Compositional Imprecise Probability

Jack Liell-Cock [1]   Sam Staton [1]

[1] University of Oxford

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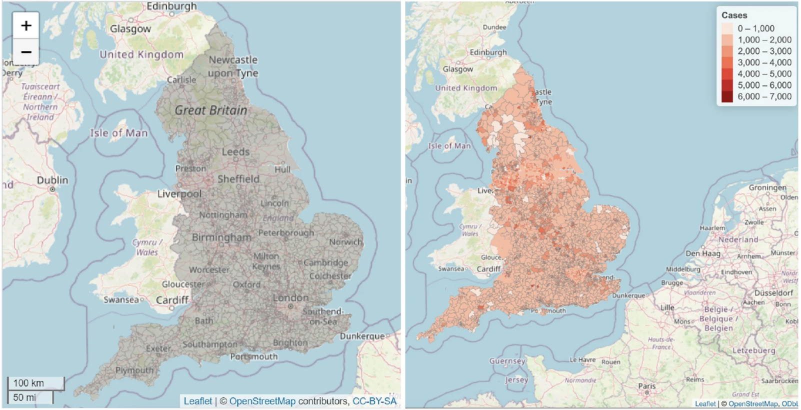
Context

- Probabilistic programming for modelling
- Programs model phenomena allowing automatic inference
- Successes in areas like UK covid predictions



Overview

This work:

- Design a fully compositional probabilistic programming language with Bernoulli and Knightian uncertainty
- Using graded monads and named Knightian choices
- **Theorem 1:** This gives a refined bound on uncertainty
- **Theorem 2:** It is maximal among compositional accounts

Overview

This work:

- Design a fully compositional probabilistic programming language with Bernoulli and Knightian uncertainty
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- **Theorem 1:** This gives a refined bound on uncertainty
- **Theorem 2:** It is maximal among compositional accounts

Not this work:

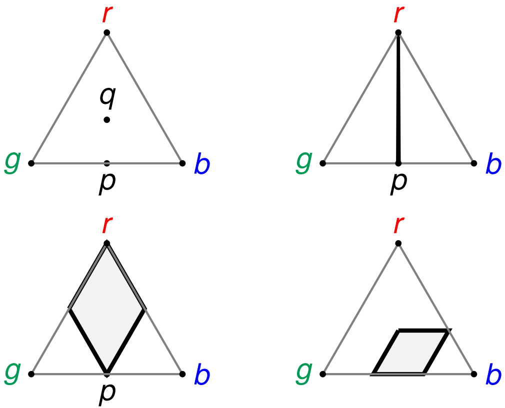
- Wider interest in probability with non-determinism [Dash and Staton 2020; Dahlqvist et al. 2018; Keimel et al. 2017; Dash 2024; Kozen et al. 2023; Varacca et al. 2006; Jacobs 2021]
- Our focus is in the setting of imprecise probability

Contents

Outline

Imprecise Probability

- **Probability** = point in simplex
- **Imprecise probability** = convex set of points



Applications of Imprecise Probability

Wider applicataions:

- Statistical robustness [Walley 1991; Huber 1981],
- Economics [Amarante et al. 2007; Battigalli, Cerreia-Vioglio, et al. 2015; Battigalli, Corrao, et al. 2019; Cerreia-Vioglio et al. 2023; Vicig 2008; Pavlidis et al. 2011]
- Engineering [Ferson et al. 2003; Aslett et al. 2022; Vasile 2021; Krpelík et al. 2021]

In machine learning:

- Bayesian learning [Caprio et al. 2024]
- Reinforcement learning [Oren et al. 2024; Zanger et al. 2023]
- Conformal prediction [Stutz et al. 2023; Javanmardi et al. 2024]
- Foundations of safe AI [Dalrymple 2024; Dalrymple et al. 2024]

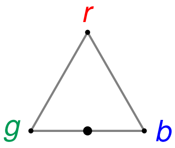
A First Language for Imprecise Probability

Our prototype language for imprecise probability will have:

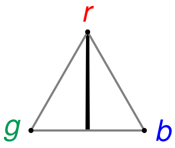
- If-then-else statements;
  - if b then t else u
- Let bindings;
  - $x \leftarrow t \; ; \; u$
- Two commands returning booleans:
  - ***bernoulli***: a fair Bernoulli choice;
  - ***knight***: a Knightian choice.

Examples with the Language

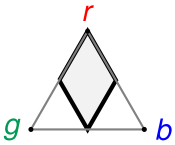
$z \leftarrow$ ***bernoulli*** ;
if z then g else b



$x \leftarrow$ ***knight*** ; $z \leftarrow$ ***bernoulli*** ;
if z then (if x then r else g)
else (if x then r else b)



$x \leftarrow$ ***knight*** ; $y \leftarrow$ ***knight*** ; $z \leftarrow$ ***bernoulli*** ;
if z then (if x then r else g)
else (if y then r else b)



Outline

Desiderata of the Language

Desideratum (1)

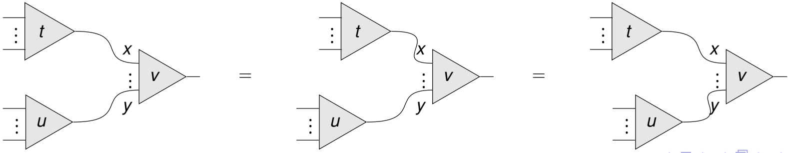
The language should be commutative:

$$x \leftarrow t \ ; \ y \leftarrow u \ ; \ v \quad = \quad y \leftarrow u \ ; \ x \leftarrow t \ ; \ v$$

for $x \notin \mathrm{free}(u)$ *and* $y \notin \mathrm{free}(t)$*; and affine:*

$$x \leftarrow t \ ; \ u \quad = \quad u$$

for $x \notin \mathrm{free}(u)$.



Desideratum (2)

Standard equational reasoning about if-then-else should apply:

$$\text{if } \; b \; \text{then } (x \leftarrow t \; ; \; u) \; \text{else } (x \leftarrow t \; ; \; v)$$

$$=$$

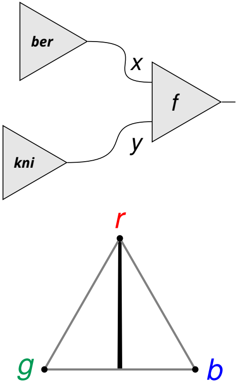
$$x \leftarrow t \; ; \; \text{if } \; b \; \text{then } u \; \text{else } v$$

for $x \notin \mathrm{free}(b)$.

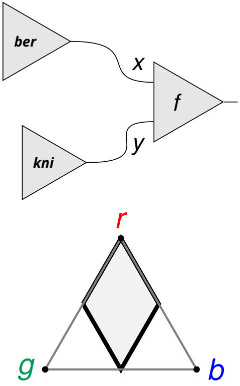
Outline

The Problem: Convex Powerset doesn't work

bernoulli interpreted as $\left\{ \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \right\}$



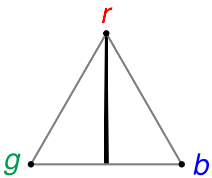
knight interpreted as $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

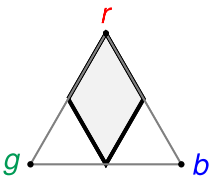


$$f(x,y) = \text{if } x \text{ then } (\text{if } y \text{ then } r \text{ else } g) \text{ else } (\text{if } y \text{ then } r \text{ else } b)$$

The Problem: Nothing Works

Theorem: Any semantic model that satisfies our desiderata cannot distinguish the following convex sets of distributions.



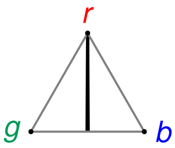


$z \leftarrow$ ***bernoulli*** ;
if z then (if $x \leftarrow$ ***knight*** ; x then *r* else *g*)
else (if $x \leftarrow$ ***knight*** ; x then *r* else *b*)

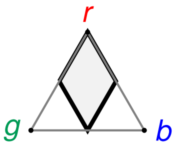
Outline

The Solution: Named Knightian choices

$z \leftarrow$ ***bernoulli*** ;
if z then (if $x \leftarrow$ ***knight***(a_1) ; x then return r else return g)
else (if $x \leftarrow$ ***knight***(a_1) ; x then return r else return b)



$z \leftarrow$ ***bernoulli*** ;
if z then (if $x \leftarrow$ ***knight***(a_1) ; x then return r else return g)
else (if $x \leftarrow$ ***knight***(a_2) ; x then return r else return b)



The Solution: Named Knightian choices

- Reader monad transformer of finite distributions monad

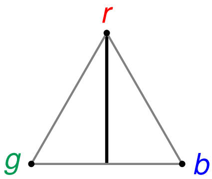
$$T_{2^A}(X) = [2^A \Rightarrow D(X)]$$

- **Bernoulli** choices given by *distributions*
- **Knightian** choices given by *reading*
- We generalise the Knightian choices 2^A to arbitrary sets B

$$T_B(X) = [B \Rightarrow D(X)]$$

- Convex powerset recovered by taking convex hull of the image

$$\begin{array}{r} r: \\ g: \\ b: \end{array}\begin{pmatrix} 1 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{pmatrix} \longrightarrow$$



A Graded Monad for Imprecise Probability

Definition

T is the $\mathbf{FinStoch}_{\mathrm{Surj}}$-graded version of:

$$T_A(X) = [A \Rightarrow D(X)]$$

T supports finite probability and finite non-determinism:

- $\boldsymbol{bernoulli} \in T_{2^{\emptyset}}(2)$ given by $\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$
- $\boldsymbol{knight}(a) \in T_{2^{\{a\}}}(2)$ given by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

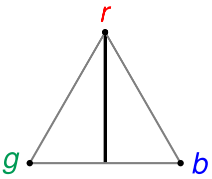
Function and monoidal composition use independent non-deterministic branches by the graded multiplication:

$$\mu_{A,B} : T_A \circ T_B \to T_{A \times B}$$

Outline

Theorem 1: Improved Uncertainty Bounds

$$\begin{array}{c} r: \\ g: \\ b: \end{array} \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{pmatrix} \longrightarrow$$



- There is a mapping $T_A(X) \to \mathrm{CP}(X)$ by taking the convex hull of the image
- This induces an "op-lax functor":

$$R : [X \Rightarrow T_A(Y)] \to [X \Rightarrow \mathrm{CP}(Y)]$$

- That is, composition in our framework gives tighter bounds on the *Knightian uncertainty* than composition with CP

$$R(g \circ f) \subseteq R(g) \circ R(f)$$

Theorem 2: Maximality of the Theory

- The mapping into convex powersets is not injective
- There is ”more stuff” in our theory than in convex powersets
- So can we quotient our theory any further?
- Not without:
  - Breaking connection with convex subsets (and imprecise probability); or
  - Compromising the compositional structure

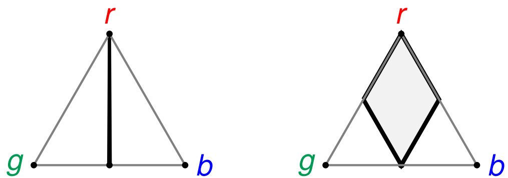
Conclusion

This work:

- Using graded monads and named Knightian choices we give a fully compositional account of Bernoulli and Knightian uncertainty together
- **Theorem 1:** This gives a refined bound on uncertainty
- **Theorem 2:** It is maximal among compositional accounts

Future work:

- Iteration and infinite dimensional structures
- Function spaces via quasi-Borel spaces
- Implementation and approximation of bounds



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