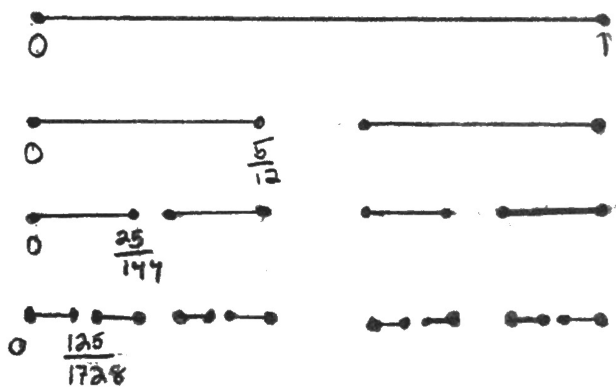
Chaotic Dynamics - Problem Set 3

Jack Lambert*
University of Colorado Boulder, CSCI 4446
Submitted 02/05/2019

Problem 1:

n	ϵ	$N(\epsilon)$
0	1	1
1	5/12	2
2	25/144	4
3	125/1728	8
N	$\left(\frac{5}{12}\right)^N$	2^N



$$d_{cap} = \lim_{\substack{\epsilon \to \infty \\ N \to \infty}} \frac{\log(N(\epsilon))}{\log(1/\epsilon)} = \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \frac{\log(2^N)}{\log(1/(5/12)^N)}$$

$$= \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \frac{\log(2^N)}{\log((12/5)^N)} = \lim_{\substack{\epsilon \to 0 \\ N \to \infty}} \frac{\cancel{N}\log(2)}{\cancel{N}\log(12/5)} = \frac{\log(2)}{\log(12/5)}$$

$$\boxed{= 0.792}$$

Figure 1. Capacity Dimension of a middle-sixth-removed Cantor set

Problem 2:

Part a.)

Starting with the following third-order ODE:

$$2x'''(t) - 3\tan\left(\frac{1}{2}x''(t)\right) + 16\log\left(x'(t)\right) - x(t) = 0 \tag{1}$$

*SID: 104414093

A transformation into 3 first-order ODE's can be done using the following substitution of variables:

$$x'(t) = x_1(t) \tag{2}$$

$$x_1'(t) = x_2(t) \tag{3}$$

$$x_2'(t) = \frac{3}{2}\tan\left(\frac{1}{2}x_2(t)\right) - 8\log\left(x_1(t)\right) - \frac{1}{2}x(t) \tag{4}$$

Part b.)

Starting with the set of first-order ODE's:

$$x' = y \tag{5}$$

$$y' = z \tag{6}$$

$$z' = yz + \log(y) \tag{7}$$

A transformation into a single third-order differential equation can be done using the following relations:

$$z = y' = x'' \;\; \rightarrow \;\; \dot{z} = x''' \tag{8}$$

$$x''' = x'x'' + logx' \;\; \rightarrow \;\; x''' - x'x'' - log\,(x') = 0 \tag{9}$$

Part c.)

The systems described by the equations in parts (a) and (b) are both nonlinear equations. In part (a) the equation has the nonlinear terms tan and log, while part (b) has the nonlinear terms consisting of $x'x''$ and log. These terms are nonlinear as they do not satisfy the condition that the functions can be written in the form $Lx = f$, where L is a linear operator. In the obvious cases of tan and log this is easy to see as these terms are obviously not linear operators, but the term $x'x''$ is less obvious.

Problem 3:

Part a.)

Creating a fractal tree with left and right branches emanating from the trunk at right angles scaling each branch by a scale factor of $s = 0.6$, gives the following plot after iterating 12 times (when successive segments could no longer be differentiated).

