Graphical CSS Code Transformation Using ZX Calculus

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[5] Quantinuum

[6] Perimeter Institute

$[\![4, 2, 2]\!]$
square code

$[\![7, 1, 3]\!]$
Steane code

$[\![15, 1, 3]\!]$
quantum
Reed-Muller code

$[\![5, 1, 3]\!]$ code

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Stabilizer codes

Stabilizers an Abelian subgroup $\mathcal{S} < \mathcal{P}_n$

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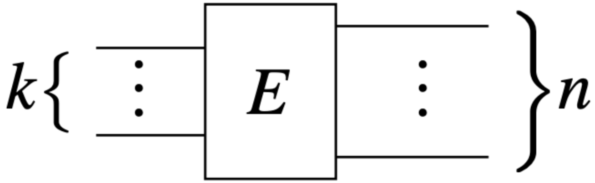
(Clifford) Encoders

$$\begin{aligned} E \quad : \quad \mathcal{H}^{\otimes k} &\to \mathcal{C} \\ |\psi\rangle &\hookrightarrow |\overline{\psi}\rangle := E|\psi\rangle \end{aligned}$$

logical states



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(Clifford) Encoders $E \cdot E^\dagger : \mathcal{U}\left(\mathcal{H}^{\otimes k}\right) \to \mathcal{U}(\mathcal{C})$

$U \hookrightarrow \overline{U} := EUE^\dagger$ **logical operators**

Logical operators

$$\overline{U} := EUE^\dagger$$

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Logical Paulis $\overline{P}_i = EP_iE^\dagger, \forall P_i \in \mathcal{P}_k$

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$$\Rightarrow EU = \overline{U}E$$

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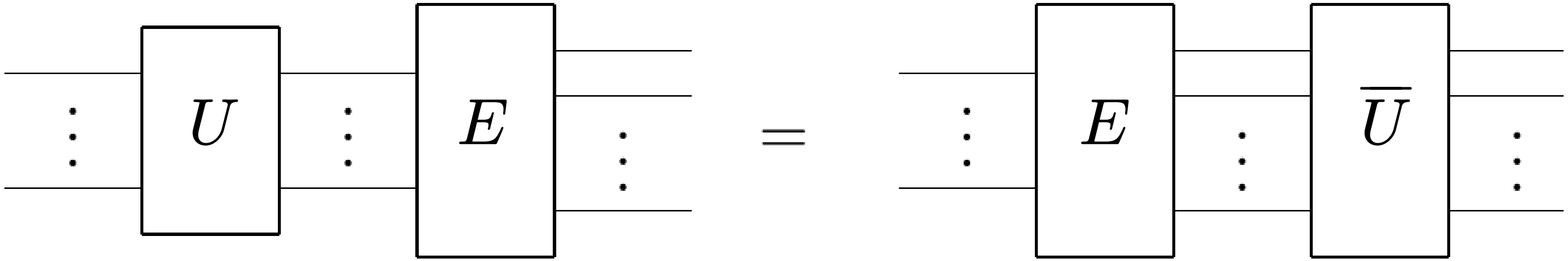

Encoders as ZX diagrams

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CSS codes stabilizer codes whose stabilizers can be divided into 2 types: **X-type** or **Z-type**, i.e.,

$$\mathcal{S} = \{\mathcal{X}_1, \mathcal{X}_2 ...\} \cup \{\mathcal{Z}_1, \mathcal{Z}_2 ...\}$$

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Every CSS code encoder is a phase-free ZX diagram.

Kissinger, A. (2022). Phase-free ZX diagrams are CSS codes (... or how to graphically grok the surface code). arXiv preprint arXiv:2204.14038.

Encoders as ZX diagrams

Stabilizers of the **Steane code**:

$$
\begin{array}{ll}
Z_1Z_3Z_5Z_7 & X_1X_3X_5X_7 \\
Z_2Z_3Z_6Z_7 & X_2X_3X_6X_7 \\
Z_4Z_5Z_6Z_7 & X_4X_5X_6X_7
\end{array}
$$

Logical Pauli operators:

$$\overline{Z} = Z_1Z_4Z_5 \qquad \overline{X} = X_1X_4X_5$$

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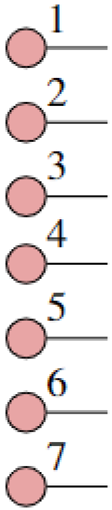

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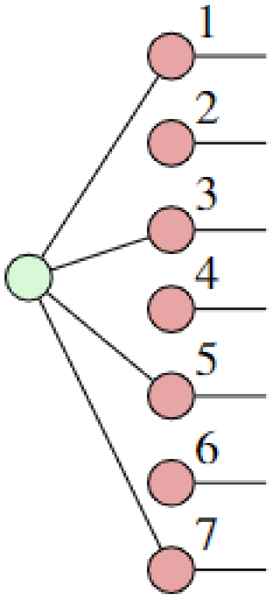

Encoders as ZX diagrams

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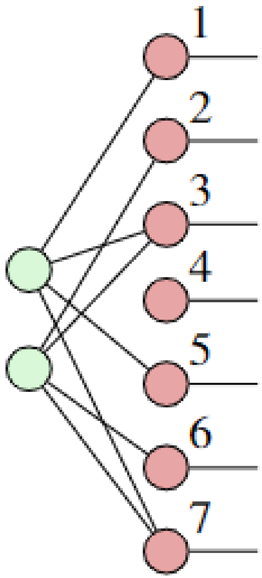

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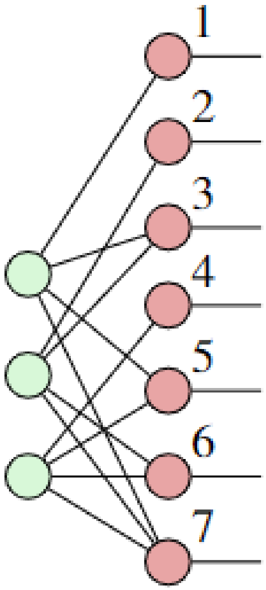

Encoders as ZX diagrams

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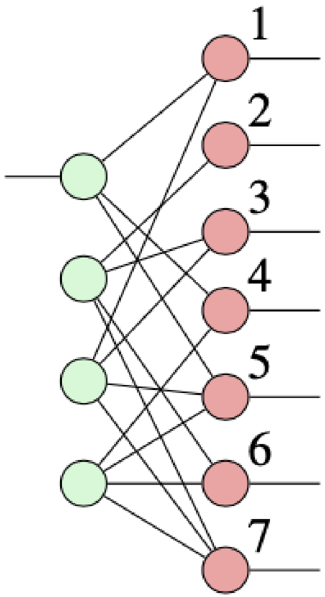
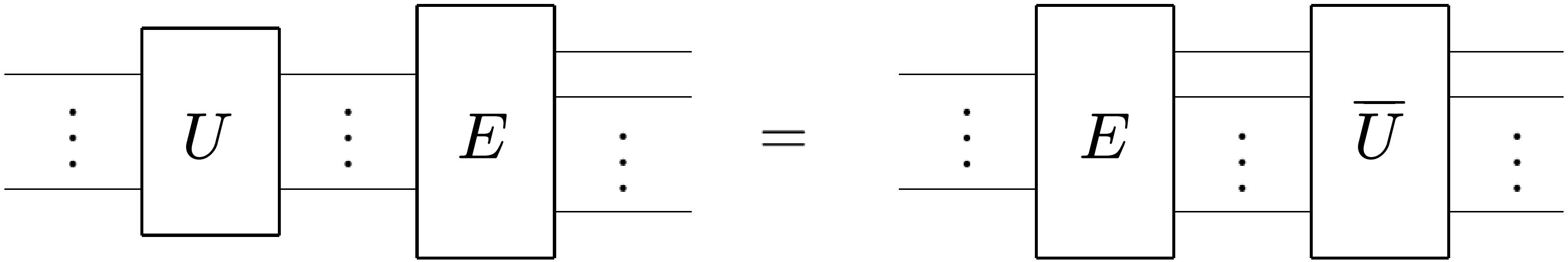


Pushing Through the Encoder (PTE)



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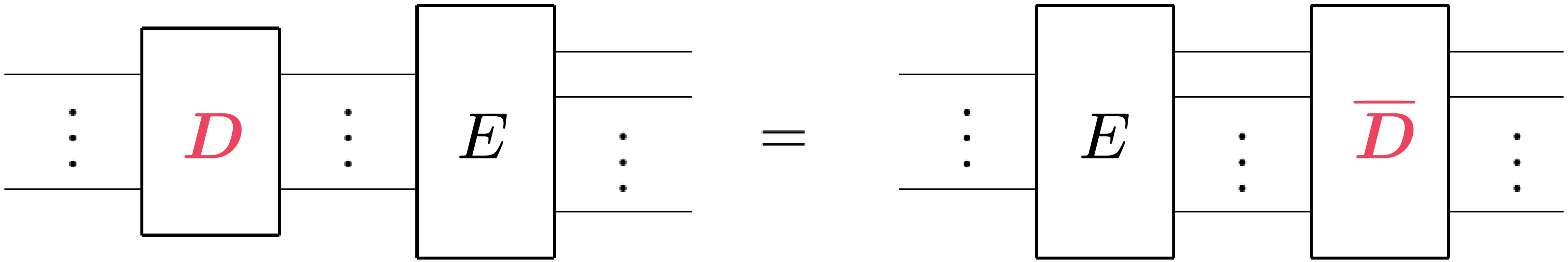
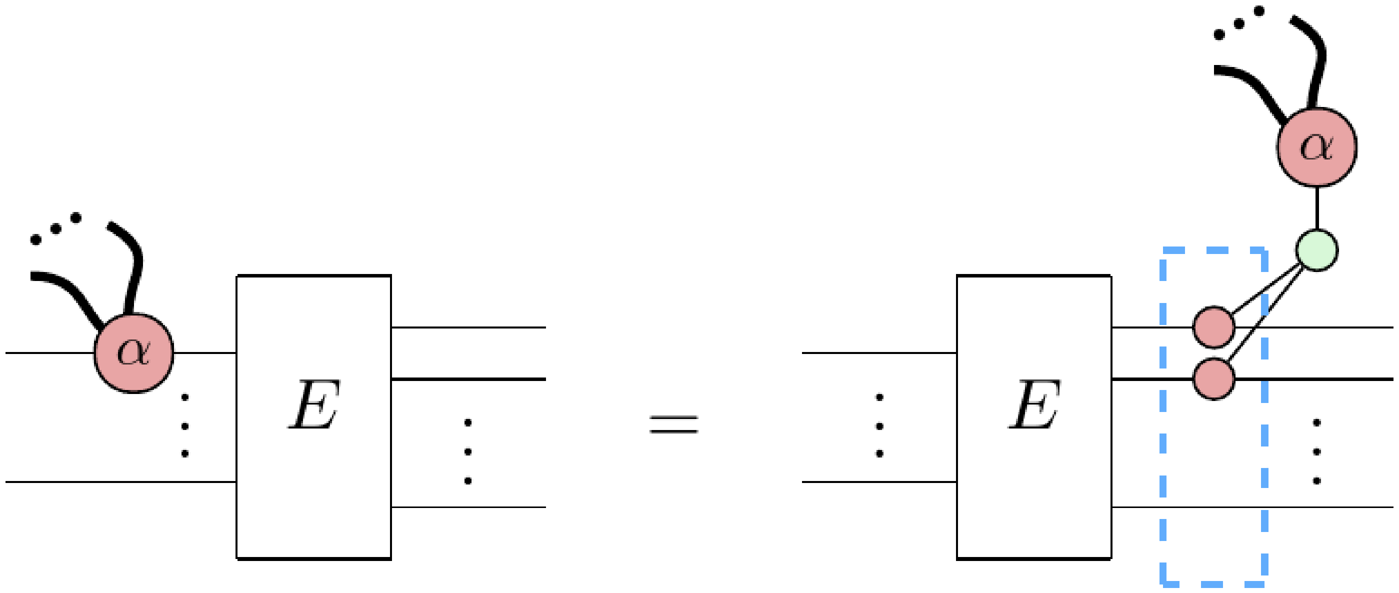


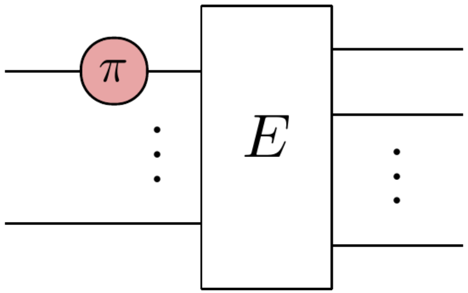
Given a ZX diagram D, what is the corresponding diagram $\overline{D}$, such that the above equation holds?

