2 < w) = P(|W| < \sqrt{w}) = P(-\sqrt{w} < W < \sqrt{w})$