Assume a non-empty convex set $C = \{x \mid Mx + c \geq 0\}$

Prove that for a point x either:

(i) x lies in C, i.e. $Mx + C \geq 0$

or

(ii) There exists a hyperplane h such that for all $y \in C$, $h(y) \geq 0$ i.e. $h(C) \geq 0$ and $h(x) < 0$

Proving (i → ii): If $Mx + C \geq 0$ then there is no hyperplane h such that $h(C) \geq 0$ and $h(x) < 0$.

Proof by contradiction: Assume $Mx + C \geq 0$ and there does exist a hyperplane h such that $h(C) \geq 0$ and $h(x) < 0$.

Contradiction: $Mx + C \geq 0$ means that $x \in C$, so both $h(x) \geq 0$ and $h(x) < 1$ are impossible.

Proving (ii → i): If there exists a hyperplane h such that $h(C) \geq 0$ and $h(x) < 0$, then $Mx + C < 0$.

Proof by Contradiction: Assume existance of such a hyperplane and $Mx + C \geq 0$.

Contradiction: For all $y \in C$, $h(y) \geq 0$. For $Mx + C < 0$ to be true, $h(x) \geq 0$, but $h(x) < 0$.