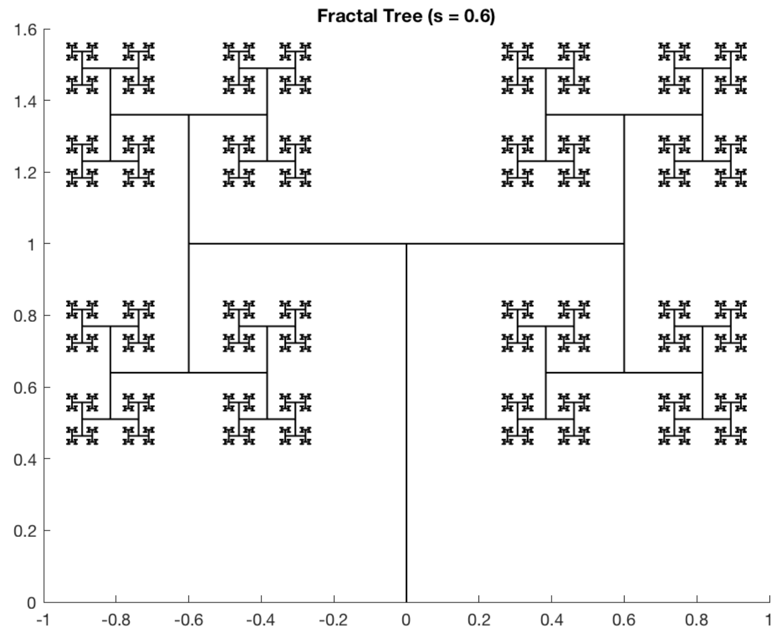


Figure 2. Fractal Tree ($s = 0.8$)

Part b.)

When the scale factor is decreased to exactly half, where each preceding line segment is half as long as the line segment before it, the tree becomes more sparse. By this I mean, the leaves on the tree become more concentrated near the first two branches from the trunk. As the scale factor is reduced even further towards zero, the leaves approach points around the first two branches where these branches are too approaching points. When the scale factor is increased, the tree expands and takes up a much larger portion of the plotting space. Since each line segment is larger as the scale factor is now larger, the naked eye can visualize more of the self similar branches, which are now expanded. This shows that the larger the scale factor - the easier it is for our eyes to witness each iteration of the new branches, where smaller scale factors shrink the image allowing the naked eye to see less branches.

Part c.)

Now applying different angles between the left and right branches (θ_l and θ_r) and different segment length ratios between the left and right branches (s_l and s_r) yields the following result:

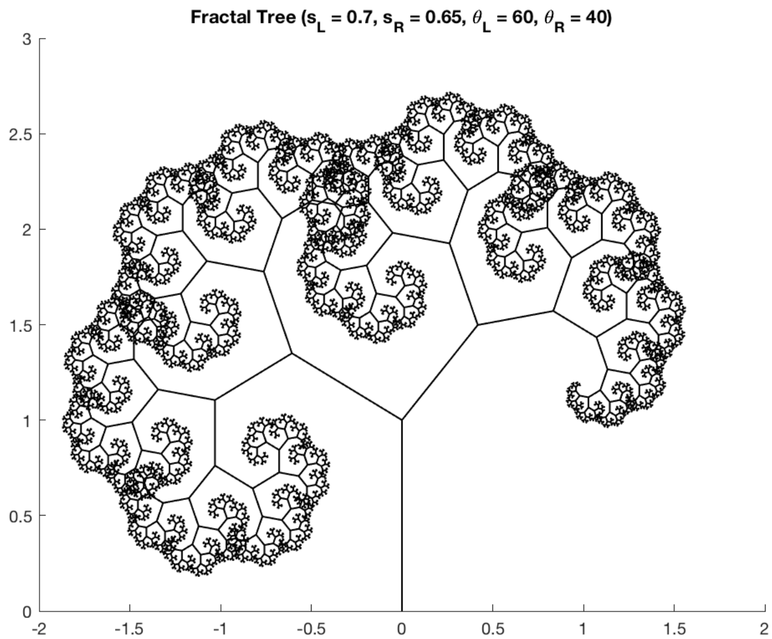


Figure 3. Fractal Tree ($\theta_l = 60°$, $\theta_r = 40°$, $s_l = 0.7$, $s_l = 0.7$)

Part d.)

Some interesting plots were also found imputing random angles and ratios and are plotted below: