Notes on a convex monad transformer

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Consider a finitary strong monad T on **Set**. Denote its Kleisli category $\mathrm{Kl}(T)$. We say the Kleisli map $f : m \to T(n)$ is surjective if the induced map $f^* : T(m) \to T(n)$ is surjective. Define $\mathrm{Kl}(T)_{\mathrm{Surj}}$ as the wide subcategory of $\mathrm{Kl}(T)$ with only surjective maps.

Both $\mathrm{Kl}(T)$ and $\mathrm{Kl}(T)_{\mathrm{Surj}}$ are monoidal with action on objects given by the Cartesian project and action on morphisms given by the monoidal action of the monad T induced by the left-strength.

1 A labelling monad transformer

Definition 1. The *label transformer*, L, is a $\mathrm{Kl}(T)_{\mathrm{Surj}}$-graded monad transformer defined by

$$L_m T(n) = m \Rightarrow T(n).$$

The unit and multiplication are given via abstraction by 1 and currying/strength respectively.

$$\begin{aligned}
\eta &: n \xrightarrow{\eta_T} T(n) \cong 1 \Rightarrow T(n) \\
\mu_{m,m'} &: [m \Rightarrow T(m' \Rightarrow T(n))] \to [m \Rightarrow m' \Rightarrow (m' \times T(m' \Rightarrow T(n)))] \to [m \Rightarrow m' \Rightarrow T(m' \times m' \Rightarrow T(n))] \\
&\to [m \times m' \Rightarrow T(T(n))] \to [m \times m' \Rightarrow T(n)]
\end{aligned}$$

Regrading is given by Kleisli composition. For $g : m' \to T(m)$,

$$g^\star T(n) : [m \Rightarrow T(n)] \to [m' \Rightarrow T(n)].$$

2 An affine monad transformer

Definition 2. A subset $S \subseteq T(n)$ is *convex* if it is closed under substitutions: for all $m \in \mathbf{Set}$, $t \in T(m)$ and $f : m \to S$, then $t \gg\!=_T f \in S$.

A convex subset $S \subseteq T(n)$ is *finitely generated* if there is an $m \in \mathbb{N}$ and $f : m \to S$ such that for any $s \in S$, there is a $t \in T(m)$ making $s = t \gg\!=_T f$.

We write $\overline{P}T(n)$ for the finitely generated convex subsets of $PT(n)$:

$$\overline{P}T(n) = \{S \subseteq T(n) \mid \forall m \in \mathbb{N}, t \in T(m), f : m \to S.\ t \gg\!=_T f \in S\}$$

It has a monad structure when T is affine. The unit is given by

$$\eta_n^{\overline{P}T}(i) = \{\eta_n^T(i)\}.$$

Note, this is only a singleton if and only if T is affine. The monad laws depend on this set being singular.

The bind is given by

$$t \gg=_{\overline{P}T} f = \{ s \gg=_T g \mid s \in t, g \in \kappa(f) \} ,$$

where $\kappa : PX^Y \to P(X^Y)$ is defined by $\kappa(f) = \{g : X \to Y \mid \forall x \in X.\ g(x) \in f(x)\}$.

There is moreover an ordering given by subset, and the join is a convex closure of the union:

$$S \oplus S' := \{ t \gg=_T f \mid n \in \mathbb{N}, t \in T(n), f : n \to S \cup S' \}$$

Proposition 3. *There is a family of functions $\phi_{m,n} : L_m T(n) \to \overline{P}T(n)$, that takes $f \in L_m T(n)$ to its substitutive closure,*

$$\phi_{m,n}(f) = \{ t \gg=_T f \mid t \in T(m) \} ,$$

and the family is natural in $m \in \mathrm{Kl}(T)^{\mathrm{op}}_{\mathrm{Surj}}$ and $n \in \mathbf{Set}$.

Proof. For naturality in m, suppose $g \in \mathrm{Kl}(T)^{\mathrm{op}}_{\mathrm{Surj}}(m', m)$. Then naturality in m amounts to the fact that the substitutive close of $f \circ g$ is the same as the substitutive closure of f, which is true since g is surjective. For naturality in n, suppose $h \in \mathbf{Set}(n, n')$. Then naturality amounts to the fact that... □

Definition 4. A function $f : m \to T(n)$ is extremal if for $i \in m$ and $t \in T(m)$, $t \gg=_T f = f(i)$ only if there is no such $t' \in T(m \setminus \{i\})$ with $t = T(\iota)(t')$, where $\iota : m \setminus \{i\} \to m$ is the inclusion morphism. In other words, no elements in the range of f are a convex combination of the others.

Lemma 5. *Each convex subset $S \subseteq T(n)$ is generated by an extremal function, and this extremal function is unique up to isomorphisms of the domain.*

Proof. Let $S \subseteq T(n)$ be a convex and finitely generated. So there is some $f : m \to T(n)$ such that

$$S = \{ t \gg= f \mid t \in T(m) \}$$

for finite m. If f is not extremal, then there is some $i \in m$ and $t' \in T(m \setminus \{i\})$ with $t = T(\iota)(t')$. So, for any $s = t \gg= f$, we have

$$s \;=\; T(\iota)(t') \gg= f \;=\; t' \gg= f \circ \iota$$

meaning S is generated by $f \circ \iota : m \setminus \{i\} \to T(n)$. We may continue this process until the generating function is extremal, and it is guaranteed to terminate because m is finite.

Next, we assume $f : m \to T(n)$ and $f' : m' \to T(n)$ are extremal generating functions for S. Then for $i \in m$, we must have $f(i) = t \gg= f = t' \gg= f'$ for some $t \in T(m)$ and $t' \in T(m')$. As f is extremal, $t \gg= f$ is not a convex combination of the remaining elements of S. So there is some $i' \in m'$ such that $f'(i') = f(i)$. Repeating for each $i \in m$ gives a function $p : m \to m'$ with $f = f' \circ p$. The converse gives a function $p' : m' \to m$ with $f' = f \circ p'$. As extremal functions are injective, p and p' must form an isomorphism. □

2.1 Connection to Kan Extensions

Fritz and Perrone [1, 2] propose a method to extract a canonical monad from a graded monad, by taking the left Kan extension. The Kan extension of our graded monad $L_a T$ gives the finitely-generated convex powerset $\overline{P}T$ as a functor, not as a monad.

Proposition 6. *The family $\phi_{m,n} : L_m T(N) \to \overline{P}T(n)$ exhibits $\overline{P}T : \mathbf{FinSet} \to \mathbf{Set}$ as the Kan extension of*

$$L_{(-)}T(=) : \mathrm{Kl}(T)^{\mathrm{op}}_{\mathrm{Surj}} \to [\mathbf{FinSet}, \mathbf{Set}]$$

along the unique functor $! : \mathrm{Kl}(T)^{\mathrm{op}}_{\mathrm{Surj}} \to 1$.

$$\begin{tikzcd}
\mathrm{Kl}(T)^{\mathrm{op}}_{\mathrm{Surj}} \arrow[r, "!"] \arrow[dr, "L_{(-)}T(=)"'] & 1 \arrow[d, "\overline{P}T"] \\
& {[\mathbf{FinSet}, \mathbf{Set}]}
\end{tikzcd}$$

Proof. Kan extensions in [**FinSet**, **Set**] can be computed pointwise, and for any $n \in \mathbf{FinSet}$ the Kan extension of $L_{(-)}T(n) : \mathrm{Kl}(T)^{\mathrm{op}}_{\mathrm{Surj}} \to \mathbf{Set}$ along $\mathrm{Kl}(T)^{\mathrm{op}}_{\mathrm{Surj}} \to 1$ is simply the colimit of the functor. Thus it suffices to show that the canonical function

$$\Phi : \operatorname*{colim}_{m \in \mathrm{Kl}(T)^{\mathrm{op}}_{\mathrm{Surj}}} L_m T(n) \to \overline{P}T(n)$$

(induced by ϕ) is a bijection.

This function Φ is given by $\Phi[m, f \in \text{Ł}_m T(n)] = \phi_{m,n}(f)$. It is trivially surjective because any convex subset of $T(n)$ is by definition generated by a function f of this form.

To see that it is injective we suppose that $\phi_{m,n}(f) = \phi_{m',n}(f')$, for $f \in L_m T(n)$ and $f' \in L_{m'}T(n)$. We must show that $[m, f] = [m', f']$ in the colimit. It suffices to find m'' with $h \in L_{m''}T(n)$ and surjections $g \in \mathrm{Kl}(T)_{\mathrm{Surj}}(m, m'')$ and $g' \in \mathrm{Kl}(T)_{\mathrm{Surj}}(m', m'')$ such that the following diagram commutes:

$$\begin{tikzcd}
m \arrow[dr, "g"] \arrow[ddr, bend right, "f"'] & & m' \arrow[dl, "g'"'] \arrow[ddl, bend left, "f'"] \\
& m'' \arrow[d, "h"'] & \\
& n &
\end{tikzcd}$$

The finitely generated convex set $\phi_{m,n}(f) = \phi_{m',n}(f')$ must have a unique extremal generator $h : m'' \to T(n)$ (Lemma 5). We construct g by noting that $f(i)$ must be a convex combination from the m'' extremal points, so we let $g(i)$ be the corresponding substitutive term. We construct g' from f' similarly. To see that g is surjective we note that since f is surjective onto its image we must have points in m that map onto the extremal points, and hence onto all the points of m'' via g. Similarly, g' is surjective. ☐

2.2 An Op-lax Functor and Tighter Uncertainty Bounds

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3 Examples

- The construction given in [4] follows the case for when T is the finite distributions monad D. It gives an account of Bernoulli uncertainty with Knightian uncertainty. The convex transformer produces the well-known convex powerset of distributions monad, $\overline{P}D = CP$.
- When T is the powerset monad P, this gives an account of the covariant powerset functor distributing with itself. The convex powerset of powerset functors $\overline{P}P$ appears as a result of a weak distributive law of P over P in [3], and is referred to as the *monad of upclosed sets of subsets*.
- When T is the reader monad R over some set r, the above gives an account of *imprecise possibility*. I am not aware of this previously appearing in literature.
- It would be interesting to investigate what this construction looks like for D_S or M_S – the monads of distributions or multisets over some semiring S.
- Can we generalise this construction beyond **Set** to say ω**Cpo** to talk about recursion in an imprecise setting.

References

[1] Fritz, T., and Perrone, P. A criterion for Kan extensions of lax monoidal functors. arxiv:1809.10481, 2018.

[2] Fritz, T., and Perrone, P. A probability monad as the colimit of spaces of finite samples. *Theory and Applications of Categories 34* (2019).

[3] Goy, A. *On the compositionality of monads via weak distributive laws.* PhD thesis, Université Paris-Saclay, Gif-sur-Yvette, France, Oct. 2021.

[4] Liell-Cock, J., and Staton, S. Compositional Imprecise Probability: A Solution from Graded Monads and Markov Categories. *Proc. ACM Program. Lang. 9*, POPL (Jan. 2025), 54:1596–54:1626.