Lemma: PTE



assuming $\overline{X}_1 = X_1 X_2$

Example: $\overline{X}_1$



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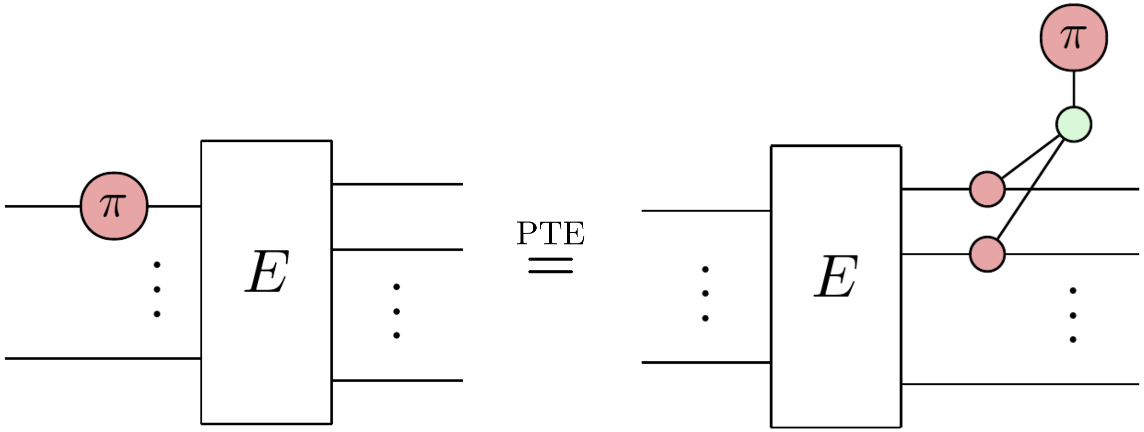

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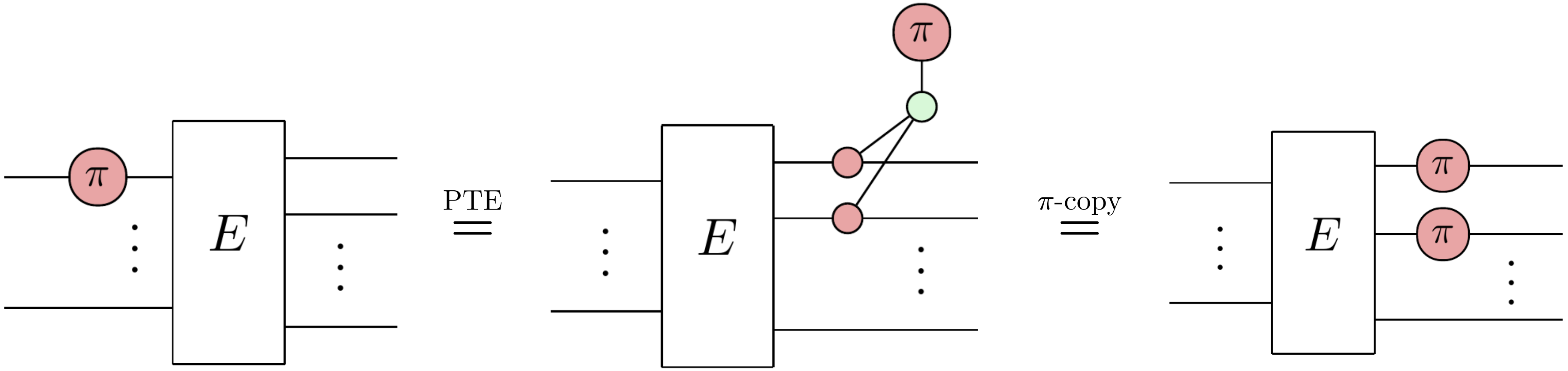

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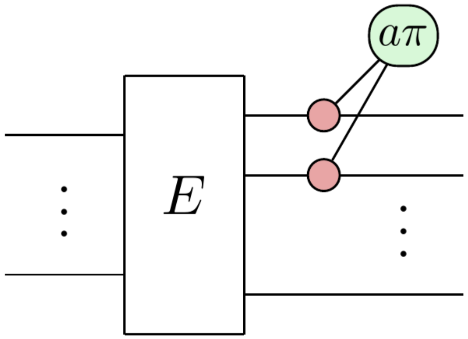
Example: $X_1 X_2$ measurement



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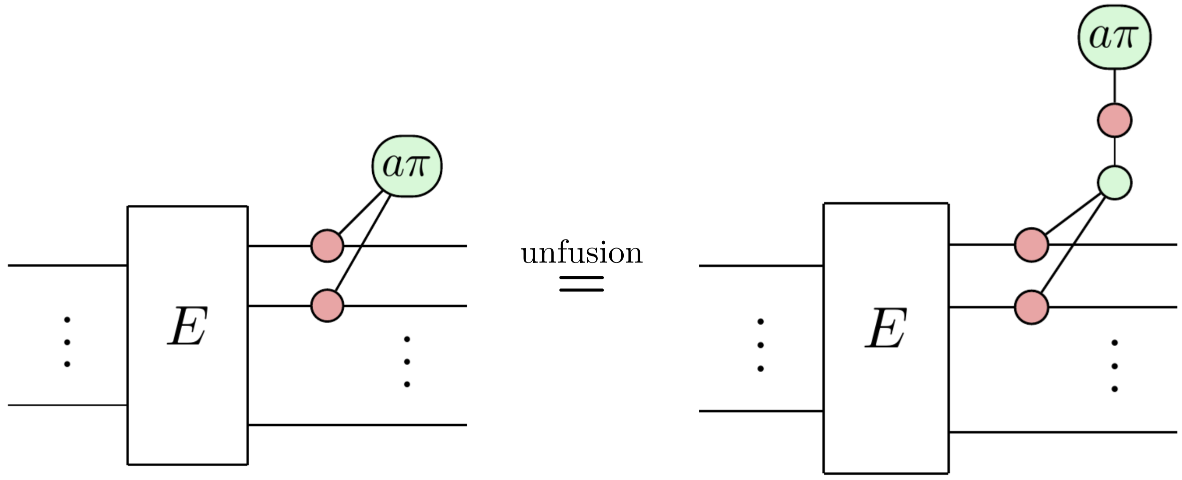

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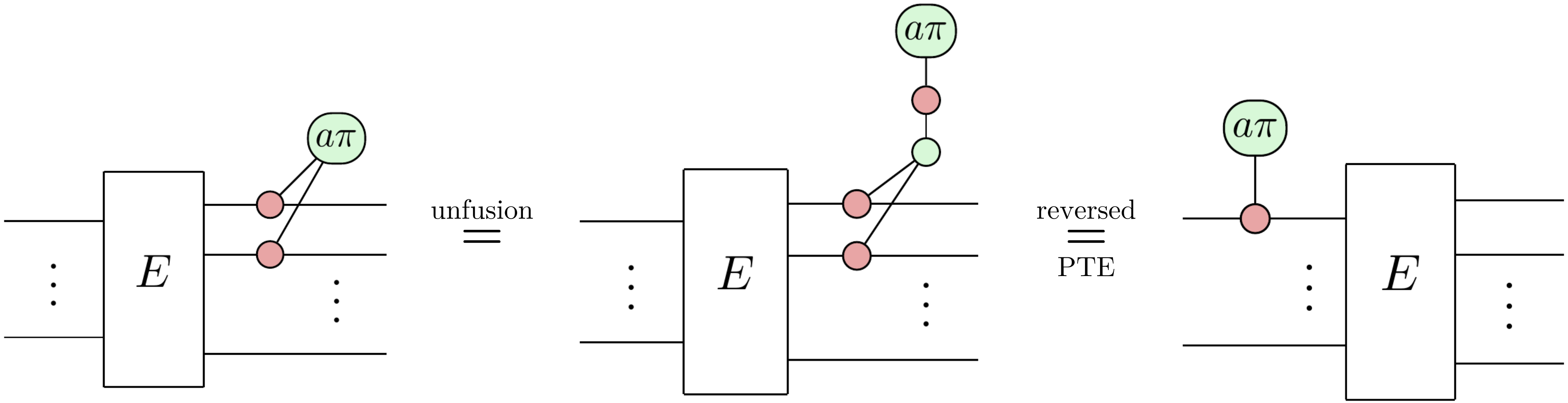
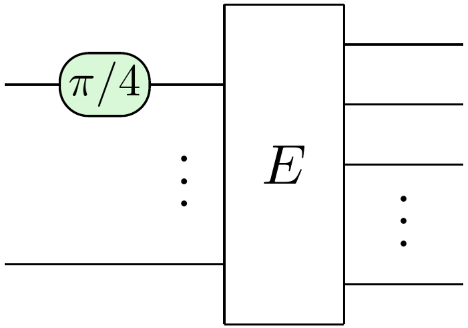


assuming $\overline{X}_1 = X_1 X_2$

Example: $\overline{T}$ gate



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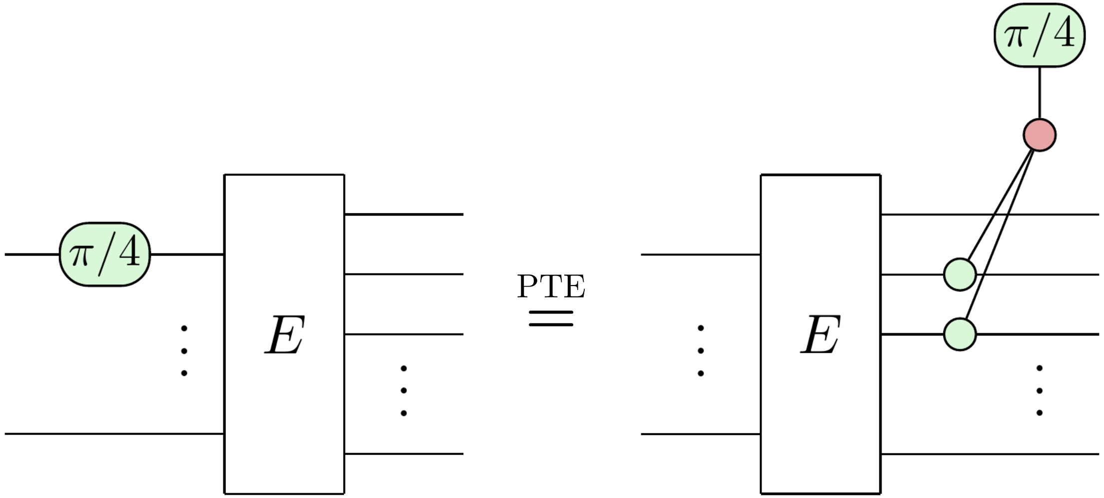

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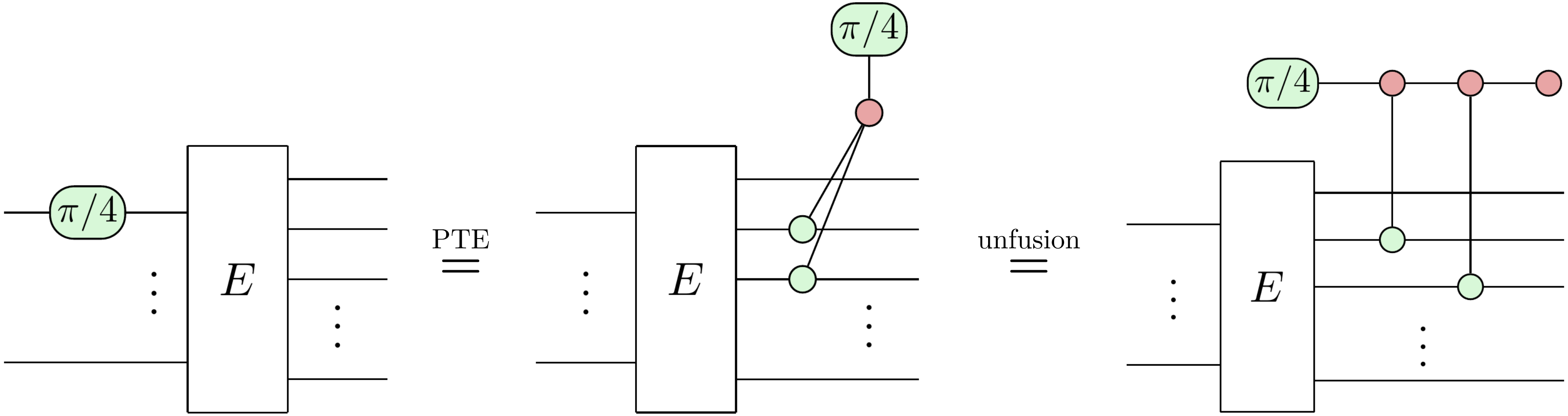

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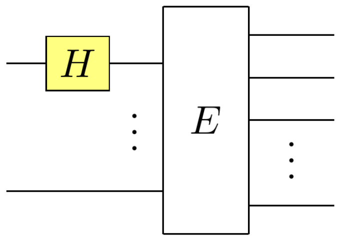
Example: $\overline{H}$ gate



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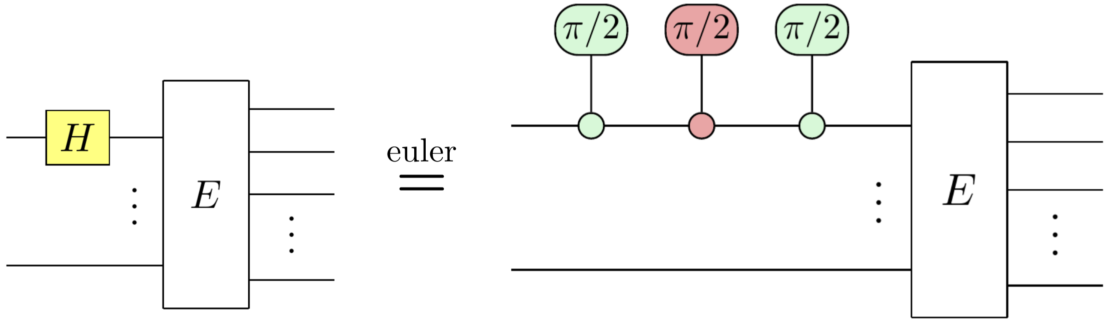

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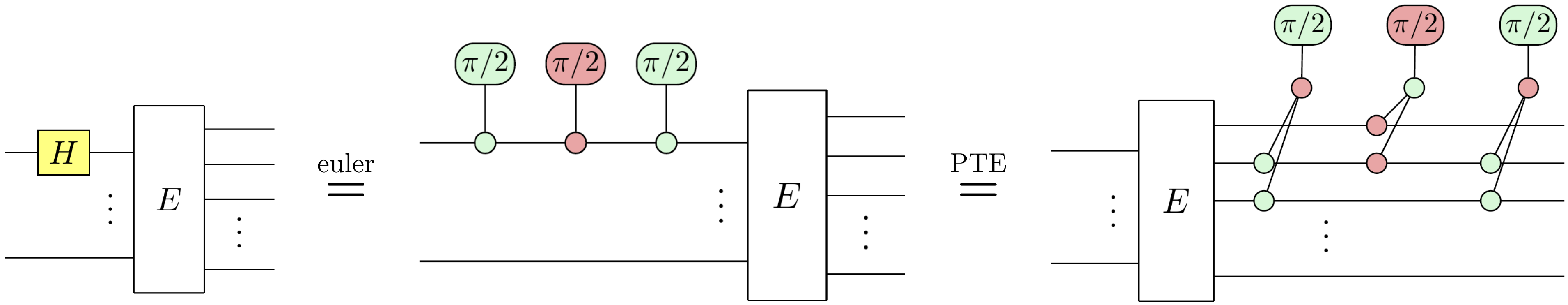

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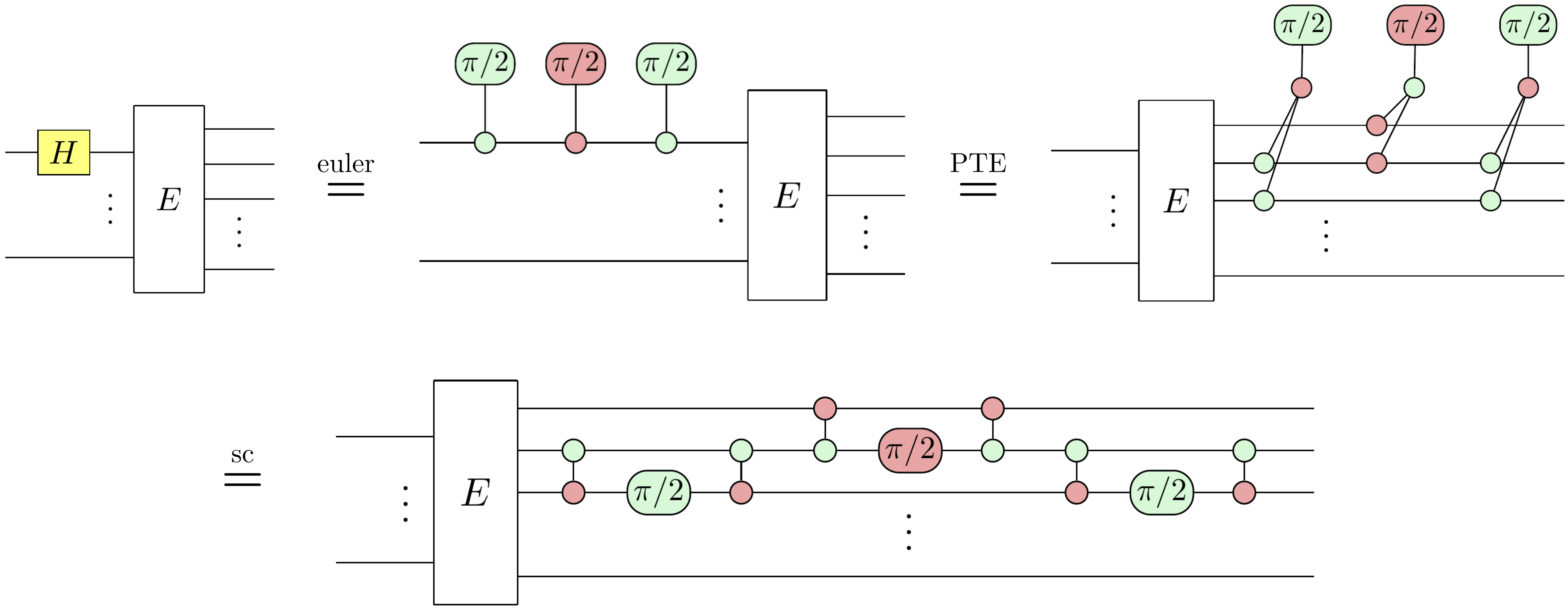


Code morphing

Code morphing

Given a **parent code** $\mathcal{C}_{\text{parent}}$, with stabilizers $\mathcal{S}$ and physical qubits Q,

choose a subset $R \subset Q$.

child code $\mathcal{C}_{\text{child}}$ whose stabilizers are

$$\mathcal{S}_{\text{child}} := \{S \in \mathcal{S} : \operatorname{supp}(S) \subset R\}.$$

Code morphing

Consider the encoders E_{parent} and E_{child}, which are Clifford,

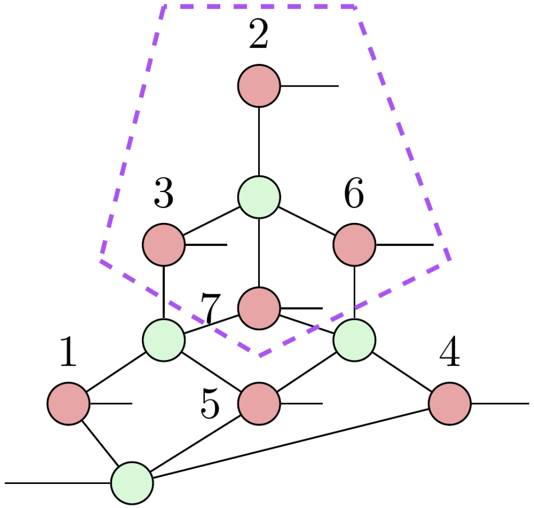
morphed code $\mathcal{C}_{\text{morphed}}$ whose encoder is

$$E_{\text{morphed}} := \left(I^{\otimes|Q\setminus R|} \otimes E_{\text{child}}^{\dagger}\right) E_{\text{parent}}.$$

Code morphing

Given an encoder E_{parent} and a subset $R \subset Q$,

$R = \{2, 3, 6, 7\}$



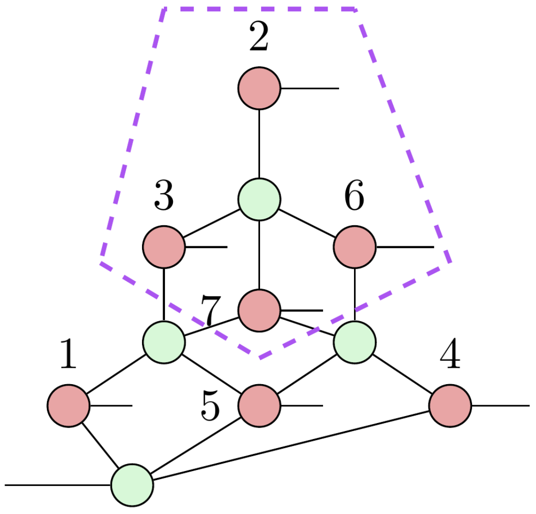
E_{parent}

Code morphing

Given an encoder E_{parent} and a subset $R \subset Q$,

1. Unfuse all green spiders which are supported both on R and $Q \setminus R$.

$R = \{2, 3, 6, 7\}$

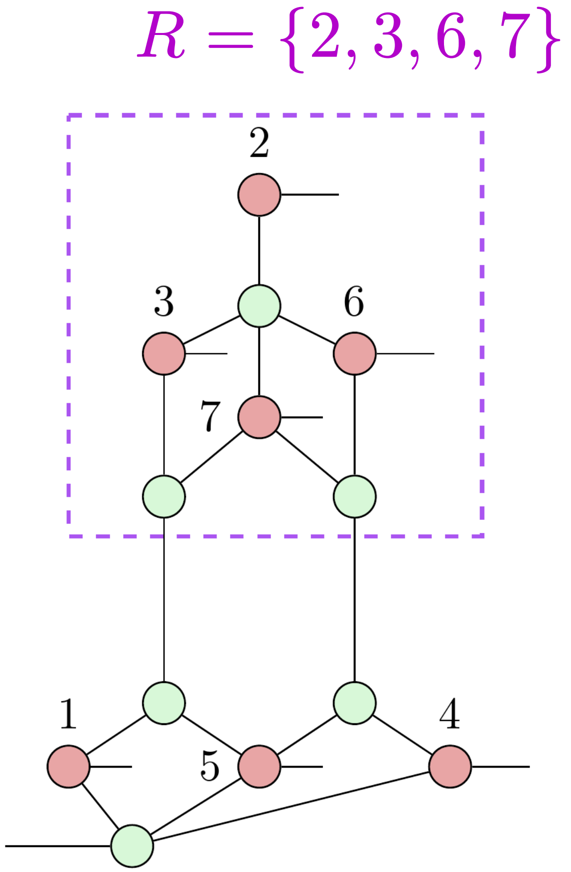


E_{parent}

Code morphing

Given an encoder E_{parent} and a subset $R \subset Q$,

2. Add an identity red spider between each pair of unfused green spiders.

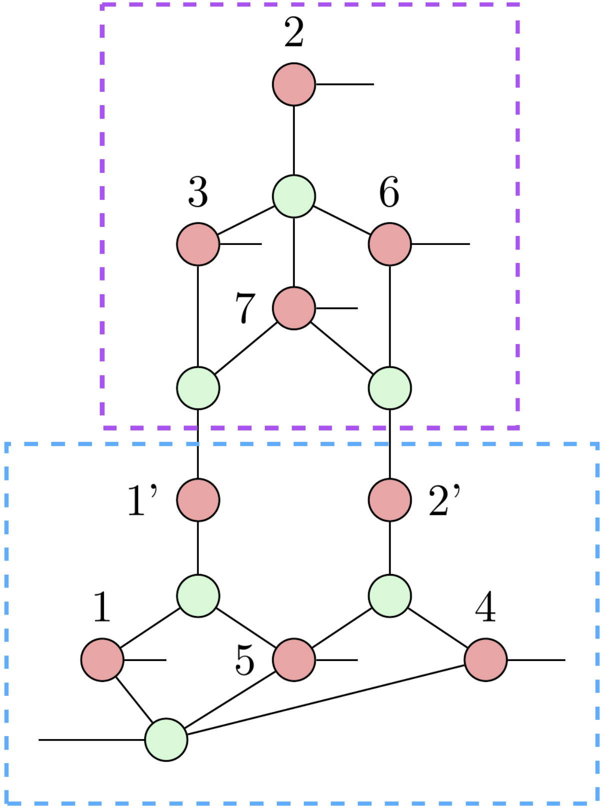


Code morphing

$R = \{2, 3, 6, 7\}$

Given an encoder E_{parent} and a subset $R \subset Q$,

3. let E_{child} be the subdiagram enclosed by R;
   let E_{morphed} be the subdiagram enclosed by $Q \setminus R$



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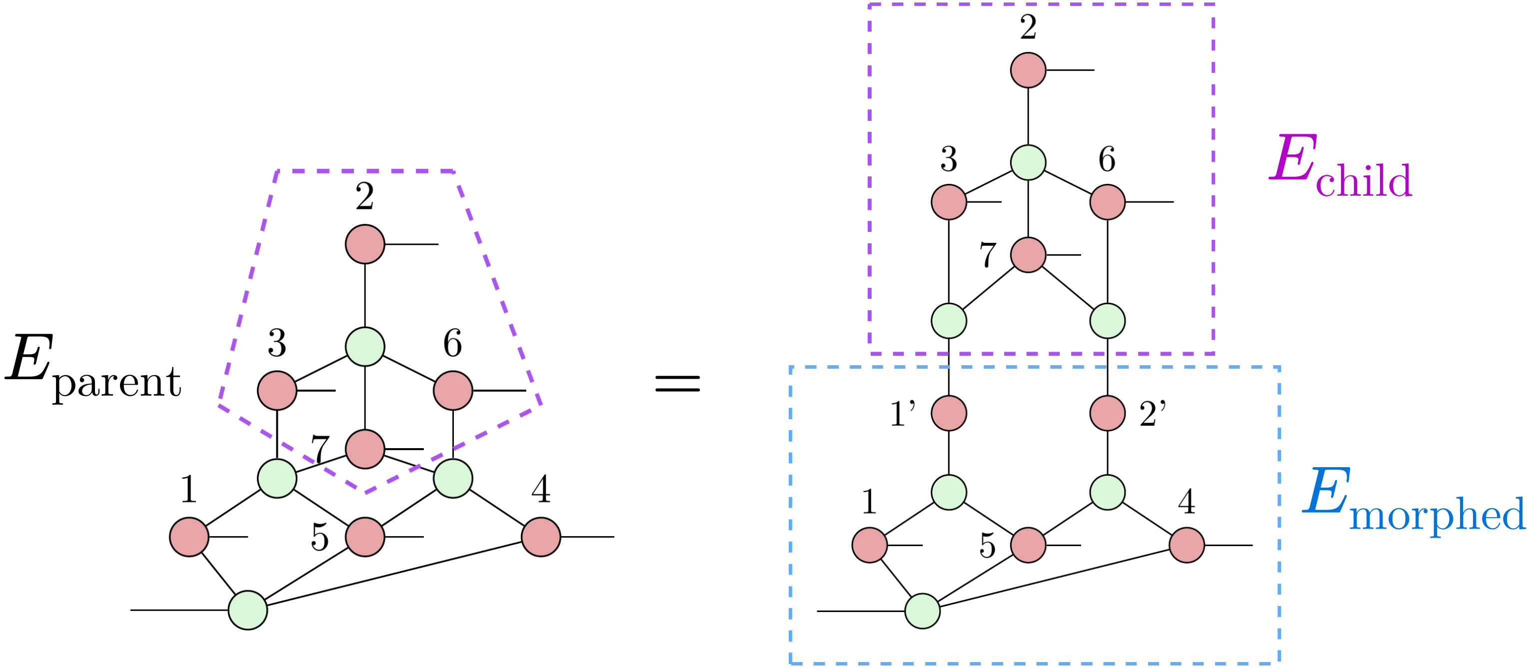

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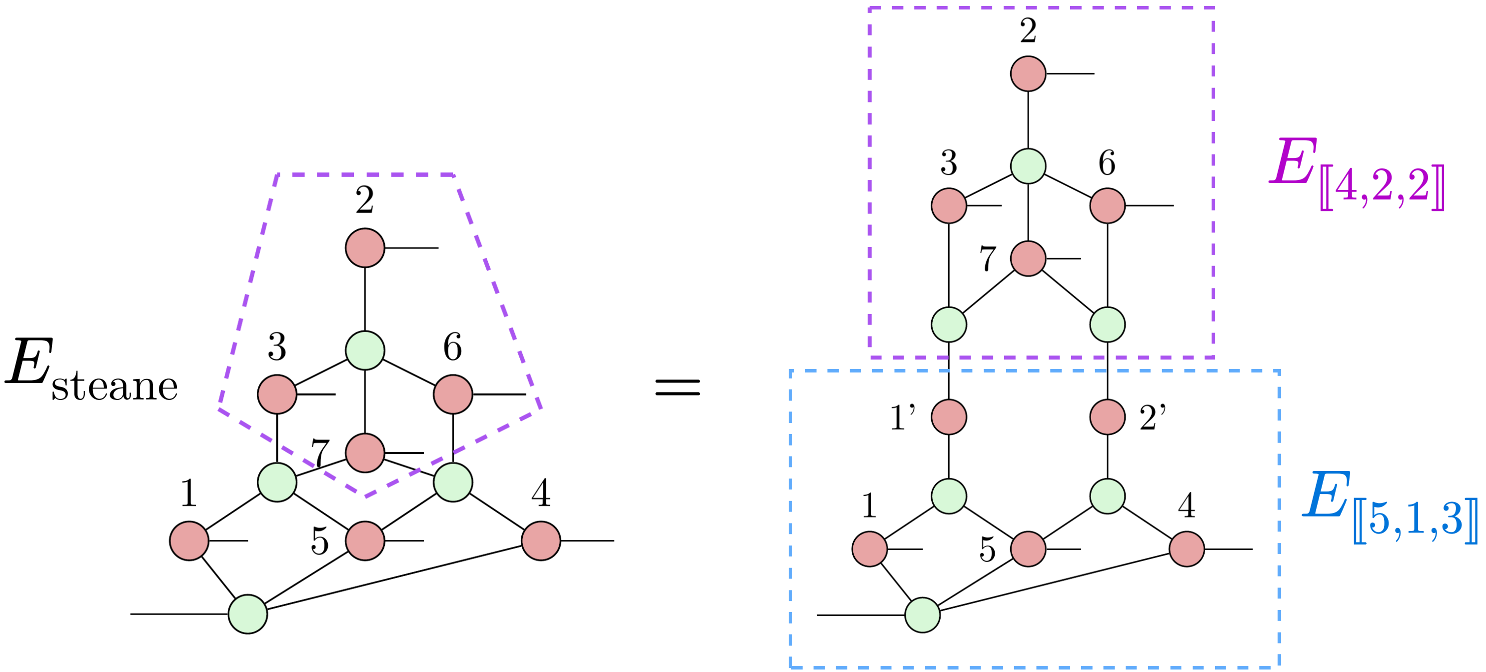
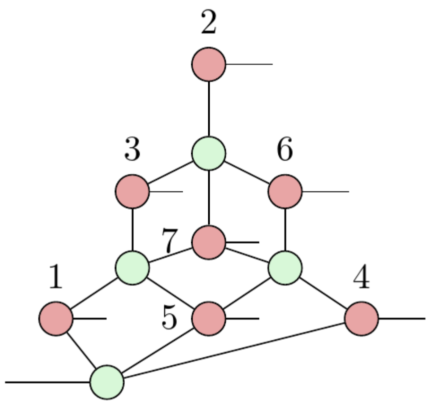


Code morphing

$R = \{4, 5, 6, 7\}$



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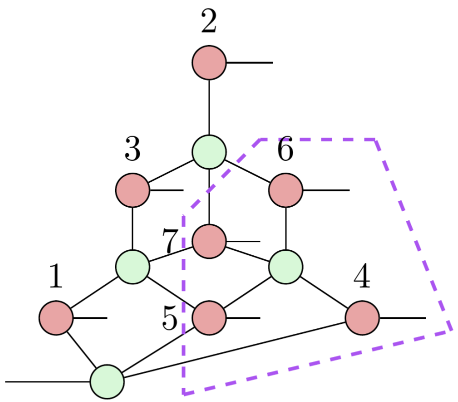

E_{steane}

Code morphing

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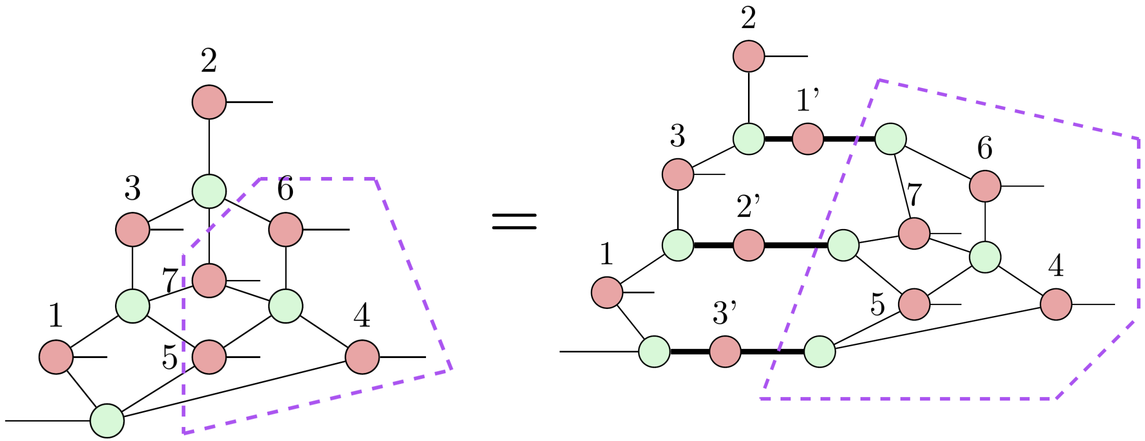

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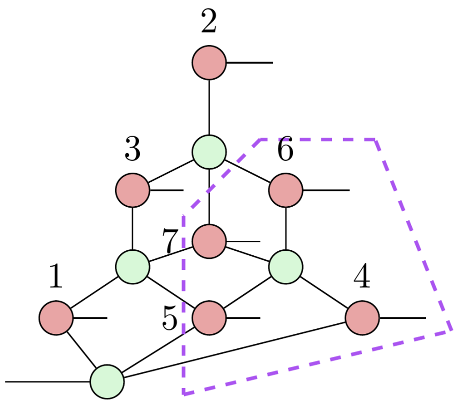
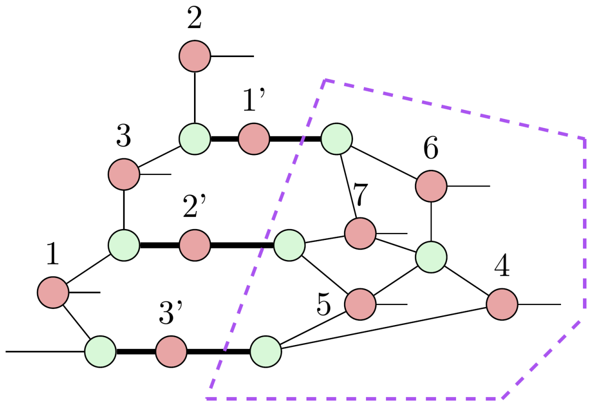


$=$



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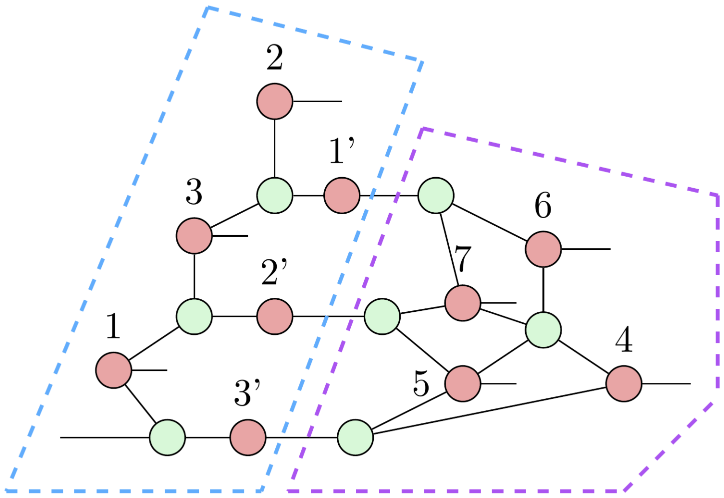


E_{steane} $E_{[\![6,1,1]\!]}$ $E_{[\![4,3,1]\!]}$

Steane & quantum Reed-Muller(QRM) code

	Steane	QRM
qubits	7	15
# stabilizers	6	14

Steane & quantum Reed-Muller(QRM) code

	Steane	ExSteane	QRM
qubits	7	15	15
# stabilizers	6	14	14

$E_{\text{ex}} = E_{\text{steane}} \otimes |\Psi\rangle$, where

$$|\Psi\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes (E_{\text{steane}}|0\rangle) + |1\rangle \otimes (E_{\text{steane}}|1\rangle))$$

Anderson, J. T., Duclos-Cianci, G., & Poulin, D. (2014). Fault-tolerant conversion between the steane and reed-muller quantum codes. Physical review letters, 113(8), 080501.

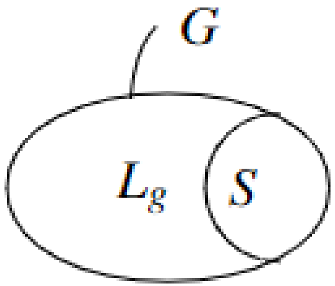
Quantum subsystem code

Gauge group any subgroup $\mathcal{G} < \mathcal{P}_n$

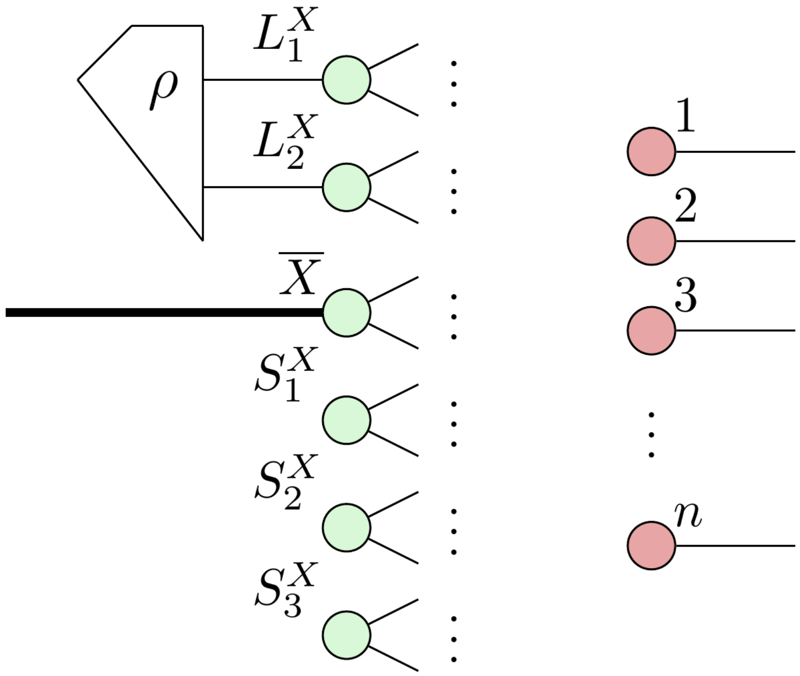
Stabilizer group $\mathcal{S} := \mathcal{N}(\mathcal{G}) \cap \mathcal{G} = \{S \in \mathcal{G} : SG = GS, \forall G \in \mathcal{G}\}$

Gauge operators $\mathcal{L}_g := \mathcal{G} \,/\, \mathcal{S}$

$$\cong \langle L_1^X, L_1^Z, \ldots, L_t^X, L_t^Z \rangle < \mathcal{P}_n$$



Subsystem code encoders as ZX diagrams

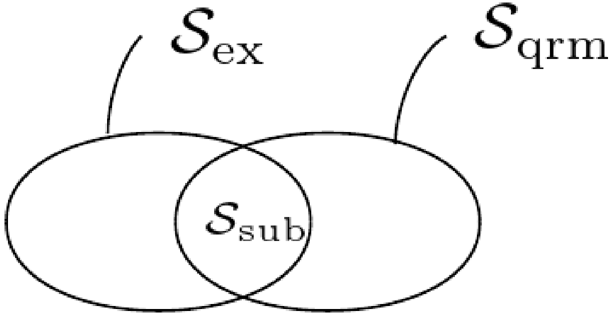


The $[\![15, 1, 3, 3]\!]$ subsystem code

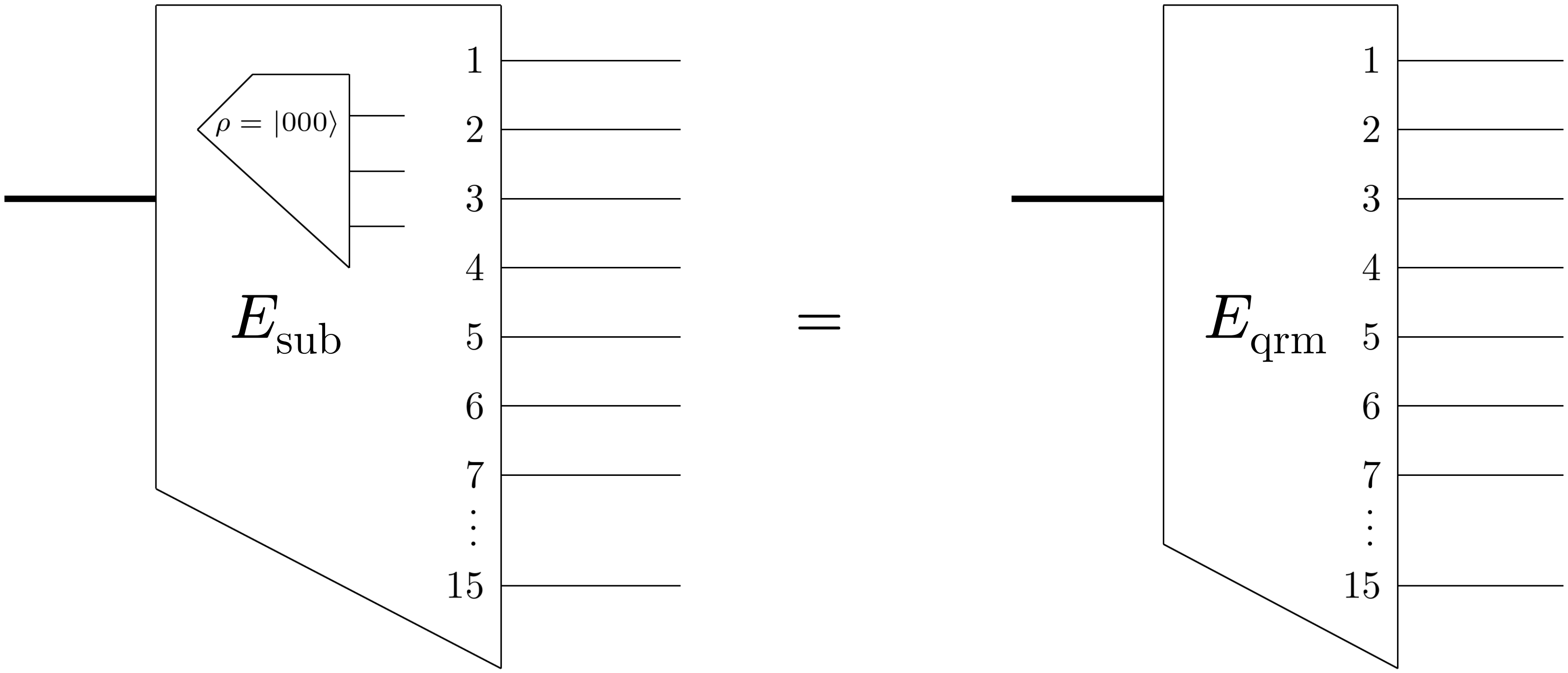
$$\mathcal{G}_{\mathrm{sub}} := \mathcal{S}_{\mathrm{ex}} \cup \mathcal{S}_{\mathrm{qrm}}$$

$$\Rightarrow \mathcal{S}_{\mathrm{sub}} = \mathcal{S}_{\mathrm{ex}} \cap \mathcal{S}_{\mathrm{qrm}}$$

$$\Rightarrow \mathcal{L}_g = \mathcal{S}_{\mathrm{ex}} \odot \mathcal{S}_{\mathrm{qrm}} = \langle L_1^X, L_2^X, L_3^X, L_1^Z, L_2^Z, L_3^Z \rangle$$



QRM



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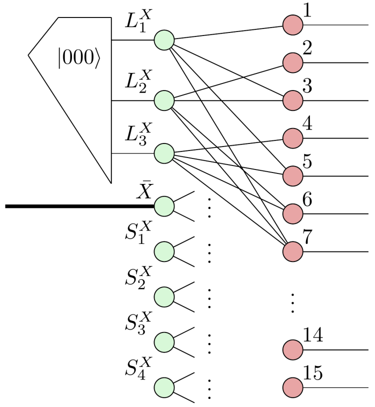

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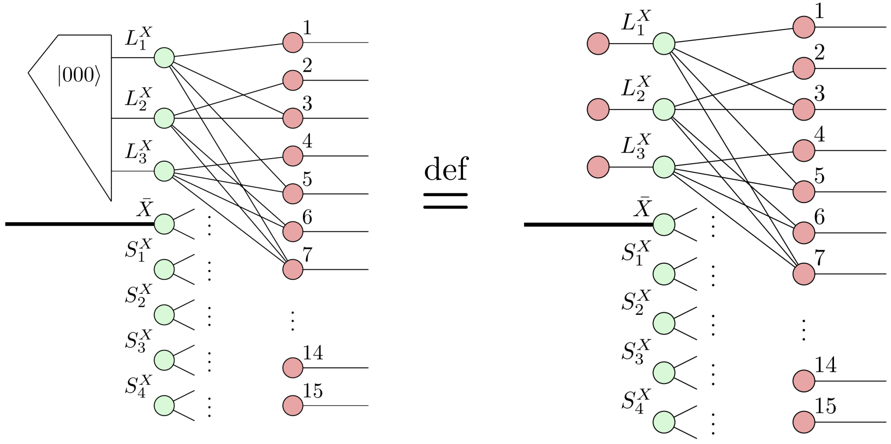

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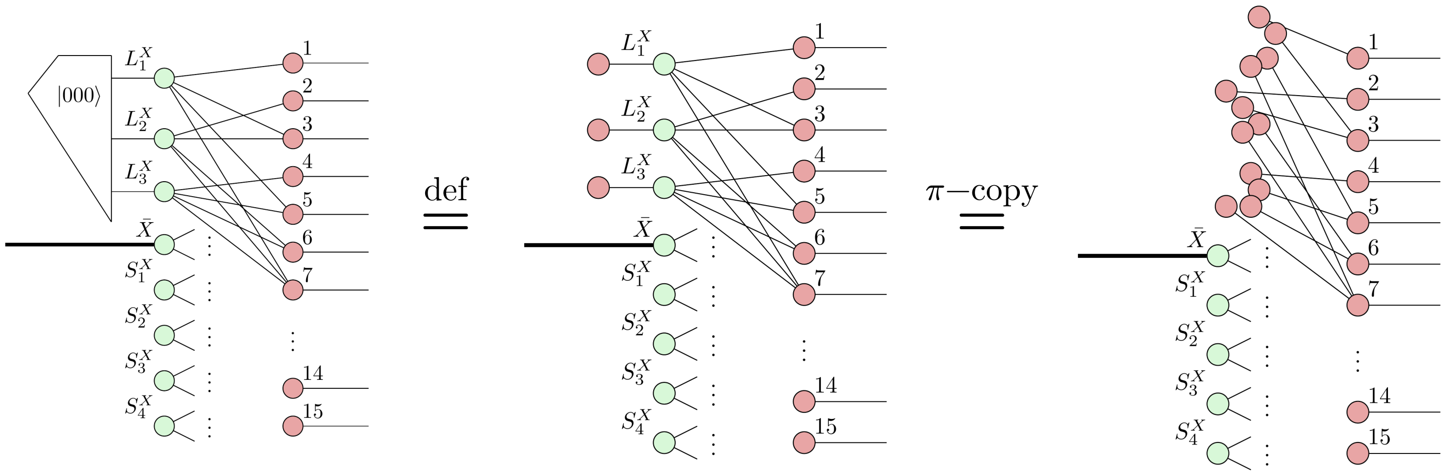
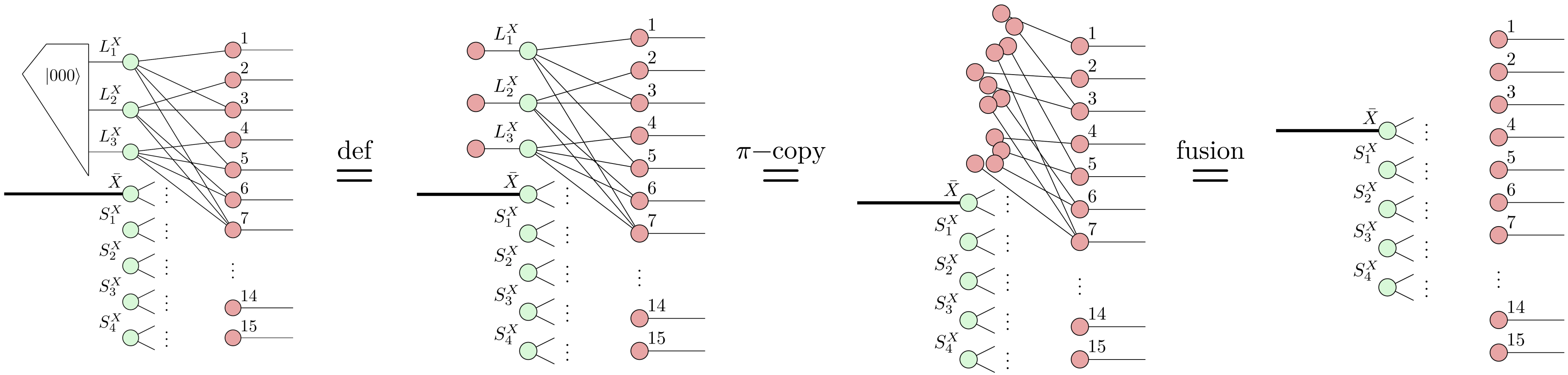
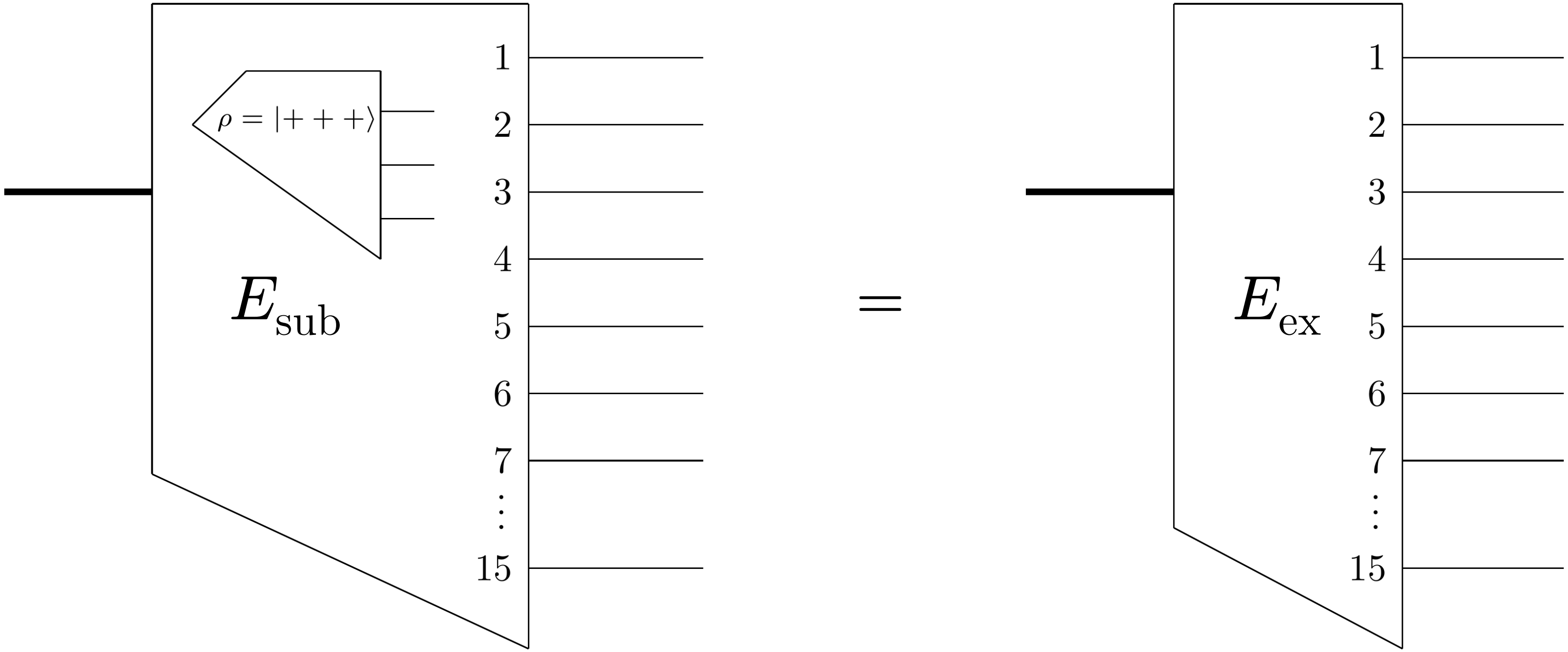


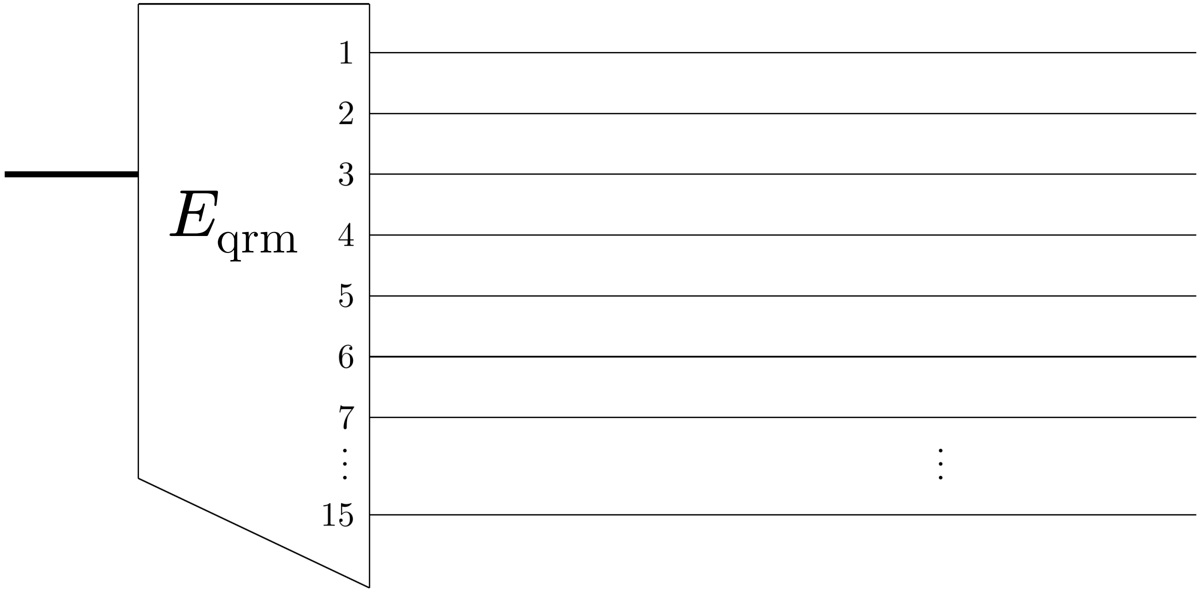
QRM


$|000\rangle$
L_1^X L_2^X L_3^X $\bar{X}$ S_1^X S_2^X S_3^X S_4^X
1 2 3 4 5 6 7 14 15
def
$\pi-$copy
fusion

ExSteane



Gauge fixing (QRM → ExSteane)



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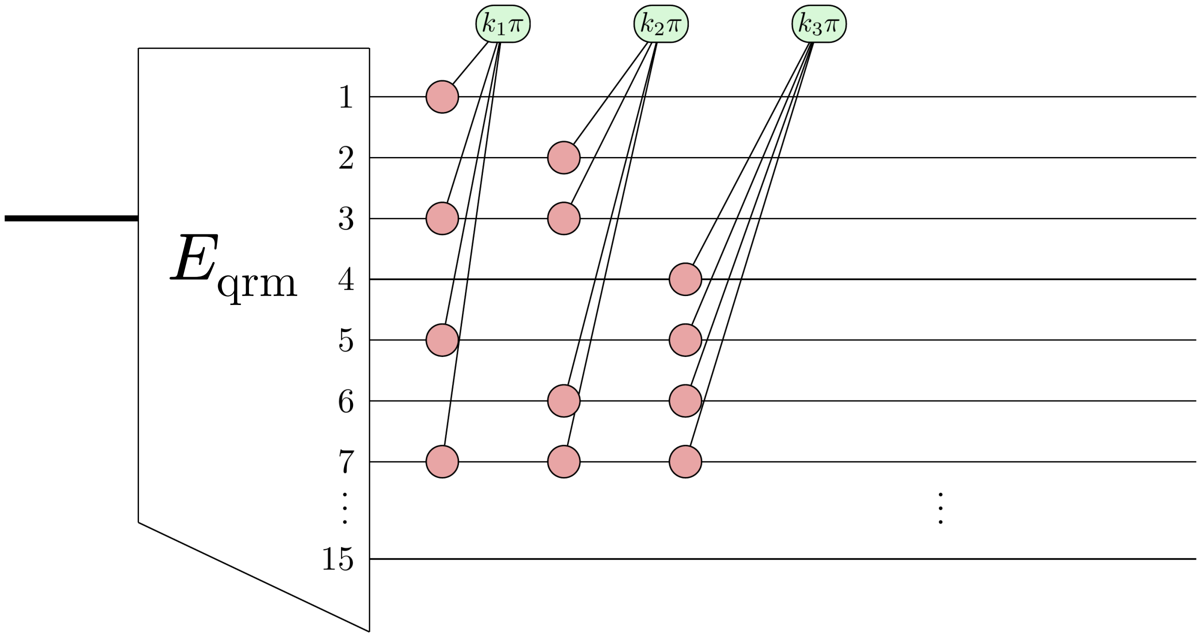

Step 1 measure gauge operators L_1^X, L_2^X, L_3^X, obtaining outcomes $k_1, k_2, k_3 \in \mathbb{Z}_2$.

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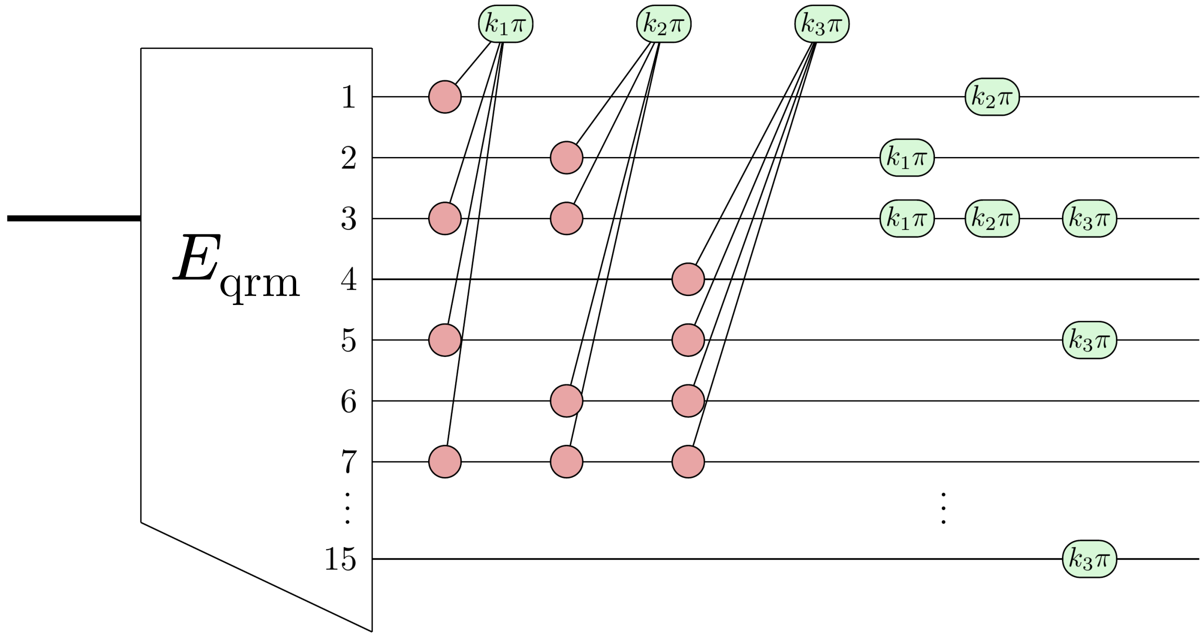

Step 2 For each $k_i = 1$, apply L_i^Z.

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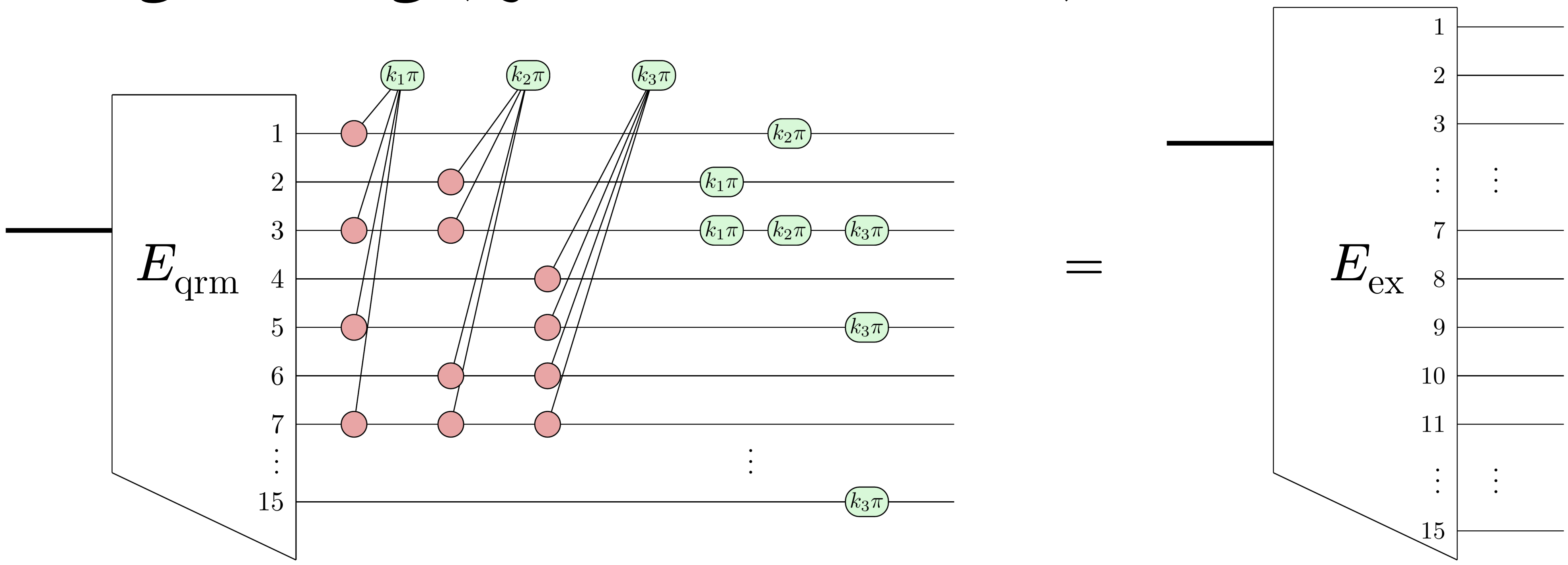
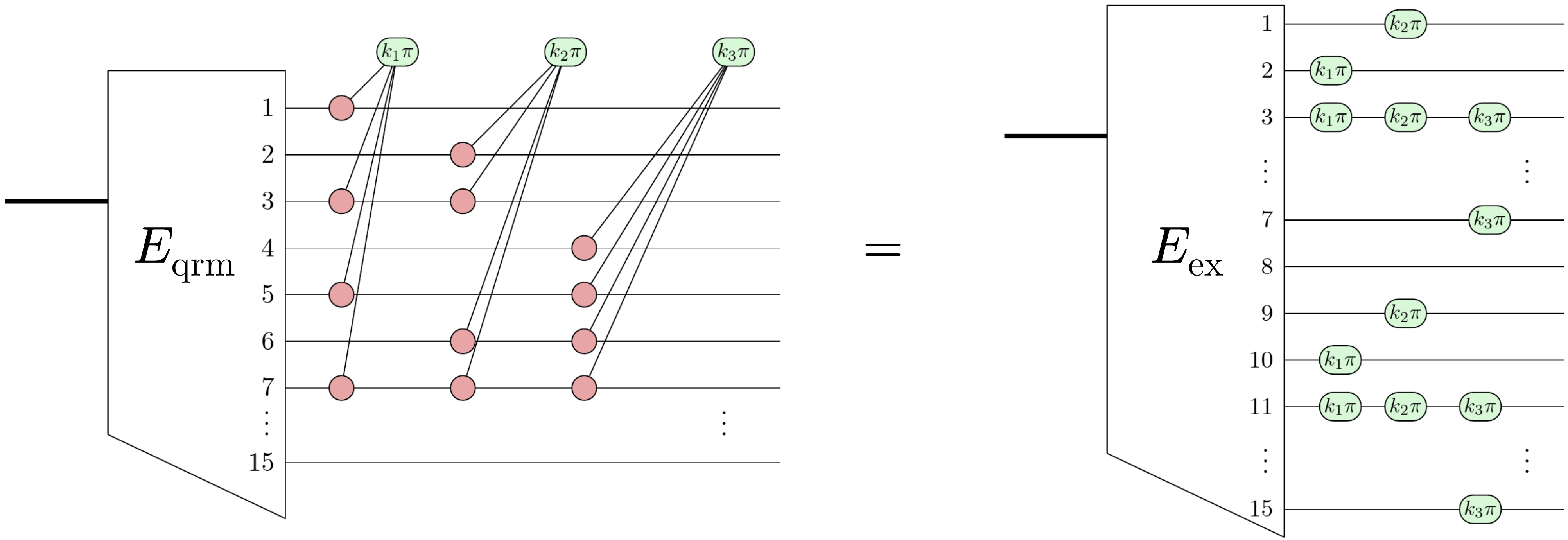
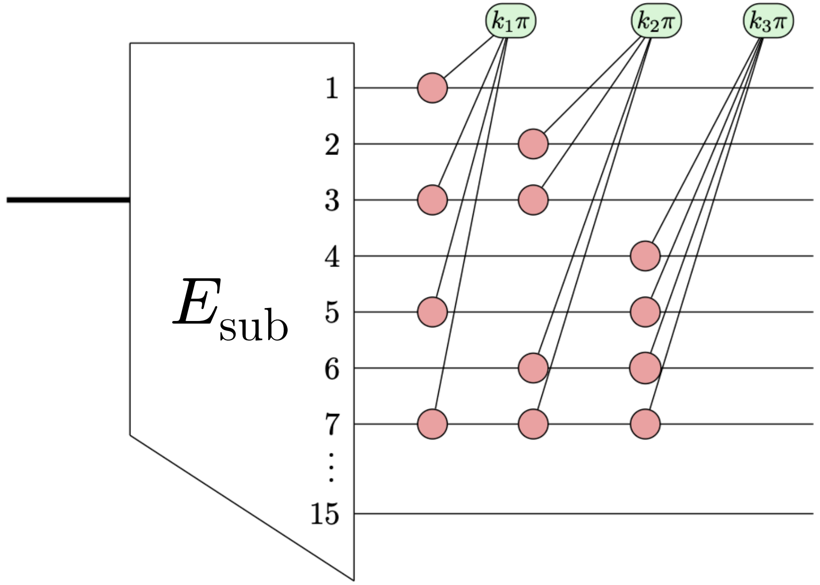


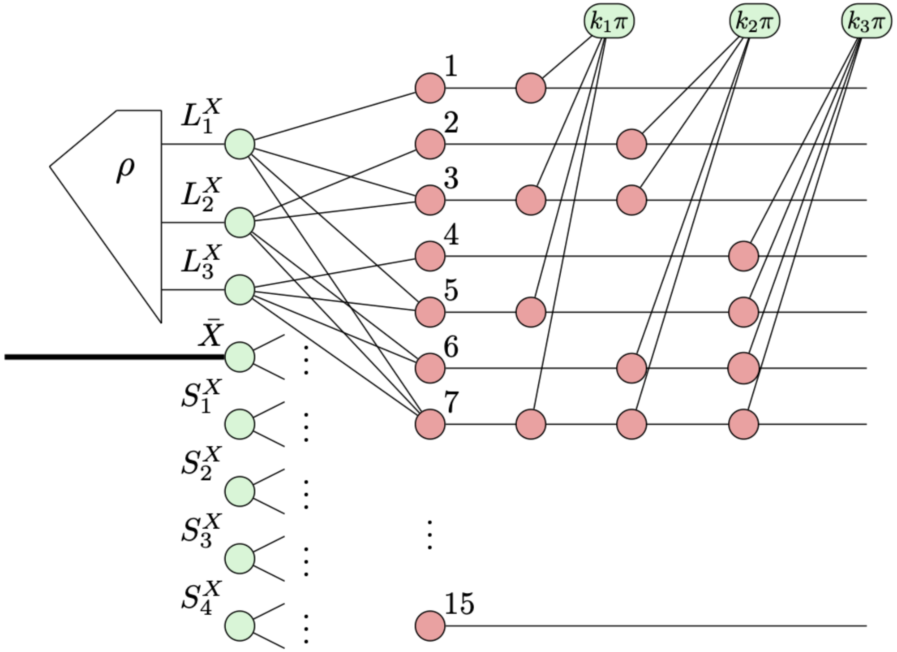
Gauge fixing (QRM → ExSteane)



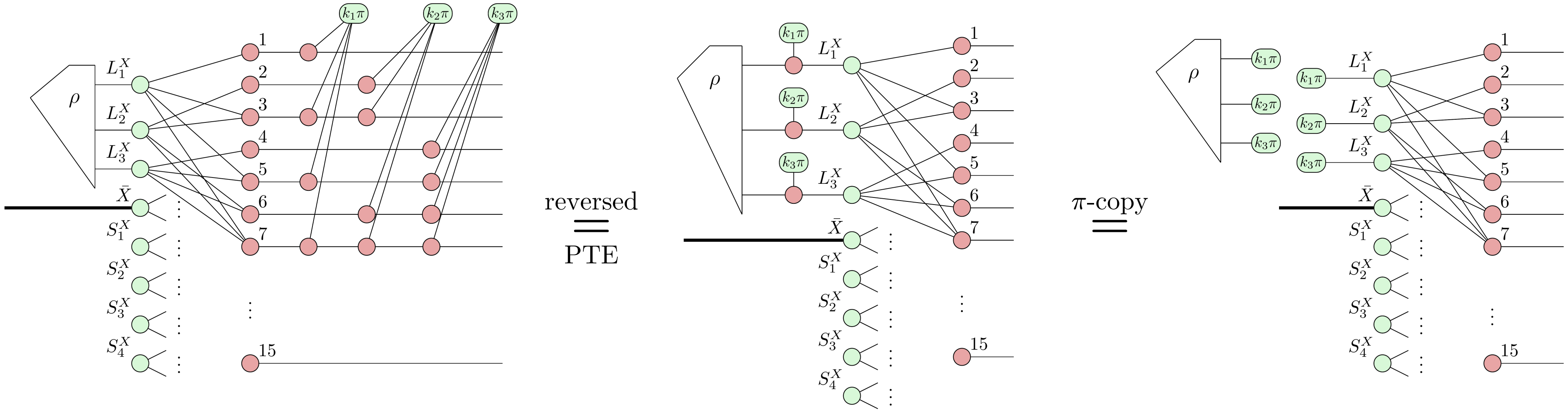
Gauge fixing (QRM $\to$ ExSteane)



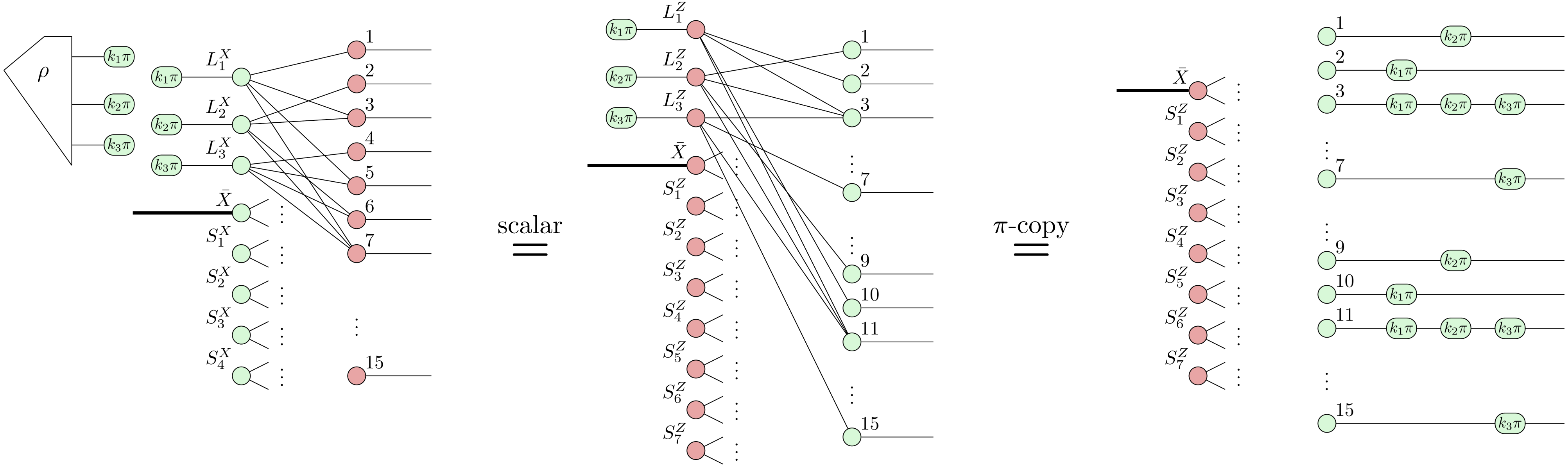
$\stackrel{\text{def}}{=}$



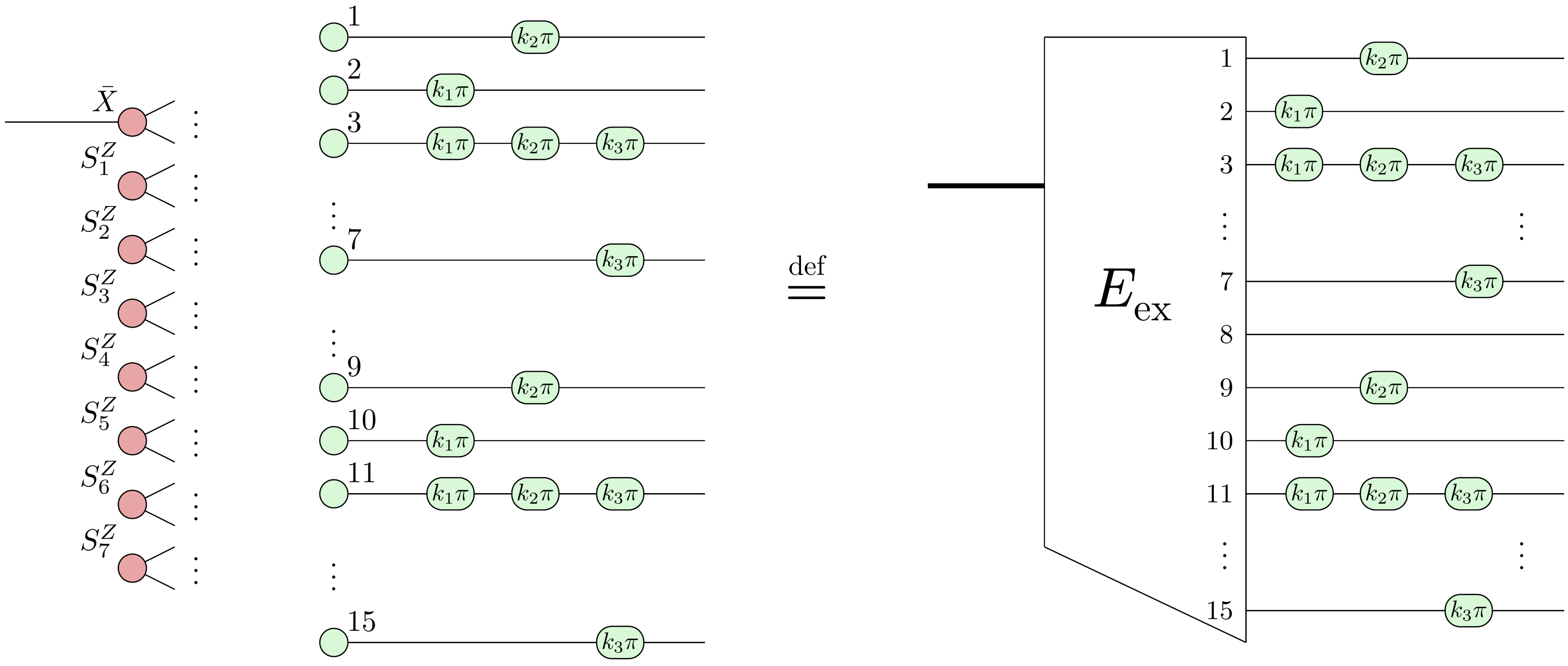
Gauge fixing (QRM → ExSteane)



Gauge fixing (QRM → ExSteane)



Gauge fixing (QRM → ExSteane)



Conclusion & outlook

- pushing through the encoder: discover / verify conditions for transversality.

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Conclusion & outlook

- pushing through the encoder: discover / verify conditions for transversality.
- code morphing: generic procedure for code search
- gauge fixing:
  - preservation of error-correcting properties during code transformation
  - code deformation

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Thank you.

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