Notes on grades for arbitrary distributive laws

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Consider a finitary strong monad $T : \mathbf{FinSet} \to \mathbf{Set}$. The category of elements of T, denoted $\mathrm{El}(T)$ are pairs of objects and computations over that object, $\langle n \in \mathbf{FinSet}, t \in T(n) \rangle$. Morphisms are those between the objects that preserve the computational structure:

$$\mathrm{El}(T)(\langle m, s \rangle, \langle n, t \rangle) = \{ f \in \mathbf{FinSet}(m, n) \mid T(f)(s) = t \}$$

This category is monoidal, with a monoidal product on the objects given by the Cartesian product and the left-monoidal action of T induced by the strength.

$$T(n) \times T(m) \xrightarrow{\theta} T(n \times T(m)) \xrightarrow{\widehat{\theta}^*} T(n \times m)$$

Now, consider a monoidal category $(\mathcal{D}, I, \otimes)$ and a strong monoidal functor $F : (\mathcal{D}, I, \otimes) \to (\mathbf{Set}, 1, \times)$. Then we can build a $\mathcal{D}$-graded label monad via the following construction.

$$L_d(x) = [F(d) \Rightarrow x]$$

The identity is given by

$$\eta : x \xrightarrow{\sim} [1 \Rightarrow x] \xrightarrow{\sim} [F(I) \Rightarrow x].$$

The multiplication is given by currying,

$$\mu_{d,d'}(x) : [F(d) \Rightarrow F(d') \Rightarrow x] \xrightarrow{\sim} [F(d) \times F(d') \Rightarrow x] \xrightarrow{\sim} [F(d \otimes d') \Rightarrow x].$$

There is a $\mathcal{D}$-graded distributive law, $\delta_d : TL_d \to L_d T$, between L and an arbitrary trivially graded strong monad T induced by the strength of T.

$$\delta_d(x) : T(F(d) \Rightarrow x) \to (F(d) \Rightarrow F(d) \times T(F(d) \Rightarrow x)) \to (F(d) \Rightarrow T(F(d) \times (F(d) \Rightarrow x))) \to (F(d) \Rightarrow T(x))$$

The first and last arrows are given by the unit and counit of the closed structure on $\mathbf{Set}$, respectively. The distributive law coherence diagrams follow from the coherence axioms for the strength of T with diagram chases.

1 Abstract Effects

We now look at the case where $\mathcal{D} = \mathrm{El}(S)$ for some finitary monad $S : \mathbf{FinSet} \to \mathbf{Set}$, and $F : \mathrm{El}(S) \to \mathbf{FinSet}$ is the left projection. The monad takes the form:

$$L_{\langle x, t \in S(x) \rangle}(y) = (x \Rightarrow y)$$

We interpret this as the following: objects of $\mathrm{El}(S)$ are some abstract object x along with a computation over that object $t \in S(x)$. Then, regradings $g : \langle x', t' \rangle \to \langle x, t \rangle$ are abstract relabellings $x' \to x$ that preserve the computational structure of t. The reader aspect of the graded monad specifies how to instantiate the abstract object, which we can use to 'push forward' our abstract effect into a real effect at any time.

$$\begin{aligned} \kappa : L_{\langle x,t\rangle}(y) &\to S(y) \\ f &\mapsto T(f)(t) \end{aligned}$$

In fact, this recovery of the monad is the left Kan extension along the unique functor $\mathrm{El}(S) \to 1$. This method is a well-known technique for extracting a classical monad from a graded one [1, 2]. The proof is an instance of the density of representables theorem. Note that this Kan extension does not necessarily produce a graded monad morphism between the functors L and S (e.g. see [3]) but it will always give S as a functor.

$$\begin{tikzcd} \mathrm{El}(S) \arrow[r, "!"] \arrow[dr, "L_{(-)}"'] & 1 \arrow[d, "S"] \\ & {[\mathbf{FinSet}, \mathbf{Set}]} \end{tikzcd}$$

The takeaway point is that we have shown there is a distributive law between L and an arbitrary trivially graded strong monad T. Hence, this gives a way to define a distributive law between any two monads. The price to be paid is that one of the monads has to be 'abstract' with the computation traced via a grading.

2 Interactions of Effects

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References

[1] Fritz, T., and Perrone, P. A criterion for Kan extensions of lax monoidal functors. arxiv:1809.10481, 2018.

[2] Fritz, T., and Perrone, P. A probability monad as the colimit of spaces of finite samples. *Theory and Applications of Categories 34* (2019).

[3] Liell-Cock, J., and Staton, S. Compositional Imprecise Probability: A Solution from Graded Monads and Markov Categories. *Proc. ACM Program. Lang. 9*, POPL (Jan. 2025), 54:1596–54:1626.